05_Support_Vector_Machine

5.0 A Quick View

What does it do?

- Find an optimized separating plane to
  - Separate samples of 2 classes
  - Maximize the margins

Summary

	G	Constraint	Solution
Reg	$G=\sum_{i=1}^{N}\lambda_i-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\lambda_i\lambda_j)\cdot(y_iy_j)\cdot\mathbf{x}_i^\top\mathbf{x}_j$	$\begin{cases}\lambda_i\geq 0\\ \sum_{i=1}^{N}\lambda_iy_i=0\end{cases}$	$f(\mathbf{x})=(\sum_{i=1}^{N}\lambda_iy_i$
SfMg	$G=\sum_{i=1}^{N}\lambda_i-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\lambda_i\lambda_j)\cdot(y_iy_j)\cdot\mathbf{x}_i^\top\mathbf{x}_j$	$\begin{cases}0\leq\lambda_i\leq C\\ \sum_{i=1}^{N}\lambda_iy_i=0\end{cases}$	$f(\mathbf{x})=(\sum_{i=1}^{N}\lambda_iy_i$
NLin	$G=\sum_{i=1}^{N}\lambda_i-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\lambda_i\lambda_j)\cdot(y_iy_j)\cdot K(\mathbf{x}_i,\mathbf{x}_j)$	$\begin{cases}\lambda_i\geq 0\\ \sum_{i=1}^{N}\lambda_iy_i=0\end{cases}$	$f(\mathbf{x})=\sum_{i=1}^{N}\lambda_iy_iK$

5.1 Margin

5.1.1 Motive

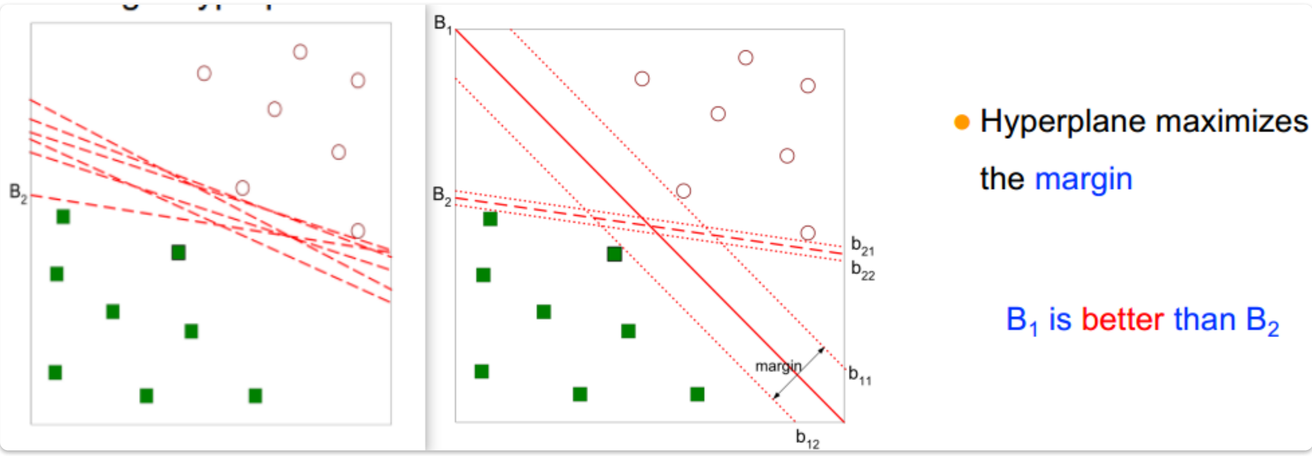
Given

- A set of multi-dimensional linearly separable classes
  - $\mathcal{X}=\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_N\}$
- Two classes to categorize samples in $\mathcal{X}$:
  - $\omega=\{\omega_{(1)},\omega_{(2)}\}$
- A mapping relation of $\mathcal{D}\in\mathcal{X}\times\omega$
  - $\mathcal{D}=\{\langle\mathbf{x}_1,\omega_1\rangle\ \langle\mathbf{x}_2,\omega_2\rangle\ \cdots,\langle\mathbf{x}_N,\omega_N\rangle\}$

  #### Do
- Find a hyperplane $\mathbf{w}^\top\mathbf{x}+b=0$ that separates the two classes.

- $\mathbf{w}$ is the normal vector of this hyperplane.

From multiple possible solutions, we want the one that maximizes the margin.



