Associating Uncertainty to Extended Poses - Supplementary Material

Martin Brossard, Axel Barrau, Paul Chauchat, and Silvère Bonnabel

The present supplementary material provides detailed proofs along with comprehensive technical derivations for the referred paper [1]. The latter paper is self-contained, but this supplementary material might prove useful for the reader not being familiar with this kind of mathematical derivations. The present document also contains additional results and information for the interested reader.

A. Organization of the Supplementary Material

Section I details the derivations of proofs for some theoretical results of the paper. Section II provides the proofs of our results for IMU preintegration with rotating Earth. Section III shows how integrating IMU factor with piecewise constant global acceleration assumption or piecewise constant IMU measurements assumption. Section IV details preintegration residual and Jacobian with respect to state estimates in flat Earth model. Section V extends the approach of Section IV for tightly incorporating bias errors. Python scripts are available for numerically checking some parts of this document.

I. Additional Details of Proofs of Paper Results

A. Details of Proof of Equation (32) for Revisiting IMU Equations

We extensively used the property of equation (32) along the paper, that is recapped here for convenience as

$$\Phi\big(\mathbf{T}\exp(\boldsymbol{\xi})\big) = \Phi\left(\mathbf{T}\right)\exp\left(\mathbf{F}\boldsymbol{\xi}\right), \tag{1}$$

where

$$\mathbf{F} := \mathbf{F}_{\Delta t} := \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{I}_3 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \Delta t\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}, \Phi(\mathbf{T}) = \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{v} & \mathbf{p}+\Delta t\mathbf{v} \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]. \tag{2}$$

We proof (1) as follows. First, matrix computations bring the property that $\Phi(\cdot)$ is a group automorphism, i.e. $\Phi(\mathbf{T}^1\mathbf{T}^2) = \Phi(\mathbf{T}^1)\Phi(\mathbf{T}^2)$, as on the left we have

$$\begin{aligned} \Phi(\mathbf{T}^1\mathbf{T}^2) &= \Phi\left(\left[\begin{array}{c|c|c} \mathbf{R}^1\mathbf{R}^2 & \mathbf{R}^1\mathbf{v}^2+\mathbf{v}^1 & \mathbf{R}^1\mathbf{p}^2+\mathbf{p}^1 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]\right) \\ &= \left[\begin{array}{c|c|c} \mathbf{R}^1\mathbf{R}^2 & \mathbf{R}^1\mathbf{v}^2+\mathbf{v}^1 & \mathbf{R}^1\mathbf{p}^2+\mathbf{p}^1+(\mathbf{R}^1\mathbf{v}_2+\mathbf{v}_1)\Delta t \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right], \end{aligned} \tag{3}$$

such that

$$\begin{aligned} \Phi(\mathbf{T}^1)\Phi(\mathbf{T}^2) &= \left[\begin{array}{c|c|c} \mathbf{R}^1 & \mathbf{v}^1 & \mathbf{p}^1+\Delta t\mathbf{v}^1 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]\left[\begin{array}{c|c|c} \mathbf{R}^2 & \mathbf{v}^2 & \mathbf{p}^2+\Delta t\mathbf{v}_2 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right] \\ &= \left[\begin{array}{c|c|c} \mathbf{R}^1\mathbf{R}^2 & \mathbf{R}^1\mathbf{v}^2+\mathbf{v}^1 & \mathbf{p}^1+\Delta t\mathbf{v}^1+\mathbf{R}^1(\mathbf{p}^2+\Delta t\mathbf{v}^2) \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right] \\ &\overset{(3)}{=} \Phi(\mathbf{T}^1\mathbf{T}^2). \end{aligned} \tag{4}$$

M. Brossard, A. Barrau and S. Bonnabel are with MINES ParisTech, PSL Research University, Centre for Robotics, 60 Boulevard Saint-Michel, 75006 Paris, France (e-mail: martin.brossard@mines-paristech.fr; axel.barrau@safrangroup.com; paul.chauchat@isae-supaero.fr; silvere.bonnabel@unc.nc). A. Barrau is with Safran Tech, Groupe Safran, Rue des Jeunes Bois-Châteaufort, 78772, Magny Les Hameaux Cedex, France. P. Chauchat is with ISAE, 31400 Toulouse, France. S. Bonnabel is with University of New Caledonia, ISEA, 98851, Noumea Cedex, New Caledonia.

Second, we show $\Phi(\exp(\boldsymbol{\xi})) = \exp(\mathbf{F}\boldsymbol{\xi})$. Developing the exponential and propagating the obtained quantity, we have

$$
\begin{aligned}
\Phi(\exp(\boldsymbol{\xi})) &= \Phi\left(\left[\begin{array}{c|c|c} \exp(\boldsymbol{\phi}) & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\nu} & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\rho} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]\right) \\
&= \left[\begin{array}{c|c|c} \exp(\boldsymbol{\phi}) & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\nu} & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\rho} + \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\nu}\Delta t \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right] \\
&= \left[\begin{array}{c|c|c} \exp(\boldsymbol{\phi}) & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\nu} & \mathcal{J}_{\boldsymbol{\phi}}\left(\boldsymbol{\rho} + \boldsymbol{\nu}\Delta t\right) \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right],
\end{aligned} \tag{5}
$$

and then we compute the right part as

$$
\begin{aligned}
\exp(\mathbf{F}\boldsymbol{\xi}) &= \exp\left(\begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\nu} \\ \boldsymbol{\rho} + \Delta t \boldsymbol{\nu} \end{bmatrix}\right) \\
&= \left[\begin{array}{c|c|c} \exp(\boldsymbol{\phi}) & \mathcal{J}_{\boldsymbol{\phi}}\boldsymbol{\nu} & \mathcal{J}_{\boldsymbol{\phi}}\left(\boldsymbol{\rho} + \boldsymbol{\nu}\Delta t\right) \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right] \\
&\overset{(5)}{=} \Phi(\exp(\boldsymbol{\xi})).
\end{aligned} \tag{6}
$$

Injecting (6) in (4) with $\mathbf{T}^1 = \mathbf{T}$ and $\mathbf{T}^2 = \exp(\boldsymbol{\xi})$, we obtain the desired results as

$$
\boxed{\Phi\big(\mathbf{T}\exp(\boldsymbol{\xi})\big) = \Phi\left(\mathbf{T}\right)\exp\left(\mathbf{F}\boldsymbol{\xi}\right).}
$$

B. Third- and Fourth-Order Contributions

We detail how we compute the quantity $\mathbf{S}_{4\text{th}}$ in equation (48) of the referred paper along the lines of [7]. Les us define the operators

$$
\ll \mathbf{A} \gg := -\operatorname{trace}(\mathbf{A})\mathbf{I}_3 + \mathbf{A}, \tag{7}
$$

$$
\ll \mathbf{A}, \mathbf{B} \gg := \ll \mathbf{A} \gg \ll \mathbf{B} \gg + \ll \mathbf{B}, \mathbf{A} \gg, \tag{8}
$$

that provide the identity

$$
-\boldsymbol{\phi}_\times \mathbf{D} \boldsymbol{\omega}_\times = \ll \boldsymbol{\omega}\boldsymbol{\phi}^T, \mathbf{D} \gg, \tag{9}
$$

where $\boldsymbol{\phi} \in \mathbb{R}^3$, $\boldsymbol{\omega} \in \mathbb{R}^3$, and $\mathbf{D} \in \mathbb{R}^{3\times 3}$. The covariance of the propagated extended pose is given as

$$
\begin{aligned}
\boldsymbol{\Sigma}_{i+1} &:= \mathrm{E}[\boldsymbol{\xi}_{i+i}\boldsymbol{\xi}_{i+i}^T] \\
&\simeq \underbrace{\mathrm{E}[\boldsymbol{\xi}\boldsymbol{\xi}^T]}_{\mathbf{A}_i\boldsymbol{\Sigma}_i\mathbf{A}_i^T} + \underbrace{\mathrm{E}[\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T]}_{\mathbf{Q}} + \mathbf{S}_{4\text{th}},
\end{aligned} \tag{10}
$$

where

$$
\mathbf{S}_{4\text{th}} = \frac{1}{4}\underbrace{\mathrm{E}\left[\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\boldsymbol{\xi}^{\curlywedge T}\right]}_{\mathbf{B}} + \frac{1}{12}\mathrm{E}\left[\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T + \boldsymbol{\eta}_i^\curlywedge\boldsymbol{\eta}_i^\curlywedge\boldsymbol{\xi}\boldsymbol{\xi}^T + \boldsymbol{\xi}\boldsymbol{\xi}^T\boldsymbol{\eta}_i^\curlywedge\boldsymbol{\eta}_i^\curlywedge + \boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\right], \tag{11}
$$

and where we have omitted showing any terms that have an odd power in either $\boldsymbol{\xi}$ or $\boldsymbol{\eta}_i$ since these will by definition have expectation zero, and we recall the above result makes us of

$$
\begin{aligned}
\boldsymbol{\xi}_{i+1} &= \log(\exp(\boldsymbol{\xi})\exp(\boldsymbol{\eta}_i)) \\
&= \boldsymbol{\xi} + \boldsymbol{\eta}_i + \frac{1}{2}\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i + \frac{1}{12}\left(\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i + \boldsymbol{\eta}_i^\curlywedge\boldsymbol{\eta}_i^\curlywedge\boldsymbol{\xi}\right) - \frac{1}{24}\boldsymbol{\eta}_i^\curlywedge\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i + O(\|\boldsymbol{\xi}_{i+1}\|^5).
\end{aligned} \tag{12}
$$

We compute the expression (11) term by term. Looking at the second right term in (11), we have

$$
\mathrm{E}\left[\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\right] = \mathrm{E}\left[\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge\right]\underbrace{\mathrm{E}\left[\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\right]}_{\mathbf{Q}}, \tag{13}
$$

as noises are not correlated. We first compute

$$
\boldsymbol{\xi}^\curlywedge\boldsymbol{\xi}^\curlywedge = \begin{bmatrix} \boldsymbol{\phi}_\times\boldsymbol{\phi}_\times & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \boldsymbol{\nu}_\times\boldsymbol{\phi}_\times + \boldsymbol{\phi}_\times\boldsymbol{\nu}_\times & \boldsymbol{\phi}_\times\boldsymbol{\phi}_\times & \mathbf{0}_{3\times 3} \\ \boldsymbol{\rho}_\times\boldsymbol{\phi}_\times + \boldsymbol{\phi}_\times\boldsymbol{\rho}_\times & \mathbf{0}_{3\times 3} & \boldsymbol{\phi}_\times\boldsymbol{\phi}_\times \end{bmatrix}. \tag{14}
$$

