Local version of the Riemann-Roch-Grothendieck theorem

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Let's recall Characteristic classes and Riemann-Roch-Grothendieck (RRG) theorem.

Let X be a compact complex manifold. (Most of the time, we won't need X to be compact), let E be a holomorphic vector bundle over X.

We would like to define $\mathrm{ch}(E) \in H^{\bullet}_{\mathrm{DR}}(X)$ and $\mathrm{Td}(E)$.

Let L be a line bundle. Associated to it is the first Chern class $c_1(L) \in H^2(X)$. This is compatible with pullbacks.

When $E = \bigoplus L_i$, define $\mathrm{Ch}(E) = \sum \mathrm{Ch}(L_i) = \sum e^{c_1(L_i)} \in H^{2\bullet}(X)$, where the exponential function is defined by the usual formal power series.

The Todd class $\mathrm{Td}(E)$ is defined as $\prod_{i=1}^{n} \frac{c_1(L_i)}{1-e^{-c(L_i)}}$

Given a short exact sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0,$$

we have

$$\mathrm{Ch}(E_1) = \mathrm{Ch}(E_2) + \mathrm{Ch}(E_0)$$

and

$$\mathrm{Td}(E_1) = \mathrm{Td}(E_0) \cdot \mathrm{Ch}(E_2)$$

Grauert's theorem

- Given a proper holomorphic map $f : X \to Y$ of compact manifolds and a vector bundle E/X, the sheaves $R^{\bullet}\pi_* E$[1] are coherent sheaves over Y.
- If X is a projective manifold, and $\mathcal{E}$ is a coherent sheaf on X, then there is a global resolution

$$0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to \mathcal{E} \to 0$$

  by vector bundles.

In particular, we can define the Chern character $\mathrm{Ch}(E) = \sum_i \mathrm{Ch}(E_i)(-1)^i$. This definition is not dependent on the choice of global resolution and it is a well defined class in $H^{\bullet}(X)$.

Rrg theorem

Let $f : X \to Y$ be a holomorphic and smooth map of smooth projective manifolds and let E/X be a vector bundle. Then (in $H^\bullet(Y)$), we have

$$\sum_{i=1}^{n} (-1)^i \operatorname{Ch}(R^i f_* E) \cdot \operatorname{Td}(TY) = f_*(\operatorname{Ch}(E) \cdot \operatorname{Td}(TX))$$

[1-1]

Example

For example, let $i : D \to Y$ be a smooth divisor in Y. Let us apply the RRG theorem for $X = D$, $E = \mathcal{O}_D$, and $f = i$. Now i is a finite map, so $R^j \pi_* \mathcal{O}_Y = \begin{cases} 0 & \text{if } g \geq 1 \\ i_* \mathcal{O}_Y & \text{otherwise} \end{cases}$

The RRG theorem says that

$$\operatorname{ch}(i_* \mathcal{O}_D) \cdot \operatorname{Td}(TY) = i_*(\operatorname{Td}(TD)).$$

Let us verify this by hand. We have the short exact sequence

$$0 \to \mathcal{O}(-D) \xrightarrow{s_D(\text{multiplication by the canonical section})} \mathcal{O}_Y \to i_* \mathcal{O}_D \to \operatorname{Res}.$$

We have $\operatorname{Ch}(i_* \mathcal{O}_D) = 1 - e^{-c_1(\mathcal{O}(D))}$. We also have

$$c_1(D) \cdot \frac{1 - e^{-c_1(\mathcal{O}(D))}}{c_1(D)} \cdot \operatorname{Td}(TY) = i_*(\operatorname{Td}(TD)).$$

Moreover, the LHS above equals

$$i_* \left(\frac{1 - e^{c_1(D)|_D}}{c_1(D)|_D} \cdot \operatorname{Td}(TY|_D) \right)$$

Writing $TY|_D = TD \oplus N$ and $N = c_1(D)|_D$, we have that

$$\operatorname{Td}(TY|_D) = \operatorname{Td}(TY) \cdot \operatorname{Td}(N)$$

Example

As for another example, let Y be a point. The theorem says that

$$\sum (-1)^i \dim H^i(X, E) = \int \operatorname{Ch}(E) \operatorname{Td}(TX))$$

which is the Riemann-Roch-Hirzebruch formula.

