Review

- 变量替换下二重积分的计算

$$u = u(x,y),\ v = v(x,y)$$

$$(x,y) \in D \leftrightarrow (u,v) \in \Omega$$

$$\iint_D f(x,y)\,dxdy = \iint_\Omega f(x(u,v),y(u,v)) \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

- $\det \dfrac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \det \dfrac{\partial(u,v)}{\partial(x,y)}$

§4. 三重积分

- 三重积分的几何与物理背景
- 三重积分在直角坐标系下的计算
- 三重积分在柱坐标下的计算
- 三重积分的变量替换

1. 三重积分的几何与物理背景

设Ω为$\mathbb{R}^3$中有界闭区域. 与二重积分一样, 通过分划, 取点, 求*Riemann*和与取极限的过程, 可以定义三重积分.

体积微元

•Ω的体积

$$\iiint_{\Omega} dV = \iiint_{\Omega} dxdydz$$

•Ω的质量

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

$f(x,y,z)$为(x,y,z)处的点密度.

2. 三重积分在直角坐标系下的计算

1) 化为"先一后二"型累次积分

若Ω为$\mathbb{R}^3$中柱体, 分别以$z_2(x,y)$和$z_1(x,y)$为上顶下底, 在平面oxy上的投影为D_{xy},即Ω可表示为

$$\Omega:\begin{cases} z_1(x,y)\le z\le z_2(x,y) \\ (x,y)\in D_{xy} \end{cases}.$$

设密度函数为$f(x,y,z)$, 为计算Ω的质量, 想象把Ω压缩成平行于xy平面的薄片, 则有

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

$$= \iint_{D_{xy}} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right] dxdy$$

$$\triangleq \iint_{D_{xy}} dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz.$$

若D_{xy}又可表示为$D_{xy}:\begin{cases} a \le x \le b, \\ y_1(x) \le y \le y_2(x), \end{cases}$则

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

$$=\int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

其意义是先固定x和y对z积分,再固定x,对y积分,最后对x积分.

2)化为"先二后一"型累次积分

用Ω_z表示平行于oxy坐标的平面截Ω得到的截面, Ω可以表示为

$$\Omega:\begin{cases} c \le z \le d \\ (x, y) \in \Omega_z \end{cases}.$$

设密度函数为$f(x, y, z)$, 为计算Ω的质量, 想象把Ω压缩成平行于z轴的细线, 则有

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

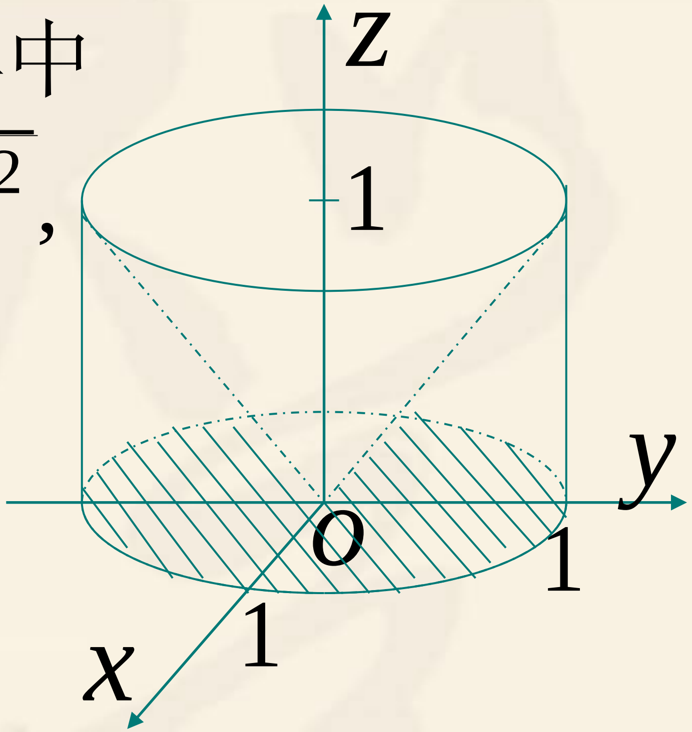
$$= \int_c^d \left[\iint_{\Omega_z} f(x,y,z)dxdy \right] dz$$

$$\triangleq \int_c^d dz \iint_{\Omega_z} f(x,y,z)dxdy.$$

例: $I=\iiint_{\Omega}(x^2+y^2)z\,dxdydz$, 其中

Ω由$x^2+y^2=1$, 曲面$z=\sqrt{x^2+y^2}$,

和$z=0$围成.