Making use of the property

$$\phi_\times \phi_\times = -(\phi^T \phi)\mathbf{I}_3 + \phi\phi^T, \tag{15}$$

and taking the expectation of (14), we obtain

$$\begin{aligned}\mathbf{A}_{\boldsymbol{\Sigma}} &= \mathrm{E}\left[\boldsymbol{\xi}^{\curlywedge}\boldsymbol{\xi}^{\curlywedge}\right] \\ &= \begin{bmatrix} \ll \boldsymbol{\Sigma}_{\phi\phi} \gg & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \ll \boldsymbol{\Sigma}_{\nu\phi} + \boldsymbol{\Sigma}_{\phi\nu} \gg & \ll \boldsymbol{\Sigma}_{\phi\phi} \gg & \mathbf{0}_{3\times 3} \\ \ll \boldsymbol{\Sigma}_{\rho\phi} + \boldsymbol{\Sigma}_{\phi\rho} \gg & \mathbf{0}_{3\times 3} & \ll \boldsymbol{\Sigma}_{\phi\phi} \gg \end{bmatrix}.\end{aligned} \tag{16}$$

The last third blocks of (11) are similarly computed and we obtain

$$\mathbf{A}_{\mathbf{Q}} = \begin{bmatrix} \ll \mathbf{Q}_{\phi\phi} \gg & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \ll \mathbf{Q}_{\nu\phi} + \mathbf{Q}_{\phi\nu} \gg & \ll \mathbf{Q}_{\phi\phi} \gg & \mathbf{0}_{3\times 3} \\ \ll \mathbf{Q}_{\rho\phi} + \mathbf{Q}_{\phi\rho} \gg & \mathbf{0}_{3\times 3} & \ll \mathbf{Q}_{\phi\phi} \gg \end{bmatrix}, \tag{17}$$

$$\mathrm{E}\left[\boldsymbol{\xi}^{\curlywedge}\boldsymbol{\xi}^{\curlywedge}\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T + \boldsymbol{\eta}_i^{\curlywedge}\boldsymbol{\eta}_i^{\curlywedge}\boldsymbol{\xi}\boldsymbol{\xi}^T + \boldsymbol{\xi}\boldsymbol{\xi}^T\boldsymbol{\eta}_i^{\curlywedge}\boldsymbol{\eta}_i^{\curlywedge} + \boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\boldsymbol{\xi}^{\curlywedge}\boldsymbol{\xi}^{\curlywedge}\right] = \mathbf{A}_{\boldsymbol{\Sigma}}\mathbf{Q} + \mathbf{Q}\mathbf{A}_{\boldsymbol{\Sigma}}^T + \mathbf{A}_{\mathbf{Q}}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}_{\mathbf{Q}}^T. \tag{18}$$

It now remains to compute

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{\phi\phi} & \mathbf{B}_{\phi\nu} & \mathbf{B}_{\phi\rho} \\ \mathbf{B}_{\nu\phi} & \mathbf{B}_{\nu\nu} & \mathbf{B}_{\nu\rho} \\ \mathbf{B}_{\rho\phi} & \mathbf{B}_{\rho\nu} & \mathbf{B}_{\rho\rho} \end{bmatrix}. \tag{19}$$

As variables are not correlated, we have

$$\begin{aligned}\mathbf{B} &= \mathrm{E}\left[\boldsymbol{\xi}^{\curlywedge}\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\boldsymbol{\xi}^{\curlywedge T}\right] \\ &= \mathrm{E}\left[\boldsymbol{\xi}^{\curlywedge}\underbrace{\mathrm{E}\left[\boldsymbol{\eta}_i\boldsymbol{\eta}_i^T\right]}_{\mathbf{Q}}\boldsymbol{\xi}^{\curlywedge T}\right] \\ &= \mathrm{E}\left[\boldsymbol{\xi}^{\curlywedge}\mathbf{Q}\boldsymbol{\xi}^{\curlywedge}\right].\end{aligned} \tag{20}$$

Developing the above expression and taking the expectation by using the relation (9), we obtain

$$\mathbf{B}_{\phi\phi} = \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\phi\phi} \gg, \tag{21}$$

$$\begin{aligned}\mathbf{B}_{\nu\phi} &= \mathbf{B}_{\phi\nu}^T \\ &= \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\phi\nu} \gg + \ll \boldsymbol{\Sigma}_{\nu\phi}, \mathbf{Q}_{\phi\phi} \gg,\end{aligned} \tag{22}$$

$$\begin{aligned}\mathbf{B}_{\rho\phi} &= \mathbf{B}_{\phi\rho}^T \\ &= \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\phi\rho} \gg + \ll \boldsymbol{\Sigma}_{\rho\phi}, \mathbf{Q}_{\phi\phi} \gg,\end{aligned} \tag{23}$$

$$\mathbf{B}_{\nu\nu} = \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\nu\nu} \gg + \ll \boldsymbol{\Sigma}_{\phi\nu}, \mathbf{Q}_{\nu\phi} \gg + \ll \boldsymbol{\Sigma}_{\nu\phi}, \mathbf{Q}_{\nu\phi} \gg + \ll \boldsymbol{\Sigma}_{\nu\nu}, \mathbf{Q}_{\phi\phi} \gg, \tag{24}$$

$$\begin{aligned}\mathbf{B}_{\nu\rho} &= \mathbf{B}_{\rho\nu}^T \\ &= \ll \boldsymbol{\Sigma}_{\nu\rho}, \mathbf{Q}_{\phi\phi} \gg + \ll \boldsymbol{\Sigma}_{\phi\rho}, \mathbf{Q}_{\nu\phi} \gg + \ll \boldsymbol{\Sigma}_{\nu\phi}, \mathbf{Q}_{\rho\phi} \gg + \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\nu\rho} \gg,\end{aligned} \tag{25}$$

$$\mathbf{B}_{\rho\rho} = \ll \boldsymbol{\Sigma}_{\phi\phi}, \mathbf{Q}_{\rho\rho} \gg + \ll \boldsymbol{\Sigma}_{\phi\rho}, \mathbf{Q}_{\rho\phi} \gg + \ll \boldsymbol{\Sigma}_{\rho\phi}, \mathbf{Q}_{\phi\rho} \gg + \ll \boldsymbol{\Sigma}_{\rho\rho}, \mathbf{Q}_{\phi\phi} \gg. \tag{26}$$

We finally end with the formula for $\mathbf{S}_{\text{4th}}$ as

$$\boxed{\mathbf{S}_{\text{4th}} = \frac{1}{12}\left(\mathbf{A}_{\boldsymbol{\Sigma}}\mathbf{Q} + \mathbf{Q}\mathbf{A}_{\boldsymbol{\Sigma}}^T + \mathbf{A}_{\mathbf{Q}}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}_{\mathbf{Q}}^T\right) + \frac{1}{4}\mathbf{B}.}$$

II. IMU Preintegration with Rotating Earth

This section provides the proofs of the results for IMU preintegration with rotating Earth, see Section VI of the referred paper. We start recalling the dynamical model with rotating Earth and IMU preintegration factors.

A. IMU Equations with Rotating Earth

To proof the validity of the solution

$$\mathbf{T}'_t = \mathbf{\Gamma}'_t \Phi_t(\mathbf{T}'_0)\mathbf{\Upsilon}_t, \tag{27}$$

(see equation (76) of the referred paper, Section VI), it is sufficient to work on a continuous noise-free model. Accounting for Earth rotation, we have the dynamic model [2]

$$\dot{\mathbf{R}} = -\mathbf{\Omega}_\times \mathbf{R} + \mathbf{R}\boldsymbol{\omega}_\times, \tag{28}$$

$$\dot{\mathbf{v}} = \mathbf{R}\mathbf{a} + \mathbf{g} - 2\mathbf{\Omega}_\times \mathbf{v} - \mathbf{\Omega}_\times^2 \mathbf{p}, \tag{29}$$

$$\dot{\mathbf{p}} = \mathbf{v}. \tag{30}$$

We recall the differential equations that satisfy the IMU preintegration factors, see e.g. [3,4],

$$\dot{\Delta\mathbf{R}} = \Delta\mathbf{R}\boldsymbol{\omega}_\times, \tag{31}$$

$$\dot{\Delta\mathbf{v}} = \Delta\mathbf{R}\mathbf{a}, \tag{32}$$

$$\dot{\Delta\mathbf{p}} = \Delta\mathbf{v}, \tag{33}$$

which are initialized at

$$\Delta\mathbf{R}_0 = \Delta\mathbf{R}_{ii} = \mathbf{I}_3, \tag{34}$$

$$\Delta\mathbf{v}_0 = \Delta\mathbf{v}_{ii} = \mathbf{0}_3, \tag{35}$$

$$\Delta\mathbf{p}_0 = \Delta\mathbf{p}_{ii} = \mathbf{0}_3. \tag{36}$$

We note the left variables of (34)-(36) are used when integrating from t_0 to time t while the middles variables are used when integrating between t_i and t_j for preintegration.

