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Homework # 6

3.1 Random Variables Introduction

3.1.13

(a) *Proof.* Let us think of each k separately.
When $k = 2$ we have $1 - \frac{2}{2n} = \frac{2n-2}{2n}$
When $k = 3$ we have $1 - \frac{4}{2n} = \frac{2n-4}{2n}$
...
When $k = k$ we have $\frac{2n-2(k-1)}{2n}$
$\therefore$ the equation becomes $\frac{2n-2}{2n} * \frac{2n-4}{2n} * .. \frac{2n-2(k-1)}{2n}$ $\square$

3.2 Expectations

3.2.7

(a) For this problem we can take a look at the expectation of an indicator. $E(I_A) = P(A)$. We are given that the probability that the switch is closed is p_i. We can then say that $E(I_i) = P(i) = p_i$. We can say that $X = \sum_{i=1}^{n}[I_i]$. Therefore, $E(x) = E(\sum_{i=1}^{n}[I_i]) = \sum_{i=1}^{n}[E(I_i)] = \mathbf{\sum_{i=1}^{n}[P_i]}$

3.2.8

(a)

We are given the following conditions...
$E(X^2) = 3$
$E(Y^2) = 4$
$E(XY) = 2$
We must find $E[(X+Y)^2]$
$E[(X+Y)^2]$ $E[(X^2+2XY+Y^2]$
Using addition rule for expectation...
$E(X^2) + E(XY) + E(XY) + E(Y^2)$
$3 + 2 + 2 + 4 = \mathbf{11}$

3.2.11

(a) Let us assume that p_i is the probability of ticket i being the winning ticket. This $p_i = 0.1$ because $\frac{100}{1000}$. Let us say that $X = X_1 + X_2 + X_3$, which means that X is the total number of prize tickets in 3 tickets if we assume that X gets 1 for prize and 0 for no prize. For the addition rule for expectation and expectation of an indicator we can see that $E(X) = \sum_{i=1}^{n} P_i = 3 * 0.1 = 0.3$
Using Markov we can see the probability of winning at least one.

$P(X \geq 1) \leq \frac{E(x)}{1} = 0.3$
We can also look at the actual probability. Winning at least one is the complimentary of winning none, which is shown by ...

$$1 - P(x = 0)$$
$$1 - \frac{900*899*898}{1000*999*998}$$
$$0.271$$

The upper bound for the probability of winning at least one prize is 0.3, while the actual is 0.271. The probabilities are close due to the large denominator. Winning is actually pretty hard.

3.2.14

(a) Let us say that I_{A_i} is an indicator of event A_i happening. A_i is the event that at least one person needs to get off the elevator at floor i.

$$\begin{aligned} E(X) &= E\left[\sum_{i=1}^{10} I_{A_i}\right] \\ &= \sum_{i=1}^{10} E[I_{A_i}] \\ &= \sum_{i=1}^{10} 1 - \left(\frac{9}{10}\right)^{12} \\ &= 10 * \left[1 - \left(\frac{9}{10}\right)^{12}\right] \\ &= 7.176 \end{aligned}$$

3.3 Standard Deviation and Normal Approximation

3.3.2

(a) Let us see what combinations are possible...

TTT
TTH
THT
HTT
THH
HHT
HTH
HHH

There are a total of 8 possibilities.

$$P(\text{no head}) = \frac{1}{8}$$
$$P(\text{one head }) = \frac{3}{8}$$

$$P(\text{ two head }) = \tfrac{3}{8}$$
$$P(\text{ three head }) = \tfrac{1}{8}$$

$$\begin{aligned}
\mu &= E(Y^2) \\
&= 0 * \frac{1}{8} + 1 * \frac{3}{8} + 4 * \frac{3}{8} + 9 * \frac{1}{8} \\
&= 3 \\
\sigma^2 &= E(X^2) - \mu^2 \\
&= \left[0 * \frac{1}{8} + 1 * \frac{3}{8} + 16 * \frac{3}{8} + 81 * \frac{1}{8}\right] - 9 \\
&= 7.5
\end{aligned}$$

