Homework 1

Probability Review and Priors

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Question 1

(15) You are a data scientist and are choosing between three approaches, A, B, and C, to a problem. With approach A you will spend a total of 4 days coding and running an algorithm and it will not produce useful results. With approach B you will spend a total of 3 days coding and running an algorithm and it will not produce useful results. With approach C you will spend 1 day coding and running an algorithm and it will produce useful results. You are equally likely to choose among unselected options. What is the expected time in days for you to obtain the results you are looking for, if you continue to select an unselected option when you do not obtain useful results? What is the variance on this time?

Solution 1

For this problem you can think of the the possible choices as shown below.

Starting with A

- $A \rightarrow B \rightarrow C$
- $A \rightarrow C$

Starting with B

- $B \rightarrow A \rightarrow C$
- $B \rightarrow C$

Starting with C

- C

We always end at C given the information that the algorithm will produce useful results. We are also given that the approaches A, B, and C are all equally likely. First lets compute how many days each option (steps) will take

Starting with A

- $A \rightarrow B \rightarrow C = 4 \rightarrow 3 \rightarrow 1 = 8$ days
- $A \rightarrow C = 4 \rightarrow 1 = 5$ days

Starting with B

- $B \to A \to C = 3 \to 4 \to 1 = 8$ days
- $B \to C = 3 \to 1 = 4$ days

Starting with C

- $C \to 1$ days

Additionally we need to compute the probabilities that each approach will be. We are alos given the informatino taht we will choose among the unselected options.

Starting with A

- $A \to B \to C = \frac{1}{3} \to \frac{1}{2} \to 1 = \frac{1}{6}$
- $A \to C = \frac{1}{3} \to \frac{1}{2} = \frac{1}{6}$

Starting with B

- $B \to A \to C = \frac{1}{3} \to \frac{1}{2} \to 1 = \frac{1}{6}$
- $B \to C = \frac{1}{3} \to \frac{1}{2} = \frac{1}{6}$

Starting with C

- $C \to \frac{1}{3}$

Given all the days and propbabilities we can find the expected number of days with the given probabilites.

$D(A_1)$ = The days for the first option starting with A

$P(A_1)$ = The probability for the first option starting with A

$$E(X) = D(A_1) \cdot P(A_1) + D(A_2) \cdot P(A_2) + D(B_1) \cdot P(B_1) + D(B_2) \cdot P(B_2) + D(C_1) \cdot$$

$$E(X) = 8 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 8 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 1 \cdot \frac{1}{3}$$

$$E(X) = \frac{8}{6} + \frac{5}{6} + \frac{8}{6} + \frac{4}{6} + \frac{2}{6}$$

$$E(X) = \frac{27}{6} = 4.5$$

In order to find the variance let us map the data into data point values.

Expected Days = 4.5

Paths = [ABC, AC, BAC, BC, C]

Proabilities = [$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}$]

Actual Days = [8, 5, 8, 4, 1]

We can compute the variance of X to be the following formulation.

$$VAR[X] = E[(X - E[X])^2] = \Sigma_{x \in dom(X)}(x - E[x])^2 p(x)$$

In [1]:

```
probs = [1/6, 1/6, 1/6, 1/6, 1/3]
days = [8,5,8,4,1]
expected = 0
for i in range(len(days)):
    expected = probs[i]*days[i] + expected
expected
```

Out[1]:

```
4.5
```

In [2]:

```
probs = [1/6, 1/6, 1/6, 1/6, 1/3]
actual = 8,5,8,4,1
var = 0
for i in range(len(actual)):
    var = var + (actual[i] - expected)**2 * probs[i]
var
```

Out[2]:

```
8.25
```

From the computation above and code computation we can see that for this problem the expected time to produce results is **4.5** with a variance of **8.25**

Question 2

(15) Suppose if it is sunny or not in Charlottesville depends on the weather of the last three days. Show how this can be modeled as a Markov chain.

Solution 2

Since the weather condition is dependent on the last three days the following options are possible for the three day sequence.

1. Sunny, Sunny, Sunny
2. Sunny, Sunny, Not Sunny
3. Sunny, Not Sunny, Sunny
4. Sunny, Not Sunny, Not Sunny
5. Not Sunny, Sunny, Sunny
6. Not Sunny, Sunny, Not Sunny
7. Not Sunny, Not Sunny, Sunny
8. Not Sunny, Not Sunny, Not Sunny

Let's say that our previous sequence of days for weather is S, S, S. The next day will be the following. *, S, S. The* can either be S or N with assumption that S and N has a 0.5 probability of occurence for the most flexible setting. From this information we can make a transition Matrix like the following where the rows indicate what stage the weather is and rows represent what the new weather state will be.

States	1	2	3	4	5	6	7	8
1	0.5	0	0	0	0.5	0	0	0
2	0.5	0	0	0	0.5	0	0	0
3	0	0.5	0	0	0	0.5	0	0
4	0	0.5	0	0	0	0.5	0	0
5	0	0	0.5	0	0	0	0.5	0
6	0	0	0.5	0	0	0	0.5	0
7	0	0	0	0.5	0	0	0	0.5
8	0	0	0	0.5	0	0	0	0.5

The table shown above will the the transition matrix that can be utilized for the transition matrix in a markov chain problem.

Question 3

(15) Assume a Gaussian distribution for observations, Xi, i = 1, . . . , N with unknown mean, M, and known variance 5. Suppose the prior for M is Gaussian with variance 10. How large a random sample must be taken (i.e., what is the minimum value for N) to specify an interval having unit length of 1 such that the probability that M lies in this interval is 0.95?

Solution 3

In [3]:
```
prior_variance = 10
known_variance = 5
```

We know that the interval of 0.95 is the following

$\mu_x \pm 2\sigma_x$

Given this we know that $4\sigma_x = 1 \rightarrow \sigma_x = \frac{1}{4}$

We also know that $Var[M|x,\sigma^2] = \frac{\sigma_0^2\sigma^2}{\sigma^2+N\sigma^2}$. which we can solve for N

$\sigma^2 + N\sigma^2 = \frac{\sigma_0^2\sigma^2}{Var[M|x,\sigma^2]}$

$N\sigma^2 = \frac{\sigma_0^2\sigma^2}{Var[M|x,\sigma^2]} - \sigma^2$

$N = \frac{\sigma_0^2\sigma^2}{Var[M|x,\sigma^2]\cdot\sigma^2} - \sigma^2 \cdot \frac{1}{\sigma^2}$

$N = \frac{10*5}{(\frac{1}{4})^2\cdot 10} - 5 \cdot \frac{1}{10}$

$N = 5 * 16 - \frac{1}{2}$

In [4]:
```
5*16 - 0.5
```
Out[4]:
```
79.5
```

In order to satisify that the interval of 0.95 will have a unit length of 1, with the given conditions we will need a sample size of **79.5 = 80** or any sample size where **n > 80**

Question 4

(15) You have started an online business selling books that are of interest to your customers. A publisher has just given you a large book with photos from famous 20th century photographers. You think this book will appeal to people who have bought art books, history books and coffee table books. In an initial offering of the new book you collect data on purchases of the new book and combine these data with data from the past purchases (see ArtHistBooks.csv).