5.1.2 Distance to Hyperplane

The distance from each sample $\mathbf{x}_i$ to the hyperplane is:

$$r = \frac{\mathbf{w}^\top \mathbf{x}_i + b}{||\mathbf{w}||}$$

Proof. Suppose that $\mathbf{x}_p$ is the projection of a data sample $\mathbf{x}_i$ on the hyperplane $\mathbf{w}^\top \mathbf{x} + b = 0$. Therefore,

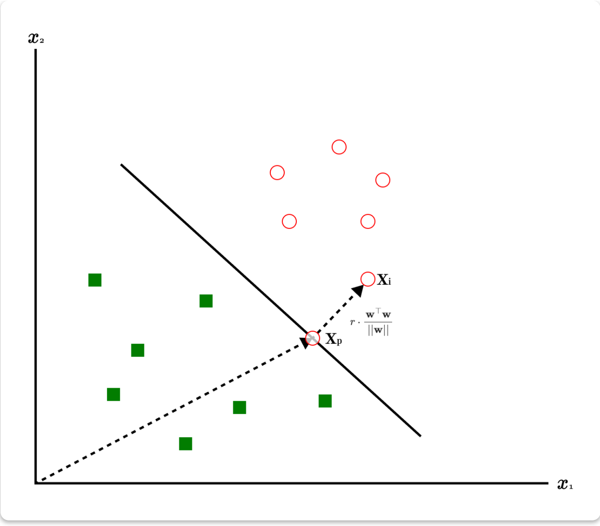
$$\mathbf{x}_i = \mathbf{x}_p + r \cdot \frac{\mathbf{w}}{||\mathbf{w}||}$$

namely,

$$\begin{aligned}
\mathbf{w}^\top \mathbf{x}_i + b &= \mathbf{w}^\top (\mathbf{x}_p + r \cdot \frac{\mathbf{w}}{||\mathbf{w}||}) + b \\
&= \mathbf{w}^\top \mathbf{x}_p + b + r \cdot \frac{\mathbf{w}^\top \mathbf{w}}{||\mathbf{w}||} \\
&= 0 + r \cdot \frac{\mathbf{w}^\top \mathbf{w}}{||\mathbf{w}||} \\
&= r \cdot \frac{\mathbf{w}^\top \mathbf{w}}{||\mathbf{w}||}
\end{aligned}$$

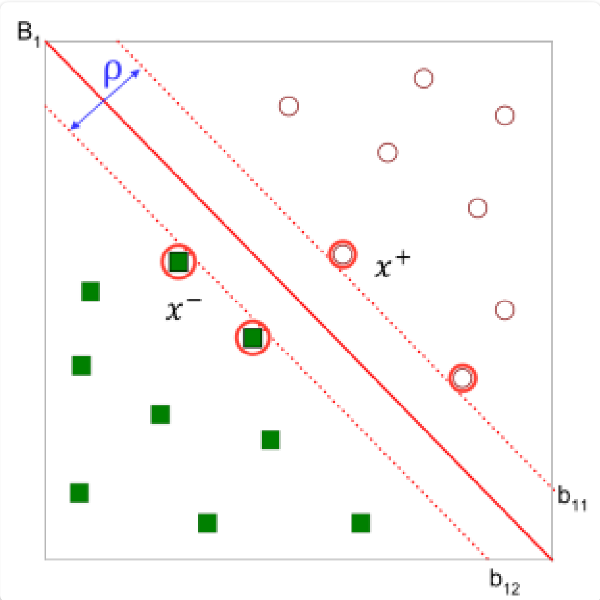
Thus,

$$\begin{aligned}
r &= (\mathbf{w}^\top \mathbf{x}_i + b) \cdot \frac{||\mathbf{w}||}{\mathbf{w}^\top \mathbf{w}} \\
&= (\mathbf{w}^\top \mathbf{x}_i + b) \cdot \frac{||\mathbf{w}||}{||\mathbf{w}||^2} \\
&= \frac{\mathbf{w}^\top \mathbf{x}_i + b}{||\mathbf{w}||}
\end{aligned}$$



5.1.3 Support Vectors and Margin

- Support Vectors are:
  - A subset of training samples
  - Samples closes to the hyperplane
- Margin ρ is the distance between support vectors.
  - The hyperplane is to maximize the margin ρ.



