

IASD M2 at Paris Dauphine

# Deep Reinforcement Learning

## 11: Optimal Control and Planning

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# Acknowledgement

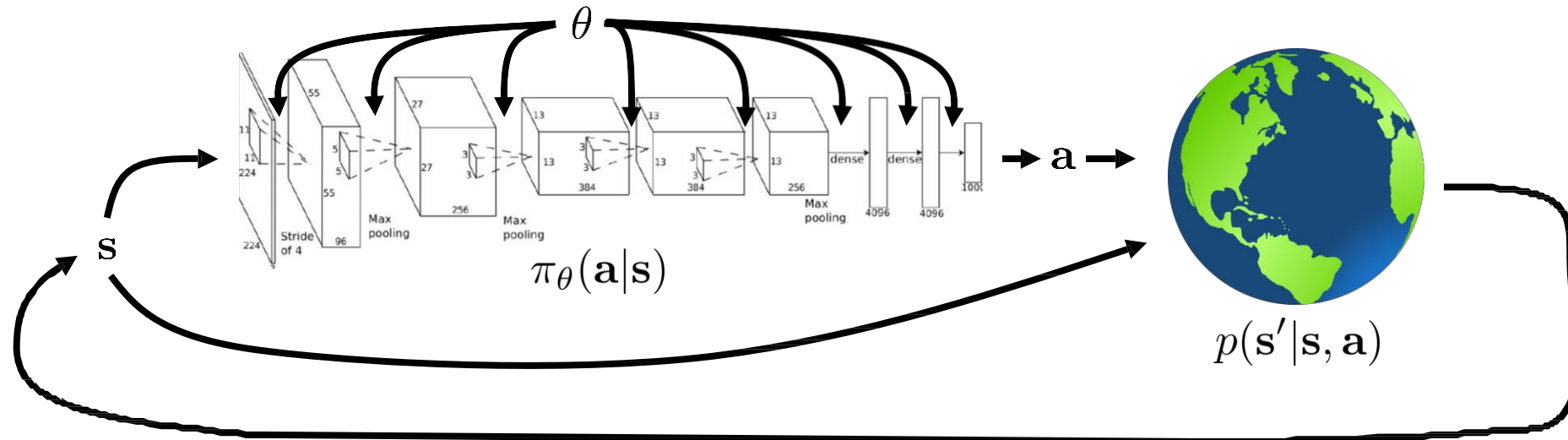
These materials are based on the seminal course of Sergey Levine  
CS285



# Today's Lecture

1. Introduction to model-based reinforcement learning
  2. What if we know the dynamics? How can we make decisions?
  3. Stochastic optimization methods
  4. Monte Carlo tree search (MCTS)
  5. Trajectory optimization
- Goals:
    - Understand how we can perform planning with known dynamics models in discrete and continuous spaces

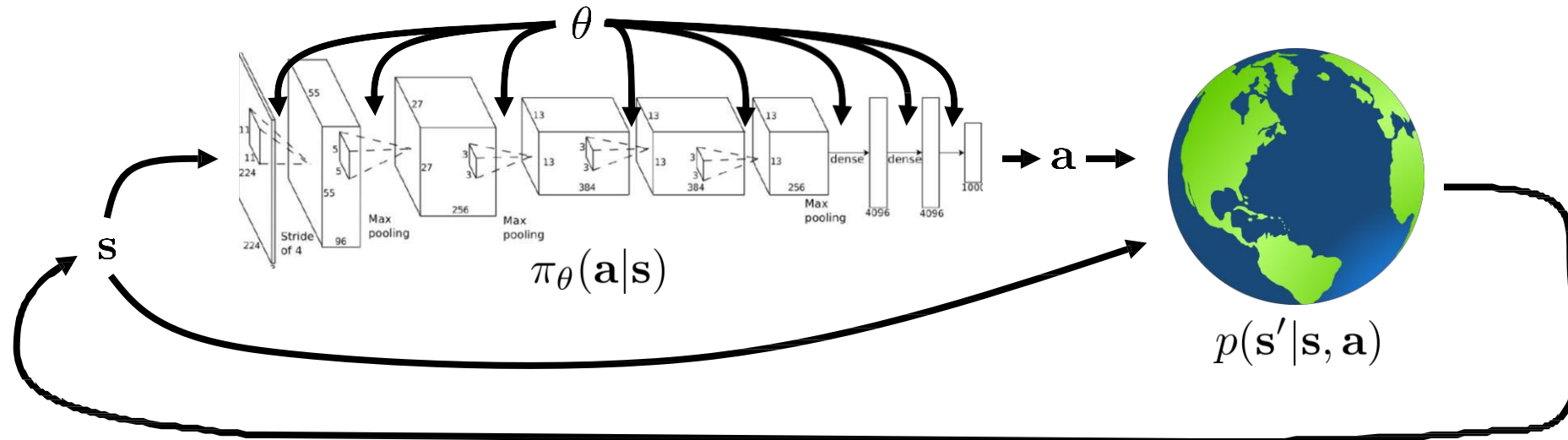
# Recap: the reinforcement learning objective



$$\underbrace{p_\theta(s_1, a_1, \dots, s_T, a_T)}_{\pi_\theta(\tau)} = p(s_1) \prod_{t=1}^T \pi_\theta(a_t | s_t) p(s_{t+1} | s_t, a_t)$$

$$\theta^* = \arg \max_{\theta} E_{\tau \sim p_\theta(\tau)} \left[ \sum_t r(s_t, a_t) \right]$$

# Recap: model-free reinforcement learning



$$\underbrace{p_\theta(s_1, a_1, \dots, s_T, a_T)}_{\pi_\theta(\tau)} = p(s_1) \prod_{t=1}^T \pi_\theta(a_t | s_t) \cancel{p(s_{t+1} | s_t, a_t)}$$

assume this is unknown  
don't even attempt to learn it

$$\theta^* = \arg \max_{\theta} E_{\tau \sim p_\theta(\tau)} \left[ \sum_t r(s_t, a_t) \right]$$

# What if we knew the transition dynamics?

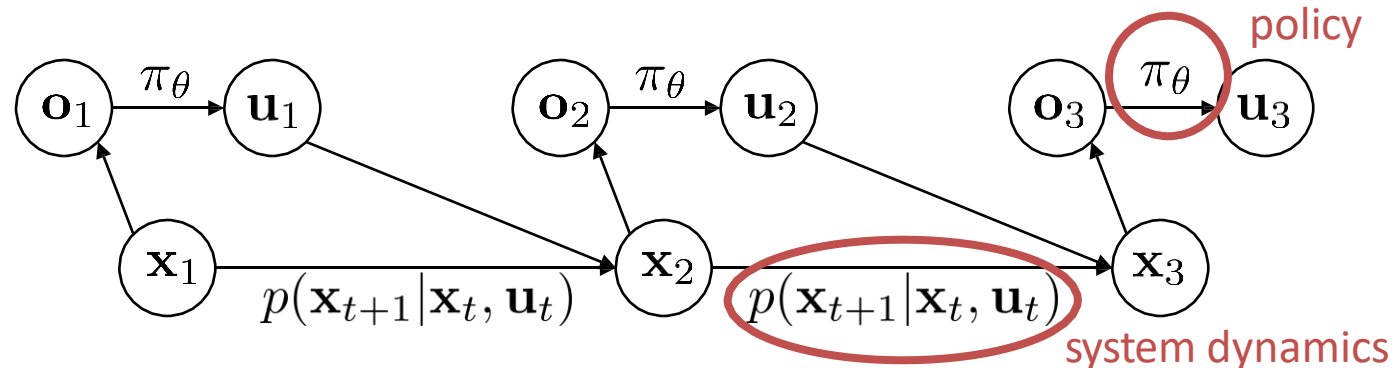
- Often we do know the dynamics
  1. Games (e.g., Atari games, chess, Go)
  2. Easily modeled systems (e.g., navigating a car)
  3. Simulated environments (e.g., simulated robots, video games)
- Often we can learn the dynamics
  1. System identification – fit unknown parameters of a known model
  2. Learning – fit a general-purpose model to observed transition data

Does knowing the dynamics make things easier?