Use Bayesian analysis to give the posterior probabilities for purchases of art books, history books and coffee table books, as well as, the separate probabilities for purchases of the new book given each possible combination of prior purchases of art books, history books and coffee table books. Do this by first using beta priors with values of the hyperparameters that represent lack of prior information. Then compute these probabilities again with beta priors that show strong weighting for low likelihood of a book purchase. Compare your results.

Solution 4

In [5]:
```
import pandas as pd
books = pd.read_csv("datasets/ArtHistBooks.csv")
books
```
Out[5]:

	ArtBooks	HistoryBooks	TableBooks	Purchase
0	0	0	1	0
1	0	1	0	0
2	0	0	0	0
3	1	0	1	0
4	1	1	1	0
...	...	...	...	...
995	1	1	0	1
996	0	1	0	0
997	1	0	1	0
998	1	1	0	0
999	0	1	0	0

1000 rows × 4 columns

In [6]:
```
a_lack = 1
b_lack = 1

a_prior = 10
b_prior = 200
```

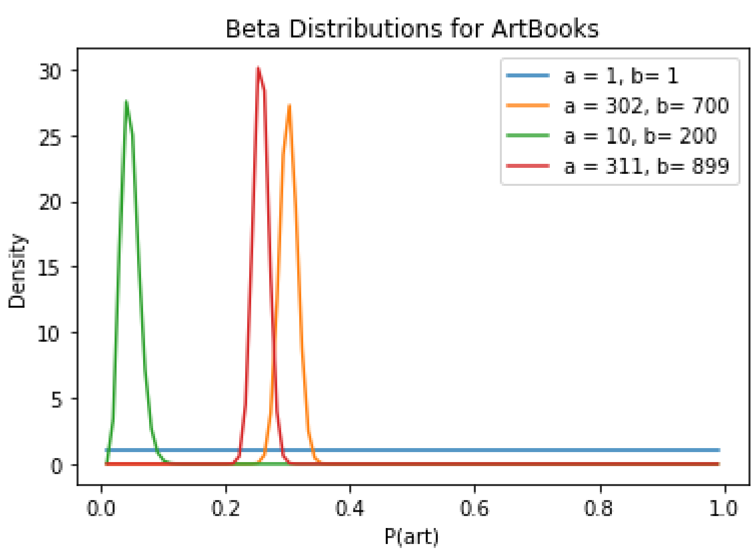
In [7]:
```
k_art = len(books[books["ArtBooks"]>0])
k_history = len(books[books["HistoryBooks"]>0])
k_table = len(books[books["TableBooks"]>0])
n = len(books)
print(k_art, k_history, k_table, n)
```

```
301 543 380 1000
```

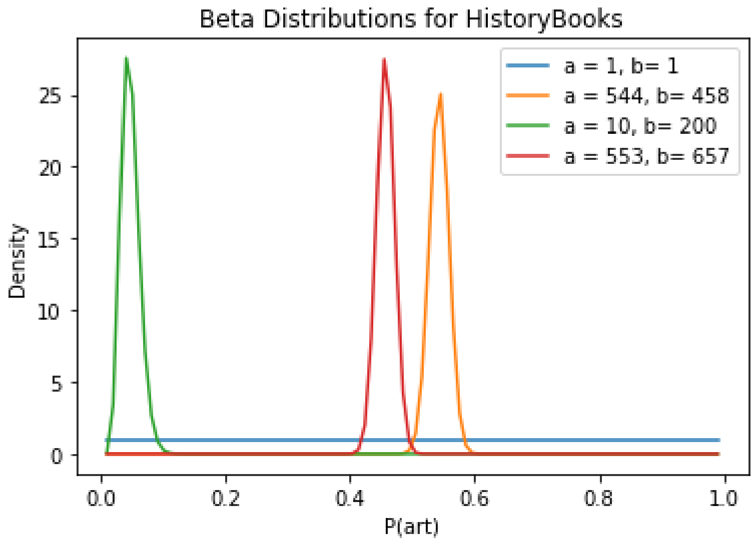
In [8]:
```
import numpy as np
x = np.linspace(0, 1, 100)[1:-1]
```

In [9]:
```
import matplotlib.pyplot as plt
from scipy.stats import beta

def createPlot(priors, bookType):
    for a,b in priors:
        prior_prob = beta.pdf(x = x, a=a, b=b)
        plt.plot(x, prior_prob, label='a = {}, b= {}'.format(a, b))
    plt.legend()
    plt.xlabel("P(art)")
    plt.ylabel("Density")
    plt.title("Beta Distributions for " + bookType)
    plt.show()

def createPriors(type_addition, n):
    return [(a_lack, b_lack),
            (a_lack + type_addition, b_lack+n-type_addition),
            (a_prior, b_prior),
            (a_prior + type_addition, b_prior+n-type_addition)]
```

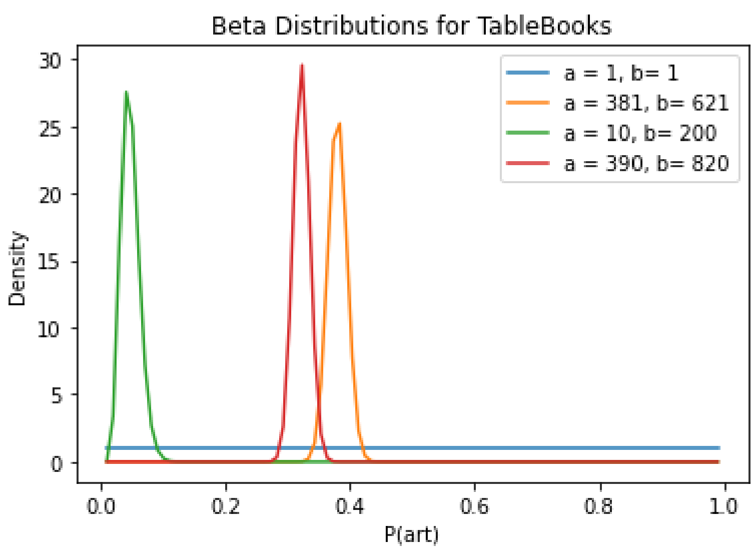
In [10]:
```
## For ArtBooks
artPriors = createPriors(k_art, len(books))
createPlot(artPriors,"ArtBooks")
```



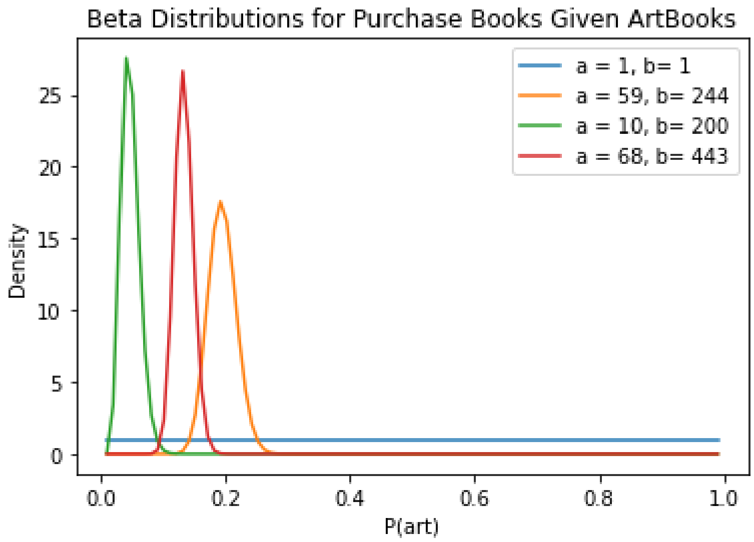
In [11]:
```python
## For History Books
historyPriors = createPriors(k_history, len(books))
createPlot(historyPriors,"HistoryBooks")
```



