Online Learning in Games

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Last lectures : Zero-sum games

Definition

- Two players, actions sets S^1 and S^2, payoff function $g(s^1, s^2) \to \mathbb{R}$
- Player 1 can guarantee $\underline{v} = \sup_{s^1} \inf_{s^2} g(s^1, s^2)$
- Player 2 can guarantee $\overline{v} = \inf_{s^2} \sup_{s^1} g(s^1, s^2)$
- Value exists iff $\underline{v} = \overline{v} = v$

Theorem (von Neumann, Sion)

If S^1 and S^2 are convex, g is quasi-concave in s^1, quasi-convex in s^2 and other regularity conditions (semi-continuity, compactness of S^1 or S^2), the game has a value and each payer has (ε-)optimal strategy to guarantee $v(\pm\varepsilon)$

Contents

Nash Equilibrium

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- Thus, a Nash equilibrium is a strategy profile s.t. no player has a profitable deviation.
- In zero-sum games, Nash equilibrium coincides with the saddle point.

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Exemple : Cournot Equilibrium.

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Lemma (Sperner, 1928)

Every labeling of any simplicial subdivision of a simplex Δ has an odd number of completely labeled sub-simplices.

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- Thus, Δ has an odd number of completely labeled rooms.

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Sperner lemma implies existence of a completely labelled simplex. By tending ϵ to zero and using compactness of Δ we deduce existence of $v \in \Delta$ such that for all i, $f^i(v) \leq v^i$. Thus $f(v) = v$. □

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Corollary (Brouwer for convex compact sets)

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Proof : Apply Kakutani.

Application : existence Nash Equilibrium

Theorem (Glicksberg 1952, pure stratregies)

Let $G = (N, (S^i)_{i \in N}, (g^i)_{i \in N})$ be a game s.t. for all $i \in N$:

*S^i **is a convex compact subset** of a topological vector space*

*$g^i : S \to \mathbb{R}$ is **continuous***

*for all $s^{-i} \in S^{-i}$, $s^i \mapsto g^i(s^i, s^{-i})$ is **quasi-concave**.*

Then the set of Nash equilibria of G is compact and non-empty.

Proof : Apply Kakutani.

Contents

Examples

1, -1	-1, 1
-1, 1	1, -1

3, 3	-1, 4
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3. Battle of the sexes : Two pure equilibria (with different payoffs)

Notations for Finite Games

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- **The game is finite if N and S^i is finite $\forall i$.**

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Mixed extension

- Given a mixed strategy profile $\sigma = (\sigma^i)_{i \in N}$, the expected payoff of player i is :

$$g^i(\sigma) = \sum_{s \in S} (\prod_{i \in N} \sigma^i(s^i)) g^i(s)$$

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- This defines an extended game g^i from $\prod_{i \in N} S^i$ to $\prod_{i \in N} \Delta(S^i)$ that we denote also by g^i : and it is called the **mixed extension of** g^i.

Best responses

Let $\sigma^i \in \Delta(S^i)$ and $\theta^{-i} \in \Delta(S^{-i})$

σ^i **is a best response against** θ^{-i} **if**

$$g^i(\sigma^i, \theta^{-i}) \geq g^i(\tau^i, \theta^{-i}) \quad \forall \tau^i \in \Delta(S^i)$$

$BR(\theta^{-i})$ **: is the set of all best responses against à** θ^{-i}.

Properties

1. $\sigma^i \in BR(\theta^{-i}) \Leftrightarrow g^i(\sigma^i, \theta^{-i}) \geq g^i(s^i, \theta^{-i}) \; \forall s^i \in S^i$

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3. $\sigma^i \in BR(\theta^{-i}) \Leftrightarrow \left(\forall s^i \in S^i, \sigma^i(s^i) > 0 \Rightarrow s^i \in BR(\theta^{-i})\right).$

Nash equilibrium

Mixed Nash equilibrium : a $\sigma \in \prod_i \Delta(S^i)$ such that :

$$\forall i \in N, \sigma^i \in BR(\sigma^{-i})$$

Pure Nash equilibrium : mixed Nash equilibrium where all players play a pure strategy.

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Every finite game has a mixed Nash equilibrium

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Proof 2 : Kakutani

Exercises

Compute all pure and mixed Nash equilibria of the following finite games :

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M	(0, 0)	(4, 4)	(0, 0)
B	(0, 8)	(0, 0)	(6, 6)

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- A symmetric three-player finite game where each player chooses one of two rooms and wins 1 if he is alone and zero otherwise.

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- Impact of common knowledge on stability / instability.

General games (infinite actions)

Theorem (Glicksberg 1952, mixed startegies)

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then its set of mixed equilibria is compact and non-empty.

