# 03_Naïve_Bayes

# 3.0 Why Naïve Bayes?

- Uncertainty
  - Lack of exact knowledge that could enable us to reach a perfectly reliable conclusion.
  - But classical logic only allow *exact reasoning*. It assumes that *the law of the excluded middle* can always be applied:
    - `IF A is true THEN A is not false`, and
    - `IF A is false THEN A is not true`
- Weak Implications
  - Hard to establish concrete correlations between $\text{IF}$ and $\text{THEN}$.
  - Handle vague associations.
- Imprecise Language
  - Natural language is ambiguous.
  - We describe facts with: sometimes, often, frequently, hardly, ...
  - Difficult to establish IF-THEN rules based on NL.

# 3.1 Basic Probability Theory

## 3.1.1 Probability 概率

- The probability of an event
  - *Scientific Measure of Chance*.
  - = the proportion of cases in which the event occurs.
  - Expression: From $0$ (absolute impossible) $\to$ Unity (Absolute certain).
  - Mostly strictly between $0$ and $1$. Each event has *at least two* outcomes: $\text{success}$ or $\text{failure}$. It's stated that:
    - $P(\text{success}) = p = \dfrac{s}{s+f}$, and
    - $P(\text{failure}) = q = \dfrac{f}{s+f}$, where
    - $p + q = 1$.

## 3.1.2 Conditional Probability 条件概率

- Let: $A, B$ be an Event.
  - Supposed that $A$ and $B$ are *not mutual exclusive*.
  - That is, $A$ and $B$ can occur at the same time.

- Conditional Probability of $A$ over $B$ is:
  - The probability that: If $B$ occur, then $A$ occur.
  - $P(A|B) = \dfrac{\#.\,(A \text{ and } B \text{ occur})}{\#.\,(B \text{ occur})}$
- Bayesian Rule:
  - We know that:
    - $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \implies P(A|B) \cdot P(B) = P(A \cap B)$,
    - $P(B|A) = \dfrac{P(A \cap B)}{P(A)} \implies P(B|A) \cdot P(A) = P(A \cap B)$.
  - Therefore, we can conclude that:
    ★ $P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)}$ ,
  - which yields the Bayesian Rule.

# 3.2 Bayesian Reasoning

## 3.2.1 Bayesian Rule

### From Rules

Suppose that all rules in the knowledge base are represented in this form:

```
IF E is true
THEN H is true, with probability p
```

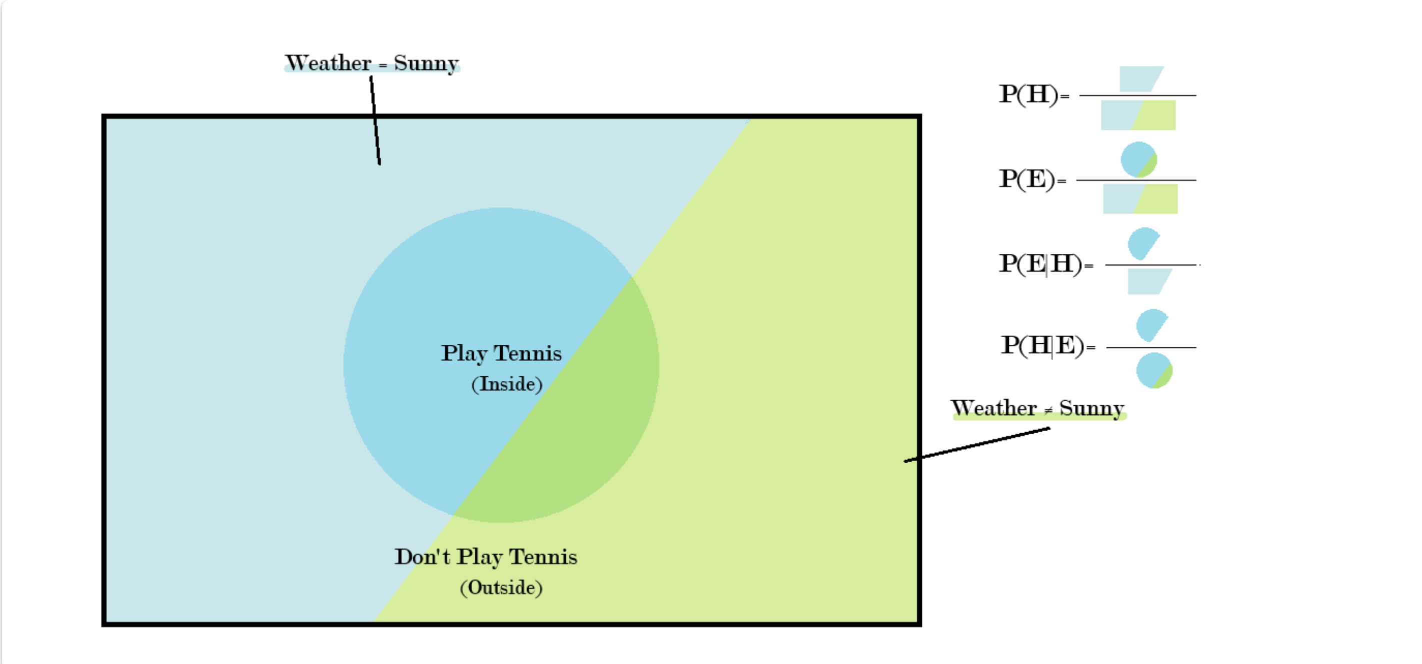
For instance,

```
IF weather is sunny
THEN play_tennis = true, p=0.9
```

