수학적 모델 설계

Heating and Cooling of buildings

Heating and Cooling : Goal

- Mathematical Model $\boldsymbol{T(t)}$ describing the 24-hour temperature profile inside a building

- Factors
  1. Outside temperature,
  2. Heat source inside
  3. Heating and Cooling

Three questions to answer

I. How long does it take to change $T(t)$ substantially?

II. How does $T(t)$ vary during spring and fall?
(No heating nor cooling season)

III. How does $T(t)$ vary during summer and winter?
(Cooling or heating season)

Notations

- $T(t)$: temperature inside the Building at t
- $H(t)$: rate of increase in temperature due to the heat produced
  by people, lights, machines at t
- $U(t)$: rate of increase or decrease in temperature
  due to air conditioning at t
- $M(t)$: temperature outside the building at t

- Remark: Usually $\boldsymbol{U(t)}$ and $\boldsymbol{H(t)}$ are described in terms of energy per unit time, we can express $\boldsymbol{U(t)}$ and $\boldsymbol{H(t)}$ in terms of temperature per unit time.

  Temperature/time = (Heat Capacity) x (Energy/time)

Newton's Cooling Law

- The rate of change in ***T*(*t*)** is proportional to the difference between the outside temperature ***M*(*t*)** and the inside temperature ***T*(*t*)**

$$\frac{dT(t)}{dt} = K\left[M(t) - T(t)\right] \quad \cdots\cdots\cdots \quad (1)$$

K is positive constant: dependent on the physical properties of the building

Governing Equation and Solution

$$\frac{dT(t)}{dt} = K\left[M(t) - T(t)\right] + H(t) + U(t) \quad \cdots\cdots \; (2)$$

$$\left.\begin{aligned} &\frac{dT(t)}{dt} + P(t)T(t) = Q(t) \\ &P(t) := K, \quad Q(t) := KM(t) + H(t) + U(t) \end{aligned}\right\} \quad \cdots\cdots \; (3)$$

- **Solution to (2): (3) is linear !!**

$$T(t) = e^{-Kt}\left\{\int e^{Kt}\left[KM(t) + H(t) + U(t)\right]dt + C\right\} \quad \cdots\cdots \; (4)$$

Question I

- Suppose that at the end of the day (at time t_0) when the people leave the building, the outside temperature stays constant at M_0, the additional heating rate H inside the building is zero, and the furnace/air conditioner rate U is zero. Determine $T(t)$, given initial condition $T(t_0)=T_0$.
- ***Solution!!***

$$T(t) = M_0 + (T_0 - M_0)e^{-K(t-t_0)} \quad \cdots\cdots \quad (5)$$

Discussion to Question I

$$T(t) = M_0 + (T_0 - M_0)e^{-K(t-t_0)} \quad \cdots\cdots \quad (5)$$

- $M_0 < T_0$:

- T(t) decreases exponentially.
- How long does it take to change $T(t)$ Substantially?
- Determine a measure of the time it takes for temperature to change substantially.

Time Constant

$$\frac{dA(t)}{dt} = -\alpha A(t), \quad A(t) = A(0)e^{-\alpha t}$$

- If $\alpha > 0$, $A(t)$ decays exponentially as $t \to \infty$.
- **Time constant for the building**:

  The time it takes for $A(t)$ to change from $A(0)$ to $A(0)/e \approx 0.368\, A(0)$ is $1/\alpha$
- For general linear equation: $dA/dt = -\alpha A + g(t)$

The time constant: $1/|\alpha|$

Time Constant for a building

$$\frac{dT}{dt}(t) = -KT(t) + KM_0, \frac{d(T - M_0)}{dt}(t) = -K\left[T(t) - M_0\right]$$

- ***Time constant for the building**: 1/K*

  The time it takes for the temperature difference $T_0 - M_0$ to change from $T_0 - M_0$ to $(T_0 - M_0)/e$
- *A typical value for the time constant for a building: 2 to 4 hours*
- *It is a physical property of the building*

Question II (Spring or Fall)

- Find the building temperature $T(t)$ if the additional heating rate $H(t)$ is equal to the constant H_0, there is no heating or cooling ($U(t)=0$), and the outside temperature $M(t)$ varies as a sine wave over a 24-hr period, with its minimum at $t=0$ at (midnight) and its maximum at $t=12$ (noon) that is

$$M(t)=M_0-B\cos\omega t,$$

where B is a positive constant, M_0 is the avearge outside temperature, and $\omega=2\pi/24=\pi/12$ radians/hr.