B. Preliminary Results with Change of Variables

We first rewrite the model (28)-(30) with an alternative variable that makes the proof easier to follow. Let us introduce the auxiliary variable

$$\boxed{\mathbf{v}' = \mathbf{v} + \mathbf{\Omega}_\times \mathbf{p},} \tag{37}$$

such that

$$\begin{aligned}
\dot{\mathbf{v}}' &= \underbrace{\dot{\mathbf{v}}}_{\overset{(29)}{=}\mathbf{R}\mathbf{a}+\mathbf{g}-2\mathbf{\Omega}_\times\mathbf{v}-\mathbf{\Omega}_\times^2\mathbf{p}} + \mathbf{\Omega}_\times \underbrace{\dot{\mathbf{p}}}_{\overset{(30)}{=}\mathbf{v}} && (38)\\
&= \mathbf{R}\mathbf{a} + \mathbf{g} - 2\mathbf{\Omega}_\times \mathbf{v} - \mathbf{\Omega}_\times^2 \mathbf{p} + \mathbf{\Omega}_\times \mathbf{v}\\
&= \mathbf{R}\mathbf{a} + \mathbf{g} - \mathbf{\Omega}_\times \mathbf{v} - \mathbf{\Omega}_\times^2 \mathbf{p}\\
&= \mathbf{R}\mathbf{a} + \mathbf{g} - \mathbf{\Omega}_\times (\underbrace{\mathbf{v} - \mathbf{\Omega}_\times \mathbf{p}}_{\overset{(37)}{=}\mathbf{v}'})\\
&= \mathbf{R}\mathbf{a} + \mathbf{g} - \mathbf{\Omega}_\times \mathbf{v}', && (39)\\
\dot{\mathbf{p}} &= \mathbf{v}\\
&\overset{(37)}{=} \mathbf{v}' - \mathbf{\Omega}_\times \mathbf{p}. && (40)
\end{aligned}$$

The model (28)-(30) thus unexpectedly simplifies to

$$\boxed{\begin{aligned}
\dot{\mathbf{R}} &= -\mathbf{\Omega}_\times \mathbf{R} + \mathbf{R}\boldsymbol{\omega}_\times, && (41)\\
\dot{\mathbf{v}}' &= \mathbf{R}\mathbf{a} + \mathbf{g} - \mathbf{\Omega}_\times \mathbf{v}', && (42)\\
\dot{\mathbf{p}} &= \mathbf{v}' - \mathbf{\Omega}_\times \mathbf{p}. && (43)
\end{aligned}}$$

In the following, we show how the solution $\mathbf{T}'_t$ satisfies the differential system (41)-(43), and thus equivalently satisfy (28)-(30).

C. Proof of IMU Equations Revisited

Let us introduce the auxiliary extended pose

$$\mathbf{T}'_t := \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{v}' & \mathbf{p} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right], \tag{44}$$

as done in the paper, see equation (75). We have to prove that the solution $\mathbf{T}'_t$ written in the form (27) agrees with differential equations (41)-(43). We recall that we have defined in the paper

$$\mathbf{\Gamma}'_t := \left[\begin{array}{c|cc} \mathbf{\Gamma}^{\mathbf{R}} & \mathbf{\Gamma}^{\mathbf{v}} & \mathbf{\Gamma}^{\mathbf{p}} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right] \tag{45}$$

through the following differential equations

$$\dot{\mathbf{\Gamma}}^{\mathbf{R}} = -\mathbf{\Omega}_\times \mathbf{\Gamma}^{\mathbf{R}}, \tag{46}$$
$$\dot{\mathbf{\Gamma}}^{\mathbf{v}} = \mathbf{g} - \mathbf{\Omega}_\times \mathbf{\Gamma}^{\mathbf{v}}, \tag{47}$$
$$\dot{\mathbf{\Gamma}}^{\mathbf{p}} = \mathbf{\Gamma}^{\mathbf{v}} - \mathbf{\Omega}_\times \mathbf{\Gamma}^{\mathbf{p}}, \tag{48}$$

which are initialized at

$$\mathbf{\Gamma}^{\mathbf{R}}_0 = \mathbf{\Gamma}^{\mathbf{R}}_{ii} = \mathbf{I}_3, \tag{49}$$
$$\mathbf{\Gamma}^{\mathbf{v}}_0 = \mathbf{\Gamma}^{\mathbf{v}}_{ii} = \mathbf{0}_3, \tag{50}$$
$$\mathbf{\Gamma}^{\mathbf{p}}_0 = \mathbf{\Gamma}^{\mathbf{p}}_{ii} = \mathbf{0}_3, \tag{51}$$

and we recall for convenience the IMU preintegration measurement that depend on IMU measurements and the $\Phi_t(\cdot)$ function that only depend on t as

$$\mathbf{\Upsilon}_t = \left[\begin{array}{c|cc} \Delta\mathbf{R} & \Delta\mathbf{v} & \Delta\mathbf{p} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right], \Phi_t(\mathbf{T}') = \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{v}' & \mathbf{p} + t\mathbf{v}' \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]. \tag{52}$$

Developing (27) with matrix products, using (52), the state solution is given as

$$\boxed{\begin{aligned} \mathbf{R} &= \mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0\Delta\mathbf{R}, && (53)\\ \mathbf{v}' &= \mathbf{\Gamma}^{\mathbf{R}}\left(\mathbf{R}_0\Delta\mathbf{v} + \mathbf{v}'_0\right) + \mathbf{\Gamma}^{\mathbf{v}}, && (54)\\ \mathbf{p} &= \mathbf{\Gamma}^{\mathbf{R}}\left(\mathbf{R}_0\Delta\mathbf{p} + \mathbf{p}_0 + t\mathbf{v}'_0\right) + \mathbf{\Gamma}^{\mathbf{p}}, && (55) \end{aligned}}$$

where $\mathbf{v}'_0 = \mathbf{v}_0 + \mathbf{\Omega}_\times \mathbf{p}_0$.

We now proof that (53)-(55) describe a valid solution of (41)-(43). Initial conditions are immediately checked as

$$\begin{aligned} \mathbf{R}_0 &= \underbrace{\mathbf{\Gamma}^{\mathbf{R}}_0}_{\overset{(49)}{=}\mathbf{I}_3} \mathbf{R}_0 \underbrace{\Delta\mathbf{R}_0}_{\overset{(34)}{=}\mathbf{I}_3} \\ &= \mathbf{R}_0, && (56)\\ \mathbf{v}'_0 &= \underbrace{\mathbf{\Gamma}^{\mathbf{R}}_0}_{\overset{(49)}{=}\mathbf{I}_3}\left(\mathbf{R}_0 \underbrace{\Delta\mathbf{v}_0}_{\overset{(35)}{=}\mathbf{0}_3} + \mathbf{v}'_0\right) + \underbrace{\mathbf{\Gamma}^{\mathbf{v}}_0}_{\overset{(50)}{=}\mathbf{0}_3} \\ &= \mathbf{v}'_0, && (57)\\ \mathbf{p}_0 &= \underbrace{\mathbf{\Gamma}^{\mathbf{R}}_0}_{\overset{(49)}{=}\mathbf{I}_3}\left(\mathbf{R}_0 \underbrace{\Delta\mathbf{p}_0}_{\overset{(36)}{=}\mathbf{0}_3} + \mathbf{p}_0 + 0\mathbf{v}'_0\right) + \underbrace{\mathbf{\Gamma}^{\mathbf{p}}_0}_{\overset{(51)}{=}\mathbf{0}_3} \\ &= \mathbf{p}_0. && (58) \end{aligned}$$

We now verify each of the above three differential equations one by one. We differentiate (53) with matrix rule to obtain

$$\begin{aligned} \dot{\mathbf{R}} &\overset{(53)}{=} \underbrace{\dot{\mathbf{\Gamma}}^{\mathbf{R}}}_{\overset{(46)}{=}-\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{R}}} \mathbf{R}_0\Delta\mathbf{R} + \mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0 \underbrace{\Delta\dot{\mathbf{R}}}_{\overset{(31)}{=}\Delta\mathbf{R}\boldsymbol{\omega}_\times} \\ &= -\mathbf{\Omega}_\times \underbrace{\mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0\Delta\mathbf{R}}_{\overset{(53)}{=}\mathbf{R}} + \underbrace{\mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0\Delta\mathbf{R}}_{\overset{(53)}{=}\mathbf{R}}\boldsymbol{\omega}_\times \\ &= -\mathbf{\Omega}_\times\mathbf{R} + \mathbf{R}\boldsymbol{\omega}_\times, && (59) \end{aligned}$$

which matches with (41). We differentiate (54) to obtain

$$\begin{aligned}
\dot{\mathbf{v}}' &\overset{(54)}{=} \underbrace{\dot{\mathbf{\Gamma}}^{\mathbf{R}}}_{\overset{(46)}{=}-\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{R}}} (\mathbf{R}_0\Delta\mathbf{v} + \mathbf{v}'_0) + \mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0 \underbrace{\dot{\Delta\mathbf{v}}}_{\overset{(32)}{=}\Delta\mathbf{R}\mathbf{a}} + \underbrace{\dot{\mathbf{\Gamma}}^{\mathbf{v}}}_{\overset{(47)}{=}\mathbf{g}-\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{v}}} \\
&= -\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{R}}\mathbf{R}_0\Delta\mathbf{v} + \mathbf{\Gamma}^{\mathbf{R}}(\mathbf{R}_0\Delta\mathbf{R}\mathbf{a} + \mathbf{v}'_0) + \mathbf{g} - \mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{v}} \\
&= -\mathbf{\Omega}_\times \underbrace{\mathbf{\Gamma}^{\mathbf{R}}(\mathbf{R}_0\Delta\mathbf{v} + \mathbf{v}'_0 + \mathbf{\Gamma}^{\mathbf{v}})}_{\overset{(54)}{=}\mathbf{v}'} + \mathbf{R}\mathbf{a} + \mathbf{g} \\
&= -\mathbf{\Omega}_\times\mathbf{v}' + \mathbf{R}\mathbf{a} + \mathbf{g},
\end{aligned} \tag{60}$$

