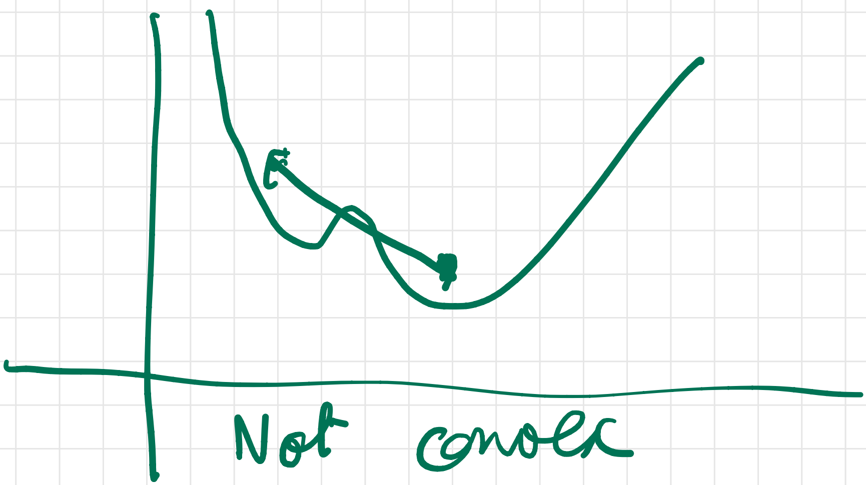
Convex Functions

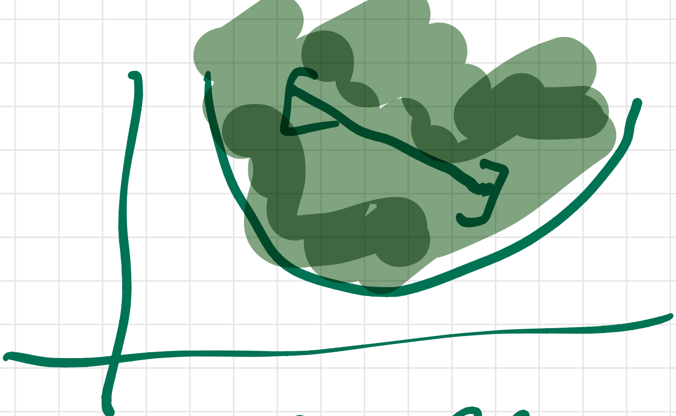
I Definition

Def: let $f : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ $= X$

We say that f is convex if its epigraph is convex

$$\text{epi } f = \{(x, r) \in X \times \mathbb{R} \ / \ r \geq f(x)\}$$





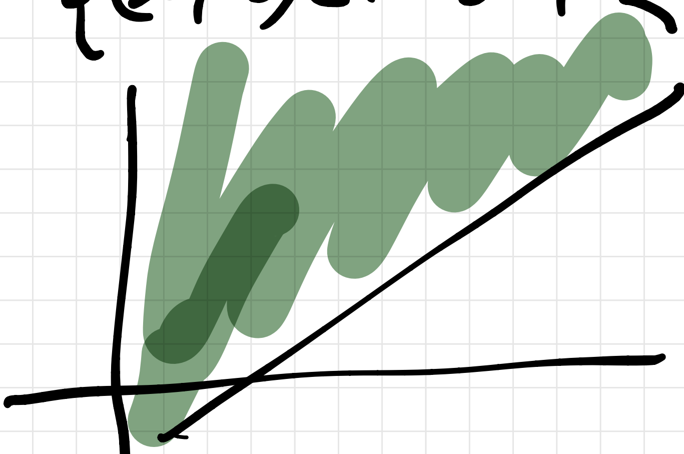
Convex

<u>Examples:</u> Affine functions

$$f(x) = \langle p, x \rangle - \alpha$$

(epi f is a half-space

$$\left\{ (x, r) \in X \times \mathbb{R} \,/\, \left\langle \begin{pmatrix} p \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ r \end{pmatrix} \right\rangle \leq \alpha \right\}$$