Question II: *Governing Equation*

$$\frac{dT(t)}{dt} = K\left[M(t) - T(t)\right] + H_0 \;\Rightarrow$$

$$\left.\begin{aligned} &\frac{dT(t)}{dt} + P(t)T(t) = Q(t) \\ &P(t) := K, \quad Q(t) := K(M_0 - B\cos\omega t) + H_0 \end{aligned}\right\} \cdots\cdots (6)$$

- Set $B_0 := M_0 + H_0/K$

$$Q(t) = K(B_0 - B\cos\omega t)$$

$$KB_0 = \frac{1}{24}\int_0^{24} Q(t)\,dt$$

Question II: *Solution*

$$\left.\begin{array}{l}\dfrac{dT(t)}{dt}+P(t)T(t)=Q(t)\\ P(t):=K,\quad Q(t):=K_0(B_0-B\cos\omega t)\end{array}\right\}\cdots\cdots\ (6')$$

$$T(t)=B_0-BF(t)+Ce^{-Kt}\quad\cdots\cdots\cdots\ (7)$$

$$F(t):=\frac{\cos\omega t+(\omega/K)\sin\omega t}{1+(\omega/K)^2}$$

$$C=T_0-B_0+BF(0)=T_0-B_0+\frac{B}{1+(\omega/K)^2}$$

$$B_0=M_0+\frac{H_0}{K}$$

Question II: *discussion (1)*

$$T(t) = B_0 - BF(t) + Ce^{-Kt} \quad \cdots\cdots\cdots \; (7)$$

- *Daily average temeprature inside the building*

$$B_0 \approx \frac{1}{24}\int_0^{24} T(t)\ dt := M_0 + \frac{H_0}{K} \quad \textit{(neglecting exponential term)}$$

$$B_0 = M_0 : \ \textit{when}\ H_0 = 0$$

- BF *in (7) represents the sinusoidal variation of temperature inside the building*

$$F(t) := \frac{\cos\omega t + (\omega / K)\sin\omega t}{1 + (\omega / K)^2} = \frac{\cos(\omega t - \phi)}{\left[1 + (\omega / K)^2\right]^{1/2}}, \quad \tan\phi = \omega / K$$

Question II: *discussion(2)*

$$T(t) = B_0 - BF(t) + Ce^{-Kt} \quad \cdots\cdots\cdots \ (7)$$

- *Typical value s:* $K \in \left[\frac{1}{4}, \frac{1}{2}\right], \ \omega = \frac{2\pi}{24}, \ \frac{\omega}{K} \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$
- $B_0 = 20^o C, \ B = 10^o C, \omega / K = \frac{\pi}{6}, \ T_0 = 15^o C, \ K = 1/2$
- $B_0 = 20^o C, \ B = 10^o C, \omega / K =, \frac{\pi}{3} \ T_0 = 15^o C, \ K = 1/4$
- $C = T_0 - B_0 + \dfrac{B}{1 + (\omega / K)^2} = 2.8483(K = 0.5), \ \text{-}0.2304(K = 0.25)$
- $F(t) = \dfrac{\cos(\omega t - \tan^{-1} \frac{\omega}{K})}{\left[1 + (\omega / K)^2\right]^{1/2}}, M(t) = B_0 - B\cos\omega t \quad (H_0 = 0)$

