Review

- 二重积分的几何与物理意义
- 二重积分的定义

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{\lambda(T)\to 0}\sum_{i=1}^{n}\sum_{j=1}^{k} f(\xi_{ij},\eta_{ij})\Delta x_i \Delta y_j .$$

$$\iint\limits_{D} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{I=[a,b]\times[c,d](\supset D)} f_I(x,y)\mathrm{d}x\mathrm{d}y.$$

- 二重积分的性质

- 可积条件

Thm. $D=[a,b]\times[c,d]$,则

(1) $f\in R(D)\Rightarrow f$在D上有界;

(2) $f\in C(D)\Rightarrow f\in R(D)$;

(3) f在D上的间断点集为零面积集$\Rightarrow f\in R(D)$.

Thm. $D\subset\mathbb{R}^2$为有界闭集, f为D上有界函数.若f在D上的间断点集为零面积集,∂D为零面积集,则$f\in R(D)$.

§2. 二重积分的计算

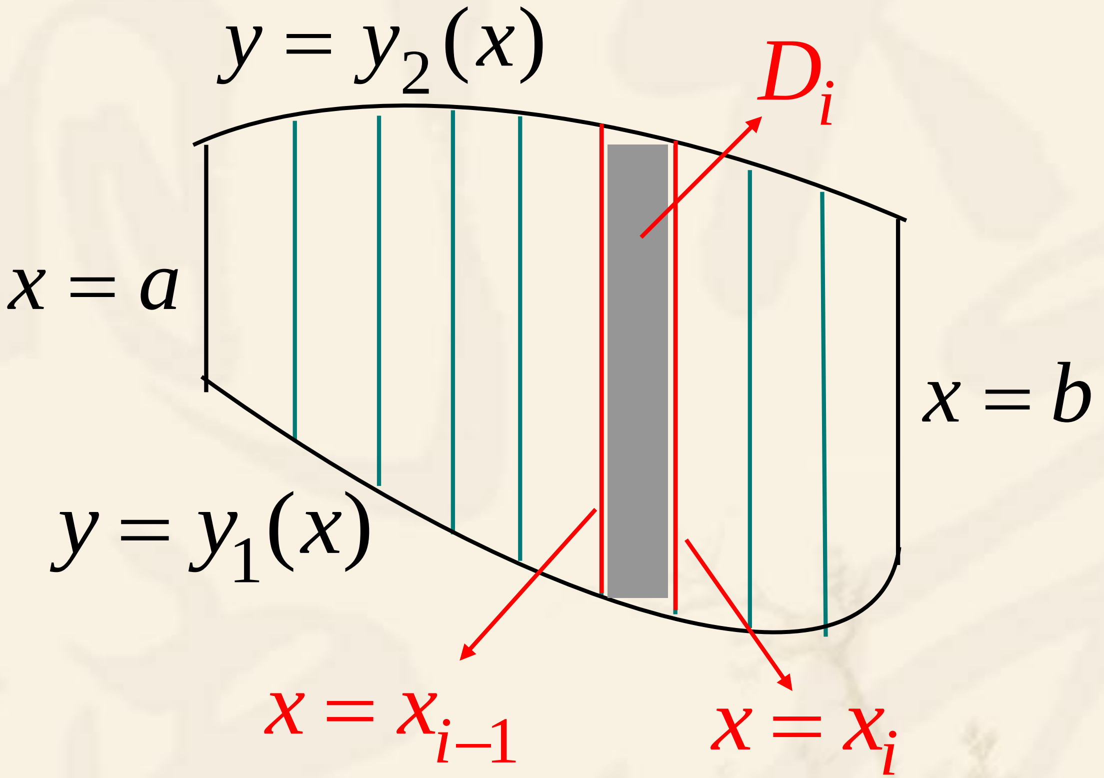
- 直角坐标下二重积分的计算及例题
- 极坐标下二重积分的计算及例题
- 补充例题

1. 用直角坐标系计算二重积分

$$S: z = f(x,y), (x,y) \in D.$$

换一个思路来计算以D为下底,以S为顶的曲顶柱体Ω的体积 $V(\Omega) = \iint_D f(x,y)\mathrm{d}x\mathrm{d}y.$

设$D=\{(x,y)\,|\,a\le x\le b, y_1(x)\le y\le y_2(x)\}$.



- Step1.对D进行分划: $a=x_0<x_1<\cdots<x_n=b$,

将D分成平行于y轴的细条$D_1, D_2, \cdots, D_n$.

相应地,Ω被平行于OYZ平面的平面$x = x_i$切成薄片$\Omega_1, \Omega_2, \cdots, \Omega_n$.

- Step2.求近似和

曲顶柱体Ω中截面$x = x$的面积为

$$\mathrm{A}(x) = \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y.$$

于是薄片Ω_i的体积近似为

$$\mathrm{V}(\Omega_i) \approx \mathrm{A}(x_i)(x_{i+1} - x_i) = \mathrm{A}(x_i)\Delta x_i.$$

曲顶柱体的体积近似为 $\mathrm{V}(\Omega) \approx \sum_{i=1}^{n} \mathrm{A}(x_i)\Delta x_i.$

•Step3.取极限　当分划越来越细时,

$$\sum_{i=1}^{n} \mathrm{A}(x_i)\Delta x_i \to \mathrm{V}(\Omega).$$

综上,

$$\mathrm{V}(\Omega) = \int_a^b \mathrm{A}(x)\mathrm{d}x = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y \right)\mathrm{d}x,$$

即

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y \right)\mathrm{d}x$$

$$\triangleq \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y. \qquad (*)$$

Remark:等式后两项的意义是, 先固定x(视x为常数), 对变量y求定积分

$$A(x)=\int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y,$$

再让x变起来, 对变量x求定积分

$$\int_a^b A(x)\mathrm{d}x.$$

正因为如此,(*)式右端的积分也称为先y后x的累次积分.

