Lanchester’s Combat Theory

Construction of the models

- Suppose that an x-force and y-force are engaged in combat.
- The strengths of the two forces are measured by the number of their combatants.
- $x(t)$ and $y(t)$ denote the number of combatants (strengths) of the x and y forces at time t, which is measured in days from the start of the combat.

- <u>The rate of change of each of $x(t)$ and $y(t)$ equals</u>

(its reinforcement rate)-(its operational loss rate)-(its combat loss)

Reinforcement rate and Operational loss rate

- *The reinforcement rate* of a combat force is the rate at which new combatants enter or are withdrawn from the battle. We denote the reinforcement rates of x-force and y-force by $f(t)$ and $g(t)$, respectively.
- *The operational lass rate* of a combat force is its loss rate due to non-combat mishap; i.e., desertions, diseases, etc. It is very difficult to quantify, so *we neglect this factor.* (See handout or next slide #4)

More on the operational loss rate

- Lanchester proposed that the operational loss rate of a combat force is proportional to its strength. However, this does not appear to be very realistic. (For example, the desertion rate in a combat force depends on a host of psychological and other tangible factors which are difficult even to describe, let alone quantify.)
- We will take the easy way out here and consider only those engagements in which the operational loss rates are negligible.

Combat loss rate : Conventional combat

- Suppose that the x-force is a conventional force which operates in the open, comparatively speaking, and that every member of this force is within “kill range” of the enemy y. We also assume that as soon as the conventional force suffers a loss, fire is concentrated on the remaining combatants.

1. Under this “ideal” conditions, the combat loss rate of a conventional force x equals $ay(t)$ for some positive constant a. This constant is called the combat effectiveness coefficient of the y-force.

Combat loss rate : Guerilla combat

- Suppose that x is a guerilla force, **invisible** to its opponent y and occupying a region R. The y-force fires into R, but cannot know when a kill has been made.

1. It is certainly plausible that the combat loss rate for guerilla force x should be proportional to $x(t)$, for the larger $x(t)$, the greater the probability that an opponent's shot will kill.
2. On the other hand, the combat loss rate for the force x is also proportional to $y(t)$, for the larger $y(t)$, the greater the number of x-casualties.
3. The **combat loss rate for a guerilla force x equals $cx(t)y(t)$**, where the constant c is called the combat effectiveness coefficient of the opponent y.

$$\frac{1}{x(t)}\frac{dx}{dt} = -cy(t)$$

Lanchestrian models

- Converntional combat: $\begin{cases} \dfrac{dx}{dt} = -ay + f(t) \\ \dfrac{dy}{dt} = -bx + g(t) \end{cases} \quad \cdots \quad (1a)$

- Conventional-guerilla combat: $\begin{cases} \dfrac{dx}{dt} = -cxy + f(t) \\ \dfrac{dy}{dt} = -hx + g(t) \end{cases} \quad \cdots \quad (1b)$

  ($x =$ guerilla)

- (1a) is a linear system and can be solved explicitly once a, b, f and g are known.
- (1b) is a nonlinear system, and its solution is much more difficult to get.
- <u>Two speacial cases: no reinforcement.</u>

Two special cases: no reinforcement

- Converntional combat:

$$\begin{cases} \dfrac{dx}{dt} = -ay \\ \dfrac{dy}{dt} = -bx \end{cases} \qquad \cdots \quad (2a)$$

- Conventional-guerilla combat:

  (x = guerilla)

$$\begin{cases} \dfrac{dx}{dt} = -cxy \\ \dfrac{dy}{dt} = -hx \end{cases} \qquad \cdots \quad (2b)$$

Conventional combat: The square law

- From (2a), we have

$$\frac{dy}{dx} = \frac{bx}{ay} \quad \text{or} \quad ay\frac{dy}{dx} = bx$$

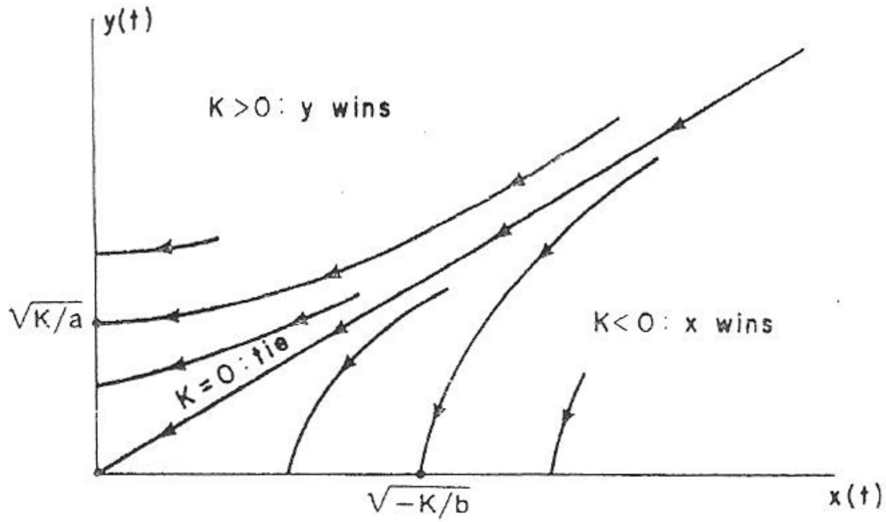
Integrating this equation gives

$$ay^2 - bx^2 = ay_0{}^2 - bx_0{}^2 = K \qquad \cdots \; (3)$$

- The curve (3) define a family of hyperbolas in the x - y plane (See the figure in the next slide)