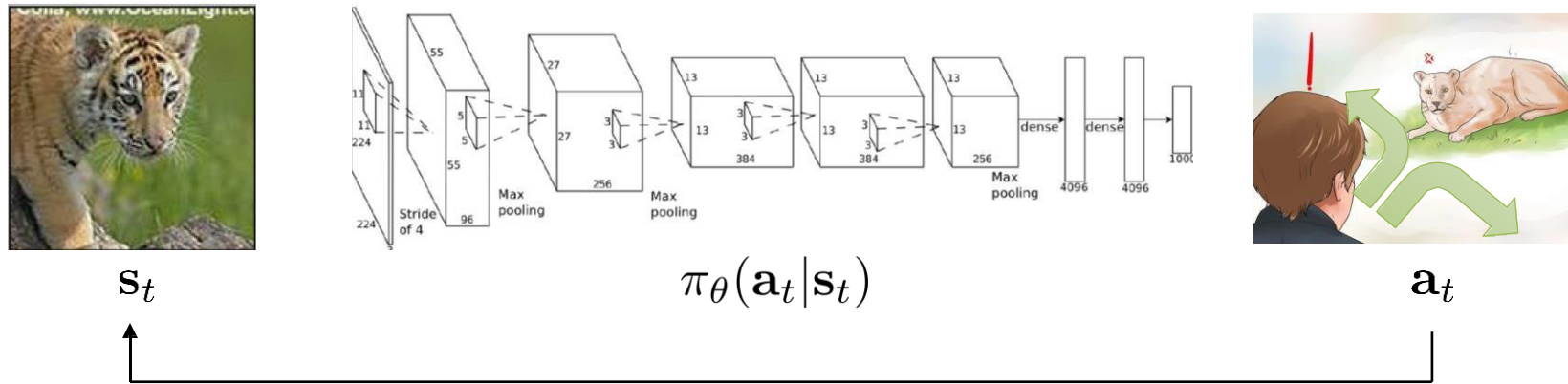
Often, yes!

# Model-based reinforcement learning

1. Model-based reinforcement learning: learn the transition dynamics, then figure out how to choose actions
2. Today: how can we make decisions if we *know* the dynamics?
  - a. How can we choose actions under perfect knowledge of the system dynamics?
  - b. Optimal control, trajectory optimization, planning
3. Next lecture: how can we learn *unknown* dynamics?
4. How can we then also learn policies? (*e.g. by imitating optimal control*)



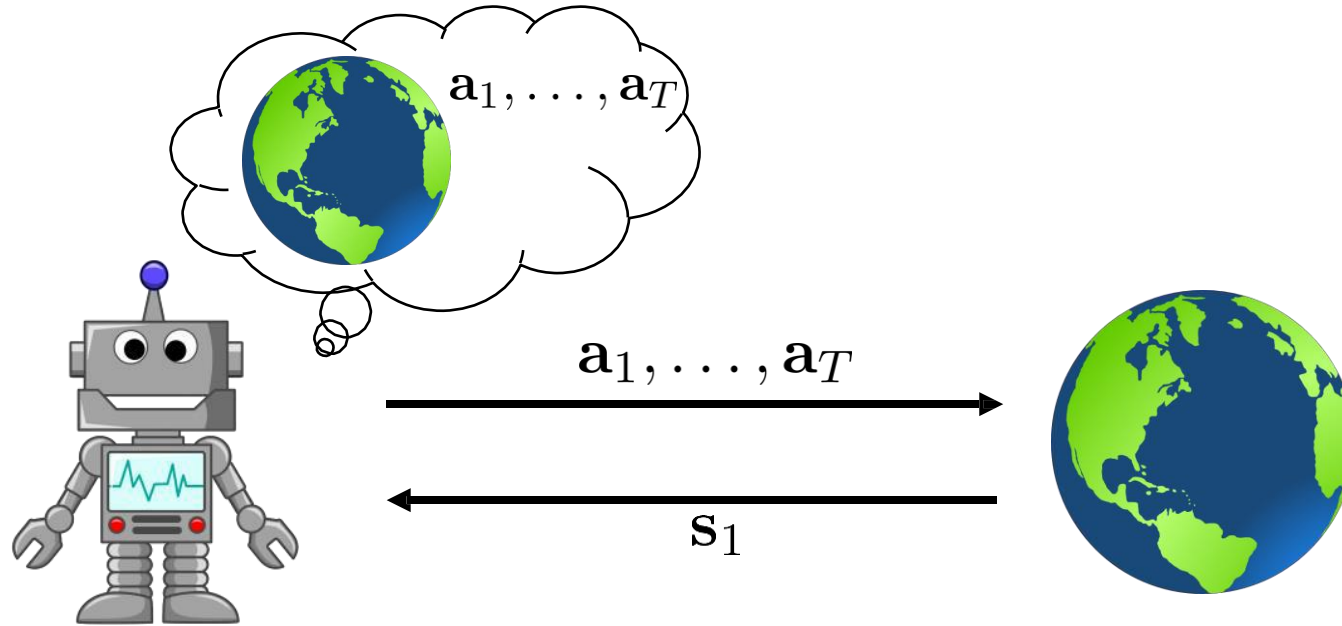
# The objective



$$\min_{\mathbf{a}_1, \dots, \mathbf{a}_T} \sum_{t=1}^T \log p(\text{seen by tiger} | \mathbf{s}_{t-1}, \mathbf{a}_{t-1})$$

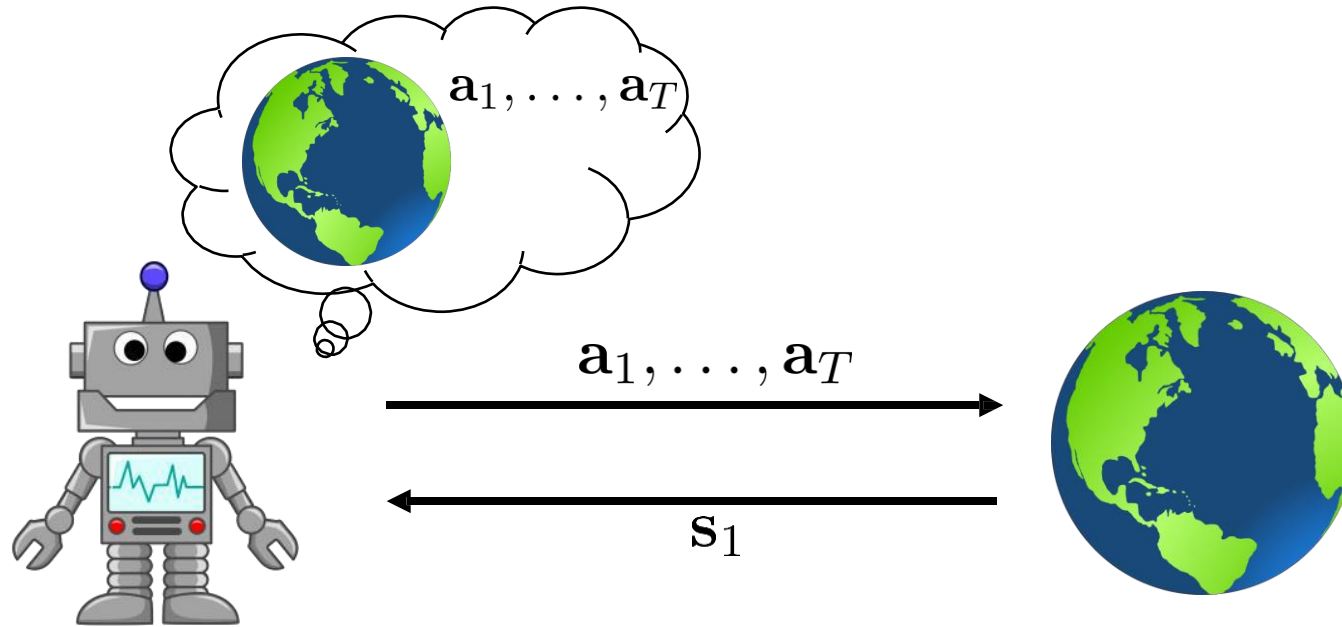


# The deterministic case



$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} \sum_{t=1}^T r(\mathbf{s}_t, \mathbf{a}_t) \text{ s.t. } \mathbf{a}_{t+1} = f(\mathbf{s}_{t+1}, \mathbf{a}_t)$$

# The stochastic open-loop case



$$p_{\theta}(\mathbf{s}_1, \dots, \mathbf{s}_T | \mathbf{a}_1, \dots, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^T p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

