PG 45

P2.1:

$E(E(E(y|x_1,x_2,x_3)|x_1,x_2)|x_1) = E(E(y|x_1,x_2)|x_1) = E(y|x_1)$

P2.2:

$E(yx) = E[E(yx|x)] = E[xE[y|x]] = E[x(a+bx)] = E(ax+bx^2) = aE(x) + bE(x^2)$

P2.3:

Proof: $E(h(x)e) = E[E(h(x)\cdot e|x)] = E[h(x)E[e|x]] = 0$

P2.4:

$E[y|x] = \begin{cases} 0.8 & x=0 \\ 0.6 & x=1 \end{cases}$. $E(y^2|x) = \begin{cases} 0.8 & x=0 \\ 0.6 & x=1 \end{cases}$ $Var(y|x) = \begin{cases} 0.16 & x=0 \\ 0.24 & x=1 \end{cases}$.

P2.5:

(a) $\min E(e^2-h(x))^2$

(c) $E(e^2-\sigma^2(x))^2$ Let $E(e^2|x) = \sigma^2(x)$

$= E(e^4) - 2E(e^2E(e^2|x)) + E(E^2(e^2|x)) = E(e^4) - \underbrace{2E(\sigma^2(x)E(e^2|x))}_{=2E(E(e^2\sigma^2(x)|x))} + E(\sigma^4(x))$

$E(e^2E(e^2|x))$
$= E[E(e^2E(e^2|x)|x)]$
$= E[E(e^2|x)\cdot E(e^2|x)]$
$= E(\sigma^4(x))$

~~Let~~ $= E(e^4) - 2E(\sigma^4(x)) + E(\sigma^4(x)) = E(e^4) - E(\sigma^4(x))$

$E(e^2-h(x))^2$

$= E(e^4) - 2E(e^2h(x)) + E(h^2(x)) = E(e^4) - 2E[E(e^2h(x)|x)] + E(h^2(x))$

$= E(e^4) - 2E[h(x)\sigma^2(x)] + E[h^2(x)] = E(e^4) - E(\sigma^4(x)) + E(\sigma^4(x)) - 2E[h(x)\sigma^2(x)] + E[h^2(x)]$

$= E(e^2-\sigma^2(x))^2 + E[\sigma^2(x) - h(x)]^2 \geq E(e^2-\sigma^2(x))^2$ Proof completed.

P2.6:

Proof: $Var(y) = Var[m(x)+e] = Var(m(x)) + Var(e) + 2Cov(m(x),e)$

$= Var(m(x)) + \sigma^2 + E(\underset{0}{m(x)\cdot e}) - \underset{0}{E(m(x))\cdot E(e)} = Var(m(x)) + \sigma^2$ Proof completed

P2.7:

Proof: $\sigma^2(x) = E[e^2|x] = E[(y-m(x))^2|x] = E[y^2 - 2m(x)y + m^2(x)|x] = E[y^2|x] - 2m(x)E[y|x] + m^2(x)$

$= E[y^2|x] - 2E[y|x]E[y|x] + E^2[y|x] = E[y^2|x] - E^2[y|x]$ Proof completed.

P2.8:

Conditional on x, we can treat $x'\beta$ to be constant. So the distribution is actually

$Pr(y=j|x) = \frac{exp(-x'\beta)(x'\beta)^j}{j!} = \frac{exp(-a)a^j}{j!}$ $a = x'\beta$.

So $E(y|x) = a = x'\beta$; $Var(y|x) = a = x'\beta$. This justifies a linear regression model.

P2.9: Let $x_3 = \begin{cases} 1 & \text{if } A \text{ or } B \\ 0 & \text{if } C \end{cases}$; $x_4 = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } \neg A \end{cases}$

$E[y|x_1,x_2] = \alpha_1x_1 + \alpha_3x_3 + \alpha_4x_4$.

P2.10: $E[x^2e] = E[E(x^2e|x)] = E[x^2E(e|x)] = 0$ True

P2.11: ~~E(xe)~~ False. $E(x^2e) = E(x\cdot xe)$ No way to figure it out.

P2.12: False. e.g. $e = x\varepsilon$, $x \sim N(0,1)$, $\varepsilon \sim N(0,1)$, x and ε are independent.

P2.13: False. ~~$E(e|x) = E$~~ $E(xe) = E[E[xe|x]] = E[xE(e|x)] = 0$ No way to figure it out.

Problem Set 2

Pass

Zibin Huang

P2.14.

