Problem Set 7 - Life Cycle Model

ECON 7395-I, Spring 2018
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Problem 1

- *Agent's finite horizon problem in recursive form:*

$$
\begin{aligned}
V^t(a) = \max_{n,a'} \left[\frac{c^{1-\sigma}}{1-\sigma}\right] + \beta V^{t+1}(a') \\
s.t. \quad c + a' = (1+r)a + w\epsilon \\
a_0 = 0, a_{T+1} = 0
\end{aligned}
$$

  Factor prices (wages w and interest rate r). Work efficiency ϵ_t changes deterministically with age. Agent dies at $T+1$.

- *Parameter Values:*
  $\sigma = 2$, $\beta = 1$, $r = 0.03$, $w = 1$
  Agent lives from age 16 – 90

Determining Work Efficiency, ϵ_t

- Convert weekly salary data for each age from Bureau of Labor Statistics to hourly wage.
- Get coefficients $q0$, $q1$ and $q2$ of quadratic that fits this data.
- Compute for each t, $\epsilon_t = q_0 + q_1 t + q_2 t^2$ to get the hourly wage series.

Characterizing Equations

Lagrangian

$$L = \max_{c_t, a_{t+1}} \sum_{t=0}^{T} \beta^t \left[\left(\frac{c^{1-\sigma}}{1-\sigma} \right) + \lambda_t [(1+r)a_t + w\epsilon_t - c_t - a_{t+1}] \right]$$

First Order Conditions

$$c_t : \qquad \beta^t c_t^{-\sigma} - \beta^t \lambda_t^{-\sigma} = 0 \qquad (1)$$

$$a_{t+1} : \qquad \beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1+r) = 0 \qquad (2)$$

$$\lambda_t : \qquad (1+r)a_t + w\epsilon_t - c_t - a_{t+1} = 0 \qquad (3)$$

Simplifying the f.o.cs to get:

$$((1+r)a_{t+1} + w\epsilon_{t+1} - a_{t+2})^{\sigma} = \beta(1+r)((1+r)a_t + w\epsilon_t - a_{t+1})^{\sigma} \qquad (4)$$

Forward Iteration

Solution Method

1. Guess (a_1), given $(a_0 = 0)$ and the parameter values.
2. Compute
   i a_{t+2} using equation (4)
   $$a_{t+2} = (1+r)a_{t+1} + w\epsilon_{t+1} - (\beta(1+r))^{\frac{1}{\sigma}}((1+r)a_t + w\epsilon_t - a_{t+1})$$
   ii c_t using the budget constraint
   $$c_t = (1+r)a_t + w\epsilon_t - a_{t+1}$$
3. Do step 2 recursively to get consumption and asset sequences: $\{c_t\}_{t=0}^{T}$ and $\{a_t\}_{t=1}^{T+1}$
4. Check if the last element of sequence a_t, satisfies the boundary condition, i.e. $a_{T+1} = 0$
5. Thus $F : a_1 \rightarrow a_{T+1}$. Find roots of F using bisection method.

Brute Force

Solution Method

1. Vectorize the system of Euler equations

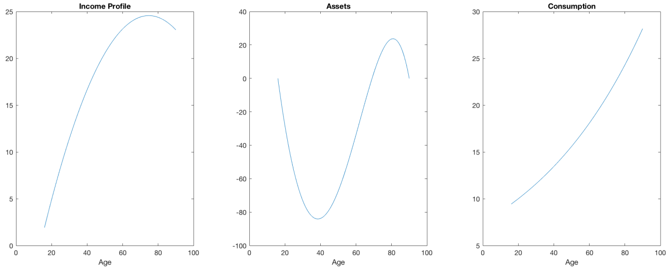
$$(1+r)\boldsymbol{a_{t+1}} + w\boldsymbol{\epsilon_{t+1}} - \boldsymbol{a_{t+2}} - (\beta(1+r))^{\frac{1}{\sigma}}((1+r)\boldsymbol{a_t} + w\boldsymbol{\epsilon_t} - \boldsymbol{a_{t+1}}) = 0$$

2. Initial and final asset levels are given $a_0 = 0$ and $a_{T+1} = 0$

$$\boldsymbol{a_t} = \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{T-1} \end{bmatrix}, \quad \boldsymbol{a_{t+1}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_T \end{bmatrix}, \quad \boldsymbol{a_{t+2}} = \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ 0 \end{bmatrix}.$$

3. Use *fsolve* to solve the system and get the vector of assets. Use budget constraint to retrieve the consumption vector.

Solution



Problem 2

- *Agent's finite horizon problem in recursive form:*

$$
\begin{aligned}
V^t(h,a) = \max_{n,a'} \left[\frac{c^{1-\sigma}}{1-\sigma} + \frac{(1-n)^{1-\gamma}}{1-\gamma} \right] + \beta V^{t+1}(h',a') \\
s.t. \quad c + a' = (1+r)a + hn \\
h' = G(h,n), \quad n \in [0,1] \\
a_0 = 0, a_{T+1} = 0, h_0 \equiv 1
\end{aligned}
$$

  Three learning technologies:
  (1) $h' = (1-\delta)h + \theta n^{\alpha},$ [learning-by-doing]
  (2) $h' = (1-\delta)h + \theta(1-n)^{\alpha},$ [learning-by-not-doing]
  (3) $h' = (1-\delta)h + \theta n^{\alpha}(1-n)^{1-\alpha},$ [Both]

- *Parameter Values:*
  $\sigma = 2,\ \beta = 1,\ r = 0.03,\ w = 1$
  $\delta = 0.025,\ \theta = 0.8,\ \alpha = 0.75,\ \gamma = 0.8$
  Agent lives from age 16 – 90

