

Population Growth Modelling

- *How does one predict the growth of a population?*
 - $P(t)$: the population at time t
 - assumed to be *continuous function*
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A Mathematical for a population of bacteria

- A population of bacteria that reproduce by simple cell division
- The **death rate** is zero: food is sufficient
- Assume that the growth rate is proportional to the population present

$$\frac{dP(t)}{dt} = k_1 P(t), \quad P(0) = P_0 \quad \dots \dots \dots \quad (1)$$

A Mathematical Model for human population

Death rate k_1 is nonzero

$$\frac{dP(t)}{dt} = (k_2 - k_1)P(t) = kP(t) \dots \dots \dots \quad (2)$$

$k_2 > k_1$, k is proportionality constant

Malthusian Model

$$\left. \begin{array}{l} \frac{dP(t)}{dt} = kP(t) \\ P(t_0) = P_0 \end{array} \right\} \dots\dots\dots (3)$$

$$P(t) = P_0 e^{k(t-t_0)}$$

Exponential Model

Malthusian Model

- Is it reasonably good as a human population model?
- Example: Next Slide !!

$$\left. \begin{array}{l} \frac{dP(t)}{dt} = kP(t) \\ P(t_0) = P_0 \end{array} \right\} \dots\dots\dots (4)$$

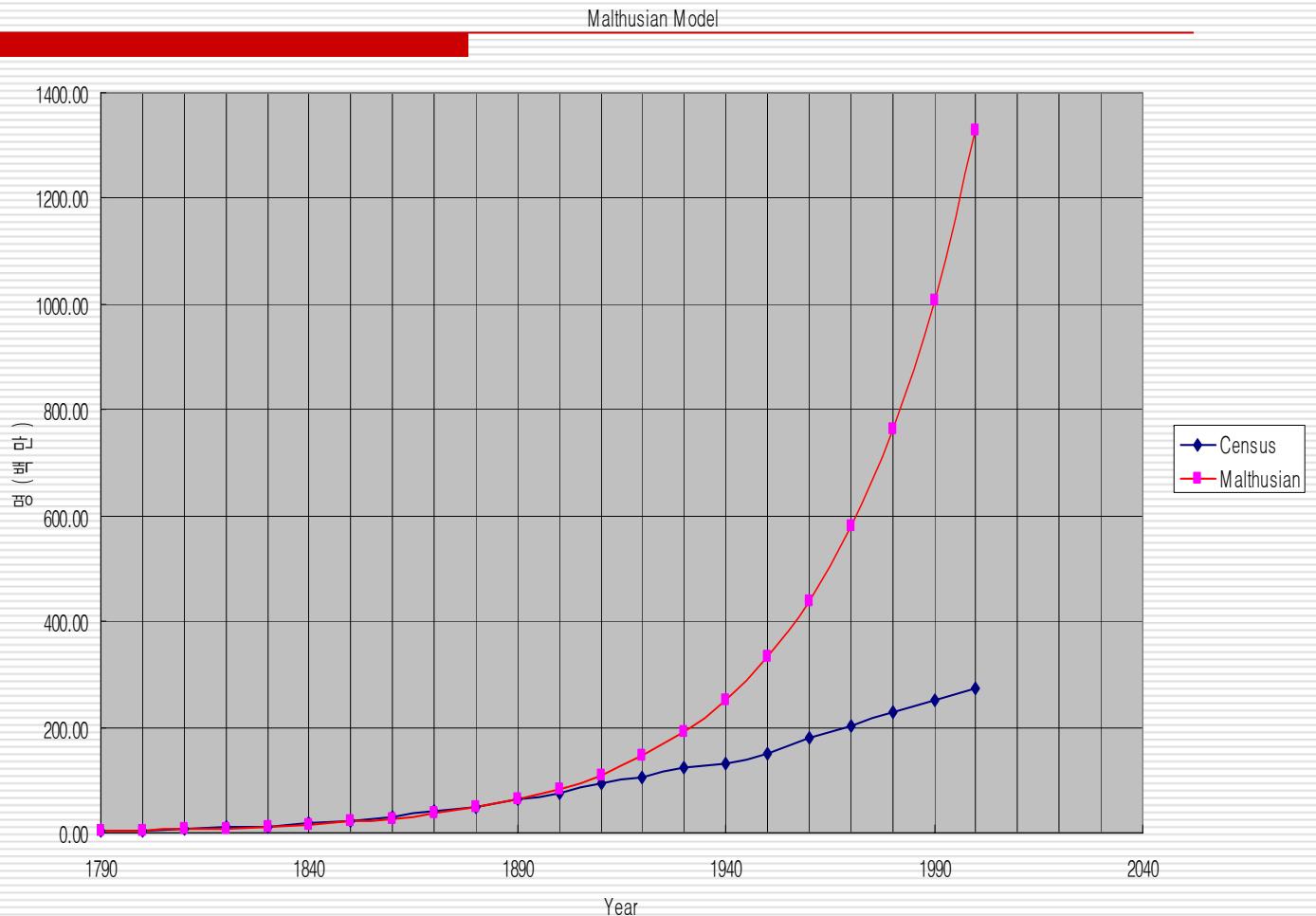
$$P(t) = P_0 e^{k(t-t_0)} \quad t_0 = 1790, P(t_0) = 3.93(\text{백만})$$

