# Review

- 三重积分化累次积分

 （先一后二）

$$\Omega : \begin{cases} z_1(x, y) \le z \le z_2(x, y), \\ (x, y) \in D_{xy}, \end{cases}$$

$$\iiint\limits_{\Omega} f(x, y, z)dxdydz = \iint\limits_{D_{xy}} dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz.$$

（先二后一）

$$\Omega : \begin{cases} c \le z \le d, \\ (x, y) \in \Omega_z, \end{cases}$$

$$\iiint_{\Omega} f(x, y, z)dxdydz = \int_c^d dz \iint_{\Omega_z} f(x, y, z)dxdy.$$

●投影法确定积分区域

●三重积分的变量替换

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$

$$(x, y, z) \in \Omega \leftrightarrow (u, v, w) \in \Omega^{*}.$$

$$\iiint_{\Omega} f(x, y, z) dxdydz$$

$$= \iiint_{\Omega^{*}} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)$$

$$\cdot \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw.$$

# §5. 重积分的应用

- 曲面面积
- 质心
- 转动惯量
- 万有引力

原则：微元法

## 1. 曲面的面积

设曲面$S$有参数方程

$$x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D,$$

简记为
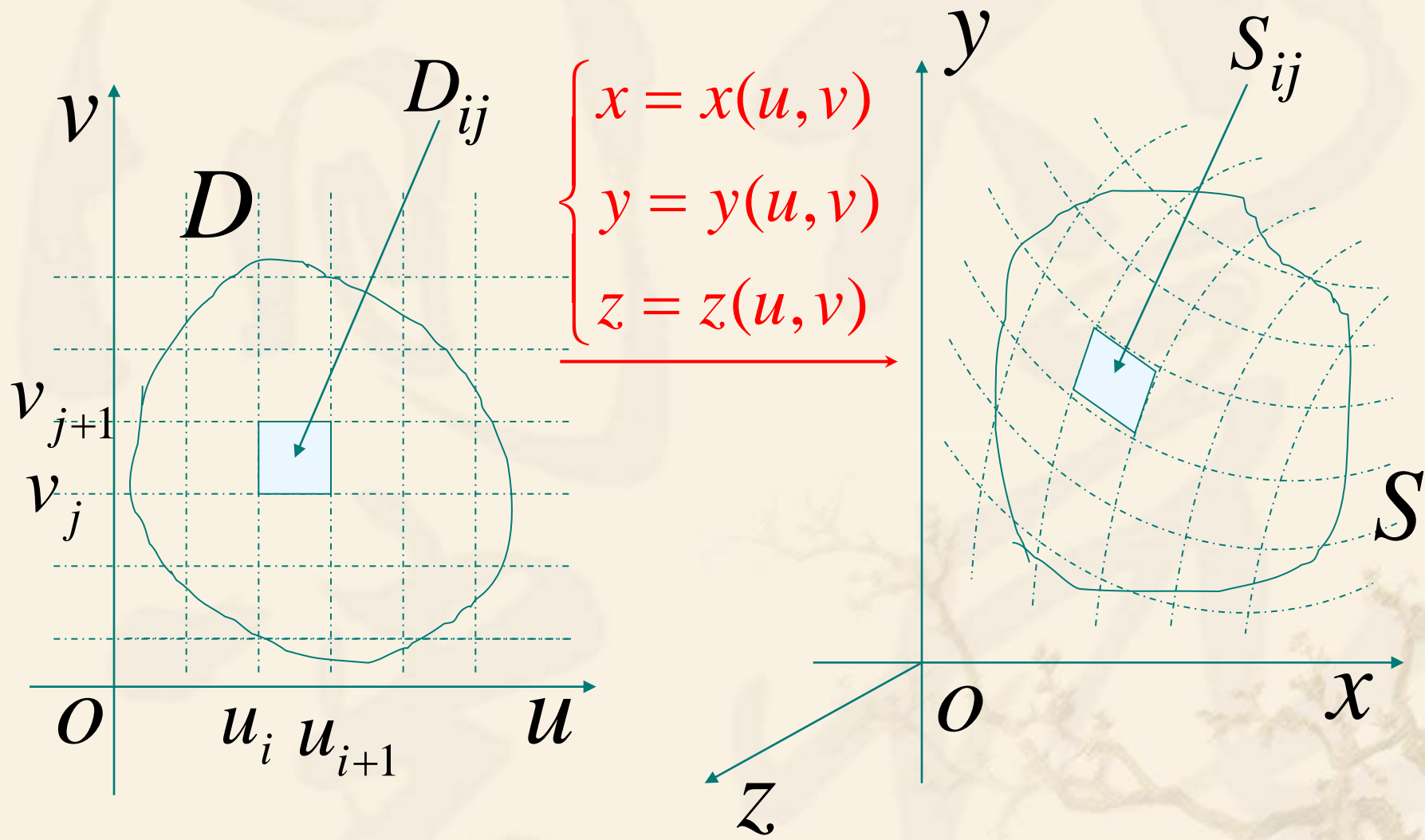
$$S : r = r(u,v), (u,v) \in D.$$

在$ouv$平面上,用平行于坐标轴的直线

$$u = u_i (i = 1, 2, \cdots, n), v = v_j (j = 1, 2, \cdots, m)$$

将区域$D$分割成若干小矩形$D_{ij}$.

$v$

$D$

$D_{ij}$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$v_{j+1}$

$v_j$

$O$ $u_i$ $u_{i+1}$ $u$

$y$

$S_{ij}$

$S$

$O$ $x$

$z$

$D_{ij}$的顶点为$(u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1}), (u_{i+1}, v_{j+1})$.

对应地,空间曲边四边形$S_{ij}$的四个顶点为

$$P_{ij}(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j)),$$

$$P_{i+1,j}(x(u_{i+1}, v_j), y(u_{i+1}, v_j), z(u_{i+1}, v_j)),$$

