Derivation of
Restorating Force Equation

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Prerequisites

1. Characteristic equation

Characteristic equation is an algebraic equation of degree n upon which depends the solution of a given nth-order differential equation.
For example, for the following DE problem,

$$y'' - 4y' + 3y = 0$$

by change of function (which is valid due to Picard - Lindelof theorem of initial value problems),

$$y(t) := c \cdot e^{\theta t}$$

we can obtain

$$\theta^2 - 4\theta + 3 = 0$$

By solving for θ, we get the solution of DE,

$$y(t) = c_1 e^t + c_2 e^{3t}$$

2. Euler's formula

$$e^{ix} = \cos(x) + i\sin(x)$$

Proof) By Maclaurin series expansion,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos(x) + i\sin(x) \end{aligned}$$

3. Product rule

$$\frac{d}{dx}(u \cdot v) = \left(\frac{du}{dx}\right) v + u \left(\frac{dv}{dx}\right)$$

4. Chain rule

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Basic approach

$$m\ddot{x} + kx = 0$$

The **characteristic equation** for the above ODE is

$$m\theta^2 + k = 0$$

which has complex roots

$$\theta = \pm i\sqrt{k/m}$$

Thus, the solution is given by

$$\begin{aligned}
x(t) &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\
&= c_1(\cos(\omega t) + i\sin(\omega t)) + c_2(\cos(\omega t) - i\sin(\omega t)) \\
&= d_1\cos(\omega t) + d_2\sin(\omega t) \\
&= A\cos(\omega t + \delta)
\end{aligned}$$

where $\omega := \sqrt{k/m}$. Note that $d_1, d_2 \in \mathbb{R}$.

The last line is due to the fact that the **linear combination of sinusoids are again sinusoids**, but with a new amplitude and phase shift.

Interpretation : $\omega = \sqrt{k/m}$ as **frequency**, $A = \sqrt{d_1^2 + d_2^2}$ as **amplitude**, and δ as **phase angle**.

Advanced approach

$$m\ddot{x} + kx = 0$$

By multiplying $\dot{x}$ to the above equation, we can see

$$m\ddot{x}\dot{x} + kx\dot{x} = 0$$

$$m\frac{d}{dt}\left(\frac{dx}{dt}\right) \cdot \frac{dx}{dt} + kx\frac{dx}{dt} = 0$$

Since

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) \cdot \frac{dx}{dt} = \frac{1}{2}\left[\frac{d}{dt}\left(\frac{dx}{dt}\right) \cdot \frac{dx}{dt} + \frac{dx}{dt} \cdot \frac{d}{dt}\left(\frac{dx}{dt}\right)\right] = \frac{d}{dt}\left(\left(\frac{dx}{dt}\right)^2\right)$$

and

$$x\frac{dx}{dt} = \frac{d(x^2/2)}{dx} \cdot \frac{dx}{dt} = \frac{d(x^2/2)}{dt}$$

we get the following **energy conservation law** formula, with the first term being the kinetic energy, and the second term being the potential energy of the spring.

$$\int \frac{m}{2}\frac{d}{dt}\left(\left(\frac{dx}{dt}\right)^2\right) dt + \int k\frac{d(x^2/2)}{dt} dt = E$$

$$m\frac{(\dot{x})^2}{2} + k\frac{x^2}{2} = E$$

If we proceed,

$$m\frac{(\dot{x})^2}{2} + k\frac{x^2}{2} = E$$

$$\dot{x} = \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}$$

$$\frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = \pm dt$$

$$\therefore \quad \arccos\left(x\sqrt{\frac{k}{2E}}\right) = \pm\sqrt{\frac{k}{m}}t + \phi$$

$$x = \sqrt{\frac{2E}{k}}\cos(\omega t + \phi)$$

since

$$\frac{d}{dx}\arccos(x) = \frac{1}{\sqrt{1-x^2}}$$