Lanchester's square law

- The arrow head on the curves indicate the direction of changing strengths as time passes.
- One force wins the battle if the other vanishes first.
  - y wins if $K > 0$ since x has been annihilated by the time y(t) has decreased to $\sqrt{K/a}$
  - Similarly x wins if $K < 0$
- y always seeks to establish a setting in which $K > 0$, i.e., $ay_0^{\ 2} > bx_0^{\ 2}$:
  This can be done by increasing a; i.e.,
  by using stronger and more accurate weapons, or
  by increasing the initial force y_0.



- Note that though that
  a double of a results in a doubling of ay^2
  while a doubling of y_0 results in a *four - fold* increase of $ay_0^{\ 2}$.

Conventional-guerilla combat

- From (2a), we have

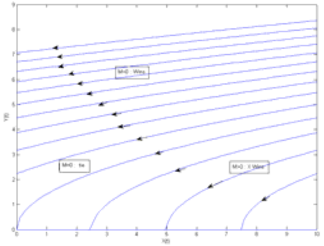
$$\frac{dy}{dx} = \frac{dx}{cxy} = \frac{d}{cy} \quad \cdots \quad (4)$$

Multipying both sides of (4) by cy and integrating gives

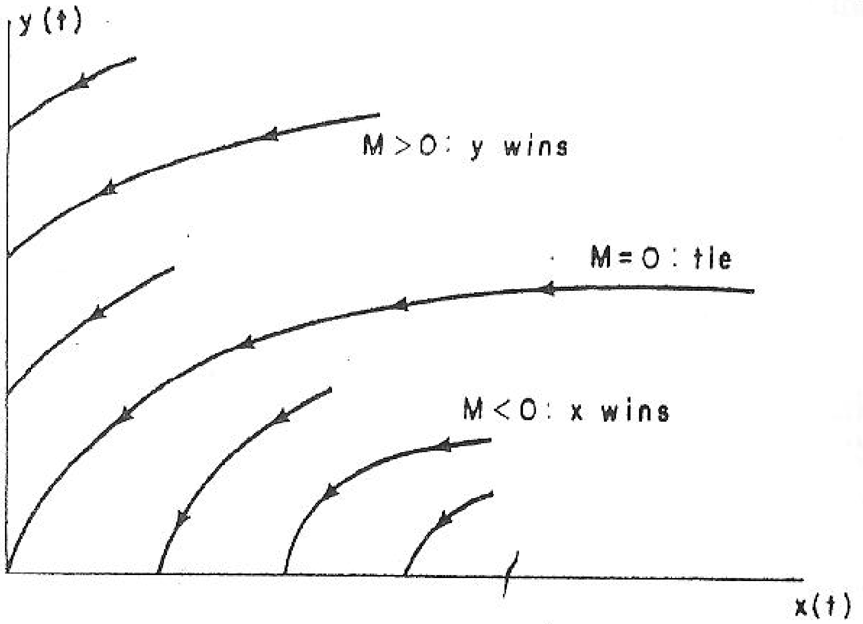
$$cy^2 - 2hx = cy_0{}^2 - 2hx_0 = M \qquad \cdots \ (5)$$

- The curve (5) define a family of parabolas in the x - y plane
  (See the figure in the next slide)

Conventional-guerilla combat



- The arrow head on the curves indicate the direction of changing strengths as time passes.
- One force wins the battle if the other vanishes first.
  - y wins if $M > 0$ since x has been annihilated by the time y(t) has decreased to $\sqrt{M/c}$
  - Similarly x wins if $M < 0$
- y always seeks to establish a setting in which $M > 0$, i.e., $cy_0^2 > 2hx_0$: This can be done by increasing c; i.e., by using stronger and more accurate weapons, or by increasing the initial force y_0.
- x always seeks to establish a setting in which $M < 0$, i.e., $cy_0^2 < 2hx_0$: This can be done by increasing h; i.e., by using stronger and more accurate weapons, or by increasing the initial force x_0.



- Note that x is a guerilla force. Thus it is very important for x to use stronger and more accurate weapons.

Some final remarks

- It is usually impossible to determine, *a priori*, the numerical value of the combat coefficients a, b, c and h. Thus it would appear that Lanchester's combat models have little or no applicability to real-life engagements.
- However this is not so. It is often possible to determine suitable values of a and b (c or h) – *see the handout!!*

Projects

- Population model: 한국
- Pursuit curves:
- Theories of war: analysis and exercises
- Heating and cooling: some exercises
- Traffic flow?
- What else?

- One team: 3 person
- Due date: 6/23