$$P_{i,j+1}(x(u_i, v_{j+1}), y(u_i, v_{j+1}), z(u_i, v_{j+1})),$$

$$P_{i+1,j+1}(x(u_{i+1}, v_{j+1}), y(u_{i+1}, v_{j+1}), z(u_{i+1}, v_{j+1})).$$

$$\overrightarrow{P_{ij}P_{i+1,j}} \approx (x_u'(u_i, v_j), y_u'(u_i, v_j), z_u'(u_i, v_j))\Delta u_i$$

$$= r_u'(u_i, v_j)\Delta u_i$$

$$\overrightarrow{P_{ij}P_{i,j+1}} \approx r'_v(u_i, v_j)\Delta v_j.$$

当分划很细时, 空间曲面$S_{ij}$可近似地看成以线段

$P_{ij}P_{i+1,j}, P_{ij}P_{i,j+1}$为邻边的平行四边形, 其面积

$$\Delta S_{ij} \approx \left\| r'_u(u_i, v_j) \times r'_v(u_i, v_j) \right\| \Delta u_i \Delta v_j$$

$$= \left| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} \right|_{(u_i, v_j)} \Delta u_i \Delta v_j$$

即 $\Delta S_{ij} \approx \sqrt{A^2 + B^2 + C^2}\,\Delta u_i \Delta v_j$，其中

$$A = \det \left. \frac{\partial(y,z)}{\partial(u,v)} \right|_{(u_i,v_j)}, \quad B = \det \left. \frac{\partial(z,x)}{\partial(u,v)} \right|_{(u_i,v_j)},$$

$$C = \det \left. \frac{\partial(x,y)}{\partial(u,v)} \right|_{(u_i,v_j)}.$$

● 曲面$S : x = x(u,v),\, y = y(u,v),\, z = z(u,v),\, (u,v) \in D,$

的面积为

$$\iint_D \|r_u' \times r_v'\|\, dudv = \iint_D \sqrt{A^2 + B^2 + C^2}\, dudv.$$

● 若曲面$S$的方程为$z = f(x, y), (x, y) \in D$, 则

$$S: x = x, y = y, z = f(x, y), (x, y) \in D.$$

$$r'_x \times r'_y = \det \begin{pmatrix} i & j & k \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{pmatrix} = (-f'_x, -f'_y, 1)$$

$$A = \det \begin{pmatrix} 0 & f'_x \\ 1 & f'_y \end{pmatrix} = -f'_x, B = -f'_y, C = 1.$$

