Homework 10 Max Ryoo

6.1 special problem.

1) Find the conditional distribution of Y given $X=2$.

So $P(Y=y \mid X=2) = \dfrac{P(X=y, X=2)}{P(X=2)}$

Where joint probabilities

$P(Y=y, X=2) = P(X=2 \mid Y=y)\,P(Y=y)$

It will be clear to draw joint distribution table.

Possible Y

possible X	1	2	3	4	
0	$\frac{1}{2}\cdot\frac{1}{4}$	$\frac{1}{4}\cdot\frac{1}{4}$	$\frac{1}{8}\cdot\frac{1}{4}$	$\frac{1}{16}\cdot\frac{1}{4}$	
1	$\frac{1}{2}\cdot\frac{1}{4}$	$\frac{2}{4}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{4}{16}\cdot\frac{1}{4}$	
2	0	$\frac{1}{4}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{6}{16}\cdot\frac{1}{4}$	marginal dst = $\frac{9}{32}$
3	0	0	$\frac{1}{8}\cdot\frac{1}{4}$	$\frac{4}{16}\cdot\frac{1}{4}$	
4	0	0	0	$\frac{1}{16}\cdot\frac{1}{4}$	

We know that conditional distribution is found by dividing by $P(X=2)$

Conditional dis of Y give $X=2$

$Y=$	1	2	3	4
$P(Y=y \mid X=2)$	0	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{3}$

6.2

4a) We can infer that $f_{Y|X}(x) = \frac{1}{x}$ since Y will be between $\{1, \dots, x\}$

$E(Y|X) = \sum y f_{Y|X}(x) = \sum_{y=1}^{x} \frac{y}{x} = \frac{1+x}{2}$

We know $E(Y) = E[E(Y|X)] = E(\frac{1}{2}) + E(\frac{X}{2}) = \frac{E(X)}{2} + \frac{1}{2}$

$E(X) = \sum x \cdot \frac{1}{n} = \frac{n+1}{2}$ so substitute, $\frac{1}{2}\cdot\frac{n+1}{2} + \frac{1}{2} = \boxed{\frac{n+3}{4}}$

12a) We can see that the pattern observed is at n^{th} step the number of black balls will be $3+n-1 = n+2$

white balls will be $n+2-B_n$ where B_n is number of black balls.

→

$$E(B_{n+1} \mid B_n) = B_n \left(\frac{n+2-B_n}{n+2}\right) + (B_n+1)\left(\frac{B_n}{n+2}\right)$$
$$= \frac{B_n}{n+2}\left(n+2-B_n+B_n+1\right)$$
$$= \frac{B_n}{n+2}\cdot(n+3)$$
$$= \frac{n+3}{n+2}B_n$$

12b) $E(B_1)=1 = 3/3 \quad n=1$
$E(B_2) = 2\times\frac{1}{3} + 1\times\frac{2}{3} = 4/3 \quad n=2$
$E(B_3) = 3\times\frac{1}{3}\times\frac{1}{2} + 2\times\frac{1}{3}\cdot\frac{1}{2} + 2\times\frac{1}{4}\frac{1}{3} + 1\times\frac{3}{4}\cdot\frac{1}{3}$
$+ 2\times\frac{1}{4}\frac{1}{3} + 1\times\frac{3}{4}\cdot\frac{1}{3} = 5/3 \quad n=3$

Pattern is $\frac{n+2}{3}$. Induction from this the we can say $E(B_n) = \frac{n+2}{3}$

Section 6.3

1) X is uniform $(0,1)$ we can thus say $f_X(x)=1$ for $x\in(0,1)$
We have $P(A|X=x) = x^2$
$P(A) = \int P(A|X=x)\, f_X(x) = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$

4a) The marginal density for Y is $f_Y(y) = \int_0^y \lambda^3 x e^{-\lambda y}\,dx$
$= \lambda^3 e^{-\lambda y}\int_0^y x\,dx$
$= \lambda^3 e^{-\lambda y}\frac{x^2}{2}\Big|_0^y$
$= \lambda^3 e^{-\lambda y}\frac{y^2}{2}$

Y has gamma $(3,\lambda)$ and expectation is $E(Y) = 3/\lambda$

4b) We know $f_X(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^3 x e^{-\lambda y}}{\lambda^3 e^{-\lambda y}\frac{y^2}{2}} = 2x$

