Calculating π

Introduction

- The constant π : the ratio of the circumferenc e of a circle to its diameter
- Its approximated numerical value has been st udied for many millenia. [P. Beckman, A His tory of π, Golum press, 1971]
- How to calculate π: it is a non-algebraic (tra nscendental), i.e. irrational number
- Four different ways to calculate π
  1) A definite integral
  2) An ODE method
  3) Series Techniques; Taylor, Fourier
  4) Monte Carlo method

A chronology of π

[P. Beckman, A History of π, Golum press, 1971]

When	Approximation of π	Who
Ca. 2000BC	3 1/8	Babylonians
Ca. 2000BC	$(16/9)^2 \approx 3.161$	Egyptians
Ca. 550 BC	3	I Kings vii
3rd century BC	3.14163	Archimedes
3rd century AD	$\sqrt{10} \approx 3.16$	Chung Hing
6th century AD	$\sqrt{10} \approx 3.16$	Brahmogupta
1220	3.141818	Fibonacci
1665-6	To 16 decimal places	Newton
1706	100 places	Machin
1873-74	707 places	Shanks
1949	2037 places	Eniac computer
1967	500,000 places	CDC 6600(Paris, France)
2012	?	?

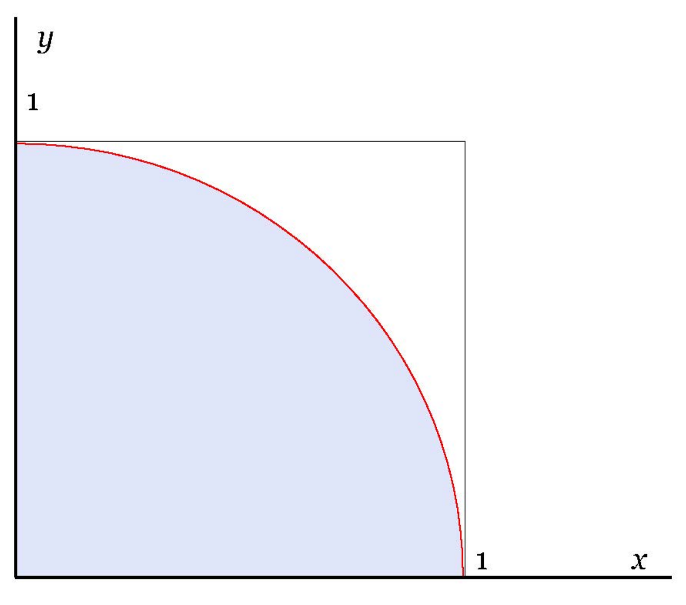
Monte Carlo Method (1)

- Monte Carlo : a Mediterranean city famous for gambling
- A Monte Carlo method first used to calculate π by George Louis Leclerc, Compte de Buffon in 18th century:
- A model : Tossing a needle into a horizontal surface with parallel lines. The probability that the needle crosses one of the lines proportional to π.
- This method can be used to experimentally calculate π.

Monte Carlo Method (2)

- N: the number of points that fall any where in the square.
- N_c : the number of points that fall into the first quadrant of the unit circle.

$$\frac{\pi}{4} = \frac{N_c}{N} \quad ?$$



- How to generate points or trials?
- Create an $N \times 2$ matrix of random numbers. Each row in the matrix will be the x and y coordinate of a point in the plane.
- Compute the distance of each point from the origin.
- Count the number of points whose distance from the origin le ss than or equal to 1.

Mote Carlo Method (3)

- Excel functions to be used
  1. Random number function: rand()
  2. Conditional selection: IF(cond, value_if_true, value_if_flase)
  3. Counting the number of cells containing nume rics : COUNT(range)
- Try several values of $N=100, 1,000, 10,000$...
- Does approximations of π become more accura te as we increase N?
- Discussion!! Accuracy and effectiveness

An ODE method (1)

- Consider the following initial value problem

$$y'' = -y, \quad y(0) = 1, y'(0) = 0$$

- The solution: $y=\cos x$
  1. At $x = \pi/2$, $\cos(x) = 0$.
  2. Find an approximated value x for which $\cos(x) \approx 0$.
  3. Then $\pi \approx 2x$
- Use a numerical ODE solver!!
- Intermediate value theorem

An ODE method (2)

- Taylor Series

$$y(x+h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2 + \frac{1}{3!}y^{(3)}(x)h^3 + \cdots + \frac{1}{n!}y^{(n)}(x)h^n + \cdots$$

$$y(x-h) = y(x) - y'(x)h + \frac{1}{2}y''(x)h^2 - \frac{1}{3!}y^{(3)}(x)h^3 + \cdots + \frac{(-1)^n}{n!}y^{(n)}(x)h^n + \cdots$$

- Difference formulae for some derivatives

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h)$$

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h)$$

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2)$$

An ODE method (3)

$$y''(x) + y(x) = 0 \Rightarrow \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + y(x) \approx 0$$

$$\therefore \;\; y(x+h) \approx (2-h^2)y(x) - y(x-h) \quad \mathrm{L} \quad (1)$$

For given $h > 0$,

$$y(0) = 1 \quad \mathrm{L\;L\;L\;L\;L} \quad (2)$$

$$y'(0) = \frac{y(0) - y(-h)}{h} = 0 \Rightarrow y(-h) = 1 \quad \mathrm{L} \quad (3)$$

Use (1), (2), (3) to find $y(h), y(h+h), \ldots, y(nh)$

$$y(h) = (2-h^2)y(0) - y(-h)$$

$$y(h+h) = (2-h^2)y(h) - y(0)$$

$$y(3h) = (2-h^2)y(2h) - y(h)$$

M

$$y(nh) = (2-h^2)y((n-1)h) - y((n-2)h)$$

M

Find the first n such that $y(nh) \geq 0$ and $y((n+1)h) \leq 0$!!