$$t = 1890, P(1890) = 62.95$$

$$k \approx 0.027737$$

Results: Malthusian Model

Year	Census	Malthusian
1790	3.93	3.93
1800	5.31	5.19
1810	7.24	6.84
1820	9.64	9.03
1830	12.87	11.92
1840	17.07	15.73
1850	23.19	20.76
1860	31.44	27.39
1870	39.82	36.15
1880	50.16	47.7
1890	62.95	62.95
1900	75.99	83.07
1910	91.97	109.63
1920	105.71	144.67
1930	122.78	190.91
1940	131.67	251.94
1950	151.33	332.47
1960	179.32	438.75
1970	203.21	579
1980	226.50	764.08
1990	249.63	1008.32
2000	275.00	1330.63
2010		



K 0.027737

Logistic Model A (1)

- Malthusian Model considered only death by natural causes.
- What about premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes ?
- A competition within the population should be considered:
- another component of death rate that is proportional to the number of two-party interaction
- 두 집단 x와 y의 경쟁이 있으면 집단 x의 크기의 변화율은 x와 y의 곱에 비례하여 감소한다.
- 따라서 동일집단 $P(t)$ 의 인구의 변화율은 다음과 같다

$$\frac{dP(t)}{dt} = ap(t) - bP^2(t)$$

Logistic Model A(2) : Logistic function

$$\left. \begin{aligned} \frac{dP(t)}{dt} &= aP(t) - bP^2(t) \\ P(t_0) &= P_0 \end{aligned} \right\} \dots\dots\dots (5)$$

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-a(t-t_0)}} \dots (6)$$

Logistic Model A (3)