$\varepsilon \sim N(-x, \sigma^2)$

False. Consider $e = \varepsilon + x$ such that $E(\varepsilon|x) = -x$; $Var(\varepsilon|x) = \sigma^2$.

We have: $E[e|x] = 0$; $E(e^2|x) = \sigma^2$.

But here, e is not independent of x.

P2.15.

BLP:

$x = 1$

$\therefore \alpha = E^{-1}(1 \cdot 1) E(1 \cdot y)$

$= E(y)$.

BP:

$$E(y-\alpha)^2 = E(y - E(y) + E(y) - \alpha)^2 = E(y-E(y))^2 + 2E[(y-E(y))(E(y)-\alpha)] + E(E(y)-\alpha)^2$$

$$= E(y-E(y))^2 + \{E[yE(y)] - E(y\alpha) - E[E^2(y)] + E(\alpha E(y))\} \cdot 2 + E(E(y)-\alpha)^2$$

$$= E(y-E(y))^2 + \{E^2(y) - \alpha E(y) - E^2(y) + \alpha E(y)\} \cdot 2 + E(E(y)-\alpha)^2$$

$$= E(y-E(y))^2 + E(E(y)-\alpha)^2 \geq E(y-E(y))^2$$

When $\alpha = E(y)$, we have the equality.

P2.16:

$f(x) = \int_0^1 \frac{3}{2}(x^2+y^2)dy = \frac{3}{2}x^2 + \frac{1}{2}$ $\quad f(y|x) = \frac{f(x,y)}{f(x)} = \frac{3(x^2+y^2)}{3x^2+1}$

~~$E(y|x) = \int_0^1 y \cdot \frac{3}{2}(x^2+y^2)dy = \int_0^1 (\frac{3}{2}yx^2 + \frac{3}{2}y^3)dy = \frac{3}{4}x^2 + \frac{3}{8}$~~ $\quad E(y|x) = \int_0^1 y \cdot \frac{3(x^2+y^2)}{3x^2+1} dy$

$= \frac{6x^2+3}{12x^2+4}$.

For linear predictor:

$\beta = Var^{-1}(x) \cdot Cov(x,y)$ $\quad f_x(x) = \int_0^1 \frac{3}{2}(x^2+y^2)dy = \frac{3}{2}x^2 + \frac{1}{2}$

$E(xy) = \int_0^1\int_0^1 xy \cdot \frac{3}{2}(x^2+y^2)dxdy$ $\quad E(x^2) = \int_0^1 x^2 f_x(x)dx = \int_0^1 (\frac{3}{2}x^4 + \frac{1}{2}x^2)dx = \frac{7}{15}$

$= \int_0^1\int_0^1 (\frac{3}{2}x^3y + \frac{3}{2}xy^3)dxdy$ $\quad \therefore \beta = \frac{3}{8} \times \frac{15}{7} = \cancel{\frac{45}{56}}$ $\frac{45}{56}$.

$= \int_0^1 (\frac{3}{8}y + \frac{3}{4}y^3)dy = \frac{3}{8}$ $\quad f_Y(y) = \int_0^1 \frac{3}{2}(x^2+y^2)dx = \frac{3}{2}y^2 + \frac{1}{2}$

$\therefore E(x) = \int_0^1 x \cdot f_x(x)dx = \int_0^1 x(\frac{3}{2}x^2 + \frac{1}{2})dx = \frac{5}{8}$ $\quad E(y) = \int_0^1 y \cdot f_Y(y)dy = \frac{5}{8}$

~~$\frac{5}{8} = \frac{5}{8} \cdot \frac{45}{56} + \alpha \Rightarrow \alpha = \frac{55}{448}$~~ $\quad$ ~~$\therefore y = \frac{45}{56}x + \frac{55}{448}$~~.

~~They are different.~~ $\quad Var(x) = E^2(x) - E(x^2) = \frac{7}{15} - \frac{25}{64} = \frac{73}{960}$

$Cov(x,y) = E(xy) - E(x)E(y) = -\frac{1}{64}$ $\quad \therefore \beta = -\frac{15}{73}$. $\quad \alpha = \bar{y} - \beta\bar{x} = \frac{5}{8} + \frac{15}{73} \times \frac{5}{8} = \frac{55}{73}$ $\quad \therefore y = -\frac{15}{73}x + \frac{55}{73}$.

They are different.

P2.17.