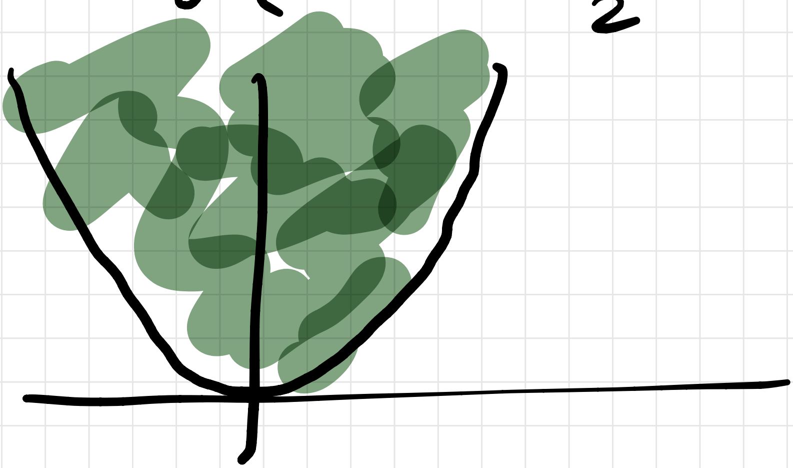


positive semidefinite

. Quadratic functions

$$f(x) = \frac{1}{2}\|x\|^2$$

$$\text{epi } f \left\{ (x, r) \;/\; \frac{1}{2}\|x\|^2 \leq r \right\}$$



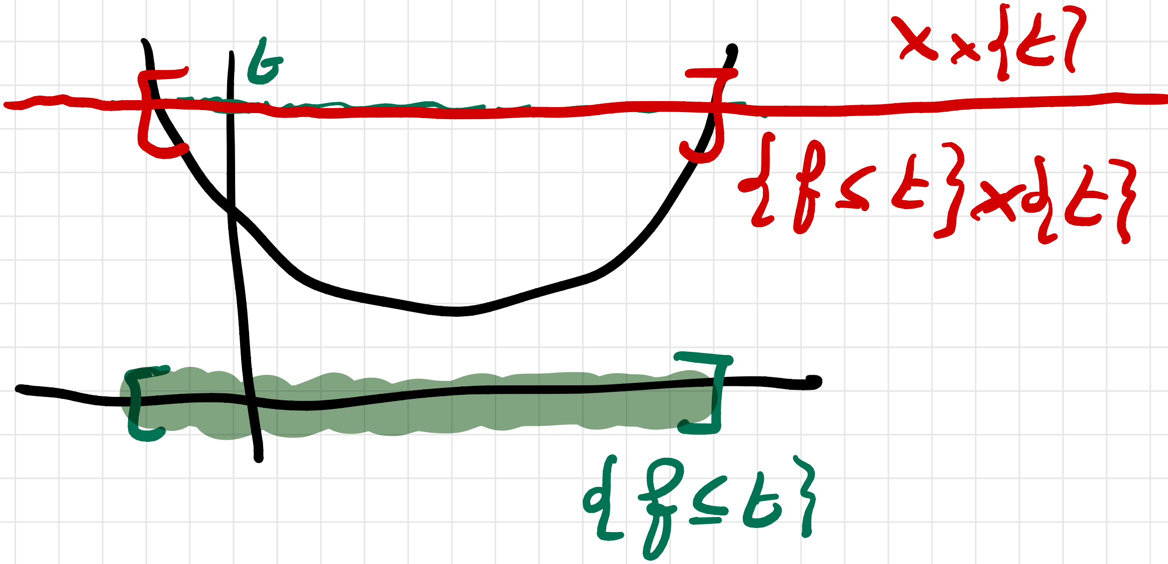
⚠ $f(x) = x_1^2 - x_2^2$

is not convex

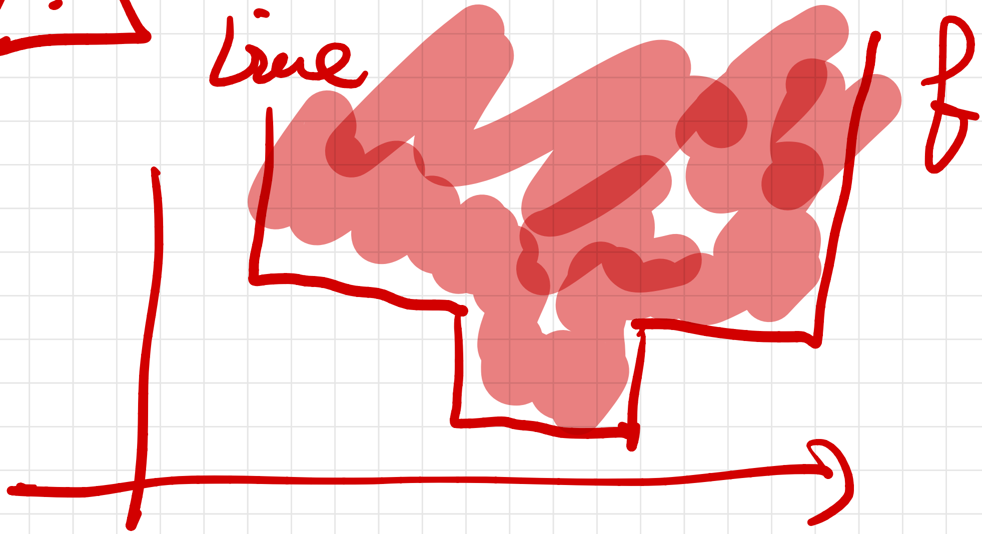
- $f(x) = +\infty$ for all x
  $\text{epi} f = \emptyset$ (convex)

- $f(x) = -\infty$ for all x
  $\text{epi} f = X \times \mathbb{R}$ convex

Prop: If f is convex
then for all $t \in \mathbb{R}$,
$\{f \leq t\} \overset{\text{def}}{=} \{x \in X \mid f(x) \leq t\}$
is convex



⚠ The converse is not true



II How to recognize convex functions?

a) Inequalités

Prop : let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$.

Then f is convex

$\iff \forall x, y \in \mathrm{dom}\, f, \ \forall \theta \in [0,1]$ ($] \quad [$)

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

$= \left\{ \begin{array}{l} \theta_i \geq 0 \\ \sum \theta_i = 1 \end{array} \right.$

$\iff \forall n \geq 2, \ \forall \theta \in \Delta_n$

$$f\left(\sum \theta_i x_i\right) \leq \sum \theta_i f(x_i)$$

Example: All norms are convex!

b) Smooth functions

Prop: let $f : U \to \mathbb{R}$, U convex open, differentiable at x (and y)

TFAE:

① f is (strictly) convex (f is below its chords)

