Review

含参广义积分的性质

$$I(y)=\int_a^{+\infty} f(x,y)dx, \quad D=[a,+\infty)\times[\alpha,\beta].$$

$$\bullet\begin{cases} f(x,y),\ f_y'(x,y)\in C(D); \\ \forall y\in[\alpha,\beta], I(y)=\int_a^{+\infty} f(x,y)dx \text{ 收敛;} \\ \int_a^{+\infty} f_y'(x,y)dx \text{ 关于} y\in[\alpha,\beta]\text{一致收敛;} \end{cases}$$

$$\Rightarrow \in C^1[\alpha,\beta], \text{且} I'(y)=\frac{d}{dy}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty} f_y'(x,y)dx.$$

$$\bullet\begin{cases} f(x,y)\in C(D); \\ \int_a^{+\infty} f(x,y)dx\text{关于}y\in[\alpha,\beta]\text{一致连续}; \end{cases}$$

$$\Rightarrow\begin{cases} I(y)\in C[\alpha,\beta],\text{即} \\ \qquad \lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\lim_{y\to y_0} f(x,y)dx. \\ I(y)\in R[\alpha,\beta], \\ \qquad \text{且}\int_\alpha^\beta dy\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\int_\alpha^\beta f(x,y)dy. \end{cases}$$

§ 4. Γ函数与B函数

Γ函数与B函数是物理和工程技术中常见的两个函数.

$$\Gamma(x)=\int_0^{+\infty} t^{x-1}e^{-t}dt, \quad (x>0)$$

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (x>0, y>0)$$

1. Γ函数（Gamma函数）

Lemma1. Γ函数在$(0,+\infty)$上有定义、连续且有任意阶可微.

Lemma2. 设$f, g \in C[a,b]$, $f(t) \geq 0, g(t) \geq 0, \forall t \in [a,b]$. 若$\alpha > 0, \beta > 0, \alpha + \beta = 1$, 则

$$\int_a^b \left(f(t)\right)^\alpha \left(g(t)\right)^\beta dt \leq \left(\int_a^b f(t)dt\right)^\alpha \left(\int_a^b g(t)dt\right)^\beta .$$

Proof. 若$u \geq 0, v \geq 0, \alpha > 0, \beta > 0, \alpha + \beta = 1$, 则

$$u^\alpha v^\beta \leq \alpha u + \beta v. \quad (\text{几何平均} \leq \text{代数平均})$$

记$F = \int_a^b f(t)dt, G = \int_a^b g(t)dt, \tilde{f}(t) = f(t)/F, \tilde{g}(t) = g(t)/G$.

往证$\int_a^b \left(\tilde{f}(t)\right)^\alpha \left(\tilde{g}(t)\right)^\beta dt \leq 1$.事实上,

$$\text{左边} \leq \int_a^b \left[\alpha \tilde{f}(t) + \beta \tilde{g}(t)\right] dt$$

$$= \alpha \int_a^b \tilde{f}(t)dt + \beta \int_a^b \tilde{g}(t)dt = \alpha + \beta = 1. \square$$

$$\Gamma(x)=\int_0^{+\infty} t^{x-1}e^{-t}dt, \quad (x>0)$$

Theorem1.Γ函数有以下基本性质

$(1)\Gamma(x)\geq 0, \forall x\in(0,+\infty)$, 且$\Gamma(1)=1$;

$(2)\Gamma(x+1)=x\Gamma(x), \forall x\in(0,+\infty)$;

$(3)\ln\Gamma(x)$在$(0,+\infty)$是凸函数.

Proof.$(1)\Gamma(1)=\int_0^{+\infty} e^{-t}dt=-e^{-t}\Big|_{t=0}^{+\infty}=1.$

$$(2)\Gamma(x+1)=\int_0^{+\infty} t^x e^{-t}dt=-\int_0^{+\infty} t^x de^{-t}$$

$$=-t^x e^{-t}\Big|_{t=0}^{+\infty}+\int_0^{+\infty} xe^{-t}t^{x-1}dt=x\Gamma(x).$$

(3)欲证$\ln\Gamma(x)$在$(0,+\infty)$是凸函数, 只要证$\forall\alpha>0$, $\beta>0, \alpha+\beta=1$, 及$x>0, y>0$, 都有

$$\ln\Gamma(\alpha x+\beta y)\le\alpha\ln\Gamma(x)+\beta\ln\Gamma(y)$$

也即

$$\Gamma(\alpha x+\beta y)\le\left(\Gamma(x)\right)^{\alpha}\left(\Gamma(y)\right)^{\beta}.$$