解法一: ("先一后二")

$$\Omega:\begin{cases} x^2+y^2\le 1, \\ 0\le z\le\sqrt{x^2+y^2} \end{cases}$$

$$I=\iint_{x^2+y^2\le 1}dxdy\int_0^{\sqrt{x^2+y^2}}(x^2+y^2)z\,dz.$$

$$=\iint_{x^2+y^2\le 1}\frac{1}{2}(x^2+y^2)^2dxdy=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^1 r^5dr=\frac{\pi}{6}.\square$$

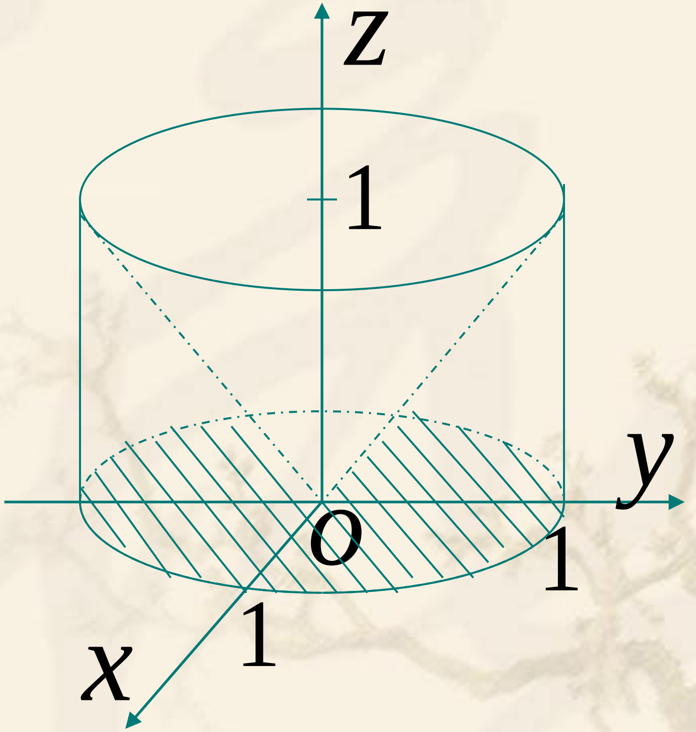
解法二: (″先二后一″) $\Omega:\begin{cases}0\le z\le 1,\\ z^2\le x^2+y^2\le 1.\end{cases}$

$$I=\int_0^1 zdz\iint_{z^2\le x^2+y^2\le 1}(x^2+y^2)dxdy$$

$$=\int_0^1 zdz\int_0^{2\pi}d\theta\int_z^1 r^2\cdot rdr$$

$$=2\pi\int_0^1 z\cdot\frac{1}{4}(1-z^4)dz$$

$$=\pi/6.\square$$



3. 用柱坐标计算三重积分

$$\Omega \leftrightarrow \Omega^*$$

$$(x, y, z) \leftrightarrow (r, \theta, z)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint_{\Omega} f(x, y, z)dxdydz$$

$$= \iiint_{\Omega^*} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$$

Remark: 当积分区域在坐标平面上的投影为圆域或圆域的一部分,而被积函数具有形如

$$f(x^2+y^2,z),\ f(x^2+z^2,y),\ f(y^2+z^2,x)$$

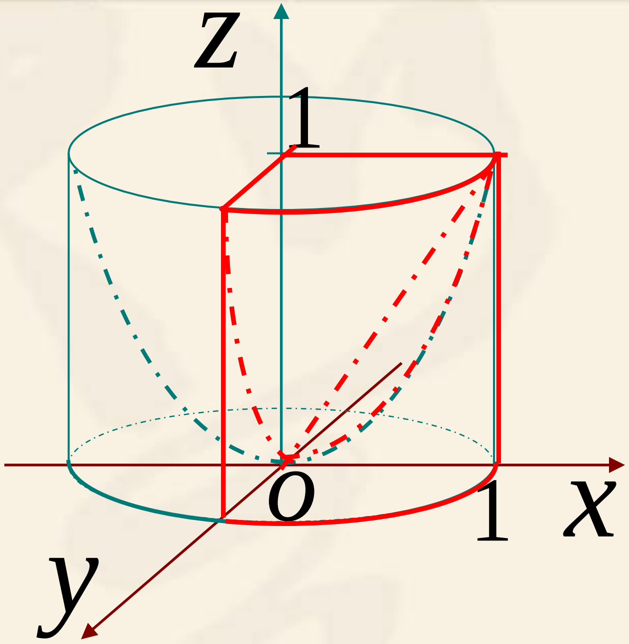
等形式时,宜用柱坐标代换.