In (6), we find a and b
using the data

$$t = 1790, P_0 = P(1790) = 3.93$$

$$t = 1840, P(1840) = 17.07$$

$$t = 1890, P(1890) = 62.95$$

Then we have a nonlinear system

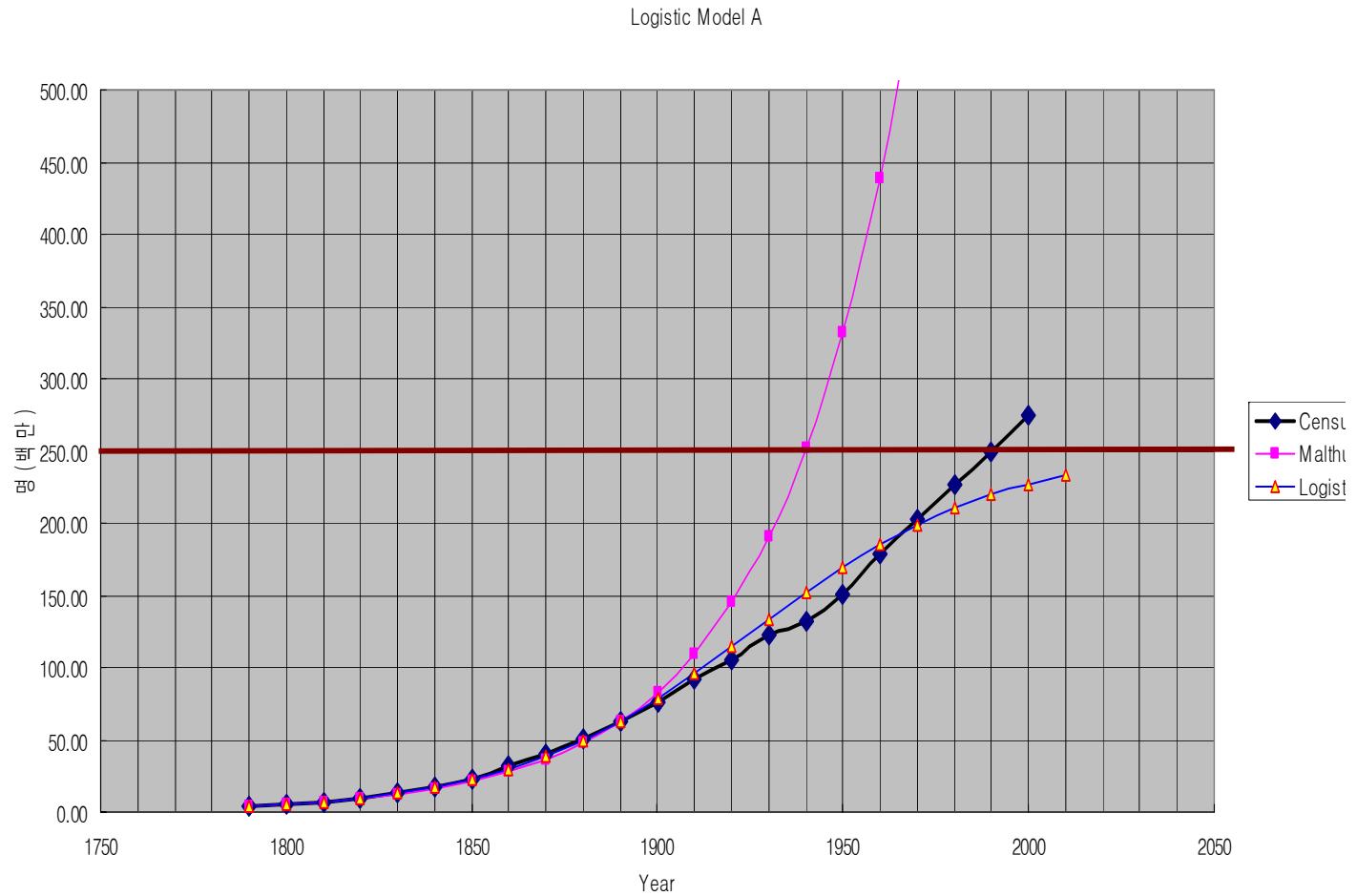
$$17.07 = \frac{3.93a}{3.93b + (a - 3.93b)e^{-50a}}$$

$$62.95 = \frac{3.93a}{3.93b + (a - 3.93b)e^{-100a}}$$

$$a = 0.0304667, b = 0.0001214$$

Logistic Model A (4)

Year	Census	Malthusian	Logistic
1790	3.93	3.93	3.93
1800	5.31	5.186242	5.300199
1810	7.24	6.844048	7.134322
1820	9.64	9.03178	9.578393
1830	12.87	11.91883	12.81574
1840	17.07	15.72874	17.06989
1850	23.19	20.7565	22.60217
1860	31.44	27.39141	29.69976
1870	39.82	36.1472	38.64893
1880	50.16	47.70181	49.6891
1890	62.95	62.94992	62.94791
1900	75.99	83.07214	78.36712
1910	91.97	109.6265	95.64196
1920	105.71	144.6692	114.205
1930	122.78	190.9133	133.2794
1940	131.67	251.9396	151.9987
1950	151.33	332.4733	169.5589
1960	179.32	438.7498	185.3483
1970	203.21	578.9982	199.0134
1980	226.50	764.0777	210.4544
1990	249.63	1008.319	219.7706
2000	275.00	1330.632	227.1862
2010			232.983
A	0.0304667		
B	0.0001214		250.9613



Logistic Model B (1)

