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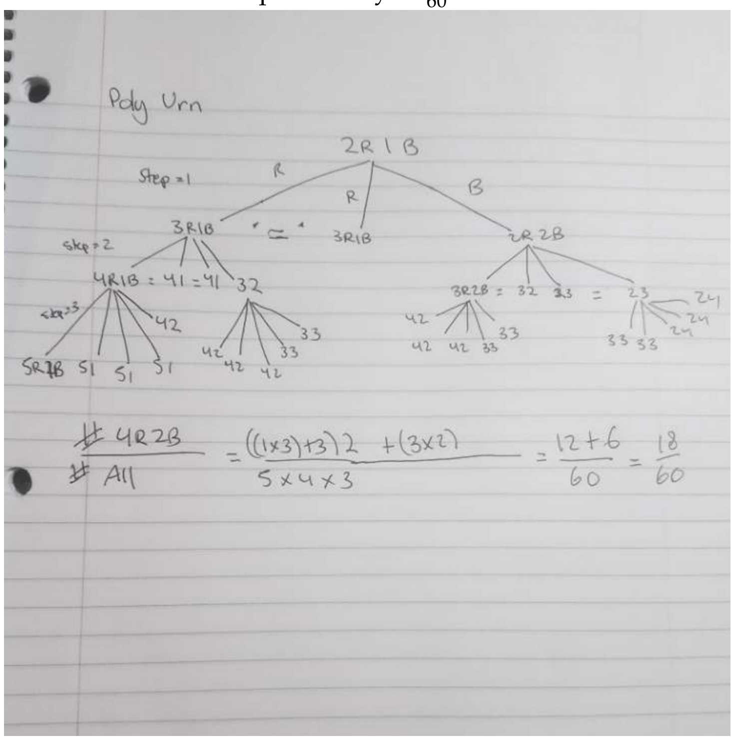
Homework # 3

A problem on the Polya urn...

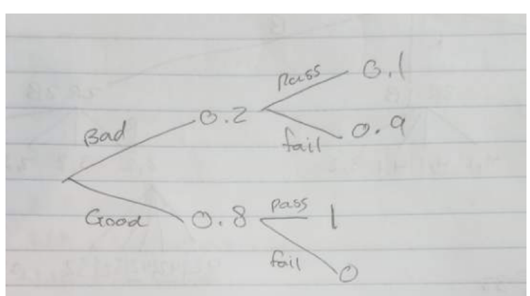
Let initially a box contain 2 red and 1 blue ball. At each step we draw a random ball from the box, and add 2 balls of the same color back. What is the probability that after 3 steps there will be 4 red and 2 blue balls in the box? (note: the answer might not be uniform because we start from 2 red and 1 blue, and not like it was in class)

The probability is $\frac{18}{60} \rightarrow \mathbf{0.3}$



1.5 Bayes' Rule

1.5.3

(a) 

$$
\begin{aligned}
P(bad) &= 0.2 \\
P(pass|bad) &= 0.1 \\
P(good|pass) &= ? \\
&= \frac{\#(goodpass)}{\#(pass)} \\
&= \frac{0.8}{(0.8 * 1) + (0.2 * 0.1)} \\
&= \frac{0.8}{0.82} \\
&= 0.97561 \\
&= \mathbf{0.98}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(bad) &= 0.2 \\
P(pass|bad) &= 0.1 \\
P(bad|pass) &= ? \\
&= \frac{\#(badpass)}{\#(pass)} \\
&= \frac{0.2 * 0.1}{(0.8 * 1) + (0.2 * 0.1)} \\
&= \frac{0.02}{0.82} \\
&= 0.02439 \\
&= \mathbf{0.02}
\end{aligned}
$$

1.5.7

(a) As shown in Example 1 we already have the probabilities for white...

$$
\begin{aligned}
P(\text{Box 1} \mid \text{white}) &= \frac{6}{23} \\
P(\text{Box 2} \mid \text{white}) &= \frac{8}{23} \\
P(\text{Box 3} \mid \text{white}) &= \frac{9}{23}
\end{aligned}
$$

Now using the same method we must find for black as well...

$$P(\text{Box } 1 \mid \text{black}) = \frac{6}{13}$$
$$P(\text{Box } 2 \mid \text{black}) = \frac{4}{13}$$
$$P(\text{Box } 3 \mid \text{black}) = \frac{3}{13}$$

If we pick the highest posterior probability for both the chance of use being right over the long run is...

$$P(\text{Box } 1 \mid \text{white}) * P(\text{white}) + P(\text{Box } 1 \mid \text{black}) * P(\text{black})$$
$$\frac{9}{23} * \frac{23}{36} + \frac{6}{13} * \frac{13}{36}$$
$$\mathbf{\frac{5}{12}}$$

(b) This is problem 7c. Sorry for the confusion on ordering...
In this case the posterior probabilities will change ...

