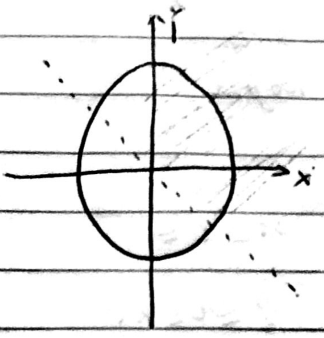
Quiz 2 Zhi Xu

See also the notebook

3. I calculate $P(X>0 \mid X+Y>0)$



$P(X+Y>0) = \frac{1}{2}$

$\Rightarrow P(X>0 \mid X+Y>0) = \frac{3}{4}$

4. (1) you should keep rolling

at 35 $\Rightarrow$ 36 w.p. $\frac{1}{6}$ $\Rightarrow$ \$0

37, 38, 39, 40, 41 w.p. $\frac{5}{6}$ $\Rightarrow$ one could at least keep rolling once more because the maximum you could get is $41+6=47$

no square number between 37 and 47

$\Rightarrow$ with another roll, one get at least $37+6=43$

$\Rightarrow E[\text{keep rolling}] \geq \frac{5}{6} \times 43 > 35$

(2) at 35 $\Rightarrow$ until excess 43, i.e. >43.

that is, one will stop at 44 45 46 47 48 49.

let $P(X)$ be the probability of getting to X

$P(49) = \frac{1}{6}P(43)$

$P(48) = \frac{1}{6}P(43) + \frac{1}{6}P(42)$

$P(47) = \frac{1}{6}P(43) + \frac{1}{6}P(42) + \frac{1}{6}P(41)$

$P(46) = \frac{1}{6}P(43) + \frac{1}{6}P(42) + \frac{1}{6}P(41) + \frac{1}{6}P(40)$

$P(45) = \frac{1}{6}P(43) + \frac{1}{6}P(42) + \frac{1}{6}P(41) + \frac{1}{6}P(40) + \frac{1}{6}P(39)$

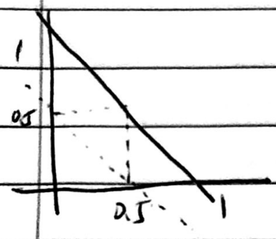
$P(44) = \frac{1}{6}P(43) + \frac{1}{6}P(42) + \frac{1}{6}P(41) + \frac{1}{6}P(40) + \frac{1}{6}P(39) + \frac{1}{6}P(38)$

$\Rightarrow 44$ is the most probable one

(3) I'm not sure. I would guess yes.

If we keep rolling, I think the probability of hitting a square number is 1? (not quite sure, need a more careful proof). That is, if we keep rolling the expected value is 0

5. Let's work out the longest one, the rest can be done similarly.
We follow the standard method. let L be the longest piece
we calculate the CDF of L, take derivative and then integral



$$L = \max\{X, Y, 1-X-Y\}$$

$$F_L(l) = P(L \le l) = \begin{cases} 1-3(1-l)^2 & 0.5 \le l \le 1 \\ (3l-1)^2 & \frac{1}{3} \le l \le 0.5 \\ 0 & l < \frac{1}{3} \end{cases}$$

$$f_L(l) = \frac{\partial F_L(l)}{\partial l} = \begin{cases} 6(1-l) & 0.5 \le l \le 1 \\ 6(3l-1) & \end{cases}$$

$$\Rightarrow E(L) = \int_{\frac{1}{3}}^{1} l \cdot f_L(l)\, dl$$

$$= \frac{11}{18}$$

6. (1) indicator variable $X_i = \begin{cases} 1 \\ 0 \end{cases}$ if the ith person gets his/her hat

$$E[Y] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = n \cdot \frac{1}{n} = 1$$

(2) $Var(Y) = Cov(Y, Y)$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} [E(X_i^2) - E(X_i)^2] + \sum_{i \neq j} E[X_i X_j] - E[X_i]E[X_j]$$

$$= n \times [\frac{1}{n} - \frac{1}{n^2}] + n(n-1)\left(\frac{1}{n(n-1)} - \frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{n} + 1 - \frac{n-1}{n}$$

$$= 1$$

(3) By induction. let $P(x)$ be the probability that exactly x people get the correct hat in the first round. Then

$$R(N) = \sum_{x=0}^{n} P(x)(1 + R(N-x))$$

similarly $S(N) = \sum_{x=0}^{n} P(x)(N + S(N-x))$

false selection $F(N) = \sum_{x=0}^{n} P(x)(N - x + F(N-x))$

I don't know how to solve the recursion, ∵ don't see an easy way to calculate $P(x)$