Exercises n°1 : Random Variables

Exercice 1. Suppose that X is a discrete random variable with $P(X = 0) = .25$, $P(X = 1) = .125$, $P(X = 2) = .125$ and $P(X = 3) = .5$. Graph the frequency (or probability mass function) and the cumulative distribution function.

Exercice 2. The following table shows the cumulative distribution function of a discrete variable. Find the frequency function.

$$\begin{pmatrix} \text{k :} & 0 & 1 & 2 & 3 & 5 \\ \text{F(k) :} & 0 & .2 & .3 & .5 & 1 \end{pmatrix}$$

Exercice 3. Sketch the probability density function (pdf) and the cumulative distribution function (cdf) of a random variable that is uniform on $[-1, 1]$.

Exercice 4. Suppose that X is a random variable such that $P(X = 2) = P(X = 3) = 1/10$ et $P(X = 5) = 8/10$.

1. Graph the CDF F of X.
2. Compute $\mathbb{P}(2 < X \leq 4.8)$ and $\mathbb{P}(2 \leq X \leq 4.8)$.
3. Compute $E(X)$.

Exercice 5. X is a random variable with PDF :

$$f_X(t) = \frac{1}{2}\exp(-\frac{t}{2}) \text{ si } t > 0, \quad 0 \text{ sinon.}$$

1. Find the CDF.
2. Compute $\mathbb{P}(X \geq 2)$.
3. Compute $E(X)$.

Exercice 6. X is a random variable with CDF :

$$F_X(t) = 0 \text{ si } t \leq 0, \quad F_X(t) = \frac{1}{2}t \text{ si } 0 \leq t \leq 2, \quad F_X(t) = 1 \text{ si } t \geq 2.$$

1. Graph the CDF.
2. Give the PDF.
3. Compute $\mathbb{P}(\frac{1}{4} < X < \frac{3}{4})$.

Exercice 7. X is a random variable with CDF :

$$F_X(t) = 0 \text{ si } t < 1, \quad F_X(t) = \frac{1}{5} \text{ si } 1 \leq t < 2, \quad F_X(t) = \frac{4}{5} \text{ si } 2 \leq t < 3, \quad F_X(t) = 1 \text{ si } t \geq 3.$$

1. Graph the CDF.
2. Give the PMF of X.

Exercice 8. Suppose that X has the density function $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Find c, the cdf, and $P(.1 \leq X < .5)$. What is the median of the distribution of X ? the quantile of order .75 ?

Exercice 9. Let $A = [-1, 1]$ and X have distribution F. Let $Y = I_A(X)$ where I_A is the indicator function for A. Find the probability function for Y, and an expression for its cdf.

Exercice 10. The Weibull cumulative distribution function is

$$F(x) = 1 - \exp(-(x/\alpha)^\beta), \quad x \geq 0, \quad \alpha, \beta > 0$$

1. Find the density function.
2. Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponentiel distribution.
3. How could Weibull random variables be generated from a uniform random number generator ?

Exercice 11. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 3$ and $\sigma = 4$. Solve the following using the Normal table or using a computer package.

1. Find $P(X < 3)$.
2. Find $P(X > -2)$ and $P(0 \leq X \leq 4)$.
3. Compute $\mathbb{E}(\frac{1}{2}X - 1)$ and $\mathrm{Var}(\frac{1}{2}X - 1)$

Exercice 12. Compute the expectation and the variance of a random variable that is uniform on $[-1, 3]$.

Exercice 13. What is the expectation of $I_A(X)$?

Exercice 14. Let X be a r. v. with pdf $f(x) = 2x$, $0 \leq x \leq 1$. Find $\mathbb{E}(X)$, $\mathbb{E}(X^2)$ and $\mathrm{Var}(X)$.

Exercice 15. Let (X, Y) be a random vector such that $\mathrm{Var}(X) = 1/2$, $\mathrm{Var}(Y) = 1$ and $\mathrm{Cov}(X, Y) = -1/4$. Let $U = X - 2Y - 3$.

Find $\mathrm{Var}(U)$ and $\mathrm{Cov}(U, Y)$.

Exercice 16. The Weibull cumulative distribution function is

$$F(x) = 1 - \exp(-(x/\alpha)^\beta), \quad x \geq 0, \quad \alpha, \beta > 0$$

1. Find the density function.
2. Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponentiel distribution.
3. Find the pth quantile of the Weibull distribution.
4. How could Weibull random variables be generated from a uniform random number generator ?