Contents

Potential games : Monderer and Shapley (1996)

Definition

A real valued function P defined on $S = \prod_{i \in N} S^i$ is a potential function for the game $G = (N, (S^i)_{i \in N}, (g^i)_{i \in N})$ if $\forall s^i, t^i \in S^i$, $u^{-i} \in S^{-i}$, $\forall i \in I$ one has

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Theorem

A potential game has a equilibrium in pure strategies.

Congestion games are potential games

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- G **is a potential game** with potential P given by

$$P(s) = \sum_{k \in K} \sum_{r=1}^{t^k(s)} u^k(r).$$

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Theorem

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Theorem

Every finite potential game $G = (g^i, S^i)_{i \in N}$ is isomorphic to a congestion game $F = (f^i, T^i)_{i \in N}$, e.g. there are isomorphisms $\phi^i : S^i \to T^i$ such that

$$g^i(s^1, ..., s^n) = f^i(\phi^1(s^1), ..., \phi^n(s^n)).$$

Improvement path algorithm in potential games

A improvement path is a sequence of strategy profiles $(s_1, s_2, ...s_k, ...)$ such that for **every period** k, there is a unique player, say i, that changes his strategy from $k-1$ to k $(s_k = (x^i, s_{k-1}^{-i}))$ and get his payoff strictly improved $(g^i(s_k) > g^i(s_{k-1}))$.

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Theorem (Monderer and Shapley (1996))

In a finite potential game, any improving path converges to a Nash equilibrium in finite time.

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Theorem (Monderer and Shapley (1996))

In a finite potential game, any improving path converges to a Nash equilibrium in finite time.

Proof : the potential keep increasing along the path !

Extensions of potential games

- Ordinal potential games :

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- Best best reply potential games :

$$\arg\max_{s^i} g^i(s^i, u^{-i}) = \arg\max_{s^i} P(s^i, u^{-i})$$

- Generalized best reply potential games :

$$\arg\max_{s^i} g^i(s^i, u^{-i}) \subset \arg\max_{s^i} P(s^i, u^{-i})$$

A characterization of ordinal potential games

Theorem

A finite game G is a generalized ordinal potential game if and only if every improvement path is finite.

A finite game G is a generalized best reply potential game if and only if every best reply path is finite (e.g. acyclic).

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Principle

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A sequence $(s_n)_{n\geq 1}$ is a FP process if for each $n \geq 1$ if :
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Theorem

Fictitious play converges in smooth and compact potential games to the set of Nash equilibria.

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Non-convergence of Fictitious play in 2-player games : Shapley (1964)

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The realised payoff up to stage n is

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- Consider the following two-player game :

$(0,0)$	(a,b)	(b,a)
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with $a > b > 0$.
Starting from a Pareto entry, the above principles imply that fictitious play does not converge to $(1/3, 1/3, 1/3)$ (the unique equilibrium of the game)

Contents

Variational Characterization

Definition

The game is smooth if $s^i \mapsto g^i(s^i, s^{-i})$ is $\mathcal{C}^1$ for all s^{-i} and $i \in I$. It is concave if $s^i \mapsto g^i(s^i, s^{-i})$ is concave for all s^{-i} and $i \in I$. The game is Hilbert if each strategy set is a subset of a Hilbert space.

Theorem

Let $G = (I, \{S^i\}_{i \in I}, \{g^i\}_{i \in I})$ be a smooth Hilbert game. Then :

1) *If s is a Nash equilibrium then*

$$\langle \vec{\nabla} g(s), s - t \rangle \geq 0, \quad \forall t \in S,$$

where $\langle \vec{\nabla} g(s), s - t \rangle := \sum_{i \in I} \langle \nabla_i g^i(s), s^i - t^i \rangle$.

2) *If the game is concave, condition 1) is sufficient for s to be a Nash equilibrium.*

Above, $\nabla_i g^i(s)$ denote the gradient of $g^i(s^i, s^{-i})$ with respect to s^i.

Monotone Games

Definition

A Hilbert smooth game is monotone if, for all $(s, t) \in S \times S$,

$$\langle \vec{\nabla} g(s) - \vec{\nabla} g(t), s - t \rangle \leq 0,$$

and the game is strictly monotone if the inequality is strict whenever $s \neq t$.

Theorem (Rosen)

For a Hilbert smooth monotone game :

1) *A profile $s \in S$ is a Nash equilibrium if and only if for all $t \in S$, $\langle \vec{\nabla} g(t), s - t \rangle \geq 0$.*
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- By strict monotonicity, we have the opposite inequality, hence equality, thus $s = t$.