第 7 次习题课　二重积分

1. 若 $f(x,y)$ 是有界闭区域 D 上的非负连续函数,且在 D 上不恒为零,则 $\iint\limits_D f(x,y)d\sigma > 0$

证: 由题设存在 $P_0(x_0,y_0)\in D$ 使得 $f(P_0)>0$. 令 $\delta = f(P_0)$,则由连续函数的局部保号性知:

$\exists\eta>0$ 使得 $f(P)>\delta/2$, $\forall P\in D_1(D_1=U(P_0,\eta)\cap D)$.又因 $f(x,y)\ge 0$ 且连续，所以

$$\iint\limits_D f(x,y)d\sigma = \iint\limits_{D_1} f(x,y)d\sigma + \iint\limits_{D-D_1} f(x,y)d\sigma \ge \frac{\delta}{2}\cdot\Delta D_1 > 0$$

故 $\qquad \iint_D f(x,y)d\sigma > 0$

2. 改变累次积分顺序 $\int_0^1 dx\int_0^{x^2} f(x,y)dy + \int_1^3 dx\int_0^{\frac{1}{2}(3-x)} f(x,y)dy$；

解: $\int_0^1 dx\int_0^{x^2} f(x,y)dy + \int_1^3 dx\int_0^{\frac{1}{2}(3-x)} f(x,y)dy = \int_0^1 dy\int_{\sqrt{y}}^{3-2y} f(x,y)dx$

3. 对积分 $\iint\limits_D f(x,y)dxdy$，$D=\{(x,y)\,|\,0\le x\le 1,\ 0\le x+y\le 1\}$ 进行极坐标变换并写出变换后不同顺序的累次积分

解: 由 $D=\{(x,y)\,|\,0\le x\le 1,\ 0\le x+y\le 1\}$，用极坐标变换后,有

$$\iint\limits_D f(x,y)dxdy = \int_{-\frac{\pi}{4}}^{0} d\theta\int_0^{\sec\theta} r f(r\cos\theta,r\sin\theta)dr + \int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}} r f(r\cos\theta,r\sin\theta)dr$$

$$=\int_0^{\frac{\sqrt{2}}{2}} rdr\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} f(r\cos\theta,r\sin\theta)d\theta + \int_{\frac{\sqrt{2}}{2}}^{1} rdr\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}-\arccos\frac{1}{\sqrt{2}r}} f(r\cos\theta,r\sin\theta)d\theta$$

$$+\int_{\frac{\sqrt{2}}{2}}^{1} rdr\int_{\frac{\pi}{4}+\arccos\frac{1}{\sqrt{2}r}}^{\frac{\pi}{4}} f(r\cos\theta,r\sin\theta)d\theta + \int_1^{2} rdr\int_{-\frac{\pi}{4}}^{-\arccos\frac{1}{r}} f(r\cos\theta,r\sin\theta)d\theta$$

4. 计算二重积分：$\iint\limits_D |xy|dxdy$,其中 D 为圆域: $x^2+y^2\le a^2$。

解: 由对称性有

$$\iint\limits_D |xy|dxdy = 4\int_0^{\frac{\pi}{2}} d\theta\int_0^a r\sin\theta\cdot r\cos\theta\cdot rdr$$

$$=4\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta\cdot\int_0^a r^3dr = 2\cdot\frac{-\cos 2\theta}{2}\Big|_0^{\frac{\pi}{2}}\cdot\frac{r^4}{4}\Big|_0^a = \frac{a^4}{2}.$$

5. 求由曲线所围的平面图形面积：$(\frac{x^2}{a^2}+\frac{y^2}{b^2})=\sqrt{x^2+y^2}$ 。

解：令 $x=ar\cos\theta, y=br\sin\theta$,则 $\left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right|=abr$,

$$D^{'}=\{(r,\theta):0\le\theta\le 2\pi, 0\le r\le\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\}.$$

于是所求面积

$$\begin{aligned}\Delta D&=\iint_D dxdy=\iint_{D^{'}} abrdrd\theta\\&=ab\int_0^{2\pi}d\theta\int_0^{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}rdr\\&=\frac{1}{2}ab\pi(a^2+b^2).\end{aligned}$$

6. 试作适当变换,计算下列积分:

(1) $\iint_D (x+y)\sin(x-y)dxdy, D=\{(x,y)\,|\,0\le x+y\le\pi, 0\le x-y\le\pi\};$

(2) $\iint_D e^{\frac{y}{x+y}}dxdy, D=\{(x,y)\,|\,x+y\le 1, x\ge 0, y\ge 0\}.$

解: (1)令 $u=x+y, v=x-y$, 则 $D^{'}=\{(u,v)\,|\,0\le u\le\pi, 0\le v\le\pi\}$,

$$\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right|=2,\qquad \left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}.$$

于是 $\iint_D (x+y)\sin(x-y)dxdy=\iint_{D^{'}} u\sin v\cdot\frac{1}{2}dudv=\frac{1}{2}\int_0^{\pi}udu\int_0^{\pi}\sin vdv=\frac{1}{2}\pi^2$.