Characterizing Equations

Lagrangian

$$L = \max_{c_t, n_t, a_{t+1}, h_{t+1}} \sum_{t=0}^{T} \beta^t \left(\frac{c^{1-\sigma}}{1-\sigma} + \frac{(1-n)^{1-\gamma}}{1-\gamma} \right) + \lambda_t[(1+r)a_t + h_t n_t - c_t - a_{t+1}]$$
$$+\mu_t[(1-\delta)h + \theta(1-n)^{\alpha} - h_{t+1}]$$

Simplifying the f.o.cs to get:

$$c_{t+1} = [\beta(1+r)]^{\frac{1}{\sigma}} c_t \tag{5}$$
$$h_{t+1} = (1-\delta)h_t + \theta n_t^{\alpha} \tag{6}$$
$$a_{t+1} = (1+r)a_t + h_t n_t - c_t \tag{7}$$

$$\frac{1}{\alpha\theta} n_t^{1-\alpha} \left((1-n_t)^{-\gamma} - c_t^{-\sigma} h_t \right) = \beta c_{t+1}^{-\sigma} n_{t+1} + \beta(1-\delta) \left[\frac{1}{\alpha\theta} n_{t+1}^{1-\alpha} \left((1-n_{t+1})^{-\gamma} - c_{t+1}^{-\sigma} h_{t+1} \right) \right] \tag{8}$$

Backward Iteration

1. Guess (c_T, h_{T+1}), given $a_{T+1} = 0$ and the parameter values.
2. Compute
   i n_t and h_t solving the two equations (6) and (8)

$$\frac{1}{\alpha\theta} n_t^{1-\alpha} \left((1-n_t)^{-\gamma} - c_t^{-\sigma} h_t \right) = \beta c_{t+1}^{-\sigma} n_{t+1} + \beta(1-\delta) \left[\frac{1}{\alpha\theta} n_{t+1}^{1-\alpha} \left((1-n_{t+1})^{-\gamma} - c_{t+1}^{-\sigma} h_{t+1} \right) \right]$$

$$h_{t+1} = (1-\delta) h_t + \theta n_t^{\alpha}$$

   - Check if corner exists: $n_t > 0$ *or* $n_t \leq 0$
   - Set value of n_t accordingly.

   ii a_t using equation (7)

$$a_t = (c_t + a_{t+1} - h_t n_t)/(1+r)$$

   iii c_t using equation (5)

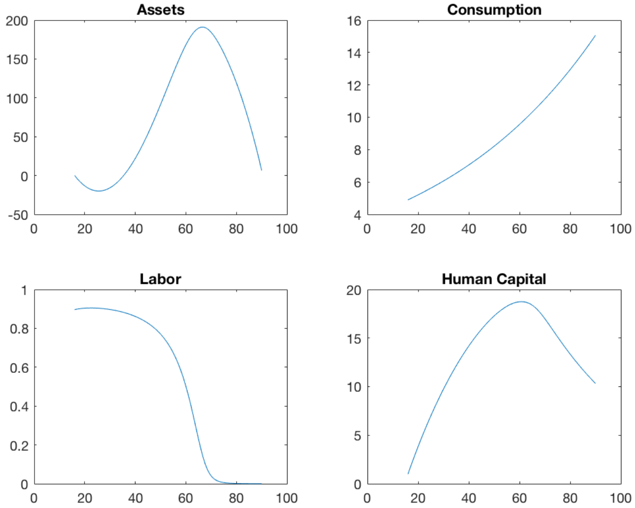
$$c_t = [\beta(1+r)]^{-\frac{1}{\sigma}} c_{t+1}$$

3. Go back recursively to get sequence of $\{c_t,\ h_t,\ a_t,\ n_t\}$, and period 0 assets and human capital (a_0, h_0).
4. Thus $F : (c_T, h_{T+1}) \rightarrow (a_0, h_0 - 1)$. Find roots of F.

Forward Iteration

1. Guess (c_0, n_0), given $(a_0 = 0, h_0 = 1)$ and the parameter values.
2. Compute
   i c_{t+1} using equation (5)
   $$c_{t+1} = [\beta(1+r)]^{\frac{1}{\sigma}} c_t$$
   ii h_{t+1} using equation (6)
   $$h_{t+1} = (1-\delta)h_t + \theta n_t^{\alpha}$$
   iii a_{t+1} using equation (7)
   $$a_{t+1} = (1+r)a_t + h_t n_t - c_t$$
   iv n_{t+1} solving the non-linear equation (8)
   $$\frac{1}{\alpha\theta} n_t^{1-\alpha}\left((1-n_t)^{-\gamma} - c_t^{-\sigma} h_t\right) = \beta c_{t+1}^{-\sigma} n_{t+1} + \beta(1-\delta)\left[\frac{1}{\alpha\theta} n_{t+1}^{1-\alpha}\left((1-n_{t+1})^{-\gamma} - c_{t+1}^{-\sigma} h_{t+1}\right)\right]$$
   - Check if corner exists: $n_{t+1} > 0$ *or* $n_{t+1} \leq 0$
   - Set value of n_{t+1} accordingly.
3. Repeat step 2 to get sequence of $\{c_t,\ h_t,\ a_t,\ n_t\}$, and period $T+1$ assets (a_{T+1}) which is the last element of sequence a_t.
4. Thus $F : (c_0, n_0) \rightarrow a_{T+1}$. Find roots of F.

Solution – Learning By Doing



Solution – Learning By Not Doing