Approximation of iid Normal and AR(1) Processes

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1 Introduction

In this note, I will present methods to approximate AR(1) process using first order Markov process. Among many methods, I will present only the methods which do not explicitly rely on the numerical integration. I will start with the method developed by Tauchen (1986), which is widely used in various economic models. Then I will show an improvement of the method, proposed in Ada and Cooper (2003).

As a preparation, I will start with an approximation of an iid normal shock. The two methods used to approximate an iid normal shock correspond to the two methods that I will show for the approximation of AR(1) process.

For methods explicitly based on numerical integration, including the method proposed by Tauchen and Hussey (1991), please see the separate note.

2 iid Normal Shock: A Motivating Example

In many applications in economics, there are iid normal shocks. Below is an example used in macro.

Problem 1 (Consumption-Savings Decision with iid Normal Shock to Labor Income)

$$V(z,a) = \max_{c\geq 0, a'\geq \underline{a}} \{u(c) + \beta EV(z',a')\}$$

subject to

$$a(1+r) + we^z = a' + c$$
$$z' \sim iidN(0,\sigma_z^2)$$
r and w are given

It is easy to solve the problem using discretized state space method or finite element approximation of the value function, once the iid normal shock to labor income is approximated by a discrete shock. That's what we want to do. Let me first of all specify our problem:

Problem 2 (Approximation of normal distribution)
We want to approximate $N(\mu,\sigma_z^2)$ *with* $\{z_i\}_{i=1}^n$ *and* $\{p_i\}_{i=1}^n$ *where* p_i *is the probability that* z_i *realizes.*

3 Approximating iid Normal Shock: Simple method

Below is the algorithm:

Algorithm 1 (Simple Method to Approximate iid Normal Shock)

1. *Set n, which is the number of potential realizations of z.*

2. *Set the upperbound $\overline{z}$ and the lowerbound $\underline{z}$ to the support of z. Considering the symmetry of the normal distribution around μ, a natural way to set the bounds is to choose λ such that:*

$$\overline{z} = \mu + \lambda\sigma_z$$

$$\underline{z} = \mu - \lambda\sigma_z$$

3. *Set $\{z_i\}_{i=1}^n$ such that, $z_1 = \underline{z}$, $z_n = \overline{z}$, and all of $\{z_i\}_{i=1}^n$ are equally distanced. In other words, for $i = 1, 2, ..., n$:*

$$z_i = \underline{z} + \frac{\overline{z} - \underline{z}}{n-1}(i-1) = \underline{z} + \frac{2\lambda\sigma_z}{n-1}(i-1)$$

4. *Construct the midpoints $\{m_i\}_{i=1}^{n-1}$. m_i is constructed as follows:*

$$m_i = \frac{z_{i+1} + z_i}{2}$$

5. *Denote the cumulative density function (CDF) of the normal distribution with mean 0 and variance 1 as $\Phi(.)$. For $i = 2, 3, ..., n-1$, p_i (probability assigned to z_i) can be computed as follows:*

$$p_i = \Phi\left(\frac{m_i - \mu}{\sigma_z}\right) - \Phi\left(\frac{m_{i-1} - \mu}{\sigma_z}\right)$$

In other words, the probability that z_i is realized is defined as the probability that a draw from the normal distribution falls into the interval $[m_{i-1}, m_i]$, which is constructed around z_i. For the two endpoints, we have to take care of the fact that we cut both tails by putting finite bounds to the support. Therefore:

$$p_1 = \Phi\left(\frac{m_1 - \mu}{\sigma_z}\right)$$

$$p_n = 1 - \Phi\left(\frac{m_{n-1} - \mu}{\sigma_z}\right)$$

Nice thing about the method is simplicity. Notice that you have a complete freedom in choosing the number of abscissas (n) and the set of abscissas $\{z_i\}_{i=1}^n$. Of course, in general, the larger n, the better, but we usually cannot afford to use a high n. Regarding how to set $\{z_i\}_{i=1}^n$, if we decide to use equal-distance abscissas, the choice of $\{z_i\}_{i=1}^n$ is replaced by the choice of λ. Still, λ is a free parameter. Especially when n is small, be careful in how to choose λ. You can choose λ such that certain property of the original distribution is replicated by the approximated distribution. For example, we can choose λ such that the variance of the approximated process is the same as the variance of the original process.

