$R(f) = \mathbb{E}_{X,Y}\left[\mathbb{1}(f(X) \neq Y)\right]$ true risk

$\mathcal{Y} = \{0,1\}$

$= \mathbb{E}_X\left[\mathbb{E}_Y\left[\mathbb{1}(f(X) \neq Y) \mid X\right]\right]$

$= \mathbb{E}_X\left[\mathbb{P}_Y\left[f(X) \neq Y \mid X\right]\right]$

$= \mathbb{E}_X\left[\mathbb{P}_Y\left[f(X)=1, Y=0 \mid X\right] + \mathbb{P}_Y\left[f(X)=0, Y=1 \mid X\right]\right]$

$= \mathbb{E}_X\left[\mathbb{P}(Y=0 \mid X) \cdot \mathbb{1}(f(X)=1) + \mathbb{P}[Y=1 \mid X] \cdot \mathbb{1}(f(X)=0)\right]$

$= \mathbb{E}_X\left[\underbrace{(1-\mathbb{P}(Y=1 \mid X)) \cdot \mathbb{1}(f(X)=1) + \mathbb{P}(Y=1 \mid X) \cdot \mathbb{1}(f(X)=0)}_{\varphi(X)}\right]$

which function f^* minimises $R(f) = \mathbb{E}_X[\varphi(x)]$?

if $f(X)=1$, $\varphi(X) = 1 - P(Y=1 \mid X)$

if $f(X)=0$, $\varphi(X) = P(Y=1 \mid X)$

$\Rightarrow f^*(X) = \begin{cases} 1 \text{ if } P(Y=1 \mid X) > P(Y=0 \mid X) \\ 0 \text{ if } P(Y=0 \mid X) > P(Y=1 \mid X) \end{cases}$

with $f^*(X) =$ we will have

$\varphi(X) = \min\left(P(Y=1 \mid X), P(Y=0 \mid X)\right)$

$$\boxed{\text{Bayes Risk: } R(f^*) = \mathbb{E}_X\left[\min\left(P(Y=1 \mid X), P(Y=0 \mid X)\right)\right]}$$

$R(f) = \mathbb{E}_{XY}\left[(f(X)-Y)^2\right]$

$= \mathbb{E}_X\left[\mathbb{E}_Y\left[(f(X)-Y)^2 \mid X\right]\right]$

$= \mathbb{E}_X \mathbb{E}_Y\left[\left(\underline{f(X)-\mathbb{E}[Y|X)} + \underline{\mathbb{E}(Y|X)-Y}\right)^2 \mid X\right)\right]$

$= \mathbb{E}_X\Big\{\mathbb{E}_Y\left[(f(X)-\mathbb{E}(Y|X))^2 \mid X\right] + \mathbb{E}_Y\left[(\mathbb{E}(Y|X)-Y)^2 \mid X\right]$

$+ \underbrace{2\,\mathbb{E}_Y\left[(\underbrace{f(X)-\mathbb{E}(Y|X)}_{cste})\cdot(\mathbb{E}(Y|X-Y)) \mid X\right]}_{\underbrace{cste \times \mathbb{E}_Y\left[Y-\mathbb{E}[Y|X] \mid X\right]}_{= cste\left(\mathbb{E}_Y[Y|X]-\mathbb{E}_Y(Y|X)\right)=0}}$

$= \mathbb{E}_X\left\{\mathbb{E}_Y\left[(f(X)-\mathbb{E}(Y|X))^2 \mid X\right] + \mathbb{E}_Y\left[(Y-\mathbb{E}[Y|X])^2 \mid X\right]\right\}$

The Bayes predictor is $f^*(x) = \mathbb{E}[Y|X]$

Bayes Risk $= R(f^*) = \mathbb{E}_X\left[\mathrm{var}(Y|X)\right]$

Q: Is the 1-NN Bayes-consistent?

Let $h_S^{NN}(x)$ be the 1-NN classifier taking neighbors from S

$R(h_S^{NN}) \xrightarrow[P]{?} R(f^*)$ when $N \to \infty$

Let us look at $R(h_S^{NN})$ in the two class setting, where $P(Y=1|X) = P(Y=1) = p$ (not needed)

$\mathbb{E}_{S\sim P^N}\left[R(h_S^{NN})\right]$

$= \mathbb{E}_{S\sim P^N}\mathbb{E}_X\left[P_Y[Y=0|X]\cdot \mathbb{1}(h_S^{NN}(X)=1) + P_Y[Y=1|X]\cdot \mathbb{1}(h_S^{NN}(X)=0)\right]$

define Y_{NN} as the class of the nearest neighbor of X in S

$$= \mathbb{E}_S \mathbb{E}_X \left[P[Y=0|X] \cdot \mathbb{1}(Y_{NN}=1) + P(Y=1|X) \cdot \mathbb{1}(Y_{NN}=0) \right]$$

$$= \mathbb{E}_X \left[P(Y=0|X) \cdot \mathbb{E}_S(Y_{NN}=1|X) + P(Y=1|X) \cdot \mathbb{E}_S(Y_{NN}=0|X) \right]$$

$$= \mathbb{E}_X \left[P(Y=0|X) \cdot P(Y_{NN}=1|X) + P(Y=1|X) \cdot P(Y_{NN}=0|X) \right]$$

$$= \mathbb{E}_X \left[p \times (1-p) + (1-p)p \right]$$

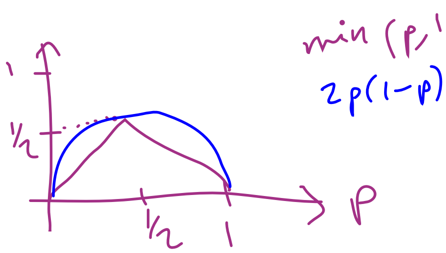
$$= 2p(1-p)$$

$\Rightarrow$ when $P(Y=1|X) = P(Y=1)$ $\quad \mathbb{E}_S\left[R(h_S^{NN})\right] = 2p(1-p)$

In general, when $N \to \infty$ $\quad \mathbb{E}_S\left[R(h_S^{NN})\right] \to \mathbb{E}_X\left[2P(Y=1|X)\left(1-P(Y=1|X)\right)\right]$

Recall the Bayes error $R(f^*) = \mathbb{E}_X\left[\min\left(P(Y=1|X), P(Y=0|X)\right)\right]$

$= \min(p, 1-p)$ if $P(Y=1|X) = P(Y=1)$



ex: if $p = \frac{1}{4}$ $\quad \min(p, 1-p) = \frac{1}{4}$

$2p(1-p) = \frac{3}{8}$

This shows that $R(h_S^{NN}) \not\to R(f^*)$ when $N \to \infty$

h_S^{NN}, the 1-nearest neighbor is <u>not</u> Bayes consistent !!!