Mathematical Modeling

Introduction

Model?

- A miniature representation of something
- A pattern of something to be made
- An example for imitation or emulation
- A description of analogy used to help visualize something (e.g., atom) that cannot be directly observed
- **A system of postulates, data and inferences presented as a mathematical description of an entity or state of affairs**

- ***Modeling is an activity, a cognitive activity in which we think about and make a models to describe how devices or objects of interest behave.***

  Using words, drawings or sketches, physical models, computer programs, or mathematical formulae.

Mathematical models

- **Representations or descriptions in mathematical terms of the behavior of real devices and objects (realities)**

We want to know

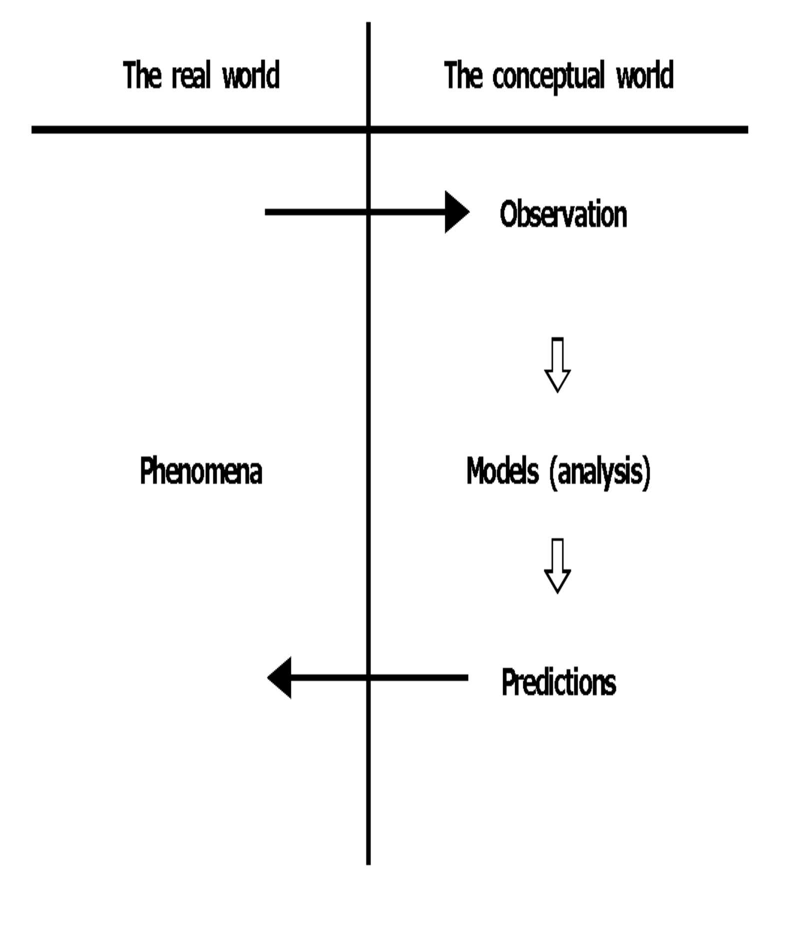
- *How to make or generate mathematical representations of reality or models*
- *How to validate them*
- *How to use them*
- *How and when their use is limited*

Why do we do Mathematical Modeling?

- Modeling devices and phenomena is essential to both engineering and science for very practical reasons. Money? Convenience?
- Mathematicians, scientists, and engineers want to experience the sheer joy of formulation and solving mathematical problems. Knowledge?

- ***Mathematical modeling and the scientific method***
- ***Mathematical modeling and the practice of Engineering***

Mathematical Modeling and the Scientific method



- **Observation part**:
  (1) measuring
  (2) gathering empirical evidence & facts
- **Modeling part**: analyzing observations because we want develop models that
  (1) describe observation
  (2) explains why it is so ..
  (3) allows us to predict
- **Prediction part:** exercise our models to tell us what will happen in (1) a yet-to-be-conducted experiments or (2) an anticipated set of events in the real world
- Valid? Adequate? Improve the model. Iterative looping procedure.

Mathematical Modeling and the Practice of Engineering

- **Engineers are interested in *designing***
  - *Devices and*
  - *Processes and*
  - *Systems*
- **They need to model devices and processes:**

They must be able to describe and analyze objects and devices in order to predict their behavior to see if that behavior is what the engineers want.

