Online Learning in Games
Lecture 2: Zero-sum games with infinitely many actions

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From previous lecture

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- Question : general sufficient conditions for existence of the value (either in pure or mixed strategies).

Contents

Berge lemma

Lemma (Berge, 1965)

Let $C_1, \ldots, C_n$ be non-empty convex compact subsets of a Euclidean space. Assume that the union $\bigcup_{i=1}^{n} C_i$ is convex and that for each $j = 1, \ldots, n$, $\bigcap_{i \neq j} C_i$ is non-empty.

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- Then C_n and $\bigcap_{i=1}^{n-1} C_i := D_n$ are $\neq \emptyset$, disjoint, convex and compact.
- By Hahn–Banach, C_n and D_n can be strictly separated by a hyperplane H.

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Proof of Berge lemma 2

- Define $\tilde{C}_i = C_i \cap H$ for $i = 1, \ldots, n-1$, and $\tilde{C} = (\bigcup_{i=1}^{n} C_i) \cap H$.

Proof of Berge lemma 2

- Since $C_n \cap H = \emptyset = D_n \cap H$, we have $\bigcup_{i=1}^{n-1} \tilde{C}_i = \tilde{C}$ and $\bigcap_{i=1}^{n-1} \tilde{C}_i = \emptyset$.

Proof of Berge lemma 2

- By the induction hypothesis, $\exists j \in \{1, \ldots, n-1\}$ such that $\bigcap_{i \neq j, n} \tilde{C}_i = \emptyset$.

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- Let $K = \bigcap_{i \neq j,n} C_i$. Then $D_n \subset K$ and $C_n \cap K \neq \emptyset$.

Proof of Berge lemma 2

- As K is convex, we must have $K \cap H \neq \emptyset$.

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- But $K \cap H = \bigcap_{i \neq j,n} \tilde{C}_i = \emptyset$, a contradiction.

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- Remark : if E is compact and f u.s.c (resp l.s.c), then f achieves its maximum on E (resp. minimum).

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Then G has a value :

$$\sup_{i \in I} \inf_{j \in J} g(i,j) = \inf_{j \in J} \sup_{i \in I} g(i,j).$$

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- Assume I compact. By contradiction suppose there is v such that

$$\sup_{i \in I} \inf_{j \in J} g(i,j) < v < \inf_{j \in J} \sup_{i \in I} g(i,j).$$

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- $(I_j)_{j\in J}$ is an open covering of I, hence $\exists$ a finite subset J_0 of J s.t.

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- Thus, $\forall i \in I_0$, $g(i, j_0) > v$ and by quasi-concavity, $g(i, j_0) > v \ \forall i \in \mathrm{co}(I_0)$.
- Similarly, $\exists s_0 \in \mathrm{co}(I_0)$ s.t. $g(i_0, j) < v$ for each j in $\mathrm{co}(J_0)$.
  A contradiction.

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We can weaken the topological conditions by strengthening the convexity hypothesis on $g(i, \cdot)$.

Proposition

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Then G has a value : $\sup_{i \in I} \inf_{j \in J} g(s, t) = \inf_{j \in J} \sup_{i \in I} g(i, j)$, and player 1 has an optimal strategy.

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- Elements of I and J are pure strategies.
- **If X has a topological structure, $\Delta(X)$ is usually endowed with the weak* topology** (the weakest topology such that $\hat{\phi} : \mu \mapsto \int_X \phi \, d\mu$ is continuous for each real continuous function ϕ on X).

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Let $G = (I, J, g)$ be a zero-sum game such that :

(i) *I and J are compact Hausdorff topological spaces ;*

(ii) *for each j in J, $g(\cdot, j)$ is u.s.c., and for each i in I, $g(i, \cdot)$ is l.s.c. ;*

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Proof : a direct consequence of Sion minmax theorem in pure strategies.

Mixed minmax theorem (second version)

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Proof :

- When I is compact and $g(\cdot, t)$ u.s.c., then $\Delta(I)$ (endowed with the weak* topology) is compact and $(\sigma \mapsto g(\sigma, j) = \int_S g(i,j)\sigma(di))$ is u.s.c.

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- Assumptions of last proposition are satisfied.

Contents

Fictitious play : Danskin (1954-1981)

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- We start from any (i_1, j_1) in $I \times J$.
- At each stage $n \geq 2$, each player will play an approximate best response to the sum of past behavior of the opponent up to stage $n - 1$.

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Definition

A sequence $(i_n, j_n)_{n \geq 1}$ is an η-FP process for some fixed $\eta \geq 0$ if for each $n \geq 1$:

- i_{n+1} is an η-best response of player 1 against $y_n := \frac{1}{n} \sum_{t=1}^{n} \delta_{j_t} \in \Delta(J)$,
- j_{n+1} is an η-best response of player 2 against $x_n := \frac{1}{n} \sum_{t=1}^{n} \delta_{i_t} \in \Delta(I)$.

Fictitious play : Theorem

Theorem (Danskin, 1954-1981)

Let $(i_n, j_n)_{n\geq 1}$ be the realization of a η-fictitious play process. If the game is compact and continuous and if $val(g)$ denotes its value in mixed strategies, then $\forall\varepsilon > 0, \exists N, \forall n \geq N, \forall x \in \Delta(I), \forall y \in \Delta(J)$

$$g(x_n, y) \geq val(g) - \varepsilon - \eta \text{ and } g(x, y_n) \leq val(g) + \varepsilon + \eta.$$

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- The payoff function for this zero-sum game is given by is :

$$\begin{aligned} \Phi(g, c) &= \int_X \log(c(x))dP(x) + \int_Z \log(1 - c(g(z))dQ(z) \qquad (1) \\ &= \int_X \log(c(x))dP(x) + \int_X \log(1 - c(x))dQ(g^{-1}(x)) \qquad (2) \end{aligned}$$

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$$= \int_X \log(c(x))dP(x) + \int_X \log(1 - c(x))dQ(g^{-1}(x)) \tag{2}$$

- The G-player minimizes with respect to g, the C-player maximizes in c.

Resolution

Proposition

If $\mathcal{G}$ is the whole set of measurable functions from Z to X, then the game has a value in pure strategies and optimal strategies are
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Proof :
Apply Sion's theorem to $I = \mathcal{C}$ and $J = \Delta_f(\mathcal{G})$. Observe that Φ is continuous, and concave in c.