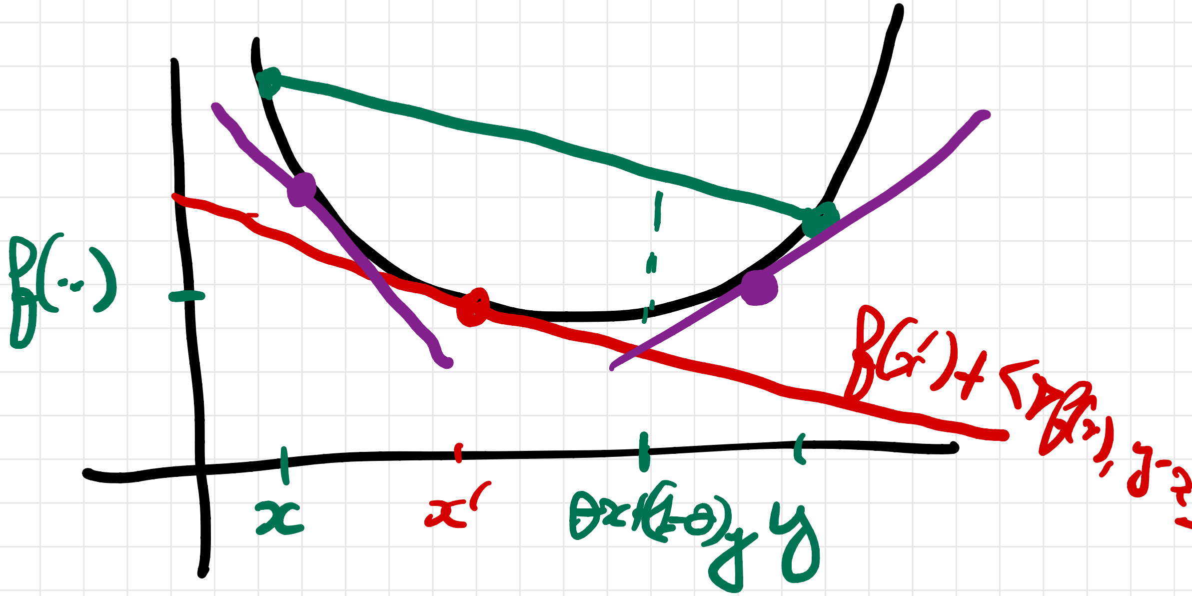
② f is above its tangent

$\forall x, y \in U$

$$f(y) \underset{>}{\geq} f(x) + \langle \nabla f(x), y-x \rangle$$

③ the slopes are nondecreasing

$\forall x, y$ $\quad \langle \nabla f(y) - \nabla f(x), y-x \rangle \underset{>}{\geq} 0$



Prop: Let $f : X \to \mathbb{R}$ twice differentiable

Then f is strictly convex iff if

$\forall x \in X$, $\nabla^2 f(x) \succeq 0$

$\succ$

$$\parallel$$

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$$

That is
$\forall h \in \mathbb{R}^N$

$$h^T (\nabla^2 f(x)) h \geq 0.$$

What about strictly convex functions ?

$\forall x, y \in \operatorname{dom} f, \quad x \neq y$

$\forall \theta \in]0, 1[$

$$f(\theta x + (1-\theta) y) < \theta \underbrace{f(x)}_{\min f} + (1-\theta) \underbrace{f(y)}_{\min f}$$

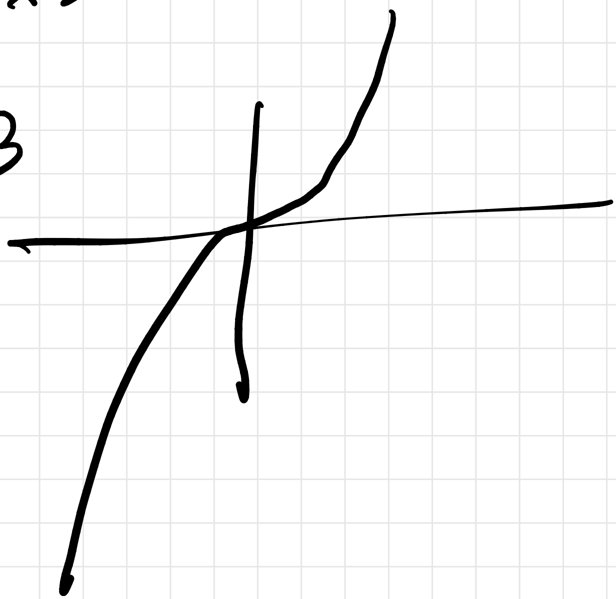
(strictly convex functions have at most one minimizer)

Example:

$f(x) = x^4$ in $\mathbb{R}$

$f''(x) = 12 \times x^2$

$f'(x) = 4x^3$



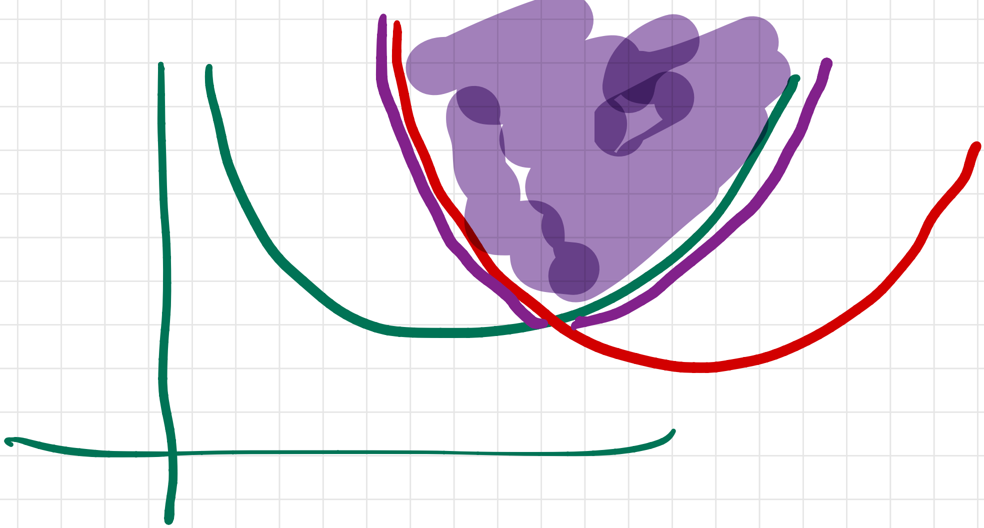
c) Operations on convex functions

Prop: The following functions are convex:

① $f_1 + f_2$, where f_1 and f_2 are convex

② αf , $\alpha > 0$, with f convex

③ $\sup_{i \in I} f_i$, $\{f_i\}_{i \in I}$ family of convex functions



④ $x_1 \longmapsto \inf_{x_2 \in \mathbb{R}^{N_2}} f(x_1, x_2)$

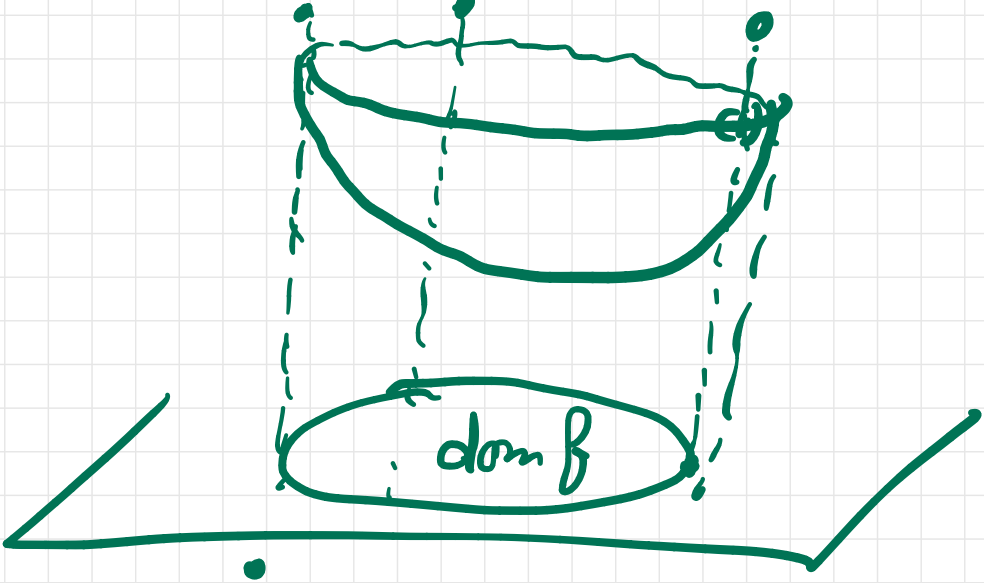
where $f : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex

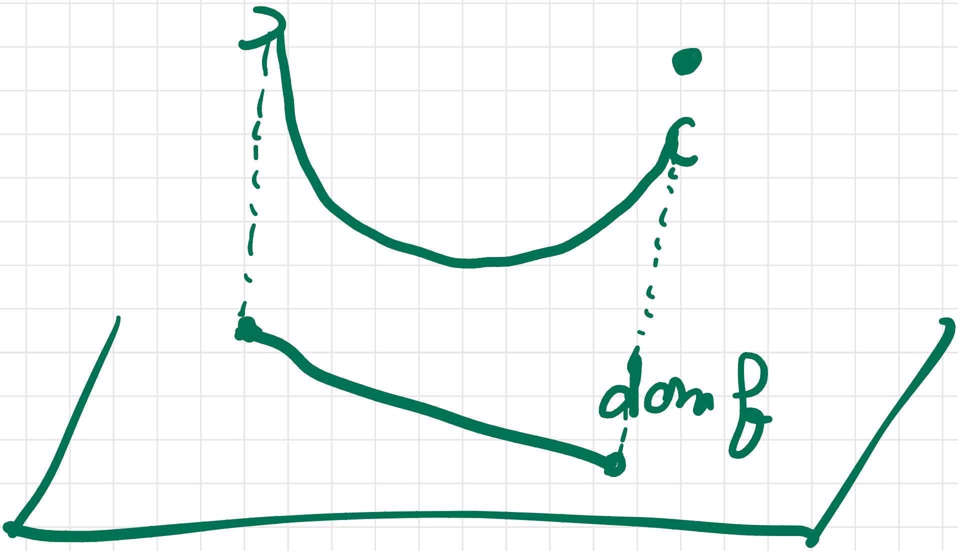
III Continuity

Prop: let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$.
convex.