### Problem Specification

#### Given

- A hypothesis $H$.
  - e.g., $\text{play tennis} = \text{true}$
- An evidence $E$.
  - which supports this hypothesis.
  - e.g., $\text{weather is sunny}$
  Do




Get the prob that event H (Hypothesis) will occur, as P.

$$p(H|E) = \frac{p(E|H) \cdot p(H)}{p(E)} = \frac{p(E|H) \cdot p(H)}{\Big[p(E|H) \cdot p(H)\Big] + \Big[p(E|\neg H) \cdot p(\neg H)\Big]}$$

where,

- $p(H)$ is the prior probability of hypothesis $H$ being true.
  - e.g., the portion of days among all that *played* tennis.
- $p(E|H)$ is the posterior probability of evidence $E$ being true under hypothesis $H$.
  - e.g., the portion of days with a sunny weather among all the days that *played* tennis.
- $p(\neg H)$ is the prior probability of hypothesis $H$ being false.
  - e.g., the portion of days among all that *didn't play* tennis.
- $p(E|\neg H)$ is the posterior probability of evidence $E$ being true under hypothesis $\neg H$.
  - e.g., the portion of days with a sunny weather among all the days that *didn't play* tennis.

## 3.2.2 Variances

Single Evidence, Multiple Hypothesis:

$$P(H_i|E) = \frac{P(E|H_i) \cdot P(H_i)}{\sum_{k=1}^{m}\Big[P(E|H_k) \cdot P(H_k)\Big]}$$

Multiple Evidence, Multiple Hypothesis:

$$P(H_i|E_1, E_2, \ldots, E_n) = \frac{P(E_1, E_2, \ldots, E_n|H_i) \cdot P(H_i)}{\sum_{k=1}^{m}\Big[P(E_1, E_2, \ldots, E_n|H_k) \cdot P(H_k)\Big]}$$

- $\approx \dfrac{\Big[P(E_1|H_i) \cdot P(E_2|H_i) \cdot \ldots \cdot P(E_n|H_i)\Big] \times P(H_i)}{\sum_{k=1}^{m} \Big[P(E_1|H_k) \cdot P(E_2|H_k) \cdot \ldots \cdot P(E_n|H_k) \times P(H_k)\Big]}$
    - if conditional independence holds.
- $= \dfrac{P(H_i) \cdot \Big[\prod_{a=1}^{n} P(E_a|H_i)\Big]}{\sum_{k=1}^{m} \Big[P(H_k) \cdot \prod_{b=1}^{n} P(E_b|H_k)\Big]}$