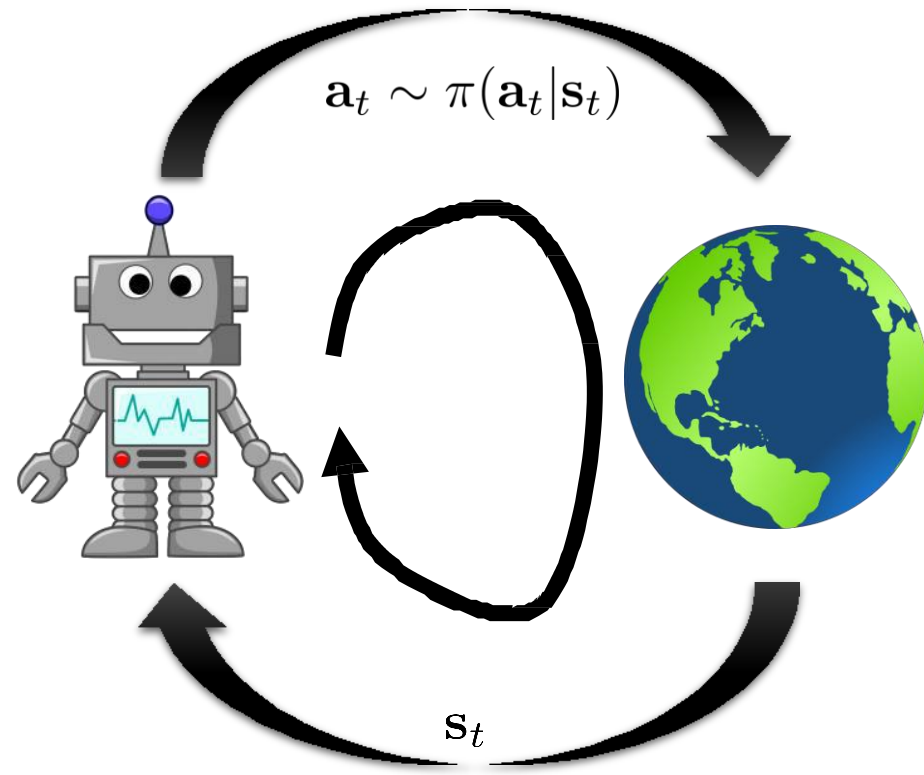
$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} E \left[ \sum_t r(\mathbf{s}_t, \mathbf{a}_t) | \mathbf{a}_1, \dots, \mathbf{a}_T \right]$$

why is this suboptimal?

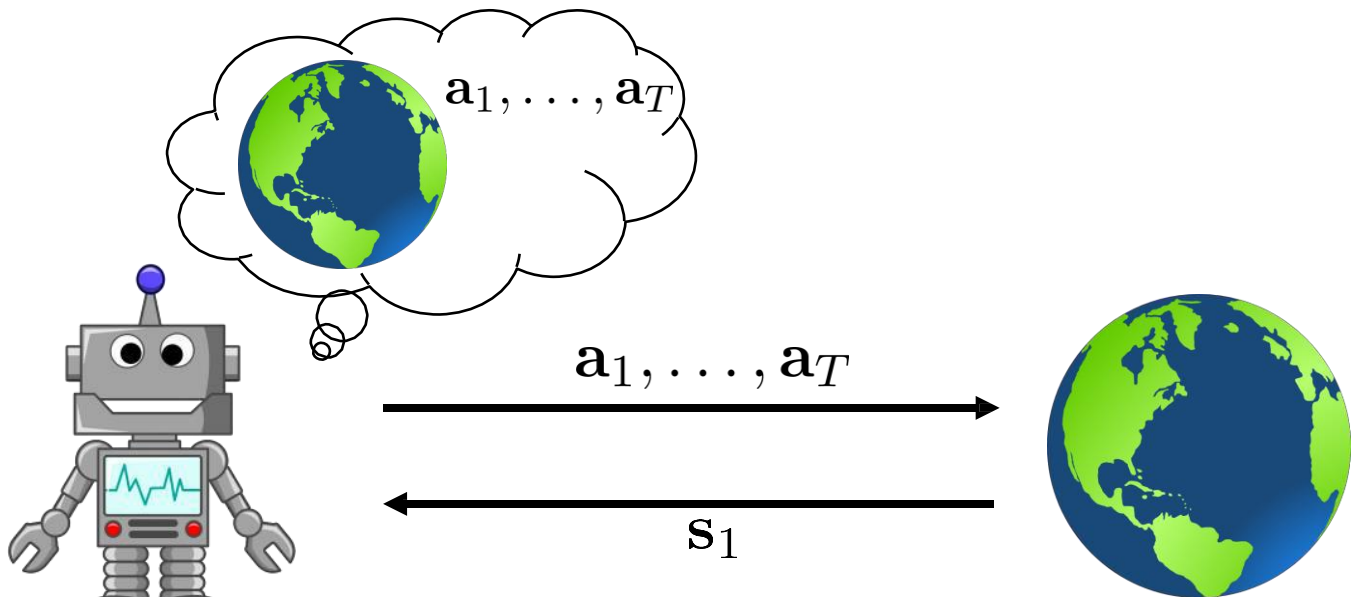
# Aside: terminology

what is this “loop”?

closed-loop

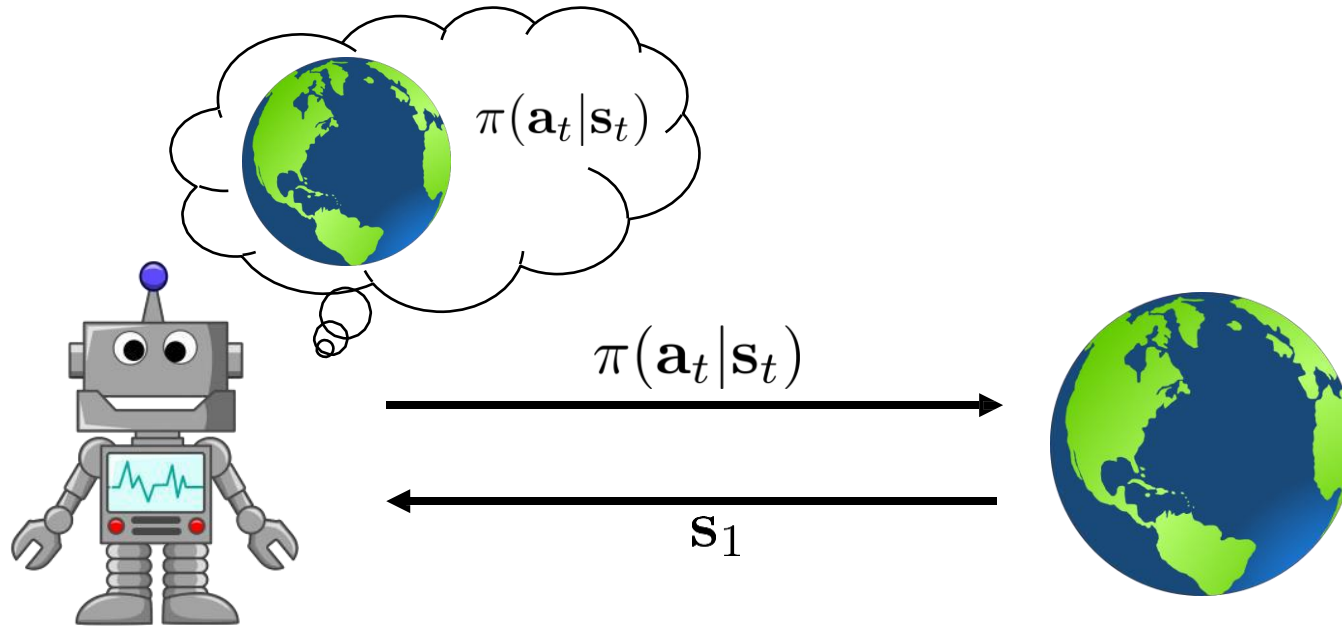


open-loop



only sent at  $t = 1$ ,  
then it's one-way!

# The stochastic closed-loop case



$$p(\mathbf{s}_1, \mathbf{a}_1, \dots, \mathbf{s}_T, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^T \pi(\mathbf{a}_t | \mathbf{s}_t) p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

$$\pi = \arg \max_{\pi} E_{\tau \sim p(\tau)} \left[ \sum_t r(\mathbf{s}_t, \mathbf{a}_t) \right]$$

form of  $\pi$ ?