Example

Degree 1 piece or RRG. Knudsen and Mumford associated to a general coherent sheaf $\mathcal{E}$ on X the determinant bundle $\det \mathcal{E}$, which is a line bundle on X.

When $\mathcal{E} = E$ is a vector bundle, $\det E = \wedge^{\text{top}} E$.

We consider $\lambda(E) = \otimes(\det R^i f_* E)^{(-1)^i}$. One thing that is true is that $\mathrm{Ch}_1(\mathcal{E}) = c_1(\det \mathcal{E})$. The RRG theorem can be re-written as

$$c_1(\lambda(E)) \cdot \mathrm{Td}(TY) = \pi_*(\mathrm{Ch}(E) \cdot \mathrm{Td}(X)) \tag{1}$$

We will refine the equation (1) under an assumption --- that f is a submersion.

Chern-Weil theory

"Pass characteristic classes to differential forms" representing them in DeRham cohomology $H^\bullet(X)$.

Given a complex vector bundle $E \to X$, let $\nabla^E : \mathcal{C}^\infty(X, E) \to \Omega^1(X, E)$ be an arbitrary connection, where $\Omega^1(X, E)$ denotes the 1-differential forms with values in E. The connection ∇^E can be extended in a unique way --- as

$$\nabla^E : \Omega^i(X, E) \to \Omega^{i+1}(X, E)$$

--- in such a way that it respects the Leibniz rule.

Now we can define curvature:

$$R^E = (\nabla^E)^2$$

✎ Proposition

$R^E \in \Omega^2(X, \mathrm{End}(E))$

Proof: Let $f \in \mathcal{C}^\infty(X)$. Verify that $[R^E, f] = 0$[2]:

$$[(\nabla^E)^2, f] = [\nabla^E, [\nabla^E, f]] = ddf = 0$$

$\square$

Now let $ f$ be a formal analytic function (there are no convergence conditions). The series $f(R^E)$ is an element of $\Omega^{2\bullet}(X, \mathrm{End}(E))$. Moreover, $\mathrm{Tr}^E[f(R^E)] \in \Omega^{2\bullet}(E)$

✎ Proposition

- $\mathrm{Tr}[f(\Omega^E)]$ is closed
- The DeRham cohomology class is independent of the choice of ∇^E.

Lemma

For $\alpha \in \Omega^\bullet(X, \mathrm{End}(E))$, $d\,\mathrm{Tr}^E[\alpha] = \mathrm{Tr}^E[[\nabla^E, \alpha]]$

Now we have $d\,\mathrm{Tr}[f(R^E)] = \mathrm{Tr}[[\nabla^E, f(R^E)]] = 0$.

Thus, $\mathrm{Tr}[f(\Omega^E)]$ is closed.

To show that the DeRham cohomology class is independent of the choice of ∇^E, we show that

$$\frac{d}{dt}\mathrm{Tr}[f(r_t^E)] = \mathrm{Tr}[(\frac{d}{dt}\nabla_t^2)f(R_t^E)].$$

The RHS equals

$$\mathrm{Tr}[[\frac{d}{dt}\nabla_t\nabla_t]f(\nabla_t^2)] = \mathrm{Tr}[[\Delta_t, \frac{d}{dt}\nabla_t f(\nabla_t^2)]] = d\,\mathrm{Tr}[\frac{d\nabla}{dt} + f'(\nabla_t^2)].$$

...

□

Now let $E \to X$ be a holomorphic vector bundle over a complex manifold. Let h^E be a Hermitian metric on E. There exists a unique connection ∇^E such that $\nabla^{E,0} = \delta$ and ∇^E is Hermitian with respect to h^E. From now on, write

$$\mathrm{ch}(E, h^E) = \mathrm{ch}(E, \nabla^E)$$

Main question

Let $\pi : X \to B$ be a holomorphic submersion. Also assume that X and B are Kahler. Let $E \to X$ be a vector bundle. Let h^E be a Hermitian metric on E. Let g^{TX}/B be a Kahler metric on the fibers of π. In other words, $TX/B = TX/\pi^*TB$.