由Lemma2,

$$\begin{aligned}\Gamma(\alpha x+\beta y)&=\int_0^{+\infty}t^{\alpha x+\beta y-1}e^{-t}dt\\&=\int_0^{+\infty}\left(t^{x-1}e^{-t}\right)^{\alpha}\left(t^{y-1}e^{-t}\right)^{\beta}dt\\&\le\left(\int_0^{+\infty}t^{x-1}e^{-t}dt\right)^{\alpha}\left(\int_0^{+\infty}t^{y-1}e^{-t}dt\right)^{\beta}\\&=\left(\Gamma(x)\right)^{\alpha}\left(\Gamma(y)\right)^{\beta}.\ \square\end{aligned}$$

Corollary1.Γ函数可以看成阶乘函数的推广.

$$\Gamma(n+1)=n!,\quad n=1,2,3,\cdots.\square$$

Theorem1中的三条性质完全确定了Γ函数.

Theorem2.(Bohr-Mollerup)若定义于$(0,+\infty)$上的函数$f(x)$满足以下条件:

(1)$f(x)\geq 0,\forall x\in(0,+\infty)$, 且$f(1)=1$,

(2)$f(x+1)=xf(x),\forall x\in(0,+\infty)$,

(3)$\ln f(x)$在$(0,+\infty)$是凸函数,

则$f(x)=\Gamma(x),\forall x\in(0,+\infty)$.

Proof: 只要证明这三个条件确定的函数是唯一的. 为此, 以下证明过程分两步:

Step1. $\forall n \in N$, $f(n)$唯一确定.

Step2. $\forall x \in (0,1)$, $f(x)$唯一确定.

这是因为由性质(2), $\forall x \in (0,1)$及$\forall n \in N$,

$$f(x+n+1) = (x+n)(x+n-1)\cdots xf(x).$$

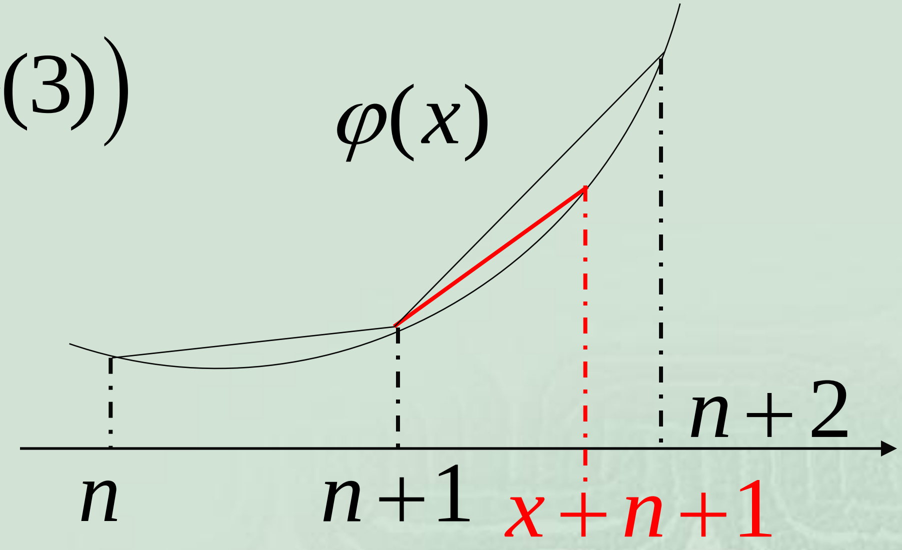
也被唯一确定了.

Step1. 由性质(1), $\forall n \in N$, $f(n+1) = n!$唯一确定.