例:$I=\iiint_{\Omega}(x^2+y^2+z)dxdydz$,

其中Ω为第一卦限中由曲面

$z=x^2+y^2, x^2+y^2=1$及三坐

标平面围成的区域.



解法一:在柱坐标$x=r\cos\theta$,

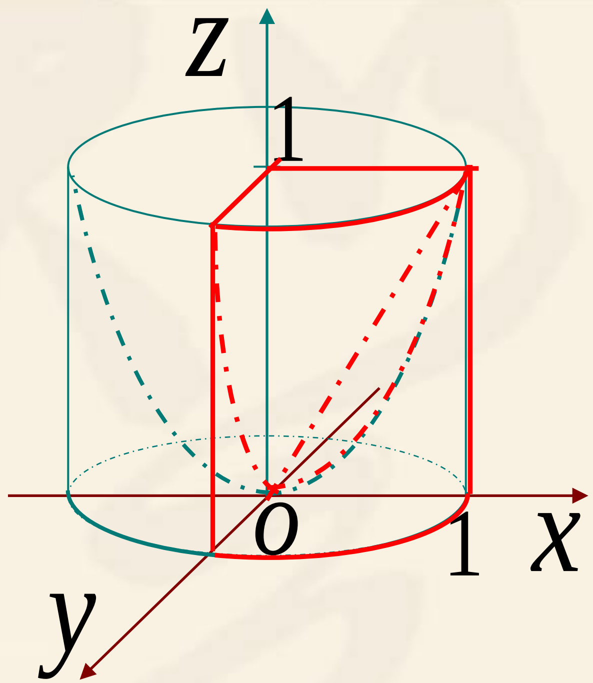
$y=r\sin\theta, z=z$下, Ω在oxy平面的投影为

$$E_{r\theta}=\left\{(r,\theta)\middle|0\le r\le 1, 0\le\theta\le\pi/2\right\},$$

Ω上下两边界面的方程为$z=0$和$z=r^2$.即

$$\Omega=\left\{(r,\theta,z)\middle|0\le r\le 1, 0\le\theta\le\pi/2, 0\le z\le r^2\right\}.$$

$$I=\iiint_{\Omega}(x^2+y^2+z)dxdydz$$
$$=\iint_{E_{r\theta}} rdrd\theta\int_0^{r^2}(r^2+z)dz$$
$$=\int_0^1 rdr\int_0^{\frac{\pi}{2}}d\theta\int_0^{r^2}(r^2+z)dz$$
$$=\frac{\pi}{2}\int_0^1 r(r^4+\frac{1}{2}r^4)dr=\pi/8.\square$$



解法二:"先二后一"

$$I=\int_0^1 dz\iint_{\Omega_z}(r^2+z)rdrd\theta=\int_0^1 dz\int_0^{\frac{\pi}{2}}d\theta\int_{\sqrt{z}}^{1}(r^2+z)rdr$$
$$=\frac{\pi}{2}\int_0^1\left[(1-z^2)/4+z(1-z)/2\right]dz=\pi/8.\square$$

例:$I = \iiint\limits_{x^2+y^2+z^2\le R^2} \dfrac{dxdydz}{\sqrt{x^2+y^2+(z-h)^2}}$, $(h > R)$.