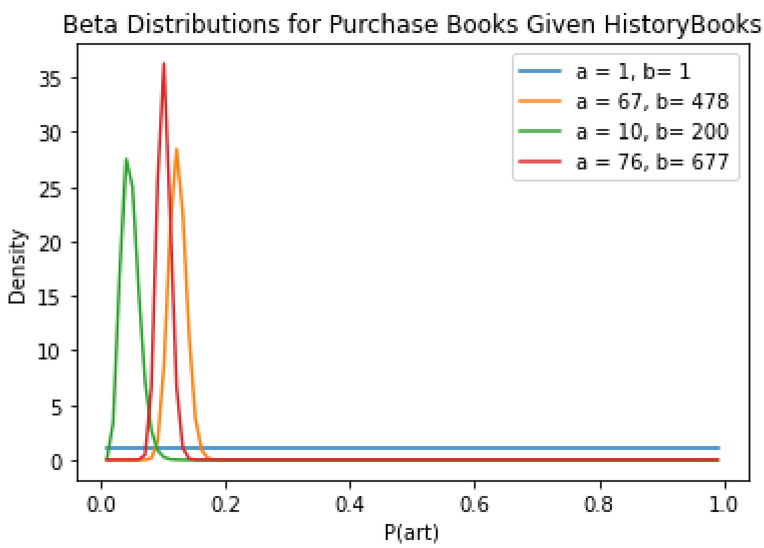
In [12]:
```python
## For Table Books
tablePriors = createPriors(k_table, len(books))
createPlot(tablePriors,"TableBooks")
```



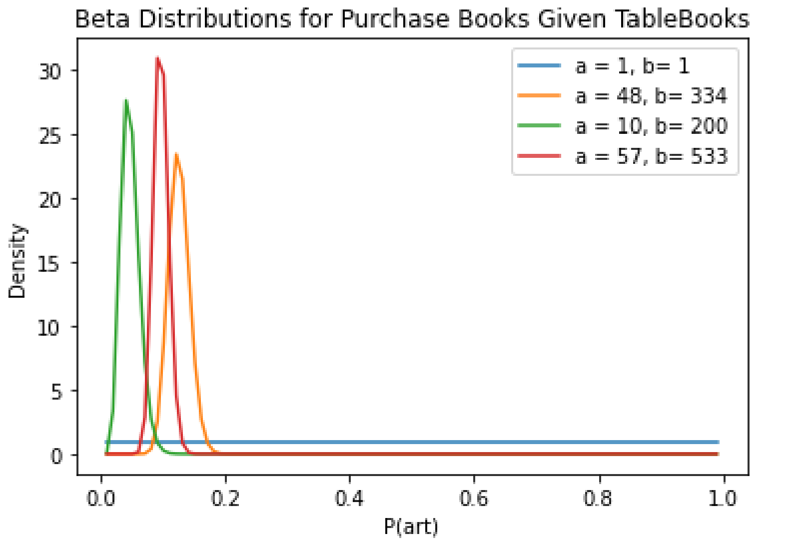
In [13]:
```python
## For Purchase given ArtBook Purchases
n_art = len(books.query("ArtBooks>0"))
purchase_given_art = len(books.query("ArtBooks>0").query("Purchase>0"))
tablePriors = createPriors(purchase_given_art, n_art)
createPlot(tablePriors,"Purchase Books Given ArtBooks")
```



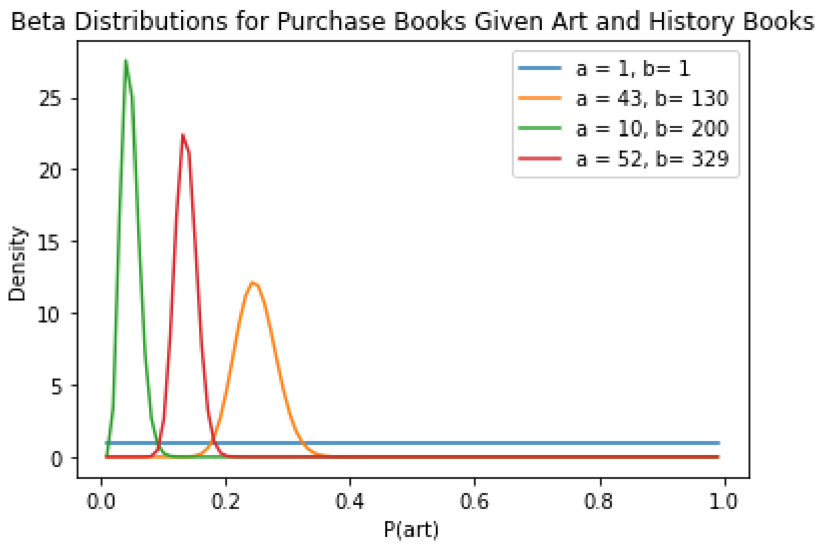
In [14]:
```
## For Purchase given HistoryBooks Purchases
n_history = len(books.query("HistoryBooks>0"))
purchase_given_history = len(books.query("HistoryBooks>0").query("Purchase>0"))
tablePriors = createPriors(purchase_given_history, n_history)
createPlot(tablePriors,"Purchase Books Given HistoryBooks")
```



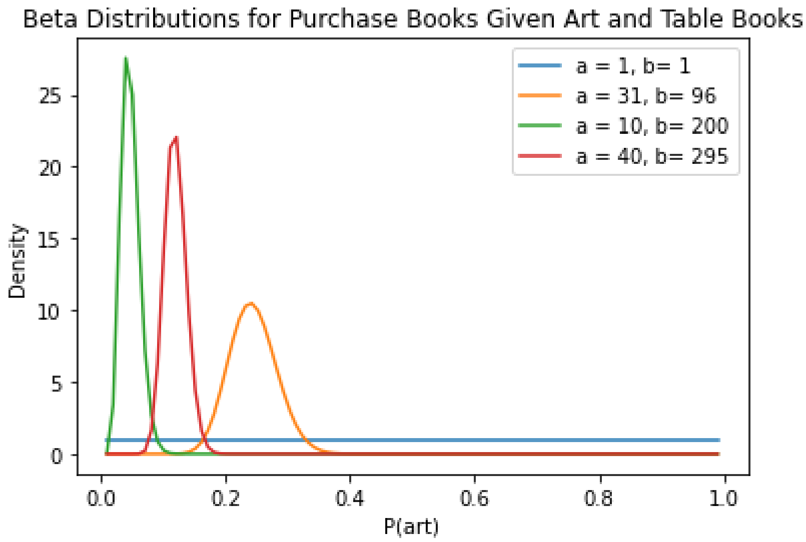
In [15]:
```
## For Purchase given TableBooks Purchases
n_table = len(books.query("TableBooks>0"))
purchase_given_table = len(books.query("TableBooks>0")\
                          .query("Purchase>0"))
tablePriors = createPriors(purchase_given_table, n_table)
createPlot(tablePriors,"Purchase Books Given TableBooks")
```