Step2. $\forall x \in (0,1)$, 令$\varphi(x) = \ln f(x)$. 则

$$\varphi(x+n+1)=\varphi(x)+\ln\left[x(x+1)\cdots(x+n)\right].$$

由于$\varphi(x)$是凸函数(条件(3))



$$\frac{\varphi(n+1)-\varphi(n)}{(n+1)-n}$$

$$\le\frac{\varphi(x+n+1)-\varphi(n+1)}{(x+n+1)-(n+1)}$$

$$\le\frac{\varphi(n+2)-\varphi(n+1)}{(n+2)-(n+1)}$$

即

$$\ln n\le\frac{\varphi(x+n+1)-\ln(n!)}{x}\le\ln(n+1)$$

$$x\ln n+\ln(n!)\le\varphi(x+n+1)\le x\ln(n+1)+\ln(n!)$$

至此, 我们得到了关于$\varphi(x)$的两个条件

$$\varphi(x+n+1)=\varphi(x)+\ln\left[x(x+1)\cdots(x+n)\right].$$

$$x\ln n+\ln(n!)\le\varphi(x+n+1)\le x\ln(n+1)+\ln(n!)$$

由此得

$$\ln\frac{n^x\cdot n!}{x(x+1)\cdots(x+n)}\le\varphi(x)\le\ln\frac{(n+1)^x\cdot n!}{x(x+1)\cdots(x+n)}$$

于是 $0\le\varphi(x)-\ln\dfrac{n^x\cdot n!}{x(x+1)\cdots(x+n)}\le x\ln\left(1+\dfrac{1}{n}\right)$

由夹挤原理得

$$\varphi(x)=\lim_{n\to+\infty}\ln\frac{n^x\cdot n!}{x(x+1)\cdots(x+n)}.\square$$

Corollary2. $\Gamma(x)=\lim\limits_{n\to+\infty}\dfrac{n^x\cdot n!}{x(x+1)(x+2)\cdots(x+n)}$.$\square$

Theorem3.(Γ函数的余元公式)

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x},\forall x\in(0,1).$$

Proof. $\Gamma(x)=\lim\limits_{n\to+\infty}\dfrac{n^x}{x(1+x)(1+\frac{x}{2})\cdots(1+\frac{x}{n})}$,

$$\Gamma(1-x)=\lim_{n\to+\infty}\frac{n^{1-x}}{(1-x)(1-\frac{x}{2})\cdots(1-\frac{x}{n})(n+1-x)}.$$

$$\Gamma(x)\Gamma(1-x)$$

$$=\lim_{n\to+\infty}\left[\frac{1}{x(1-x^2)(1-\frac{x^2}{2^2})\cdots(1-\frac{x^2}{n^2})}\cdot\frac{n}{n+1-x}\right]$$

$$=\frac{1}{\lim\limits_{n\to+\infty} x\prod\limits_{n=1}^{+\infty}(1-\frac{x^2}{n^2})}=\frac{\pi}{\sin\pi x}.$$

最后一个等式的证明见下面的引理.□

Lemma3. $\sin\pi x=\lim\limits_{n\to+\infty}\pi x\prod\limits_{n=1}^{+\infty}(1-\frac{x^2}{n^2}),\forall x\in(0,1).$

Proof.

令$\psi(x)=\ln\left[\pi\prod\limits_{n=1}^{+\infty}(1-\frac{x^2}{n^2})\right]=\ln\pi+\sum\limits_{n=1}^{+\infty}\ln(1-\frac{x^2}{n^2}).$

则$\psi(0)=\ln\pi,\psi'(x)=\sum\limits_{n=1}^{+\infty}\frac{2x}{x^2-n^2}.$

给定$x\in(0,1)$,函数$\cos xt,t\in[-\pi,\pi]$的Fourier展式为

$$\cos xt=\frac{\sin x\pi}{\pi}\left(\frac{1}{x}+\sum_{n=1}^{+\infty}(-1)^n\frac{2x\cos nt}{x^2-n^2}\right).$$

令$t=\pi$得, $\pi\cot\pi x=\frac{1}{x}+\sum\limits_{n=1}^{+\infty}\frac{2x}{x^2-n^2}.$

因此,

$$\psi'(x) = \pi\cot\pi x - \frac{1}{x},\ \ \psi(x) = \ln\frac{\sin\pi x}{x} + C.$$

令$x \to 0^+$得, $C = 0$, 即

$$\ln\left[\pi\prod_{n=1}^{+\infty}(1-\frac{x^2}{n^2})\right] = \ln\frac{\sin\pi x}{x},$$

$$\sin\pi x = \lim_{n\to+\infty}\pi x\prod_{n=1}^{+\infty}(1-\frac{x^2}{n^2}), \forall x \in (0,1).\square$$

Corollary3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\square$

2. B函数(Beta函数)

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad (x>0,y>0)$$

Lemma4. $\mathrm{B}(x,y)$对任意$x>0,y>0$有定义,且满足以下性质:

(1) $\mathrm{B}(x,y)>0,\forall x,y>0$, 且$\mathrm{B}(1,y)=1/y$,

(2) $\mathrm{B}(x+1,y)=\dfrac{x}{x+y}\mathrm{B}(x,y),\forall x,y>0,$

(3) 给定$y>0$, $\ln\mathrm{B}(x,y)$关于x在$(0,+\infty)$是凸函数. □

Lemma5. $\mathrm{B}(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \forall x,y>0.$

Proof:任意给定$y>0$, 令$f(x)=\dfrac{\mathrm{B}(x,y)\Gamma(x+y)}{\Gamma(y)}$,

往证$f(x)=\Gamma(x)$.只要证f满足Γ函数的三条性质.