Can we find a Hermitian metric $\|\cdot\|$ on X such that

$$\lambda(E) = \otimes(\det R^i\pi_*E)$$

satisfies

$$\mathrm{c}_1(\lambda(E), \|\cdot\|) = \pi_*(\mathrm{Td}(TX/B, g^{\mathrm{TX}/B})\,\mathrm{Ch}(E, h^E))$$

holds pointwise?

Answer

In relative dimension 1, Quillen
...Bismut-Gillet-Soule

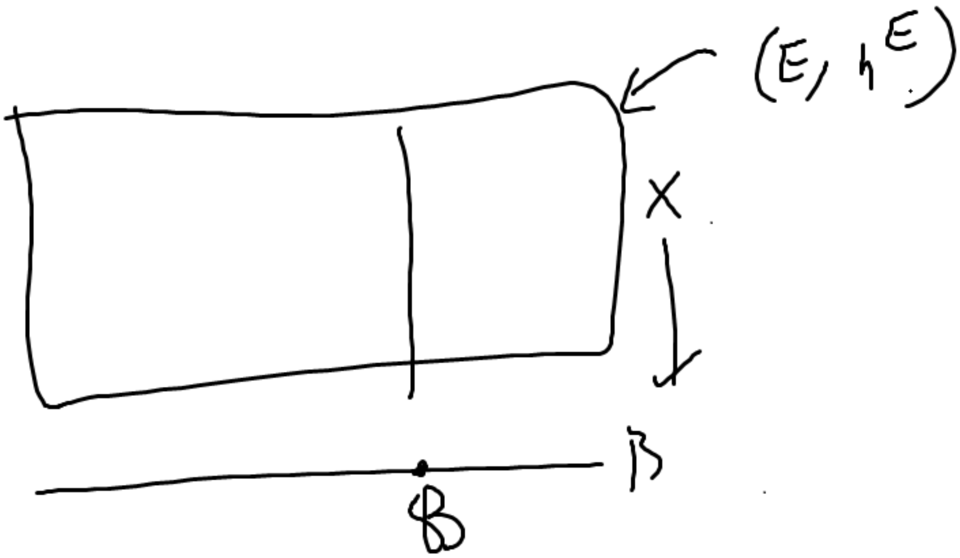
Bismut-Gillet-Soule have a series of three papers

We call $\|\cdot\| = \|\cdot\|_Q$ the **Quillen norm**

Quillen norm

Definition

$\|\cdot\|_Q = \|\cdot\|_{L^2} \cdot \text{Analytic torsion}$



$\lambda(E)_{\mathcal{B}} = \det H^i(X_{\mathcal{B}}, E)^{(-1)^i}$

Now fix Y with a Riemannian metric g^{TY} and let E be a vector bundle on Y with a Hermitian metric h^E. There is the L^2-metric given by

$$\langle s, s' \rangle_{L^2} = \int_y \langle s(X), s'(X) \rangle_{h^E} dv_{g^{\mathrm{TX}}}$$

where $s, s' \in \mathcal{C}^\infty(Y, E)$.

On $H^0(Y, E)$, $\|\cdot\|_{L^2}$ is just the restriction.

Hodge theory

The Kodaira Laplacian $\square^E$ is given as follows: take $\delta : \Omega^{0,i}(X, E) \to \Omega^{0,i+1}(X, E)$. Define $\square_i^E = \delta^*\delta + \delta\delta^*$. The Kodaira Laplacian preserves the degree of a differential form.

Hodge theory states the following:

- $\ker \square_i$ is finite dimensional.
- $\ker \square_i \cong H^i(X,E)$. More precisely, Given $s \in \ker \square_i$, we have

$$\langle \square s, s\rangle = \langle \delta s, \delta\rangle\langle \delta^*, \delta^* s\rangle$$

...

Proof.