Remark:对称地, 若区域D具有如下形式:

$$D=\{(x,y)\,|\,c\le y\le d,\, x_1(y)\le x\le x_2(y)\}.$$

则

$$\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y=\int_c^d\left(\int_{x_1(y)}^{x_2(y)} f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

$$\triangleq\int_c^d \mathrm{d}y\int_{x_1(y)}^{x_2(y)} f(x,y)\mathrm{d}x.$$

Remark:对于一般的区域D, 可以分成若干个具有以上两种形式的区域, 并将二重积分利用区域可加性化为累次积分来计算.

Thm. 设$f(x,y)$在有界闭区域D上连续, 若

$$D=\left\{(x,y)\middle|a\le x\le b, y_1(x)\le y\le y_2(x)\right\},$$

其中$y_1(x), y_2(x)\in C\left([a,b]\right)$.则

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y=\int_a^b \mathrm{d}x\int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y.$$

若$D=\left\{(x,y)\middle|c\le y\le d, x_1(y)\le x\le x_2(y)\right\}$,

其中$x_1(y), x_2(y)\in C\left([c,d]\right)$.则

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y=\int_c^d \mathrm{d}y\int_{x_1(y)}^{x_2(y)} f(x,y)\mathrm{d}x. \square$$

Remark:将二重积分化为累次积分计算时,选择不同的积分次序,难易程度可能相差很大.一般应根据被积函数和积分区域选择合适的累次积分次序.

例: 求$I=\iint_{x^2+y^2\le a^2} y^2\sqrt{a^2-x^2}\,\mathrm{d}x\mathrm{d}y.$

解:积分区域为$x\in[-a,a], y\in[-\sqrt{a^2-x^2},\sqrt{a^2-x^2}].$

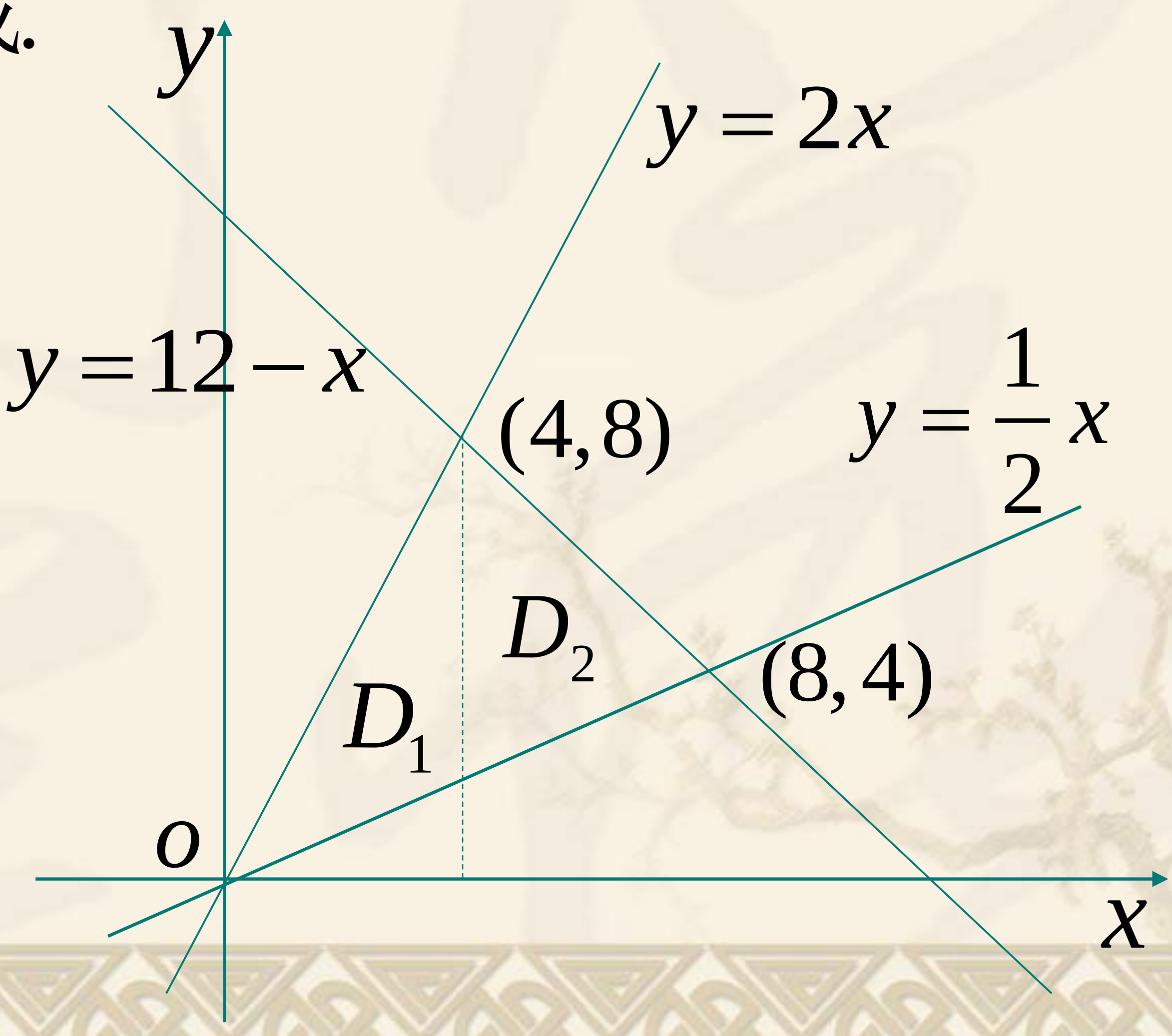
$$I=\int_{-a}^{a}\mathrm{d}x\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\sqrt{a^2-x^2}\,\mathrm{d}y$$

$$=\int_{-a}^{a}\sqrt{a^2-x^2}\,\mathrm{d}x\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,\mathrm{d}y$$

