Mathematical Modeling

Theoretical Modeling

Types of problem can be tackled by using mathematics

- ***Mathematical problems (exercises):*** *to develop and practice mathematical skills*. For example, find the two solutions of the quadratic equation $x^2 + 2x - 5 = 0$.
- ***Mathematical investigation:*** *to explore area of mathematics which might be new to learner*. For example,
  - Start with any 4 digit number where the digits are not all the same
  - Rearrange them in ascending and descending order and subtract the smaller from the larger
  - Repeat above process with the number obtained. You will know when to stop
  - Try different starting numbers and investigate how long a chain of number you can obtain.
- ***Powerful application of mathematics:*** *to solve problem set in the real world (Real World Problems)*. For examples, greenhouse effect etc. Need to obtain a mathematical model that will describe or represent some aspect of the real situation. Use the model to predict something about the future.

Real World Problems

- **Empirical modeling**

  Based on data alone

- **Theoretical modeling**

  One based more on theory than data alone

Empirical Modeling

- Reasonably straightforward: Models are easy to find, providing that we are given or can collect the data from appropriate experiment.
- An important problem solving tool.
- Severe **limitation** in validity.

Problem	Prediction	Implication/Validity ?
1. Greenhouse effect 2. World record of W. Marathon 3. Braking distance of a car with speed of 200 km/h	1. UK will be flooded in 2078 2. 120 min in year 3. 277 m	1. Maybe reasonable predictions and greenhouse effect may draw concerns. In future other kinds of fuels may be used, e.g., electricity, solar energy. 2. Zero minutes for 42.195 km!! Near future it maybe a reasonable prediction. 3. Can be validated using experimental data.

Summary: Skills we have learned

- Understand the problem: Why? Find? Given?
- Be aware of the assumptions and simplifications made in solving the problem: Assume? How?
- Question the results or predictions of the model: Predict? Valid? Verified? Improve?
- Use?

Principles: Questions and Answers

-*ways of thinking about mathematical modeling*

- **Why?** What are we looking for? Identify the need for the model.
- **Find?** What do we want to know? List the data we are seeking.
- **Given?** What do we know? Identify the available relevant data.
- **Assume?** What can we assume? Identify the circumstances that apply.
- **How?** How should we look at this model? Identify the governing physical principles.
- **Predict?** What will our model predict? Identify the equations what will be used, the calculations that will be made, and the answer that will result.
- **Valid?** Are the prediction valid? Identify tests that can be made to validate the model, i.e., is it consistent with its principles and assumptions?
- **Verified?** Are the prediction good? Identify tests that can be made to verify the model, i.e., is it useful in terms of the initial reason it was done?
- **Improve?** Can we improve the model? Identify parameter values that are not adequately known, variables that should have been included, and/or assumption/restrictions that could be lifted. Implement the iterative loop that call "model-validate-verify-improve-predict."
- **Use?** How will we exercise the model? What will we do with the model?

Theoretical Modeling

- *Some examples*
  1. *pedestrian crossing*
  2. *Icing Cakes*
  3. *Counter of a cassette tape recorder*

- *Notice that how the data is used after the model is formed to help to check the validity of the model.*

Pedestrian crossing

- Formulate a mathematical model for crossing one-way street so that a pedestrian can cross the road safely. Use your model to decide under what conditions a local council decides to install a pedestrian crossing.

Pedestrian Crossing 2

- The introduction of a pedestrian crossing will be dependent on
  1) the amount of road traffic,
  2) the amount of pedestrian traffic,
  3) cost,
  4) type of pedestrian traffic (children, elderly people, patients, etc).

- A Simple Model: Assumptions and simplifications
  1) the road is one-way, a single-carriage way with no obstruction for positioning of a pedestrian crossing,
  2) the speed of the traffic is constant and equal to the road speed limit,
  3) the density of the traffic is constant,
  4) the pedestrian walk across the road at a constant speed.