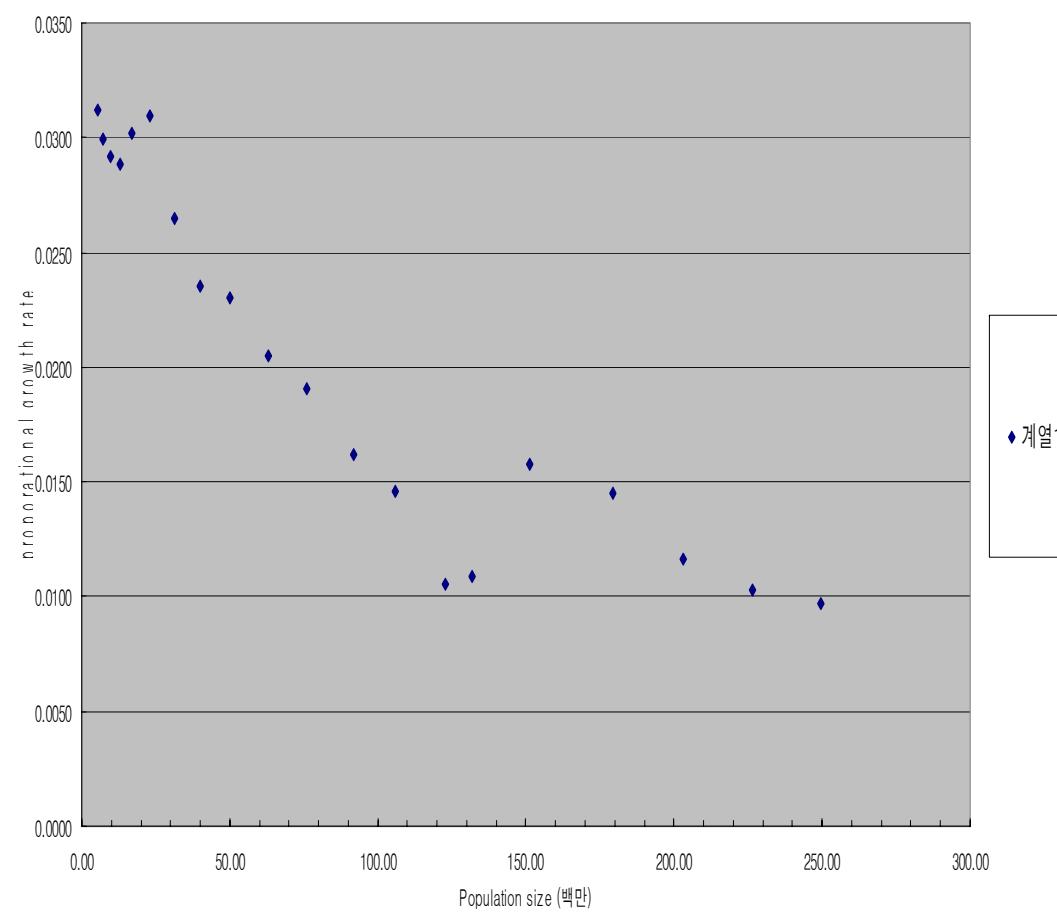
$$\frac{1}{P(t)} \frac{dP(t)}{dt} = a - bP(t)$$

$$\frac{1}{P(t)} \frac{dP(t)}{dt} \approx \frac{1}{P(t)} \frac{P(t+h) - P(t-h)}{2h}$$



Logistic Model B (2)

Year	Census	Malthusian	Logistic	$1/p \frac{dp}{dt}$
1790	3.93	3.93	3.93	
1800	5.31	5.186242	5.3002	0.0312
1810	7.24	6.844048	7.1343	0.0299
1820	9.64	9.03178	9.5784	0.0292
1830	12.87	11.91883	12.816	0.0289
1840	17.07	15.72874	17.07	0.0302
1850	23.19	20.7565	22.602	0.0310
1860	31.44	27.39141	29.7	0.0264
1870	39.82	36.1472	38.649	0.0235
1880	50.16	47.70181	49.689	0.0231
1890	62.95	62.94992	62.948	0.0205
1900	75.99	83.07214	78.367	0.0191
1910	91.97	109.6265	95.642	0.0162
1920	105.71	144.6692	114.21	0.0146
1930	122.78	190.9133	133.28	0.0106
1940	131.67	251.9396	152	0.0108
1950	151.33	332.4733	169.56	0.0157
1960	179.32	438.7498	185.35	0.0145
1970	203.21	578.9982	199.01	0.0116
1980	226.50	764.0777	210.45	0.0102
1990	249.63	1008.319	219.77	0.0097
2000	275.00	1330.632	227.19	
2010			232.98	