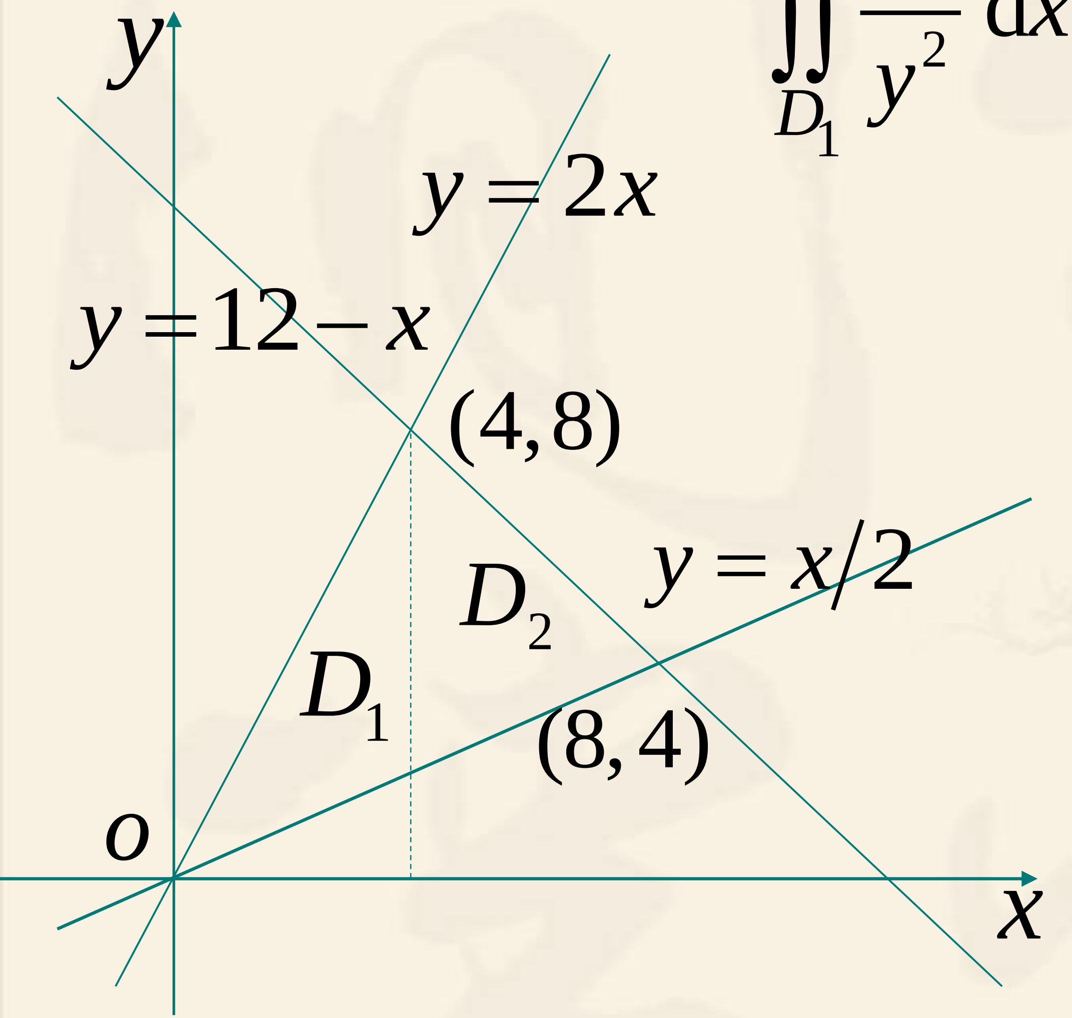
$$=\int_{-a}^{a}\sqrt{a^2-x^2}\left(\frac{1}{3}y^3\Big|_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\right)\mathrm{d}x$$

$$=\frac{2}{3}\int_{-a}^{a}\left(a^2-x^2\right)^2\mathrm{d}x=\frac{32}{45}a^5.\ \square$$

例: 求$I=\iint\limits_{D}\frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y$, 其中$D$由直线$y=2x$, $y=\frac{1}{2}x$及$y=12-x$围成.



解: 如图, 区域D可以分成D_1, D_2两部分.



$$\iint_{D_1}\frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y=\int_0^4\mathrm{d}x\int_{\frac{1}{2}x}^{2x}\frac{x^2}{y^2}\,\mathrm{d}y$$

$$=\int_0^4\left(-\frac{x^2}{y}\Bigg|_{y=\frac{1}{2}x}^{y=2x}\right)\mathrm{d}x$$

$$=\int_0^4 x^2\left(\frac{2}{x}-\frac{1}{2x}\right)\mathrm{d}x$$

$$=12,$$

$$\iint_{D_2} \frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y = \int_4^8 \mathrm{d}x \int_{\frac{1}{2}x}^{12-x} \frac{x^2}{y^2}\,\mathrm{d}y$$

$$= \int_0^4 x^2\left(\frac{2}{x} - \frac{1}{12-x}\right)\mathrm{d}x = 120 - 144\ln 2.$$