In [16]:
```
## For Purchase given Art and History Purchases
n_art_hist = len(books.query("ArtBooks>0").query("HistoryBooks>0"))
purchase_given_art_hist = len(books.query("ArtBooks>0")\
                              .query("HistoryBooks>0")\
                              .query("Purchase>0"))
tablePriors = createPriors(purchase_given_art_hist, n_art_hist)
createPlot(tablePriors,"Purchase Books Given Art and History Books")
```

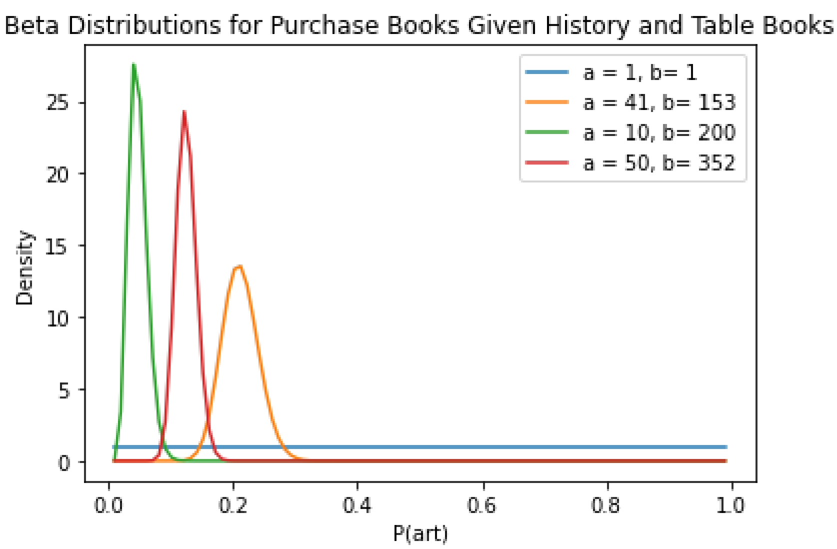


In [17]:
```
## For Purchase given Art and Table Purchases
n_art_table = len(books.query("ArtBooks>0")\
                  .query("TableBooks>0"))
purchase_given_art_table = len(books.query("ArtBooks>0")\
                               .query("TableBooks>0")\
                               .query("Purchase>0"))
tablePriors = createPriors(purchase_given_art_table, n_art_table)
createPlot(tablePriors,"Purchase Books Given Art and Table Books")
```



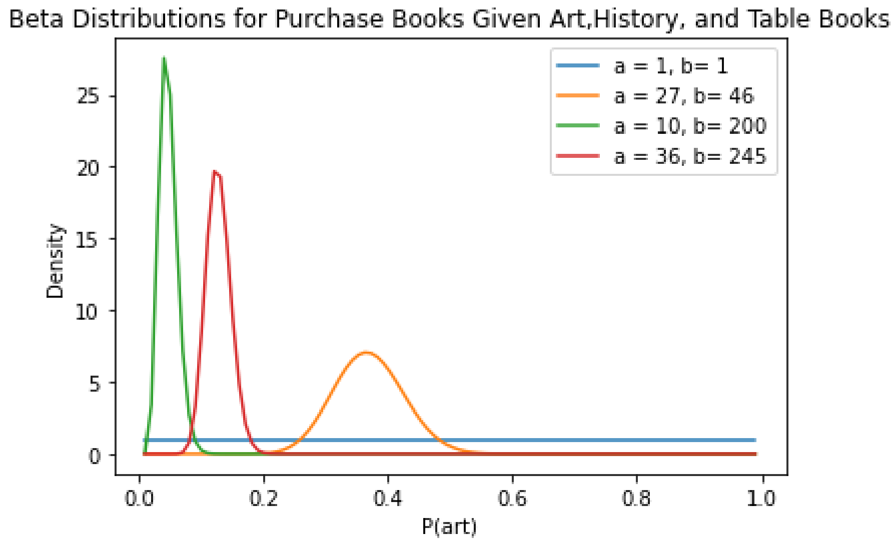
In [18]:

```python
## For Purchase given History and Table Purchases
n_hist_table = len(books.query("HistoryBooks>0")\
                   .query("TableBooks>0"))
purchase_given_hist_table = len(books\
                                .query("HistoryBooks>0")\
                                .query("TableBooks>0")\
                                .query("Purchase>0"))
tablePriors = createPriors(purchase_given_hist_table, n_hist_table)
createPlot(tablePriors,"Purchase Books Given History and Table Books")
```



In [19]:

```python
## For Purchase given Art and History and Table Purchases
n_all = len(books.query("ArtBooks>0")\
            .query("HistoryBooks>0")\
            .query("TableBooks>0"))
purchase_given_all = len(books.query("ArtBooks>0")\
                         .query("HistoryBooks>0")\
                         .query("TableBooks>0")\
                         .query("Purchase>0"))
tablePriors = createPriors(purchase_given_all, n_all)
createPlot(tablePriors,"Purchase Books Given Art,History, and Table Books")
```



Question 5

(15) The data set CHDdata.csv contains cases of coronary heart disease (CHD) and variables associated with the patient's condition: systolic blood pressure, yearly tobacco use (in kg), low density lipoprotein (ldl), adiposity, family history (0 or 1), type A personality score (typea), obesity (body mass index), alcohol use, age, and the diagnosis of CHD (0 or 1).

Perform a Bayesian analysis of these data that finds the posterior marginal probability distributions for the means for the data of patients with and without CHD. You should first standard scale (subtract the mean and divide by the standard deviation) all the numeric variables (remove family history and do not scale CHD). Then separate the data into two sets, one for patients with CHD and one for patients without CHD.

Your priors for both groups should assume means of 0 for all variables and a correlation of 0 between all pairs of variables. You should assume all variances for the variables are 1. Use a prior alpha equal to one plus the number of predictor variables. Compute and compare the Bayesian estimates for the posterior means for each group.

For 5 extra credit points, compute the probability of observing a point at least as extreme as the posterior mean of patients without coronary heart disease under the posterior distribution for the patients with coronary heart disease. Then compute the probability of observing a point at least as extreme as the posterior mean of patients with coronary heart disease under the posterior distribution for the patients without coronary heart disease.

