Project 1

MGMT 237G
Instructor: L. Goukasian

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them, and interpret the results. Code quality, speed, and accuracy will determine the grades.

Submit your codes and a PDF file of your answers to questions (including graphs, histograms, but no codes, in this PDF file) by 11PM PDT on Next Wednesday.

1. Use the Random Number generators discussed in the class to do the following:

   (a) Using the LGM method generate 10,000 Uniformly distributed random numbers on [0,1] and compute the empirical mean and the standard deviation of the sequence.

   (b) Use built-in functions of the software you are using to do the same thing as in (a).

   (c) Compare your findings in (a) and (b) and comment (be short, but precise).

2. Use the numbers of part (a) of question 1 to do the following:

   (a) Generate 10,000 random numbers with the following distribution:

$$X = \begin{cases} -1 \ \ with \ \ probability \ \ 0.30 \\ \ \ 0 \ \ with \ \ probability \ \ 0.35 \\ \ \ 1 \ \ with \ \ probability \ \ 0.20 \\ \ \ 2 \ \ with \ \ probability \ \ 0.15 \end{cases}$$

   (b) Draw the histogram and compute the empirical mean and standard deviation of the sequence of 10,000 numbers generated in part (a).

3. Using the LGM method generate Uniformly distributed random numbers on [0,1] to do the following:

   (a) Generate 1,000 random numbers with Binomial distribution with $n = 44$ and $p = 0.64$. (*Hint*: A random variable with Binomial distribution (n, p) is a sum of n Bernoulli (p) distributed random variables, so you will need to generate 44,000 Uniformly distributed random numbers, to start with).

   (b) Draw the histogram. Compute the probability that the random variable X that has Binomial (44, 0.64) distribution, is at least 40: $P(X \geq 40)$. Use any statistics textbook or online resources for the exact number for the above probability and compare it with your finding and comment.

4. Using the LGM method generate Uniformly distributed random numbers on [0,1] to:

   (a) Generate 10,000 Exponentially distributed random numbers with parameter $\lambda = 1.5$.

   (b) Compute $P(X \geq 1)$ and $P(X \geq 4)$.

(c) Compute the empirical mean and the standard deviation of the sequence of 10,000 numbers generated in part (a). Draw the histogram by using the 10,000 numbers of part (a).

5. Using the LGM method generate Uniformly distributed random numbers on [0,1] to:

    (a) Generate 5,000 Normally distributed random numbers with mean 0 and variance 1, by **Box-Muller** Method.
    (b) Now use the **Polar-Marsaglia** method to Generate 5,000 Normally distributed random numbers with mean 0 and variance 1.
    (c) Now compare the efficiencies of the two above-algorithms, by comparing the execution **times** to generate 5,000 normally distributed random numbers by the two methods. Which one is more efficient? If you do not see a clear difference, you need to increase the number of generated realizations of random variables to 10,000, 20,000, etc.

Project 2

MGMTMFE 405
Instructor: L. Goukasian

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1. Generate a series $(X_i, Y_i)$for $i = 1, \dots, n$ of Bivariate-Normally distributed random vectors, with the mean vector of (0,0) and the variance – covariance matrix of $\begin{pmatrix} 3 & a \\ a & 5 \end{pmatrix}$. Compute the following by simulation:

$$\rho(a) = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \acute{X})(Y_i - \acute{Y})}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \acute{X})^2}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \acute{Y})^2}}$$

where $\acute{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \acute{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Take $n = 1000$ and $a = -0.7$.
Inputs: a
Output: ρ.

2. Evaluate the following expected values by using Monte Carlo simulation:

$$E = \mathrm{E}\,[\max(0, (X^3 + \sin(Y) + X^2 Y))]$$

where X and Y have $N(0,1)$ distribution and a correlation of $\rho = 0.6$.
Inputs: ρ
Outputs: E

3. (a) Estimate the following expected values by simulation:

$$A(t) = E\left(W_t{}^2 + sin(W_t)\right) \text{ and } B(t) = E\left(e^{\frac{t}{2}} cos(W_t)\right) \text{ for } t = 1,\ 3,\ 5.$$

Here W_t is a Standard Wiener Process.

(b) How are the values of $B(t)$ (for the cases $t = 1,\ 3, 5$) related?

(c) Now use a variance reduction technique (whichever you want) to compute the expected value $B(t)$ for the case $t = 5$. Do you see any improvements? Comment on your findings..

Inputs: t
Outputs: 1) $A(t)$ and $B(t)$ for all 3 t's. 2) Writeup: comments for parts (b) and (c).

4. Let S_t be a Geometric Brownian Motion process: $S_t = S_0 e^{\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)}$, where $r = 0.04, \sigma = 0.2, S_0 = \$88$, W_t is a Standard Brownian Motion process (Standard Wiener process).