于是 $\iint_{D} \frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y = \iint_{D_1} \frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y + \iint_{D_2} \frac{x^2}{y^2}\,\mathrm{d}x\mathrm{d}y$

$$= 132 - 144\ln 2. \quad \square$$

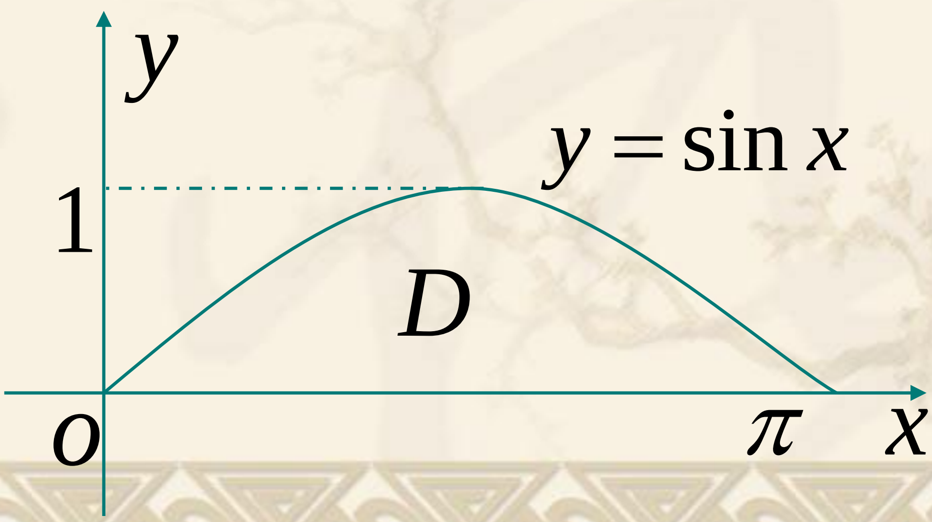
例: 求$I=\int_0^1 \mathrm{d}y\int_{\arcsin y}^{\pi-\arcsin y} x\mathrm{d}x.$

分析:按所给积分次序,内层积分容易求出,但再积分就困难了.所以尝试交换积分次序.

解: $I=\int_0^{\pi} x\mathrm{d}x\int_0^{\sin x}\mathrm{d}y$

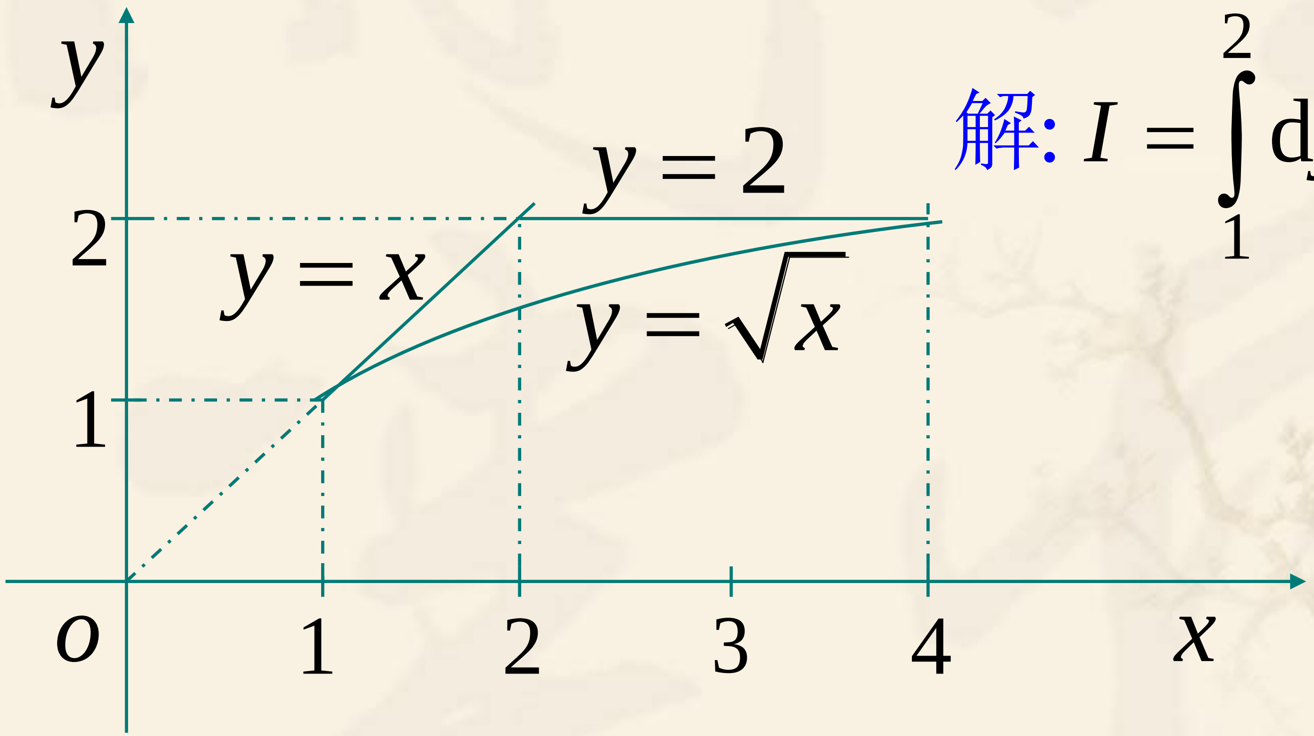
$=\int_0^{\pi} x\sin x\mathrm{d}x=-\int_0^{\pi} x\,\mathrm{d}\cos x$

$=-x\cos x\Big|_{x=0}^{\pi}+\int_0^{\pi}\cos x\mathrm{d}x=\pi.$ □



例: $I=\int_{1}^{2}\mathrm{d}x\int_{\sqrt{x}}^{x}\sin\frac{\pi x}{2y}\mathrm{d}y+\int_{2}^{4}dx\int_{\sqrt{x}}^{2}\sin\frac{\pi x}{2y}\mathrm{d}y.$

分析: 里层积分困难, 考虑交换积分次序.

