Review

- 向量值函数在一点可微及微分的定义

$$f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,\ \lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0$$

- $f=\left(f_1,f_2,\cdots,f_m\right)^{\mathrm{T}}:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$在$x_0$可微

$\Leftrightarrow n$元函数$f_i:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}$在$x_0$可微$,i=1,2,\cdots,m.$

- **Chain Rule**

$u = g(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,\ y = f(u) : g(\Omega) \subset \mathbb{R}^m \to \mathbb{R}^k,$

$g(x)$在$x_0 \in \Omega$可微, $f(u)$在$u_0 = g(x_0)$可微,则

$$J(f \circ g)\big|_{x_0} = J(f)\big|_{u_0} \cdot J(g)\big|_{x_0},$$

即$$\left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0} = \left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{u_0} \cdot \left.\frac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0}.$$

$k = 1$时,

$$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i}, i = 1, 2, \cdots, n.$$

§6. 隐函数定理与反函数定理

曲线$x^2+y^2=1$在$(0,1)$的某个邻域中可表示为

$y=\sqrt{1-x^2}$，且$y'(x)=\dfrac{-x}{\sqrt{1-x^2}}$；在$(1,0)$的某个邻域

中可表示为$x=\sqrt{1-y^2}$，且$x'(y)=\dfrac{-y}{\sqrt{1-y^2}}$.

Question: (1) $f(x,y)=0$何时确定隐函数$y=y(x)$?

(2)如何通过$f(x,y)$的性质研究隐函数$y=y(x)$的性质，如连续性，可微性?

(3)如何计算隐函数的（偏）导数和（全）微分?

1. 一个方程确定的隐函数

设$f(x,y)=0, f(x_0,y_0)=0$.若存在连续可微的隐函数$y=y(x)$, $y(x_0)=y_0$,满足

$$f(x,y(x))\equiv 0,$$

则上式两边对x求导,有

$$f_x'+f_y'\cdot y'(x)=0.$$

若$f_y'(x_0,y_0)\neq 0$,则在x_0的某个邻域中,

$$y'(x)=-\frac{\partial f(x,y)}{\partial x}\Bigg/\frac{\partial f(x,y)}{\partial y}.$$