which agrees with (54). We similarly operate for (55) as

$$\begin{aligned}
\dot{\mathbf{p}} &\overset{(55)}{=} \underbrace{\dot{\mathbf{\Gamma}}^{\mathbf{R}}}_{\overset{(46)}{=}-\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{R}}} (\mathbf{R}_0\Delta\mathbf{p} + \mathbf{p}_0 + t\mathbf{v}'_0) + \mathbf{\Gamma}^{\mathbf{R}}\big(\mathbf{R}_0 \underbrace{\dot{\Delta\mathbf{p}}}_{\overset{(33)}{=}\Delta\mathbf{v}} + \mathbf{v}'_0\big) + \underbrace{\dot{\mathbf{\Gamma}}^{\mathbf{p}}}_{\overset{(48)}{=}\mathbf{\Gamma}^{\mathbf{v}}-\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{p}}} \\
&= -\mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{R}}(\mathbf{R}_0\Delta\mathbf{p} + \mathbf{p}_0 + t\mathbf{v}'_0) + \mathbf{\Gamma}^{\mathbf{R}}(\mathbf{R}_0\Delta\mathbf{v} + \mathbf{v}'_0) + \mathbf{\Gamma}^{\mathbf{v}} - \mathbf{\Omega}_\times\mathbf{\Gamma}^{\mathbf{p}} \\
&= \underbrace{\mathbf{\Gamma}^{\mathbf{R}}(\mathbf{R}_0\Delta\mathbf{v} + \mathbf{v}'_0 + \mathbf{\Gamma}^{\mathbf{v}})}_{\overset{(54)}{=}\mathbf{v}'} - \mathbf{\Omega}_\times\Big(\underbrace{\mathbf{\Gamma}^{\mathbf{R}}\big(\mathbf{R}_0\Delta\mathbf{p} + \mathbf{p}_0 + t\mathbf{v}'_0\big) + \mathbf{\Gamma}^{\mathbf{p}}}_{\overset{(55)}{=}\mathbf{p}}\Big) \\
&= \mathbf{v}' - \mathbf{\Omega}_\times\mathbf{p},
\end{aligned} \tag{61}$$

which matches with (55). Consequently, the solution $\mathbf{T}'_t$ in (27) is a correct solution of the differential equations (41)-(43). Injecting (37) in (53)-(55) and coming to discrete-time instants between t_i and t_j, we obtains the final paper results as

$$\boxed{\begin{aligned}
\mathbf{R}_j &= \mathbf{\Gamma}^{\mathbf{R}}_{ij}\mathbf{R}_i\Delta\mathbf{R}_{ij}, \\
\mathbf{v}_j &= \mathbf{\Gamma}^{\mathbf{v}}_{ij} + \mathbf{\Gamma}^{\mathbf{R}}_{ij}(\mathbf{R}_i\Delta\mathbf{v}_{ij} + \mathbf{v}_i + \mathbf{\Omega}_\times\mathbf{p}_i) - \mathbf{\Omega}_\times\mathbf{p}_j, \\
\mathbf{p}_j &= \mathbf{\Gamma}^{\mathbf{p}}_{ij} + \mathbf{\Gamma}^{\mathbf{R}}_{ij}(\mathbf{R}_i\Delta\mathbf{p}_{ij} + (\mathbf{v}_i + \mathbf{\Omega}_\times\mathbf{p}_i)\Delta t_{ij} + \mathbf{p}_i),
\end{aligned}}$$

which are exact.

D. Closed-Form Expressions of "Gamma" Factors

We now show how resolving (46)-(48) with exact closed-form expressions (albeit numerical integration schemes such as Euler integration are possible), which are presented in equation (82)-(84) of the referred paper. We first identify (46) with the definition of the $SO(3)$ exponential map [5], thus

$$\boxed{\mathbf{\Gamma}^{\mathbf{R}}_{ij} = \exp(-\Delta t_{ij}\mathbf{\Omega}).} \tag{62}$$

For $\mathbf{\Gamma}^{\mathbf{v}}$ and $\mathbf{\Gamma}^{\mathbf{p}}$, let us define and then solve the following continuous time linear system

$$\begin{bmatrix} \dot{\mathbf{\Gamma}}^{\mathbf{v}} \\ \dot{\mathbf{\Gamma}}^{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} -\mathbf{\Omega}_\times & \mathbf{0}_{3\times3} \\ \mathbf{I}_3 & -\mathbf{\Omega}_\times \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}^{\mathbf{v}} \\ \mathbf{\Gamma}^{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ \mathbf{0}_3 \end{bmatrix}. \tag{63}$$

The solution of this linear dynamical system is given by

$$\begin{aligned}
\begin{bmatrix} \mathbf{\Gamma}^{\mathbf{v}}_{ij} \\ \mathbf{\Gamma}^{\mathbf{p}}_{ij} \end{bmatrix} &= \mathbf{\Phi}(\Delta t_{ij}, 0) \begin{bmatrix} \underbrace{\mathbf{\Gamma}^{\mathbf{v}}_{ii}}_{\overset{(50)}{=}\mathbf{0}_3} \\ \underbrace{\mathbf{\Gamma}^{\mathbf{p}}_{ii}}_{\overset{(51)}{=}\mathbf{0}_3} \end{bmatrix} + \int_0^{\Delta t_{ij}} \mathbf{\Phi}(\Delta t_{ij}, u) \begin{bmatrix} \mathbf{g} \\ \mathbf{0}_3 \end{bmatrix} du \\
&= \int_0^{\Delta t_{ij}} \mathbf{\Phi}(\Delta t_{ij}, u) \begin{bmatrix} \mathbf{g} \\ \mathbf{0}_3 \end{bmatrix} du,
\end{aligned} \tag{64}$$

where the state-transition matrix $\mathbf{\Phi}(\cdot,\cdot)$ is given by the matrix exponential

$$\begin{aligned}
\mathbf{\Phi}(\Delta t_{ij}, 0) &= \exp_m\left(\begin{bmatrix} -\mathbf{\Omega}_\times & \mathbf{0}_{3\times3} \\ \mathbf{I}_3 & -\mathbf{\Omega}_\times \end{bmatrix} \Delta t_{ij} \right) \\
&= \begin{bmatrix} \exp(-\mathbf{\Omega}\Delta t_{ij}) & \mathbf{0}_{3\times3} \\ \exp(-\mathbf{\Omega}\Delta t_{ij})\Delta t_{ij} & -\exp(-\mathbf{\Omega}\Delta t_{ij}) \end{bmatrix}.
\end{aligned} \tag{65}$$

Using the closed-form expression of the $SO(3)$ exponential

$$\exp(\boldsymbol{\phi}) = \mathbf{I}_3 + \frac{\sin\phi}{\phi}\boldsymbol{\phi}_\times + \frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}_\times^2, \tag{66}$$

where $\phi = \|\boldsymbol{\phi}\|$, we first solve for $\boldsymbol{\Gamma}_{ij}^{\mathbf{v}}$ with integration by parts as

$$\begin{aligned}\boldsymbol{\Gamma}_{ij}^{\mathbf{v}} &= \int_0^{\Delta t_{ij}} \exp(-\boldsymbol{\Omega} u)\mathbf{g} du \\ &= \int_0^{\Delta t_{ij}} \left(\mathbf{I}_3 + \frac{\sin(\phi u)}{\phi}(-\boldsymbol{\Omega})_\times + \frac{1-\cos(\phi u)}{\phi^2}(-\boldsymbol{\Omega})_\times^2\right)\mathbf{g} du \\ &= \left(\Delta t_{ij}\mathbf{I}_3 - \frac{1-\cos(\phi\Delta t_{ij})}{\phi^2}\boldsymbol{\Omega}_\times + \frac{\phi\Delta t_{ij} - \sin(\phi\Delta t_{ij})}{\phi^3}\boldsymbol{\Omega}_\times^2\right)\mathbf{g},\end{aligned} \tag{67}$$

where $\phi = \|\boldsymbol{\Omega}\|$, and that matches with the translation of the $SE(3)$ exponential map, which leverages the left-Jacobian of $SO(3)$ given as

$$\begin{aligned}\mathcal{J}_{-\Delta t_{ij}\boldsymbol{\Omega}} &= \mathbf{I}_3 - \frac{1-\cos(\phi\Delta t_{ij})}{\Delta t_{ij}^2\phi^2}(\Delta t_{ij}\boldsymbol{\Omega})_\times + \frac{\phi\Delta t_{ij} - \sin(\phi\Delta t_{ij})}{\Delta t_{ij}^3\phi^3}(\Delta t_{ij}\boldsymbol{\Omega})_\times^2 \\ &= \mathbf{I}_3 - \frac{1-\cos(\phi\Delta t_{ij})}{\Delta t_{ij}\phi^2}\boldsymbol{\Omega}_\times + \frac{\phi\Delta t_{ij} - \sin(\phi\Delta t_{ij})}{\Delta t_{ij}\phi^3}\boldsymbol{\Omega}_\times^2 \\ &= \frac{1}{\Delta t_{ij}}\left(\Delta t_{ij}\mathbf{I}_3 - \frac{1-\cos(\phi\Delta t_{ij})}{\phi^2}\boldsymbol{\Omega}_\times + \frac{\phi\Delta t_{ij} - \sin(\phi\Delta t_{ij})}{\phi^3}\boldsymbol{\Omega}_\times^2\right).\end{aligned} \tag{68}$$

We similarly solve with integration by parts for position as

$$\begin{aligned}\boldsymbol{\Gamma}_{ij}^{\mathbf{p}} &= \int_0^{\Delta t_{ij}} u\exp(-\boldsymbol{\Omega} u)\mathbf{g} du \\ &= \int_0^{\Delta t_{ij}} u\left(\mathbf{I}_3 + \frac{\sin(\phi u)}{\phi}(-\boldsymbol{\Omega})_\times + \frac{1-\cos(\phi u)}{\phi^2}(-\boldsymbol{\Omega})_\times^2\right)\mathbf{g} du \\ &= \left(\frac{\Delta t_{ij}^2}{2}\mathbf{I}_3 + \frac{\phi\Delta t_{ij}\cos(\phi\Delta t_{ij}) - \sin(\phi\Delta t_{ij})}{\phi^3}\boldsymbol{\Omega}_\times + \frac{\frac{\phi^2\Delta t_{ij}^2}{2} - \cos(\phi\Delta t_{ij}) - \phi\Delta t_{ij}\sin(\phi\Delta t_{ij}) + 1}{\phi^4}\boldsymbol{\Omega}_\times^2\right)\mathbf{g},\end{aligned} \tag{69}$$

that provides the paper's result

$$\boxed{\begin{aligned}\boldsymbol{\Gamma}_{ij}^{\mathbf{v}} &= \mathcal{J}_{-\Delta t_{ij}\boldsymbol{\Omega}}\Delta t_{ij}\mathbf{g}, & (70)\\ \boldsymbol{\Gamma}_{ij}^{\mathbf{p}} &= \left(\frac{\Delta t_{ij}^2}{2}\mathbf{I}_3 + \frac{\phi\Delta t_{ij}\cos(\phi\Delta t_{ij}) - \sin(\phi\Delta t_{ij})}{\phi^3}\boldsymbol{\Omega}_\times + \frac{\frac{\phi^2\Delta t_{ij}^2}{2} - \cos(\phi\Delta t_{ij}) - \phi\Delta t_{ij}\sin(\phi\Delta t_{ij}) + 1}{\phi^4}\boldsymbol{\Omega}_\times^2\right)\mathbf{g}. & (71)\end{aligned}}$$

For small increment $\Delta t_{ij}\phi \ll 1$, we have

$$\boldsymbol{\Gamma}_{ij}^{\mathbf{v}} \approx \left(\mathbf{I}_3 + \frac{\Delta t_{ij}}{2}\boldsymbol{\Omega}_\times\right)\Delta t_{ij}\mathbf{g}, \tag{72}$$

$$\boldsymbol{\Gamma}_{ij}^{\mathbf{p}} \approx \left(\frac{1}{2}\mathbf{I}_3 - \frac{\Delta t_{ij}}{3}\boldsymbol{\Omega}_\times\right)\Delta t_{ij}^2\mathbf{g}. \tag{73}$$

III. Closed-Form Expressions for IMU Integration

This section describes how computing the IMU increment $\boldsymbol{\Upsilon}_i$ for one time step given one of the following two assumptions:

1) piecewise constant global acceleration, as in [3], which may be easily violated in realistic navigation;
2) piecewise constant IMU measurements, as proposed in, e.g., [6].