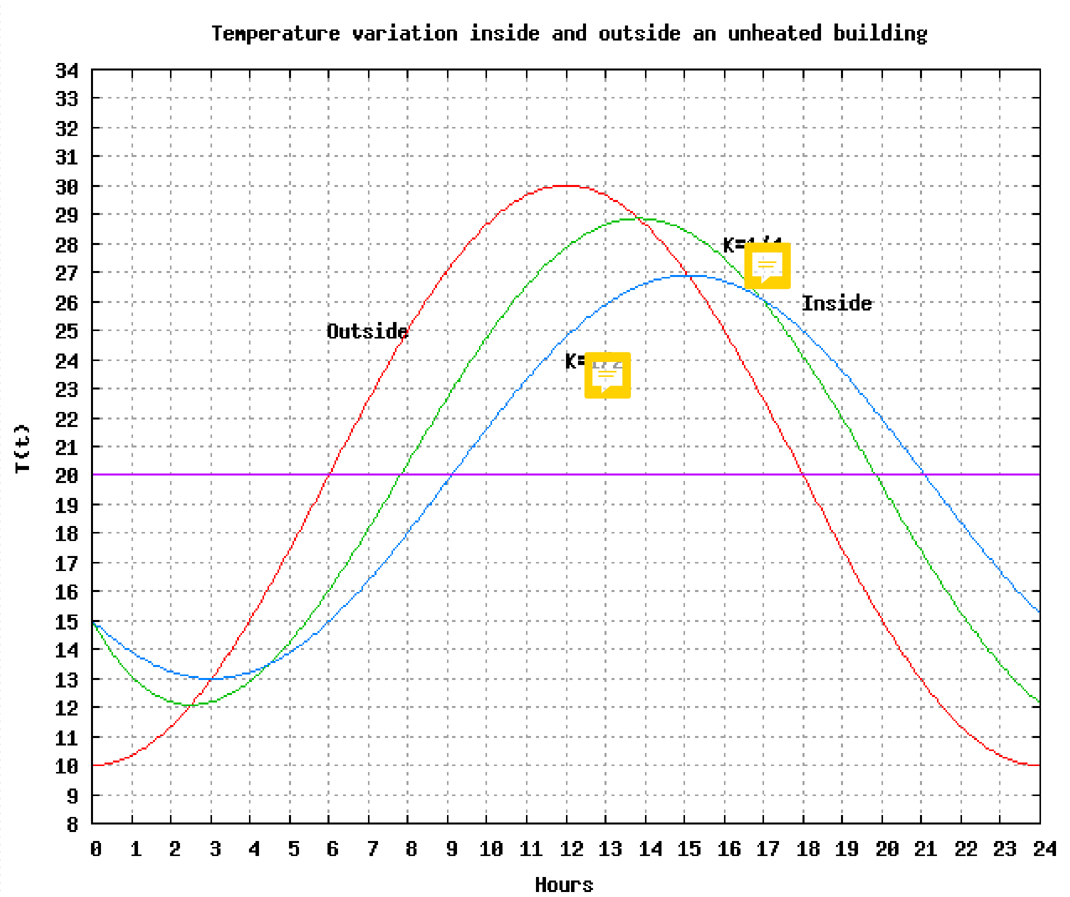
Question II: *discussion(3)*

$T(t) = B_0 - BF(t) + Ce^{-Kt}$ .... *(7)*

- *Typical values:* $K \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $\omega = \frac{2\pi}{24}$, $\frac{\omega}{K} \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$
- $|F(t)| = \dfrac{1}{\left[1 + (\omega / K)^2\right]^{1/2}} = 0.6906 \ (\frac{\omega}{K} = \frac{\pi}{6}), \quad 0.8895 \ (\frac{\omega}{K} = \frac{\pi}{3})$
- $\arctan \frac{\omega}{K} = 1.8424 (\frac{\omega}{K} = \frac{\pi}{6}), \ 3.0880 \ (\frac{\omega}{K} = \frac{\pi}{3})$

- Graphs for solutions : Next Slide!!

Question II: *discussion(4)*



Question III (Air Conditioning)

- Suppose that a simple thermostat is installed that is used to compare the actual temperature inside the building with a desired temperature T_D. If the actual temperature is below T_D, the furnace supplies heating; otherwise it is turned off. If the actual temperature is above T_D, the air conditioner supplies cooling; otherwise it is off. Assume the amount of heating or cooling supplied is proportional to the difference in temperature, that is

$$U(t) = K_U [T_D - T(t)]$$

where K_U is a positive constant. Find $T(t)$.

Question II: *Governing Equation*

$$\frac{dT(t)}{dt} = K\left[M(t) - T(t)\right] + H(t) + K_U\left[T_D - T(t)\right] \Rightarrow$$

$$\left.\begin{array}{l} \dfrac{dT(t)}{dt} + P(t)T(t) = Q(t) \\ P(t) := K + K_U, \quad Q(t) := KM(t) + H(t) + K_U T_D \end{array}\right\} \cdots\cdots (8)$$

- Assume $M(t)$ and $H(t)$ are the same as Question II

  $M(t) = K(M_0 - B\cos\omega t),\ H(t) = H_0$

- $Q(t) = K_1(B_2 - B_1\cos\omega t)$

$$\omega := \frac{\pi}{12},\ \mathrm{K}_1 := K + K_U, B_1 := \frac{BK}{K_1}, B_2 = \frac{K_U T_D + KM_0 + H_0}{K_1}$$

Question III: *Solution*

$$\left.\begin{array}{l}\dfrac{dT(t)}{dt}+P(t)T(t)=Q(t)\\ P(t):=K+K_U:=K_1,\quad Q(t):=K_1(B_2-B_1\cos\omega t)\end{array}\right\}\cdots\cdots\ (8')$$

$$T(t)=B_2-B_1F_1(t)+Ce^{-K_1t}\quad\cdots\cdots\cdots\ (9)$$

$$F_1(t):=\frac{\cos\omega t+(\omega/K_1)\sin\omega t}{1+(\omega/K_1)^2}$$

$$C=T_0-B_2+B_1F_1(0)=T_0-B_2+\frac{B_1}{1+(\omega/K_1)^2}$$

Question III: *discussion (1)*

$$T(t) = B_2 - B_1 F_1(t) + Ce^{-K_1 t} \quad \cdots\cdots\cdots \text{(9)}$$

- *Typical value of* $K_U \leq 2,\ K \in \left[\frac{1}{4}, \frac{1}{2}\right]$
- *Time constant for the building with heating and cooling*

$$\frac{1}{P} = \frac{1}{K_1} = \frac{1}{K + K_U} \leq 0.5(hr)$$

- *When the heating or cooling turned on,*

  it takes less than 30 minutes for the exponential term to die out
- *If we neglect this exponential term, the average temperature inside the building is* B_2

  $B_2 \approx T_D$ *(* $\therefore\ K_1$ *is much larger than K and* H_0 *is small)*
- *Thus after a certain period of time, the temperature inside the building*

  is roughly T_D*, which is the desired temperature + a small sinuoidal variation.*

Question III: *discussion (2)*

- **Energy saving:**

  To save energy the heating or cooling system may be left off during the night. When it is turn on in the morning it will take roughly less than 30 minutes for the inside the building to attain the desired temperature.