由Lemma3得

$$(1)\ f(1)=\frac{\mathrm{B}(1,y)\Gamma(1+y)}{\Gamma(y)}=\frac{\Gamma(1+y)}{y\Gamma(y)}=1.$$

$$(2)\ f(x+1)=\frac{\mathrm{B}(x+1,y)\Gamma(x+y+1)}{\Gamma(y)}$$

$$=\frac{x}{(x+y)}\mathrm{B}(x,y)\cdot\frac{(x+y)\Gamma(x+y)}{\Gamma(y)}=xf(x).$$

(3)对于取定的$y>0$,由于$\ln \mathrm{B}(x,y)$和$\ln\Gamma(x+y)$都是x的凸函数,所以

$$\ln f(x)=\ln \mathrm{B}(x,y)+\ln\Gamma(x+y)-\ln\Gamma(y)$$

也是x的凸函数.□

Corollary4.$\mathrm{B}(x,y)=\mathrm{B}(y,x), \forall x,y>0$.□

3. 例题　例1.$I=\int_0^1\sqrt{x-x^2}\,dx.$

解:$I=\int_0^1 x^{1/2}(1-x)^{1/2}dx=B(3/2,3/2)$

$$=\frac{\Gamma(3/2)^2}{\Gamma(3)}=\frac{[1/2\,\Gamma(1/2)]^2}{2!}=\frac{\pi}{8}.$$ □

例2. $I=\int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$.

解: 令$t=x^{1/4}$, 则$x=t^4$, $dx=4t^3dt$,

$$I=4\int_0^1 t^3(1-t)^{-1/2}dt=4\mathrm{B}(4,1/2)=\frac{4\Gamma(4)\Gamma(1/2)}{\Gamma(4+1/2)}$$

$$=4\cdot\frac{3!\Gamma(1/2)}{(3+1/2)(2+1/2)(1+1/2)1/2\,\Gamma(1/2)}=\frac{128}{35}\square$$

例3. $I=\int_0^{\pi/2}\sqrt{\tan x}dx.$

解: $I=\int_0^{\pi/2}\sin^{1/2}x\cos^{-1/2}xdx.$ 令 $t=\sin^2 x$, 则

$$dt=2\sin x\cos xdx,\ dx=\frac{dt}{2\sin x\cos x}.$$

$$I=\frac{1}{2}\int_0^1 t^{-1/4}(1-t)^{-3/4}dt=\frac{1}{2}\mathrm{B}(3/4,1/4)$$

$$=\frac{1}{2}\frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}=\frac{1}{2}\frac{\pi}{\sin\pi/4}=\frac{\pi}{\sqrt{2}}.\ \square$$

例4. $I=\int_0^{+\infty} x^{2n}e^{-x^2}dx.$

解: 令$t=x^2$, 则$dt=2xdx, dx=\dfrac{dt}{2\sqrt{t}}.$

$$I=\int_0^{+\infty} t^n e^{-t}\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^{+\infty} t^{n-1/2}e^{-t}dt$$

$$=\frac{1}{2}\Gamma(n+1/2)=\frac{1}{2}(n-1/2)(n-3/2)\cdots 1/2\,\Gamma(1/2)$$

$$=\frac{(2n-1)!!}{2^{n+1}}\sqrt{\pi}.\square$$

作业:利用Β函数和Γ函数计算积分

$$(1)\int_0^{+\infty}\frac{dx}{1+x^4}$$

$$(2)\int_0^{\pi/2}\sin^6 x\cos^4 x dx$$

$$(3)\int_0^a x^2\sqrt{a^2-x^2}dx.$$

答案:$(1)\dfrac{\pi}{2\sqrt{2}},(2)\dfrac{3\pi}{516},(3)\dfrac{\pi}{16}a^4.$