Solution 5

In [20]:
```
cor_heart = pd.read_csv("datasets/CHDdata.csv")
cor_heart
```

Out[20]:

	sbp	tobacco	ldl	adiposity	famhist	typea	obesity	alcohol	age	chd
0	160	12.00	5.73	23.11	Present	49	25.30	97.20	52	1
1	144	0.01	4.41	28.61	Absent	55	28.87	2.06	63	1
2	118	0.08	3.48	32.28	Present	52	29.14	3.81	46	0
3	170	7.50	6.41	38.03	Present	51	31.99	24.26	58	1
4	134	13.60	3.50	27.78	Present	60	25.99	57.34	49	1
...	...	...	...	...	...	...	...	...	...	...
457	214	0.40	5.98	31.72	Absent	64	28.45	0.00	58	0
458	182	4.20	4.41	32.10	Absent	52	28.61	18.72	52	1
459	108	3.00	1.59	15.23	Absent	40	20.09	26.64	55	0
460	118	5.40	11.61	30.79	Absent	64	27.35	23.97	40	0
461	132	0.00	4.82	33.41	Present	62	14.70	0.00	46	1

462 rows × 10 columns

In [21]:

```
## Standardize
cor_standard = cor_heart[["sbp", "tobacco", "ldl", "adiposity", "typea", "obesi
cor_standard = (cor_standard - cor_standard.mean())/cor_standard.std()
cor_standard["chd"] = cor_heart["chd"]
cor_standard
```

Out[21]:

	sbp	tobacco	ldl	adiposity	typea	obesity	alcohol	age
0	1.057417	1.821099	0.477894	-0.295183	-0.418017	-0.176594	3.274189	0.628654
1	0.276789	-0.789382	-0.159507	0.411694	0.193134	0.670646	-0.612081	1.381617
2	-0.991731	-0.774141	-0.608585	0.883374	-0.112441	0.734723	-0.540597	0.217947
3	1.545310	0.841352	0.806252	1.622382	-0.214300	1.411091	0.294742	1.039361
4	-0.211103	2.169453	-0.598928	0.305020	0.702427	-0.012842	1.645991	0.423301
...	...	...	...	...	...	...	...	...
457	3.692037	-0.704470	0.598614	0.811401	1.109862	0.570971	-0.696228	1.039361
458	2.130781	0.122871	-0.159507	0.860240	-0.112441	0.608942	0.068445	0.628654
459	-1.479624	-0.138395	-1.521228	-1.307946	-1.334744	-1.413043	0.391960	0.834008
460	-0.991731	0.384137	3.317227	0.691875	1.109862	0.309916	0.282897	-0.192760
461	-0.308682	-0.791559	0.038474	1.028605	0.906144	-2.692210	-0.696228	0.217947

462 rows × 9 columns

In [22]:

```
has_cor = cor_standard[cor_standard["chd"] == 1]
has_cor
```

Out[22]:

	sbp	tobacco	ldl	adiposity	typea	obesity	alcohol	age
0	1.057417	1.821099	0.477894	-0.295183	-0.418017	-0.176594	3.274189	0.628654
1	0.276789	-0.789382	-0.159507	0.411694	0.193134	0.670646	-0.612081	1.381617
3	1.545310	0.841352	0.806252	1.622382	-0.214300	1.411091	0.294742	1.039361
4	-0.211103	2.169453	-0.598928	0.305020	0.702427	-0.012842	1.645991	0.423301
7	-1.186888	0.096745	-0.072589	-1.388915	0.906144	-0.696330	-0.421730	1.039361
...	...	...	...	...	...	...	...	...
453	-0.698996	-0.443205	1.197385	1.834446	-1.742178	1.294803	-0.696228	0.560203
454	0.374368	-0.652217	0.038474	0.335866	0.702427	0.490281	-0.360050	-0.261211
455	-0.503839	-0.303863	-0.922457	0.137940	-0.519876	-0.494606	1.240780	-1.082625
458	2.130781	0.122871	-0.159507	0.860240	-0.112441	0.608942	0.068445	0.628654
461	-0.308682	-0.791559	0.038474	1.028605	0.906144	-2.692210	-0.696228	0.217947

160 rows × 9 columns

In [23]:

```
no_cor = cor_standard[cor_standard["chd"] == 0]
no_cor
```

Out[23]:

	sbp	tobacco	ldl	adiposity	typea	obesity	alcohol	age
2	-0.991731	-0.774141	-0.608585	0.883374	-0.112441	0.734723	-0.540597	0.217947
5	-0.308682	0.558314	0.835225	1.388470	0.906144	1.121558	-0.118638	0.149496
6	0.179211	0.090213	-0.656873	-1.183278	0.600569	-1.242171	-0.589206	-0.329662
8	-1.186888	-0.791559	-0.439577	-0.772004	-0.418017	-0.281016	-0.594517	-0.945722
12	-0.991731	-0.791559	-1.381193	-1.973696	0.600569	-1.061806	-0.696228	-1.767136
...	...	...	...	...	...	...	...	...
452	0.764682	0.412441	-0.743792	0.437399	0.804286	0.025129	1.051654	-0.055857
456	1.545310	-0.704470	-0.304371	2.140331	0.294993	1.674519	-0.612081	0.970910
457	3.692037	-0.704470	0.598614	0.811401	1.109862	0.570971	-0.696228	1.039361
459	-1.479624	-0.138395	-1.521228	-1.307946	-1.334744	-1.413043	0.391960	0.834008
460	-0.991731	0.384137	3.317227	0.691875	1.109862	0.309916	0.282897	-0.192760

302 rows × 9 columns

In [24]:

```
import numpy as np
posterior_data_has_cor = []
for i in has_cor.columns:
    if i != "chd":
        n = len(has_cor)
        # when we use conjugate priors, we only need to use Python like a simpl
        xbar = np.average(has_cor[i])
```

```
        tau0 = 1000 # chosen hyperparameter
        ## Pick a high tau0 because we know that the variance should be 0, whic
        mu0 = 0  # chosen hyperparameter
        tau = 1 # assumed known
        # "getting" the posterior is simply invoking the formula
        posterior_mean = xbar*(n*tau/(tau0 + n*tau )) + mu0*(tau0/(tau0 + n*tau
        posterior_precision = tau0 + n*tau
        posterior_variance = 1/posterior_precision
        posterior_data_has_cor.append([i, posterior_mean, posterior_precision,

posterior_has_cor = pd.DataFrame(posterior_data_has_cor)
posterior_has_cor.columns = ['Variable', 'Posterior Mean', 'Posterior Percisior
posterior_has_cor
```

Out[24]:

	Variable	Posterior Mean	Posterior Percision	Posterior Variance
0	sbp	0.036411	1160	0.000862
1	tobacco	0.056734	1160	0.000862
2	ldl	0.049794	1160	0.000862
3	adiposity	0.048103	1160	0.000862
4	typea	0.019527	1160	0.000862
5	obesity	0.018947	1160	0.000862
6	alcohol	0.011837	1160	0.000862
7	age	0.070601	1160	0.000862

In [25]:

```
import numpy as np
posterior_data_no_cor = []
for i in no_cor.columns:
    if i != "chd":
        n = len(no_cor)
        # when we use conjugate priors, we only need to use Python like a simpl
        xbar = np.average(no_cor[i])
        tau0 = 1000 # chosen hyperparameter
        ## Pick a high tau0 because we know that the variance should be 0, whic
        mu0 = 0  # chosen hyperparameter
        tau = 1 # assumed known
        # "getting" the posterior is simply invoking the formula
        posterior_mean = xbar*(n*tau/(tau0 + n*tau )) + mu0*(tau0/(tau0 + n*tau
        posterior_precision = tau0 + n*tau
        posterior_variance = 1/posterior_precision
        posterior_data_no_cor.append([i, posterior_mean, posterior_precision, p

posterior_no_cor = pd.DataFrame(posterior_data_no_cor)
posterior_no_cor.columns = ['Variable', 'Posterior Mean', 'Posterior Percision'
posterior_no_cor
```

Out[25]:

	Variable	Posterior Mean	Posterior Percision	Posterior Variance
0	sbp	-0.032440	1302	0.000768
1	tobacco	-0.050547	1302	0.000768
2	ldl	-0.044363	1302	0.000768
3	adiposity	-0.042857	1302	0.000768
4	typea	-0.017397	1302	0.000768
5	obesity	-0.016881	1302	0.000768
6	alcohol	-0.010546	1302	0.000768
7	age	-0.062901	1302	0.000768

From the posterior means, percision, and variance for the data with heart disease we can see that all the posterior means were positive while for the data with no heart disease the means were all negative. From our common medical knowledge we know that if a person has high values for the variables (indiciating poor health) that person may have heart disease. This is represented by positive posterior mean values of the means from the data with heart disease and negative posterior mean values of the means from the data without heart disease.