Which one would you choose as an approximated value of $\frac{\pi}{2}$? Any alternatives?

An ODE method (4)

- Try various values of *h* !

- Discussion.
  1. Accuracy test
  2. Make a table of errors for various values of *h*.
  3. Is this method good to compute π?

Numerical integration (1)

$$4\int_0^1 \frac{dx}{1+x^2} = \pi$$

- Box formulae – left point, right point, mid-point rule
- Trapezoidal rule – using a linear function
- Simpson's rule – using a quadratic function
- Boole's rule – using a quartic function(fourth order p olynomial)
- Other numerical integration technique?

Numerical Integration (2)

- Derive each integration formulae!
- Try several values of h and compare the results
- Discussions!
  1. Accuracy
  2. The speed at which the method attains a certai n precision
  3. The ease of coding

Numerical integration (3)

$$A = \int_a^b f(x)dx, \quad h = \frac{b-a}{N}, x_i = a + ih, y_i = f(x_i), \quad i = 0,\ldots,N,$$

- Left-point rule: $A \approx h\sum_{i=1}^{N} f(x_{i-1}) = h\left\{f(x_0) + f(x_1) + \cdots + f(x_{N-1})\right\}$

- Right-point rule: $A \approx h\sum_{i=1}^{N} f(x_i) = h\left\{f(x_1) + f(x_2) + \cdots + f(x_N)\right\}$

- Mid-point rule: $A \approx h\sum_{i=1}^{N} f(\frac{x_{i-1}+x_i}{2}) = h\left\{f(\frac{x_0+x_1}{2}) + f(\frac{x_1+x_2}{2}) + \cdots + f(\frac{x_{N-1}+x_N}{2})\right\}$

- Trapezoidal rule: $A \approx h\,(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{N-1} + \tfrac{1}{2}y_N)$

- Simpson's rule: $A \approx \tfrac{1}{3}h\left\{(y_0 + y_{2n}) + 2(y_2 + \cdots + y_{2n-2}) + 4(y_1 + \cdots + y_{2n-1})\right\}, \quad N = 2n$

- Boole's rule: $A \approx \tfrac{1}{45}h\{14(y_0 + y_{4n}) + 64(y_1 + y_3 + \cdots + y_{4n-3} + y_{4n-1}) +$
  $24(y_2 + y_6 \cdots + y_{4n-2}) + 28(y_4 + y_8 + \cdots + y_{4n-4})\}, \quad N = 4n$

Taylor series method (1)

- Taylor series

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + \cdots , \quad |x| < 1$$

$$\left(\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \right)$$

$$\tan^{-1}1 = \frac{\pi}{4}$$

- Binomial Theorem

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \cdots , \quad |x| < 1$$

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \quad \Rightarrow \quad \sin^{-1}1 = \frac{\pi}{2}, \qquad \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

- Convergent alternative series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = s$

$$|s - \sum_{n=1}^{N} a_n| < a_{N+1}$$

Taylor method (2)

$$\tan 2\beta = \frac{1}{5} \Rightarrow \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{5}{12} \quad \Rightarrow \quad \tan 4\beta = \frac{2\tan 2\beta}{1-\tan^2 2\beta} = \frac{120}{119} \Rightarrow \tan(4\beta - \tfrac{\pi}{4}) = \frac{\tan 4\beta - 1}{1+\tan 4\beta} = \frac{1}{239}$$

$$\pi = 16\beta - 4\tan^{-1}(\tfrac{1}{239}) = 16\tan^{-1}\tfrac{1}{5} - 4\tan^{-1}\tfrac{1}{239} \approx 16\{\tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3\} - 4\tfrac{1}{239}$$

$$= 3.15733333344 - 0.1673640167 = 3.14059693177$$

$$|\pi - 3.14059693177| \le 9.996 \times 10^{-4}$$

Use Taylor series for $\tan^{-1} x$ to get approximations of $\tan^{-1}\frac{1}{5}$ and $\tan^{-1}\frac{1}{239}$.

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + \cdots, \quad |x| < 1$$

$$\tan^{-1}\tfrac{1}{5} = \tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 + \cdots + \tfrac{(-1)^{n-1}}{2n-1}(\tfrac{1}{5})^{2n-1} + \cdots$$

$$\tan^{-1}\tfrac{1}{239} = \tfrac{1}{239} - \tfrac{1}{3}(\tfrac{1}{239})^3 + \tfrac{1}{5}(\tfrac{1}{239})^5 - \tfrac{1}{7}(\tfrac{1}{239})^7 + \cdots + \tfrac{(-1)^{n-1}}{2n-1}(\tfrac{1}{239})^{2n-1} + \cdots$$

Make a table of number of terms to accuracy needed for this formulation.

$$\left|\tan^{-1}\tfrac{1}{5} - \{\tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3\}\right| < \tfrac{1}{5}(\tfrac{1}{5})^5 \approx 6.4 \times 10^{-5}$$

$$\left|\tan^{-1}\tfrac{1}{239} = \tfrac{1}{239}\right| < \tfrac{1}{3}(\tfrac{1}{239})^3 \approx 2.44 \times 10^{-8}$$