Common points and Differences

between the scientific method & engineering design

- **Common points:** *models are applied to predict outcome of experiments or to explain phenomena (validation)*
- **Differences in motivations and approach:** *Prediction in engineering designs assumes that resources of time, imagination, and money can be invested with confidence because the predicted outcome will be a good one:* for example, a model based new airplane will fly without dire, unanticipated consequences.

Principles of Mathematical Modeling

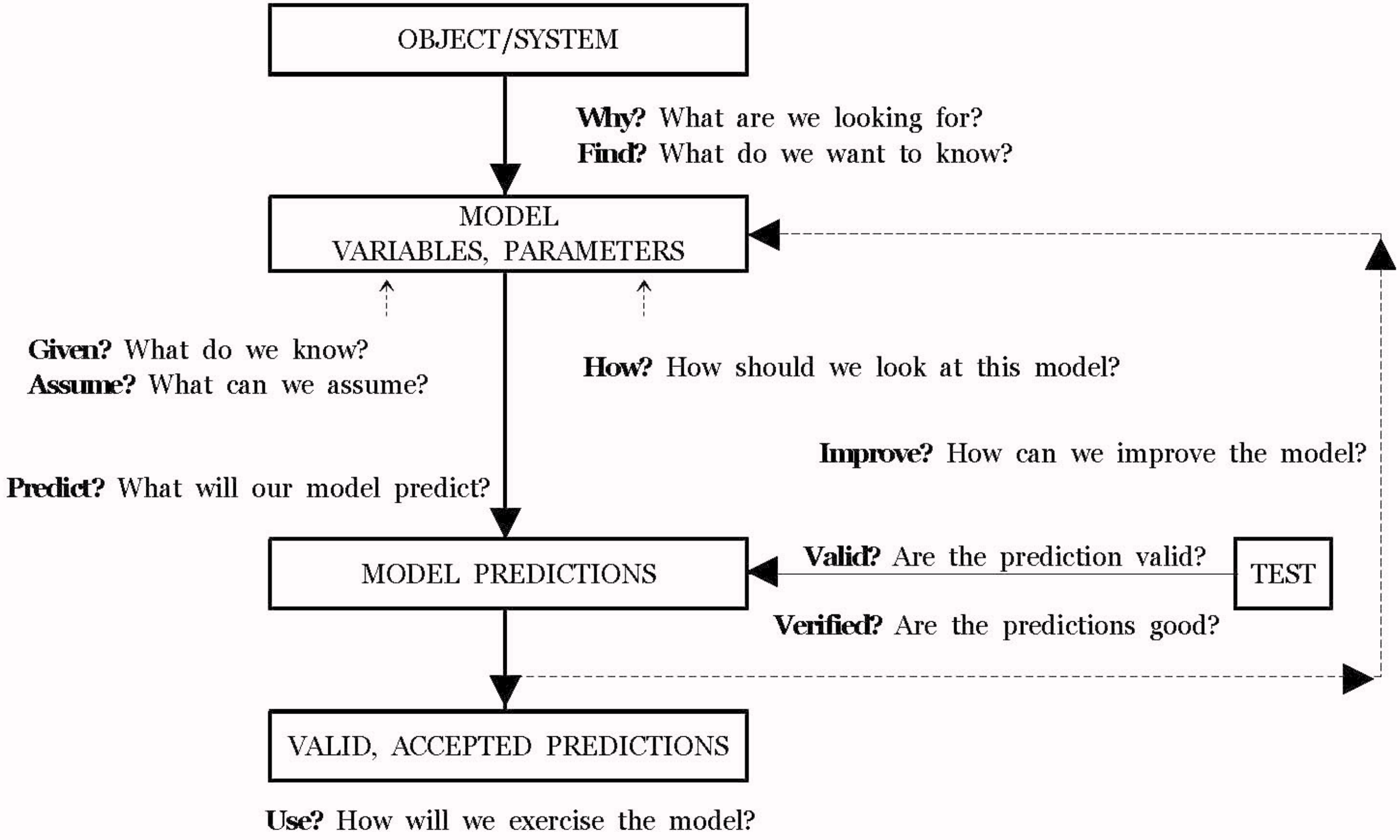
- Mathematical Modeling is a *principled activity* that has both principles behind it and methods that can be successfully applied
- Methodological modeling principles
  - **Why?**
  - **Find?**
  - **Given?**
  - **Assume?**
  - **How?**
  - **Predict?**
  - **Valid?**
  - **Verified?**
  - **Improve?**
  - **Use?**