For each model assumption, we provide a way to compute the measurement mean $\hat{\boldsymbol{\Upsilon}}_i$ and its Jacobian w.r.t. IMU noises in order to compute the noise covariance of the IMU increment in the form

$$\boldsymbol{\Upsilon}_i = \hat{\boldsymbol{\Upsilon}}_i\exp(\boldsymbol{\eta}_i), \tag{74}$$

$$\boldsymbol{\eta}_i \sim \mathcal{N}\left(\mathbf{0}_6, \mathbf{G}_i \operatorname{cov}\left(\begin{bmatrix}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} \\ \boldsymbol{\eta}_i^{\mathbf{a}}\end{bmatrix}\right)\mathbf{G}_i^T\right). \tag{75}$$

We first recall useful paper equations that are needed through this section, this are, respectively, exponential product approximation, $SO(3)$ left-Jacobian and inverse left-Jacobian

$$\log\big(\exp(\boldsymbol{\xi})\exp(\boldsymbol{\eta})\big) = \boldsymbol{\xi} + \mathcal{J}_{\boldsymbol{\xi}}^{-1}\boldsymbol{\eta} + O(\|\boldsymbol{\eta}\|^2), \tag{76}$$

$$\Rightarrow \log\big(\exp(\boldsymbol{\xi}+\boldsymbol{\eta})\big) = \boldsymbol{\xi} + \mathcal{J}_{\boldsymbol{\xi}}\boldsymbol{\eta} + O(\|\boldsymbol{\eta}\|^2), \tag{77}$$

$$\begin{aligned}\mathcal{J}_{\boldsymbol{\phi}} &= \mathbf{I}_3 + \frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}_\times + \frac{\phi-\sin\phi}{\phi^3}\boldsymbol{\phi}_\times^2 \\ &= \mathbf{I}_3 + \frac{1}{2}\boldsymbol{\phi}_\times + O(\|\boldsymbol{\phi}_\times\|^3),\end{aligned} \tag{78}$$

$$\begin{aligned}\mathcal{J}_{\boldsymbol{\phi}}^{-1} &= \mathbf{I}_3 - \frac{1}{2}\boldsymbol{\phi}_\times + \left(\phi^{-2} + \frac{1+\cos\phi}{2\phi\sin\phi}\right)\boldsymbol{\phi}_\times^2 \\ &= \mathbf{I}_3 - \frac{1}{2}\boldsymbol{\phi}_\times + O(\|\boldsymbol{\phi}_\times\|^3),\end{aligned} \tag{79}$$

where $\phi = \|\boldsymbol{\phi}\|$.

A. *Integration with Constant Global Acceleration Assumption*

This subsection assumes a piecewise constant global acceleration model as done in the discrete approximation used by [3]. Let us integrate IMU dynamics for one time step with constant global acceleration:

$$\begin{aligned}\Delta\mathbf{R}_i &= \exp\left(\int_0^{\Delta t} \underbrace{\boldsymbol{\omega}_i}_{\text{cst.}} dt\right) \\ &= \exp(\boldsymbol{\omega}_i\Delta t),\end{aligned} \tag{80}$$

$$\begin{aligned}\Delta\mathbf{v}_i &= \int_0^{\Delta t} \underbrace{\Delta\mathbf{R}_t\mathbf{a}_i}_{\overset{\text{cst.}}{=}\Delta\mathbf{R}_0\mathbf{a}_i=\mathbf{a}_i} dt \\ &= \mathbf{a}_i\Delta t,\end{aligned} \tag{81}$$

$$\begin{aligned}\Delta\mathbf{p}_i &= \int_0^{\Delta t} \underbrace{\Delta\mathbf{v}_t}_{\overset{(81)}{=}\mathbf{a}_i t} dt \\ &= \int_0^{\Delta t} t dt\mathbf{a}_i = \frac{1}{2}\Delta t^2\mathbf{a}_i.\end{aligned} \tag{82}$$

It leads to the true and measurement values

$$\boldsymbol{\Upsilon}_i = \left[\begin{array}{c|cc}\exp(\boldsymbol{\omega}_i\Delta t) & \mathbf{a}_i\Delta t & \mathbf{a}_i\Delta t^2/2 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2}\end{array}\right], \hat{\boldsymbol{\Upsilon}}_i = \left[\begin{array}{c|cc}\exp(\hat{\boldsymbol{\omega}}_i\Delta t) & \hat{\mathbf{a}}_i\Delta t & \hat{\mathbf{a}}_i\Delta t^2/2 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2}\end{array}\right], \tag{83}$$

where $\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_i^m - \hat{\mathbf{b}}_i^{\boldsymbol{\omega}}$ and $\hat{\mathbf{a}}_i = \mathbf{a}_i^m - \hat{\mathbf{b}}_i^{\mathbf{a}}$ are IMU measured quantities. We now search to express the Jacobian of the noise, i.e. finding the 9×6 matrix $\mathbf{G}_i$ such that

$$\boldsymbol{\Upsilon}_i = \hat{\boldsymbol{\Upsilon}}_i \exp\left(\mathbf{G}_i\begin{bmatrix}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} \\ \boldsymbol{\eta}_i^{\mathbf{a}}\end{bmatrix}\right). \tag{84}$$

Instead of looking at the quantity $\hat{\boldsymbol{\Upsilon}}_i^{-1}\boldsymbol{\Upsilon}_i$, we are going to express true quantity $\boldsymbol{\Upsilon}_i$ as a measurement term $\hat{\boldsymbol{\Upsilon}}_i$ multiplied by a noisy quantity. Computing term by term, this leads to the first-order in the noise terms to (with biases perfectly known)

$$\begin{aligned}\exp(\boldsymbol{\omega}_i\Delta t) &= \exp\left((\hat{\boldsymbol{\omega}}_i - \boldsymbol{\eta}_i)\Delta t\right) \\ &\overset{(77)}{=} \exp(\hat{\boldsymbol{\omega}}_i\Delta t)\exp(-\mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}^{-1}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t + O(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2)),\end{aligned} \tag{85}$$

$$\mathbf{a}_i\Delta t = (\hat{\mathbf{a}}_i - \boldsymbol{\eta}_i^{\mathbf{a}})\Delta t, \tag{86}$$

$$\frac{1}{2}\mathbf{a}_i\Delta t^2 = \frac{1}{2}(\hat{\mathbf{a}}_i - \boldsymbol{\eta}_i^{\mathbf{a}})\Delta t^2. \tag{87}$$

We thus split the mean of $\boldsymbol{\Upsilon}_i$ on the left and the perturbation on the right as

$$\boldsymbol{\Upsilon}_i \simeq \underbrace{\left[\begin{array}{c|cc}\exp(\hat{\boldsymbol{\omega}}_i\Delta t) & \hat{\mathbf{a}}_i\Delta t & \hat{\mathbf{a}}_i\Delta t^2/2 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2}\end{array}\right]}_{\overset{(83)}{=}\hat{\boldsymbol{\Upsilon}}_i} \underbrace{\left[\begin{array}{c|cc}\exp(-\mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}^{-1}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t) & -\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t & -\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t^2/2 \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2}\end{array}\right]}_{\exp(\boldsymbol{\eta}_i)}, \tag{88}$$

where $\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)$ is required to compensate for matrix multiplication. Note that in the case of $SO(3)$ we have

$$\exp(-\hat{\boldsymbol{\omega}}_i\Delta t) = \exp(\hat{\boldsymbol{\omega}}_i\Delta t)^{-1} = \exp(\hat{\boldsymbol{\omega}}_i\Delta t)^T \overset{(80)}{=} \Delta\mathbf{R}_i^T. \tag{89}$$

We now compute the logarithm of the right part of (88) as

$$\begin{aligned}\boldsymbol{\eta}_i &= \log\left(\left[\begin{array}{c|cc}\exp(-\mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t) & -\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t & -\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t^2/2 \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2}\end{array}\right]\right) \\ &= \begin{bmatrix}-\mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t \\ \boldsymbol{\nu} \\ \boldsymbol{\rho}\end{bmatrix},\end{aligned} \tag{90}$$

where

$$\begin{aligned}\boldsymbol{\nu} &= -\underbrace{\mathcal{J}^{-1}_{-\mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t}}_{\overset{(79)}{=}\mathbf{I}_3+O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2\right)}\exp\left(-\hat{\boldsymbol{\omega}}_i\Delta t\right)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t \\ &= -\exp\left(-\hat{\boldsymbol{\omega}}_i\Delta t\right)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2\right), \qquad (91) \\ \boldsymbol{\rho} &= -\underbrace{\mathcal{J}^{-1}_{-\mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\Delta t}}_{\overset{(79)}{=}\mathbf{I}_3+O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2\right)}\exp\left(-\hat{\boldsymbol{\omega}}_i\Delta t\right)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t^2/2 \\ &= -\exp\left(-\hat{\boldsymbol{\omega}}_i\Delta t\right)\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t^2/2 + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2\right). \qquad (92)\end{aligned}$$

We thus obtain the Jacobian w.r.t. noise as

$$\boldsymbol{\Upsilon}_i = \hat{\boldsymbol{\Upsilon}}_i\exp\left(\underbrace{-\begin{bmatrix}\mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\Delta t & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\Delta t \\ \mathbf{0}_{3\times 3} & \exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\Delta t^2/2\end{bmatrix}}_{\mathbf{G}_i}\begin{bmatrix}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} \\ \boldsymbol{\eta}_i^{\mathbf{a}}\end{bmatrix}\right). \tag{93}$$

This Jacobian expression may then be used to compute Jacobian of the preintegrated measurement $\boldsymbol{\Upsilon}_i$ or $\boldsymbol{\Upsilon}_{ij}$ following Section V of the paper. It is also required for update the factor when the bias estimate is updated.