Pedestrian crossing 3

- **Symbols to represent the physical quantities.**

Physical quantity	symbol	units
Width of the road	w	meters
Speed of the pedestrians	v	meters per second
Time interval between the traffic	T	seconds

- **A condition for the pedestrians to cross the road safely:**

$$w/v < T$$

- ***Install a pedestrian crossing on a road if $w/v > T$***

Pedestrian Crossing 4

- *Data & Validation*
  - *For the vehicle data, suppose that the road is in a 30mph (13.3m/s) speed limit area and safe distance between vehicles is 23m*

    $$T = 23/13.3 = 1.73 \text{ seconds}$$

  - *Suppose that the average speed (v) of the pedestrian 4mph (1.77m/s) and the width (w) of a single-carriage way 3m*

    $$w/v = 3/1.77 = 1.69 \text{ seconds}$$

    $$\boldsymbol{T > w/v}$$

- **Do not install a pedestrian crossing$\therefore$.**

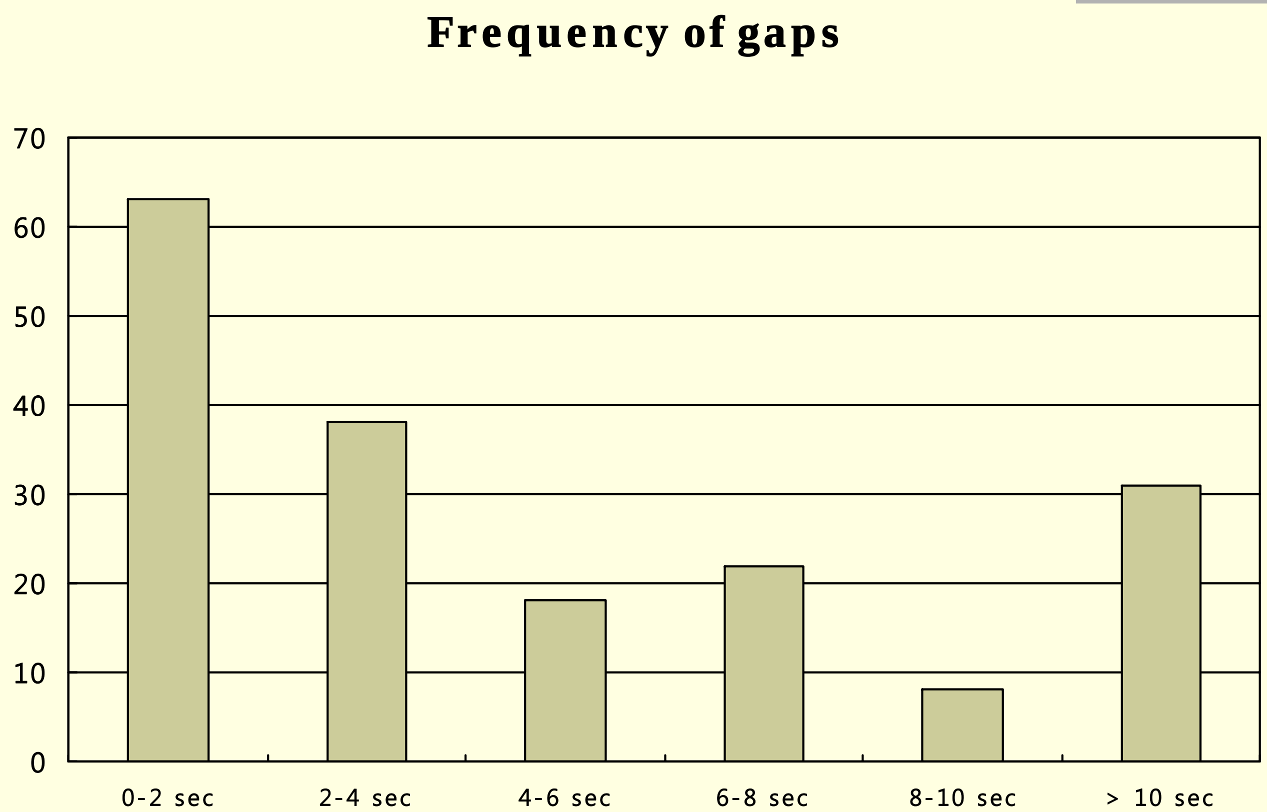
  Is it a reasonable decision?

Pedestrian Crossing 5

- Criticisms
  - The traffic is unlikely to be evenly spaced at the Highway code recommended distance
  - A safety margin of 0.04 seconds is not really realistic
  - The model suggests that you can either cross or not cross which is not realistic, the question of how long to wait does not enter the problem
  - 4mph is quite a fast walking speed, especially for elderly people.

- **A more complicated and perhaps more realistic model:**
  - Assume that a crossing will be installed when the probability $\boldsymbol{p}$ of a gap of time interval greater than w/v is less than some predetermined value $\boldsymbol{p_0}$.

Pedestrian crossing 6: A simple improvement - Data



Frequency distribution of time gaps between cars on one carriage way of a road. The data collected by a group of students on a road during a 30 minute period.