Principles: Questions and Answers

–*ways of thinking about mathematical modeling*

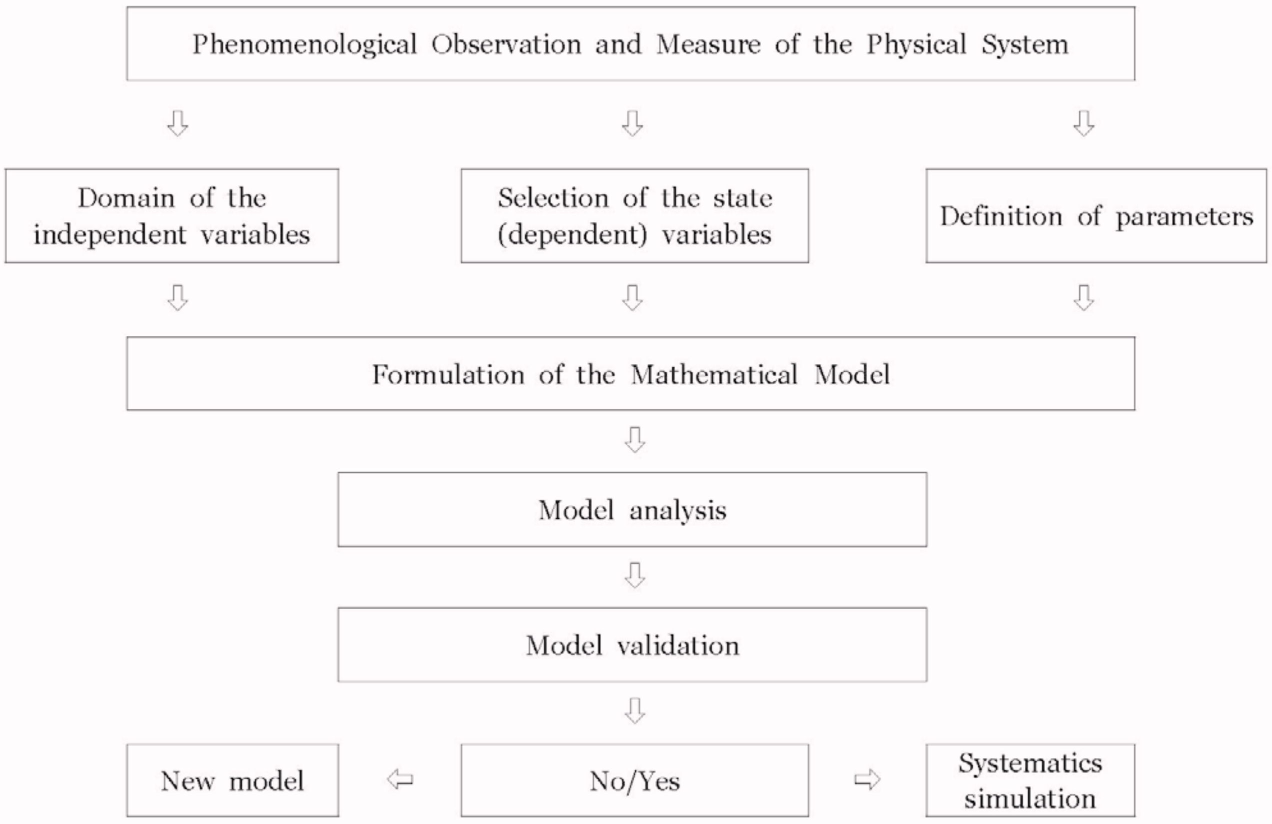
- **Why?** What are we looking for? Identify the need for the model.
- **Find?** What do we want to know? List the data we are seeking.
- **Given?** What do we know? Identify the available relevant data.
- **Assume?** What can we assume? Identify the circumstances that apply.
- **How?** How should we look at this model? Identify the governing physical principles.
- **Predict?** What will our model predict? Identify the equations what will be used, the calculations that will be made, and the answer that will result.
- **Valid?** Are the prediction valid? Identify tests that can be made to validate the model, i.e., is it consistent with its principles and assumptions?
- **Verified?** Are the prediction good? Identify tests that can be made to verify the model, i.e., is it useful in terms of the initial reason it was done?
- **Improve?** Can we improve the model? Identify parameter values that are not adequately known, variables that should have been included, and/or assumption/restrictions that could be lifted. Implement the iterative loop that call “model-validate-verify-improve-predict.”
- **Use?** How will we exercise the model? What will we do with the model?