Thm. 设F在$(x_0, y_0)\in\mathbb{R}^2$的某个邻域W中有定义,且

$(1) F(x_0, y_0) = 0,$

$(2)\ F(x, y)\in C^1(\mathrm{W})$,即$F_x', F_y'$在W中连续,

$(3) F_y{}'(x_0, y_0)\neq 0.$

则存在$\delta > 0$以及$I = (x_0 - \delta, x_0 + \delta)$上定义的函数

$y = y(x)$,满足

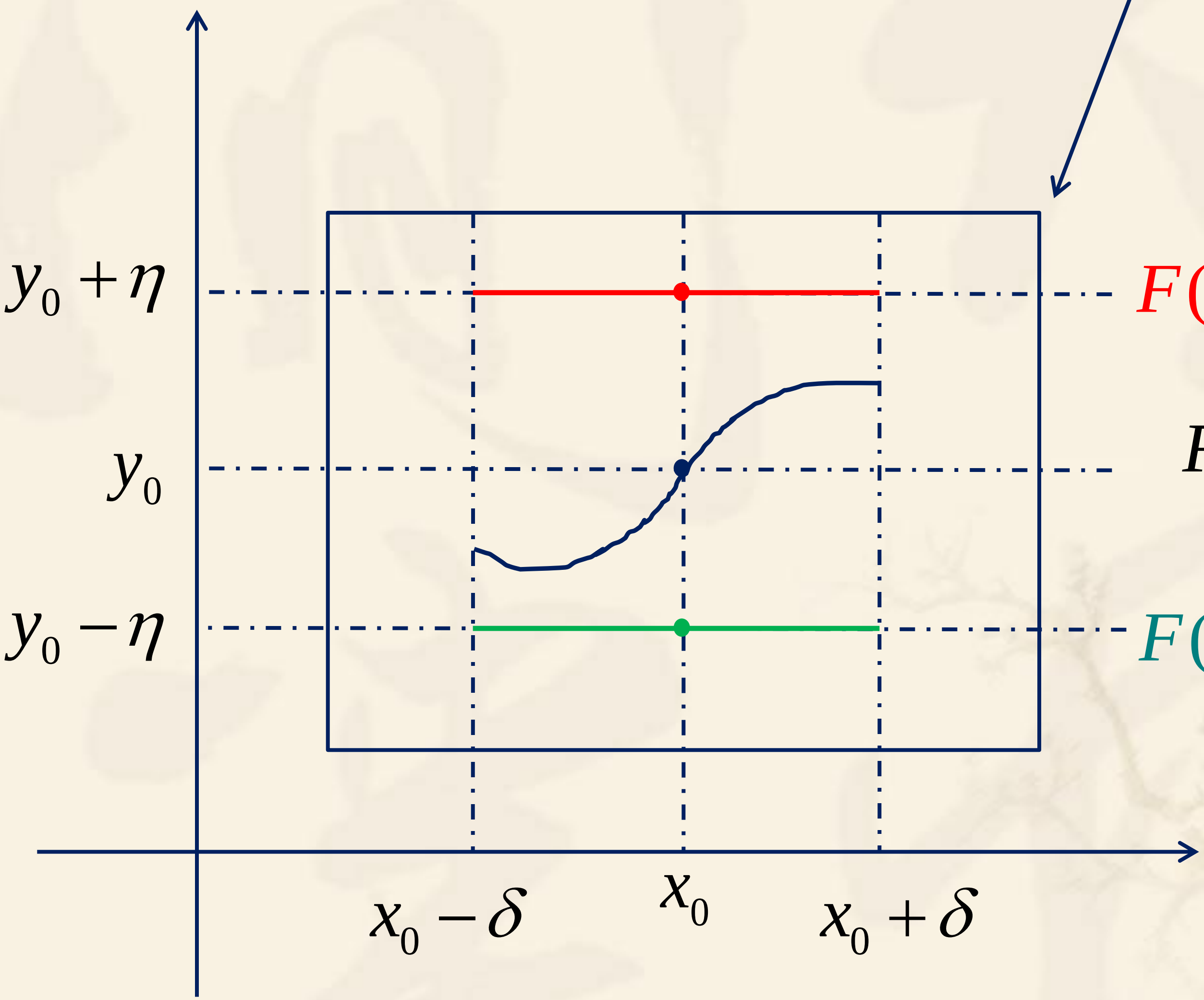
$(1) y(x_0) = y_0$,且$F(x, y(x)) \equiv 0, \forall x\in I,$

$(2) y = y(x)\in C^1(I)$,即$y'(x)$在I上连续,

$$(3)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F(x,y)}{\partial x}\bigg/\frac{\partial F(x,y)}{\partial y}, \forall x\in I.$$

Remark:$F_y{}'(x_0, y_0)\neq 0$不是隐函数存在的必要条件.

$$F'_y(x_0, y_0) > 0 \Rightarrow F'_y(x, y) > 0, \forall (x, y) \in W_1$$



$$F(x, y_0 + \eta) > 0, \quad \forall |x - x_0| < \delta$$

$$\Uparrow$$

$$F(x_0, y_0 + \eta) > 0$$

$$F(x_0, y_0) = 0$$

$$F(x_0, y_0 - \eta) < 0$$

$$\Downarrow$$

$$F(x, y_0 - \eta) < 0, \quad \forall |x - x_0| < \delta$$

Proof. (1)先证隐函数的存在性.

因$F_y'(x_0,y_0)\neq 0$,不妨设$F_y'(x_0,y_0)>0$.$F\in C^1(W)$,则

$\exists a,b>0, s.t.$ $F_y'(x,y)>0, \forall |x-x_0|<a, |y-y_0|<b.$ (*)

$F(x_0,y)$对y连续,由(*)及$F(x_0,y_0)=0$,给定$\eta\in(0,b)$,有

$$F(x_0,y_0-\eta)<0<F(x_0,y_0+\eta).$$

由F的连续性,$\exists\delta\in(0,a), s.t.$

$$F(x,y_0-\eta)<0<F(x,y_0+\eta),\ \forall |x-x_0|<\delta.$$

由(*)知,任意给定$|x-x_0|<\delta$,$F(x,y)$是y的增函数.结合连续函数的介值定理,$\forall |x-x_0|<\delta, \exists! y=y(x)\in$ $(y_0-\eta, y_0-\eta), s.t.\ F(x,y)=0.$

(2)记(1)中构造的隐函数为$y=f(x)$, 下证其连续性.