(a) Estimate the price **c** of a European Call option on the stock with $T = 5$, $X = \$100$ *by using Monte Carlo simulation*.

(b) Compute the exact value of the option **c** by the Black-Scholes formula.

(c) Now use variance reduction techniques (whichever you want) to estimate the price in part (a) again. Did the accuracy improve? Comment.

Inputs: r , σ, S_0
Outputs: 1) *Ca1 for* parts (a) and (b); 2) Writeup: comments for part (c)

5. (a) For each integer number n from 1 to 10, use 1000 simulations of S_n to estimate ES_n, where S_t is a Geometric Brownian Motion process: $S_t = S_0 e^{\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)}$, where $r = 0.04, \sigma = 0.18, S_0 =$ \$88. Plot all of the above $E(S_n)$, for n ranging from 1 to 10, in one graph.

(b) Now simulate 6 paths of S_t for $0 \le t \le 10$ (defined in part (a)) by dividing up the interval [0, 10] into 1,000 equal parts.

(c) Plot your data from parts (a) and (b) in one graph.

(d) What would happen to the ES_n graph if you increased σ from 18% to 35%? What would happen to the 6 plots of S_t for $0 \le t \le 10$, if you increased σ from 18% to 35%?
Inputs: σ
Outputs: 1) Graphs: plots in a .jpg file; 2) writeup: comments in a .pdf file for part (d)

6. Consider the following integral for computing the number π: $4\int_0^1 \sqrt{1-x^2}dx = \pi$.

(a) The integral above can be estimated by a simple numerical integration using, say Euler's discretization (or any other discretization) scheme. Estimate the integral by using the Euler's discretization scheme.
(b) Estimate the integral by Monte Carlo simulation.
(c) Now try the Importance Sampling method to improve the estimate of π in part (b). Comment on errors and improvements.
Inputs: n (number of MC simulations)
Outputs: 1) Values: *Ia* for part (a), *Ib* for part (b); 2) Writeup: comments in a .pdf file

Project 3

MGMTMFE 405
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1. Evaluate the following expected values and probabilities:

$$p1 = P(Y_2 > 5),$$
$$e1 = E\left(X_2^{\frac{1}{3}}\right), \qquad e2 = E(Y_3), \qquad e3 = E\big(X_2 Y_2 1(X_2 > 1)\big),$$

where the Ito's processes X and Y evolve according to the following SDEs:

$$dX_t = \left(\frac{1}{5} - \frac{1}{2}X_t\right)dt + \frac{2}{3}dW_t,\ X_0 = 1; \quad dY_t = \left(\left(\frac{2}{1+t}\right)Y_t + \frac{1+t^3}{3}\right)dt + \frac{1+t^3}{3}dZ_t,\ \ Y_0 = \frac{3}{4},$$

and W, Z are independent Wiener Processes.

Inputs: $X_0,\ Y_0$
Outputs: Values: p1, e1. e2, e3.

2. Estimate the following expected values:

$$e1 = E(1 + X_3)^{1/3}, \quad e2 = E(X_1 Y_1)$$

where

$$dX_t = \frac{1}{4}X_t dt + \frac{1}{3}X_t dW_t - \frac{3}{4}X_t dZ_t,\ \ X_0 = 1; \ \text{ and } \ Y_t = e^{-0.08t + \frac{1}{3}W_t + \frac{3}{4}Z_t},$$

and W, Z are independent Wiener Processes.

Inputs: X_0
Outputs: Values: e*1*, *e2*

3.

(a) Write a code to compute prices of European Call options via Monte Carlo simulation. Use variance reduction techniques (e.g. Antithetic Variates) in your estimation. The function should be generic: for any input of the 5 parameters - $S_0, T, X, r, \sigma -$ the output is the corresponding price of the European call option.

(b) Write code to compute the prices of European Call options by using the Black-Scholes formula. Use the approximation of $N(\cdot)$ described in this chapter. The code should be generic: for any input values of the 5 parameters - S_0, T, X, r, σ - the output is the corresponding price of the European call option.

(c) Estimate the greeks –delta, gamma, theta, and vega of European Call options (all five Greeks) and graph them as functions of the initial stock price S_0. Use $X = 20$, $\sigma = 0.25$, $r = 0.04$ and $T = 0.5$ in your estimations. Use the range [15, 25] forS_0, with a step size of 1. You will have 4 different graphs for each of the 4 greeks.

In all cases, dt (time-step) should be user-defined. Use dt=0.004 (a day) as a default value.

Inputs: *dt, S_0, T, X, r, Sigma*
Outputs: Values: *C1* for part (a), *C2* for part (b), *D, G T, V* for part (c).