In the above graph, there are 3 hyperplanes:

$$
\begin{aligned}
B_1 &: \mathbf{w}^\top \mathbf{x} + b = 0 \\
b_{11} &: \mathbf{w}^\top \mathbf{x} + b = +1 \\
b_{12} &: \mathbf{w}^\top \mathbf{x} + b = -1
\end{aligned}
$$

Where $\mathbf{x}^+$ and $\mathbf{x}^-$ lies on the hyperplanes b_{11} and b_{12}. Then:

$$
\begin{aligned}
\mathbf{w}^\top(\mathbf{x}^+ - \mathbf{x}^-) &= \mathbf{w}^\top \mathbf{x}^+ - \mathbf{w}^\top \mathbf{x}^- \\
&= (\mathbf{w}^\top \mathbf{x}^+ + b) - (\mathbf{w}^\top \mathbf{x}^- + b) \\
&= (1) - (-1) \\
&= 2
\end{aligned}
$$

The margin would be:

$$\rho = \frac{\mathbf{w}^\top(\mathbf{x}^+ - \mathbf{x}^-)}{\|\mathbf{w}\|} = \frac{2}{\|\mathbf{w}\|}$$

5.2 Quadratic Optimization

5.2.1 Formulation

Let

- $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ be the data set
- $y = \{y_1, y_2, \cdots, y_N\} \subset \{-1, 1\}^N$ be the class labels of the corresponding data in $\mathcal{X}$.
  Do
- Find the optimal $\mathbf{w}$ such that:
  - $\rho = \frac{2}{\|\mathbf{w}\|}$ is maximized, and
  - $\begin{cases} \mathbf{w}^\top\mathbf{x}_i + b \geq 1 & \text{if } y_i = +1 \\ \mathbf{w}^\top\mathbf{x}_i + b \leq -1 & \text{if } y_i = -1 \end{cases} \text{for } i = 1, 2, \cdots, N$

Maximizing the margin $\rho = \frac{2}{\|\mathbf{w}\|}$ is equivalent to minimizing:

$$\frac{1}{2}\|\mathbf{w}\|^2 = \frac{1}{2}\mathbf{w}^\top\mathbf{w}$$

★ The formulated quadratic optimization problem of SVM is:
- Minimize: $\frac{1}{2}\|\mathbf{w}\|^2$
- With constraint: $y_i(\mathbf{w}^\top + b) \geq 1, \ \forall\, 1, 2, \cdots, N$

5.2.2 Lagrangian of Quadratic Optimization

★ The Lagrangian of the quadratic optimization problem is:

$$L(\mathbf{w}, b) = \frac{1}{2}\mathbf{w}^\top\mathbf{w} + \sum_{i=1}^{N}\lambda_i(1 - y_i(\mathbf{w}^\top\mathbf{x}_i + b))$$

where $\lambda_1, \lambda_2, \cdots, \lambda_N \geq 0$ is the Lagrangian multiplier of all the data points in $\mathcal{X}$ respectively.

★ At the end, only the support vector's Lagrangian multiplier would be non-zero.
- That is $\lambda_i \neq 0$ if and only if $\mathbf{x}_i$ is a support vector.
- Non-support vectors won't contribute to the hyper plane.

Suppose that we have already found a series of such Lagrangian multipliers. To optimize L, we compute the partial derivatives of L with respect to $\mathbf{w}$ and b.

Optimize L w.r.t. $\mathbf{w}$.

$$
\begin{aligned}
\frac{dL}{d\mathbf{w}} &= \mathbf{w} + \frac{dL}{d\mathbf{w}} \sum_{i=1}^{N} \lambda_i - (\lambda_i y_i)\mathbf{w}^\top \mathbf{x}_i - (\lambda_i y_i b) \\
&= \mathbf{w} + \sum_{i=1}^{N} -\lambda_i y_i \mathbf{x}_i \\
&= \mathbf{w} - \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i
\end{aligned}
$$

Let $\frac{dL}{d\mathbf{w}} = 0$.

$$
\mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i
$$

Optimize L w.r.t. b

$$
\begin{aligned}
\frac{dL}{db} &= 0 + \frac{dL}{db} \sum_{i=1}^{N} \lambda_i - (\lambda_i y_i)\mathbf{w}^\top \mathbf{x}_i - (\lambda_i y_i b) \\
&= \sum_{i=1}^{N} -\lambda_i y_i \\
&= -\sum_{i=1}^{N} \lambda_i y_i
\end{aligned}
$$

Let $\frac{dL}{db} = 0$.

$$
\sum_{i=1}^{N} \lambda_i y_i = 0
$$

- ⭐ Therefore, the optimized weight and bias of this quadratic is:

$$
\mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i
$$

- ⭐ with a constraint of:

$$
\sum_{i=1}^{N} \lambda_i y_i = 0
$$

  with respect to $\lambda_1, \lambda_2, \cdots, \lambda_N \geq 0$.

