Review

曲面的切平面与法线:

曲面方程	点	法向量
$\mathrm{r}=\mathrm{r}(u,v)$	$\mathrm{r}_0=\mathrm{r}(u_0,v_0)$	$\left.(\mathrm{r}_u'\times\mathrm{r}_v')\right\|_{(u_0,v_0)}$
$z=f(x,y)$	(x_0,y_0,z_0) $z_0=f(x_0,y_0)$	$\left.(-f_x',-f_y',1)^{\mathrm{T}}\right\|_{(x_0,y_0)}$
$F(x,y,z)=0$	$\mathrm{r}_0=(x_0,y_0,z_0)$	$\mathrm{grad}F(\mathrm{r}_0)$

曲线的切向量:

曲线方程	点	切向量
$\mathrm{r}=\mathrm{r}(t)$	$\mathrm{r}_0=\mathrm{r}(t_0)$	$\mathrm{r}'(t_0)=\big(x'(t_0),y'(t_0),z'(t_0)\big)$
$\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$	$\mathrm{r}_0=(x_0,y_0,z_0)$	$\mathrm{grad}F(\mathrm{r}_0)\times\mathrm{grad}G(\mathrm{r}_0)$

§8. 多元函数的Taylor公式

对充分光滑的一元函数$f(x)$,有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$+\frac{1}{(n+1)!}f^{(n+1)}\big(x_0+\theta(x-x_0)\big)(x-x_0)^{n+1},$$

$\left(0<\theta<1\right)$, 带*Lagrange*余项的*n*阶*Taylor*公式

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$+o\left((x-x_0)^n\right)$, 带*Peano*余项的*n*阶*Taylor*公式

*Taylor*公式的应用之一是近似计算.

对二元可微函数$g(x,y)$,利用全微分也可以进行近似计算.

$$
\begin{aligned}
g(x,y) = g(x_0,y_0) &+ g'_x(x_0,y_0)(x-x_0) \\
&+ g'_y(x_0,y_0)(y-y_0) \\
&+ o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) \approx g(x_0,y_0) &+ g'_x(x_0,y_0)(x-x_0) \\
&+ g'_y(x_0,y_0)(y-y_0)
\end{aligned}
$$

Question: 如何进一步提高精度?

Lemma. 设$\mathrm{A}=(a_{ij})_{n\times n}$为$n$阶实对称矩阵，则

$$\left|\mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x}\right| \le \left(n\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\right)\cdot\|\mathrm{x}\|^{2}, \qquad \forall \mathrm{x}\in\mathbb{R}^{n}.$$

Proof. 记$\mathrm{A}=(\alpha_1,\cdots,\alpha_n)$, $\forall \mathrm{x}=(x_1,\cdots,x_n)^{\mathrm{T}}\in\mathbb{R}^n$,有

$$\begin{aligned}\|\mathrm{A}\mathrm{x}\| &= \|x_1\alpha_1+\cdots+x_n\alpha_n\| \le |x_1|\|\alpha_1\|+\cdots+|x_n|\|\alpha_n\| \\ &\le \big(|x_1|+\cdots+|x_n|\big)\big(\|\alpha_1\|+\cdots+\|\alpha_n\|\big) \\ &\le n\|\mathrm{x}\|\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|,\end{aligned}$$

于是 $\left|\mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x}\right| \le \|\mathrm{x}\|\cdot\|\mathrm{A}\mathrm{x}\| \le \left(n\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\right)\cdot\|\mathrm{x}\|^{2}$.□

Def.设n元函数f在$B(\mathrm{x}_0,\delta)$中二阶连续可微, 称实对称阵

$$H_f(\mathrm{x}_0)=\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}_{\mathrm{x}_0}$$

为f在点x_0的Hessian矩阵.