4. Consider the following 2-factor model for stock prices with stochastic volatility:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t}\, S_t dW_t^1 \\ dV_t = \alpha(\beta - V_t)dt + \sigma\sqrt{V_t}\, dW_t^2 \end{cases}$$

where the Brownian Motion processes above are correlated: $dW_t^1 dW_t^2 = \rho dt$, where the correlation ρ is a constant in $[-1,1]$.
Estimate the price of a European Call option (via Monte Carlo simulation) that has a strike price of K and matures in T years.
Use the following default parameters of the model: $\rho = -0.6,\ r = 0.03,\ S_0 = \$48,\ V_0 = 0.05,\ \sigma = 0.42,\ \alpha = 5.8,\ \beta = 0.0625$.
Use the Full Truncation, Partial Truncation and Reflection methods, and provide 3 price estimates by using the three methods.

Inputs: $\rho,\ r,\ S_0,\ K, T,\ V_0,\ \sigma,\ \alpha,\ \beta$.
Outputs: Values: *C1*, C*2*, C*3*.

5. The objective of this exercise is to compare a sample of Pseudo-Random numbers with a sample of Quasi-MC numbers of $Uniform[0,1]x[0,1]$:

a) Generate 100 2-dimensional Uniform [0,1]x[0,1] vectors by using any one of the algorithms for random number generation.
b) Generate 100 points of the 2-dimensional Halton sequences, using bases 2 and 7.
c) Generate 100 points of the 2-dimensional Halton sequences, using bases 2 and 4. (Note: 4 is a non-prime number!).
d) Draw all 3 sequences of random numbers on separate graphs. Are there differences in the three sets (visual test only)? Comment on your observations.
e) Use 2-dimensional Halton sequences to compute the following integral:

$$I = \int_0^1 \int_0^1 e^{-xy}\left(sin6\pi x + cos^{\frac{1}{3}}2\pi y\right) dxdy$$

Use N=10,000 in your simulations. Try different couples for bases: (2,4), (2,7), (5,7).

Inputs: For e) b1 and b2 (bases)
Outputs: Values: *I* for part (e); Graphs: 3 plots for part (d); Writeup: comments for part (d).

6. ***OPTIONAL [NOT for grading]***

(a) You hold two European Call options with similar characteristics, but one (the first) matures in 1 year and the other (the second) matures in 3 months. Which will have higher delta? Higher Gamma? Use explicit formulas to answer. Use simulations to answer.
(b) Which is more expensive: an ATM European call or an ATM European put on the same stock with the same maturity? Use explicit formulas to answer. Use simulations to answer.
(c) Which is more expensive: a 5% OTM European call or a5% OTM European put on the same stock with the same maturity? Use explicit formulas to answer. Use simulations to answer.
(d) Which is higher: the Gamma of an ATM European call or an ATM European put on the same stock with the same maturity? What if they both are 10% ITM or 10% OTM? Use explicit formulas to answer. Use simulations to answer.

Project 4

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1. Compare the convergence rates of the four methods below by doing the following:
Use the Binomial Method to price a 6-month European Call option with the following information: the risk-free interest rate is 5% per annum and the volatility is 24%/annum, the current stock price is \$32 and the strike price is \$30. Divide the time interval into n parts to estimate the price of this option. Use $n = 10,20,40,80,100,200,$ and 500 to estimate the price and draw them all in one graph, where the horizontal axis measures n, and the vertical one the price of the option.

(a) Use the binomial method in which

$$u = \frac{1}{d}, \qquad d = c - \sqrt{c^2 - 1}, \qquad c = \frac{1}{2}\left(e^{-r\Delta} + e^{(r+\sigma^2)\Delta}\right), \qquad p = \frac{e^{r\Delta} - d}{u - d}$$

(b) Use the binomial method in which

$$u = e^{r\Delta}\left(1 + \sqrt{e^{\sigma^2\Delta} - 1}\right), \qquad d = e^{r\Delta}\left(1 - \sqrt{e^{\sigma^2\Delta} - 1}\right), \qquad p = 1/2$$

(c) Use the binomial method in which

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}}, \qquad d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta - \sigma\sqrt{\Delta}}, \qquad p = 1/2$$

(d) Use the binomial method in which

$$u = e^{\sigma\sqrt{\Delta}}, \qquad d = e^{-\sigma\sqrt{\Delta}}, \qquad p = \frac{1}{2} + \frac{1}{2}\left(\frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta}}{\sigma}\right)$$

Outputs: Graphs: 4 plots in one graph.

2. Take the current price of AMZN. Use risk-free rate of 2% per annum, and strike price that is the closest integer to 110% of the current price (divisible by 10). Estimate the price of the call option that expires in January of next year, using the Binomial Method. AMZN does not pay dividends. To estimate the historical volatility, use 60 months of historical stock price data on the company. You may use *Bloomberg* or *finance.yahoo.com* to obtain historical prices and the current stock price of AMZN.

(a) Compare your estimated option price with the one you can get from *Bloomberg* or *finance.yahoo.com* and comment.
(b) If the two are different in part (a), find the volatility that would make your estimated price equal to the market price and comment.