B. Integration with Constant IMU Measurements Assumption

Piecewise constant global acceleration may be violated in practice and we thus opt in this subsection for piecewise constant IMU measurements. It requires more derivation to compute the mean and noise terms but this approach has been proven more accurate in [6].

Let us thus integrate IMU dynamics for one time step with constant IMU Measurements, i.e. constant angular velocity and acceleration measurements, as

$$\begin{aligned}\Delta\mathbf{R}_i &= \exp\left(\int_0^{t_i}\underbrace{\boldsymbol{\omega}_i}_{\text{cst.}}dt\right) \\ &= \exp\left(\boldsymbol{\omega}_i\Delta t\right), \qquad (94) \\ \Delta\mathbf{v}_i &= \int_0^{t_i}\underbrace{\Delta\mathbf{R}_t}_{\overset{(94)}{=}\exp(\boldsymbol{\omega}_i t)}\underbrace{\mathbf{a}_i}_{\text{cst.}}dt \\ &= \int_0^{t_i}\exp\left(\boldsymbol{\omega}_i t\right)\mathbf{a}_i dt \\ &= \mathcal{J}_{\boldsymbol{\omega}\Delta t}\mathbf{a}_i\Delta t, \qquad (95) \\ \Delta\mathbf{p}_i &= \int_0^{t_i}\underbrace{\Delta\mathbf{v}_t}_{\overset{(95)}{=}\mathcal{J}_{\boldsymbol{\omega}t}\mathbf{a}_i t}dt \\ &= \int_0^{t_i}\mathcal{J}_{\boldsymbol{\omega}t}t dt\mathbf{a}_i \\ &= \left(\frac{\Delta t^2}{2}\mathbf{I}_3 + a(\boldsymbol{\omega}_i)_\times + b(\boldsymbol{\omega}_i)_\times^2\right)\mathbf{a}_i, \qquad (96)\end{aligned}$$

where $a = \frac{\phi\Delta t\cos(\phi\Delta t)-\sin(\phi\Delta t)}{\phi^3}$, $b = \frac{\frac{\phi^2\Delta t^2}{2}-\cos(\phi\Delta t)-\phi\Delta t\sin(\phi\Delta t)+1}{\phi^4}$ and $\phi = \|\boldsymbol{\omega}_i\|$. It leads to the true and measurement values

$$\boldsymbol{\Upsilon}_i = \left[\begin{array}{c|cc} \exp(\boldsymbol{\omega}_i\Delta t) & \mathcal{J}_{\boldsymbol{\omega}_i\Delta t}\mathbf{a}_i\Delta t & \mathbf{C}\mathbf{a}_i \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right], \hat{\boldsymbol{\Upsilon}}_i = \left[\begin{array}{c|cc} \exp(\hat{\boldsymbol{\omega}}_i\Delta t) & \mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}\hat{\mathbf{a}}_i\Delta t & \hat{\mathbf{C}}\hat{\mathbf{a}}_i \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right], \tag{97}$$

where $\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_i^m - \hat{\mathbf{b}}_i^{\boldsymbol{\omega}}$, $\hat{\mathbf{a}}_i = \mathbf{a}_i^m - \hat{\mathbf{b}}_i^{\mathbf{a}}$,

$$\mathbf{C} = \frac{\Delta t^2}{2}\mathbf{I}_3 + \frac{\phi\Delta t\cos(\phi\Delta t) - \sin(\phi\Delta t)}{\phi^3}(\boldsymbol{\omega}_i)_\times + \frac{\frac{\phi^2\Delta t^2}{2} - \cos(\phi\Delta t) - \phi\Delta t\sin(\phi\Delta t) + 1}{\phi^4}(\boldsymbol{\omega}_i)_\times^2, \tag{98}$$

$$\hat{\mathbf{C}} = \frac{\Delta t^2}{2}\mathbf{I}_3 + \frac{\hat{\phi}\Delta t\cos(\hat{\phi}\Delta t) - \sin(\hat{\phi}\Delta t)}{\hat{\phi}^3}(\hat{\boldsymbol{\omega}}_i)_\times + \frac{\frac{\hat{\phi}^2\Delta t^2}{2} - \cos(\hat{\phi}\Delta t) - \hat{\phi}\Delta t\sin(\hat{\phi}\Delta t) + 1}{\hat{\phi}^4}(\hat{\boldsymbol{\omega}}_i)_\times^2, \tag{99}$$

$\phi = \|\boldsymbol{\omega}_i\|$ and $\hat{\phi} = \|\hat{\boldsymbol{\omega}}_i\|$. The equations are obtained with analogy to the IMU preintegration with Coriolis, see Section II, and thus substitute the values obtained in (83).

Computing the noise Jacobian $\mathbf{G}_i$ is more tedious. First, we note the computation of $\Delta\mathbf{R}_i$ is the same as constant global acceleration, leading to same Jacobian for $\Delta\mathbf{R}_i$. Regarding accelerometer noise, the noise is linear in the expressions of the factors so the Jacobian w.r.t. $\boldsymbol{\eta}_i^{\mathbf{a}}$ is simple to obtain.

To compute the Jacobian of $\Delta\mathbf{v}_i$, we indeed obtain

$$\begin{aligned} \Delta\mathbf{v}_i &= \mathcal{J}_{\boldsymbol{\omega}_i\Delta t}\mathbf{a}_i\Delta t \\ &= \mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}\hat{\mathbf{a}}_i\Delta t + \mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t + \frac{\partial\Delta\mathbf{v}_i}{\partial\boldsymbol{\phi}}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2, \|\boldsymbol{\eta}_i^{\mathbf{a}}\|^2\right) \\ &= \Delta\hat{\mathbf{v}}_i + \mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}\boldsymbol{\eta}_i^{\mathbf{a}}\Delta t + \frac{\partial\Delta\mathbf{v}_i}{\partial\boldsymbol{\phi}}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2, \|\boldsymbol{\eta}_i^{\mathbf{a}}\|^2\right), \end{aligned} \tag{100}$$

To compute $\frac{\partial\Delta\mathbf{v}_i}{\partial\boldsymbol{\phi}}$, we apply the chain rule as follows

$$\frac{\partial\Delta\mathbf{v}}{\partial\boldsymbol{\phi}} = -A(\mathbf{a}_i)_\times - B(\boldsymbol{\omega}_i)_\times\left[(\mathbf{a}_i)_\times + \mathbf{a}_i\right] + (\boldsymbol{\omega}_i)_\times\mathbf{a}_i\frac{\partial A}{\partial\boldsymbol{\phi}} + (\boldsymbol{\omega}_i)_\times^2\mathbf{a}_i\frac{\partial B}{\partial\boldsymbol{\phi}}, \tag{101}$$

where

$$A = \frac{1 - \cos(\phi\Delta t)}{\phi^2}, \tag{102}$$

$$B = \frac{\phi\Delta t - \sin(\phi\Delta t)}{\phi^3}, \tag{103}$$

$$\begin{aligned} \frac{\partial A}{\partial\boldsymbol{\phi}} &= \frac{d\frac{1-\cos(\phi\Delta t)}{\phi^2}}{d\boldsymbol{\phi}}|_{\boldsymbol{\phi}} = \underbrace{\frac{d\phi}{d\boldsymbol{\phi}}}_{\phi^{-1}\boldsymbol{\phi}^T}\underbrace{\frac{d\frac{1-\cos(\phi\Delta t)}{\phi^2}}{d\phi}}_{\phi^{-3}(\phi\Delta t\sin(\phi\Delta t)-2-2\cos(\phi\Delta t))} \\ &= \frac{\boldsymbol{\phi}^T}{\phi^4}\left(\phi\Delta t\sin(\phi\Delta t) - 2 - 2\cos(\phi\Delta t)\right), \end{aligned} \tag{104}$$

$$\begin{aligned} \frac{\partial B}{\partial\boldsymbol{\phi}} &= \frac{d\frac{\phi\Delta t-\sin(\phi\Delta t)}{\phi^3}}{d\boldsymbol{\phi}}|_{\boldsymbol{\phi}} = \underbrace{\frac{d\phi}{d\boldsymbol{\phi}}}_{\phi^{-1}\boldsymbol{\phi}^T}\underbrace{\frac{d\frac{\phi\Delta t-\sin(\phi\Delta t)}{\phi^3}}{d\boldsymbol{\phi}}}_{\phi^{-4}(-2\phi\Delta t-\phi\Delta t\cos(\phi\Delta t)+3\sin(\phi\Delta t))} \\ &= \frac{\boldsymbol{\phi}^T}{\phi^5}\left(-2\phi\Delta t - \phi\Delta t\cos(\phi\Delta t) + 3\sin(\phi\Delta t)\right). \end{aligned} \tag{105}$$