Principles of Mathematical Modeling
- diagram -



Mathematical Modeling

Modeling and model analysis



Flow chart- Modeling and model analysis

Solution techniques and analytical theories (methods) of mathematical models

- Relevant Mathematical subjects: Calculus, Linear Algebra, Numerical Analysis including computing software, Ordinary differential equations, Partial Differential equations, Statistics and probability, Number theory (cryptology), Differential Geometry, discrete mathematics, Hilbert spaces, etc.
- Least square method, root finding methods, Numerical solver of ODE and PDE, stability, integral transform, complex technique
- Computer languages and Software: Fortran, Matlab, Maple, Mathematica, Ruby, C++, java, lisp, etc.

A Classification of mathematical Modeling

- **Empirical Modeling**

  Based on data alone

- **Theoretical Modeling**

  One based more on theory than data alone

Some Methods of Mathematical Modeling

1. Dimensional Homogeneity and Consistency
2. Abstraction and Scaling
3. Conservation and Balance Principles
4. Constructing Linear Models

Some Methods of Mathematical Modeling – **Dimensional homogeneity and consistency**

- A basic, yet very powerful idea
- *Every equation we use must be dimensionally homogeneous or dimensionally consistent.*

  <u>(Dimensional Analysis Technique)</u>

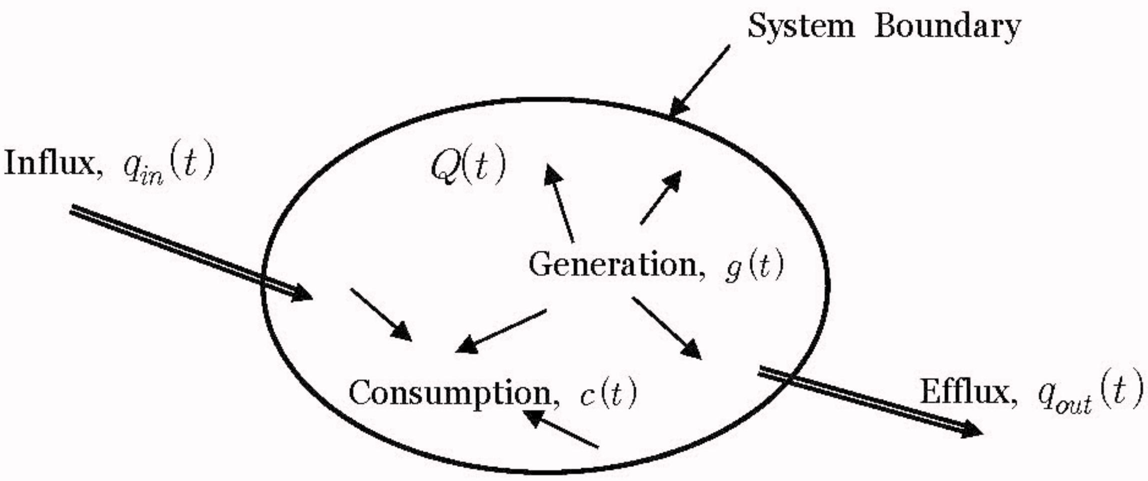
Dimensions: fundamental physical quantities, e.g., mass, time, length, area, degree

Units: numerical expressions of a quantity's dimensions expressed in terms of a given physical standard, e.g., 100m

Some Methods of Mathematical Modeling - **Abstraction and scaling**

- **Abstraction:** the process choosing an appropriate level of detail for the model (requires thoughtful approach to identifying the phenomena on which we want to focus – why a model is being sought or developed). A linear elastic spring model - a coiled simple spring system, automobile, response of bridges or skyscrapers to earthquake or to wind loading.
- **Scaling:** Abstraction requires finding the right scale for the model. For example, the spring model can be used at micro scale to model atomic bond, or the macro level for buildings.
  - The notion of scaling includes ideas such as the effect of geometry on scale, the relationship of function to scale, the role of size in determining limit - all of which are needed to choose the right scale for a model in relation to the reality we want to capture.

Some Methods of Mathematical Modeling – **Conservation and Balance principles**



$$\frac{dQ(t)}{dt} = q_{in}(t) + g(t) - q_{out}(t) - c(t)$$

- Heating and cooling, Population dynamics, Traffic flow

Some Methods of Mathematical Modeling – **Constructing Linear models**

Models of devices or systems are *linear* when their basic equations are such that the magnitude of their behavior or response produced is *directly proportional* to the excitation or input that drives them.

- ***Principle of Superposition:*** The response of a linear system to the sum of individual inputs is obtained by adding (*superposing*) the separated response of the system to each individual input. *c.f., Decomposing into a simpler inputs*
- Nonlinearity & Linearization - Linearity