



Review

- Leibnitz判别法

$$a_n \searrow 0 \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n \text{ 收敛.}$$

- Dirichlet判别法

$$\left. \begin{array}{l} \text{数列}\{a_n\}\text{单调趋于}0; \\ \left| \sum_{k=1}^n b_k \right| \leq M, \forall n; \end{array} \right\} \Rightarrow \sum_{k=1}^{+\infty} a_n b_n \text{ 收敛.}$$



- Abel判别法

$$\left. \begin{array}{l} \text{数列}\{a_n\}\text{单调且有界,} \\ \sum_{k=1}^{+\infty} b_k \text{收敛} \end{array} \right\} \Rightarrow \sum_{k=1}^{+\infty} a_n b_n \text{收敛.}$$

- Taylor展开在级数判敛中的应用
- 非负项级数的比较、比值判敛法不适用于一般项级数



• 无穷和运算的结合律

(1) $\sum a_n$ 收敛 \Rightarrow

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) \\ + \cdots + (a_{n_{k-1}+1} + \cdots + a_{n_k}) + \cdots$$

收敛到同一和.

$$(2) (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) \\ + \cdots + (a_{n_{k-1}+1} + \cdots + a_{n_k}) + \cdots$$

收敛, 且同一括号中各项有相同的正负号

$\Rightarrow \sum a_n$ 也收敛到同一和.



• 无穷和运算的交换律

Thm $\sum a_n$ 绝对收敛

\Rightarrow 任意重排 $\sum a'_n$ 也绝对收敛到同一和.

Thm (Riemann定理) $\sum a_n$ 条件收敛, 则

$\forall \lambda \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}, \exists$ 重排 $\sum a'_n, s.t., \sum a'_n = \lambda.$



§ 4. 无穷乘积

无穷乘积: $\prod_{1 \leq k < +\infty} p_k = p_1 p_2 \cdots p_n \cdots,$

部分乘积: $P_n = \prod_{1 \leq k \leq n} p_k.$

Def. 若 $\lim_{n \rightarrow \infty} P_n = P \in \mathbb{R}$, 则称 $\prod_{1 \leq k < +\infty} p_k$ 收敛, 记为

$$\prod_{1 \leq k < +\infty} p_k = P;$$

若数列 $\{P_n\}$ 发散, 则称 $\prod_{1 \leq k < +\infty} p_k$ 发散.

Remark. $\prod_{1 \leq k < +\infty} p_k$ 收敛 $\Leftrightarrow \{P_n\}$ 收敛.



例. 证明 $\prod_{2 \leq n < +\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}$.

Proof. 记 $a_n = \frac{n^2 - n + 1}{n(n-1)}$, 则

$$\begin{aligned} p_n &= \frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)} \\ &= \frac{n^2 + n + 1}{\textcolor{red}{n}(n+1)} \bigg/ \frac{n^2 - n + 1}{\textcolor{red}{n}(n-1)} = a_{n+1} / a_n \end{aligned}$$

$$\prod_{n=2}^m p_n = \prod_{n=2}^m \frac{a_{n+1}}{a_n} = \frac{a_{m+1}}{a_2} = \frac{2}{3} a_{m+1}, \quad \prod_{2 \leq n < +\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}. \quad \square$$



Thm.(无穷乘积收敛的必要条件)

$$\prod_{1 \leq k < +\infty} p_k = P \neq 0 \Rightarrow \lim_{n \rightarrow \infty} p_n = 1.$$

Proof. 记 $P_n = \prod_{1 \leq k \leq n} p_k$, 则 $\lim_{n \rightarrow \infty} P_n = P \neq 0$.

$$\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} = \frac{\lim_{n \rightarrow \infty} P_n}{\lim_{n \rightarrow \infty} P_{n-1}} = \frac{P}{P} = 1. \square$$

Corollary. $\prod_{1 \leq k < +\infty} (1 + a_k)$ 收敛到非零实数 $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$.



Thm. 设 $p_n > 0, |a_n| < 1$, 则

$$\prod_{1 \leq k < +\infty} p_n = P > 0 \Leftrightarrow \sum_{n=1}^{+\infty} \ln p_n \text{ 收敛}$$

$$\prod_{1 \leq k < +\infty} (1 + a_n) = P > 0 \Leftrightarrow \sum_{n=1}^{+\infty} \ln(1 + a_n) \text{ 收敛}.$$

Proof. $P > 0$, 则

$$\prod_{1 \leq k < +\infty} p_n = P \Leftrightarrow \lim_{n \rightarrow \infty} \prod_{1 \leq k \leq n} p_k = P$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \sum_{1 \leq k \leq n} \ln p_k = \ln P \Leftrightarrow \sum_{n=1}^{+\infty} \ln p_n = \ln P. \square$$

Remark. $\prod_{1 \leq k \leq n} p_k = e^{\sum_{1 \leq k \leq n} \ln p_k}$, 因此 $\prod_{1 \leq k < +\infty} p_n = e^{\sum_{1 \leq k < +\infty} \ln p_k}$.



例. $\sum_{n=1}^{+\infty} u_n^2 < +\infty$, 证明 $\prod_{1 \leq n < +\infty} \cos u_n$ 收敛.

Proof. $\sum_{n=1}^{+\infty} u_n^2 < +\infty$, 则 $\lim_{n \rightarrow \infty} u_n = 0$.

$\exists N$, 当 $n \geq N$ 时, $\cos u_n > 0$,

$$0 \geq \ln \cos u_n = \ln \sqrt{1 - \sin^2 u_n} = \frac{1}{2} \ln(1 - \sin^2 u_n)$$

$$\sim -\frac{1}{2} \sin^2 u_n \sim -\frac{1}{2} u_n^2, \quad n \rightarrow +\infty \text{ 时.}$$

$\sum_{n=N}^{+\infty} u_n^2 < +\infty$, 则 $\sum_{n=N}^{+\infty} \ln \cos u_n$ 收敛, 从而 $\prod_{1 \leq n < +\infty} \cos u_n$ 收敛. \square



作业：习题5.4, No. 1, 2