neural net

time-varying linear

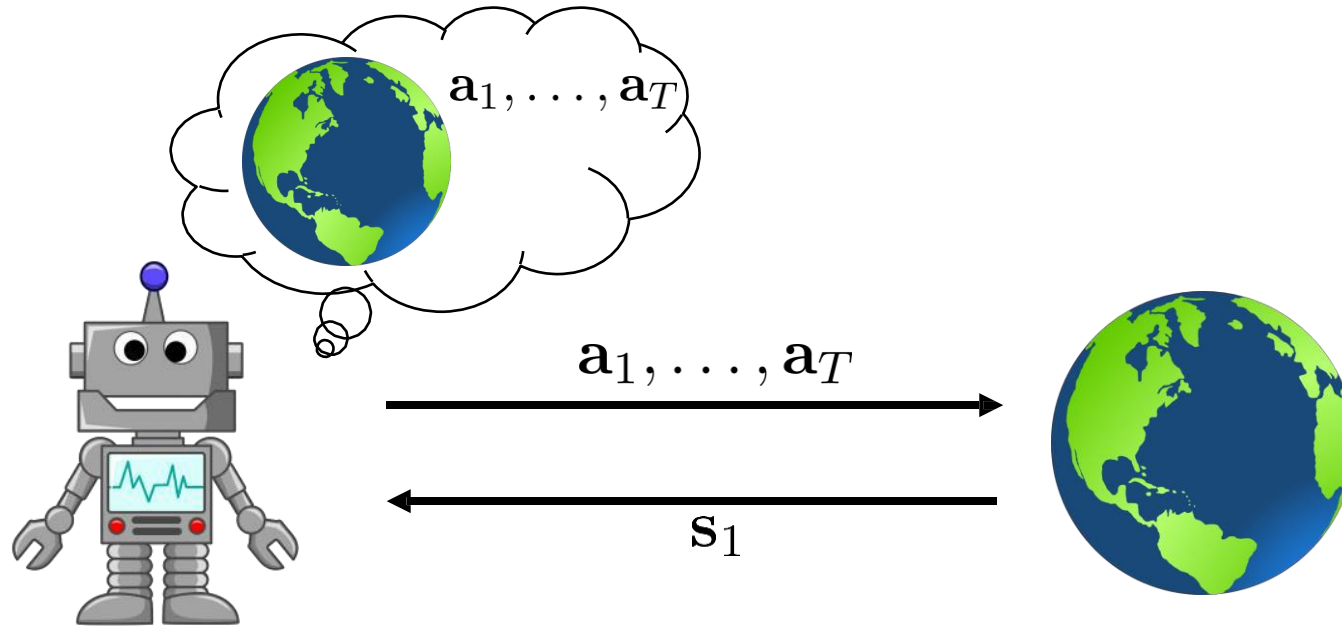
$\mathbf{K}_t \mathbf{s}_t + \mathbf{k}_t$

global  
local

(more on this later)

# Open-Loop Planning

# But for now, open-loop planning



$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} \sum_{t=1}^T r(\mathbf{s}_t, \mathbf{a}_t) \text{ s.t. } \mathbf{a}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t)$$

# Stochastic optimization

abstract away optimal control/planning:

$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} \underbrace{J(\mathbf{a}_1, \dots, \mathbf{a}_T)}$$

don't care what this is

$$\mathbf{A} = \arg \max_{\mathbf{A}} J(\mathbf{A})$$

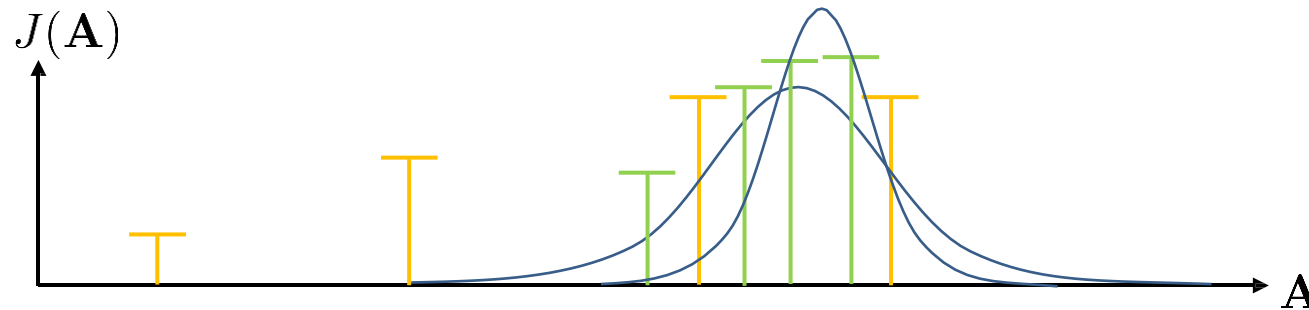
simplest method: guess & check      “random shooting method”

1. pick  $\mathbf{A}_1, \dots, \mathbf{A}_N$  from some distribution (e.g., uniform)
2. choose  $\mathbf{A}_i$  based on  $\arg \max_i J(\mathbf{A}_i)$

# Cross-entropy method (CEM)

1. pick  $\mathbf{A}_1, \dots, \mathbf{A}_N$  from some distribution (e.g., uniform)
2. choose  $\mathbf{A}_i$  based on  $\arg \max_i J(\mathbf{A}_i)$

can we do better?



typically use Gaussian distribution

see also: CMA-ES (sort of like CEM with momentum)

cross-entropy method with continuous-valued inputs:

1. sample  $\mathbf{A}_1, \dots, \mathbf{A}_N$  from  $p(\mathbf{A})$
2. evaluate  $J(\mathbf{A}_1), \dots, J(\mathbf{A}_N)$
3. pick the *elites*  $\mathbf{A}_{i_1}, \dots, \mathbf{A}_{i_M}$  with the highest value, where  $M < N$
4. refit  $p(\mathbf{A})$  to the elites  $\mathbf{A}_{i_1}, \dots, \mathbf{A}_{i_M}$



# What's the upside?

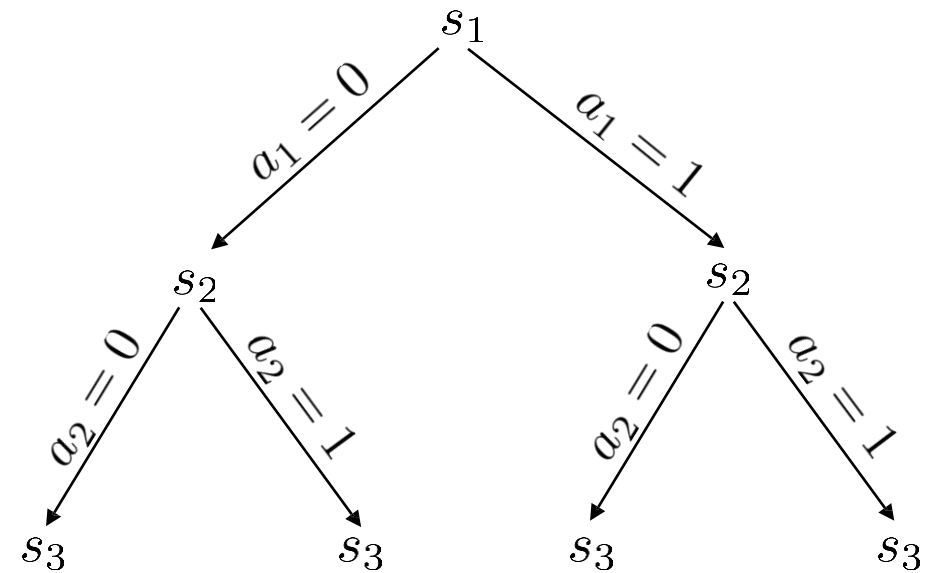
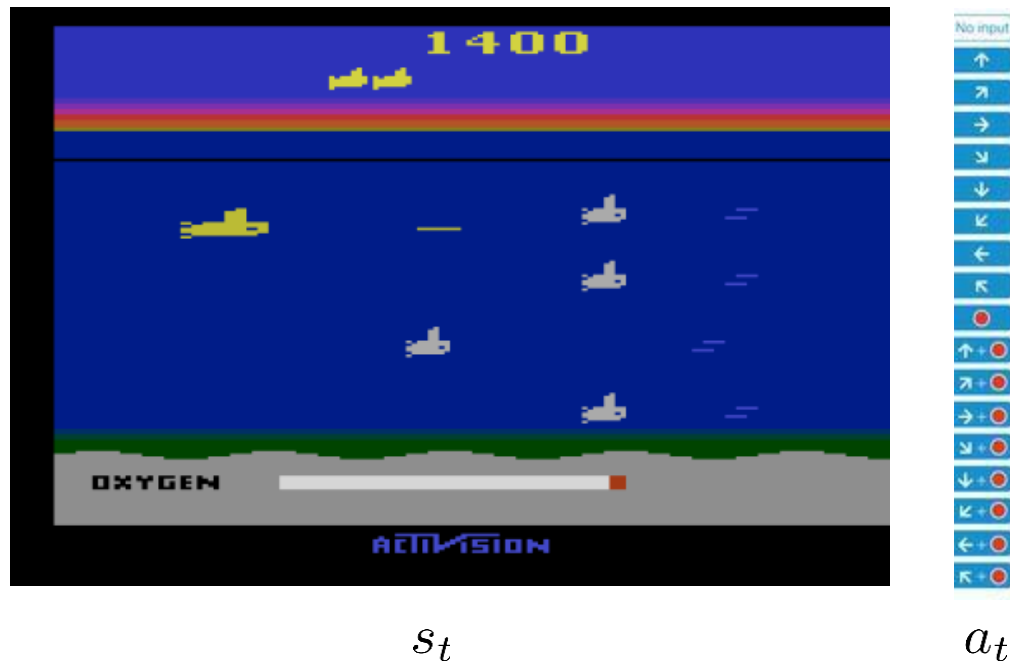
1. Very fast if parallelized
2. Extremely simple

# What's the problem?

1. Very harsh dimensionality limit
2. Only open-loop planning

# Discrete case: Monte Carlo tree search (MCTS)

discrete planning as a search problem



# Discrete case: Monte Carlo tree search (MCTS)

how to approximate value without full tree?