Question 6

(10) For each of the following types of distributions, state the support type (single or multivariable and discrete or continuous), the formula for the PMF or PDF, the parameters, the support, the mean, and some typical uses of the distribution. You may use whatever source(s) you want, including for example Wikipedia.

Solution A

Bernoulli Distribution

- Support Type: Single, Discrete
- Formula for PMF or PDF: PMF = $P(k|p) = p^k(1-p)^{1-k}$
- Parameters: $\begin{cases} p \in [0,1] & \text{probability of success} \\ q = 1-p \end{cases}$
- Support: $k \in \{0,1\}$
- Mean: p
- Typical Uses: An example of this distribution is flipping a coin. This distribution is used to model any random variable that has two possible outcomes (success, failure)

Sources:

- https://en.wikipedia.org/wiki/Bernoulli_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution B

Binomial Distribution

- Support Type: Single, Discrete
- Formula for PMF or PDF: PMF = $\binom{n}{k} p^k (1-p)^{n-k}$
- Parameters: $\begin{cases} p \in [0,1] & \text{probability of success} \\ n \in \{1,2,3,\dots\} & \#\text{ trials} \\ q = 1-p \end{cases}$
- Support: $k \in \{0,1,\dots,n\}$
- Mean: np
- Typical Uses: The binomial distribution is to model the number of success from multiple independent bernouli trials, such as the number of heads in multiple coin flips.

Sources:

- https://en.wikipedia.org/wiki/Binomial_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution C

Poisson Distribution

- Support Type: Single, Discrete
- Formula for PMF or PDF: PMF = $\frac{\lambda^k e^{-\lambda}}{k!}$
- Parameters: $\lambda \in (0, \infty)$
- Support: $k \in \mathbb{N}_0$ (Natural numbers starting from 0)
- Mean: λ
- Typical Uses: A famous example for this distribution is the number of accidents on a stretch of road per month. This distribution is used to model the number of occurences over a certain fixed amount of time. However, the conditions must be that the events are independent and that there is a fixed mean rate.

Sources:

- https://en.wikipedia.org/wiki/Poisson_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution D

Uniform Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $f(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
- Parameters: $-\infty < a < b < \infty$
- Support: $x \in [a,b]$
- Mean: $\frac{b-a}{2}$

- Typical Uses: This distribution is used to represent when a random variable has values that are equally possible. For example, spinning a spinner which will come to rest at any angle in $[0, 2\pi]$

Sources:

- https://en.wikipedia.org/wiki/Continuous_uniform_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution E

Beta Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $\begin{cases} f(x|\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad \text{where} \\ B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{cases}$
- Parameters: $\begin{cases} \alpha \in (0,\infty) \\ \beta \in (0,\infty) \end{cases}$
- Support: $x \in [0,1]$ (possibly excluding endpoints)
- Mean: $\frac{\alpha}{\alpha+\beta}$
- Typical Uses: This distribution is used as a prior probability distribution in Bayesian methods. It is used to model a random variable on an interval of finite length.

Sources:

- https://en.wikipedia.org/wiki/Beta_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution F

Gamma Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $f(x|k,\theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$
- Parameters: $\begin{cases} k \in (0,\infty) \\ \theta \in (0,\infty) \end{cases}$
- Support: $x \in (0,\infty)$ (possibly excluding endpoints)
- Mean: $k\theta$
- Typical Uses: A typical example is rainfall accumulated in a reservoir. The Gamma distribution is used to model accumulation over a unit of time.

Sources:

- https://en.wikipedia.org/wiki/Gamma_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution G

Gaussian Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)$
- Parameters: $\begin{cases} \mu \in \mathbb{R} & \text{mean} \\ \sigma > 0 & \text{standard deviation} \end{cases}$
- Support: $x \in \mathbb{R}$
- Mean: μ
- Typical Uses: This distribution is the most commenly used for biological and physical measurments.

Sources:

- https://en.wikipedia.org/wiki/Normal_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution H

t Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $f(x|\nu,\mu,\sigma) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\sigma^2\Gamma(\frac{\nu}{2})}(1+\frac{1}{\nu}\frac{(x-\mu)^2}{\hat{\sigma}^2})^{-\frac{\nu+1}{2}}$
- Parameters: $\begin{cases} \nu > 0 & \text{(d.o.f)} \\ \mu \in \mathbb{R} & \text{(location)} \\ \sigma > 0 & \text{(scale)} \end{cases}$
- Support: $x \in \mathbb{R}$
- Mean: $\begin{cases} \mu & \text{if } \nu > 0 \\ \text{else} & undef. \end{cases}$
- Typical Uses: This distribution is used when a set of samples is used to assume to have a normal distribution. This distribution is similar to normal distribution, but its shape does differ a slight bit.

Sources:

- https://en.wikipedia.org/wiki/Student%27s_t-distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution I

Cauchy Distribution

- Support Type: Single, Continous
- Formula for PMF or PDF: PDF = $\frac{1}{\pi\gamma\left[1+(\frac{x-x_0}{\gamma})^2\right]}$

- Parameters: $\begin{cases} x_0 \in \mathbb{R} & \text{(location)} \\ \gamma > 0 & \text{(location)} \end{cases}$
- Support: $x \in (-\infty, \infty)$
- Mean: $undefined$
- Typical Uses: The typical use case for the Cauchy distribution is for the use as the canonical example of a pathological distribution.

Sources:

- https://en.wikipedia.org/wiki/Cauchy_distribution

Solution J

Multinomial Distribution

- Support Type: Single, Discrete
- Formula for PMF or PDF: PMF = $\frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$
- Parameters: $\begin{cases} n > 0 & \text{(number of trials)} \\ k > 0 & \text{(numbeer of mutually exclusive events (integer))} \\ p_1, \ldots, p_k & \text{event probabilities)} \ (\Sigma p_i = 1) \end{cases}$
- Support: $\begin{cases} x_1 \in \{0, \ldots, n\} \\ i \in \{1, \ldots, k\} \\ \text{with } \Sigma_i x_i = n \end{cases}$
- Mean: $E(X_i) = np_i$
- Typical Uses: This distribution is a generalization of the binomial distribution.

Sources:

- https://en.wikipedia.org/wiki/Multinomial_distribution

Solution K

Dirichlet Distribution

- Support Type: Multivariate, Continous
- Formula for PMF or PDF: PDF = $\begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$
- Parameters: $\begin{cases} \mu \in \mathbb{R} & \text{(mean)} \\ \Sigma & \text{(covariance matrix)} \end{cases}$
- Support: $x \in [a, b]$
- Mean: $\frac{b-a}{2}$
- Typical Uses: This distribution is the multivariate generalizatino of the beta distribution. It is commonly used as prior distributions in Bayesian statistics. Also, the Dirichlet distribution is the conjugate prior of the categorical distribution and multinomial distribution.

