第九周作业参考答案

习题 4.2

3.

(1) $I=\int_0^1\sqrt{x_t'^2+y_t'^2+z_t'^2}dt=\int_0^1\sqrt{9+36t^2+36t^4}dt=5$

(2) $\sqrt{3}$

4. $I=\int_{\sqrt{3}}^{\sqrt{15}}x^2\sqrt{1+\frac{1}{x^2}}dx=\int_{\sqrt{3}}^{\sqrt{15}}x\sqrt{x^2+1}dx=\frac{1}{3}\left(x^2+1\right)\frac{3}{2}\Big|_{\sqrt{3}}^{\sqrt{15}}=\frac{56}{3}$

5. $I=\int\limits_{x^2+y^2=a^2}\left(a+\frac{x^2}{a}\right)dl=\int_0^{2\pi}(a+a\cos^2\varphi)ad\varphi=3\pi a^2$

6. $dx=a(1-\cos t)dt, dy=a\sin tdt,$

故 $dl=\sqrt{dx^2+dy^2}=2a\sin\frac{t}{2}dt, 0\le t\le\pi$

$M=\int_0^{\pi}2a\sin\frac{t}{2}dt=4a$

$M_x=\int_0^{\pi}a(1-\cos t)\cdot 2a\sin\frac{t}{2}dt=\frac{16}{3}a^2,\ \overline{y}=\frac{M_x}{M}=\frac{4}{3}a$

$M_y=\int_0^{\pi}a(t-\sin t)\cdot 2a\sin\frac{t}{2}dt=\frac{16}{3}a^2,\ \overline{x}=\frac{M_y}{M}=\frac{4}{3}a$

所以，质心为 $\left(\frac{4}{3}a,\frac{4}{3}a\right)$

7. $dl=\sqrt{a^2+\frac{b^2}{4\pi^2}}dt,\ J_x=\int_L(y^2+z^2)dl=\sqrt{\pi^2a^2+\frac{b^2}{4}}\left(a^2+\frac{2}{3}b^2\right)$

习题 4.3

2. 令 $x=\rho\cos\varphi, y=\rho\sin\varphi,$ 则 $\rho=a\cos\varphi, \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ 得

$x=a\cos^2\varphi, y=a\cos\varphi\sin\varphi, -a|\sin\varphi|\le z\le a|\sin\varphi|, \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$A=\frac{D(y,z)}{D(\varphi,z)}=a\cos 2\varphi, B=\frac{D(z,x)}{D(\varphi,z)}=a\sin 2\varphi, C=\frac{D(x,y)}{D(\varphi,z)}=0$

$dS=\sqrt{A^2+B^2+C^2}d\varphi dz=ad\varphi dz,$ 所以

$S=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}ad\varphi\int_{-a|\sin\varphi|}^{a|\sin\varphi|}dz=4a^2$

3. $M=\iint\limits_{x^2+y^2\le 2}\frac{x^2+y^2}{2}\sqrt{1+x^2+y^2}dxdy=\frac{2}{15}\pi\left(1+6\sqrt{3}\right)$

6.

① $M=\frac{\pi}{2}a^2, M_{yz}=M_{zx}=M_{xy}=\frac{\pi}{4}a^3, \overline{x}=\overline{y}=\overline{z}=\frac{a}{2},$

② $\overline{x}=\overline{y}=0, \overline{z}=\frac{a}{2}$

10. 令 $x=a\sin\theta\cos\varphi, y=b\sin\theta\sin\varphi, z=c\cos\theta, 0\le\theta\le\pi, 0\le\varphi\le 2\pi$

$A=\dfrac{D(y,z)}{D(\theta,\varphi)}=bc\sin^2\theta\cos\varphi, B=\dfrac{D(z,x)}{D(\theta,\varphi)}=ac\sin^2\theta\sin\varphi, C=\dfrac{D(x,y)}{D(\theta,\varphi)}=ab\sin\theta\cos\theta$

$L(x,y,z)=\dfrac{|Ax+By+Cz|}{\sqrt{A^2+B^2+C^2}}, dS=\sqrt{A^2+B^2+C^2}d\theta d\varphi$, 所以

$$\iint\limits_S L(x,y,z)dS=\iint\limits_{D_{\theta\varphi}}|Ax+By+Cz|d\theta d\varphi=\int_0^{\pi}d\theta\int_0^{2\pi}|abc\sin\theta|d\varphi=4\pi abc$$

习题 4.4

2.

(1) $-\dfrac{56}{15}$

(2) 令 $x=a\cos\varphi, y=a\sin\varphi$, 得 $I=-2\pi$

(3) 由 x,y 轮换对称性, $\oint\limits_{L^+}\dfrac{dx+dy}{|x|+|y|}=-\oint\limits_{L^+}\dfrac{dy+dx}{|y|+|x|}=0$

(4) 由轮换对称性 $\int\limits_{L^+}\left(z^2-x^2\right)dy=\int\limits_{L^+}\left(x^2-y^2\right)dz=\int\limits_{L^+}\left(y^2-z^2\right)dx=-\dfrac{4}{3}$, 所以 $I=-4$

(5) L 满足 $x^2+2y^2=1, z=y$, 令 $x=\cos\varphi, z=y=\dfrac{1}{\sqrt{2}}\sin\varphi$,

$$\oint\limits_{L^+}xyzdz=\int_0^{2\pi}\cos\varphi\left(\frac{1}{\sqrt{2}}\sin\varphi\right)^2\cdot\frac{1}{\sqrt{2}}\cos\varphi d\varphi=\frac{\sqrt{2}}{16}\pi$$

4.

(1) $W=\int_{(a,0)}^{(0,b)}-xdx-ydy=\dfrac{a^2-b^2}{2}$

(2) $W=0$