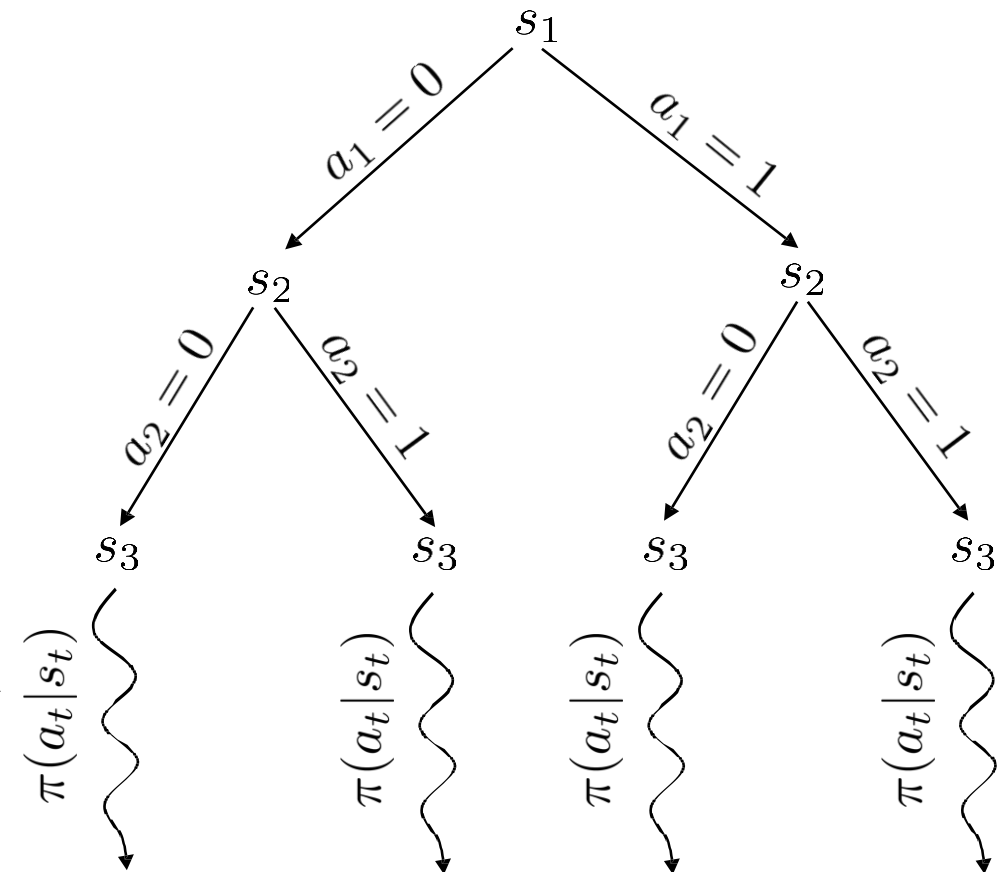


$s_t$



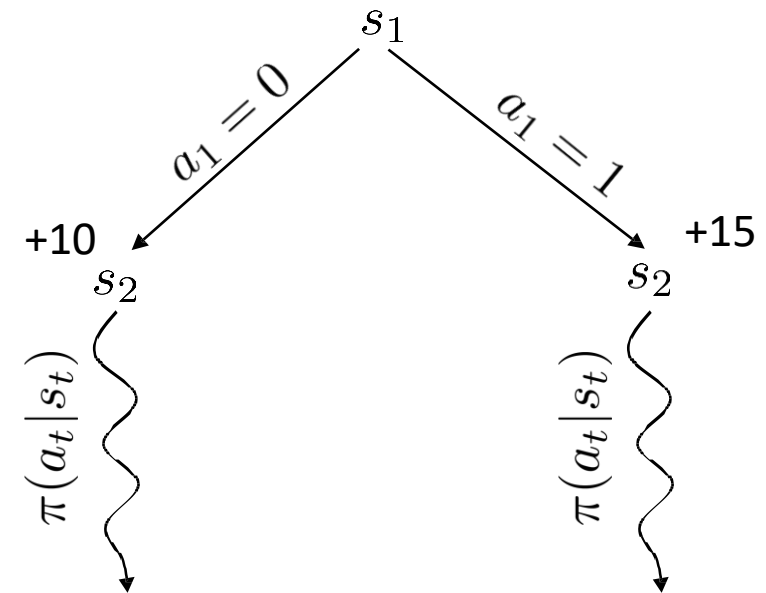
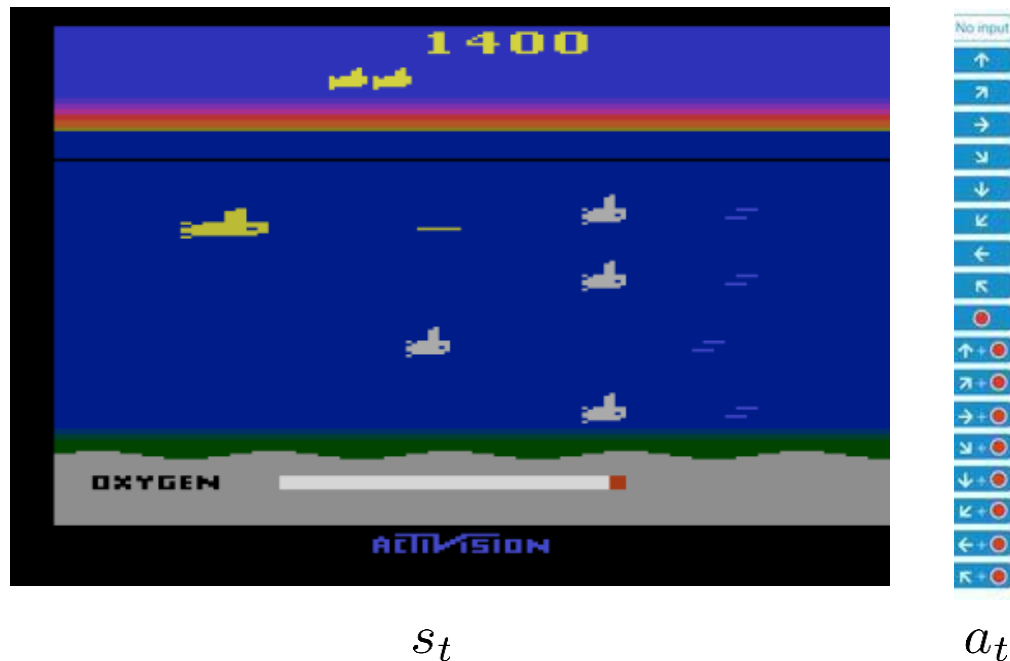
$a_t$

e.g., random policy



# Discrete case: Monte Carlo tree search (MCTS)

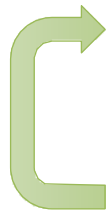
can't search all paths – where to search first?



intuition: choose nodes with best reward, but also prefer rarely visited nodes

# Discrete case: Monte Carlo tree search (MCTS)

generic MCTS sketch



1. find a leaf  $s_l$  using  $\text{TreePolicy}(s_1)$
2. evaluate the leaf using  $\text{DefaultPolicy}(s_l)$
3. update all values in tree between  $s_1$  and  $s_l$