4 Approximating iid Normal Shock: Ada and Cooper (2003)

The method proposed by Ada and Cooper (2003) is as follows. Basically it's an improvement over the simple method presented above. The key difference is that the method divides the original support into n intervals in a way that the probability that a random draw falls into an interval is $\frac{1}{n}$ for any internal. Let's see the algorithm.

Algorithm 2 (Approximation of iid Normal Shock by Ada and Cooper (2003))

1. *Set n, which is the number of potential realizations of z.*

2. *Denote the cumulative density function (CDF) of $N(0,1)$ as $\Phi(.)$. First of all, construct $\{m_i\}_{i=1}^{n-1}$ such that the following is satisfied:*

$$\Phi\left(\frac{m_i-\mu}{\sigma_z}\right)=\frac{i}{n}$$

3. *Using the inverse function of $\Phi(.)$, m_i can be explicitly defined as follows:*

$$m_i=\Phi^{-1}\left(\frac{i}{n}\right)\sigma_z+\mu$$

4. *The points $\{m_i\}_{i=1}^{n-1}$ define the intervals $\{Z_i\}_{i=1}^{n}$. Specifically, interval Z_i with $i=2,3,...,n-1$ is characterized by $Z_i=[m_{i-1},m_i]$. The interval at the two ends (Z_1 and Z_n) are characterized as $Z_1=(-\infty,m_1]$ and $Z_n=[m_{n-1},\infty)$.*

5. *Next step is to construct the abscissas $\{z_i\}_{i=1}^{n}$. z_i is chosen such that z_i is the expected value of the truncated normal distribution with interval Z_i. Formally:*

$$z_i=E[z|z\in Z_i]$$

Notice the formula for the expected value of the truncated normal distribution $N(\mu,\sigma^2)$ is as follows:

$$E[y|y\in[\underline{y},\overline{y}]]=\mu-\sigma\frac{\phi\left(\frac{\overline{y}-\mu}{\sigma}\right)-\phi\left(\frac{\underline{y}-\mu}{\sigma}\right)}{\Phi\left(\frac{\overline{y}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{y}-\mu}{\sigma}\right)}$$

where $\Phi(.)$ and $\phi(.)$ are the CDF and PDF of $N(0,1)$. Applying the formula to the current problem, z_i can be computed as follows:

$$\begin{aligned}z_i&=E[z|z\in Z_i]\\&=\mu-\sigma_z\frac{\phi(\frac{m_i-\mu}{\sigma_z})-\phi(\frac{m_{i-1}-\mu}{\sigma_z})}{\Phi(\frac{m_i-\mu}{\sigma_z})-\Phi(\frac{m_{i-1}-\mu}{\sigma_z})}\\&=\mu-\sigma_z n\left(\phi\left(\frac{m_i-\mu}{\sigma_z}\right)-\phi\left(\frac{m_{i-1}-\mu}{\sigma_z}\right)\right)\qquad i=2,3,...,n-1\end{aligned}$$

$$z_1 = \mu - \sigma_z n \ \phi\left(\frac{m_1 - \mu}{\sigma_z}\right)$$

$$z_n = \mu + \sigma_z n \ \phi\left(\frac{m_{n-1} - \mu}{\sigma_z}\right)$$

We can directly apply the formula to compute $\{z_i\}_{i=1}^n$*, if we can compute CDF for normal distribution.*

6. *Finally, obviously by construction:*

$$p_i = \frac{1}{n} \quad \forall i$$

Notice there is no degree of freedom with respect to this method, apart from the choice of n, and the abscissas are generally not equally distanced. The method automatically puts more abscissas to the neighborhood of the center of the distribution where the probability is higher.

5 AR(1) Shock: A Motivating Example

Now let's look at the methods to approximate AR(1) process, using first order Markov chain. The methods are direct extension of the methods to approximate iid normal shock that we have just seen. We start by a motivating example.