### Example

Given the prior an conditional probs as follows:

| | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| $P(H_i)$ | $0.40$ | $0.35$ | $0.25$ |
| $P(E_1\|H_i)$ | $0.3$ | $0.8$ | $0.5$ |
| $P(E_2\|H_i)$ | $0.9$ | $0.0$ | $0.7$ |
| $P(E_3\|H_i)$ | $0.6$ | $0.7$ | $0.9$ |

- Want $P(H_3|E_3)$.
- $P(H_3|E_3) = \dfrac{P(E_3|H_3)P(H_3)}{P(E_3)}$, where
    - $P(E_3|H_3) \cdot P(H_3) = 0.9 \times 0.25 = 0.36$
    - $P(E_3) = \Big[P(E_3|H_1) \cdot P(H_1)\Big] \times \Big[P(E_3|H_2) \cdot P(H_2)\Big] \times \Big[P(E_3|H_3) \cdot P(H_3)\Big]$
        - $= 0.6 \times 0.4 + 0.7 \times 0.35 + 0.9 \times 0.25 = 0.2838$
- $\implies P(H_3|E_3) = \dfrac{0.36 \times 0.25}{0.2838} = 0.3171$

# 3.3 Naïve Bayes Classifiers

## 3.3.1 Maximum A Posteriori

In Naïve Bayes Classifiers, we denote "Classes" as Hypothesis, and "Features" as Evidence. To find the best class, we find the hypothesis that gives the best $p(h|E)$.

$$h_{\text{conclusion}} = \text{argmax}_{h \in H} P(h|E)$$

- $= \text{argmax}_{h \in H} \dfrac{P(E|h) \cdot P(h)}{P(E)}$
- $= \text{argmax}_{h \in H} \Big[P(E|h) \cdot P(h)\Big]$

Omit the $P(E)$ since it's constant, which is independent from the hypothesis.

- Here, $P(E|h) \cdot P(h)$ is the Maximum A Posteriori **最大后验**.

### 3.3.2 Naïve Bayes Estimation

- Given:
  - A conjunctive test sample: $x_1, x_2, \dots, x_n$
- $c_{MAP} = \text{argmax}_{c_j \in C}\left[P(c_j|x_1, x_2, \dots x_n)\right]$
  - $= \text{argmax}_{c_j \in C}\left[\frac{P(x_1, x_2, \dots, x_n|c_j) \cdot P(c_j)}{P(x_1, x_2, \dots, x_n)}\right]$
  - $= \text{argmax}_{c_j \in C}\left[P(x_1, x_2, \dots, x_n|c_j) \cdot P(c_j)\right]$
  - $= \text{argmax}_{c_j \in C}\left[P(x_1|c_j) \times P(x_2|c_j) \times \dots \times P(x_n|c_j) \times P(c_j)\right]$
    - assumed the conditional independency holds.
  - ★ $= \text{argmax}_{c_j \in C}\left[P(c_j) \times \prod_{k=1}^{n} P(x_k|c_j)\right]$

#### Naïve Bayes Classifier: An Example.

| Day | Outlook | Temp | Humitity | Wind | Play Tennis |
|---|---|---|---|---|---|
| 1 | Sunny | Hot | High | Weak | No |
| 2 | Sunny | Hot | High | Strong | No |
| 3 | Overcast | Hot | High | Weak | Yes |
| 4 | Rain | Mild | High | Weak | Yes |
| 5 | Rain | Cool | Normal | Weak | Yes |
| 6 | Rain | Cool | Normal | Strong | No |
| 7 | Overcast | Cool | Normal | Strong | Yes |
| 8 | Sunny | Mild | High | Weak | No |
| 9 | Sunny | Cool | Normal | Weak | Yes |
| 10 | Rain | Mild | Normal | Weak | Yes |
| 11 | Sunny | Mild | Normal | Strong | Yes |
| 12 | Overcast | Mild | High | Strong | Yes |
| 13 | Overcast | Hot | Normal | Weak | Yes |
| 14 | Rain | Mild | High | Strong | No |