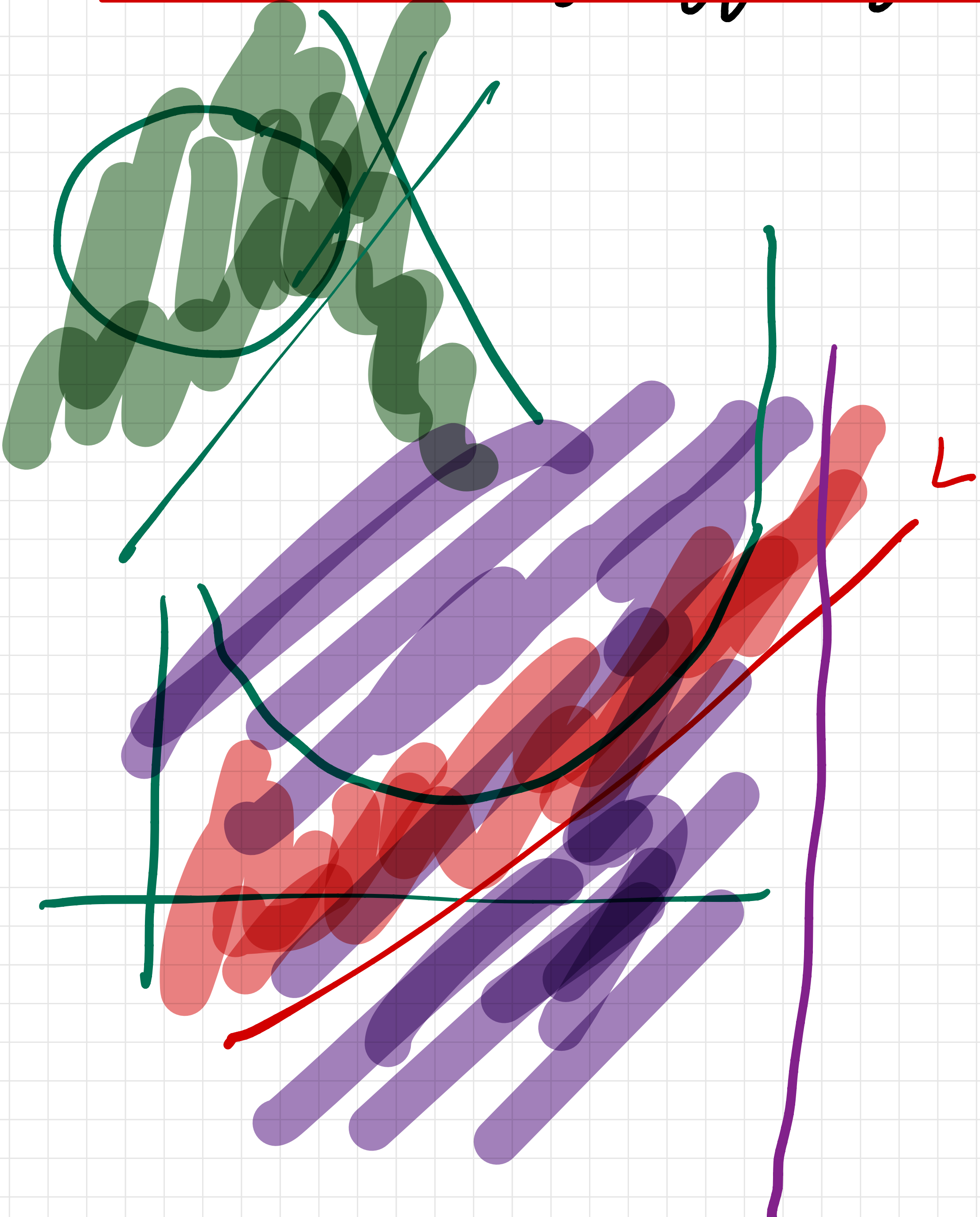
1) If $\text{Int}(\text{dom} f) \neq \emptyset$ then f is continuous on $\text{Int}(\text{dom} f)$

2) More generally, the restriction of f to $\text{Aff}(\text{dom} f)$ is continous on $\text{rint}(\text{dom} f)$


dom f


dom f

IV Envelope of affine functi

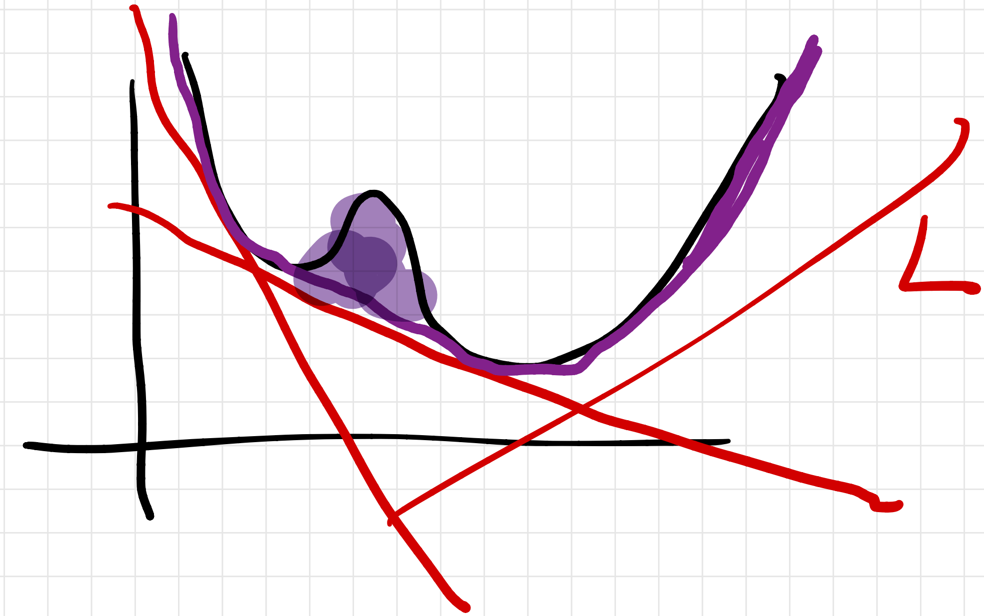


Theorem: Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, l.s.c. proper.

