1. Isotypicity and non-unitarity

The main result of this section is **??**, which states that counterexamples to Putman-Wieland in genus ≥ 3 cannot be isotypic, i.e., there exists an element of $H^1(\Sigma_{g'}, \mathbb{C})^\rho$ with infinite orbit under the action of a finite index subgroup of the mapping class group. We show more, namely **??**: if $X \to Y$ is an H-cover, where Y has genus at least 3, the virtual action of the mapping class group of Y on an H-isotypic component of the cohomology of X is non-unitary.

In Corollary 1.4 we use this to show how a result from the retracted paper of Boggi-Looijenga [**?**] would imply the Putman-Wieland conjecture.

Our main tool for proving this is a natural bilinear pairing, which we next introduce. Let C be a smooth proper connected curve of genus g, $D \subset C$ a reduced divisor, $E_\star$ a parabolic vector bundle on (C, D), and $E := E_0$. As described in [**?**, (4.5)], there is a bilinear pairing

$$B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \to \omega_C^{\otimes 2}(D) \tag{1.1}$$

given as the composition

$$B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \xrightarrow{\otimes} (E \otimes E^\vee) \otimes \omega_C^{\otimes 2}(D) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}} \omega_C^{\otimes 2}(D),$$

where tr denotes the trace pairing $E \otimes E^\vee \to \mathscr{O}_C$. By restriction to $H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \subset H^0(C, E \otimes \omega_C(D))$, we also obtain an induced pairing

$$H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \times H^0(C, E^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)).$$

Theorem 1.1. *Let $E_\star$ be a semistable parabolic bundle on (C, D) of parabolic degree zero, with underlying vector bundle $E := E_0$. Suppose $g \geq 2$ and that the pairing $H^0(\widehat{E}_0 \otimes \omega_C(D)) \otimes H^0(E^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D))$ vanishes. Then $g = 2$.*

Proof. First, we may assume $\operatorname{rk} E > 1$ by [**?**, Proposition 4.2.3], as $g \geq 2$. If $F_\star$ is a parabolic sheaf, we call the image of $H^0(C, F_0) \otimes \mathscr{O}_C \to F_0 \subset F_\star$ the globally generated subsheaf of $F_\star$, which is by definition also a subsheaf of F_0. Let $U \subset \widehat{E}_\star \otimes \omega_C(D)$ denote the globally generated subsheaf of $\widehat{E}_\star \otimes \omega_C(D)$ and V denote the globally generated subsheaf of $\widehat{((E_\star)^\vee \otimes \omega_C(D))_0}$. Under the identifications

$$\left(\widehat{E}_\star \otimes \omega_C(D)\right)_0 = \widehat{E}_0 \otimes \omega_C(D)$$
$$\widehat{((E_\star)^\vee \otimes \omega_C(D))_0} = E^\vee \otimes \omega_C,$$

(see [**?**, Definition 2.6.1] for the notion of a dual of a parabolic bundle) U and V are also the globally generated subsheaves of $\widehat{E}_0 \otimes \omega_C(D)$ and $E^\vee \otimes \omega_C$ respectively.

Let $c_V := \operatorname{rk} E - \operatorname{rk} V$ and $c_U := \operatorname{rk} E - \operatorname{rk} U$. Using **??** applied to the bundles $E_\star \otimes \omega_C(D)$ and $E_\star^\vee \otimes \omega_C(D)$, we know $\operatorname{rk} E \geq g c_V$ and $\operatorname{rk} E \geq g c_U$.

On the other hand, we claim $\operatorname{rk} U + \operatorname{rk} V \leq \operatorname{rk} E$. Granting this claim, we find $c_V + c_U \geq \operatorname{rk} E$. Since $\operatorname{rk} E \geq g c_V$ and $\operatorname{rk} E \geq g c_U$, adding these gives

$$2 \operatorname{rk} E \geq g(c_V + c_U) \geq g \operatorname{rk} E,$$

implying $2 \geq g$.

It remains to show $\operatorname{rk} U + \operatorname{rk} V \leq \operatorname{rk} E$. We will argue this using the fact that an isotropic subsheaf for a non-degenerate quadratic form on a vector bundle of rank $2 \operatorname{rk} E$ has rank at most $\operatorname{rk} E$. Indeed, consider the quadratic form q_E on $E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C$ associated to the nondegenerate bilinear form B_E of (1.1):

$$\begin{aligned} q_E : E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C &\to \omega_C^{\otimes 2}(D) \\ (v, w) &\mapsto B_E(v, w). \end{aligned}$$

Any vector bundle subsheaf of $E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C$ isotropic for this quadratic form has rank at most $\operatorname{rk} E$, as may be verified on the generic fiber using that isotropic subspaces of a rank $2 \operatorname{rk} E$ non-degenerate quadratic space have dimension at most $\operatorname{rk} E$. Therefore, it is enough to show $U \oplus V$ is killed under q_E. Using $q_E(U \oplus V) = B_E(U \times V)$, it is enough to show $B_E(U \times V) = 0$. We have a commutative diagram

$$\begin{tikzcd} (H^0(C,U) \otimes \mathscr{O}_C) \times (H^0(C,V) \otimes \mathscr{O}_C) \arrow[r] \arrow[d] & H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathscr{O}_C \arrow[d] \\ U \times V \arrow[r, "B_E"] & \omega_C^{\otimes 2}(D) \end{tikzcd} \tag{1.2}$$

where the top horizontal map vanishes by assumption. Since the vertical maps are surjective, the bottom horizontal map satisfies $B_E(U \times V) = 0$, as desired. □