Similarly applying the chain rule for $\Delta\mathbf{p}_i$, we obtain

$$\begin{aligned} \Delta\mathbf{p}_i &= \mathbf{C}\mathbf{a}_i \\ &= \hat{\mathbf{C}}\hat{\mathbf{a}}_i + \hat{\mathbf{C}}\boldsymbol{\eta}_i^{\mathbf{a}} + \frac{\partial\Delta\mathbf{p}}{\partial\boldsymbol{\phi}}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2, \|\boldsymbol{\eta}_i^{\mathbf{a}}\|^2\right) \\ &= \Delta\hat{\mathbf{p}}_i + \hat{\mathbf{C}}\boldsymbol{\eta}_i^{\mathbf{a}} + \frac{\partial\Delta\mathbf{p}}{\partial\boldsymbol{\phi}}\boldsymbol{\eta}_i^{\boldsymbol{\omega}} + O\left(\|\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\|^2, \|\boldsymbol{\eta}_i^{\mathbf{a}}\|^2\right), \end{aligned} \tag{106}$$

where

$$\frac{\partial\Delta\mathbf{p}}{\partial\boldsymbol{\phi}} = -a(\mathbf{a}_i)_\times - b(\boldsymbol{\omega}_i)_\times\left[(\mathbf{a}_i)_\times + \mathbf{a}_i\right] + (\boldsymbol{\omega}_i)_\times\mathbf{a}_i\frac{\partial a}{\partial\boldsymbol{\phi}} + (\boldsymbol{\omega}_i)_\times^2\mathbf{a}_i\frac{\partial b}{\partial\boldsymbol{\phi}}, \tag{107}$$

$$\frac{\partial a}{\partial \boldsymbol{\phi}} = \frac{d\frac{\phi\Delta t\cos(\phi\Delta t)-\sin(\phi\Delta t)}{\phi^3}}{d\boldsymbol{\phi}}|_{\boldsymbol{\phi}} = \underbrace{\frac{d\phi}{d\boldsymbol{\phi}}}_{\phi^{-1}\boldsymbol{\phi}^T} \underbrace{\frac{d\frac{\phi\Delta t\cos(\phi\Delta t)-\sin(\phi\Delta t)}{\phi^3}}{d\phi}}_{\phi^{-4}(-\phi^2\Delta t^2\sin(\phi\Delta t)-3\phi\Delta t\cos(\phi\Delta t)+3\sin(\phi\Delta t))}$$

$$= \frac{\boldsymbol{\phi}^T}{\phi^5}\left(-\phi^2\Delta t^2\sin(\phi\Delta t) - 3\phi\Delta t\cos(\phi\Delta t) + 3\sin(\phi\Delta t)\right), \tag{108}$$

$$\frac{\partial b}{\partial \boldsymbol{\phi}} = \frac{d\frac{\frac{\phi^2\Delta t^2}{2}-\cos(\phi\Delta t)-\phi\Delta t\sin(\phi\Delta t)+1}{\phi^4}}{d\boldsymbol{\phi}}|_{\boldsymbol{\phi}} = \underbrace{\frac{d\phi}{d\boldsymbol{\phi}}}_{\phi^{-1}\boldsymbol{\phi}^T} \underbrace{\frac{d\frac{\frac{\phi^2\Delta t^2}{2}-\cos(\phi\Delta t)-\phi\Delta t\sin(\phi\Delta t)+1}{\phi^4}}{d\boldsymbol{\phi}}}_{\phi^{-5}\left(\frac{3}{2}\phi^2\Delta t^2-\phi^2\Delta t^2\cos(\phi\Delta t)-4(-\cos(\phi\Delta t)-\phi\Delta t\sin(\phi\Delta t)+1)\right)}$$

$$= \frac{\boldsymbol{\phi}^T}{\phi^6}\left(-\frac{3}{2}\phi^2\Delta t^2 - \phi^2\Delta t^2\cos(\phi\Delta t) - 4\left(-\cos(\phi\Delta t) - \phi\Delta t\sin(\phi\Delta t) + 1\right)\right). \tag{109}$$

Taking into account matrix multiplication, we thus have

$$\boldsymbol{\Upsilon}_i = \hat{\boldsymbol{\Upsilon}}_i \exp\left(\underbrace{-\begin{bmatrix} \mathcal{J}^{-1}_{\hat{\boldsymbol{\omega}}_i\Delta t}\Delta t & \mathbf{0}_{3\times3} \\ -\exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\frac{\partial\Delta\mathbf{v}_i}{\partial\boldsymbol{\phi}} & \exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\mathcal{J}_{\hat{\boldsymbol{\omega}}_i\Delta t}\Delta t \\ \exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\frac{\partial\Delta\mathbf{p}_i}{\partial\boldsymbol{\phi}} & \exp(-\hat{\boldsymbol{\omega}}_i\Delta t)\hat{\mathbf{C}} \end{bmatrix}}_{\mathbf{G}_i} \begin{bmatrix} \boldsymbol{\eta}_i^{\boldsymbol{\omega}} \\ \boldsymbol{\eta}_i^{\mathbf{a}} \end{bmatrix}\right). \tag{110}$$

IV. Preintegrated Measurement Residual & Jacobians

In this section, we provide analytic expressions for the residual errors of the preintegrated measurements along with their Jacobian matrices w.r.t. state variables $\mathbf{T}_i$ and $\mathbf{T}_j$. These Jacobians are required when using iterative optimization techniques.

A. Preintegrated Measurement Residual

Based on our uncertainty representation

$$\boldsymbol{\Upsilon}_{ij} = \hat{\boldsymbol{\Upsilon}}_{ij}\exp(\boldsymbol{\eta}_{ij}) = \left(\boldsymbol{\Gamma}_{ij}\Phi_{ij}\left(\mathbf{T}_i\right)\right)^{-1}\mathbf{T}_j, \tag{111}$$

the residual is given as

$$\begin{aligned}
\hat{\mathbf{r}}_{ij} &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\left(\boldsymbol{\Gamma}_{ij}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)\right)^{-1}\hat{\mathbf{T}}_j\right) \\
&= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right) \\
&= \log\left(\left[\begin{array}{c|c|c} \Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j & \Delta\hat{\mathbf{R}}^T\left(\hat{\mathbf{R}}_i^T\left(\hat{\mathbf{v}}_j - \mathbf{g}\Delta t - \hat{\mathbf{v}}_i\right) - \Delta\hat{\mathbf{v}}_{ij}\right) & \Delta\hat{\mathbf{R}}^T\left(\hat{\mathbf{R}}_i^T\left(\hat{\mathbf{p}}_j - \frac{1}{2}\Delta t_{ij}^2\mathbf{g} - \mathbf{p}_i - \mathbf{v}_i\Delta t\right) - \Delta\hat{\mathbf{p}}_{ij}\right) \\ \hline \mathbf{0}_{2\times3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array}\right]\right) \\
&= \begin{bmatrix} \log\left(\Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j\right) \\ \mathcal{J}^{-1}_{\log\left(\Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j\right)}\Delta\hat{\mathbf{R}}^T\left(\hat{\mathbf{R}}_i^T\left(\hat{\mathbf{v}}_j - \mathbf{g}\Delta t - \hat{\mathbf{v}}_i\right) - \Delta\hat{\mathbf{v}}_{ij}\right) \\ \mathcal{J}^{-1}_{\log\left(\Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j\right)}\Delta\hat{\mathbf{R}}^T\left(\hat{\mathbf{R}}_i^T\left(\hat{\mathbf{p}}_j - \frac{1}{2}\Delta t_{ij}^2\mathbf{g} - \mathbf{p}_i - \mathbf{v}_i\Delta t\right) - \Delta\hat{\mathbf{p}}_{ij}\right) \end{bmatrix},
\end{aligned} \tag{112}$$

where the preintegration factor has been corrected w.r.t. bias as

$$\hat{\boldsymbol{\Upsilon}}_{ij} = \hat{\boldsymbol{\Upsilon}}_{ij}(\hat{\mathbf{b}}_i)\exp\left(\frac{\partial\hat{\boldsymbol{\Upsilon}}_{ij}}{\partial\mathbf{b}}|_{\hat{\mathbf{b}}_i}\delta\mathbf{b}\right). \tag{113}$$

B. Preintegrated Measurement Jacobian

The error of the system is defined as

$$\mathbf{e} = \begin{bmatrix} \boldsymbol{\xi}_i \\ \boldsymbol{\xi}_j \end{bmatrix}. \tag{114}$$

We first compute the Jacobian of the residual w.r.t. $\hat{\mathbf{T}}_j$ by perturbing the residual as follows

$$\begin{aligned}\mathbf{r}_{ij} &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\exp(\boldsymbol{\xi}_j)\right)\\ &= \underbrace{\log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right)}_{\overset{(112)}{=}\hat{\mathbf{r}}_{ij}} + \mathcal{J}_{\hat{\mathbf{r}}_{ij}}^{-1}\boldsymbol{\xi}_j + O(\|\boldsymbol{\xi}_j\|^2)\\ &= \hat{\mathbf{r}}_{ij} + \mathcal{J}_{\hat{\mathbf{r}}_{ij}}^{-1}\boldsymbol{\xi}_j + O(\|\boldsymbol{\xi}_j\|^2).\end{aligned} \tag{115}$$

We similarly process for the Jacobian of the residual w.r.t. $\hat{\mathbf{T}}_i$ as

$$\begin{aligned}\mathbf{r}_{ij} &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\exp(\boldsymbol{\xi}_j)\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right)\\ &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\left(\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)\exp(\mathbf{F}_{\Delta t_{ij}}\boldsymbol{\xi}_j)\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right)\\ &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\exp(-\mathbf{F}_{\Delta t_{ij}}\boldsymbol{\xi}_j)\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right)\\ &= \log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\exp(-\operatorname{Ad}_{\Phi_{ij}(\hat{\mathbf{T}}_i)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j}\mathbf{F}_{\Delta t_{ij}}\boldsymbol{\xi}_j)\right)\\ &= \underbrace{\log\left(\hat{\boldsymbol{\Upsilon}}_{ij}^{-1}\Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j\right)}_{\overset{(112)}{=}\hat{\mathbf{r}}_{ij}} - \mathcal{J}_{\hat{\mathbf{r}}_{ij}}^{-1}\operatorname{Ad}_{\Phi_{ij}(\hat{\mathbf{T}}_i)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j}\mathbf{F}_{\Delta t_{ij}}\boldsymbol{\xi}_i + O(\|\boldsymbol{\xi}_j\|^2)\\ &= \hat{\mathbf{r}}_{ij} - \mathcal{J}_{\hat{\mathbf{r}}_{ij}}^{-1}\operatorname{Ad}_{\Phi_{ij}(\hat{\mathbf{T}}_i)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j}\mathbf{F}_{\Delta t_{ij}}\boldsymbol{\xi}_i + O(\|\boldsymbol{\xi}_j\|^2).\end{aligned} \tag{116}$$

The 9×18 Jacobian w.r.t. system error is thus obtained as

$$\frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{e}} = \mathcal{J}_{\hat{\mathbf{r}}_{ij}}^{-1}\begin{bmatrix}-\operatorname{Ad}_{\Phi_{ij}(\hat{\mathbf{T}}_i)^{-1}\boldsymbol{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j}\mathbf{F}_{\Delta t_{ij}} & \mathbf{I}_9\end{bmatrix}. \tag{117}$$

Considering rotating Earth consists only on correctly modified the residual (112) after the third line.