$$P(\text{Box } 1 \mid \text{white}) = \frac{9}{23}$$
$$P(\text{Box } 2 \mid \text{white}) = \frac{6}{23}$$
$$P(\text{Box } 3 \mid \text{white}) = \frac{27}{92}$$

Now using the same method we must find for black as well...

$$P(\text{Box } 1 \mid \text{black}) = \frac{9}{13}$$
$$P(\text{Box } 2 \mid \text{black}) = \frac{3}{13}$$
$$P(\text{Box } 3 \mid \text{black}) = \frac{9}{52}$$

If we pick the highest posterior probability for both the chance of use being right over the long run is...

$$P(\text{Box } 1 \mid \text{white}) * P(\text{white}) + P(\text{Box } 1 \mid \text{black}) * P(\text{black})$$
$$\frac{27}{92} * \frac{23}{36} + \frac{9}{13} * \frac{13}{36}$$
$$\mathbf{\frac{7}{16}}$$

1.6 Sequence of Events

1.6.6

(a) First we can assume that the probability for $r = 1$ and $r > 8$ is 0.
This is because in the case of $r = 1$ there is no way of rolling the same number since you have not rolled at least twice.
The case for $r > 8$ is 0 as well since there are only six sides of a dice. There has to be a number that is repeated in this case since there isn't more (different options) the roll can output. . .
All other options are computed below . . .

Table for all rolls...

r	computation	result
1	0	0
2	$(\frac{6}{6}) * (\frac{1}{6})$	$\frac{1}{6}$
3	$(\frac{6}{6}) * (\frac{5}{6}) * (\frac{2}{6})$	$\frac{5}{18}$
4	$(\frac{6}{6}) * (\frac{5}{6}) * (\frac{4}{6}) * (\frac{3}{6})$	$\frac{5}{18}$
5	$(\frac{6}{6}) * (\frac{5}{6}) * (\frac{4}{6}) * (\frac{3}{6}) * (\frac{4}{6})$	$\frac{5}{27}$
6	$(\frac{6}{6}) * (\frac{5}{6}) * (\frac{4}{6}) * (\frac{3}{6}) * (\frac{2}{6}) * (\frac{5}{6})$	$\frac{25}{324}$
7	$(\frac{6}{6}) * (\frac{5}{6}) * (\frac{4}{6}) * (\frac{3}{6}) * (\frac{2}{6}) * (\frac{1}{6}) * (\frac{6}{6})$	$\frac{5}{324}$
$r \geq 8$	0	0

The logic behind this is for example take $r = 4$. We can see that the first roll will not matter so there are 6 possible outcomes out of 6, which is represented by the first fraction. However, the next roll must be anything **except** that number chosen. This means the next probability is $\frac{5}{6}$. This process is done again to get $\frac{4}{6}$. However, for the last step we want to repeat one of the numbers already chosen. This means that we currently have three options to choose from, which gives us $\frac{3}{6}$. Multiply all these probabilities will give the result of $\frac{5}{18}$.

(b) $p_1 + p_2 + ... + p_10 = \mathbf{1}$
This must be true because for a six-sided die there are only six options. The number must repeat for rolls of 7 above and it is zero for the middle layers of r. This equation covers all possibilities which is why it must equate to 1.

(c)

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_10 = 1$$
$$0 + \frac{1}{6} + \frac{5}{18} + \frac{5}{18} + \frac{5}{27} + \frac{25}{324} + \frac{5}{324} + 0 + 0 + 0 = 1$$

1.6.8

(a) We can solve this problem by looking at this equation...

Proof.

$$P(B_{12} \cap B_{23}) = P(\text{all have the same birthday})$$

Let's look at it seperatly...

$$\begin{aligned} P(B_{12}) &= \frac{365}{365} * \frac{1}{365} \\ P(B_{23}) &= \frac{365}{365} * \frac{1}{365} \\ P(\text{all have the same birthday}) &= \frac{365}{365} * \frac{1}{365} * \frac{1}{365} \end{aligned}$$

Based on the multiplication rule for independent events...

$$\begin{aligned} P(B_{12} \cap B_{23}) &= P(\text{all have the same birthday}) \\ \frac{365}{365} * \frac{1}{365} * \frac{365}{365} * \frac{1}{365} &= \frac{365}{365} * \frac{1}{365} * \frac{1}{365} \\ \frac{1}{365^2} &= \frac{1}{365^2} \end{aligned}$$

Because this equality holds for independent events we can say that they are independent.
□

(b) *Proof.* This is not independent. The B_{12} and B_{23} implies that B_{13} based on the definition. Also using the same logic as above the equation will be ...
$\frac{1}{365^2} = \frac{1}{365^3}$ which is not true.
□

2.1 Distributions

2.1.7

(a) We can do this with Binomial Distribution. However, we are missing some information. We have n, k, but we must calculate p...

Ways to win...