Sources:

- https://en.wikipedia.org/wiki/Dirichlet_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution L

Multivariate Gaussian Distribution

- Support Type: Multivariate, Continous
- Formula for PMF or PDF: PDF = $\frac{1}{(2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}}}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$
- Parameters: $\begin{cases} \mu \in \mathbb{R}^k & \text{(mean)} \\ \Sigma & \text{(covariance matrix)} \end{cases}$
- Support: $x \in \mathbb{R}^k$
- Mean: μ
- Typical Uses: This distribution is the multivariate generalization of the normal distribution. The distribution is used to describe a set of corrleated real-valued random variables that cluster around a mean value.

Sources:

- https://en.wikipedia.org/wiki/Multivariate_normal_distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution M

Multivariate t Distribution

- Support Type: Multivariate, Continous
- Formula for PMF or PDF: PDF = $f(x|\nu, \mu, \Sigma) = \frac{\Gamma(\nu+k)}{\Gamma(\frac{\nu}{2})(\nu\pi)^{\frac{k}{2}}|\Sigma|^{\frac{1}{2}}}[1 + \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)]^{-\frac{\nu+k}{2}}$
- Parameters: $\begin{cases} \mu \in \mathbb{R}^k & \text{(mean)} \\ \Sigma & \text{(covariance matrix)} \end{cases}$
- Support: $x \in \mathbb{R}^k$
- Mean: μ
- Typical Uses: This distribution is the multivariate generalization of the t distribution.

Sources:

- https://en.wikipedia.org/wiki/Multivariate_t-distribution
- https://github.com/wbasener/BayesianML/blob/main/BasenerBrownBML_1_18_2021.pdf

Solution N

Wishart Distribution

- Support Type: Multivariate, Continous

- Formula for PMF or PDF: PDF = $\begin{cases} f_X(x) = \dfrac{|x|^{\frac{n-p-1}{2}} e^{-\frac{tr(V^{-1}x)}{2}}}{2^{\frac{np}{2}}|V|^{\frac{n}{2}}\Gamma_p(\frac{n}{2})} \\ \Gamma_p \text{ is the multivariate gamma function} \\ tr \text{ is the trace function} \end{cases}$
- Parameters: $\begin{cases} n > p-1 & \text{(degrees of freedom)} \\ V > 0 & \text{(scale matrix)}(p \times p \text{ pos. def}) \end{cases}$
- Support: $X(p \times p)$ positive definite matrix
- Mean: $E[X] = nV$
- Typical Uses: This distribution is the multivariate generalization of the gamma distribution. The wishart distribution is the conjugate prior of the inverse covariance-matrix of a multivariate-normal random-vector

Sources:

- https://en.wikipedia.org/wiki/Wishart_distribution

Question 7

(15) Using the Python Notebook https://www.kaggle.com/billbasener/pt2-probabilities-likelihoods-and-bayes-theorem, complete the challenge question from Section 6: Modify the code from Section 5 to and add the ability to use the posterior from conjugate prior function to output the posterior probability parameters given parameters and for a Gaussian Likelihood with known variance σ2, and use your modified function to create the Prior, Likelihood, Posterior plots as in Section 5 of the notebook.