3.3.3

(a)

$$\begin{aligned}
E(2X + 3Y) &= 2E(X) + 3E(Y) \\
&= 2 * 1 + 3 * 1 \\
&= 5
\end{aligned}$$

(b)

$$\begin{aligned}
Var(2X + 3Y) &= 2Var(X) + 3Var(Y) \\
&= 4 * 2 + 9 * 2 \\
&= 26
\end{aligned}$$

(c)

$$\begin{aligned}
E(XYZ) &= E(X)E(Y)E(Z) \\
&= 1
\end{aligned}$$

(d)

$$\begin{aligned}
Var(XYZ) &= Var(XY) * Var(Z) \\
Var(XY) &= Var(X)Var(Y) + Var(X)E(Y)^2 + Var(Y)E(X)^2 \\
&= 4 + 2 + 2 \\
&= 8 \\
Var(XYZ) &= Var(XY)Var(Z) + Var(XY)E(Z)^2 + Var(Z)E(XY)^2 \\
&= 8 * 2 + 8 * 1 + 2 * 1 \\
&= 26
\end{aligned}$$

3.3.8

(a) Let us say that I_{A_1} is the indicator that event A_1 happens, I_{A_2} is the indicator that event A_2 happens, and that I_{A_3} is the indicator that event A_3 happens. Then $N = I_{A_1} + I_{A_2} + I_{A_3}$

(b) Due to the expectation of an indicator we have that $E(I_A) = P(A)$

$$\begin{aligned} E(N) &= E(I_{A_1} + I_{A_2} + I_{A_3}) \\ &= E(I_{A_1}) + E(I_{A_2}) + E(I_{A_3}) \\ &= \frac{1}{5} + \frac{1}{4} + \frac{1}{3} \\ &= \mathbf{\frac{47}{60}} \end{aligned}$$

(c) Let us take into consideration that A_1, A_2, A_3 are all disjoint.

$$\begin{aligned} Var(N) &= E(N^2) - [E(N)]^2 \\ &= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - [E(I_{A_1} + I_{A_2} + I_{A_3})]^2 \\ &= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - \left[\frac{47}{60}\right]^2 \\ &= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - \frac{2209}{3600} \\ &= E({I_{A_1}}^2 + {I_{A_2}}^2 + {I_{A_3}}^2 + 2I_{A_1}I_{A_2} + 2I_{A_2}I_{A_3} + 2I_{A_1}I_{A_3}) - \frac{2209}{3600} \end{aligned}$$

However, A_1, A_2, A_3 are disjoint which means that the combination terms are not allowed...The equation thus becomes

$$\begin{aligned} &= E(I_{A_1} + I_{A_2} + I_{A_3}) - \frac{2209}{3600} \\ &= \frac{47}{60} - \frac{2209}{3600} \\ &= \mathbf{\frac{611}{3600}} \end{aligned}$$

(d) Let us look at the condition where they are independent.

The addition rule for variances for independent is ...
$Var(X + Y) = Var(X) + Var(Y)$

$$\begin{aligned} Var(N) &= Var(I_{A_1} + I_{A_2} + I_{A_3}) \\ &= Var(I_{A_1}) + Var(I_{A_2}) + Var(I_{A_3}) \\ &= \frac{1}{5}\left(1 - \frac{1}{5}\right) + \frac{1}{4}\left(1 - \frac{3}{4}\right) + \frac{1}{3}\left(1 - \frac{1}{3}\right) \\ &= \frac{4}{25} + \frac{3}{16} + \frac{2}{9} \\ &= \frac{2051}{3600} \end{aligned}$$

(e) Let us look at the condition of $A_1 \subset A_2 \subset A_3$

$$\begin{aligned}
Var(N) &= E(N^2) - [E(N)]^2 \\
&= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - [E(I_{A_1} + I_{A_2} + I_{A_3})]^2 \\
&= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - \left[\frac{47}{60}\right]^2 \\
&= E((I_{A_1} + I_{A_2} + I_{A_3})^2) - \frac{2209}{3600} \\
&= E({I_{A_1}}^2 + {I_{A_2}}^2 + {I_{A_3}}^2 + 2I_{A_1}I_{A_2} + 2I_{A_1}I_{A_3} + 2I_{A_2}I_{A_3}) - \frac{2209}{3600}
\end{aligned}$$