Problem 3 (Consumption-Savings Decision with AR(1) Shock to Labor Income)

$$V(z,a) = \max_{c\geq 0, a'\geq \underline{a}} \left\{u(c) + \beta E_{z'|z} V(z',a')\right\}$$

subject to

$$a(1+r) + we^z = a' + c$$
$$z' = (1-\rho)\mu + \rho z + \epsilon' \qquad \epsilon' \sim iidN(0,\sigma_\epsilon^2)$$
r *and* w *are given*

Problem 4 (Approximation of AR(1) process)
We want to approximate the following AR(1) process with $\{z_i\}_{i=1}^n$ *and* $\{p_{ij}\}_{i=1,j=1}^{n,n}$ *where* p_{ij} *is the transition probability from state* z_i *to state* z_j*:*

$$z' = (1-\rho)\mu + \rho z + \epsilon' \qquad \epsilon' \sim iidN(0,\sigma_\epsilon^2)$$

6 Approximating AR(1) Shock: Tauchen (1986)

Below is the algorithm:

Algorithm 3 (Approximating AR(1) Shock: Tauchen (1986))

1. *Set n, which is the number of potential realizations of z.*
2. *Notice that the stationary distribution of z is $N(\mu, \sigma_z^2)$ where $\sigma_z = \frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}$.*
3. *Set the upperbound $\overline{z}$ and the lowerbound $\underline{z}$ to the support of z. Considering the symmetry of the normal distribution around μ, a natural way to set the bounds is to choose λ such that:*

$$\overline{z} = \mu + \lambda\sigma_z$$

$$\underline{z} = \mu - \lambda\sigma_z$$

4. *Set $\{z_i\}_{i=1}^n$ such that, $z_1 = \underline{z}$, $z_n = \overline{z}$, and all of $\{z_i\}_{i=1}^n$ are equally distanced. In other words, for $i = 1, 2, ..., n$:*

$$z_i = \underline{z} + \frac{\overline{z} - \underline{z}}{n-1}(i-1) = \underline{z} + \frac{2\lambda\sigma_z}{n-1}(i-1)$$

5. *Construct the midpoints $\{m_i\}_{i=1}^{n-1}$. m_i is constructed as follows:*

$$m_i = \frac{z_{i+1} + z_i}{2}$$

6. *Let's construct intervals $\{Z_i\}_{i=1}^n$ as follows:*

$$Z_1 = (-\infty, m_1]$$

$$Z_i = [m_i, m_{i+1}] \quad i = 2, 3, ..., n-1$$

$$Z_n = [m_n, \infty)$$

7. *We will approximate the transition probability p_{ij} as the probability that, conditional on z_i, $z' = (1-\rho)\mu + \rho z_i + \epsilon'$ falls into the interval j. p_{ij} can be easily computed as follows:*

$$p_{ij} = \Phi\left(\frac{m_j - (1-\rho)\mu - \rho z_i}{\sigma_\epsilon}\right) - \Phi\left(\frac{m_{j-1} - (1-\rho)\mu - \rho z_i}{\sigma_\epsilon}\right) \qquad j = 2, 3, ..., n-1$$

$$p_{i1} = \Phi\left(\frac{m_1 - (1-\rho)\mu - \rho z_i}{\sigma_\epsilon}\right)$$

$$p_{in} = 1 - \Phi\left(\frac{m_{n-1} - (1-\rho)\mu - \rho z_i}{\sigma_\epsilon}\right)$$

where $\Phi(.)$ is CDF of $N(0,1)$.

Nice thing about the method is simplicity. Notice that you have a complete freedom in choosing the number of abscissas (n) and the set of abscissas $\{z_i\}_{i=1}^n$. Of course, in general, the larger n, the better, but we usually cannot afford to use a high n. Regarding how to set $\{z_i\}_{i=1}^n$, if we decide to use equally-distanced abscissas, the choice of $\{z_i\}_{i=1}^n$ is replaced by the choice of λ. Still, λ is a free parameter. How should we determine n and λ? There is no single answer. Tauchen (1986) implements Monte Carlo experiments and finds that $n = 9$ and $\lambda = 3$ works really well even if the original process has a very high persistence. Again, after choosing n such that n is the largest number with which the computational time is bearable, λ can be pinned down such that some statistics of the approximated process match the counterparts of the original process. For example, if you are using a life-cycle economy, and trying to discretize the shocks to earnings, λ can be chosen such that the life-cycle profile of variances of earnings, conditional on age, associated with the discretized process replicates the life-cycle profile of earnings variance implied by the original process.