Outputs: Writeup: comments for parts (a) and (b).

3. Consider the following information on the stock of a company and options on it:

$S_0 = \$49,\ K = \$50,\ r = 0.03,\ \sigma = 0.2,\ T = 0.3846$ (20 weeks), $\mu = 0.14.$

Using the Binomial Method (any one of them) estimate the following and draw the graphs:

(i) Delta of the call option as a function of S_0, for S_0ranging from \$20 to \$80, in increments of \$2.

(ii) Delta of the call option, as a function of T (time to expiration), from 0 to 0.3846 in increments of 0.01.

(iii) Theta of the call option, as a function of S_0, for S_0ranging from \$20 to \$80 in increments of \$2.

(iv) Gamma of the call option, as a function of S_0, for S_0ranging from \$20 to \$80 in increments of \$2.

(v) Vega of the call option, as a function of S_0, for S_0ranging from \$20 to \$80 in increments of \$2.

(vi) Rho of the call option, as a function of S_0, for S_0ranging from \$20 to \$80 in increments of \$2.

Outputs: Graphs: 6 plots all in one graph.

4. Consider 12-month put options on a stock of company XYZ. Assume the risk-free rate is 5%/annum and the volatility of the stock price is 30 % /annum and the strike price of the option is \$100. Use a Binomial Method to estimate the prices of European and American Put options with current stock prices varying from \$80 to \$120 in increments of \$4. Draw them all in one graph, compare and comment.

Outputs: i. Graphs: plot in a graph, and ii. writeup: comments.

5. Compare the convergence rates of the two methods below by doing the following:
Use the Trinomial Method to price a 6-month European Call option with the following information: the risk-free interest rate is 5% per annum and the volatility is 24%/annum, the current stock price is \$32 and the strike price is \$30. Divide the time interval into n parts to estimate the price of this option. Use n = 10, 15, 20, 40, 70, 80, 100, 200 and 500 to compute the approximate price and draw them in one graph, where the horizontal axis measures n, and the vertical one measures the price of the option.
The two methods are in (a) and (b) below:

(a) Use the trinomial method applied to the stock price-process (S_t) in which

$$u = \frac{1}{d}, \qquad d = e^{-\sigma\sqrt{3\Delta}},$$

$$p_d = \frac{r\Delta(1-u)+(r\Delta)^2+\sigma^2\Delta}{(u-d)(1-d)},\ \ p_u = \frac{r\Delta(1-d)+(r\Delta)^2+\sigma^2\Delta}{(u-d)(u-1)},\ \ p_m = 1 - p_u - p_d$$

(b) Use the trinomial method applied to the Log-stock price-process (X_t) in which

$$\Delta X_u = \sigma\sqrt{3\Delta},\quad \Delta X_d = -\sigma\sqrt{3\Delta}$$

$$p_d = \frac{1}{2}\left(\frac{\sigma^2\Delta + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta^2}{\Delta {X_u}^2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta}{\Delta X_u}\right),\ p_u = \frac{1}{2}\left(\frac{\sigma^2\Delta + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta^2}{\Delta {X_u}^2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta}{\Delta X_u}\right),\ p_m = 1 - p_u - p_d$$

Outputs: Graphs: plot in a graph.

6. Use Halton's Low-Discrepancy Sequences to price European call options. The code should be generic: it will prompt for the user inputs for S_0, K, T, r, σ, N (number of points), b_1 (base 1) and b_2 (base 2). Use the Box-Muller method to generate Standard Normal Variats as follows:

$$\begin{cases} Z_1 = \sqrt{-2ln(H_1)}cos(2\pi H_2) \\ Z_2 = \sqrt{-2ln(H_1)}sin(2\pi H_2) \end{cases}.$$

where H_1 and H_2will be the Halton's numbers with base b_1 and base $b_2$accordingly.

For the price of the call option you may use the following formula:

$$C = \boldsymbol{E}f(W_T) = e^{-(rT)}\boldsymbol{E}\left(max\left(0, S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T} - K\right)\right)$$

Inputs: S_0, K, T, r, σ, N, b_1, b_2
Outputs: Values: C

7. **[*Optional –NOT for grading*]** How much are you willing to pay to play this game: You toss a fair coin. If it is a Tail then you get \$7 in 18 months. If it is a Head then you lose \$2 immediately. The one and two-year zero-coupon rates are 4% and 6% respectively. Would the amount you are willing to pay to play this game increase or decrease if the payoff (in case of Tails) happens in 36-months?

8. **[*Optional –NOT for grading*]** Value an American Put option that has no maturity (perpetual option). What's the delta of the option if it is at-the-money?

