

Bayesian Machine Learning: Assignment #1

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Problem 1

(a) Using Integration by parts,

$$E[X] = \int_{-\infty}^{\infty} xf(x) dx = \int_{-\infty}^{\theta} x \cdot 0 dx + \int_{\theta}^{\infty} xe^{\theta-x} dx = 1 + \theta$$

$$\text{Var}[X] = E[X^2] - E[X]^2 = \int_{\theta}^{\infty} x^2 e^{\theta-x} dx - (1 + \theta)^2 = \theta^2 + 2(1 + \theta) - (1 + \theta)^2 = 1$$

(b) To show unbiasedness, compare θ with the expectation. Note X_i 's are i.i.d., and $E[X_i] = 1 + \theta$ in (a):

$$\begin{aligned} E[\hat{\theta}_n] &= E\left[\frac{1}{n} \sum_{i=1}^n (X_i - 1)\right] \\ &= \frac{1}{n} \sum_{i=1}^n E[X_i - 1] \\ &= \frac{1}{n} \sum_{i=1}^n (E[X_i] - 1) \\ &= \frac{1}{n} \sum_{i=1}^n ((1 + \theta) - 1) \\ &= \frac{1}{n} \sum_{i=1}^n \theta \\ &= \frac{1}{n} (n\theta) \\ &= \theta \end{aligned}$$

(c) Let $Y_i = X_i - 1$. Then, $E[Y_i] = \theta$ and $\text{Var}[Y_i] = 1$. Now, according to (b) and Central Limit Theorem,

$$\frac{\hat{\theta}_n - \theta}{1/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1)$$

holds. Then, $100(1 - \alpha)\%$ confidence interval of θ is $[\hat{\theta}_n - \frac{\Phi^{-1}(1-\frac{\alpha}{2})}{\sqrt{n}}, \hat{\theta}_n + \frac{\Phi^{-1}(\frac{\alpha}{2})}{\sqrt{n}}]$ where Φ is Cumulative Density Function (CDF) of unit normal distribution.

(d) According to given fact and that unit normal distribution is an even function, $\Phi^{-1}(0.025) = -1.96$ and $\Phi^{-1}(0.975) = 1.96$. Thus, 95% confidence interval of θ is $[11.33 - \frac{1.96}{\sqrt{3}}, 11.33 + \frac{1.96}{\sqrt{3}}] = [10.20, 12.46]$. It is weird that the observed data 10.0 is contradictory for any θ in the obtained confidence interval according to the PDF of X .

Problem 2

1. The proof is straightforward:

$$\begin{aligned} \mu(B) &= \mu(B \cap (A \cup A^c)) \\ &= \mu((B \cap A) \cup (B \cap A^c)) && \because \text{distributive law} \\ &= \mu(B \cap A) + \mu(B \cap A^c) && \because \text{countable additivity of } \mu \\ &= \mu(A) + \mu(B \cap A^c) && \because A \subset B \implies B \cap A = A \\ &\geq \mu(A) && \because \mu(B \cap A^c) \geq 0 : \text{nonnegativity of } \mu \end{aligned}$$

2. Use induction. (Base case) We first show it holds for $n = 1$:

$$\begin{aligned}\mu\left(\bigcup_{i=1}^1 A_i\right) &= \mu(A_1) \\ &= \sum_{i=1}^1 \mu(A_i) \\ &\leq \sum_{i=1}^1 \mu(A_i)\end{aligned}$$

(Inductive case) Assume it holds for $n = k$. Want to show it holds for $n = k + 1$:

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{k+1} A_i\right) &= \mu\left(\bigcup_{i=1}^k A_i \cup A_{k+1}\right) \\ &= \mu\left(\bigcup_{i=1}^k A_i \cup (A_{k+1} - \bigcup_{i=1}^k A_i)\right) && \because \text{simple Venn Diagram argument} \\ &= \mu\left(\bigcup_{i=1}^k A_i\right) + \mu\left(A_{k+1} - \bigcup_{i=1}^k A_i\right) && \because \text{countable additivity of } \mu \\ &\leq \mu\left(\bigcup_{i=1}^k A_i\right) + \mu(A_{k+1}) && \because (A_{k+1} - \bigcup_{i=1}^k A_i) \subseteq A_{k+1} \text{ and } 1. \\ &\leq \sum_{i=1}^k \mu(A_i) + \mu(A_{k+1}) && \because \text{inductive hypothesis} \\ &= \sum_{i=1}^{k+1} \mu(A_i)\end{aligned}$$

□

Problem 3

Give an appropriate positive constant c such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 1, g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2, g(n) = n^2$
3. $f(n) = n^2 - n + 1, g(n) = n^2/2$

Solution

We solve each solution algebraically to determine a possible constant c .