(2) 令 $u=y, v=x+y$, 则 $D^{'}=\{(u,v)\,|\,0\le u\le v, 0\le v\le 1\}$,

$$\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right|=1,\quad \left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|=1.$$

于是 $\iint_D e^{\frac{y}{x+y}}dxdy=\iint_{D^{'}} e^{\frac{u}{v}}dudv=\int_0^1 dv\int_0^v e^{\frac{u}{v}}du=\frac{1}{2}(e-1).$

7. 设 $f(x,y)$ 为连续函数,且 $f(x,y)=f(y,x)$. 证明:

$$\int_0^1 dx\int_0^x f(x,y)dy=\int_0^1 dx\int_0^x f(1-x,1-y)dy.$$

证：令 $u=1-x, v=1-y$,则

$$0\le v\le 1, 0\le u\le v,\qquad \left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}=1.$$

于是

$$\int_0^1 dx\int_0^x f(1-x,1-y)dy=\int_0^1 dv\int_0^v f(u,v)du=\int_0^1 dv\int_0^v f(v,u)du.$$

再令 $x=v,u=y,$ 得 $\quad \int_0^1 dv\int_0^v f(v,u)du=\int_0^1 dx\int_0^x f(x,y)dy.$

于是 $\quad \int_0^1 dx\int_0^x f(x,y)dy=\int_0^1 dx\int_0^x f(1-x,1-y)dy.$

8. 计算 $\mathrm{I}=\int_0^1\frac{\ln(1+x)}{(2-x)^2}dx.$

解：$\mathrm{I}=\int_0^1\frac{1}{(2-x)^2}\left(\int_0^x\frac{1}{1+y}dy\right)dx=\int_0^1\frac{1}{(2-x)^2}dx\int_0^x\frac{1}{1+y}dy$

$$=\int_0^1\frac{1}{1+y}dy\int_y^1\frac{1}{(2-x)^2}dx \text{ (交换积分次序)}$$

$$=\int_0^1\frac{(1-y)dy}{(1+y)(2-y)}=\frac{2}{3}\int_0^1\frac{dy}{1+y}+\frac{1}{3}\int_0^1\frac{dy}{2-y}=\frac{1}{3}\ln 2.$$

9. 证明:$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2(x)dx.$

证明：记 $D=[a,b]\times[a,b]$.则

$$0\le\iint_D\left[f(x)g(y)-f(y)g(x)\right]^2dxdy$$

$$=\iint_D f^2(x)g^2(y)dxdy+\iint_D f^2(y)g^2(x)dxdy-2\iint_D f(x)f(y)g(x)g(y)dxdy$$

$$=2\int_a^b f^2(x)dx\int_a^b g^2(y)dy-2\int_a^b f(x)g(x)dx\int_a^b f(y)g(y)dy$$

$$=2\int_a^b f^2(x)dx\int_a^b g^2(x)dx-2\left(\int_a^b f(x)g(x)dx\right)^2.\ \square$$

10. $f(x)\in C[0,1],\ f>0,\ f\downarrow.$ 求证:$\dfrac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx}\le\dfrac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}.$

证明：只要证 $\mathrm{I}=\int_0^1 xf^2(x)dx\int_0^1 f(x)dx-\int_0^1 xf(x)dx\int_0^1 f^2(x)dx\le 0.$

$$I=\int_0^1 xf^2(x)dx\int_0^1 f(y)dy-\int_0^1 xf(x)dx\int_0^1 f^2(y)dy$$
$$=\iint_{0\le x,y\le 1} xf^2(x)f(y)dxdy-\iint_{0\le x,y\le 1} xf(x)f^2(y)dxdy$$
$$=\iint_{0\le x,y\le 1} xf(x)f(y)[f(x)-f(y)]dxdy.$$

由于积分区域关于直线$y=x$对称,所以

$$I=\iint_{0\le x,y\le 1} yf(x)f(y)[f(y)-f(x)]dxdy.$$

两式相加,由$f>0$, $f\downarrow$,得

$$2I=\iint_{0\le x,y\le 1}(x-y)f(x)f(y)[f(x)-f(y)]dxdy\le 0.$$

11. 设$D=\{(x,y)\,|\,0\le x,y\le 1\}$, $z=f(x,y)\in C^2(D)$. $\left|\dfrac{\partial^2 f(x,y)}{\partial x\partial y}\right|\le 4,\forall(x,y)\in D;$

$f(x,y)\equiv f_x'(x,y)\equiv 0,\forall(x,y)\in\partial D$. 证明: $\left|\iint\limits_D f(x,y)dxdy\right|\le 1.$

证明：$\iint_D f(x,y)dxdy=\int_0^1 dy\int_0^1 f(x,y)dx$

$$=\int_0^1\left[x\,f(x,y)\Big|_{x=0}^1-\int_0^1 x\frac{\partial f(x,y)}{\partial x}dx\right]dy=-\int_0^1 dy\int_0^1 x\frac{\partial f}{\partial x}dx=-\int_0^1 xdx\int_0^1\frac{\partial f}{\partial x}dy$$

$$=-\int_0^1 x\left[y\frac{\partial f}{\partial x}\Big|_{y=0}^1-\int_0^1 y\frac{\partial^2 f}{\partial x\partial y}dy\right]dx=\int_0^1 xdx\int_0^1 y\frac{\partial^2 f}{\partial x\partial y}dy=\iint\limits_D xy\frac{\partial^2 f}{\partial x\partial y}dxdy$$

$$\left|\iint\limits_D f(x,y)dxdy\right|=\left|\iint\limits_D xy\frac{\partial^2 f}{\partial x\partial y}dxdy\right|\le\iint\limits_D\left|xy\frac{\partial^2 f}{\partial x\partial y}\right|dxdy$$

$$\le 4\iint\limits_D xydxdy=4\int_0^1 xdx\int_0^1 ydy=1.\square$$