Project 5

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1. Consider the following information on the stock of company XYZ: The current stock price is \$40, and the volatility of the stock price is $\sigma = 20\%$ per annum. Assume the prevailing risk-free rate is $r = 6\%$ per annum. Use the following method to price the specified option:

(a) Use the **LSMC** method with N=100,000 paths simulations (50,000 plus 50,000 antithetic) and time step of $\Delta = \frac{1}{\sqrt{N}}$ to price an American put option with strike price of $X = \$40$ and maturity of 0.5-years, 1-year, 2-years, and. Use the first k of the **Laguerre Polynomials** for $k = 2, 3, 4$. (That is, you will compute 9 prices here).
Compare the prices for the 3 cases $k = 2, 3, 4$ and comment on the choice of k.

(b) Use the **LSMC** method with N=100,000 paths simulations (50,000 plus 50,000 antithetic) and time step of $\Delta = \frac{1}{\sqrt{N}}$ to price an American put option with strike price of $X = \$40$ and maturity of 0.5-years, 1-year, 2-years, and. Use the first k of the **Hermite Polynomials** for $k = 2, 3, 4$. Use time step of $\Delta = \frac{1}{\sqrt{N}}$. (That is, you will compute 9 prices here).
Compare the prices for the 3 cases $k = 2, 3, 4$ and comment on the choice of k.

(c) Use the **LSMC** method with N=100,000 paths simulations (50,000 plus 50,000 antithetic) and time step of $\Delta = \frac{1}{\sqrt{N}}$ to price an American put option with strike price of $X = \$40$ and maturity of 0.5-years, 1-year, 2-years, and. Use the first k of the **Simple Monomials** for $k = 2, 3, 4$. (That is, you will compute 9 prices here).
Compare the prices for the 3 cases $k = 2, 3, 4$ and comment on the choice of k.

(d) Compare all your findings above and comment.

Note: You will need to use weighted-polynomials as done by the authors of the method.

Inputs: S_0, X, T, r, σ, N
Outputs: Values of Option Prices; writeup: comments.

Project 6

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1. Consider a 12-month **Fixed Strike Lookback Call and Put** options, when the interest rate is 3% per annum, the current stock price is $98 and the strike price is $100. Use the MC simulation method to estimate the prices of Call and Put options for the following range of volatilities: from 12% to 48%, in increments of 4%.
The payoff of the Call option is $(S_{max} - X)^+$, where $S_{max} = \max\{S_t \colon t \in [0, T]\}$, and the payoff of the Put option is: $(X - S_{min})^+, where\ S_{min} = \min\ \{S_t : t \in [0, T]\}$.

Inputs: S_0, X, T, r, σ, N
Outputs: Graphs: Call and Put options prices as a function of the volatility. Place the Call graph in Proj6_1a.jpg and the Put graph in Proj6_1b.jpg.

2. Assume that the value of a collateral follows a jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with intensity λ_1, independent of the Brownian Motion process W.
V_t^- is the value process before jump occurs at time t (if any).

Consider a collateralized loan, with a contract rate per period r and maturity T on the above-collateral, and assume the outstanding balance of that loan follows this process:

$$L_t = a - bc^{12t}$$

where $a > 0, b > 0\ , c > 1, and\ \ L_0$ are given. We have that $L_T = 0$.

Define the following stopping time:

$$Q = min\{t \geq 0\colon\ V_t \leq q_t L_t\}$$

This stopping time is the first time when the relative value of the collateral (with respect to the outstanding loan balance) crosses a threshold which will be viewed as the “optimal exercise boundary” of the option to default.

Define another stopping time, which is the first time an adverse event occurs:

$$S = min\{t \geq 0 \colon N_t > 0\}$$

Assume that N_t is a Poisson process with intensity of λ_2.

Define $\tau = \min\{Q, S\}$.

We assume the embedded default option will be exercised at time τ, if and only if $\tau < T$.

If the option is exercised at time Q then the payoff to the borrower is: $\left(L_Q - \epsilon V_Q\right)^+$.
If the option is exercised at time S then the payoff to the borrower is: $abs(L_S - \epsilon V_S)$, where *abs(.)* is the absolute value function.

Notes:

1. If $\min\{Q, S\} > T$ then there is no default option exercise.

2. ϵ Should be viewed as the recovery rate of the collateral, so $(1-\epsilon)$ can be viewed as the legal and administrative expenses.

Assume J has intensity λ_1 and N has intensity λ_2. N is independent of J and W.

Assume the APR of the loan is $R = r_0 + \delta\lambda_2$ where r_0 is the "risk-free" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate per period: *r*.

We have monthly compounding here, so $r = R/12$.

Assume that $q_t = \alpha + \beta t$, where $\beta > 0$, $\alpha < V_0/L_0$ and $\beta = \frac{\epsilon - \alpha}{T}$.