Opponent	Me	#ways
1	2, 3, 4, 5, 6	5
2	3, 4, 5, 6	4
3	4, 5, 6	3
4	5, 6	2
5	6	1
6	N/A	0

$$\text{Total number of ways to win} = 15$$
$$\text{Total combinations} = 36$$
$$P(winning) = p = \frac{5}{12}$$

Let us apply Binomial Distribution ...

$$P(k \text{ success in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}$$
$$P(4 \text{ success in 5 trials}) = \binom{5}{4} \left(\frac{5}{12}\right)^4 \left(\frac{7}{12}\right)^1$$

However we must also look at 5 since the question was at least 4...

$$P(k \text{ success in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}$$
$$P(4 \text{ success in 5 trials}) = \binom{5}{5} \left(\frac{5}{12}\right)^5 \left(\frac{7}{12}\right)^0$$

The two Probabilities added together is 0.10047
The probability of winning at least 4 times is **0.10047**

2.1.10

(a)

$$
\begin{aligned}
&P(k-1 \text{ heads } | k-1 \text{ or } k \text{ heads})\\
&= \frac{P(k-1 \text{ heads }) \cap P(k-1 \text{ or } k \text{ heads})}{P(k-1 \text{ or } k \text{ heads})}\\
&= \text{ the numerator will equate to just } P(k-1 \text{ heads})\\
&\text{this is because the definition of intersection}\\
&= \frac{P(k-1 \text{ heads })}{P(k-1 \text{ or } k \text{ heads})}\\
&= \text{The denominator will equate to the addition of the two or cases}\\
&= \frac{P(k-1 \text{ heads })}{P(k-1 \text{ heads }) + P(k \text{ heads })}\\
&= \frac{\binom{n}{k-1}}{\binom{n}{k-1} + \binom{n}{k}}\\
&= \frac{\frac{n!}{(k-1)!(n-k-1)!}}{\frac{n!}{(k-1)!(n-k-1)!} * \frac{n!}{(k)!(n-k)!}}\\
&= \frac{k}{k+n-k+1}\\
&= \frac{k}{n+1}
\end{aligned}
$$

(b) This is $1-$ above ...

$$
\begin{gathered}
1 - \frac{k}{n+1}\\
\frac{n+1-k}{n+1}
\end{gathered}
$$

2.5 Random Sampling

2.5.2

(a)

$$
\begin{aligned}
P(\text{ First is red }) &= \frac{26}{52}\\
P(\text{ Second is Black }) &= \frac{26}{51}\\
P(\text{ Third is Black }) &= \frac{25}{50}\\
P(\text{ First card is red and the second two black }) &= \frac{13}{102}\\
&= \mathbf{0.12745}
\end{aligned}
$$

(b) For this problem we can use the formula given on page 125 for sampling without replacement!

$$
\begin{aligned}
P(1 \text{ red and } 2 \text{ black}) &= \frac{\binom{26}{1} * \binom{26}{2}}{\binom{52}{3}} \\
&= \frac{\frac{26*26*25}{2}}{\frac{52*51*50}{3*2}} \\
&= \frac{39}{102} \\
&= \frac{13}{34} \\
&= \mathbf{0.38235}
\end{aligned}
$$

(c) We can use the above procedure to find the Probability when there is 1 red, 2 red, or 3 red.

$$
\begin{aligned}
P(1 \text{ red and } 2 \text{ black}) &= \frac{\binom{26}{1} * \binom{26}{2}}{\binom{52}{3}} \\
&= \frac{\frac{26*26*25}{2}}{\frac{52*51*50}{3*2}} \\
&= \frac{39}{102} \\
&= \frac{13}{34} \\
&= \mathbf{0.38235}
\end{aligned}
$$

$$
\begin{aligned}
P(2 \text{ red and } 1 \text{ black}) &= \frac{\binom{26}{2} * \binom{26}{1}}{\binom{52}{3}} \\
&= \frac{\frac{26*26*25}{2}}{\frac{52*51*50}{3*2}} \\
&= \frac{39}{102} \\
&= \frac{13}{34} \\
&= \mathbf{0.38235}
\end{aligned}
$$

the next 3 red is just ...

$$
\begin{aligned}
P(3 \text{ red and } 0 \text{ black}) &= \frac{26}{52} * \frac{25}{51} * \frac{24}{50} \\
&= \frac{2}{17} \\
&= \mathbf{0.11764}
\end{aligned}
$$

Summing all those probabilities together you get **0.88235**

2.5.7

(a) This is the probability of choosing consecutive 4 black balls.
$\frac{50}{80} * \frac{49}{79} * \frac{48}{78} * \frac{47}{77} = \mathbf{0.1456}$

(b) This the probability of choosing exactly three black balls is ...
$\frac{50}{80} * \frac{49}{79} * \frac{48}{78} * \frac{30}{77} * 4 = \mathbf{0.3717}$

(c) The probability that the first red ball appears on the last draw is ...
$\frac{50}{80} * \frac{49}{79} * \frac{48}{78} * \frac{30}{77} = \mathbf{0.0929}$