Pedestrian Crossing 7: a simple improvement - probability of gaps

- An average time needed for a pedestrian to cross the road safely: $w/v = 1.69$ *seconds*
- A gap of time interval greater than 2 seconds would be a ***good*** safety margin.
- The probability (***p***) of a gap of time interval greater than 2

$$\boxed{\boldsymbol{p} \quad = \quad \frac{\text{Number of gaps greater than 2 seconds}}{\text{Total number of gaps}} \quad = \quad \frac{117}{180} \quad = \quad 0.65}$$

- The total time of gaps greater than 2 seconds is 65% of 30 minutes, 19.5 minutes. We may say that for 65% of a long enough time period, e.g., *for 1 minutes 18 seconds out of 2 minutes, pedestrians can cross the road safely*. Note average time gap between vehicles: 10 seconds (30x60/180) or there comes one car every 10 seconds.
- **A reasonable threshold probability (p_0)** of a gap of time interval greater that 2 seconds: ***$p_0 = 0.5$ i.e., for 1 minute out of 2 minutes, pedestrians can cross the road safely.***
- **For this road, we do not need to install a pedestrian crossing, since $p > p_0$!!!**

Pedestrian crossing 8 Discussion

- The model could be developed further by considering
  - The variability of the speed of pedestrians: 4mph is too fast?
  - A statistical model involving the arrival times of cars and pedestrians.
  - The heaviest traffic period during a day.
- Several points to notice about this problem solving activity
  - ***"A simple model is better than no model at all."*** The first model is very simple and straightforward, however it does allow us to obtain a better understanding of the problem and help us to 'get into the problem'. In the first model there is a need for data to validate the model and in the improved model the data for a particular road is an important part of the formulation of the model.
  - Each model formulated depends on certain assumptions and simplifications chosen by the problem solver. This allows different people to formulate different models and how good the models are can be tested at the validation stage with appropriate data.
  - The mathematical model starts with a word description or **a word model** in each case. We then move from the words to the symbols which have been defined along their units.
  - The process of solving the problem is an iterative one; i.e., we start with a simple approach and then gradually refine it by looking back at the assumptions and simplifications.

Icing Cakes: Problem

- A wedding cake is to be baked in a square cake tin and will have a volume (before icing) of 4000 cm^3. Determine the dimensions of he cake which will give minimum surface a[illegible]e for icing (i.e. the top and the four sides). Also find the dimensions if the cake is baked in a circular tin.
- *Purpose: there is a rule of thumb in cookery that one third of the marzipan should be used for the top of the cake and the remain two thirds for the sides. Investigate the validity of this rule of thumb.*

Icing cake: Assumptions and simplifications

- The cake fills the tin exactly after baking and does not crumble or stick to the sides,
- each cake is perfectly flat on top and bottom so that it does not rise above top of the cake tin,
- The mixture of volume 4000m^3 includes any air bubbles etc.,
- The marzipan goes on the cake before the icing.

Icing cake: Square Cake Tin

- The sides of the cake tin have length *x cm* and the depth of the cake have length *y cm*,
- The volume of mixture in the tin is

  $V = x^2y = 4000$

- The surface area to be marzipanned and iced is

  $S = x^2 + 4xy = x^2 + 16000/x$

- The mathematical problem is to find the value of *x* which gives the minimum value for *S*. *(Using Calculus…)*
- ***The minimum value of S occurs when x=20cm, y =10 cm***

Icing cake: Circular Cake Tin

- The radius of the circular tin have *r cm* and the depth of the cake have length *y cm*,
- The volume of mixture in the tin is

  $$V = \pi r^2 y = 4000$$

- The surface area to be marzipanned and iced is

  $$S = \pi r^2 + 2\pi r y = \pi r^2 + 8000/r$$

- The mathematical problem is to find the value of *r* which gives the minimum value for *S. (Using Calculus…)*
- ***The minimum value of S occurs when $r = y = 10\,(4/\pi)^{1/3}$ cm which is about 10.84 cm***

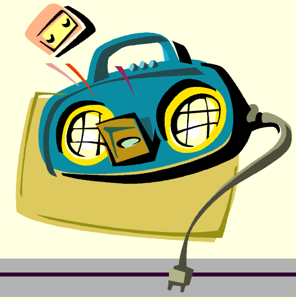
Discussions

- The test for the rule of thumb

Shape	Area of top (cm^2)	Area of sides (cm^2)	ratio
Square	*20x20=400*	*4x(10x20)=800*	*2*
circular	$\pi r^2=369.05$	$2\pi ry=738.11$	*2*