解:令$x = r\cos\theta, y = r\sin\theta, z = z$.

$$I = \int_{-R}^{R} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2-z^2}} \frac{rdr}{\sqrt{r^2+(z-h)^2}}$$

$$= \pi \int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} \frac{dr^2}{\sqrt{r^2+(z-h)^2}}$$

$$= 2\pi \int_{-R}^{R} \left[\sqrt{R^2+h^2-2hz} - (h-z)\right] dz = \frac{4\pi R^3}{3h}.\square$$

Question: $a^2+b^2+c^2>R^2$时,

$$I=\iiint\limits_{x^2+y^2+z^2\le R^2}\frac{dxdydz}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}=?$$

$$\frac{4\pi R^3}{3\sqrt{a^2+b^2+c^2}}.$$

4. 三重积分的变量替换

与二重积分类似,对三重积分引入一一映射

$$\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w), \end{cases} \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0, \forall (u,v,w) \in \Omega^*.$$

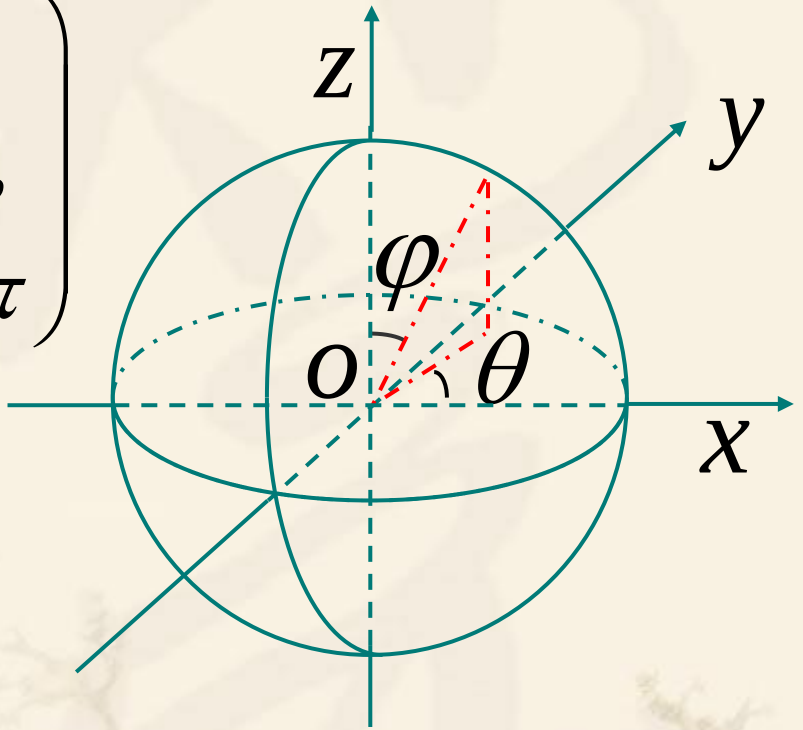
将$ouvw$空间的区域Ω^*映成$oxyz$空间的区域Ω.则

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

$$= \iiint_{\Omega^*} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right) \cdot \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw.$$

特别地, 在球坐标变换

$$\begin{cases} x = \rho \sin\varphi \cos\theta, \\ y = \rho \sin\varphi \sin\theta, \\ z = \rho \cos\varphi, \end{cases} \quad \left(\begin{array}{l} \rho \geq 0, \\ 0 \leq \varphi \leq \pi, \\ 0 \leq \theta < 2\pi \end{array} \right)$$



下,
$$\det \frac{\partial(x, y, z)}{\partial(u, v, w)} = \rho^2 \sin\varphi.$$

于是 $\iiint_{\Omega} f(x, y, z) dxdydz$

$$= \iiint_{\Omega^*} f\left(x(\rho, \varphi, \theta), y(\rho, \varphi, \theta), z(\rho, \varphi, \theta)\right)$$

$$\cdot \rho^2 \sin\varphi d\rho d\varphi d\theta.$$

例: $I=\iiint\limits_{\Omega}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2 dxdydz$,其中

$$\Omega=\left\{(x,y,z)\middle|\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}.$$

解: Ω关于oxy平面对称,故z的奇函数$\frac{xz}{ac},\frac{yz}{bc}$,在$\Omega$上的积分都为0.同理$\frac{xy}{ab}$在$\Omega$上的积分也为0.于是

$$I=\iiint\limits_{\Omega}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz.$$

作椭球坐标变换 $\begin{cases} x = a\rho\sin\varphi\cos\theta, \\ y = b\rho\sin\varphi\sin\theta, \\ z = c\rho\cos\varphi, \end{cases} \begin{pmatrix} 0 \le \rho \le 1, \\ 0 \le \varphi \le \pi, \\ 0 \le \theta < 2\pi \end{pmatrix}$

则 $\det \dfrac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} = abc\rho^2 \sin\varphi.$

$$I = \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dxdydz.$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 \rho^2 \cdot abc\rho^2 \sin\varphi d\rho$$

$$= 2\pi abc \int_0^{\pi} \sin\varphi d\varphi \int_0^1 \rho^4 d\rho = 4\pi abc/5. \square$$

例: $I=\iiint_{\Omega}(x^2+2y^2)dxdydz$, 其中$\Omega: 0\le z\le\sqrt{R^2-x^2-y^2}$.