$E(X|Y=1) = \int_0^1 x\cdot 2x\,dx = 2\int_0^1 x^2 = \boxed{\frac{2}{3}}$

8)a) $E(Y|X=x) = np = 5x$
$E(Y) = E(5X) = 5E(X) = \boxed{\frac{5}{2}}$
For $E(Y^2) = E(E(Y^2|X)) = Var(Y|X=x) + [E(Y|X=x)]^2$
$= np(1-p) + (np)^2$
$= 5x(1-x) + (5x)^2$
$E(Y^2) = E[(5X)^2 + 5X(1-X)]$
$= 20E(X^2) + 5E(X)$
$= 20[\frac{1}{12} + (\frac{1}{2})^2] + 5(\frac{1}{2}) = \frac{55}{6}$

8b) $P(Y=y \text{ and } x < X < x+dx) = P(Y=y \text{ and } x \in dx)$

$= P(Y=y | X=x) P(x \in dx)$

$= P(Y=y | X=x) f_X(x)\, dx$

$= \binom{5}{y} x^y (1-x)^{5-y} \cdot dx$

8c) $P(x < X < x+dx | Y=y) = \dfrac{P(x<X<x+dx, Y=y)}{P(Y=y)}$

$= \dfrac{5C_y\, x^y (1-x)^{5-y}\, dx}{\int_0^1 (5C_y) x^y (1-x)^{5-y}\, dx}$

Beta$(y+1, 6-y)$ distribution.

13a) $P(X \geq Y) = \sum_{y=0}^{\infty} P(X \geq y | Y=y) P(Y=y)$

$= P(X \geq 0) P(Y=0) + P(X \geq 1) P(Y=1) + P(X \geq 2) P(Y=2)$

$P(X \geq x) = 3-x/3$

$P(Y=y) = \dfrac{e^{-\lambda} \lambda^y}{y!}$

so $P(X \geq Y) = \left(\frac{3-0}{3} \cdot e^{-\lambda} \frac{\lambda^0}{0!}\right) + \left(\frac{3-1}{3} \cdot e^{-\lambda} \cdot \frac{\lambda^1}{1!}\right) + \left(\frac{3-2}{3} \cdot e^{-\lambda} \frac{\lambda^2}{2!}\right)$

$= e^{-\lambda}[1 + 2\lambda/3 + \lambda^2/6]$

So $P(X<Y) = 1 - P(X \geq Y)$

$1 - e^{-\lambda}\left(1 + \frac{2\lambda}{3} + \frac{\lambda^2}{6}\right)$

b) Conditional density $P(X \in dx | X<Y) = \dfrac{P(x \in dx, x<Y)}{P(X<Y)} = \dfrac{P(X \in dx) P(Y>x)}{P(X<Y)}$

$$P(X \in dx | x<y) = \begin{cases} \dfrac{1-e^{-\lambda}}{3-e^{-\lambda}(3+2\lambda+\frac{\lambda^2}{2})}\, dx, & 0 \leq x < 1 \\ \dfrac{1-e^{-\lambda}(1+\lambda)}{3-e^{-\lambda}(3+2\lambda+\frac{\lambda^2}{2})}\, dx, & 1 \leq x < 2 \\ \dfrac{1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})}{3-e^{-\lambda}(3+2\lambda+\frac{\lambda^2}{2})}\, dx, & 2 \leq x < 3 \end{cases}$$

c) $E(X|X<Y) = \int x P(x \in dx | X<Y) = \int_0^3 x \dfrac{P(x \in dx, Y>x)}{P(X<Y)} = \int_0^3 x \dfrac{P(x \in dx) P(Y>x)}{P(X<Y)}$

$E(X|X<Y) = \dfrac{P(Y \geq 1)\int_0^1 x\, dx + P(Y \geq 2)\int_1^2 x\, dx + P(Y \geq 3)\int_2^3 x\, dx}{3P(X<Y)}$

$P(Y \geq x) = 1 - P(Y < x) = 1 - \sum_{y=0}^{x} \dfrac{e^{-\lambda}(\lambda)^y}{y!} \Rightarrow$

$$E(X|X<Y) = \frac{P(Y\geq 1) + 3P(Y\geq 2) + 5P(Y\geq 3)}{6P(X<Y)}$$

$$E(X|X<Y) = \frac{(1-e^{-\lambda}) + 3[1-e^{-\lambda}(1+\lambda)] + 5[1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})]}{2[3-e^{-\lambda}(3+2\lambda+\frac{\lambda^2}{2})]}$$

$$= \frac{9-e^{-\lambda}(9+8\lambda+\frac{5}{2}\lambda^2)}{6-2e^{-\lambda}(3+2\lambda+\frac{\lambda^2}{2})}$$