HW 8 Max Ryoo

C4.2 10a

10a) We know that

$P(T_1=0) = 1-P(T>1) = [1-e^{-\lambda}]$

We can use this same method for

$P(T=k) = P(T>k) - P(T>(k+1))$

$= [e^{-\lambda k} - e^{-\lambda(k+1)}]$

$e^{-\lambda k}(1-e^{-\lambda})$

this is a gemometric distribution

Chapter 4.4 # 3, 4, 5, 8

3) u^2

Let $Y = u^2$ so: $u = \pm\sqrt{y}$

We can use $f_y(y) = f_x(x) / \left|\frac{dy}{dx}\right|$

$= \frac{1}{2|u|} = \boxed{\frac{1}{2\sqrt{y}}}$ for $0<y<1$

4) In this case we need to consider that $x = \pm\sqrt{y}$

$f_y(y) = \frac{f_x(x)}{|dy/dx|_+} + \frac{f_x(x)}{|dy/dx|_-}$ $\quad f_x(x) = \begin{matrix} \frac{1}{2} & -1<x<0 \\ \frac{1}{2} & 0<x<1 \end{matrix}$

$\frac{1/2}{|2\sqrt{y}|} + \frac{1/2}{|-2\sqrt{y}|} = \boxed{\frac{1}{2\sqrt{y}}}$

5) $Y = x^2$ so $x = \pm\sqrt{y}$

$f_y(y) = \frac{f_x(\sqrt{y}) + f_x(-\sqrt{y})}{|2x|}$

The density of x will be $f_x(x) = \frac{1}{2+1} = \frac{1}{3}$

We must compute density of y for different regions.

Region 1 $= [0, 1]$, $f_y(y) = \frac{\frac{1}{3}+\frac{1}{3}}{2\sqrt{y}} = \boxed{\frac{1}{3\sqrt{y}}}$

Region 2 $= (1, 4)$, $f_y(y) = \frac{\frac{1}{3}+0}{2\sqrt{y}} = \boxed{\frac{1}{6\sqrt{y}}}$

8) a) $Z = \frac{1}{1+y^2}$ rearrange this = $z(1+y^2) = 1$

$$1+y^2 = \frac{1}{z}$$
$$y^2 = \frac{1}{z} - 1$$
$$y^2 = \frac{1-z}{z}$$
$$y = \pm\sqrt{\frac{1-z}{z}}$$

$\frac{1}{1+y^2}$ $\frac{dz}{dy} = \frac{(1+y^2)(0)-(1)2y}{(1+y^2)^2} = \frac{-2y}{(1+y^2)^2}$

so... $f_z(z) = \frac{f_y(y)}{dz/dy} = f_y\left(\sqrt{\frac{1-z}{z}}\right) + f_y\left(-\sqrt{\frac{1-z}{z}}\right)$ $\frac{-2y}{(1+y^2)^2}$

$f_y\left(-\sqrt{\frac{1-z}{z}}\right) = \frac{1}{\pi[1+(-\sqrt{\frac{1-z}{z}})^2]} z = \frac{z}{\pi}$

Aswell as $f_y\left(+\sqrt{\frac{1-z}{z}}\right) = \frac{z}{\pi}$

So we can say that

$$\frac{\frac{z}{\pi}+\frac{z}{\pi}}{2z\sqrt{z-z^2}} = \frac{2(\frac{z}{\pi})}{2z\sqrt{z(1-z)}} = \boxed{\frac{1}{\pi\sqrt{z(1-z)}}}$$ ✓

b) $P(z<x) = \int_0^z f_z(z)\,dz$

$$= \int_0^z \frac{1}{\pi\sqrt{z(1-z)}}$$

let us say $z = \sin^2 t$ where $dz = 2\sin t \cos t$

$$= \int_0^{\sin^{-1}\sqrt{x}} \frac{2\sin\cos t}{\pi\sqrt{\sin^2(1-\sin^2 t)}}\,dt = \frac{1}{\pi}\int_0^{\sin^{-1}\sqrt{x}} \frac{2\sin t\cos t}{\sqrt{\sin^2 \cos^2 t}}\,dt$$

so, $P(z \le x) = \left[\frac{2\sin^{-1}\sqrt{x}}{\pi}\right]$ for $0<x<1$ ✓

c) $E(z) = \int_0^\infty z f_z(z)\,dz$

$$\int_0^1 \frac{z}{\pi\sqrt{z(1-z)}}$$

using above steps where $z = \sin^2 t$ we get

$$E(z) = \int_0^{\pi/2} \frac{\sin^2 t \cdot 2\sin t\cos t}{\sin t\cos t} = 2\left\{\frac{1}{2}\left[t - \frac{1}{2}\sin(2t)\right]\right\}\Big|_0^{\frac{\pi}{2}}$$
$$= \boxed{\frac{1}{2}}$$

d) $Var(Z) = E(X^2) - E(X)^2$

$\quad\quad\quad\quad\ \ \| \quad\quad\ \ \|$
$\quad\quad\quad\quad\quad\quad\ \ \frac{1}{2}$

We will use same steps as in b and c so,

$E(z^2) = \int_0^1 \frac{z^2\, dz}{\pi\sqrt{z(1-z)}}$

$= \int_0^{\pi/2} \frac{2\sin^4 t}{\pi}\, dt = \frac{3}{8}$

$\frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4} = \boxed{\frac{1}{8}}$

4.5 #5

5) $f_X(x) = \frac{1}{2}e^{-|x|}$

If x has value $-\infty < x \le 0$

$f(x) = \int_{-\infty}^{x} \frac{1}{2}e^{-|y|}\, dy = \int_{-\infty}^{x} \frac{1}{2}e^{y}\, dy = \frac{1}{2}e^{x}$

If x has value $0 \ge x > \infty$

Then it will just be $1 - \frac{1}{2}e^{-x}$

If $x = 0$ then $f_X(x)$ will simply be $\frac{1}{2}$

Special Problem

The joint density will become

$f(x,y) = (\lambda e^{-\lambda x})(\mu e^{-\mu y}) = \lambda\mu e^{-\lambda x - \mu y}$

Since it is independent $P(X<Y)$ is found by taking the integration

$P(X<Y) = \int_{x<y} \lambda\mu e^{-\lambda x - \mu y}\, dx\, dy$

$= \int_{x=0}^{\infty} dx \int_{y=x}^{\infty} \lambda\mu e^{-\lambda x - \mu y}\, dy$

$= \int_{x=0}^{\infty} \lambda\mu e^{-\lambda x - \mu x}\, dx$

$= \boxed{\frac{\lambda}{\lambda+\mu}}$