Proof:

When $m = \mu$, $s = \sigma^2$:

$$E\,g(x|m,s) = E\,g(x|\mu,\sigma^2) = \begin{pmatrix} E(x)-\mu \\ E(x-\mu)^2 - \sigma^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

When $E(g|m,s) = 0$:

$E(x-m) = 0 \Rightarrow E(x) - E(m) = 0 \Rightarrow m = E(x) = \mu$;

$E[(x-\mu)^2 - s] = 0 \Rightarrow E(x-\mu)^2 - E(s) \Rightarrow s = E(x-\mu)^2 = \sigma^2$. $\quad$ Proof completed.

P2.18:

(a) $xx' = \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix} [1 \ x_2 \ x_3] = \begin{bmatrix} 1 & x_2 & x_3 \\ x_2 & x_2^2 & x_2x_3 \\ x_3 & x_2x_3 & x_3^2 \end{bmatrix}$ We can see that row1 $\cdot x_2$ = row 2,

~~So xx' is a singular matrix, which means that it is not invertible.~~

$E(xx') = \begin{bmatrix} 1 & E(x_2) & E(x_3) \\ E(x_2) & E(x_2^2) & E(x_2x_3) \\ E(x_3) & E(x_2x_3) & E(x_3^2) \end{bmatrix}$ When $x_3 = a_1 + a_2x_2$, we have:

$$E(xx') = \begin{bmatrix} 1 & E(x_2) & a_1+a_2E(x_2) \\ E(x_2) & E(x_2^2) & a_1E(x_2)+a_2E(x_2^2) \\ a_1+a_2E(x_2) & a_1E(x_2)+a_2E(x_2^2) & a_1^2+2a_1a_2E(x_2)+a_2^2E(x_2^2) \end{bmatrix}$$

$\therefore$ Row 1 $\times a_1$ + Row 2 $\times a_2$ = Row 3; So xx' is a singular matrix, which means that it is not invertible

(b) $P(y|x_2, x_3) = \beta_1 + \beta_2 x_2 + \beta_3 x_3$ $\quad P(y|x_2) = \underbrace{(\beta_1 + \beta_3 a_1)}_{\gamma_1} + \underbrace{(\beta_2 + \beta_3 a_2)}_{\gamma_2} x_2$

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ a_1+a_2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} a + cx_2 + ea_1 + a_2ex_2 = 1 \\ b + dx_2 + fa_1 + fa_2x_2 = x_2 \end{cases} \Rightarrow \begin{cases} a=1, c=0, e=0 \\ b=0, d=1, f=0 \end{cases}$$

So we have $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ such that $Tx = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}$.

$$\therefore \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = E((Tx)(Tx)') \, E(Txy)^{-1}.$$

P2.19:

$\min_\beta E[(E(y|x) - x'\beta)^2]$ $\quad E(xy) = E[E(xy|x)] = E[xE(y|x)]$

$= E[E^2(y|x)] - 2E(E(y|x) \cdot x'\beta) + \beta' E(xx')\beta$

FOC: ~~$E(E(y|x)) = E(xx') \cdot \beta$~~ $\quad E(xE(y|x)) = E(xx') \beta$

$\hat{\beta} =$ ~~$\frac{E(x) \cdot E(y|x)}{E(xx')}$~~ $E^{-1}(xx') E(xE(y|x)) = E^{-1}(xx') E(xy)$.

P2.20:

$E(1(x \in X) y) = \int_X \int_R y \cdot f(x, y) dx dy$

$E(1(x \in X) m(x)) = \int_X E(1(x \in X) \cdot \int_R y f_{y|x}(y|x) dy) = \int_X \int_R y f_{y|x}(y|x) dy \cdot f_x(x) dx$

$= \int_X \int_R y \cdot f(x, y) dx dy = E(1(x \in X) y)$ Proof completed.

P3.1:

$$g_n(\hat{\mu}, \hat{\sigma}^2) = 0 \Rightarrow \begin{cases} \frac{1}{n} \Sigma(y_i - \hat{\mu}) = 0 \\ \frac{1}{n} \Sigma[(y_i - \mu)^2 - \hat{\sigma}^2] = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu} = \frac{1}{n} \Sigma y_i \\ \hat{\sigma}^2 = \frac{1}{n} \Sigma (y_i - \mu)^2 \end{cases}$$

~~P3.2: $\hat{\beta}_x = (X'X)^{-1}X'y$ $\quad \hat{\beta}_z = (C'X'XC)^{-1}C'X'y$~~

~~$\hat{e}_x = y - X(X'X)^{-1}X'y$ $\quad \hat{e}_z = y - X(C'X'XC)^{-1}C'X'y$~~