5.2.3 Dual Problem

Get λ with optimized L

Substitute $\mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i$ and $\sum_{i=1}^{N} \lambda_i y_i = 0$ into $L(\mathbf{w}, b)$ would result in:

$$
\begin{aligned}
L(\mathbf{w}, b) =& \frac{1}{2}\Big(\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i\Big)^\top \Big(\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i\Big) \\
&+ \sum_{i=1}^{N} \lambda_i (1 - y_i \Bigg(\Big(\sum_{j=1}^{N} \lambda_j y_j \mathbf{x}_j\Big)^\top \mathbf{x}_i + b\Bigg)) \\
\\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j \\
&+ \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} (\lambda_i y_i) \cdot \Big(\sum_{j=1}^{N} (\lambda_j y_j) \cdot \mathbf{x}_j^\top \mathbf{x}_i + b\Big) \\
\\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j \\
&+ \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} (\lambda_i y_i) \cdot (\lambda_j y_j) \cdot \mathbf{x}_j^\top \mathbf{x}_i + b(\lambda_i y_i)\Big) \\
\\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j \\
&+ \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i y_i) \cdot (\lambda_j y_j) \cdot \mathbf{x}_j^\top \mathbf{x}_i \\
&+ b \sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i y_i) \\
\\
=& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j \\
&+ \sum_{i=1}^{N} \lambda_i \\
&+ b \sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i y_i) \\
\\
=& \sum_{i=1}^{N} \lambda_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j
\end{aligned}
$$

The original criterion function is now with respect to only λ. That is:

$$
G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j
$$

Optimize $G(\lambda)$ by computing:

$$
\frac{dG}{d\lambda_i},\ \forall i = 2, 3, \cdots, N
$$

Check the solutions if they satisfy the constraint of:

$$
\begin{cases}
\lambda_i > 0 \\
\sum_{i=1}^{N} \lambda_i y_i = 0
\end{cases}
$$

5.2.4 Solutions: Regular

Maximize:

$$
G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j
$$

with constraint:

$$
\begin{cases}
\lambda_i \geq 0 \\
\sum_{i=1}^{N} \lambda_i y_i = 0
\end{cases}
$$

The dual solution is:

$$
f(\mathbf{x}) = (\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i)^\top \mathbf{x} + b
$$

with:

$$
\begin{cases}
\mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i \\
b = y_k - (\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i^\top) \mathbf{x}_k
\end{cases}
$$

5.3 Soft Margin Classification

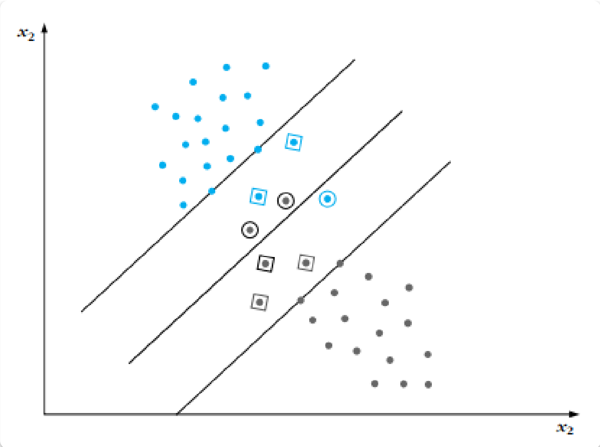
5.3.1 Problem Setup

There exists conditions that training samples can't be separated.

- In this case, *no* hyperplane could satisfy $y_i(\mathbf{w}^\top \mathbf{x} + b) > 1,\ \forall \mathbf{x}$.
- i.e., $\neg \exists \mathbf{w}, b, \forall \mathbf{x},\ y_i(\mathbf{w}^\top \mathbf{x} + b) > 1$

Training samples belong to one of the three possible categories.

- Correctly Classified: Samples outside the margin.
  - $y_i(\mathbf{w}^\top + b) > 1$
- Margin Violation: Samples within the margin, but correctly classified.
  - $y_i(\mathbf{w}^\top + b) > 1$
- Misclassified samples:
  - $y_i(\mathbf{w}^\top + b) < 0$



5.3.2 Slack Variables & Parameter C 松弛因子与C参数

Assignment of ξ_i

Assign slack variables $\xi_1, \xi_2, \cdots, \xi_N \geq 0$ to all the samples in $\mathcal{X}$.

- Correctly Classified: $\xi_i = 0$
- Margin Violation: $0 \leq \xi_i \leq 1$
- Misclassified Variables: $\xi_i > 1$

ℹ About slack variables ξ_i.

- ξ_i allows misclassification of difficult or noisy samples.
  - The resulting is called a Soft Margin.
  - If ξ_i is sufficiently large, every constraint will be forced to be satisfied.
- ξ_i is based on the output of the discriminant function $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$.
- ξ_i approximates the number of mis-classified samples.

Intuitive Optimization

The optimization problem becomes:

- Minimize: $\frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{N} \xi_i$
- With constraint: $y_i(\mathbf{w}^\top + b) \geq 1 - \xi_i, \ \forall\, 1, 2, \cdots, N$

ℹ The parameter C is a user-selected regularization parameter.