7 Approximating AR(1) Shock: Ada and Cooper (2003)

Algorithm 4 (Approximating AR(1) Shock: Ada and Cooper (2003))

1. *Set n, which is the number of potential realizations of z.*
2. *Notice that the stationary distribution of z is $N(\mu, \sigma_z^2)$ where $\sigma_z = \frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}$.*
3. *Denote the cumulative density function (CDF) of $N(0,1)$ as $\Phi(.)$. First of all, construct $\{m_i\}_{i=1}^{n-1}$ such that the following is satisfied:*

$$\Phi\left(\frac{m_i - \mu}{\sigma_z}\right) = \frac{i}{n}$$

4. *Using inverse function of $\Phi(.)$, m_i can be explicitly defined as follows:*

$$m_i = \Phi^{-1}\left(\frac{i}{n}\right)\sigma_z + \mu$$

5. *The points $\{m_i\}_{i=1}^{n-1}$ define the intervals $\{Z_i\}_{i=1}^n$. Specifically, interval Z_i with $i = 2, 3, ..., n-1$ is characterized by $Z_i = [m_{i-1}, m_i]$. The interval at the two ends (Z_1 and Z_n) are characterized as $Z_1 = (-\infty, m_1]$ and $Z_n = [m_{n-1}, \infty)$.*
6. *Next step is to construct the abscissas $\{z_i\}_{i=1}^n$. z_i is chosen such that z_i is the expected value of the truncated normal distribution with interval Z_i. Formally:*

$$z_i = E[z|z \in Z_i]$$

Notice the formula for the expected value of the truncated normal distribution $N(\mu, \sigma)$ is as follows:

$$E[y|y \in [\underline{y}, \overline{y}]] = \mu - \sigma\frac{\phi\left(\frac{\overline{y}-\mu}{\sigma}\right) - \phi\left(\frac{\underline{y}-\mu}{\sigma}\right)}{\Phi\left(\frac{\overline{y}-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{y}-\mu}{\sigma}\right)}$$

where $\Phi(.)$ and $\phi(.)$ are the CDF and PDF of $N(0,1)$, respectively. We can directly apply the formula to compute $\{z_i\}_{i=1}^n$, if we can compute both CDF and PDF of normal distribution. Applying the formula to the current problem, z_i can be computed as follows:

$$\begin{aligned} z_i &= E[z|z \in Z_i] \\ &= \mu - \sigma_z \frac{\phi(\frac{m_i-\mu}{\sigma_z}) - \phi(\frac{m_{i-1}-\mu}{\sigma_z})}{\Phi(\frac{m_i-\mu}{\sigma_z}) - \Phi(\frac{m_{i-1}-\mu}{\sigma_z})} \\ &= \mu - \sigma_z n \left(\phi\left(\frac{m_i-\mu}{\sigma_z}\right) - \phi\left(\frac{m_{i-1}-\mu}{\sigma_z}\right)\right) \qquad i = 2,3,...,n-1 \end{aligned}$$

$$z_1 = \mu - \sigma_z n \ \phi\left(\frac{m_1-\mu}{\sigma_z}\right)$$

$$z_n = \mu + \sigma_z n \ \phi\left(\frac{m_{n-1}-\mu}{\sigma_z}\right)$$

If you can compute PDF for normal distribution, it's easy to compute z_i.

7. *Finally, the transition probability can be computed as follows:*

$$\begin{aligned} p_{ij} &= prob(z' \in Z_j | z \in Z_i) \\ &= \frac{prob(z \in Z_i, z' \in Z_j)}{prob(z \in Z_i)} \\ &= n \ prob(z \in Z_i, z' \in Z_j) \\ &= n \int_{m_{i-1}}^{m_i} Prob(z' \in Z_j) \phi\left(\frac{z-\mu}{\sigma_z}\right) dz \\ &= n \int_{m_{i-1}}^{m_i} \left[\Phi\left(\frac{m_j - \mu(1-\rho) - \rho z}{\sigma_z}\right) - \Phi\left(\frac{m_{j-1} - \mu(1-\rho) - \rho z}{\sigma_z}\right)\right] \phi\left(\frac{z-\mu}{\sigma_z}\right) dz \end{aligned}$$

I omit the small modifications needed in cases $i = 1, n$ or $j = 1, n$. Just be careful in these cases. m_i or m_j cannot be defined but it's easy to see how to modify the formula.

Integrant can be easily computed but there is no nice formula for the outside integration. Therefore, we need to rely on a numerical integration method to compute p_{ij}.

Notice again that *(i)* there is no degree of freedom with respect to this method, apart from the choice of n, and *(ii)* the abscissas are generally not equally distanced. The method automatically puts more abscissas to the neighborhood of the center of the distribution where the probability is higher.

References

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