由(1)中证明知, 不论$\eta>0$取多小, 都$\exists\delta>0$, 当$|x-x_0|<\delta$时, 必有$|y-y_0|<\eta$. 因此$y=f(x)$在x_0连续.

任给$x_1\in(x_0-\delta,x_0+\delta)$, 记$y_1=f(x_1)$, 则$|y_1-y_0|<\eta$, $F(x_1,y_1)=0$, $F_y'(x_1,y_1)>0$. 即F在(x_1,y_1)与(x_0,y_0)满足相同的条件. 由前面的证明, F在(x_1,y_1)的充分小邻域中确定了同一个隐函数$y=f(x)$, 且f在x_1连续.

(3)最后证隐函数$y=y(x)$的可导公式及连续可微性.

任意给定$x\in(x_0-\delta,x_0+\delta)$,由隐函数的连续性, 当$\Delta x\to 0$时,$\Delta y=y(x+\Delta x)-y(x)\to 0.$由隐函数的定义及$F$的连续可微性知,

$$0=F(x+\Delta x,y(x)+\Delta y)-F(x,y(x))$$

$$=F_x'(x,y(x))\Delta x+F_y'(x,y(x))\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y,$$

其中,$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_i=0,i=1,2.$

而$F_y'(x,y(x))>0,$于是有

$$\lim_{\Delta x\to\infty}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\to\infty}\frac{F_x'(x,y(x))+\varepsilon_1}{F_y'(x,y(x))+\varepsilon_2}$$

$$=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall\left|x-x_0\right|<\delta.$$

即 $$y'(x)=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall\left|x-x_0\right|<\delta.$$

由F的连续可微性知,$y'(x)$在$(x_0-\delta,x_0+\delta)$上连续.□

设$f(x_1,x_2,\cdots,x_n,y)=0,\ f(x_1^0,x_2^0,\cdots,x_n^0,y_0)=0.$

若存在连续可微的隐函数

$$y=y(x_1,x_2,\cdots,x_n),\ y_0=y(x_1^0,x_2^0,\cdots,x_n^0),$$

满足 $f\left(x_1,x_2,\cdots,x_n,y(x_1,x_2,\cdots,x_n)\right)\equiv 0,$

则上式两边对x_i求偏导,有$\dfrac{\partial f}{\partial x_i}+\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial x_i}=0.$

若$\left.\dfrac{\partial f}{\partial y}\right|_{(x_1^0,x_2^0,\cdots,x_n^0,y_0)}\neq 0$,则在$(x_1^0,x_2^0,\cdots,x_n^0)$的某邻域中,

$$y'_{x_i}(x_1,x_2,\cdots,x_n)=-\frac{f'_{x_i}(x_1,x_2,\cdots,x_n,y)}{f'_y(x_1,x_2,\cdots,x_n,y)}.$$

Thm. 设函数$F(x_1, x_2, \cdots, x_n, y)$在点$(x_1^0, x_2^0, \cdots, x_n^0, y_0)$ $\in \mathbb{R}^{n+1}$的某个邻域W中有定义, 且

$$(1) F(x_1^0, x_2^0, \cdots, x_n^0, y_0) = 0,$$

$$(2) F(x_1, x_2, \cdots, x_n, y) \in C^1(\mathrm{W}),$$

$$(3) \left.\frac{\partial F}{\partial y}\right|_{(x_1^0, x_2^0, \cdots, x_n^0, y_0)} \neq 0.$$

则存在点$(x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$的一个邻域U,以及定义在U上的$n$元函数$y = y(x_1, x_2, \cdots, x_n)$,满足

(1) $y_0 = y(x_1^0, x_2^0, \cdots, x_n^0)$, 且当 $(x_1, x_2, \cdots, x_n) \in U$ 时,

$$F\left(x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)\right) \equiv 0;$$

(2) $y = y(x_1, x_2, \cdots, x_n) \in C^1(\mathrm{U})$, 即 y'_{x_i} 在U中连续,

$i = 1, 2, \ldots \cdots, n;$

(3) $$y'_{x_i}(x_1, x_2, \cdots, x_n) = -\frac{F'_{x_i}(x_1, x_2, \cdots, x_n, y)}{F'_y(x_1, x_2, \cdots, x_n, y)}.$$

Remark: $F'_y(x_1^0, x_2^0, \cdots, x_n^0, y_0) \neq 0$ 不是隐函数存在的必要条件.