Then

$$f(x) = \sup \left\{ L(x) \,/\, L \text{ affine function } L \leq f \right\}.$$

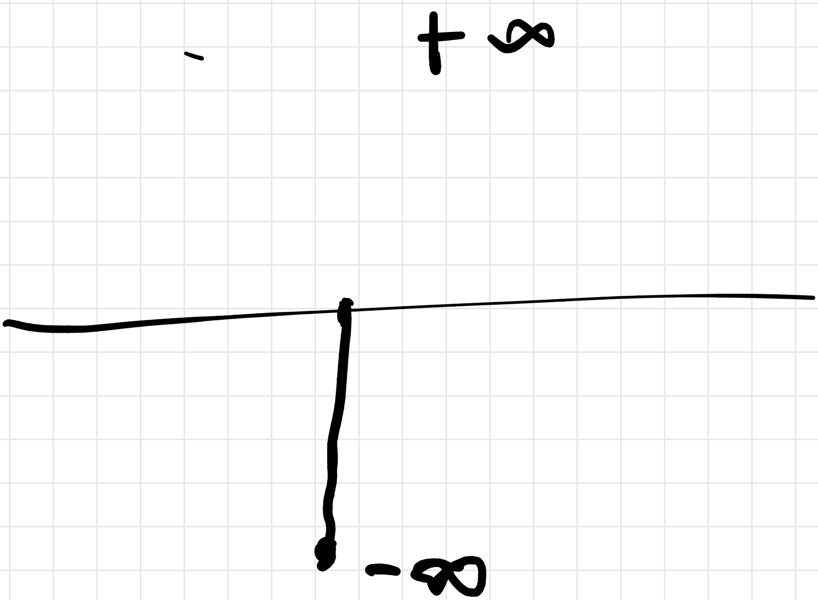
⚠ If f is not convex



We get the largest convex, l.s.c. function which is below f.

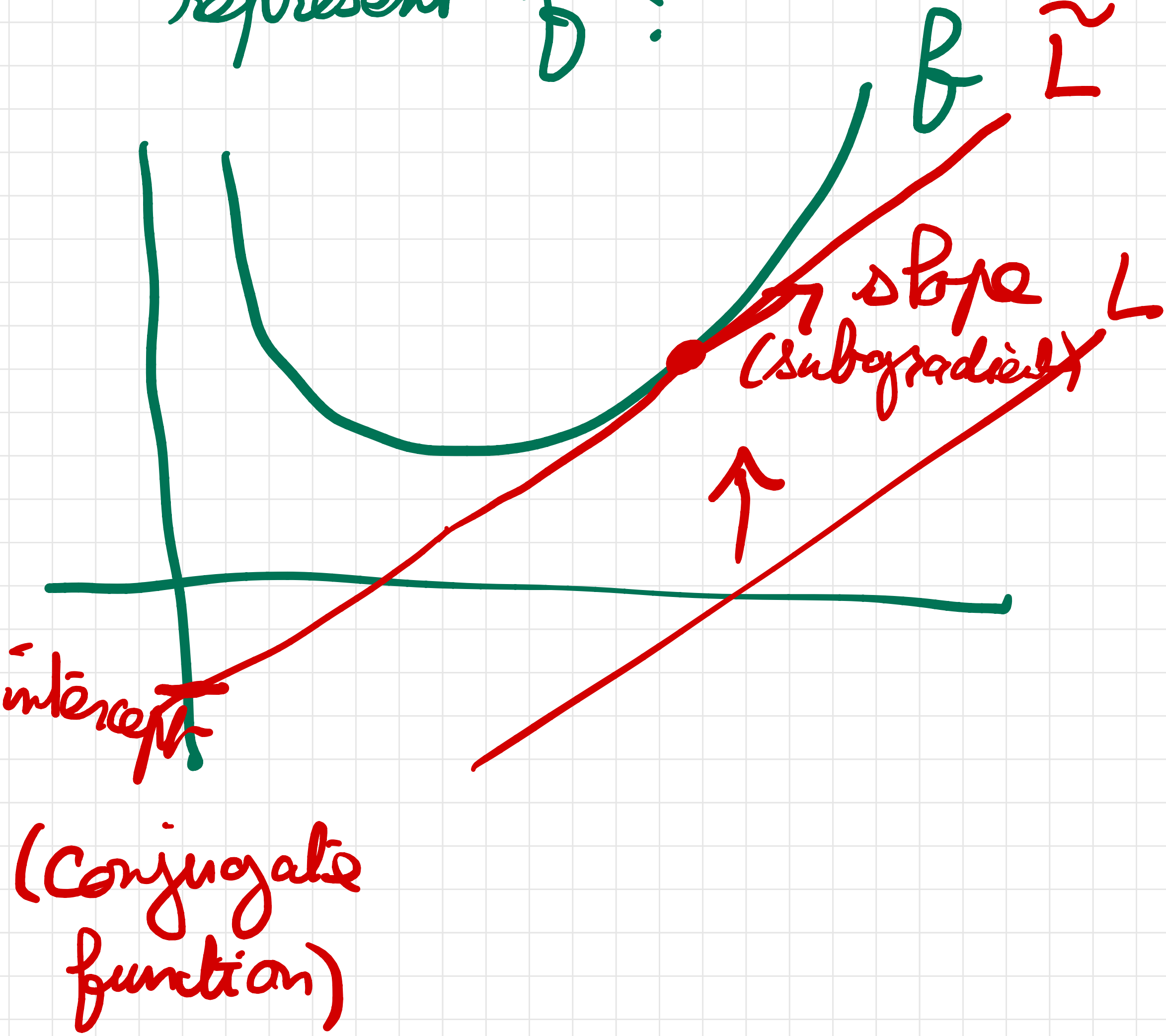
⚠ If $f(x_0) = -\infty$ at some $x_0 \in X$

$\{L \text{ affine and } L \le f\} = \emptyset$



The theorem is not valid

Question : What are the best affine functions to represent f ?