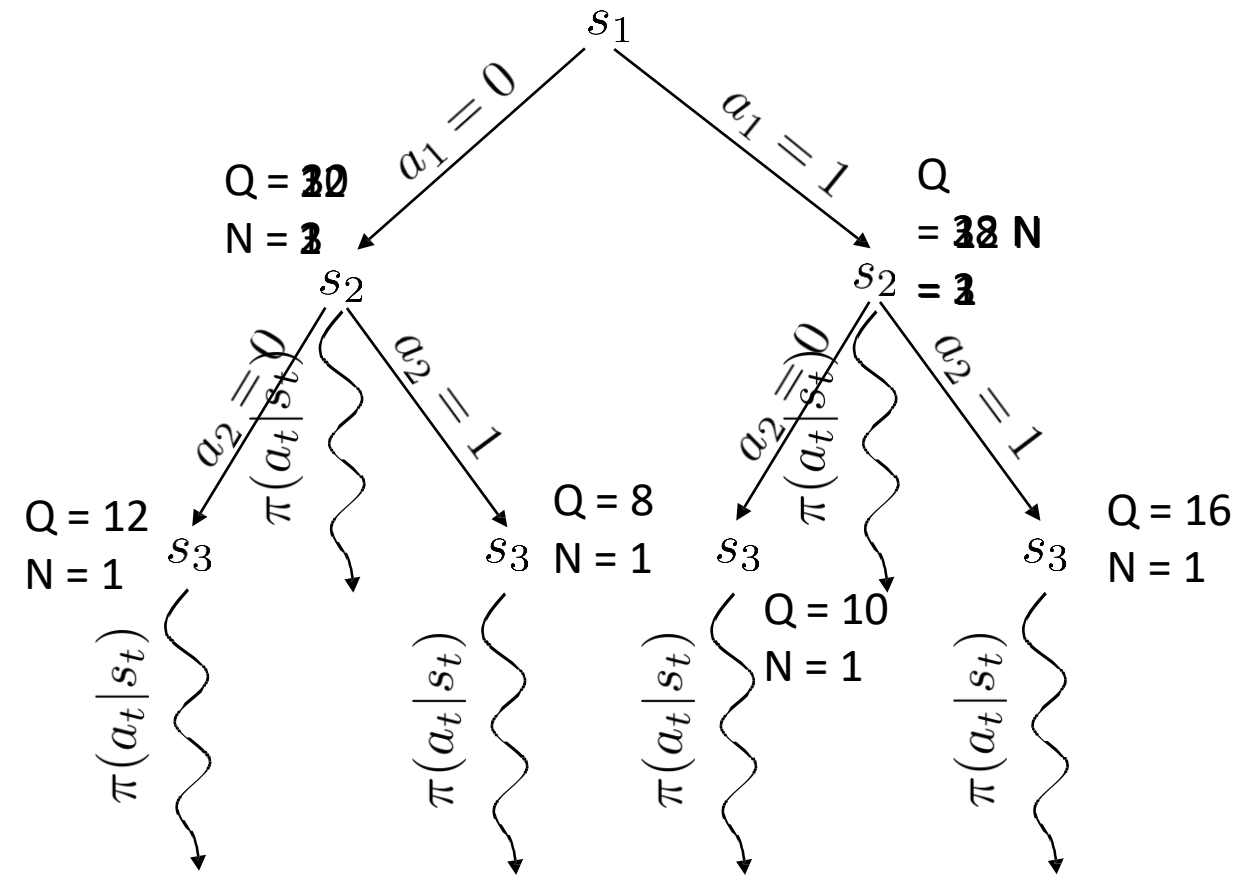
take best action from  $s_1$

UCT  $\text{TreePolicy}(s_t)$

if  $s_t$  not fully expanded, choose new  $a_t$

else choose child with best  $\text{Score}(s_{t+1})$

$$\text{Score}(s_t) = \frac{Q(s_t)}{N(s_t)} + 2C \sqrt{\frac{2 \ln N(s_{t-1})}{N(s_t)}}$$



# Additional reading

1. Browne, Powley, Whitehouse, Lucas, Cowling, Rohlfshagen, Tavener, Perez, Samothrakis, Colton. (2012). A Survey of Monte Carlo Tree Search Methods.
  - Survey of MCTS methods and basic summary.

# Trajectory Optimization with Derivatives

# Can we use derivatives?

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} \sum_{t=1}^T c(\mathbf{x}_t, \mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$

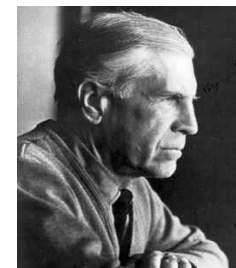
usual story: differentiate via backpropagation and optimize!

$$\text{need } \frac{df}{d\mathbf{x}_t}, \frac{df}{d\mathbf{u}_t}, \frac{dc}{d\mathbf{x}_t}, \frac{dc}{d\mathbf{u}_t}$$

$\mathbf{s}_t$  – state  
 $\mathbf{a}_t$  – action

$\mathbf{x}_t$  – state  
 $\mathbf{u}_t$  – action

in practice, it really helps to use a 2<sup>nd</sup> order method!

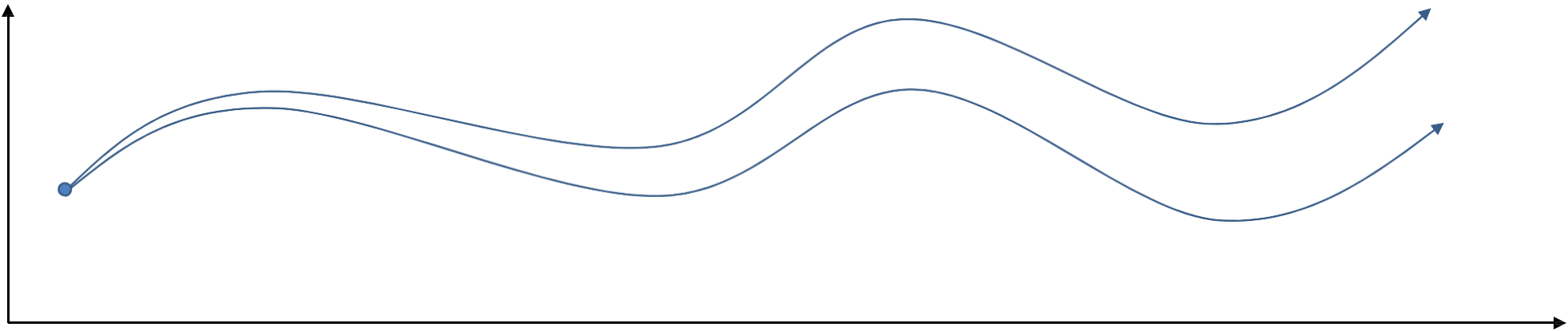




# Shooting methods vs collocation

shooting method: optimize over actions only

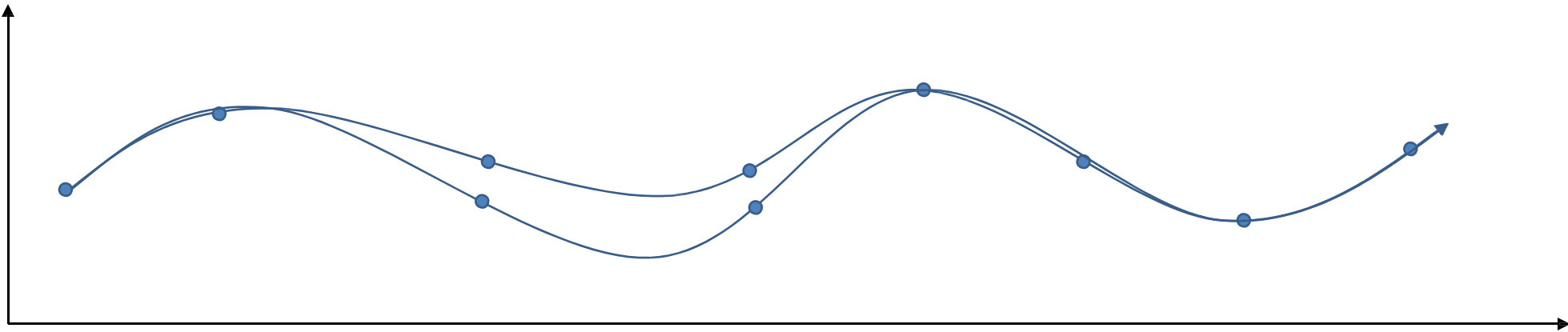
$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots)\dots), \mathbf{u}_T)$$



# Shooting methods vs collocation

collocation method: optimize over actions and states, with constraints

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T, \mathbf{x}_1, \dots, \mathbf{x}_T} \sum_{t=1}^T c(\mathbf{x}_t, \mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$



# Linear case: LQR

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots)\dots), \mathbf{u}_T)$$

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

linear

$$c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

quadratic

# Linear case: LQR

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + \underbrace{c(\overbrace{f(f(\dots))}^{\mathbf{x}_T \text{ (unknown)}}, \mathbf{u}_T)}_{\text{only term that depends on } \mathbf{u}_T}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

only term that depends on  $\mathbf{u}_T$

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

Base case: solve for  $\mathbf{u}_T$  *only*

$$\mathbf{C}_T = \begin{bmatrix} \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} & \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \\ \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} & \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \end{bmatrix}$$

$$\mathbf{c}_T = \begin{bmatrix} \mathbf{c}_{\mathbf{x}_T} \\ \mathbf{c}_{\mathbf{u}_T} \end{bmatrix}$$

$$Q(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{c}_T$$

$$\nabla_{\mathbf{u}_T} Q(\mathbf{x}_T, \mathbf{u}_T) = \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{u}_T + \mathbf{c}_{\mathbf{u}_T}^T = 0$$

$$\mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T}$$

$$\mathbf{u}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} (\mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \mathbf{c}_{\mathbf{u}_T}) \quad \mathbf{u}_T = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \quad \mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{c}_{\mathbf{u}_T}$$

# Linear case: LQR

$$\mathbf{u}_T = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \quad \mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \quad \mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{c}_{\mathbf{u}_T}$$

$$Q(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{c}_T$$

Since  $\mathbf{u}_T$  is fully determined by  $\mathbf{x}_T$ , we can eliminate it via substitution!