2. 方程组确定的隐函数

设 $F_i(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, i = 1, 2, \cdots, m.$

$$F_i(x_1^0, x_2^0, \cdots, x_n^0, y_1^0, y_2^0, \cdots, y_m^0) = 0, i = 1, 2, \cdots, m.$$

若存在连续可微的隐函数

$$y_i = y_i(x_1, x_2, \cdots, x_n), i = 1, 2, \cdots, m.$$

满足

$$y_i(x_1^0, x_2^0, \cdots, x_n^0) = y_i^0, i = 1, 2, \cdots, m.$$

$$F_i(x_1, x_2, \cdots, x_n, y_1(x_1, x_2, \cdots, x_n), y_2(x_1, x_2, \cdots, x_n),$$
$$\cdots, y_m(x_1, x_2, \cdots, x_n)) = 0, \qquad i = 1, 2, \cdots, m$$

简记为 $F(x,y)=0, F(x_0,y_0)=0,$

$$F(x,y(x))=0, y(x_0)=y_0.$$

$$(x\in\mathbb{R}^n, y\in\mathbb{R}^m, F:\mathbb{R}^{n+m}\to\mathbb{R}^m.)$$

由复合隐射的链式法则, 有 $\dfrac{\partial F}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial x}=0,$

即 $\begin{pmatrix}\dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y}\end{pmatrix}\begin{pmatrix}\dfrac{\partial x}{\partial x}\\ \dfrac{\partial y}{\partial x}\end{pmatrix}=0,\quad \dfrac{\partial F}{\partial x}I_n+\dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial x}=0,$

若 $\dfrac{\partial F}{\partial y}$ 可逆, 则 $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}.$

Thm. $F(x,y):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$在$(x_0,y_0)$的邻域W中有定义,且满足 (1)$F(x_0,y_0)=0$, (2)$F\in C^q(\mathrm{W})$,即$F$的各分量函数在W中$q$阶连续可微, (3) $\dfrac{\partial F}{\partial y}(x_0,y_0)$可逆,

则存在x_0的某个邻域$\mathrm{U}\in\mathbb{R}^n$,以及定义在U上的向量值函数$y=y(x)$,满足

(1)$y(x_0)=y_0, F(x,y(x))=0, \forall x\in \mathrm{U}$;

(2)$y(x)$在U上q阶连续可微;

(3) $$\frac{\partial y}{\partial x}=-\left(\frac{\partial F}{\partial y}\right)^{-1}\frac{\partial F}{\partial x}.$$

Remark: $\dfrac{\partial F}{\partial y}(x_0, y_0)$可逆不是隐函数存在的必要条件.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. φ可微, $x^2+z^2=y\varphi\left(\dfrac{z}{y}\right)$确定隐函数$z=z(x,y)$.求$z'_x, z'_y$.

解: 视$x^2+z^2=y\varphi(z/y)$中$z=z(x,y)$为隐函数. 两边分别对x, y求偏导, 有

$$2x+2zz'_x=y\varphi'(z/y)\cdot\frac{1}{y}z'_x,$$

$$2zz'_y=\varphi(z/y)+y\varphi'(z/y)\cdot\frac{1}{y^2}(yz'_y-z).$$

求解得

$$z'_x=\frac{2x}{\varphi'(z/y)-2z},\quad z'_y=\frac{y\varphi(z/y)-\varphi'(z/y)}{2yz-y\varphi'(z/y)}.\square$$

例. $u=f(x,y,z)$有连续偏导数, 且$z=z(x,y)$由方程 $xe^x-ye^y=ze^z$所确定, 求du.