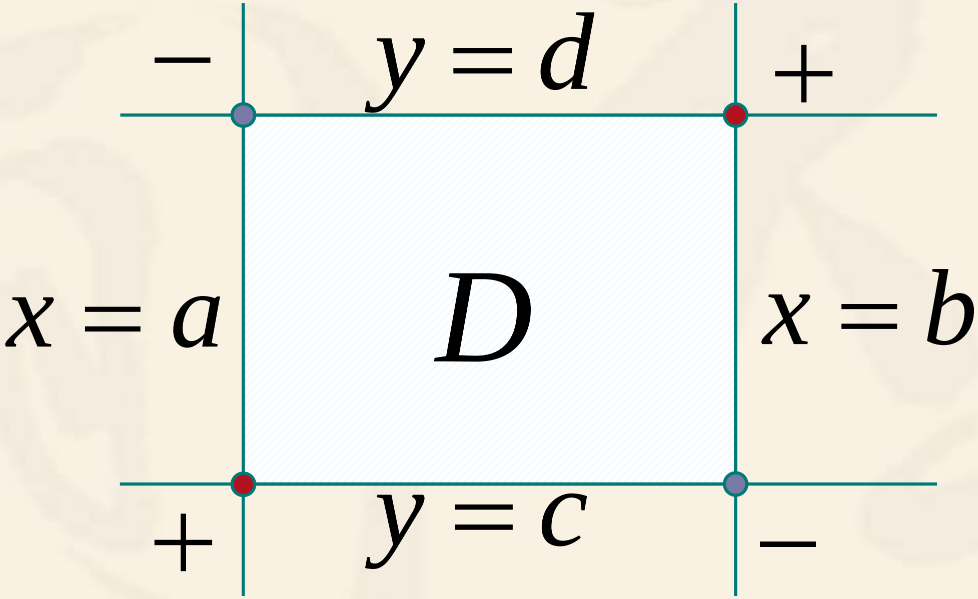


解: $I=\int_{1}^{2}\mathrm{d}y\int_{y}^{y^2}\sin\frac{\pi x}{2y}\mathrm{d}x$

$$I=\int_1^2 \mathrm{d}y\int_y^{y^2}\sin\frac{\pi x}{2y}\mathrm{d}x$$

$$=\frac{2}{\pi}\int_1^2 y\left(\cos\frac{\pi}{2}-\cos\frac{\pi y}{2}\right)\mathrm{d}y$$

$$=-\frac{2}{\pi}\int_1^2 y\cos\frac{\pi y}{2}\mathrm{d}y=4(2+\pi)/\pi^3\ .\square$$



例: 设 $\dfrac{\partial^2 f}{\partial x\partial y}$ 在 $D=[a,b]\times[c,d]$ 上可积, 则

$$\iint_D \frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}x\mathrm{d}y = f(b,d)-f(b,c)-f(a,d)+f(a,c).$$

证明: $$\iint_D \frac{\partial^2 f}{\partial x \partial y} \mathrm{d}x\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b \frac{\partial^2 f}{\partial x \partial y} \mathrm{d}x$$

$$= \int_c^d \left[\left. \frac{\partial f(x,y)}{\partial y} \right|_{x=a}^{b} \right] \mathrm{d}y$$

$$= \int_c^d \frac{\partial f(b,y)}{\partial y} \mathrm{d}y - \int_c^d \frac{\partial f(a,y)}{\partial y} \mathrm{d}y$$