Thm. 设n元函数f在$B(\mathrm{x}_0,\delta)$中二阶连续可微, 则

$\forall \mathrm{x}_0+\Delta\mathrm{x}\in B(x_0,\delta), \exists\theta\in(0,1), s.t.$

$$f(\mathrm{x}_0+\Delta\mathrm{x})=f(\mathrm{x}_0)+J_f(\mathrm{x}_0)\Delta x$$
$$+\frac{1}{2}(\Delta\mathrm{x})^{\mathrm{T}}H_f(\mathrm{x}_0+\theta\Delta\mathrm{x})\Delta\mathrm{x}$$

(称为带*Lagrange*余项的一阶*Taylor*公式), 且

$$f(\mathrm{x}_0+\Delta\mathrm{x})=f(\mathrm{x}_0)+J_f(\mathrm{x}_0)\Delta\mathrm{x}$$
$$+\frac{1}{2}(\Delta\mathrm{x})^{\mathrm{T}}H_f(\mathrm{x}_0)\Delta\mathrm{x}+o(\|\Delta\mathrm{x}\|^2), \Delta\mathrm{x}\to 0\text{时}$$

(称为带*Peano*余项的二阶*Taylor*公式).

Proof. 构造一元函数

$$g(t)=f(x_0+t\Delta x)=f(x_0^{(1)}+t\Delta x_1,\cdots,x_0^{(n)}+t\Delta x_n),$$

$$f\in C^2,\text{则}\ g'(t)=\sum_{i=1}^{n}f_i'(x_0+t\Delta x)\Delta x_i=J_f(x_0+t\Delta x)\Delta x$$

$$g''(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}f_{ij}''(x_0+t\Delta x)\Delta x_i\Delta x_j=(\Delta x)^{T}H_f(x_0+t\Delta x)\Delta x$$

$$g(t)=g(0)+g'(0)t+\frac{1}{2}g''(\theta t)t^2,\theta\in(0,1).$$

$$f(x_0+\Delta x)=g(1)=g(0)+g'(0)+\frac{1}{2}g''(\theta)$$

$$=f(x_0)+Jf(x_0)\Delta x+\frac{1}{2}(\Delta x)^{T}H_f(x_0+\theta\Delta x)\Delta x,\theta\in(0,1).$$

带*Lagrange*余项的一阶*Taylor*公式得证.

记 $P(\Delta \mathrm{x}) = H_f(\mathrm{x}_0 + \theta \Delta \mathrm{x}) - H_f(\mathrm{x}_0) = (h_{ij})_{n\times n}$,

$$\alpha(\Delta \mathrm{x}) = (\Delta \mathrm{x})^{\mathrm{T}} H_f(\mathrm{x}_0 + \theta \Delta \mathrm{x}) \Delta \mathrm{x} - (\Delta \mathrm{x})^{\mathrm{T}} H_f(\mathrm{x}_0) \Delta \mathrm{x}$$

$$= (\Delta \mathrm{x})^{\mathrm{T}} P(\Delta \mathrm{x}) \Delta \mathrm{x}.$$

已知 $f \in C^2$,则 $\lim\limits_{\Delta x \to 0} h_{ij} = 0$. 由前面的引理得

$$|\alpha(\Delta \mathrm{x})| \le \left(n \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}| \right) \cdot \|\Delta \mathrm{x}\|^2,$$

因此 $\alpha(\Delta \mathrm{x}) = o(\|\Delta \mathrm{x}\|^2)$, 当 $\Delta \mathrm{x} \to 0$ 时.

故带*Peano*余项的二阶*Taylor*公式得证.□

Question. 能否用一元函数带Peano余项的Taylor公式直接证明多元函数带Peano余项的Taylor公式?