A. Let $\square_i$ be an elliptic operator. For $s \in \mathcal{C}^\infty(X,E)$, $\|s\|_{H^2} \subseteq C \cdot (\|\square \cdot s\|_{L^2} + \|s\|_{L^2})$. In particular, if $\square s = 0$, then $\ker \square_i \cap B_{L^2}(0,1) \subset B_{H^2}(0,C)$ for some constant C. We know that $B_{H^2}(0,C)$ embeds compactly into $B(0,C)$.

B. We can decompose

$$\mathcal{C}^\infty(X,E) = \ker \square \oplus_\perp \operatorname{Im} \square.$$

It is also the case that

$$\operatorname{Im} \square = \operatorname{Im} \delta \oplus \operatorname{Im} \delta^*.$$

Moreover, $\operatorname{Im} \delta \subset \ker \delta$ and $\ker \delta \perp \operatorname{Im} \delta^*$, and if $\alpha = \delta^* \beta$, then

$$0 = \langle \delta\delta^* \beta, \beta\rangle = \langle \delta^* \beta, \delta^* \beta\rangle.$$

Analytic torsion

Analytic torsion is a spectral invariant of the fibers.

Given $E \to Y$ a vector bundle , g^{TY} a Riemannian metric, and h^E a Hermitian metric, $\square$ is an operator $\Omega^{0,i}(X,E)$ and $\square$ are essentially self-adjoint.

$$\operatorname{Spec}(\square)$$

is discrete, so it consists of positive real numbers $\lambda_1, \ldots, \lambda_n$.

Weyl's law

Weyl's laws tells you about the growth of the λ_i's:

$$\lambda_i \sim C \cdot \operatorname{vol}(M) \cdot i^{\frac{2}{n}}.$$

Spin geometry

Let s be an eigenvalue of $\square$. We have

$$\|s\|_{H^2} \leq (\|\square s\|_{L^2} + \|s\|_{L^2}) \leq c. \ldots \|s\|_{L^2}$$

$$\|s\|_{H^{2k}} \subseteq \Lambda^k \|s\|_{L^2}$$

Now write

$$\det \Box = \prod_{i=1}^{\infty} \lambda_i$$

But this does not make sense because the eigenvalues are growing. We use the zeta function to consolidate this:

$$\zeta(s) = \sum \frac{1}{\lambda_i^s}$$

If converges and is holomorphic for $\mathrm{Re}\, s > \frac{n}{2}$$

Fact

ζ has a meromorphic extension to $\mathbb{C}$ and 0 is a holomorphic point of this extension.

This is important because

$$\zeta'(0) = \sum -\log(\lambda_i)$$

and we can define

$$\det \Box := \exp(-\zeta(0)).$$

Roy-Singer made this definition.

The Quillen norm is then defined by

$$\|\cdot\|_Q = \|\cdot\|_{L^2} \cdot T(X_{\mathcal{B}}, E) = \prod (\det \Box_i)^{\frac{i\cdot(-1)^i}{2}}$$

Here, $X \to B$ is a fibration of Kahler manifolds, $g^{\mathrm{TX}/\Delta}$ is a rests of a Kahler metric

Theorem

(Quillen, Bismut-Gillet-Soule)

- $\|\cdot\|_Q$ is smooth
- $C_d(\lambda(E), \|\cdot\|_Q) = \pi_*(\mathrm{Td}(TX/B, g^{\mathrm{TX}}) \,\mathrm{Ch}(E, h^E))$

Remark

RRG is a trivial consequence of this theorem; The above theorem says that certain characters coincide point-by-point whereas RRG is a statement that characters coincide globally.

Why do we care?

There are a bunch of applications.

- It all started with arithmetic geometry --- with the arithmetic Riemann-Roch theorem of Gillet-Soule, Faltings, Bost
- Related to the theory of automorphic forms (Yoshikawa)
- Applications to mirror symmetry (there is something called BCOV torsion)
- Applications to dynamical systems (Kontsevich-Zorich)
- Applications to probability (Work of Dubidot)

See Also

Meta

References

Citations and Footnotes

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1. I think that Siarhei means f when he write π.↩↩
2. The commutator of R^E and f↩