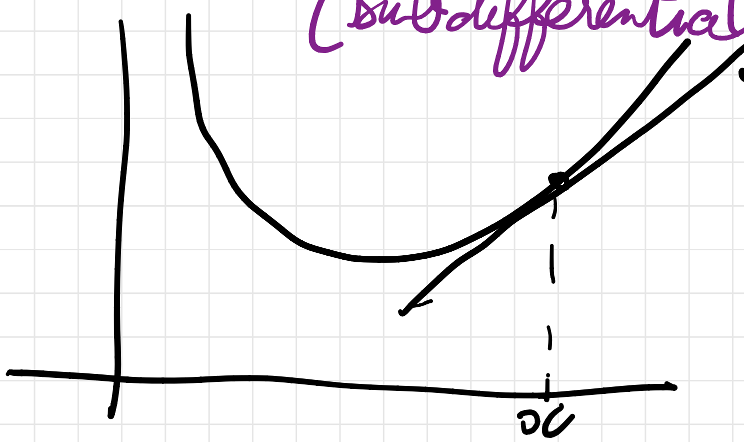
V Subgradients

Def: let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $p \in \mathbb{R}^N$. x

We say that p is a subgradient of f at x if

$$\forall y \in X, \quad f(y) \geq f(x) + \langle p, y - x \rangle$$

(We write $p \in \partial f(x)$) (subdifferential of f at x)



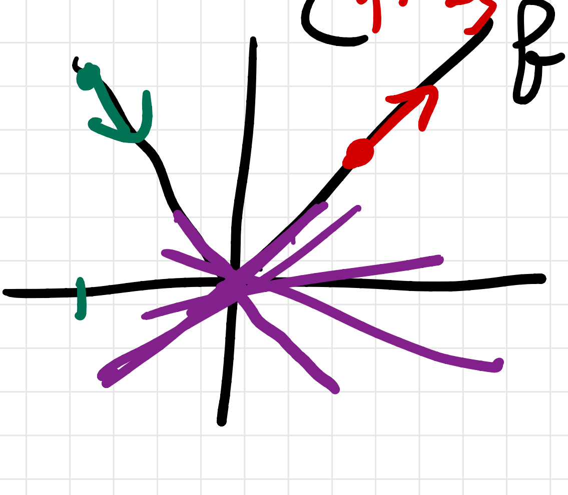
<u>Examples:</u> · If $f : X \to \mathbb{R}$ is differentiable and <u>convex</u>

$$\nabla f(x) \in \partial f(x).$$

In fact $\partial f(x) = \{\nabla f(x)\}$

· If $f(x) = |x|$ for $x \in \mathbb{R}$

$$\partial f(x) = \begin{cases} \{-1\} & \text{for } x<0 \\ [-1,1] & \text{for } x=0 \\ \{+1\} & \text{for } x>1 \end{cases}$$



. In dimension 1

$$\partial f(x) = [f'_\ell(x), f'_r(x)]$$

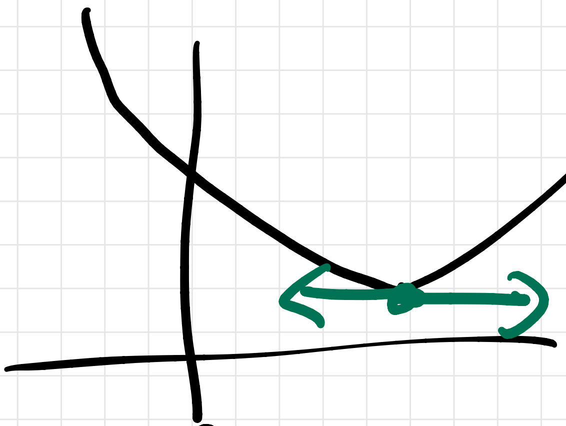
left derivative right derivative

Prop: The set $\partial f(x)$ is closed and convex

⚠ It might be empty

Theorem (Fermat's rule): let $f: X \to \mathbb{R} \cup \{+\infty\}$ and $x \in X$.

Then $x \in \operatorname{argmin} f \iff 0 \in \partial f(x)$.



Proof:

$x \in \operatorname{argmin} f \iff \forall y \; f(y) \geq f(x) + \langle 0, y - x \rangle$

$\iff \quad 0 \in \partial f(x)$

qed.

<u>Prop</u>: Let f convex and $\alpha > 0$

$\forall x \in X \quad \partial(\alpha f)(x) = \alpha \partial f(x)$

<u>Prop</u>: Let f, g two proper convex functions.