- A trade-off parameter between:
  - error and
  - margin
- Effects of parameter C:
  - Smaller C = Large Margin & More error, allows constraints to be easily ignored.
  - Larger C = Narrow Margin & Less error, constraints is hard to ignore.
  - $C = \infty$ = Hard Margin, which enforces all constraints.

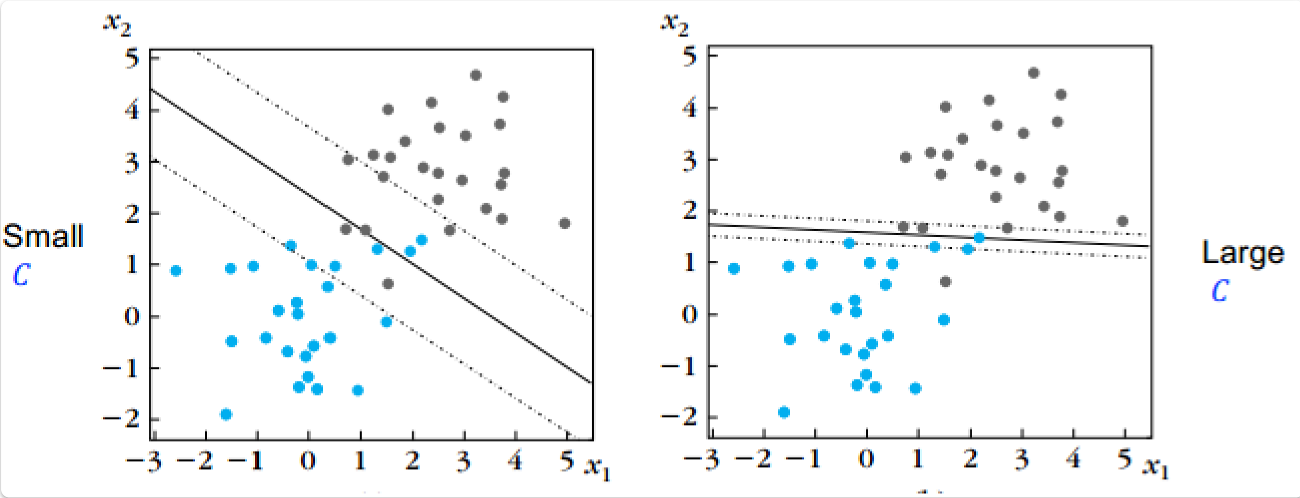
Corresponding Dual Problem

Which transfer to the dual problem as

$$G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j$$

with the constraints of:

$$\begin{cases} 0 \leq \lambda_i \leq C \\ \sum_{i=1}^{N} \lambda_i y_i = 0 \end{cases}$$



5.3.3 Solutions: Soft Margin

Maximize:

$$G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j$$

with constraint:

$$\begin{cases} 0 \leq \lambda_i \leq C \\ \sum_{i=1}^{N} \lambda_i y_i = 0 \end{cases}$$

The dual solution is:

$$f(\mathbf{x}) = (\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i)^\top \mathbf{x} + b$$

with:

$$\begin{cases} \mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i \\ b = y_k(1 - \xi_k) - (\sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i^\top)\mathbf{x}_k \end{cases}$$

5.4 Non-Linear SVM

5.4.0 Basics

Why Non-Linear SVM?

- Datasets may be too hard for linear separation.

What does it do?

- Transform data into a higher dimensional space $\mathbf{H}$,
  - via a mapping function $\mathbf{\Phi}$
  - such that the data appears of the form $\mathbf{\Phi}(\mathbf{x}_i)\mathbf{\Phi}(\mathbf{x}_j)$.
- Linear separation in $\mathbf{H}$ is *equivalent* to non-linear separation in the original input space.

Problems

- High dimensionality results in high computation burden.
- Hard to obtain a good estimation.

5.4.1 Kernel Function

i A kernel of two data is:
- The inner product between the vectors
- $K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^\top \mathbf{x}_j$

Suppose that each data point is mapped into high-dimensional space via transformation of:

$$\mathbf{\Phi} : \mathbf{x} \mapsto \phi(\mathbf{x})$$

Then, the kernel of two data becomes:

$$K(\mathbf{x}_i, \mathbf{x}_j) = \phi(\mathbf{x}_i)^\top \phi(\mathbf{x}_j)$$

We could compute $K(\mathbf{x}_i, \mathbf{x}_j)$ without computing $\phi(\mathbf{x}_i)$ and $\phi(\mathbf{x}_j)$ explicitly.