Thm. 设函数$f(x,y)$在区域D中$n+1$阶连续可微，$M_0(x_0,y_0)\in D, M(x,y)\in D$，且线段$\overline{M_0M}$完全包含在$D$中. 记

$$h=x-x_0, k=y-y_0,$$

记算子

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m \triangleq \sum_{i=0}^{m} C_m^i h^i k^{m-i}\frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则f在点(x_0,y_0)有

(1)带*Lagrange*余项的*n*阶*Taylor*公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$$

$$\left(0<\theta<1\right)$$

(2)带$Peano$余项的$n+1$阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(x_0,y_0)$$

$$+o\left(\left(\sqrt{h^2+k^2}\right)^{n+1}\right).$$

例: 写出$f(x,y)=x^y$在点(1,1)的邻域内带*Peano*余项的3阶*Taylor*公式, 并求$(1.1)^{1.02}$.

解: $f(x,y)=x^y,\ f'_x=yx^{y-1},\ f'_y=x^y\ln x,$

$$f''_{xx}=y(y-1)x^{y-2},\ f''_{yy}=x^y\ln^2 x,\ f''_{xy}=x^{y-1}+yx^{y-1}\ln x,$$

$$f'''_{xxx}=y(y-1)(y-2)x^{y-3},\quad f'''_{yyy}=x^y\ln^3 x$$

$$f'''_{xxy}=(2y-1)x^{y-2}+y(y-1)x^{y-2}\ln x,$$

$$f'''_{xyy}=yx^{y-1}\ln^2 x+2x^{y-1}\ln x.$$

令$x_0=y_0=1, h=x-1, k=y-1.$

$$f(1,1)=f'_x(1,1)=f''_{xy}(1,1)=f'''_{xxy}(1,1)=1,$$

$$f'_y(1,1)=f''_{xx}(1,1)=f''_{yy}(1,1)$$

$$=f'''_{xxx}(1,1)=f'''_{yyy}(1,1)=f'''_{xyy}(1,1)=0$$

$$f(x,y)=f(x_0,y_0)+(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(x_0,y_0)$$

$$+\frac{1}{2!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2 f(x_0,y_0)$$

$$+\frac{1}{3!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^3 f(x_0,y_0)+o((\sqrt{h^2+k^2})^3)$$

$$f(x,y)=1+(x-1)+(x-1)(y-1)$$

$$+\frac{1}{2}(x-1)^2(y-1)$$

$$+o\left(\left(\sqrt{(x-1)^2+(y-1)^2}\right)^3\right)$$

$$(1.1)^{1.02}\approx 1.1+0.1\times 0.02+\frac{1}{2}\times 0.01\times 0.02$$

$$=1.1021. \square$$

Question. 二元函数在一点的Taylor多项式是否唯一？如何证明？

唯一！

例. $\cos(x^2+y^2)$在$(0,0)$的8阶带Peano余项的Taylor展开式.

解: $$\cos t = 1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}+o(t^{2n}),\ t\to 0\text{时}.$$

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\cdots+(-1)^n\frac{(x^2+y^2)^{2n}}{(2n)!}$$

$$+o((x^2+y^2)^{2n}),\ x^2+y^2\to 0\text{时}.$$

令$n=2$得

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\frac{(x^2+y^2)^4}{4!}+o((x^2+y^2)^4),$$

$$x^2+y^2\to 0\text{时}.\square$$

例. $\ln(2+x+y+xy)$在$(0,0)$带Peano余项的2阶Taylor展开.

解: $x+y+xy\to 0$ 时,

$$\ln(2+x+y+xy)=\ln 2+\ln(1+\frac{x+y+xy}{2})$$

$$=\ln 2+\frac{x+y+xy}{2}-\frac{1}{2}\left(\frac{x+y+xy}{2}\right)^2+o\left((x+y+xy)^2\right)$$

$x^2+y^2\to 0$时, 必有$x+y+xy\to 0$ 时, 因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2}=\frac{o((x+y+xy)^2)}{(x+y+xy)^2}\cdot\frac{(x+y+xy)^2}{x^2+y^2}\to 0,$$

$$\ln(2+x+y+xy)$$

$$=\ln 2+\frac{x+y}{2}-\frac{x^2+y^2-2xy}{8}+o(x^2+y^2).\square$$

作业：习题1.8 No.1,2(2)