解:被积函数x^2+2y^2是z的偶函数,可将积分扩展到整个球域$\Omega_1: x^2+y^2+z^2\le R^2$.

$$I=\frac{1}{2}\iiint_{\Omega_1}(x^2+2y^2)dxdydz.$$

由Ω_1的轮换对称性

$$\iiint_{\Omega_1}x^2dxdydz=\iiint_{\Omega_1}y^2dxdydz=\iiint_{\Omega_1}z^2dxdydz.$$

于是, $I=\frac{1}{2}\iiint_{\Omega_1}(x^2+y^2+z^2)dxdydz$

$$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^{R}\rho^4d\rho=2\pi R^5/5.\square$$

例. $I = \iiint\limits_{x^2+y^2+z^2 \le R^2} \dfrac{dxdydz}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$,

$(a^2+b^2+c^2 > R^2)$

解. 作正交变换$Oxyz \leftrightarrow Ouvw$, 使$w$轴过$(a,b,c)$.

记$h=\sqrt{a^2+b^2+c^2}$, 则$(a,b,c) \leftrightarrow (0,0,h)$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathrm{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} * & * & a/h \\ * & * & b/h \\ * & * & c/h \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

其中A为正定阵, 即$\mathrm{AA}^{\mathrm{T}} = \mathrm{I}$. 于是

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \mathrm{A} = \pm 1;$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2$$

$$= \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\| = \left\| \mathrm{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \mathrm{A} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \right\|$$

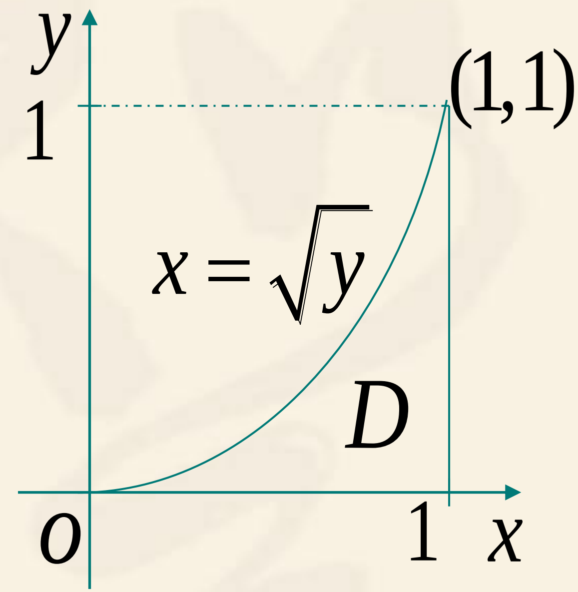
$$= (u, v, w-h) \mathrm{A}^{\mathrm{T}} \mathrm{A} \begin{pmatrix} u \\ v \\ w-h \end{pmatrix} = u^2 + v^2 + (w-h)^2;$$

$$I=\iiint\limits_{u^2+v^2+w^2\le R^2}\frac{dudvdw}{\sqrt{u^2+v^2+(w-h)^2}}$$

$$=\frac{4\pi R^3}{3h}=\frac{4\pi R^3}{3\sqrt{a^2+b^2+c^2}}.\square$$

Remark.画出积分区域Ω的立体图是化重积分为累次积分的关键. 但是有时Ω的界面复杂, 其立体图难以作出. 这时候就得寻求不画立体图, 而只画投影区域或截面区域的平面图的方法来确定累次积分的积分限.

例: $I=\iiint_{\Omega}\sqrt{x^2-y}dxdydz$, 其中$\Omega$由$y=0, z=0, x+z=1, x=\sqrt{y}$围成.