V. Preintegration with Biases

We augment the state-space of the previous section with bias dynamic model, leading to the system dynamic

$$\dot{\mathbf{R}} = \mathbf{R}\left(\boldsymbol{\omega}^m - \mathbf{b}^{\boldsymbol{\omega}} - \boldsymbol{\eta}^{\boldsymbol{\omega}}\right)_\times, \tag{118}$$
$$\dot{\mathbf{v}} = \mathbf{R}\left(\mathbf{a}^m - \mathbf{b}^{\mathbf{a}} - \boldsymbol{\eta}^{\mathbf{a}}\right) + \mathbf{g}, \tag{119}$$
$$\dot{\mathbf{p}} = \mathbf{v}, \tag{120}$$
$$\dot{\mathbf{b}}^{\boldsymbol{\omega}} = \mathbf{n}^{\boldsymbol{\omega}}, \tag{121}$$
$$\dot{\mathbf{b}}^{\mathbf{a}} = \mathbf{n}^{\mathbf{a}}, \tag{122}$$

where $\mathbf{n}^{\boldsymbol{\omega}}$ and $\mathbf{n}^{\mathbf{a}}$ are zero-mean Gaussian noises such that biases follow a random-walk dynamic. To preintegration with bias factor, we augment the preintegration factor as follow

$$\boldsymbol{\Upsilon}_{ij} = \hat{\boldsymbol{\Upsilon}}_{ij}\exp(\boldsymbol{\eta}_{ij}), \tag{123}$$
$$\Delta\mathbf{b}_{ij} = \underbrace{\Delta\hat{\mathbf{b}}_{ij}}_{=\mathbf{0}_3} + \boldsymbol{\eta}_{ij}^{\mathbf{b}}, \tag{124}$$

where $\Delta\mathbf{b}_{ij} = \mathbf{b}_i - \mathbf{b}_i$ correspond to the bias dynamic model. The covariance of the combined factor

$$\boldsymbol{\Sigma}_{ij} := \operatorname{cov}\left(\begin{bmatrix}\boldsymbol{\eta}_{ij}\\ \boldsymbol{\eta}_{ij}^{\mathbf{b}}\end{bmatrix}\right) \tag{125}$$

is incrementally computed as

$$\boldsymbol{\Sigma}_{i(j+1)} = \begin{bmatrix}\operatorname{Ad}_{\hat{\boldsymbol{\Upsilon}}_j^{-1}}\mathbf{F} & \mathbf{G}_j\\ \mathbf{0}_{6\times6} & \mathbf{I}_6\end{bmatrix}\boldsymbol{\Sigma}_{i(j+1)}\begin{bmatrix}\operatorname{Ad}_{\hat{\boldsymbol{\Upsilon}}_j^{-1}}\mathbf{F} & \mathbf{G}_j\\ \mathbf{0}_{6\times6} & \mathbf{I}_6\end{bmatrix}^T + \begin{bmatrix}\mathbf{G}_j & \mathbf{0}_{6\times6}\\ \mathbf{0}_{6\times6} & \mathbf{I}_6\end{bmatrix}\operatorname{cov}\left(\begin{bmatrix}\boldsymbol{\eta}_i^{\boldsymbol{\omega}}\\ \boldsymbol{\eta}_i^{\mathbf{a}}\\ \mathbf{n}_i^{\mathbf{a}}\\ \mathbf{n}_i^{\mathbf{a}}\end{bmatrix}\right)\begin{bmatrix}\mathbf{G}_j & \mathbf{0}_{6\times6}\\ \mathbf{0}_{6\times6} & \mathbf{I}_6\end{bmatrix}^T, \tag{126}$$

and starting from initial condition $\mathbf{\Sigma}_{ii} = \mathbf{0}_{15\times 15}$. The above system incorporates bias errors that captures the drift in a given bias over an interval. Note that these bias error terms describe the deviation of the bias over the interval due to the random-walk drift, rather than the error of the current bias estimate.

Consequently, the augmented residual is given as

$$\hat{\mathbf{r}}_{ij}^{+} = \begin{bmatrix} \log\left(\Delta\hat{\mathbf{R}}^T \hat{\mathbf{R}}_i^T \hat{\mathbf{R}}_j\right) \\ \mathcal{J}^{-1}_{\log\left(\Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j\right)} \Delta\hat{\mathbf{R}}^T \left(\hat{\mathbf{R}}_i^T \left(\hat{\mathbf{v}}_j - \mathbf{g}\Delta t - \hat{\mathbf{v}}_i\right) - \Delta\hat{\mathbf{v}}_{ij}\right) \\ \mathcal{J}^{-1}_{\log\left(\Delta\hat{\mathbf{R}}^T\hat{\mathbf{R}}_i^T\hat{\mathbf{R}}_j\right)} \Delta\hat{\mathbf{R}}^T \left(\hat{\mathbf{R}}_i^T \left(\hat{\mathbf{p}}_j - \frac{1}{2}\Delta t_{ij}^2 \mathbf{g} - \mathbf{p}_i - \mathbf{v}_i \Delta t\right) - \Delta\hat{\mathbf{p}}_{ij}\right) \\ \hat{\mathbf{b}}_j - \hat{\mathbf{b}}_i \end{bmatrix}, \tag{127}$$

where the preintegration factor has been corrected w.r.t. bias as

$$\hat{\mathbf{\Upsilon}}_{ij} = \hat{\mathbf{\Upsilon}}_{ij}(\hat{\mathbf{b}}_i) \exp\left(\frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \delta\mathbf{b}\right). \tag{128}$$

The error of the augmented system is defined as

$$\mathbf{e} = \begin{bmatrix} \boldsymbol{\xi}_i \\ \mathbf{e}_i^{\mathbf{b}} \\ \boldsymbol{\xi}_j \\ \mathbf{e}_j^{\mathbf{b}} \end{bmatrix}, \tag{129}$$

whose 15×30 residual Jacobian is given as

$$\frac{\partial \mathbf{r}_{ij}^{+}}{\partial \mathbf{e}} = \begin{bmatrix} -\mathcal{J}^{-1}_{\hat{\mathbf{r}}_{ij}} \operatorname{Ad}_{\Phi_{ij}(\hat{\mathbf{T}}_i)^{-1}\mathbf{\Gamma}_{ij}^{-1}\hat{\mathbf{T}}_j} \mathbf{F}_{\Delta t_{ij}} & -\mathcal{J}^{-1}_{\hat{\mathbf{r}}_{ij}} \operatorname{Ad}_{\hat{\mathbf{r}}_{ij}} \frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}} & \mathcal{J}^{-1}_{\hat{\mathbf{r}}_{ij}} & \mathbf{0}_{9\times 6} \\ \mathbf{0}_{6\times 9} & -\mathbf{I}_6 & \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix}, \tag{130}$$

where the bias at time t_j is not used during IMU factor computation. To compute the Jacobian of the residual w.r.t. $\mathbf{e}_i^{\mathbf{b}}$, we first known that bias error affect preintegration measurement as

$$\hat{\mathbf{\Upsilon}}_{ij} = \hat{\mathbf{\Upsilon}}_{ij} \exp\left(\frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}}\right). \tag{131}$$

We finally obtain the Jacobian by perturbing the residual as follows

$$\begin{aligned} \mathbf{r}_{ij} &= \log\left(\left(\hat{\mathbf{\Upsilon}}_{ij} \exp\left(\frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}}\right)\right)^{-1} \Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1} \mathbf{\Gamma}_{ij}^{-1} \hat{\mathbf{T}}_j\right) \\ &= \log\left(\exp\left(-\frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}}\right) \hat{\mathbf{\Upsilon}}_{ij}^{-1} \Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1} \mathbf{\Gamma}_{ij}^{-1} \hat{\mathbf{T}}_j\right) \\ &= \log\left(\hat{\mathbf{\Upsilon}}_{ij}^{-1} \Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1} \mathbf{\Gamma}_{ij}^{-1} \hat{\mathbf{T}}_j \exp\left(-\operatorname{Ad}_{\underbrace{\hat{\mathbf{\Upsilon}}_{ij}^{-1} \Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1} \mathbf{\Gamma}_{ij}^{-1} \hat{\mathbf{T}}_j}_{\hat{\mathbf{r}}_{ij}}} \frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}}\right)\right) \\ &= \underbrace{\log\left(\hat{\mathbf{\Upsilon}}_{ij}^{-1} \Phi_{ij}\left(\hat{\mathbf{T}}_i\right)^{-1} \mathbf{\Gamma}_{ij}^{-1} \hat{\mathbf{T}}_j\right)}_{\hat{\mathbf{r}}_{ij}} - \mathcal{J}^{-1}_{\hat{\mathbf{r}}_{ij}} \operatorname{Ad}_{\hat{\mathbf{r}}_{ij}} \frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}} + O(\|\mathbf{e}_i^{\mathbf{b}}\|^2) \end{aligned} \tag{132}$$

$$= \hat{\mathbf{r}}_{ij} - \mathcal{J}^{-1}_{\hat{\mathbf{r}}_{ij}} \operatorname{Ad}_{\hat{\mathbf{r}}_{ij}} \frac{\partial \hat{\mathbf{\Upsilon}}_{ij}}{\partial \mathbf{b}}|_{\hat{\mathbf{b}}_i} \mathbf{e}_i^{\mathbf{b}} + O(\|\mathbf{e}_i^{\mathbf{b}}\|^2). \tag{133}$$

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