Logistic Model B: Least Squares

$$y = Ax + B$$

$$A = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x S_x}, \quad B = \frac{S_{xx} S_y - S_{xy} S_y}{nS_{xx} - S_x S_x}$$

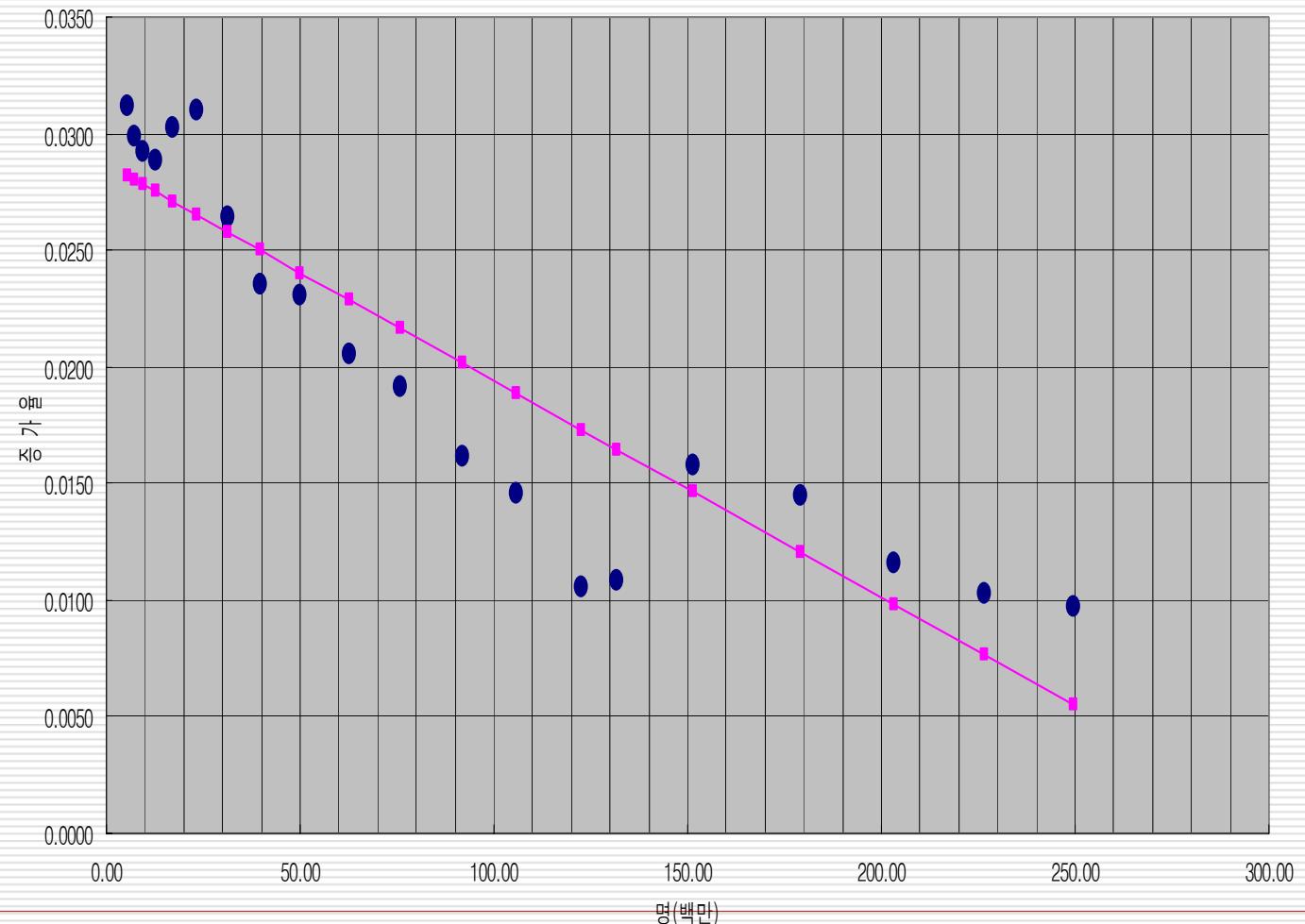
$$\frac{1}{P(t)} \frac{dP(t)}{dt} = a - bP(t)$$