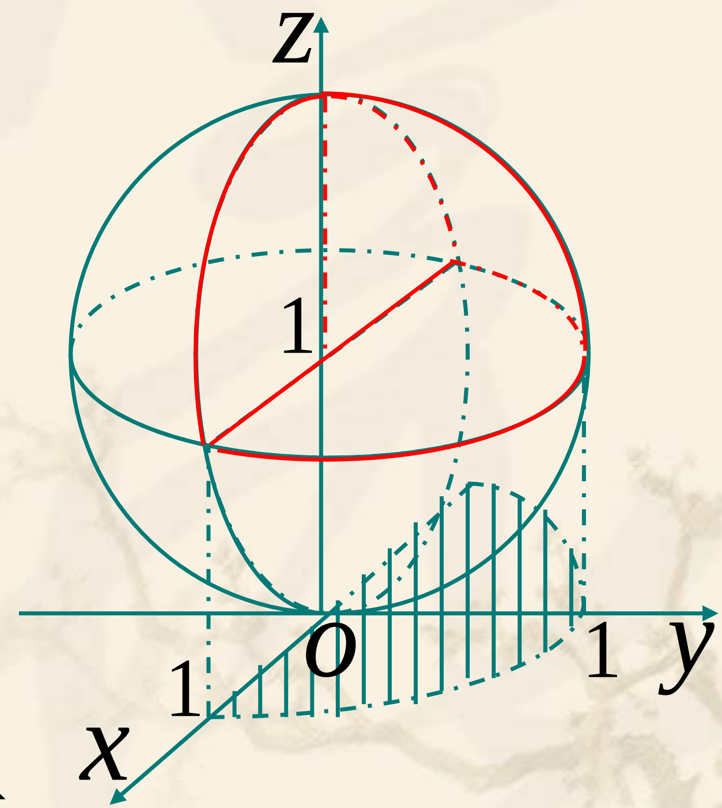
解: 被积函数不含z,先对z积分比较方便.故先将Ω向oxy投影,设投影区域为D.则D应由$y=0, x=\sqrt{y}$, 及由$z=0, x+z=1$消去z后得到的$x=1$所围成. 于是

$$I=\iint_D dxdy\int_0^{1-x}\sqrt{x^2-y}dz=\int_0^1 dx\int_0^{x^2}dy\int_0^{1-x}\sqrt{x^2-y}dz$$

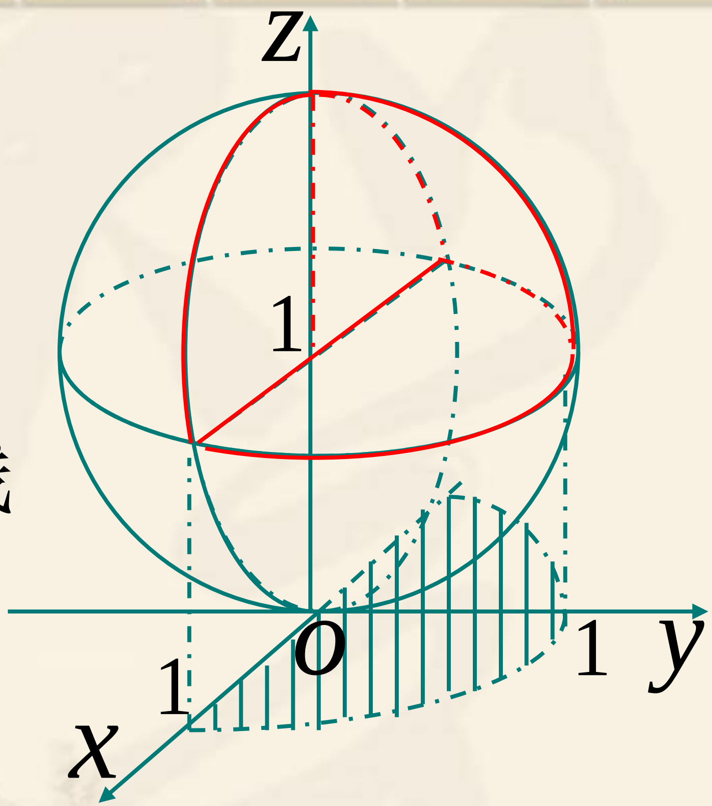
$$=\int_0^1(1-x)dx\int_0^{x^2}\sqrt{x^2-y}dy=\frac{2}{3}\int_0^1(1-x)x^3dx=1/30.\square$$

例: $I=\int_{-1}^{1}dx\int_{0}^{\sqrt{1-x^2}}dy\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}dz.$

解: 按给定积分次序积分十分困难的情况下,可以改变积分的顺序.但本例中积分区域是球域$x^2+y^2+(z-1)^2\le 1$在平面$z=1$之上满足$y\ge 0$的部分, 改变积分次序后积分仍很困难. 故考虑球坐标变换.