1.2. **Proof of isotypicity.** We now deduce our desired isotypicity consequence, **??** and **??**, for the Putman-Wieland conjecture. With setup as in **??**, we have a cover $\Sigma_{g',n'} \to \Sigma_{g,n}$ and an action of a finite index subgroup $\Gamma \subset \mathrm{Mod}_{g,n+1}$ on $H^1(\Sigma_{g',n'}, \mathbb{C})$. We aim to show that if ρ is any irreducible H representation so that every element of the characteristic subspace $H^1(\Sigma_{g'}, \mathbb{C})^\rho \subset H^1(\Sigma_{g',n'}, \mathbb{C})$ has finite orbit under Γ, then $g \leq 2$.

Proof of **??** *and* **??**. We first prove **??**.

Assume to the contrary that the virtual action of the mapping class group on $H^1(\Sigma_{g'}, \mathbb{C})^\rho$ is unitary. Let $\mathscr{X}^\circ \xrightarrow{\widetilde{f}^\circ} \mathscr{C}^\circ \xrightarrow{\pi^\circ} \mathscr{M}$ be a versal family of H-covers, and $\mathscr{X} \to \mathscr{C} \to \mathscr{M}$ the associated families of proper curves. Let $m \in \mathscr{M}$ be a point, and $X \to Y$ the fiber over $\mathscr{M}$ Let $E_\star^\rho$ be the parabolic bundle on Y corresponding to the representation ρ as in **??**.

We claim the map $\overline{\nabla}_m^\rho$ vanishes identically, as $W^1R^1\pi_*^\circ\rho$, the variation of Hodge structure on $\mathscr{M}$ associated to $H^1(X, \mathbb{C})^\rho$, is unitary by assumption. Indeed, by [**?**, 1.13], we may write

$$W^1R^1\pi_*^\circ\rho = \bigoplus_i \mathbb{V}_i \otimes W_i,$$

where the $\mathbb{V}_i$ are complex variations of Hodge structure with irreducible monodromy and the W_i are constant variations. The $\mathbb{V}_i$ carry a unique structure of a $\mathbb{C}$-VHS, up to renumbering. As $W^1R^1\pi_*^\circ\rho$ is unitary by assumption, the same is true of each $\mathbb{V}_i$, and so the Hodge filtration of each $\mathbb{V}_i$ has length at most one. This proves the claim that $\overline{\nabla}_m^\rho = 0$.

Note that $\overline{\nabla}_m^\rho$ is the weight 1 part of the map adjoint to the multiplication map

$$H^0(Y, E_0^\rho \otimes \omega_Y(D)) \otimes H^0((E_0^\rho)^\vee \otimes \omega_Y) \to H^0(Y, \omega_Y^{\otimes 2}(D))$$

by [**?**, Theorem 4.1.6]. Since the subspace of $H^0(Y, E_0^\rho \otimes \omega_Y(D))$ corresponding to $W^1 \cap F^1$ is $H^0(Y, \widehat{E}_0^\rho \otimes \omega_Y(D))$ by **??**, we obtain that the multiplication map

$$H^0(Y, \widehat{E}_0^\rho \otimes \omega_Y(D)) \otimes H^0((E_0^\rho)^\vee \otimes \omega_Y) \to H^0(Y, \omega_Y^{\otimes 2}(D))$$

also vanishes. Using Theorem 1.1, this implies $g \leq 2$.

?? is immediate, as representations with finite image are unitary. □

As a consequence, we show how a claimed result from a paper of Boggi-Looijenga (which has since been retracted by the authors) implies the Putman-Wieland conjecture.

Corollary 1.3. *Suppose* [**?**, Theorem B(i)] *were true. Then the Putman-Wieland conjecture,* **??**, *would hold for all* $g \geq 3$.

Remark 1.4. We note that, unfortunately, there is a fatal error in the proof of [**?**, Theorem B(i)], appearing in [**?**, Lemma 1.6] (which is used in the proof of [**?**, Theorem 1.1], and hence of [**?**, Theorem B(i)]). The error comes in the penultimate sentence of the proof of [**?**, Lemma 1.6], where is it claimed that

"It follows. . . ," but in fact no argument is given for this claim. It is for this reason that the authors of [**?**] retracted that paper.

Proof. Suppose $g \geq 3$, and we are given a finite étale H-cover of Riemann surfaces $f : \Sigma_{g',n'} \to \Sigma_{g,n}$ furnishing a counterexample to Putman-Wieland. This means some subrepresentation $\chi \subset H^1(X, \mathbb{C})$ has finite orbit under the action of the mapping class group $\mathrm{Mod}_{g,n+1}$. The isotypicity statement of [**?**, Theorem B(i)] implies that if $\chi \subset H^1(X, \mathbb{C})$ has finite orbit under the action of the mapping class group $\mathrm{Mod}_{g,n+1}$, every element of the χ-isotypic component $H^1(X, \mathbb{C})^\chi$ has finite orbit under the action of $\mathrm{Mod}_{g,n+1}$. By definition, this means $f : \Sigma_{g',n'} \to \Sigma_{g,n}$ is χ-isotypic in the sense of **??**, which contradicts **??**. $\square$