5.4.2 Compute Kernel

We could compute kernel $K(\mathbf{x}_i, \mathbf{x}_j)$ in the original space. Suppose that the original space is of 2 dimensions. Let $K(\mathbf{x}_i, \mathbf{x}_j) = (1 + \mathbf{x}_i^\top \mathbf{x}_j)^2$, then $K(\mathbf{x}_i, \mathbf{x}_j) = (1 + \mathbf{x}_i^\top \mathbf{x}_j)^2$.
Proof.

$$
\begin{aligned}
K(\mathbf{x}_i, \mathbf{x}_j) &= (1 + \mathbf{x}_i^\top \mathbf{x}_j)^2 \\
&= (1 + \begin{bmatrix} x_{i1} & x_{i2} \end{bmatrix} \begin{bmatrix} x_{j1} \\ x_{j2} \end{bmatrix})^2 \\
&= \Big(1 + (x_{i1}x_{j1} + x_{i2}x_{j2})\Big)^2 \\
&= 1 + (x_{i1}x_{j1} + x_{i2}x_{j2})^2 - 2((x_{i1}x_{j1} + x_{i2}x_{j2})) \\
&= x_{i1}^2 x_{j1}^2 + x_{i2}^2 x_{j2}^2 + 2x_{i1}x_{j1}x_{i2}x_{j2} + 2x_{i1}x_{j1} + 2x_{i2}x_{j2} + 1 \\
&= \begin{bmatrix} x_{i1}^2 & x_{i2}^2 & \sqrt{2}x_{i1}x_{i2} & \sqrt{2}x_{i1} & \sqrt{2}x_{i2} & 1 \end{bmatrix} \begin{bmatrix} x_{j1}^2 \\ x_{j2}^2 \\ \sqrt{2}x_{j1}x_{j2} \\ \sqrt{2}x_{j1} \\ \sqrt{2}x_{j2} \\ 1 \end{bmatrix} \\
&= \phi(\mathbf{x}_i)^\top \phi(\mathbf{x}_j)
\end{aligned}
$$

where

$$
\phi(\mathbf{x}) = \phi(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ 1 \end{bmatrix}
$$

5.4.3 Kernel Examples

	Kernel	Mapping
Linear	$K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^\top \mathbf{x}_j$	$\mathbf{\Phi} : \mathbf{x} \mapsto \phi(\mathbf{x}) = \mathbf{x}$
Polynomial	$K(\mathbf{x}_i, \mathbf{x}_j) = (1 + \mathbf{x}_i^\top \mathbf{x}_j)^p$	$\mathbf{\Phi} : \mathbf{x} \mapsto \phi(\mathbf{x}) \in \mathbb{R}^{\frac{(d+p)!}{p!d!}}$
Sigmoid	$K(\mathbf{x}_i, \mathbf{x}_j) = \tanh(\beta_0 \mathbf{x}_i^\top \mathbf{x}_j + \beta_1)$	
Gaussian	$K(\mathbf{x}_i, \mathbf{x}_j) = e^{\frac{\lVert x_i - x_j \rVert^2}{2\sigma}}$	$\mathbf{\Phi} : \mathbf{x} \mapsto \phi(\mathbf{x}) \in \mathbb{R}^\infty$

5.4.4 Solutions: Non-Linear

Maximize:

$$
G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot K(\mathbf{x}_i, \mathbf{x}_j)
$$

with constraint:

$$\begin{cases} \lambda_i \geq 0 \\ \sum_{i=1}^{N} \lambda_i y_i = 0 \end{cases}$$

The dual solution:

$$f(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i y_i K(\mathbf{x}_i, \mathbf{x}) + b$$

5.5 Examples

Non-Linear

Given

- Suppose we have five 1-D datapoints:
  - $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 5, x_5 = 6$
- Their corresponding classes are:
  - $y_1 = 1, y_2 = 1, y_3 = -1, y_4 = -1, y_5 = 1$
- Use the degree-2 polynomial kernel:
  - $K(x_i, x_j) = (x_i x_j + 1)^2$

  ### Do

  Step 1. Find λ_i.
  Maximize:

$$G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot (1 + x_i x_j)^2$$

  with constraint:

$$\sum_{i=1}^{N} \lambda_i y_i = 0$$

  Using a quadratic problem solver, we obtain:

$$\lambda_1 = 0,\ \lambda_2 = 2.5,\ \lambda_3 = 0,\ \lambda_4 = 7.333,\ \lambda_5 = 4.833$$

  Step 2. Calculate.
  The support vectors are:

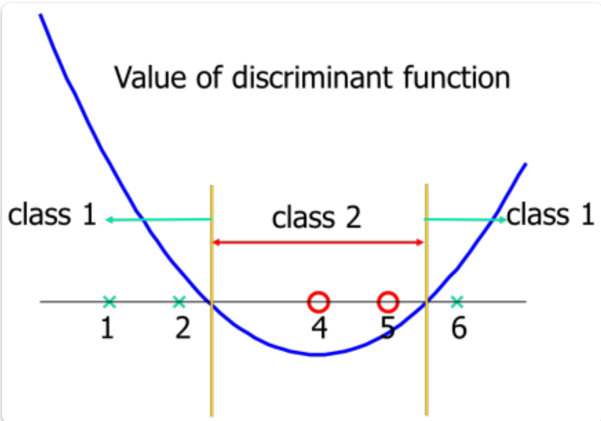
$$x_2 = 2,\ x_4 = 5,\ x_5 = 6$$

  The discriminant function is:

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{N} \lambda_i y_i K(x_i, x) + b \\
&= \sum_{i=1}^{N} \lambda_i y_i (1 + x_i x)^2 + b \\
&= 2.5 \cdot 1 \cdot (1 + 2x)^2 + 7.333 \cdot (-1) \cdot (1 + 5x)^2 + 4.833 \cdot 1 \cdot (1 + 6x)^2 + b \\
&= 0.6667x^2 - 5.333x + b
\end{aligned}
$$

Given that $f(6) = 1$, $b = 9$.
We obtained the discriminant function of:

$$f(x) = 0.6667x^2 - 5.333x + 9$$



Linear

Given

- Suppose we have three 2-D data points.
  - $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$
- Their corresponding labels are:
  - $y_1 = 1,\ y_2 = -1,\ y_3 = -1$
- Use the trivial kernel:
  - $K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^\top \mathbf{x}_j$

  ### Do

  Step 1. Find λ.

$$G(\lambda) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i \lambda_j) \cdot (y_i y_j) \cdot \mathbf{x}_i^\top \mathbf{x}_j$$

1-1. Constraint:

$$
\begin{aligned}
\sum_{i=1}^{N} \lambda_i y_i = 0 &\implies \lambda_1 - \lambda_2 - \lambda_3 = 0 \\
&\implies \lambda_1 = \lambda_2 + \lambda_3
\end{aligned}
$$

1-2. Express $G(\lambda)$ with λ_1, λ_2 and λ_3.

$$\frac{1}{2}\sum_{i=1}^{N}(\lambda_i\lambda_j)\cdot(y_iy_j)\cdot\mathbf{x}_i^\top\mathbf{x}_j=\sum_{i=1}^{N}\lambda_i-G(\lambda)$$

$$\begin{aligned}\implies &\frac{1}{2}\sum_{i=1}^{N}\Big(\\ &(\lambda_i\lambda_1)\cdot(y_iy_1)\cdot\mathbf{x}_i^\top\mathbf{x}_1+\\ &(\lambda_i\lambda_2)\cdot(y_iy_2)\cdot\mathbf{x}_i^\top\mathbf{x}_2+\\ &(\lambda_i\lambda_3)\cdot(y_iy_3)\cdot\mathbf{x}_i^\top\mathbf{x}_3\\ &\Big)=\sum_{i=1}^{N}\lambda_i-G(\lambda)\end{aligned}$$

$$\begin{aligned}\implies &\frac{1}{2}\Big(\\ &(\lambda_1\lambda_1)\cdot(y_1y_1)\cdot\mathbf{x}_1^\top\mathbf{x}_1+(\lambda_1\lambda_2)\cdot(y_1y_2)\cdot\mathbf{x}_1^\top\mathbf{x}_2+(\lambda_1\lambda_3)\cdot(y_1y_3)\cdot\mathbf{x}_1^\top\mathbf{x}_3+\\ &(\lambda_2\lambda_1)\cdot(y_2y_1)\cdot\mathbf{x}_2^\top\mathbf{x}_1+(\lambda_2\lambda_2)\cdot(y_2y_2)\cdot\mathbf{x}_2^\top\mathbf{x}_2+(\lambda_2\lambda_3)\cdot(y_2y_3)\cdot\mathbf{x}_2^\top\mathbf{x}_3+\\ &(\lambda_3\lambda_1)\cdot(y_3y_1)\cdot\mathbf{x}_3^\top\mathbf{x}_1+(\lambda_3\lambda_2)\cdot(y_3y_2)\cdot\mathbf{x}_3^\top\mathbf{x}_2+(\lambda_3\lambda_3)\cdot(y_3y_3)\cdot\mathbf{x}_3^\top\mathbf{x}_3\\ &\Big)=\sum_{i=1}^{N}\lambda_i-G(\lambda)\end{aligned}$$

