05_Decision_Tree_Learning

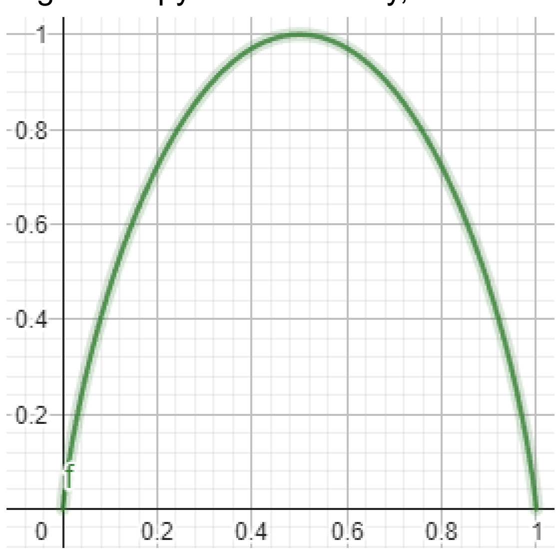
5.2 Entropy

5.2.1 Attributes & Purity of Decision

- We want to know which attribute is the best classifier.
- Each attribute can be represented as a node, splitting test sample into 2 classes (in the binary case)
- The *purer* the splitting, the better the decision.

5.2.2 Entropy 熵

[DEF] Entropy 熵

- Given **a split** from collection S with **portions** of both positive and negative p_+, p_-, the entropy of the collection would be:
    - $Entropy(S) = -p_+ \log_2 p_+ - p_- \log_2 p_-$
- The higher the entropy, the fewer the information the sample contains.
    - High Entropy = More Messy, Lower Entropy = Less Messy
    - 

    - $f(p) = -p \log_2 p - (1-p) \log_2 (1-p)$

[Prop] Entropy 熵的性质

- Domain is $[0, 1]$;
    - Monotonically, Increase in $p_+ \in [0, \frac{1}{2}]$, Decrease in $p_+ \in [\frac{1}{2}, 1]$ (symmetric in p_-);

- Range is $[0,1]$;
  - Reaches the peak at $p=\frac{1}{2}$, reaches the lowest point at $p_+=0$ and $p_+=1$.

5.2.3 Information Gain

By calculating the difference between

- The entropy **BEFORE** splitting, and
- The entropy **AFTER** splitting,
  we can calculate the "Information Gained" from the splitting.
- When calculating, we should also consider the weights, i.e., the significance of each branch.

Basics

- Suppose that a collection S has $x=|S|$ samples, separated into n branches $S_1, S_2, \cdots, S_n$. For a specific branch S_i, it contains $x_i=|S_i|$ samples.
  - $x=x_1+x_2+\cdots+x_n$
- Among the x samples, there are x_+ positive samples and x_- negative samples.
  - $x=x^++x^-$
- For each branch x_i, there are x_+^i positive samples and x_-^i negative samples.
  - $x_i=x_i^++x_i^-$
  - Obviously, $x^+=\sum_{i=1}^n x_i^+$, and $x^-=\sum_{i=1}^n x_i^-$

Entropy Before Split

- $p_+=\frac{x^+}{x^++x^-}$, $p^-=\frac{x^-}{x^++x^-}$
- $Entropy(S)=-p^+\log_2 p^+-p^-\log_2 p^-$

Weighted Entropy After Split

- For each branch S_i:
  - $p_i^+=\frac{x_i^+}{x_i^++x_i^-}$, $p_i^-=\frac{x_i^-}{x_i^++x_i^-}$
  - Weight of this branch $w_i=\frac{x_i}{x}=\frac{|S_i|}{|S|}$
  - Entropy of this branch $Entropy(S_i)=-p_i^+\log_2 p_i^+-p_i^-\log_2 p_i^-$
- Total weighted entropy after split:
  - $\sum_{i=1}^n \frac{x_i}{x}(-p_i^+\log_2 p_i^+-p_i^-\log_2 p_i^-)$

- $= \sum_{i=1}^{n} \frac{|S_i|}{|S|} Entropy(S_i)$

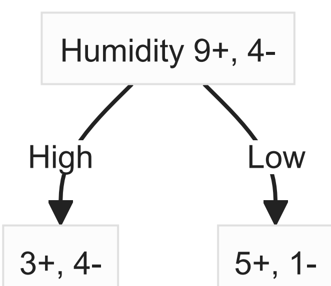
Information Gain

- $Gain(S) = Entropy\ Before\ Split\ -\ Entropy\ After\ Split$
- $= Entropy(S) - \sum_{i=1}^{n} \frac{|S_i|}{|S|} Entropy(S_i)$

5.2.4 From Entropy to Information Gain

Given:

- A decision node, with the amounts in each class:
  - Collection 1 (S_1):
    - Positive: n_1^+
    - Negative: n_1^-
  - Collection 2 (S_2):
    - Positive :n_2^+
    - Negative: n_2^-
- Example:
  - $n_1^+ = 3, n_1^- = 4$
  - $n_2^+ = 6, n_2^- = 1$



Do:

- Get the proportion of positive & negative samples in both collections.
  - S_1:
    - $p_1^+ = \frac{n_1^+}{n_1^+ + n_1^-} = \frac{3}{3+4} = \frac{3}{7}$
    - $p_1^- = \frac{n_-^1}{n_+^1 + n_-^1} = \frac{4}{3+4} = \frac{4}{7}$
  - S_2:

- $p_2^+ = \frac{n_2^+}{n_2^+ + n_2^-} = \frac{5}{5+1} = \frac{5}{6}$
- $p_2^- = \frac{n_2^-}{n_2^+ + n_2^-} = \frac{1}{5+1} = \frac{1}{6}$

- Calculate Entropies:
  - Calculate entropy **AFTER** splitting:
    - $Entropy(S_1) = -p_1^+ \log_2 p_1^+ - p_1^- \log_2 p_1^- = -\frac{3}{7}\log_2 \frac{3}{7} - \frac{4}{7} log_2 \frac{4}{7} = 0.9852$
    - $Entropy(S_2) = -p_2^+ \log_2 p_2^+ - p_2^- \log_2 p_2^- = -\frac{5}{6}\log_2 \frac{5}{6} - \frac{1}{6} log_2 \frac{1}{6} = 0.6500$
  - Calculate overall **BEFORE** splitting:
    - $p^+ = \frac{n^+}{n^+ + n^-} = \frac{9}{9+4} = \frac{9}{13}$
    - $p^- = \frac{n^-}{n^+ + n^-} = \frac{4}{9+4} = \frac{4}{13}$
    - $Entropy(S) = -p_+ log_2 p_+ - p_- log_2 p_- = -\frac{9}{13}\log_2 \frac{9}{13} - \frac{4}{13}\log_2 \frac{4}{13} = 0.8905$
- Calculate Information Gain:
  - $Gain(S) = Entropy(S) - \sum_{i \in \{1,2\}} \frac{|S_i|}{|S|} Entropy(S_i)$
  - $= 0.8905 - (\frac{7}{13} \times 0.9852 + \frac{6}{13} \times 0.6500)$
  - $= 0.0600$