解: 方程$xe^x-ye^y=ze^z$两边分别对x,y求偏导, 有

$$\left.\begin{aligned} e^x+xe^x&=z'_xe^z+zz'_xe^z \\ -e^y-ye^y&=z'_ye^z+zz'_ye^z \end{aligned}\right\}\Rightarrow\begin{cases} z'_x=\dfrac{1+x}{1+z}e^{x-z}, \\ z'_y=\dfrac{-(1+y)}{1+z}e^{y-z}. \end{cases}$$

于是,

$$du=u'_xdx+u'_ydy=\left(f'_x+f'_zz'_x\right)dx+\left(f'_y+f'_zz'_y\right)dy$$

$$=\left(f'_x+\frac{1+x}{1+z}e^{x-z}f'_z\right)dx+\left(f'_y-\frac{1+y}{1+z}e^{y-z}f'_z\right)dy. \square$$

Remark: $du = f'_x dx + f'_y dy + f'_z dz$

$$= f'_x dx + f'_y dy + f'_z (z'_x dx + z'_y dy)$$

$$= \left(f'_x + f'_z z'_x \right) dx + \left(f'_y + f'_z z'_y \right) dy.$$

一阶微分的形式不变性

例. $u = f\left(x-ut, y-ut, z-ut\right), g(x,y,z)=0$, 求$u'_x, u'_y$.

分析: 五个变量x, y, z, t, u, 两个方程, 确定两个隐函数 $z = z(x,y,t) = z(x,y), u = u(x,y,t)$.

解: 视$g(x,y,z)=0$中$z=z(x,y)$, 两边对x, y求偏导, 有

$$\begin{cases} g'_x + g'_z z'_x = 0, \\ g'_y + g'_z z'_y = 0, \end{cases} \Rightarrow \begin{cases} z'_x = -g'_x / g'_z, \\ z'_y = -g'_y / g'_z. \end{cases}$$

视$u = f\left(x-ut, y-ut, z-ut\right)$中$z = z(x,y)$为隐函数, 两边分别对$x, y$求偏导, 有

$$u'_x = (1-tu'_x) f_1' + (-tu'_x) f_2' + (z'_x - tu'_x) f_3',$$

$$u'_y = (-tu'_y) f_1' + (1-tu'_y) f_2' + (z'_y - tu'_y) f_3'.$$

求解得

$$u'_x = \frac{f_1' + f_3' z_x'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_1' g_z' - f_3' g_x'}{[1 + t(f_1' + f_2' + f_3')] g_z'}$$

$$u'_y = \frac{f_2' + f_3' z_y'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_2' g_z' - f_3' g_y'}{[1 + t(f_1' + f_2' + f_3')] g_z'}. \square$$

3. 逆映射定理

Thm. (逆映射的微分) $f:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^n$连续可微, $x_0\in\Omega$.若 $J(f)|_{x_0}$ 可逆,则存在$y_0=f(x_0)$的某个邻域U,使得U上定义了映射$y=f(x)$的逆映射$x=f^{-1}(y)$, $x_0=f^{-1}(y_0)$,且$x=f^{-1}(y)$在y_0可微,

$$\frac{\partial\left(x_1,x_2,\cdots,x_n\right)}{\partial\left(y_1,y_2,\cdots,y_n\right)}=\left(\frac{\partial\left(y_1,y_2,\cdots,y_n\right)}{\partial\left(x_1,x_2,\cdots,x_n\right)}\right)^{-1},$$

即$J(f^{-1})=\left(J(f)\right)^{-1}$.

Proof: 考虑方程组$F(x,y)\triangleq f(x)-y=0$, 有

$$F(x_0,y_0)=0, \text{且} \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{\partial f}{\partial x}\right|_{x_0} \text{可逆.}$$

由隐函数定理, 存在$y_0=f(x_0)$的邻域U及U上定义的函数$x=x(y)\triangleq f^{-1}(y)$, 满足

$$f(x(y))-y\equiv 0, x(y_0)=x_0,$$

由复合映射的链式法则, 有

$$\frac{\partial f}{\partial x}(x(y))\cdot\frac{\partial x}{\partial y}(y)-I=0, \quad \forall y\in U.$$

即$J(f)\cdot J(f^{-1})=\mathrm{I}$, $J(f^{-1})=\left(J(f)\right)^{-1}$.□

作业：习题1.6 No. 4,5,7,9.