令 $\begin{cases} x=\rho\sin\varphi\cos\theta, \\ y=\rho\sin\varphi\sin\theta, \\ z=\rho\cos\varphi. \end{cases}$



则 •$0\le\theta\le\pi$.(这是因为$y\ge 0$.)

•$0\le\varphi\le\pi/4$. 这是因为交线

$\begin{cases} z=1 \\ x^2+y^2+(z-1)^2=1 \end{cases}$ 上

$\begin{cases} z=\rho\cos\varphi=1 \\ x^2+y^2=\rho^2\sin^2\varphi=1, \end{cases}$ 此时$\varphi=\pi/4$.

•$1/\cos\varphi\le\rho\le 2\cos\varphi$. 因为平面$z=1$上, $\rho=1/\cos\varphi$,

球面$x^2+y^2+(z-1)^2=1$上, $\rho=2\cos\varphi$.

故变量替换后积分区域为

$$\left\{(\rho,\theta,\varphi)\left|\begin{array}{l}0\le\theta\le\pi,0\le\varphi\le\pi/4,\\1/\cos\varphi\le\rho\le 2\cos\varphi.\end{array}\right.\right\}$$

$$I=\int_{-1}^{1}dx\int_{0}^{\sqrt{1-x^2}}dy\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}dz$$

$$=\int_0^{\pi}d\theta\int_0^{\pi/4}d\varphi\int_{1/\cos\varphi}^{2\cos\varphi}\rho\sin\varphi d\rho$$

$$=\pi\int_0^{\pi/4}\frac{1}{2}\sin\varphi\left[4\cos^2\varphi-1/\cos^2\varphi\right]d\varphi$$

$$=\frac{2\pi}{3}(1-\frac{1}{2\sqrt{2}})-\frac{\pi}{2}(\sqrt{2}-1).\square$$

例: 设f可导, 且$f(0)=0, \Omega: x^2+y^2+z^2 \le t^2$.

求 $\lim\limits_{t\to0^+} \dfrac{1}{\pi t^4} \iiint_\Omega f(\sqrt{x^2+y^2+z^2})dxdydz$.

解: $\iiint_\Omega f(\sqrt{x^2+y^2+z^2})dxdydz$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^t f(\rho)\rho^2 d\rho = 4\pi \int_0^t f(\rho)\rho^2 d\rho$$

$\to 0, (t\to 0^+$时.) 故可用L'Hospital法则求极限.

$$原式 = \lim_{t\to0^+} \frac{4\pi\int_0^t f(\rho)\rho^2 d\rho}{\pi t^4} = \lim_{t\to0^+} \frac{4\pi f(t)t^2}{4\pi t^3}$$

$$= \lim_{t\to0^+} \frac{f(t)-f(0)}{t-0} = f'_+(0) = f'(0). \square$$

例: 设 $f \in C([0,1])$, 证明:

$$\int_0^1 dx \int_x^1 dy \int_x^y f(x)f(y)f(z)dz = \frac{1}{6}\left(\int_0^1 f(x)dx\right)^3 .$$