- A test for this model by baking a real cake of square shape
  - The cake was not flat on top – this changed the shape of the cake
  - The cake was rounded off on the corner of the square tin – the was not a perfect square
  - The cake contained a lot of bubbles
  - In fact, the area on top was more than 1/3 (33%) of the total surface area!!! (about 40 %)

An audio cassette tape counter model

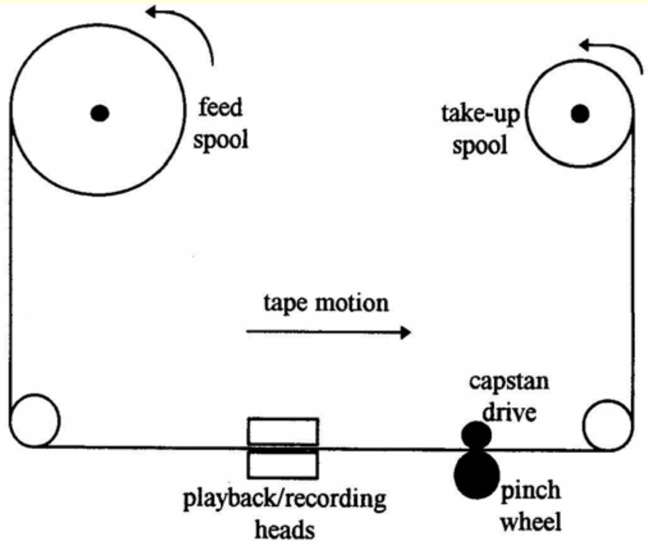


- Most audio cassette recorders have a numerical tape counter which allows the user to create a numerical index for the items on the cassette tape for playback purposes. Furthermore, it is often convenient to be able to relate the number displayed on the tape counter with the playing time remaining, for example when needing to use the cassette to record a known length of music.
- For a cassette equipped with a tape counter, *formulate a mathematical model* that describes *the relationship between the counter reading and the amount of playing time that has elapsed.*

A simple view of the mechanism of an audio cassette player

- How the counter mechanism functions!

- The tape leaves the supply spool, passes over the playback/recording heads at a constant speed and is collected up by the take-up spool.
- The constant speed is maintained by the capstan drive and pinch wheel.
- Since the tape speed across the heads needs to be constant, the supply and the take-up spools change speed during the playing(or recording) of a tape.
- We will assume that the tape counter is directly connected to the take-up spool.



A simple view of the mechanism of an audio cassette player

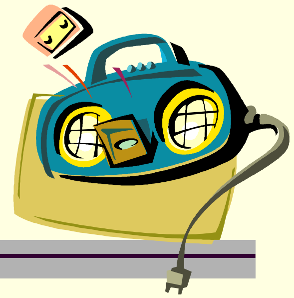
Features important in formulating a mathematical model

- Time elapsed
- Length of tape on the take-up spool
- Radius of the take-up spool when empty
- Radius of the take-up spool at general time
- Thickness of the tape
- Speed of the tape across the heads
- Counter reading
- Number of turns of the take-up spool
- Angle turned through by the take-up spool

Assumptions and simplifications

1. The speed of the tape across the heads is constant.
2. The tape has constant thickness.
3. The counter reading is a continuous variable.
4. The counter reading number is proportional to the number of turns of the take-up spool.

Variables and parameters

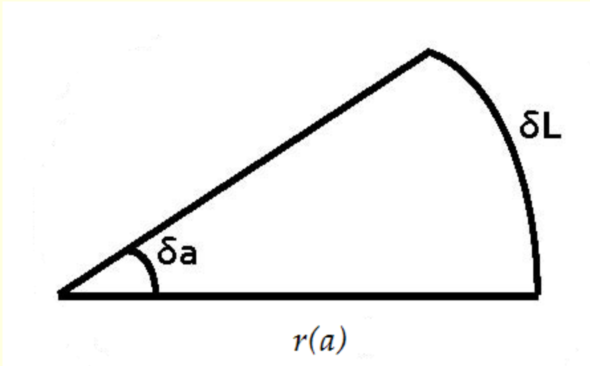


Physical quantity	symbol	units
time elapsed	t (variable)	seconds
length of tape on the take-up spool	L (variable)	cm
radius of empty spool	r_0 (parameter)	cm
radius of the take-up spool at time t	r (variable)	cm
thickness of the tape	h (parameter)	cm
speed of the tape across the heads	v (parameter)	cm/s
counter reading	c (variable)	
angle turned through by the take-up spool at time t	a (variable)	radians

A mathematical model relating *the counter reading c and the amount of playing time that has elapsed t.*

1. As the tape passes over the head at constant speed v, we have

$$L = vt \quad \cdots \qquad (1)$$



2. Assumption 4 and the number of turns is the total angle turnes through,

$$c = ka \quad \cdots \qquad (2)$$

where k is a constant of proportionality.