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{c}_T$$

$$V(\mathbf{x}_T) = \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \\ \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{k}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{k}_T + \mathbf{x}_T^T \mathbf{c}_{\mathbf{x}_T} + \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{c}_{\mathbf{u}_T} + \text{const}$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T$$

$$\mathbf{V}_T = \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{K}_T + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{K}_T$$

$$\mathbf{v}_T = \mathbf{c}_{\mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{k}_T + \mathbf{K}_T^T \mathbf{c}_{\mathbf{u}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{k}_T$$

# Linear case: LQR

Solve for  $\mathbf{u}_{T-1}$  in terms of  $\mathbf{x}_{T-1}$

$\mathbf{u}_{T-1}$  affects  $\mathbf{x}_T$ !

$$f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \mathbf{x}_T = \mathbf{F}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \mathbf{f}_{T-1}$$

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{C}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{c}_{T-1} + \underbrace{V(f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}))}_{V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T}$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}}_{\text{quadratic}} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1}}_{\text{linear}} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{v}_T}_{\text{linear}}$$

# Linear case: LQR

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{C}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{c}_{T-1} + V(f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}))$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}}_{\text{quadratic}} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1}}_{\text{linear}} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{v}_T}_{\text{linear}}$$

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{Q}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{q}_{T-1}$$

$$\mathbf{Q}_{T-1} = \mathbf{C}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}$$

$$\mathbf{q}_{T-1} = \mathbf{c}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{v}_T$$

$$\nabla_{\mathbf{u}_{T-1}} Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{x}_{T-1}} \mathbf{x}_{T-1} + \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}} \mathbf{u}_{T-1} + \mathbf{q}_{\mathbf{u}_{T-1}}^T = 0$$

$$\mathbf{u}_{T-1} = \mathbf{K}_{T-1} \mathbf{x}_{T-1} + \mathbf{k}_{T-1}$$

$$\mathbf{K}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}}^{-1} \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{x}_{T-1}}$$

$$\mathbf{k}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}}^{-1} \mathbf{q}_{\mathbf{u}_{T-1}}$$

# Linear case: LQR

Backward recursion

for  $t = T$  to 1:

$$\mathbf{Q}_t = \mathbf{C}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{F}_t$$

$$\mathbf{q}_t = \mathbf{c}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{f}_t + \mathbf{F}_t^T \mathbf{v}_{t+1}$$

$$Q(\mathbf{x}_t, \mathbf{u}_t) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{Q}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{q}_t$$

$$\mathbf{u}_t \leftarrow \arg \min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

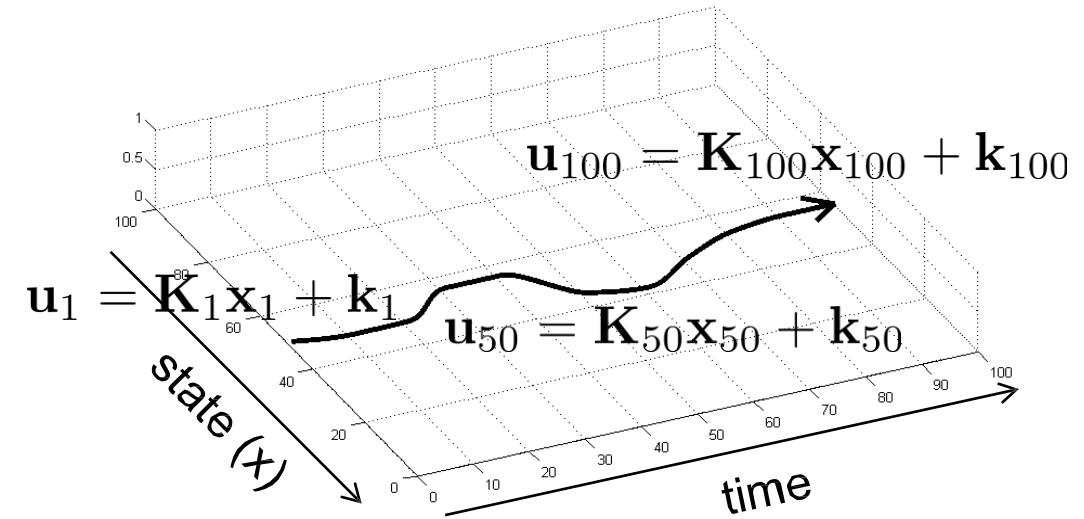
$$\mathbf{K}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t}$$

$$\mathbf{k}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{q}_{\mathbf{u}_t}$$

$$\mathbf{V}_t = \mathbf{Q}_{\mathbf{x}_t, \mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{K}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{K}_t$$

$$\mathbf{v}_t = \mathbf{q}_{\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{k}_t + \mathbf{K}_t^T \mathbf{q}_{\mathbf{u}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{k}_t$$

$$V(\mathbf{x}_t) = \text{const} + \frac{1}{2} \mathbf{x}_t^T \mathbf{V}_t \mathbf{x}_t + \mathbf{x}_t^T \mathbf{v}_t$$



we know  $\mathbf{x}_1$ !

Forward recursion

for  $t = T$  to 1:

$$\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$$



# Linear case: LQR

## Backward recursion

for  $t = T$  to 1:

$$\mathbf{Q}_t = \mathbf{C}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{F}_t$$

$$\mathbf{q}_t = \mathbf{c}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{f}_t + \mathbf{F}_t^T \mathbf{v}_{t+1}$$

$$Q(\mathbf{x}_t, \mathbf{u}_t) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{Q}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{q}_t$$

$$\mathbf{u}_t \leftarrow \arg \min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

$$\mathbf{K}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t}$$

$$\mathbf{k}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{q}_{\mathbf{u}_t}$$

$$\mathbf{V}_t = \mathbf{Q}_{\mathbf{x}_t, \mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{K}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{K}_t$$

$$\mathbf{v}_t = \mathbf{q}_{\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{k}_t + \mathbf{K}_t^T \mathbf{q}_{\mathbf{u}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{k}_t$$

$$V(\mathbf{x}_t) = \text{const} + \frac{1}{2} \mathbf{x}_t^T \mathbf{V}_t \mathbf{x}_t + \mathbf{x}_t^T \mathbf{v}_t$$

total cost from now until end if we take  $\mathbf{u}_t$  from state  $\mathbf{x}_t$

total cost from now until end from state  $\mathbf{x}_t$   
 $V(\mathbf{x}_t) = \min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t)$

# LQR for Stochastic and Nonlinear Systems

# Stochastic dynamics

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

$$\mathbf{x}_{t+1} \sim p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t)$$

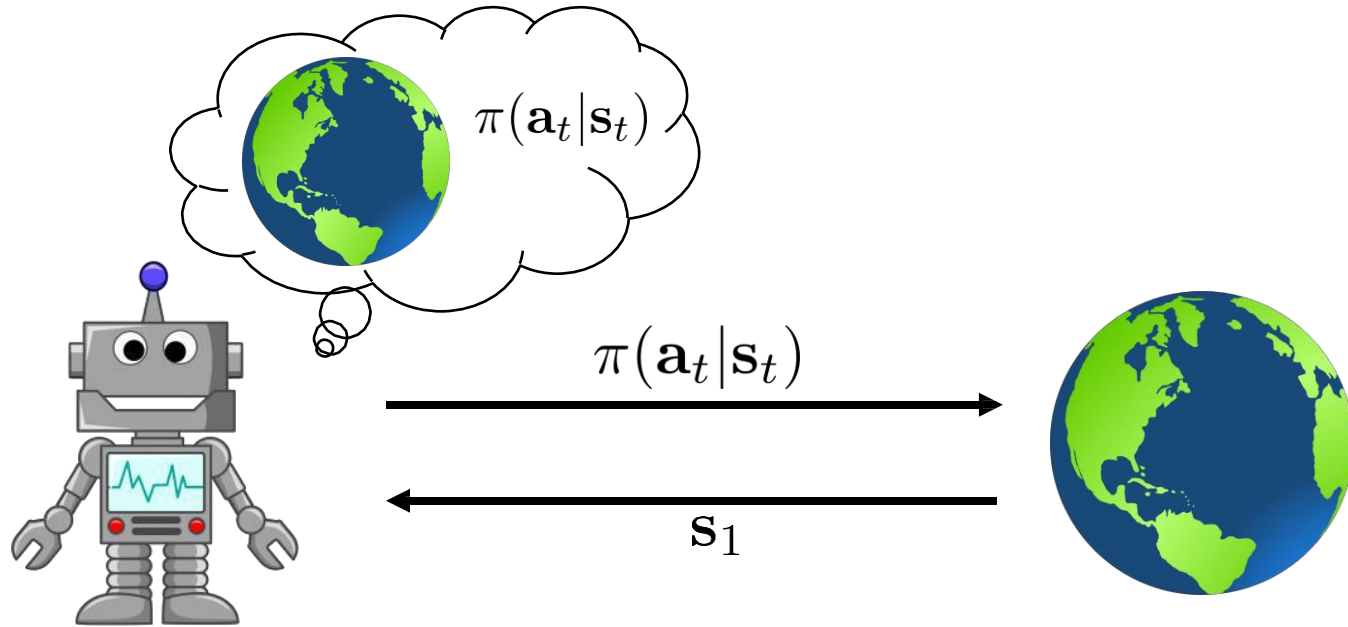
$$p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t) = \mathcal{N} \left( \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t, \Sigma_t \right)$$

