Amemiya

Exercises

P3.2

$P(X<0.5, Y<0.5) = \int_0^{0.5}\int_0^{x} 2(x+y)\,dy\,dx = 0.125$

$P(X<0.5) = 0.125$ $\quad P(Y<0.5) = \int_0^{0.5}\int_y^{1} 2(x+y)\,dx\,dy = 0.625$

P3.3

$f(x,y) = f(y|x)\cdot f(x) = 2xy + 2(1-x)(1-y) = 2-2x-2y+4xy$

$f = f(y|x\in(0.8,1)) = \dfrac{\int_{0.8}^{1} f(x,y)dx}{\int_{0.8}^{1}\int_0^1 f(x,y)\,dy\,dx} = \cancel{0.6}\ 0.2+1.6y$

$\therefore P = \int_{0.8}^{1}(0.2+1.6y)dy = 0.328$

P3.4

(a) $f(x) = \int_0^{1-x} f(x,y)dy = 2(1-x)$

(b) $P(0<Y<\frac{3}{4}|X=0.5) = \int_0^{0.5} 2dy = 1$

$f(y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{2}{2(1-x)} = \dfrac{1}{1-x}$ $\quad \therefore f(y|x=0.5) = 2$

P3.5

(a) $g(y) = \dfrac{e^{-\frac{y-1}{2}}}{2}$ $\quad$ (b) $g(y) = \dfrac{e^{-\sqrt{y}}}{2\sqrt{y}}$ $\quad$ (c) $g(y) = \dfrac{e^{-\frac{1}{y}}}{y^2}$ $\quad$ (d) $g(y) = e^{y}\cdot e^{-e^{y}}$

P3.6

$g(y) = \dfrac{0.5}{1} + \left|\dfrac{0.5}{-2}\right| = 0.75$

P3.7

$Z = X-Y \Rightarrow Y = X-Z$

$Pr(Z\le z) = \begin{cases} 1-\frac{1}{2}(1-z)^2 & z\ge 0 \\ \frac{1}{2}(1+z)^2 & z<0 \end{cases}$ $\quad \therefore f(z) = \begin{cases} -(1-z) & z\ge 0 \\ 1+z & z<0 \end{cases}$

P3.8

$f(u) = e^{-u}$ $\quad f(v) = e^{-v}$ $\quad f(u,v) = e^{-(u+v)}$ $\quad \begin{cases} X=U \\ Y=U+V \end{cases}$ $\quad \begin{cases} U=X \\ V=Y-X \end{cases}$

$f(x,y) = \dfrac{f(x,y-x)}{|1-0|} = e^{-(x+y-x)} = e^{-y}$ $\quad f(x) = e^{-x}$ $\quad f(y) = e^{-y}\int_0^{y}dx = e^{-y}\cdot y$

$f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{1}{y}$

P4.1

P4.7:

$$Vx = E(X^2) - E^2(X)$$
$$= \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx - \left[\int_0^1 x^2 dx + \int_1^2 x(2-x)dx\right]^2$$
$$= \frac{1}{6}$$

P4.11:

$E(X) = P$ $\quad E(Y) = E(E(Y|X)) = \frac{1}{2}P + (1-P) = 1 - \frac{1}{2}P$

$$E(Y|X) = \begin{cases} \frac{1}{2} & X=1 \\ 1 & X=0 \end{cases} \qquad E(Y|X)\cdot X = \begin{cases} \frac{1}{2} & X=1 \\ 0 & X=0 \end{cases}$$

$$E(XY) = E[E(XY|X)] = E[XE(Y|X)] = \frac{1}{2}P$$

$$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{2}P - P(1-\frac{1}{2}P) = \frac{1}{2}P^2 - \frac{1}{2}P$$

P4.15:

$$\beta^* = \frac{Cov(Y,Z)}{Var(Z)} = \frac{Cov(Y, X+Y)}{Var(X+Y)} = \frac{Cov(X,Y) + Var(Y)}{Var(X)+Var(Y)+2Cov(X,Y)} = \frac{0.5+1}{1+1+1} = \frac{1}{2}$$

$$\alpha^* = E(Y) - \beta^* E(Z) = 0 - \frac{1}{2}\times 0 = 0$$

P4.19:

$$X = \begin{cases} 1 & P \\ 0 & 1-P \end{cases} \qquad Y = \begin{cases} 1 & P \\ 0 & 1-P \end{cases} \qquad X+Y = \begin{cases} 1 & X-Y=1 \\ 2 \text{ or } 0 & X-Y=0 \\ 1 & X-Y=-1 \end{cases}$$

$$\therefore BP: E(X+Y|X-Y) = \begin{cases} 1 & X-Y=1 \\ 2P^2/P^2+(1-P)^2 & X-Y=0 \\ 1 & X-Y=-1 \end{cases}$$

$$BLP: \beta^* = \frac{Cov(X+Y, X-Y)}{Var(X-Y)} = \frac{Var(X)-Var(Y)}{Var(X)+Var(Y)-2Cov(X,Y)} = 0$$

$$\alpha^* = E(X+Y) - \beta^* E(X-Y) = 2P$$

$$MSE_{BP}: E(X+Y - E(X+Y|X-Y))^2 = \frac{4P^4}{(P^2+(1-P)^2)^2}\cdot(1-P)^2 + \left(\frac{2-2P^2}{P^2+(1-P)^2}\right)^2 P^2$$

$$MSE_{BLP} = E(X+Y-2P)^2 = (2-2P)^2\cdot P^2 + (1-2P)^2\cdot(1-P)P\cdot 2 + 4P^2(1-P)^2$$

P5.2:

$$U = \begin{cases} X & W=1 \\ Y & W=0 \end{cases} \qquad P(U<5) = P(U<5|W=1)\cdot P(W=1) + P(U<5|W=0)\cdot P(W=0)$$

$=$

P5.3:

$$E(Y|X) = E(T+TS^2|S) = E(T|S) + S^2E(T|S) = E(T) + S^2E(T) = 1+S^2$$

$$\beta^* = \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(S, T+TS^2)}{Var(S)} = \frac{E(S^3)}{Var(S)} = 0 \qquad \alpha^* = E(Y) = 1+E(S^2) = 2$$

$$MSE_{BP}: E[T+TS^2-1-S^2]^2 = E[T^2+T^2S^2-T-TS^2+T^2S^2+T^2S^4-TS^2-TS^4-T-TS^2+1+S^2-S^2T$$