曲面$S$的面积为$\iint_D \sqrt{1 + f'^2_x + f'^2_y} \, dxdy.$

例: 求球面$S: x^2 + y^2 + z^2 = R^2$的面积.

解: 球面$S$的参数方程为

$$x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\varphi,$$

$$(0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi)$$

则 $A = \det\dfrac{\partial(y,z)}{\partial(\varphi,\theta)} = \det\begin{pmatrix} R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \\ -R\sin\varphi & 0 \end{pmatrix}$

$$= R^2\sin^2\varphi\cos\theta,$$

$B = \det\dfrac{\partial(z,x)}{\partial(\varphi,\theta)} = \det\begin{pmatrix} -R\sin\varphi & 0 \\ R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta \end{pmatrix}$

$$= R^2\sin^2\varphi\sin\theta,$$

$$C = \det \frac{\partial(x, y)}{\partial(\varphi, \theta)} = \det \begin{pmatrix} R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta \\ R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \end{pmatrix}$$

$$= R^2 \sin\varphi\cos\varphi,$$

球面S的面积为

$$\iint\limits_{S} dS = \iint\limits_{\substack{0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi}} \sqrt{A^2 + B^2 + C^2}\, d\varphi d\theta$$

$$= \iint\limits_{\substack{0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi}} R^2 \sin\varphi\, d\varphi d\theta$$

$$= R^2 \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{2\pi} d\theta = 4\pi R^2 \square$$

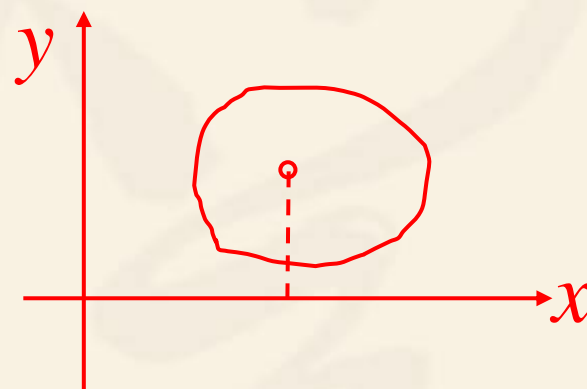## 2. 物体的质心

关于$x$轴的力矩微元为

$$y\,\mu(x,y)dxdy$$

●平板$D$的质心$(\bar{x},\bar{y})$

平板密度$\mu(x,y)$

平板质量$M = \iint_D \mu(x,y)dxdy$

平板关于$x$轴的静力矩为$M\bar{y} = \iint_D y\mu(x,y)dxdy$

故 $\bar{y} = \dfrac{\iint_D y\mu(x,y)dxdy}{\iint_D \mu(x,y)dxdy}$, $\bar{x} = \dfrac{\iint_D x\mu(x,y)dxdy}{\iint_D \mu(x,y)dxdy}$

●空间物体$\Omega$的质心$(\bar{x}, \bar{y}, \bar{z})$

密度$\mu(x, y, z)$, 质量$M = \iiint_\Omega \mu(x, y, z)dxdydz$

$\Omega$关于$yz$平面的静力矩为

$$M\bar{x} = \iiint_\Omega x\mu(x, y, z)dxdydz$$

故

$$\bar{x} = \frac{\iiint_\Omega x\mu(x, y, z)dxdydz}{\iiint_\Omega \mu(x, y, z)dxdydz},$$

$$\bar{y} = \frac{\iiint_\Omega y\mu(x, y, z)dv}{\iiint_\Omega \mu(x, y, z)dv}, \bar{z} = \frac{\iiint_\Omega z\mu(x, y, z)dv}{\iiint_\Omega \mu(x, y, z)dv}.$$

## 3. 转动惯量

● 位于 $(x, y, z)$ 处质量为 $m$ 的质点, 绕 $x, y, z$ 轴的转动惯量分别为 $m(y^2 + z^2), m(z^2 + x^2), m(x^2 + y^2)$.