$$\begin{aligned}\implies &\frac{1}{2}\Big(\\ &\lambda_1^2\cdot(1\cdot 1)\cdot\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}+\lambda_1\lambda_2\cdot(1\cdot -1)\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+\lambda_1\lambda_3\cdot(1\cdot -1)\cdot\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}+\\ &\lambda_2\lambda_1\cdot(-1\cdot 1)\cdot\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}+\lambda_2^2\cdot(-1\cdot -1)\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+\lambda_2\lambda_3\cdot(-1\cdot -1)\cdot\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}+\\ &\lambda_3\lambda_1\cdot(-1\cdot 1)\cdot\begin{bmatrix}3 & 3\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}+\lambda_3\lambda_2\cdot(-1\cdot -1)\begin{bmatrix}3 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+\lambda_3^2\cdot(-1\cdot -1)\cdot\begin{bmatrix}3 & 3\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}\\ &\Big)=\sum_{i=1}^{N}\lambda_i-G(\lambda)\end{aligned}$$

$$\begin{aligned}\implies &\frac{1}{2}\Big(\\ &5\lambda_1^2-4\lambda_1\lambda_2-9\lambda_1\lambda_3\\ &-4\lambda_1\lambda_2+5\lambda_2^2+9\lambda_2\lambda_3\\ &-9\lambda_1\lambda_3+9\lambda_3\lambda_3+18\lambda_3^2\\ &\Big)=\lambda_1+\lambda_2+\lambda_3-G(\lambda)\end{aligned}$$

$$\begin{aligned}\implies &G(\lambda)=-\frac{1}{2}(5\lambda_1^2+5\lambda_2^2+18\lambda_3^2-8\lambda_1\lambda_2-18\lambda_1\lambda_3+18\lambda_2\lambda_3)\\ &+(\lambda_1+\lambda_2+\lambda_3)\end{aligned}$$

$$\begin{aligned}\implies &G(\lambda)=-\frac{1}{2}(5(\lambda_2+\lambda_3)^2+5\lambda_2^2+18\lambda_3^2-8(\lambda_2+\lambda_3)\lambda_2-18(\lambda_2+\lambda_3)\lambda_3+18\lambda_2\lambda_3)\\ &+((\lambda_2+\lambda_3)+\lambda_2+\lambda_3)\end{aligned}$$

$$\implies G(\lambda)=-\lambda_2^2-\frac{5}{2}\lambda_3^2-\lambda_2\lambda_3+2\lambda_2+2\lambda_3$$

1-3. Minimize $G(\lambda)$

$$
\begin{aligned}
&\frac{\partial G}{\partial \lambda_3} = 0 \\
&\implies \frac{\partial}{\partial \lambda_3}\left(-\lambda_2^2 + (2-\lambda_3)\lambda_2 + (2\lambda_3 - \frac{5}{2}\lambda_3^2)\right) = 0 \\
&\implies 2\lambda_2 + \lambda_3 - 2 = 0 \\
&\frac{\partial G}{\partial \lambda_3} = 0 \\
&\implies \frac{\partial}{\partial \lambda_3}\left(-\frac{5}{2}\lambda_3^2 + (2-\lambda_2)\lambda_3 + (2\lambda_2 - \lambda_2^2)\right) = 0 \\
&\implies \lambda_2 + 5\lambda_3 - 2 = 0
\end{aligned}
$$

1-4. Summarize.

$$
\begin{cases}
\lambda_1 - \lambda_2 - \lambda_3 = 0 \\
2\lambda_2 + \lambda_3 = 2 \\
\lambda_2 + 5\lambda_3 = 2
\end{cases}
$$

$$
\implies \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}
$$

$$
\implies \begin{cases}
\lambda_1 = \frac{10}{9} \\
\lambda_2 = \frac{8}{9} \\
\lambda_3 = \frac{2}{9}
\end{cases}
$$

Step 2. Calculate.

The discriminant function is

$$
\begin{aligned}
f(\mathbf{x}) &= \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i^\top \mathbf{x} + b \\
&= (\frac{10}{9}[2 \quad 1] - \frac{8}{9}[1 \quad 2] - \frac{2}{9}[3 \quad 3])\mathbf{x} + b \\
&= \left[\tfrac{2}{3} \quad -\tfrac{4}{3}\right]\mathbf{x} + b
\end{aligned}
$$

Get b:

$$
\begin{aligned}
& f(\mathbf{x}_1) = 1 \\
\implies & f(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 1 \\
\implies & \begin{bmatrix} \frac{2}{3} & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b = 1 \\
\implies & \frac{4}{3} - \frac{4}{3} + b = 1 \\
\implies & b = 1
\end{aligned}
$$

Therefore, the discriminant function would be:

$$f(\mathbf{x}) = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3} \end{bmatrix} \mathbf{x} + 1$$