3. Relation between angle a and the length L of tape on the take-up spool.

From the figure, $\delta L = r(a)\delta a \quad \cdots \quad (3)$

4. Each time the take-up spool makes a complete revolution, the angle a increases by 2π and the radius of the tape on the take-up spool increases by h. Thus if the angle increses by δa, then the radius r will increase by $\delta r = h\delta a/(2\pi)$. Thus we have

$$\frac{dr}{da} = \frac{h}{2\pi} \quad \Rightarrow \quad r(a) = \frac{h}{2\pi}a + r_0$$

5. Substituing $r(a)$ in the equation (3) and letting δa tends to zero, we have, provided that $L = 0$ when $a = 0$

$$\frac{dL}{da} = \frac{h}{2\pi}a + r_0 \Rightarrow L = \frac{h}{4\pi}a^2 + r_0 a \quad \cdots \ (4)$$

6. Substituting a of (1) and L of (2) into the equation (4), we have

$$t = \frac{h}{4\pi v k^2}c^2 + \frac{r_0}{kv}c \quad \cdots \quad (5)$$

This is a mathematical model relating the elapsed time t and the counter reading c.

- How to determine the coefficients of the equation (5) for a real cassette player?

An experiment and validation

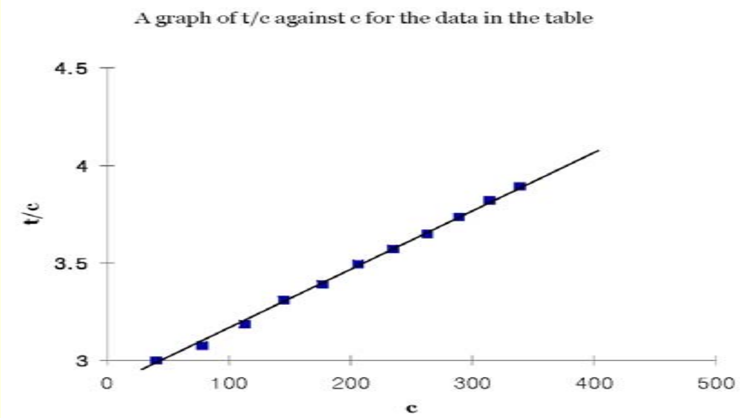
time, t (minutes)	counter reading, c	ratio, t/c (seconds/turns)
0	000	0
2	040	3.000
4	078	3.077
6	113	3.186
8	145	3.310
10	177	3.390
12	206	3.495
14	235	3.574
16	263	3.650
18	289	3.737
20	314	3.822
22	339	3.894

radius of the emty take-up spoool $r_0 = 1.1$ cm

speed of tape across heads $v = 4.76$ cm/s

tape thickness $h = 0.0013$ cm

constant of proportionality $k = 0.0838$ rad^{-1}



- From the equation (5), we have $\dfrac{t}{c} = \dfrac{h}{4\pi v k^2} c + \dfrac{r_0}{kv} \quad \cdots \quad (6)$
- Using linear least square method for the table we have

$$\frac{t}{c} \approx 0.003053\, c + 2.85645$$

- From the parameters h, v, k and r_0 used in the experiment, we have

$$\frac{h}{4\pi v k^2} \approx 0.00309, \quad \frac{r_0}{kv} \approx 2.75$$

- The agreement of the model (5) and the data is remarkably good!!

$$t = \frac{h}{4\pi v k^2} c^2 + \frac{r_0}{kv} c \approx 0.003053\, c^2 + 2.85645\, c$$

- Do you have any comment or criticisms for improvements?

(k in $c = ka$ was calculated from the number of turns of the take-up spool for the counter reading to increase by 100, about 190 compelet turns)

Length of a Toilet Roll (A project)

- You are provided a roll of toilet paper. Formulate a mathematical model to predict its length in terms of thickness of the roll.

1. List the important features that you think will be involved in formulating the model.
2. List the assumptions and simplifications.
3. Criticise your model making appropriate suggestions for improvements.