However, this time we know that A_1 is a subset of A_2, which is a subset of A_3, this means that $I_{A_1}I_{A_2} = I_{A_1}$ and $I_{A_1}I_{A_2} = I_{A_1}$ as well as $I_{A_2}I_{A_3} = I_{A_2}$.

$$\begin{aligned}
&= E({I_{A_1}}^2 + {I_{A_2}}^2 + {I_{A_3}}^2 + 2I_{A_1}I_{A_2} + 2I_{A_1}I_{A_3} + 2I_{A_1}I_{A_3}) - \frac{2209}{3600} \\
&= E({I_{A_1}}^2 + {I_{A_2}}^2 + {I_{A_3}}^2 + 2I_{A_1} + 2I_{A_1} + 2I_{A_2}) - \frac{2209}{3600} \\
&= E(5I_{A_1} + 3I_{A_2} + I_{A_3}) - \frac{2209}{3600} \\
&= 5 * \frac{1}{5} + 3 * \frac{1}{4} + \frac{1}{3} - \frac{2209}{3600} \\
&= \frac{25}{12} - \frac{2209}{3600} \\
&= \mathbf{\frac{5291}{3600}}
\end{aligned}$$

6.1 Conditional Distributions: Discrete Case

6.1.2

(a) We can find the probabilities and distribution of G due to the $p = 0.5$. We can also use binomial distribution to calculate the individual probabilities.
$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k} p^k q^{n-k}$, which for this case is ...
$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k} p^n$

The distribution of G

$T =$	0	1	2	3	4
$G = 0$, which is $P(G = 0 \mid T = x)$	1	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$
$G = 1$, which is $P(G = 1 \mid T = x)$	0	$\frac{1}{2}$	$2\left(\frac{1}{2}\right)^2$	$3\left(\frac{1}{2}\right)^3$	$4\left(\frac{1}{2}\right)^4$
$G = 2$, which is $P(G = 2 \mid T = x)$	0	0	$\left(\frac{1}{2}\right)^2$	$3\left(\frac{1}{2}\right)^3$	$6\left(\frac{1}{2}\right)^4$
$G = 3$, which is $P(G = 3 \mid T = x)$	0	0	0	$\left(\frac{1}{2}\right)^3$	$4\left(\frac{1}{2}\right)^4$
$G = 4$, which is $P(G = 4 \mid T = x)$	0	0	0	0	$\left(\frac{1}{2}\right)^4$

6.2 Conditional Expectation: Discrete Case

6.2.4

(a)

$$\begin{aligned}
E(Y) &= E(E(Y|X)) \\
&= E\left(\sum_{y=1}^{x} \frac{y}{x}\right) \\
&= E\left(\frac{1}{x} * (1 + 2 + ... + X)\right) \\
&= E\left(\frac{1+X}{2}\right) \\
&= \frac{E(x)}{2} + \frac{1}{2} \\
&= \frac{n+1}{2} * \frac{1}{2} + \frac{1}{2} \\
&= \frac{n+3}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
E(Y^2) &= E(E(Y^2|X)) \\
&= E\left(\sum_{y=1}^{x} Y^2 * \frac{1}{X}\right) \\
&= E\left(\frac{x(x+1)(2x+1)}{6} * \frac{1}{x}\right) \\
&= E\left(\frac{2x^2+3x+1}{6}\right) \\
&= \frac{1}{6} * (2E(X^2) + 3E(x) + 1) \\
&= \frac{1}{6} * (\frac{2n^2+3n+1}{3} + 3\frac{n+1}{2} + 1) \\
&= \frac{4n^2+15n+17}{36}
\end{aligned}$$

(c)

$$\begin{aligned}
P(X+Y=2) &= P(Y=1|X=1)P(X=1) \\
&= \frac{1}{n}
\end{aligned}$$