$$= f(b,y)\Big|_{y=c}^{d} - f(a,y)\Big|_{y=c}^{d}$$

$$= f(b,d) - f(b,c) - f(a,d) + f(a,c). \quad \square$$

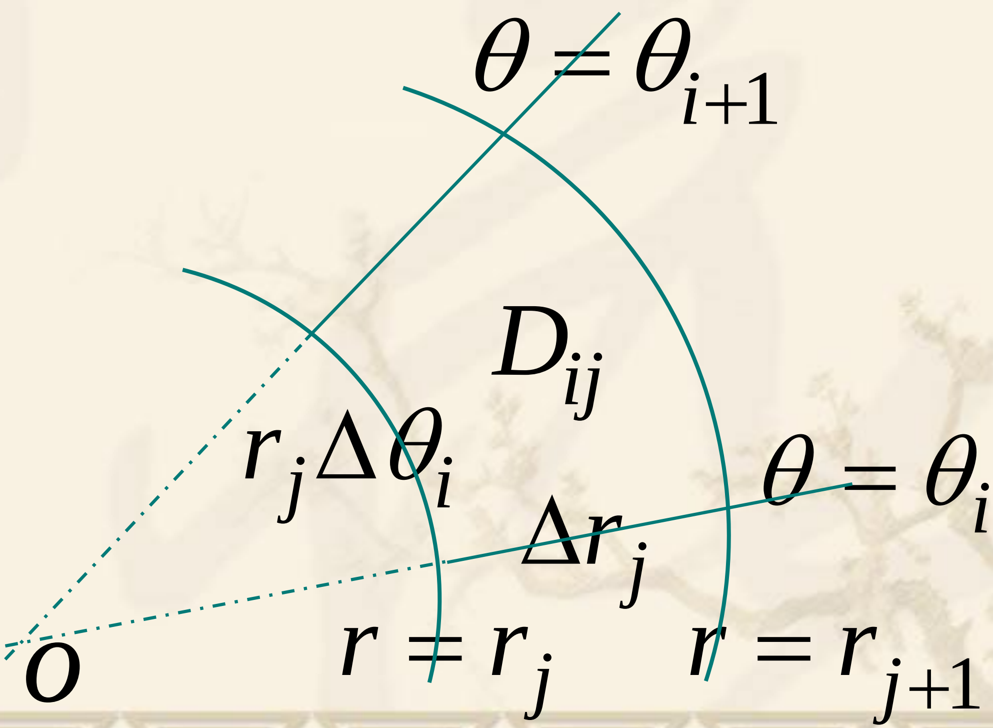
2. 用极坐标系计算二重积分

在直角坐标系下将二重积分化为累次积分来计算,如果被积区域D的形状不好,或者被积函数的表达式比较复杂,那么累次积分的计算将很复杂,甚至可能计算不出结果来.

再换一个思路来计算以D为底,以曲面$S: z = f(x, y), (x, y) \in D$为顶的曲顶柱体的$\Omega$体积$\mathrm{V}(\Omega) = \iint\limits_{D} f(x, y)\mathrm{d}x\mathrm{d}y.$

用过原点的射线$\theta=\theta_i(i=1,2,\cdots,n)$和以原点为圆心的同心圆$r=r_j(j=1,2,\cdots,m)$对区域$D$作分划.忽略位于区域$D$边界的那些不规则的小区域,考虑由$\theta=\theta_i,\theta=\theta_{i+1},r=r_j$和$r=r_{j+1}$围成的曲边四边形$D_{ij}$. 当$\Delta r_j=r_{j+1}-r_j$, $\Delta\theta_i=\theta_{i+1}-\theta_i$很小时, D_{ij}近似为矩形,边长分别为Δr_j和$r_j\Delta\theta_i$.

$$\sigma(D_{ij})\approx r_j\Delta\theta_i\Delta r_j$$



于是 $\mathrm{V}(\Omega) \approx \sum_{1\le i\le n, 1\le j\le m} \sigma(D_{ij}) f(r_j\cos\theta_i, r_j\sin\theta_i)$

$$\approx \sum_{1\le i\le n, 1\le j\le m} r_j \Delta\theta_i \Delta r_j f(r_j\cos\theta_i, r_j\sin\theta_i).$$

当分划越来越细时, 有.

$$\sum_{i,j} r_j \Delta\theta_i \Delta r_j f(r_j\cos\theta_i, r_j\sin\theta_i) \to \mathrm{V}(\Omega).$$

设E是原积分区域D在极坐标下的表示, 即

$$E = \{(r,\theta) \mid (r\cos\theta, r\sin\theta) \in D\}.$$

则 $\mathrm{V}(\Omega) = \iint_E f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta.$

即 $\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_E f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta.$

Remark:于是在极坐标系下面积微元为$\mathrm{d}\sigma = r\mathrm{d}r\mathrm{d}\theta$.

若$E = \{(r,\theta) \mid \alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta)\}$,则

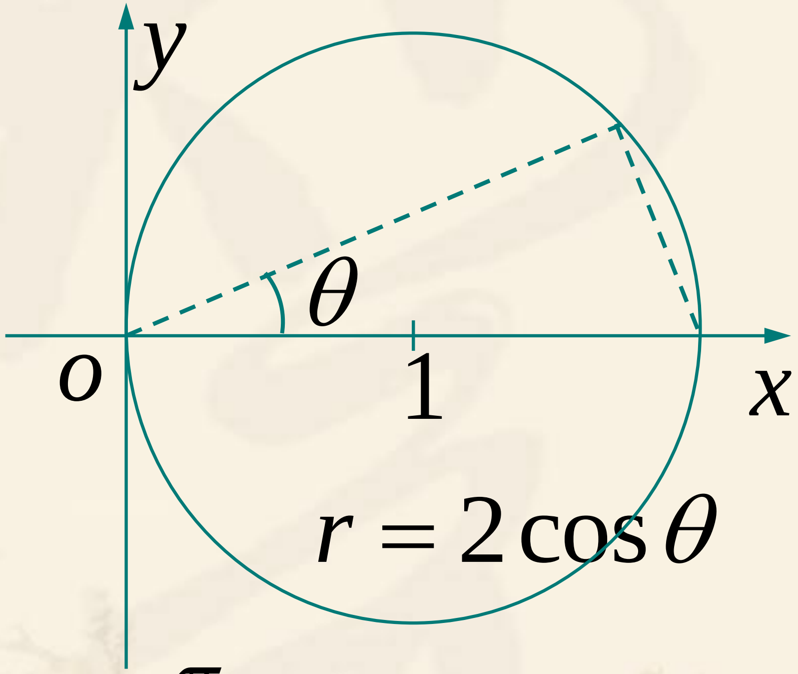
$$\iint_E f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta$$

$$= \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\mathrm{d}r.$$

于是,我们可以将二重积分化为极坐标下的累次积分来计算.