解: $\forall x \in [0,1]$

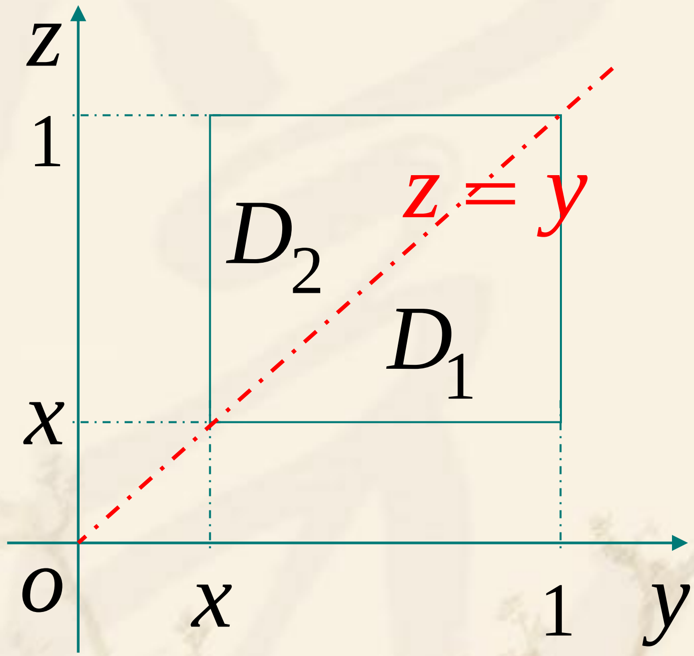
$$\int_x^1 dy \int_x^y f(y)f(z)dz$$

$$= \iint_{D_1} f(y)f(z)dydz$$

$$= \iint_{D_2} f(y)f(z)dydz$$

$$= \frac{1}{2}\iint_{D_1 \cup D_2} f(y)f(z)dydz$$

$$= \frac{1}{2}\int_x^1 dy \int_x^1 f(y)f(z)dz = \frac{1}{2}\left(\int_x^1 f(y)dy\right)^2 .$$



记$F(x)=\int_x^1 f(y)dy$，则$F'(x)=-f(x)$.于是

$$\int_0^1 dx\int_x^1 dy\int_x^y f(x)f(y)f(z)dz$$

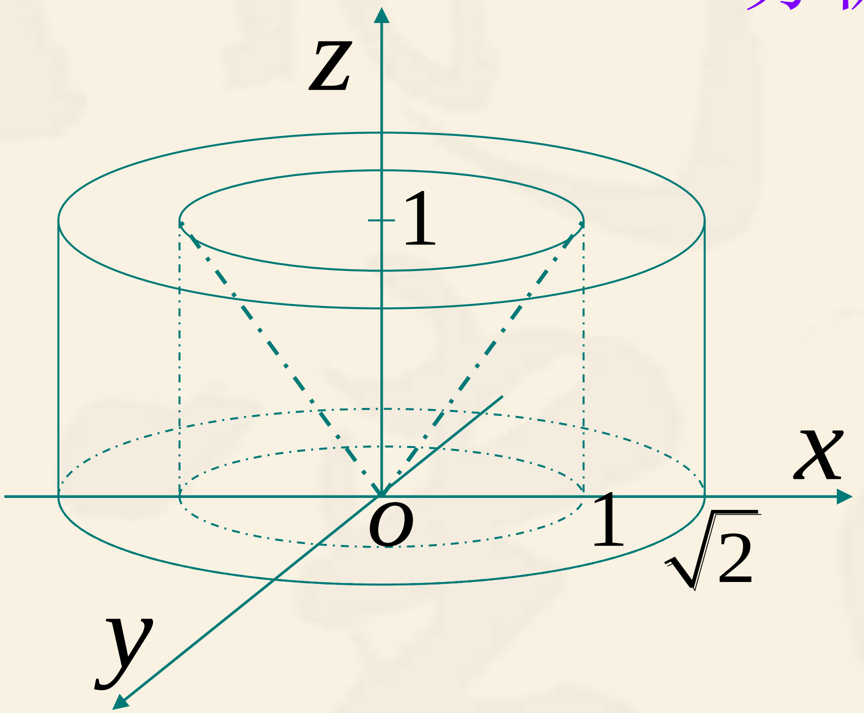
$$=\int_0^1 f(x)dx\int_x^1 dy\int_x^y f(y)f(z)dz$$

$$=\int_0^1 f(x)\cdot\frac{1}{2}\left[\int_x^1 f(y)dy\right]^2 dx=\frac{1}{2}\int_0^1 -F'(x)F^2(x)dx$$

$$=\frac{-1}{6}F^3(x)\Big|_0^1=\frac{1}{6}F^3(0)=\frac{1}{6}\left(\int_0^1 f(x)dx\right)^3 \square$$

例：求$\iiint_{\Omega}\left|z-\sqrt{x^2+y^2}\right|dxdydz$，其中$\Omega$由平面$z=0, z=1$及曲面$x^2+y^2=2$围成.

分析：关键在于去绝对值.



锥面$z=\sqrt{x^2+y^2}$将积分区域Ω分成两部分,应分别积分.

以下留为练习.

作业：习题3.4 No.5-8