Solution: choose actions according to  $\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$

$\mathbf{x}_t \sim p(\mathbf{x}_t)$ , no longer deterministic, but  $p(\mathbf{x}_t)$  is Gaussian

no change to algorithm! can ignore  $\Sigma_t$  due to symmetry of Gaussians  
(checking this is left as an exercise; hint: the expectation of a quadratic under a Gaussian has an analytic solution)

# The stochastic closed-loop case



form of  $\pi$ ?

$$p(\mathbf{s}_1, \mathbf{a}_1, \dots, \mathbf{s}_T, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^T \pi(\mathbf{a}_t | \mathbf{s}_t) p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

time-varying linear

$$\mathbf{K}_t \mathbf{s}_t + \mathbf{k}_t$$

$$\pi = \arg \max_{\pi} E_{\tau \sim p(\tau)} \left[ \sum_t r(\mathbf{s}_t, \mathbf{a}_t) \right]$$

# Nonlinear case: DDP/iterative LQR

Linear-quadratic assumptions:

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t \qquad c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

Can we *approximate* a nonlinear system as a linear-quadratic system?

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

# Nonlinear case: DDP/iterative LQR

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$\bar{f}(\delta \mathbf{x}_t, \delta \mathbf{u}_t) = \underbrace{\mathbf{F}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}$$

$$\bar{c}(\delta \mathbf{x}_t, \delta \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{C}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{c}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)}$$

$$\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$$

$$\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$$

Now we can run LQR with dynamics  $\bar{f}$ , cost  $\bar{c}$ , state  $\delta \mathbf{x}_t$ , and action  $\delta \mathbf{u}_t$

# Nonlinear case: DDP/iterative LQR

Iterative LQR (simplified pseudocode)

until convergence:

$$\mathbf{F}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{c}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{C}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

Run LQR backward pass on state  $\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$  and action  $\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$


Run forward pass with real nonlinear dynamics and  $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \mathbf{k}_t + \hat{\mathbf{u}}_t$

Update  $\hat{\mathbf{x}}_t$  and  $\hat{\mathbf{u}}_t$  based on states and actions in forward pass

# Nonlinear case: DDP/iterative LQR

Why does this work?

Compare to Newton's method for computing  $\min_{\mathbf{x}} g(\mathbf{x})$ :

 until convergence:

$$\mathbf{g} = \nabla_{\mathbf{x}} g(\hat{\mathbf{x}})$$
$$\mathbf{H} = \nabla_{\mathbf{x}}^2 g(\hat{\mathbf{x}})$$
$$\hat{\mathbf{x}} \leftarrow \arg \min_{\mathbf{x}} \frac{1}{2} (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H} (\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T (\mathbf{x} - \hat{\mathbf{x}})$$

Iterative LQR (iLQR) is the same idea: locally approximate a complex nonlinear function via Taylor expansion

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots)), \mathbf{u}_T)$$



# Nonlinear case: DDP/iterative LQR

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$

To get Newton's method, need to use *second order* dynamics approximation:

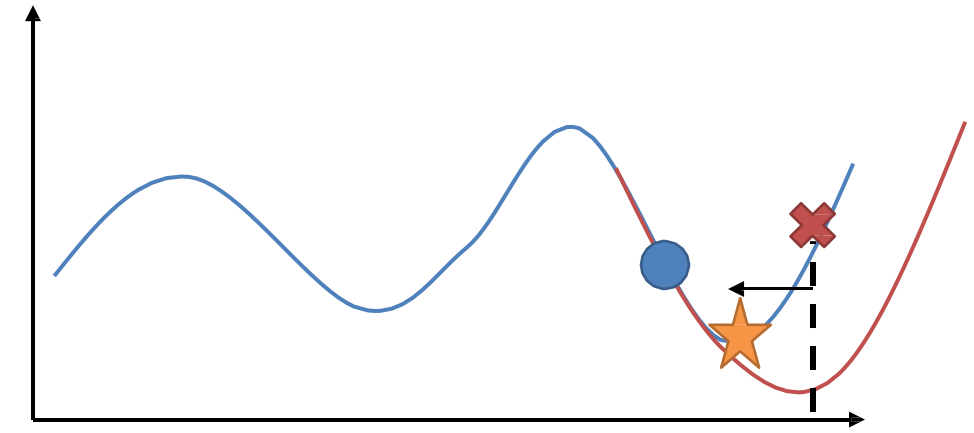
$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} + \frac{1}{2} \left( \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \cdot \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} \right) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}$$

differential dynamic programming (DDP)

# Nonlinear case: DDP/iterative LQR

$$\hat{\mathbf{x}} \leftarrow \arg \min_{\mathbf{x}} \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T(\mathbf{x} - \hat{\mathbf{x}})$$

why is this a bad idea?



until convergence:

$$\mathbf{F}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{c}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{C}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

Run LQR backward pass on state  $\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$  and action  $\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$

Run forward pass with  $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \hat{\mathbf{u}}_t$

Update  $\hat{\mathbf{x}}_t$  and  $\hat{\mathbf{u}}_t$  based on states and actions in forward pass

search over  $\alpha$   
until improvement achieved

# Case Study and Additional Readings

# Case study: nonlinear model-predictive control

## **Synthesis and Stabilization of Complex Behaviors through Online Trajectory Optimization**

Yuval Tassa, Tom Erez and Emanuel Todorov  
University of Washington

every time step:

observe the state  $\mathbf{x}_t$

use iLQR to plan  $\mathbf{u}_t, \dots, \mathbf{u}_T$  to minimize  $\sum_{t'=t}^{t+T} c(\mathbf{x}_{t'}, \mathbf{u}_{t'})$

execute action  $\mathbf{u}_t$ , discard  $\mathbf{u}_{t+1}, \dots, \mathbf{u}_{t+T}$

# Synthesis of Complex Behaviors with Online Trajectory Optimization

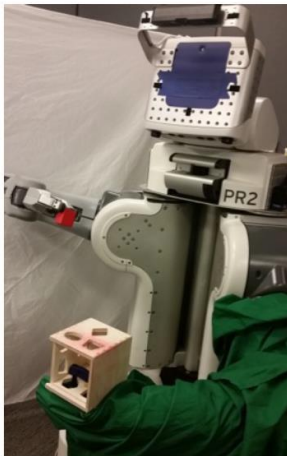
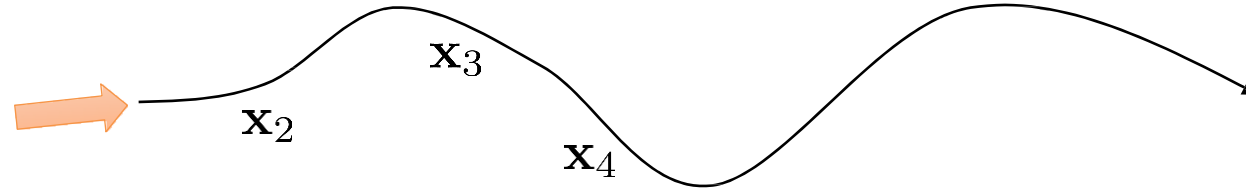
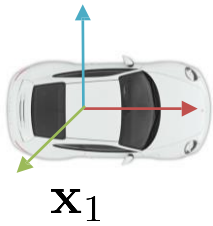
Yuval Tassa, Tom Erez & Emo Todorov

IEEE International Conference  
on Intelligent Robots and Systems  
2012

# Additional reading

1. Mayne, Jacobson. (1970). Differential dynamic programming.
  - Original differential dynamic programming algorithm.
2. Tassa, Erez, Todorov. (2012). Synthesis and Stabilization of Complex Behaviors through Online Trajectory Optimization.
  - Practical guide for implementing non-linear iterative LQR.
3. Levine, Abbeel. (2014). Learning Neural Network Policies with Guided Policy Search under Unknown Dynamics.
  - Probabilistic formulation and trust region alternative to deterministic line search.

# What's wrong with known dynamics?



Next time: learning the dynamics model