例：求$I=\iint_{x^2+y^2\le 2x}(y+\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y.$

解：积分区域关于OX轴对称，



故 $$\iint_{x^2+y^2\le 2x} y\,dxdy=0,$$

$$I=\iint_{x^2+y^2\le 2x}\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y.$$

极坐标下，积分区域为$\{-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, r^2\le 2r\cos\theta\}.$

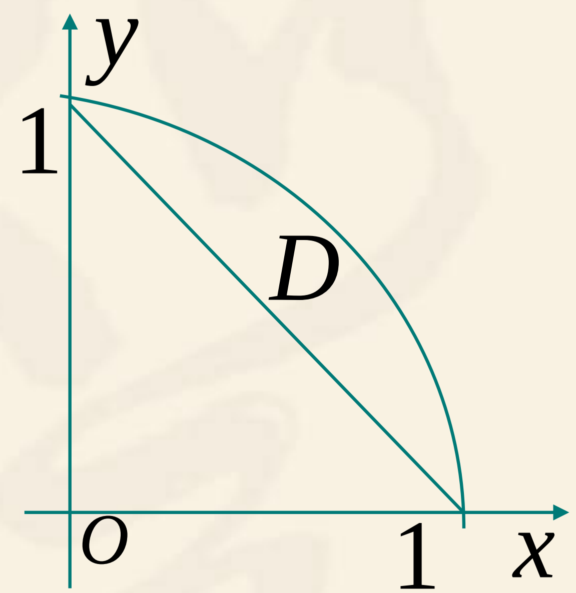
故 $$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}r^2\mathrm{d}r.$$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_{0}^{2\cos\theta}r^2\mathrm{d}r=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\left(\frac{1}{3}r^3\right)\Bigg|_{r=0}^{2\cos\theta}$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\mathrm{d}\theta=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\theta)\mathrm{d}\sin\theta$$

$$=\frac{8}{3}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{32}{9}.\ \square$$

例:求$I=\iint\limits_{x^2+y^2\le 1,x+y>1}\frac{x+y}{x^2+y^2}\mathrm{d}x\mathrm{d}y.$



解:极坐标下积分区域为

$$0\le\theta\le\frac{\pi}{2},\frac{1}{\sin\theta+\cos\theta}\le r\le 1.$$

$$I=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^{1}\frac{r\sin\theta+r\cos\theta}{r^2}\cdot r\mathrm{d}r$$

$$=\int_0^{\frac{\pi}{2}}\left(\sin\theta+\cos\theta-1\right)\mathrm{d}\theta=2-\frac{\pi}{2}.\ \square$$

例. 求$I=\iint_{x^2+y^2\le 1}(x^2+xy+2y^2)\mathrm{d}x\mathrm{d}y.$

解: $\iint_{x^2+y^2\le 1} xy\mathrm{d}x\mathrm{d}y=0$

$$\iint_{x^2+y^2\le 1} x^2\mathrm{d}x\mathrm{d}y=\iint_{x^2+y^2\le 1} y^2\mathrm{d}x\mathrm{d}y$$ (轮换不变性)

$$I=\iint\limits_{x^2+y^2\le 1}(x^2+2y^2)\mathrm{d}x\mathrm{d}y=\frac{3}{2}\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\mathrm{d}x\mathrm{d}y$$

$$=\frac{3}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r=\frac{3\pi}{4}.\square$$

例:求Poisson积分$I=\int_{-\infty}^{+\infty}e^{-x^2}\mathrm{d}x$.

解:令$I(R)=\int_{-R}^{+R}e^{-x^2}\mathrm{d}x$,则$\mathrm{I}(R)>0$.

$$I^2(R)=\int_{-R}^{+R}e^{-x^2}\mathrm{d}x\int_{-R}^{+R}e^{-y^2}\mathrm{d}y$$

$$=\iint_{-R\le x,y\le R}e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

于是,$\iint_{x^2+y^2\le R^2}e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\le\mathrm{I}^2(R)$

$$\le\iint_{x^2+y^2\le 2R^2}e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

而$\iint_{x^2+y^2\le R^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\mathrm{d}\theta\int_0^R re^{-r^2}\mathrm{d}r$

$$= 2\pi\cdot\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_{r=0}^{R} = \pi(1-e^{-R^2}).$$

同理，$\iint_{x^2+y^2\le 2R^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \pi(1-e^{-2R^2}).$

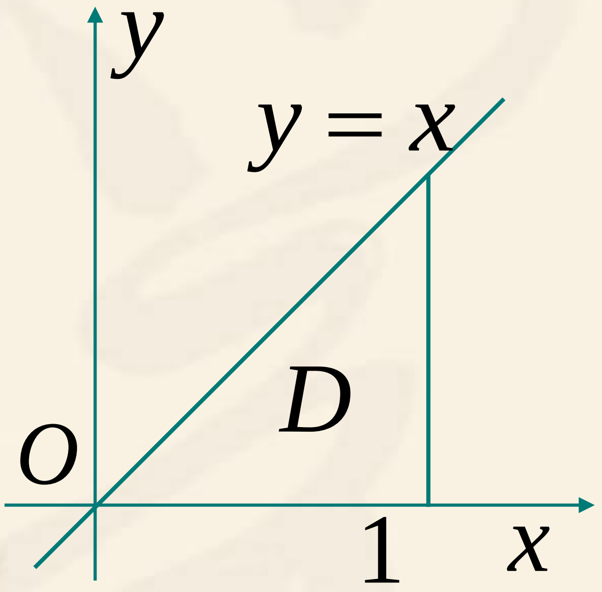
所以 $\pi(1-e^{-R^2}) \le \mathrm{I}^2(R) \le \pi(1-e^{-2R^2}).$