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-a(t-t_0)}}$$

- A=-b, B=a
- A=-0.00009283
- B=0.02051368
- a=0.02868934
- b=0.00009283
- t₀=1790
- P₀= 3.63

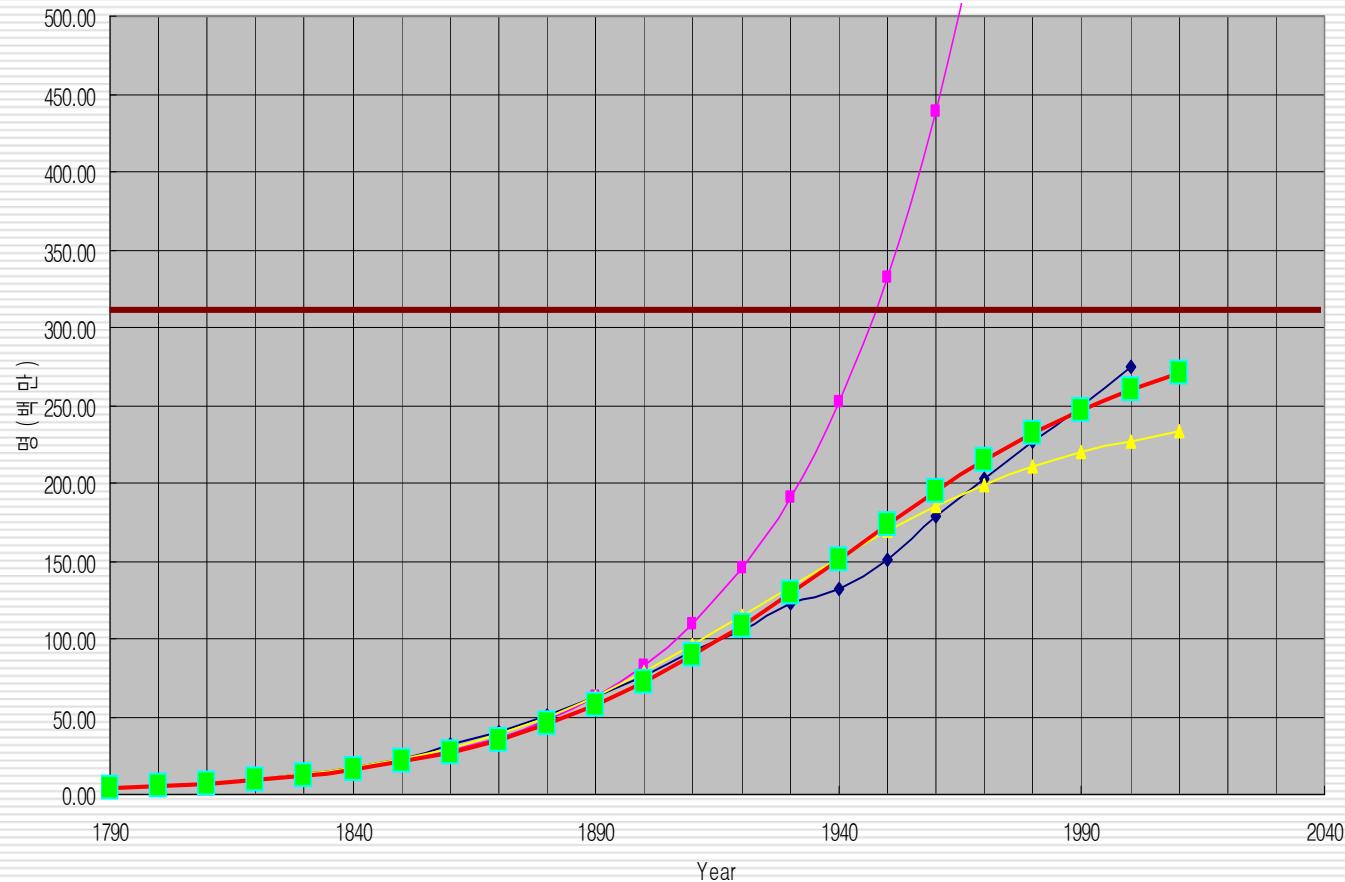
Least Square Fit

Least square fit



Logistic Model B

인구 모델



$$\frac{a}{b} = 309.0448$$

Discussion and Comments

- We have seen
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Projects

- Population of Korea
 - Spread of diseases
 - Prey-predator model
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