Known: $\text{Outlook} = \text{Sunny}, \text{Temp} = \text{Cool}, \text{Humidity} = \text{High}, \text{Wind} = \text{Strong}.$
Want: Play tennis or not?
Do:

- $MAP(\text{Yes}|\text{sunny}, \text{cool}, \text{high}, \text{strong})$
  - $= P(\text{sunny}, \text{cool}, \text{high}, \text{strong}|\text{Yes}) \times P(\text{Yes})$
  - $= P(\text{sunny}|\text{Yes}) \times P(\text{cool}|\text{Yes}) \times P(\text{high}|\text{Yes}) \times P(\text{strong}|\text{Yes}) \times P(\text{Yes})$
  - $= [\frac{2}{9} \times \frac{3}{9} \times \frac{3}{9} \times \frac{3}{9}] \times \frac{9}{14}$

- $= 0.005291005291$

- $MAP(\text{NO}|\text{sunny}, \text{cool}, \text{high}, \text{strong})$

```
- $=P(sunny, cool, high, strong|No)\times P(No)$

- $=P(sunny|No)\times P(cool|No)\times P(high|No)\times P(strong|No)\times P(No)$

- $=[\dfrac{3}{5}\times \dfrac{1}{5}\times \dfrac{4}{5}\times \dfrac{3}
{5}]\times\dfrac{5}{14}$

- $=0.02057142857$
```

**Output:**

- $c_{MAP} = argmax_{c\in\{\text{Yes},\text{No}\}} MAP(c|\text{sunny}, \text{cool}, \text{high}, \text{strong}) = \text{No}.$
- That is, don't play tennis.

# 3.4 Enhancement of NB Classifiers

## 3.4.1 Add-1 Smoothing

Initially, we have:

$$c_{\text{target}} = \text{argmax}_{c_j \in C}\left[P(c_j)\prod_{i=1}^{n} P(x_i|c_j)\right]$$

where:

$$P(x_i|c_j) = \frac{\#.\,(x \in c_j \wedge x = x_i)}{\#.\,(x \in c_j)} = \frac{n_c}{n}$$

where the number of $x$ that's in class $c_j$ could be $0$, yielding $P(x_i|c_j)$ to be $0$. In this case, even if $P(x_i|c_{j_2})$ is very large for $j_2$, the entire $MAP = P(c_j)\prod_{i=1}^{n} P(x_i|c_j)$ would be still cast to $0$.

Resolution: Add-1 smoothing, with:

- Prior: $P(c_j) = \dfrac{(\#.\,c \in C \wedge c = c_j) + m_{\text{prior}} \times p_{\text{prior}}}{(\#.\,c \in C) + m_{\text{prior}}}$, where $m_{\text{prior}} \in \mathbb{R}^+$ and $p_{\text{prior}} \in [0, 1]$
- Evidence: $P(x_i|c_j) = \dfrac{(\#.\,x_i \in c_j) + m_{\text{evid}} \times p_{\text{evid}}}{(\#.\,c_j) + m_{\text{evid}}}$, where $m_{\text{evid}} \in \mathbb{R}^+$ and $p_{\text{evid}} \in [0, 1]$

  The intuition is that:
- We create $m$ imaginary prior examples, where a portion of $p$ is the one to be examined.

## 3.4.2 Continuous $x$

Observations may be continuous. Use Gaussian Distribution instead.

$$P(x_i|c_k) = \frac{1}{\sigma_{ij}\sqrt{2\pi}} e^{\dfrac{-(x_i-\mu_{ik})^2}{2\sigma_{ik}^2}} = \mathcal{N}(x_i, \mu_{ik}, \sigma_{ik})$$

That is, for a specific class $c_j$, extract all the values $x_i \in c_j$ and form a normal distribution. This determines two variables:

- $\mu$, the mean/expectation
  - $\mu_{ik} = \frac{1}{n_k} \sum_{j=1}^{n_k} x_{ikj}$
- $\sigma$, the standard deviation
  - $\sigma_{ik} = \sqrt{\frac{1}{n_k} \sum_{j=1}^{n_k} (x_{ikj} - \mu_{ik})^2}$
  Note that:
- $i$ - the $i$-th feature,
- $k$ - the target class,
- $j$ - within a class $k$, the $j$-th data.
  After the two variables are set, the probability $P(x_i|c_j)$ can be thus calculated.
- $P(x_i|c_j)$ means that:
  - Under the given class $c_j$, there's a lot of continuous numeric input under the $i$-th feature.
  - We are given a test value $x_i$
  - $P(x_i|c_j)$ gives us the probability that $x_i$ is here according to all the numeric inputs mentioned above, with respect to a gaussian/normal distribution.

## Example

There is a set of data, having two classes:

- $\text{Play} = \text{Yes}$
- $\text{Play} = \text{No}$
  Each data sample is composed of $5$ features, listed with class-wise distribution below:

| Day | Outlook | Temp | Humitity | Wind | Play Tennis |
|---|---|---|---|---|---|
| 1 | Sunny | 85 | 85 | Weak | No |
| 2 | Sunny | 80 | 90 | Strong | No |
| 3 | Overcast | 83 | 86 | Weak | Yes |
| 4 | Rain | 70 | 96 | Weak | Yes |
| 5 | Rain | 68 | 80 | Weak | Yes |
| 6 | Rain | 65 | 70 | Strong | No |
| 7 | Overcast | 64 | 65 | Strong | Yes |

| Day | Outlook | Temp | Humitity | Wind | Play Tennis |
|---|---|---|---|---|---|
| 8 | Sunny | 72 | 95 | Weak | No |
| 9 | Sunny | 69 | 70 | Weak | Yes |
| 10 | Rain | 75 | 80 | Weak | Yes |
| 11 | Sunny | 75 | 70 | Strong | Yes |
| 12 | Overcast | 72 | 90 | Strong | Yes |
| 13 | Overcast | 81 | 75 | Weak | Yes |
| 14 | Rain | 71 | 91 | Strong | No |

Classify for a test sample:

- $\text{Outlook} = \text{Sunny}, \text{Temp} = 66, \text{Humidity} = 90, \text{Wind} = \text{Strong}$

Calculate Prior probabilities.

- $P(\text{Yes}) = \frac{9}{14}$
- $P(\text{No}) = \frac{5}{14}$

Calculate Posterior probabilities.

### Outlook

- Yes
  - $P(\text{Sunny}|\text{Yes}) = \frac{2}{9}$
- No
  - $P(\text{Sunny}|\text{Yes}) = \frac{3}{5}$

### Temperature

- Yes
  - $\mu_{\text{temp,Yes}} = \frac{1}{9}\left[83 + 70 + 68 + 64 + 69 + 75 + 75 + 72 + 81\right] = 73$
  - $\sigma_{\text{temp,Yes}} = \sqrt{\frac{1}{9}\sum_{j=1}^{n_{\text{Yes}}}(x_{\text{temp,Yes},j} - 73)^2} = \frac{4\sqrt{19}}{3}$
  - $\implies P(66|\text{Yes}) = \frac{1}{\frac{4\sqrt{19}}{3} \cdot \sqrt{2\pi}} e^{\frac{-(66-73)^2}{2 \cdot \frac{304}{9}}} = 0.0332$
- No
  - $\mu_{\text{temp,No}} = \frac{1}{5}\left[85 + 80 + 65 + 71 + 71\right] = 74.6$
  - $\sigma_{\text{temp,No}} = \sqrt{\frac{1}{5}\sum_{j=1}^{n_{No}}(x_{\text{temp,No,j}} - 74.6)^2} = \sqrt{49.84} = 7.0597$

- $\implies P(66|\text{No}) = \frac{1}{7.0597 \times \sqrt{2\pi}} e^{\frac{-(66-74.6)^2}{2 \times 49.84}} = 0.0269$