Use r_0 for discounting cash flows. Use the following base-case parameter values:

$V_0 = \$20{,}000, L_0 = \$22{,}000, \mu = -0.1, \sigma = 0.2, \gamma = -0.4, \lambda_1 = 0.2, T = 5$ years, $r_0 = 0.02, \delta = 0.25, \lambda_2 = 0.4, \alpha = 0.7, \epsilon = 0.95$. Notice that $PMT = \frac{L_0 . r}{\left[1 - \frac{1}{(1+r)^n)}\right]}$, where $r = R/12$, $n = T * 12$, and $a = \frac{PMT}{r}, b = \frac{PMT}{r(1+r)^n}, c = (1+r)$. Notice that $q_T = \epsilon$.

Write the code as a function Proj6_2func.* that takes λ_1, λ_2 and T as parameters, setting defaults if these parameters are not supplied, and outputs the default option price, the default probability and the expected exercise time. Function specification:

```
function [D, Prob, Et] = Proj6_2func(lambda1, lambda2, T)
```

(a) Estimate the value of the default option for the following ranges of parameters:

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

(b) Estimate the default probability for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

(c) Find the Expected Exercise Time of the default option, conditional on $\tau < T$. That is, estimate $E(\tau \mid \tau < T)$ for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;
λ_2 from 0.0 to 0.8 in increments of 0.1;
T from 3 to 8 in increments of 1;

Inputs: *seed*
Outputs:

i. Values: the default option D, the default probability Prob and the expected exercise time Et for parts (a), (b) and (c) with λ_1=.2, λ_2=0.4 and T=5.
ii. Graphs: For each of (a), (b) and (c) two graphs as a function of T, first with λ_1=0.2 and λ_2 from 0.0 to 0.8 in increments of 0.1, then with λ_2 = 0.4 and λ_1 from 0.05 to 0.4 in increments of 0.05. Put the two graphs in one .png file.

(d) ***[Optional – NOT for grading]*** Make additional assumptions as necessary to estimate the IRR of the investment.

Note: The drift of the V process should be a function of r_0, λ_1, σ under the risk-neutral measure, to be able to price the option, but not done so in this case.

3. *[Optional – NOT for grading]*

Compute, via MC simulation, the prices of the following options using 50,000 simulations of paths of the underlying stock price process and dividing the time-interval into 50 equal parts:

(a) **Down-and-Out- Put**: $S(0) = 50,\ X = 50,\ r = 0.04,\ T =$ 2 months, $\sigma = 0.4$, and for the following range of S_b: from \$36 to \$46, in increments of \$1.
(b) **Down-and-In - Put**: $S(0) = 50,\ X = 50,\ r = 0.04,\ T =$ 2 months, $\sigma = 0.4$, and for the following range of S_b: from \$36 to \$46, in increments of \$1.

4. *[Optional – NOT for grading]*

Compute the price of the **Asian Average Rate call** option by using:

(a) Standard MC method, where $S(0) = 50,\ X = 50,\ r = 0.04,\ T =$ 2 months, $\sigma = 0.4$.

(b) Halton's Low-discrepancy sequences to generate the paths of the stock price, where $S(0) = 50,\ X = 50,\ r = 0.04,\ T =$ 2 months, $\sigma = 0.4$.

5. *[Optional – NOT for grading]*

Assume the stock price follows an Arithmetic Brownian Motion.

(a) Derive the formula for a price of a European call option, using all the other Black-Scholes assumptions.

(b) Use MC Simulation techniques to compute the price and compare with the one implied by the formula derived.

(c) Now compare the prices of the options using this model with the Black-Scholes prices (using the same parameters). Vary your parameters over a wide range to compare the prices. Any observations?

6. *[Optional – NOT for grading] Pricing of Exchange Options.*
Consider the following information on two stocks: stock of XX and stock of YY: The current stock price of XX is \$40, and the volatility of the stock price is $\sigma_1 = 20\%$ per annum. The current stock price of YY is \$38.5, and the volatility of the stock price is $\sigma_2 = 30\%$ per annum Assume the correlation between the stock returns is $\rho = 0.67$ and the prevailing risk-free rate is $r = 6\%$ per annum. Use the following method to price the specified option:
Use the **LSMC** method with N=50,000 paths simulations and time step of $\Delta = \frac{1}{\sqrt{N}}$ to price an American Exchange Call option with strike price of $K = \$0$ and maturity of 1-year. Use the first k of the **Simple**

Project 7

MGMTMFE 405
Instructor: L. Goukasian

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them, and interpret the results. Code clarity and accuracy will determine the grades.