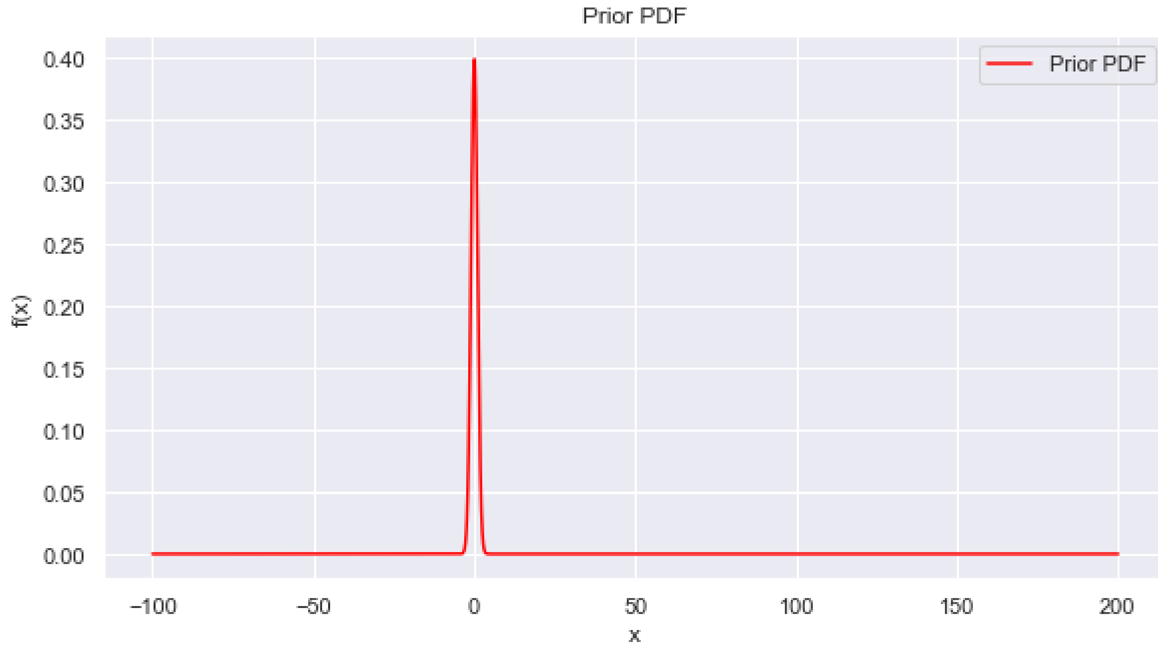
Solution 7

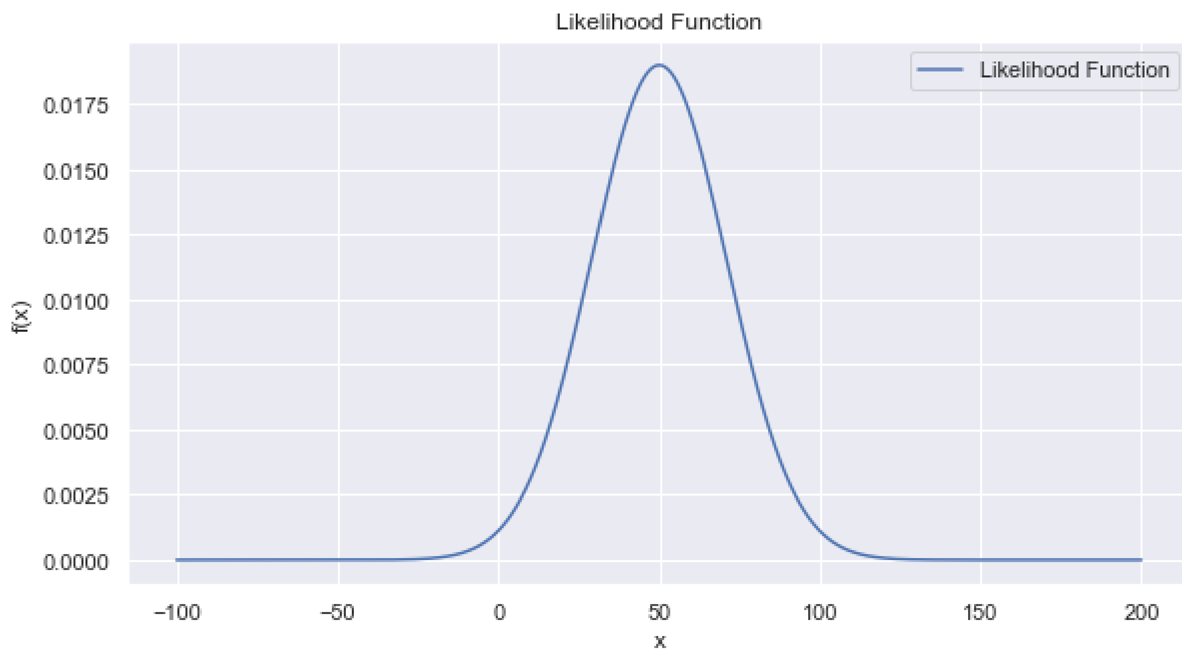
In [26]:
```
from scipy.stats import binom
from scipy.stats import beta
from scipy.stats import norm

def posterior_from_conjugate_prior(**kwargs):
    if kwargs['Likelihood_Dist_Type'] == 'Binomial':
        # Get the parameters for the likelihood and prior distribution from the
        x = kwargs['x'] # This is state space of possible values for p = 'proba
        n = kwargs['n'] # This is the number of Bernoili trials.
        k = kwargs['k'] # This is the number of 'successes'.
        a = kwargs['a'] # This is the parameter alpha for the prior Beta distri
        b = kwargs['b'] # This is the parameter beta for the prior Beta distrib

        print(f'a_prime = {k + a}.')
        print(f'b_prime = {n - k + b}.')
        Likelihood = binom.pmf(p=x, n=n, k=k)
        Prior = beta.pdf(x=x, a=a, b=b)
        Posterior = beta.pdf(x=x, a=k+a, b=n-k+b)

        return [Prior, Likelihood, Posterior]

    elif kwargs['Likelihood_Dist_Type'] == 'Gaussian_Known_Variance':
        # Get the parameters for the likelihood and prior distribution from the
        x = kwargs['x'] # This is state space of possible values for p = 'proba
```

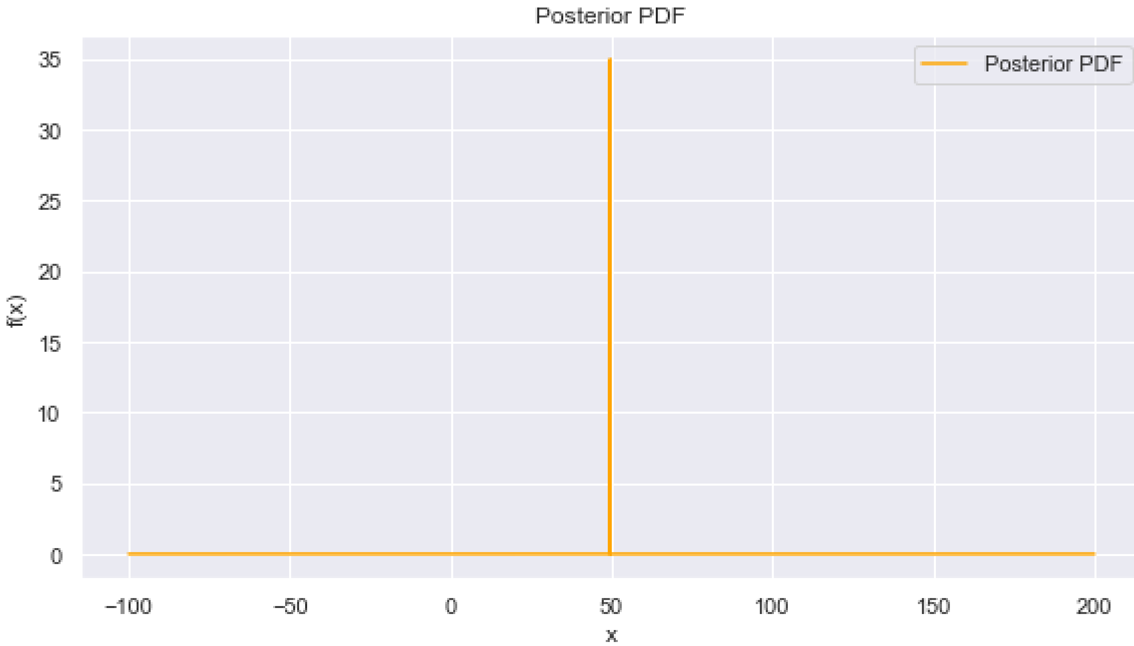
```
        mu = kwargs['mu'] # This is the number of Bernoili trials.
        var = kwargs['var'] # This is the number of 'successes'.
        n = kwargs['n'] # This is the parameter alpha for the prior Beta distri
        prior_mu = kwargs['prior_mu'] # This is the parameter beta for the pric
        prior_var = kwargs['prior_var'] # This is the parameter beta for the pr

        numerator = ((prior_mu*var) + (n*np.average(x)*prior_var))
        denominator = (var + n*prior_var)
        posterior_mu = numerator / denominator
        posterior_var = (var * prior_var) / (var + n*prior_var)


        Likelihood = norm.pdf(x=x, loc=mu, scale=var)
        Prior = norm.pdf(x=x, loc=prior_mu, scale=prior_var)
        Posterior = norm.pdf(x=x, loc=posterior_mu, scale=posterior_var)

        return [Prior, Likelihood, Posterior]

    else:
        print('Distribution type not supported.')
        return -1, -1, -1
```

In [27]:
```
x = np.arange(-100, 200, 0.01)
Prior, Likelihood, Posterior = posterior_from_conjugate_prior(Likelihood_Dist_T
```

In [28]:
```
# import matplotlib
import matplotlib.pyplot as plt
# import seaborn
import seaborn as sns
import warnings
warnings.filterwarnings('ignore')
# settings for seaborn plotting style
sns.set(color_codes=True)
# settings for seaborn plot sizes
sns.set(rc={'figure.figsize':(9.5,5)})

ax1 = sns.lineplot(x, Prior, color='red')
ax1.set(xlabel='x', ylabel='f(x)', title=f'Prior PDF');
plt.legend(labels=['Prior PDF']);
plt.show()

ax2 = sns.lineplot(x, Likelihood)
ax2.set(xlabel='x', ylabel='f(x)', title=f'Likelihood Function');
plt.legend(labels=['Likelihood Function']);
plt.show()

ax3 = sns.lineplot(x, Posterior, color='orange')
ax3.set(xlabel='x', ylabel='f(x)', title=f'Posterior PDF');
plt.legend(labels=['Posterior PDF']);
plt.show()
```


Prior PDF
Prior PDF
f(x)
x
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−100
−50
0
50
100
150
200


Likelihood Function
Likelihood Function
f(x)
x
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
0.0000
−100
−50
0
50
100
150
200



Posterior PDF
Posterior PDF
35
30
25
20
15
10
5
0
f(x)
−100
−50
0
50
100
150
200
x