**Humidity**

- Yes
  - $\mu_{\text{humid,Yes}} = \frac{1}{9}\left[86+96+80+65+70+80+70+90+75\right] = \frac{712}{9}$
  - $\sigma_{\text{humid,Yes}} = \sqrt{\frac{1}{9}\sum_{j=1}^{n_{Yes}}(x_{\text{humid,Yes},j} - \frac{712}{9})^2} = \sqrt{92.7654} = 9.6315$
  - $\implies P(90|\text{Yes}) = \frac{1}{9.6315 \times \sqrt{2\pi}} e^{\frac{-(90-\frac{712}{9})^2}{2 \times 92.7654}} = 0.0219$
- No
  - $\mu_{\text{humid,No}} = \frac{1}{5}\left[95+90+70+95+91\right] = \frac{441}{5}$
  - $\sigma_{\text{humid,No}} = \sqrt{\frac{1}{5}\sum_{j=1}^{n_{No}}(x_{\text{temp,No},j} - \frac{441}{9})^2} = \sqrt{75.76} = 8.7040$
  - $\implies P(90|\text{No}) = \frac{1}{8.7040 \times \sqrt{2\pi}} e^{\frac{-\left(90-\frac{441}{5}\right)^2}{2 \times 75.76}} = 0.0449$

**Wind**

- Yes
  - $P(\text{Strong}|\text{Yes}) = \frac{3}{9}$
- No
  - $P(\text{Strong}|\text{No}) = \frac{3}{5}$

Therefore, to sum up:

- $MAP(\text{Yes}|\text{Summy}, 66, 90, \text{Strong}) = \left(\frac{2}{9} \times 0.0332 \times 0.0219 \times \frac{3}{9}\right) \times \frac{9}{14} = 3.4623 \times 10^{-5}$
- $MAP(\text{No}|\text{Summy}, 66, 90, \text{Strong}) = \left(\frac{3}{5} \times 0.0269 \times 0.0449 \times \frac{3}{5}\right) \times \frac{5}{14} = 1.5529 \times 10^{-4}$

  Therefore, the classification result is $\text{No}$.

# 3.5 Certainty Factors Theory & Evidential Reasoning

## 3.5.1 Certainty Factor

ℹ️ A Certainty Factor is:

- A number to measure the expert's belief.
- Ranges from: $-1.0$ (*Definitely False*) ~ $+1.0$ (*Definitely True*)

## 3.5.2 Certainty Factors Theory

The certainty factor theory is an alternative to Bayesian reasoning.
Similarly to Bayesian reasoning, the knowledge base consists of a set of rules that have the following syntax:

```
IF evidence
THEN hypothesis {cf}
```

- That is, a certainty factor is assigned to the $\text{THEN}$ part, just like assigning a probability to the $\text{THEN}$ part in Bayesian Reasoning.
- The assigned certainty factor (cf) denotes the level of belief in:
  - the hypothesis $H$ being true,
  - given the evidence $E$.

### Measure of Belief and Disbelief

The certainty factors theory is based on two functions:
  - Measure of belief: $MB(H, E)$, and
  - Measure of disbelief: $MD(H, E)$
- which are defined as:

$$MB(H,E) = \begin{cases} 1, & \text{if } p(H) = 1 \\ \dfrac{max\left[p(H|E), p(H)\right] - p(H)}{max(1,0) - p(H)}, & \text{otherwise} \end{cases}$$

$$MD(H,E) = \begin{cases} 1, & \text{if } p(H) = 0 \\ \dfrac{min\left[p(H|E), p(H)\right] - p(H)}{min(1,0) - p(H)}, & \text{otherwise} \end{cases}$$

where,
- $p(H)$ is the prior probability that hypothesis $H$ is true;
- $p(H|E)$ is the posterior probability that hypothesis $H$ is true under evidence $E$.

### Certainty Factor

By analysis,

- $MB(H, E), MD(H, E) \in [0, 1]$.
- Some facts may increase the strength of belief, and vise versa.

The total strength of belief/disbelief in a hypothesis:

$$cf = \frac{MB(H,E) - MD(H,E)}{1 - min\Big[MB(H,E), MD(H,E)\Big]}$$

yielding the certainty factor.