Submit your codes and a PDF file of your answers to questions (including graphs, histograms, but no codes, in this PDF file) by 11PM PDT on Next Wednesday

1. Consider the following situation on the stock of company XYZ: The current stock price is \$10, and the volatility of the stock price is $\sigma = 20$ per annum. Assume the prevailing risk-free rate is $r = 4$ per annum. Use the $X = ln(S)$ transformation of the Black-Scholes PDE, and $\Delta t = 0.002,$ with $\Delta X = \sigma\sqrt{\Delta t};$ $then\ with\ \Delta X = \sigma\sqrt{3\Delta t};$ $then\ with\ \ \Delta X = \sigma\sqrt{4\Delta t}$, and a *uniform* grid (\$1 increments of S) to price a European Put option with strike price of $K =$ \$10, maturity of 0.5-years, and current stock prices for a range from \$4 to \$16; using the specified methods below:

(a) *Explicit Finite-Difference method,*
(b) *Implicit Finite-Difference method,*
(c) *Crank-Nicolson Finite-Difference method.*

Inputs: K, σ, T, Δt
Outputs:

i. Values: Pa, Pb and Pc for the European Put option using each of the methods (a), (b) and (c).
ii. Writeup: compare the three methods from (a), (b) and (c) and comment. To compare, calculate the relative error with respect to the exact Black-Scholes-Merton formula value. Do this for current stock prices of \$4 to \$16 in \$1 increments and put in a table. Put the table and your comments in a .pdf file.

2. Consider the following situation on the stock of company XYZ: The current stock price is \$10, and the volatility of the stock price is $\sigma = 20$ per annum. Assume the prevailing risk-free rate is $r = 4$ per annum.
Use the Black-Scholes PDE (for S) to price American Call and American Put options with strike prices of $K =$ \$10, maturity of 0.5-years, and current stock prices for a range from \$4 to \$16; using the specified methods below:

(a) *Explicit Finite-Difference method,*
(b) *Implicit Finite-Difference method,*
(c) *Crank-Nicolson Finite-Difference method.*

Choose $\Delta t = 0.002$, with $\Delta S = 0.5,$ or with $\Delta S = 1, or\ with\ \Delta S = 1.5$.

Inputs: K, σ, T, Δt
Outputs:

i. Values: Ca, Cb, Cc, Pa, Pb and Pc and for the American call and put options using each of the methods (a), (b) and (c).

ii. Graphs: Plot the American Call option price as a function of the current stock price from \$4 to \$16 in \$1 increments for methods (a), (b) and (c) on the same graph. Use a color legend or linestyles to differentiate the plots. Do the same for the American Put option in another graph. Place the two graphs in a .pdf file.

3. ***[Optional-NOT for grading]*** Consider the following situation on the stock of company XYZ: The current stock price is \$10, and the volatility of the stock price is $\sigma = 20$ per annum. Assume the prevailing risk-free rate is $r = 4$ per annum. Use the original Black-Scholes PDE

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0$$

$\Delta t = 0.002$, $u = e^{\sigma\sqrt{0.25\Delta t}}$, $u = e^{\sigma\sqrt{\Delta t}}$, $u = e^{\sigma\sqrt{4\Delta t}}$,and a *binomial/trinomial* grid to price the specified options using the specified methods below:

Use the *Explicit Finite-difference method*, the *Implicit Finite-Difference Method*, and *Crank-Nicolson Finite-Difference Method* to price a American Put option with strike price of $K =$ \$10, maturity of 0.5-years, and current stock prices for a range from \$1 to \$20, in increments of about \$1.

4. ***[Optional-NOT for grading]*** Consider the following situation on the stock of company XYZ: The current stock price is \$10, and the volatility of the stock price is $\sigma = 20$ per annum. Assume the prevailing risk-free rate is $r = 4$ per annum. Use the transformation of the Black-Scholes PDE to the **Heat Equation**, $\Delta t = 0.002$, $\Delta X = \sqrt{4\Delta t}$ or $\Delta X = \sqrt{2\Delta t}$, $\Delta X = \sigma\sqrt{1.9\Delta t}$, and a *uniform* grid to price the following options using the specified methods below:

Use the *Explicit Finite-difference method*, the *Implicit Finite-Difference Method*, and *Crank-Nicolson Finite-Difference Method* to price an American Call and Put option with strike price of $K =$ \$10, maturity of 0.5-years, and current stock prices for a range from \$1 to \$20, in increments of about \$1.

Project 8

MGMTMFE 405
Instructor: L. Goukasian

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them, and interpret the results. Code clarity and accuracy will determine the grades.

Submit your codes and a PDF file of your answers to questions (including graphs, histograms, but no codes, in this PDF file) by 11PM PDT on Next Wednesday.

1. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**Vasicek model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$$

with $r_0 = 5\%,\ \sigma = 10\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.