Part One

$$\begin{aligned}n^2 + n + 1 &= \\ &\leq n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\leq c \cdot 2n^3\end{aligned}$$

Thus a valid c could be when $c = 2$.

Part Two

$$\begin{aligned}n^2 + n\sqrt{n} &= \\&= n^2 + n^{3/2} \\&\leq n^2 + n^{4/2} \\&= n^2 + n^2 \\&= 2n^2 \\&\leq c \cdot n^2\end{aligned}$$

Thus a valid c is $c = 2$.

Part Three

$$\begin{aligned}n^2 - n + 1 &= \\&\leq n^2 \\&\leq c \cdot n^2/2\end{aligned}$$

Thus a valid c is $c = 2$.

Problem 4

Let $\Sigma = \{0, 1\}$. Construct a DFA A that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state q_k indicate the remainder of k divided by 5. For example, the remainder of 2 would correlate to state q_2 because $7 \bmod 5 = 2$.

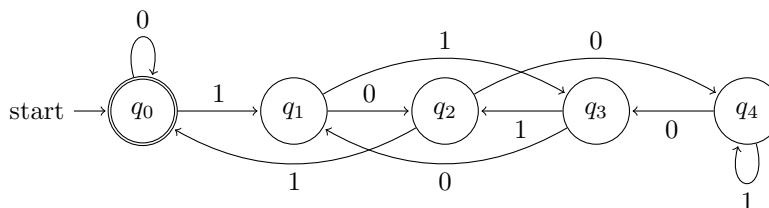


Figure 1: DFA, A , this is really beautiful, ya know?

Justification

Take a given binary number, x . Since there are only two inputs to our state machine, x can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multiplying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

	$x \bmod 5 = 0$	$x \bmod 5 = 1$	$x \bmod 5 = 2$	$x \bmod 5 = 3$	$x \bmod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state q_0 or ($x \bmod 5 = 0$), a transition line should go to state q_0 for the input 0 and a line should go to state q_1 for input 1. Continuing this gives us the Figure 1.

Problem 5

Write part of **Quick-Sort**($list, start, end$)

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1: function QUICK-SORT( $list, start, end$ )
2:   if  $start \geq end$  then
3:     return
4:   end if
5:    $mid \leftarrow \text{PARTITION}(list, start, end)$ 
6:   QUICK-SORT( $list, start, mid - 1$ )
7:   QUICK-SORT( $list, mid + 1, end$ )
8: end function
  
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Algorithm 1: Start of QuickSort

Problem 6

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \dots, n$, $E[e_i] = 0$, and $\text{Var}[e_i] = \sigma_e^2$ and $\text{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares estimator for $\hat{\beta}_1$ for the slope β_1 .

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^n (Y_i - \hat{\beta}_1 x_i)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1} (RSS) = -2 \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^n x_i Y_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^n x_i Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1 :

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

Solution

For the bias, we need to calculate the expected value $E[\hat{\beta}_1]$:

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i E[Y_i]}{\sum x_i^2} \\ &= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\ &= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \end{aligned}$$

Thus since our estimator's expected value is β_1 , we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned} \text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x_i^2} \end{aligned}$$

Problem 7

Prove a polynomial of degree k , $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \dots a_0$ are nonnegative constants.

Proof. To prove that $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, c_1 and a_k .

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^k a_i$ will give us a new constant, A . By taking this value of A , we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0 &= \\ &\leq (a_k + a_{k-1} \dots a_1 + a_0) \cdot n^k \\ &= A \cdot n^k \\ &\leq c_2 \cdot n^k \end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. □

Problem 18

Evaluate $\sum_{k=1}^5 k^2$ and $\sum_{k=1}^5 (k-1)^2$.

Problem 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem 6

Evaluate the integrals $\int_0^1 (1-x^2) dx$ and $\int_1^\infty \frac{1}{x^2} dx$.

Problem 20

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