由夹挤原理，$\lim\limits_{R\to+\infty}\mathrm{I}^2(R) = \pi.$

故$I = \lim\limits_{R\to\infty}\mathrm{I}(R) = \sqrt{\pi}.$ □

3. 补充例题

*例: 求 $\mathrm{I}=\int_0^1 \frac{\ln(1+x)}{(2-x)^2}\mathrm{d}x.$



解:

$$\mathrm{I}=\int_0^1 \frac{1}{(2-x)^2}\left(\int_0^x \frac{1}{1+y}\mathrm{d}y\right)\mathrm{d}x$$

$$=\int_0^1 \frac{1}{(2-x)^2}\mathrm{d}x\int_0^x \frac{1}{1+y}\mathrm{d}y$$

$$=\int_0^1 \frac{1}{1+y}\mathrm{d}y\int_y^1 \frac{1}{(2-x)^2}\mathrm{d}x \text{ (交换积分次序)}$$

$$=\int_0^1 \frac{(1-y)\mathrm{d}y}{(1+y)(2-y)}$$

$$=\frac{2}{3}\int_0^1 \frac{\mathrm{d}y}{1+y}+\frac{1}{3}\int_0^1 \frac{\mathrm{d}y}{2-y}=\frac{1}{3}\ln 2. \square$$

Remark:将一元函数的定积分化成二重积分计算,有时候可能会更简单.

$*$例: $\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \le \int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(x)\mathrm{d}x.$

证明: 记$D=[a,b]\times[a,b]$.

$$0 \le \iint_D [f(x)g(y)-f(y)g(x)]^2\, dxdy$$

$$= \iint_D f^2(x)g^2(y)\mathrm{d}x\mathrm{d}y + \iint_D f^2(y)g^2(x)\mathrm{d}x\mathrm{d}y$$

$$-2\iint_D f(x)f(y)g(x)g(y)\mathrm{d}x\mathrm{d}y$$

$$= 2\int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(y)\mathrm{d}y$$

$$-2\int_a^b f(x)g(x)\mathrm{d}x \int_a^b f(y)g(y)\mathrm{d}y$$

$$= 2\int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(x)\mathrm{d}x - 2\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2. \square$$

*例: $f(x)\in C[0,1],\ f>0,\ f\downarrow$.求证

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x}\le\frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

证明: 只要证 $\mathrm{I}=\int_0^1 xf^2(x)\mathrm{d}x\int_0^1 f(x)\mathrm{d}x$

$$-\int_0^1 xf(x)\mathrm{d}x\int_0^1 f^2(x)\mathrm{d}x\le 0.$$

定积分与积分变量所用字母无关, 故

$$I=\int_0^1 xf^2(x)\mathrm{d}x\int_0^1 f(y)\mathrm{d}y-\int_0^1 xf(x)\mathrm{d}x\int_0^1 f^2(y)\mathrm{d}y$$

即 $$I=\iint_{0\le x,y\le 1} xf^2(x)f(y)\mathrm{d}x\mathrm{d}y - \iint_{0\le x,y\le 1} xf(x)f^2(y)\mathrm{d}x\mathrm{d}y$$

$$=\iint_{0\le x,y\le 1} xf(x)f(y)\left[f(x)-f(y)\right]\mathrm{d}x\mathrm{d}y$$

由于积分区域关于直线$y=x$对称,

$$I=\iint_{0\le x,y\le 1} yf(x)f(y)\left[f(y)-f(x)\right]\mathrm{d}x\mathrm{d}y$$

两式相加,由$f>0$, $f\downarrow$,得

$$2I=\iint_{0\le x,y\le 1} (x-y)f(x)f(y)\left[f(x)-f(y)\right]\mathrm{d}x\mathrm{d}y \le 0. \square$$

*例: 设$D=\{(x,y)\mid 0\le x,y\le 1\}$, $z=f(x,y)$ $\in C^2(D)$.若

$$\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|\le 4,\quad \forall (x,y)\in D$$

$$f(x,y)\equiv f_x'(x,y)\equiv 0,\quad \forall (x,y)\in \partial D,$$

则$\left|\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y\right|\le 1.$

证明: $\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^1 f(x,y)\mathrm{d}x$

(分部积分) $= \int_0^1 \left[x f(x,y)\Big|_{x=0}^{1} - \int_0^1 x \frac{\partial f(x,y)}{\partial x} \mathrm{d}x \right] \mathrm{d}y$

0 $= -\int_0^1 \mathrm{d}y \int_0^1 x \frac{\partial f}{\partial x} \mathrm{d}x = -\int_0^1 x\mathrm{d}x \int_0^1 \frac{\partial f}{\partial x} \mathrm{d}y$

(分部积分) $= -\int_0^1 x \left[y \frac{\partial f}{\partial x}\Big|_{y=0}^{1} - \int_0^1 y \frac{\partial^2 f}{\partial x \partial y} \mathrm{d}y \right] \mathrm{d}x$

$$=\int_0^1 x\mathrm{d}x\int_0^1 y\frac{\partial^2 f}{\partial x\partial y}\mathrm{d}y=\iint_D xy\frac{\partial^2 f}{\partial x\partial y}\mathrm{d}x\mathrm{d}y$$

于是

$$\left|\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\right|=\left|\iint_D xy\frac{\partial^2 f}{\partial x\partial y}\mathrm{d}x\mathrm{d}y\right|$$

$$\le\iint_D\left|xy\frac{\partial^2 f}{\partial x\partial y}\right|\mathrm{d}x\mathrm{d}y\le 4\iint_D xy\mathrm{d}x\mathrm{d}y$$

$$=4\int_0^1 x\mathrm{d}x\int_0^1 y\mathrm{d}y=1.\square$$

作业：习题3. 3 No. 5, 6, 11

No.6(2) $D=\left\{(x,y)\left|\begin{array}{c}(x-a)^2+(y-a)^2\le a^2,\\ 0\le x,y\le a\end{array}\right.\right\}$

No.6(7) $D=\{(x,y)\,|\,0\le x,y\le \pi\}$