(a) Use Monte Carlo Simulation (assume each time step is a day) to find the price of a pure discount bond, with Face Value of \$1,000, maturing in $T = 0.5$ years (at time $t = 0$):

$$P(t,T) = \mathbb{E}_t^*\left[\$1{,}000 * exp\left(-\int_t^T r(s)ds\right)\right]$$

(b) Use Monte Carlo Simulation to find the price of a coupon paying bond, with Face Value of \$1,000, paying semiannual coupons of \$30, maturing in $T = 4$ years:

$$P(0,C,T) = \mathbb{E}_0^*\left[\sum_{i=1}^{8} C_i * exp\left(-\int_0^{T_i} r(s)ds\right)\right]$$

where $C = \{C_i = \$30 \text{ for } i = 1,2,\dots,7; \text{ and } C_8 = \$1{,}030\}$,
$T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \{0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4\}$.

(c) Use Monte Carlo Simulation to find the price of a European Call option on the pure discount bond in part (a). The option matures in 3 months and has a strike price of $K = \$980$. Use the explicit formula for the underlying bond price (only for the bond price).

(d) Use Monte Carlo Simulation to find the price of a European Call option on the coupon paying bond in part (b). The option matures in 3 months and has a strike price of $K = \$980$. Use Monte Carlo simulation for pricing the underlying bond.

(e) Find the price of a European Call option of part (d) by using the explicit formula for the underlying bond price, and reconcile the findings with the ones of part (d).

2. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**CIR model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

with $r_0 = 5\%, \sigma = 12\%,\ \kappa = 0.92,\ \bar{r} = 5.5\%$.

(a) Use Monte Carlo Simulation to find at time $t = 0$ the price $c(t, T, S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year:

$$c(t, T, S) = \mathbb{E}_t^*\left[exp\left(-\int_t^T r(u)du\right) * \max(P(T, S) - K, 0)\right]$$

(b) Use the *Implicit Finite-Difference Method* to find at time $t = 0$ the price $c(t, T, S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year. The PDE is given as

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 c}{\partial r^2} + \kappa(\bar{r} - r)\ \frac{\partial c}{\partial r} - rc = 0$$

with $c(T, T, S) = \max(P(T, S) - K, 0)$, and $P(T, S)$ is computed explicitly.

(c) Compute the price $c(t, T, S)$ of the European Call option above using the explicit formula, and compare it to your findings in parts (a) and (b) and comment on your findings.

3. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following system of SDE (**G2++ model**):

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \\ dy_t = -by_t dt + \eta dW_t^2 \\ r_t = x_t + y_t + \phi_t \end{cases}$$

$x_0 = y_0 = 0,\ \phi_0 = r_0 = 3\%,\ dW_t^1 dW_t^2 = \rho dt,\ \rho = 0.7, a = 0.1,\ b = 0.3, \sigma = 3\%, \eta = 8\%$.
Assume $\phi_t = const = 3\%$ for any $t \geq 0$.

Use Monte Carlo Simulation to find at time $t = 0$ the price $p(t, T, S)$ of a European Put option, with strike price of $K = \$950$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face value of \$1,000, that matures in $S = 1$ year. Compare it with the price found by the explicit formula and comment on it.

4. *[**Optional Not for grading**]*

Consider a European Put option, with strike price of $K = \$970$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1.5$ years.
Which of the two models below would result in a more expensive price for the option?

(a) The Vasicek model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ with $r_0 = 5\%,\ \sigma = 12\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.

(b) The CIR model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$ with $r_0 = 5\%, \sigma = 54\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.

Answer by using explicit formulas or by Monte Carlo simulations.
Is the answer consistent with your intuition?

Project 9

MGMTMFE 405
Instructor: L. Goukasian

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them, and interpret the results. Code clarity and accuracy will determine the grades.

Submit your codes and a PDF file of your answers to questions (including graphs, histograms, but no codes, in this PDF file) by 11PM PDT on Next Friday.

Consider a 30-year MBS with a fixed $WAC = 8\%$ (monthly cash flows starting in January of this year). The Notional Amount of the Loan is \$100,000.
Use the CIR model of interest rates $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$ with $r_0 = 0.078, k = 0.6, \bar{r} = 0.08, \sigma = 0.12$. Consider the ***Numerix Prepayment Model*** in all problems below.

1.

(a) Compute the price of the MBS. The code should be generic: the user is prompted for inputs and the program runs and gives the output.

(b) Compute the price of the MBS for the following ranges of the parameters: k in 0.3 to 0.9 (in increments of 0.1) and draw the graph of the price vs. k.

(c) Compute the price of the MBS for the following ranges of the parameters: $\bar{r}$ in 0.03 to 0.09 (in increments of 0.01) and draw the graph of the price vs. $\bar{r}$.

2. Compute the Option-Adjusted-Spread (***OAS***) if the Market Price of MBS is \$110,000.

3. Compute the ***OAS-adjusted Duration and Convexity*** of the MBS, considered in the previous question.

4. Consider the MBS described above and the ***IO*** and ***PO*** tranches. Price the ***IO*** and ***PO*** tranches for: $\bar{r}$ in 0.03 to 0.09 range, in increments of 0.01.