● $\Omega \subset \mathbb{R}^3$, 密度 $\rho(x, y, z)$, 绕坐标轴的转动惯量为

$$J_x = \iiint_\Omega (y^2 + z^2) \rho(x, y, z) dxdydz,$$

$$J_y = \iiint_\Omega (z^2 + x^2) \rho(x, y, z) dxdydz,$$

$$J_z = \iiint_\Omega (x^2 + y^2) \rho(x, y, z) dxdydz.$$

Question. $\Omega$ 绕直线 $l$ 的转动惯量?

# 4. 万有引力

- 位于$P(x, y, z), P_0(x_0, y_0, z_0)$的两质点,质量分别为$m$, $m_0$.记$r = \|PP_0\|$, $\overrightarrow{P_0P}$与$x, y, z$正半轴夹角为$\alpha, \beta, \gamma, m$对 $m_0$的万有引力的大小为$\dfrac{kmm_0}{r^2}$, 引力沿$x, y, z$轴的分

量分别为

$$F_x = \frac{kmm_0}{r^2}\cos\alpha = \frac{kmm_0(x-x_0)}{r^3},$$

$$F_y = \frac{kmm_0}{r^2}\cos\beta = \frac{kmm_0(y-y_0)}{r^3},$$

$$F_z = \frac{kmm_0}{r^2}\cos\beta = \frac{kmm_0(z-z_0)}{r^3}.$$

- 密度为$\rho(x, y, z)$的物体$\Omega$对$P_0(x_0, y_0, z_0) \notin \Omega$处 质量为$m_0$的质点的万有引力:

$$F_x = \iiint_\Omega \frac{km_0(x-x_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)^3},$$

$$F_y = \iiint_\Omega \frac{km_0(y-y_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)^3},$$

$$F_z = \iiint_\Omega \frac{km_0(z-z_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)^3},$$

例: 半径为$R$, 质量为M的均匀球体$x^2 + y^2 + z^2 \leq R^2$

对点$P(0,0,a)$ $(a > R)$处质量为$m$的质点的引力.

解: $F_x = \iiint_\Omega \dfrac{kmx\rho dxdydz}{\left(\sqrt{x^2 + y^2 + (a-z)^2}\right)^3} = 0,\ F_y = 0.$

$F_z = \iiint_\Omega \dfrac{km(a-z)\rho dxdydz}{\left(\sqrt{x^2 + y^2 + (a-z)^2}\right)^3},\quad \dfrac{4}{3}\pi R^3 \rho = M.$

在柱坐标系$x = r\cos\theta,\ y = r\sin\theta,\ z = z$下,

$F_z = \displaystyle\int_{-R}^{R} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2 - z^2}} \dfrac{km(a-z)\rho rdr}{(\sqrt{r^2 + (a-z)^2})^3}$

$$F_z = \int_{-R}^{R} dz \int_{0}^{2\pi} d\theta \int_{0}^{\sqrt{R^2-z^2}} \frac{km(a-z)\rho r dr}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= k\pi m\rho \int_{-R}^{R} (a-z)dz \int_{0}^{\sqrt{R^2-z^2}} \frac{dr^2}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= 2k\pi m\rho \int_{-R}^{R} \left( 1 - \frac{a-z}{\sqrt{R^2+a^2-2az}} \right) dz$$

$$= \frac{4k\pi m\rho R^3}{3a^2} = \frac{kMm}{a^2}. \square \qquad \text{(分部积分)}$$

Question. 密度分别为$\rho_1(x, y, z), \rho_2(x, y, z)$的两物体$\Omega_1, \Omega_2$之间的万有引力？

作业: 习题3.5 No. 1, 9