Then

$\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$

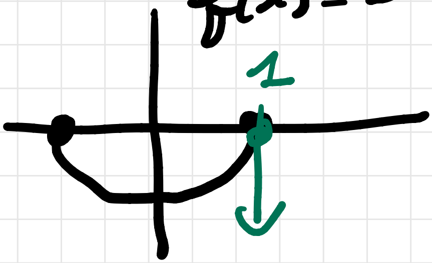
If moreover

$$\operatorname{rint}(\operatorname{dom} f) \cap \operatorname{rint}(\operatorname{dom} g) \neq \emptyset$$

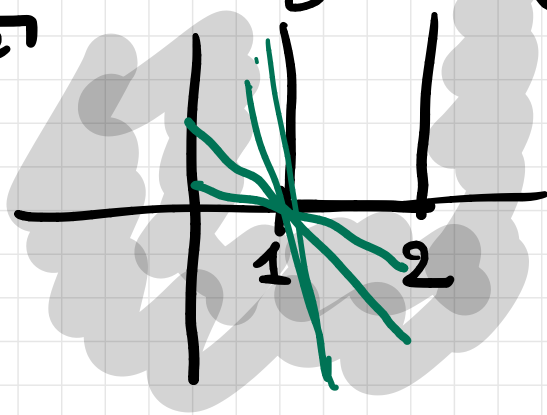
then $\partial f(x) + \partial g(x) = \partial(f+g)(x)$.

Example

$f(x) = 1-\sqrt{1-x^2}$

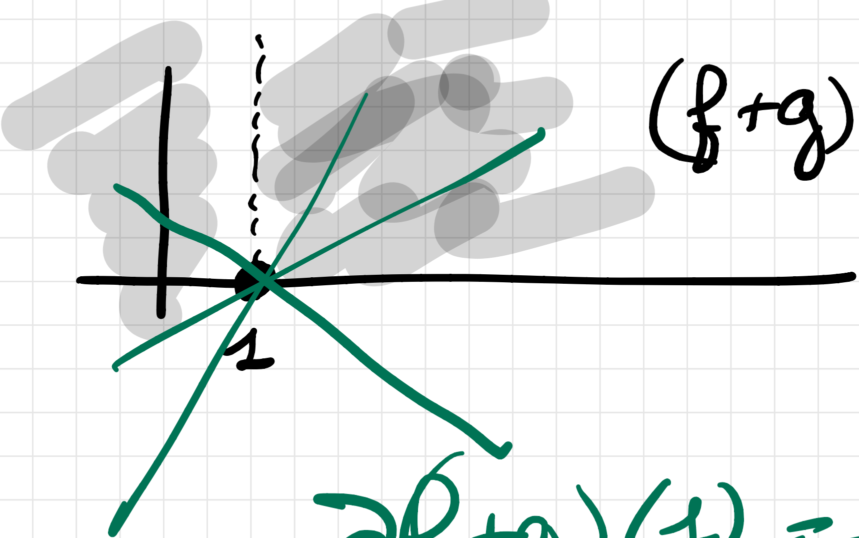


$g(x) = \chi_{[1,2]}(x)$



$$\underbrace{\partial f(1) = \emptyset \quad + \quad \partial g(1) = \mathbb{R}_-}_{= \emptyset}$$

$(f+g)(x) = \chi_{\{1\}}(x)$



$\partial(f+g)(1) = \mathbb{R}$

Prop: Let $f : x \mapsto h(Ax)$
where $h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$
is convex
and $A \in \mathbb{R}^{m \times N}$.

Then

$$A^T \partial h(Ax) \subset \partial f(x)$$

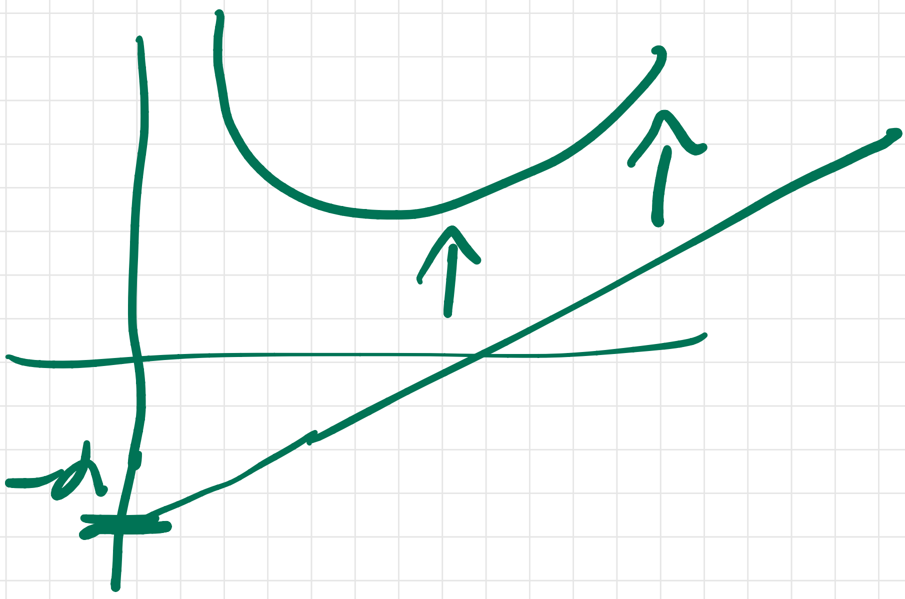
If moreover

$$\boxed{\begin{gathered}\operatorname{Im} A \cap \operatorname{rint}(\operatorname{dom} h) \neq \emptyset \\ \text{then} \quad A^T \partial h(Ax) = \partial f(x).\end{gathered}}$$

VI Conjugate functions

1) The conjugate function

We fix a slope p, what is the best intercept ?



We want

$$\forall x \in X, \quad \langle p, x \rangle - \alpha \leq f(x)$$

$\Longleftrightarrow \forall x \quad \langle p, x \rangle - f(x) \leq \alpha$

The smallest value is

$$\alpha = \sup \left(\langle p, x \rangle - f(x) \right)$$

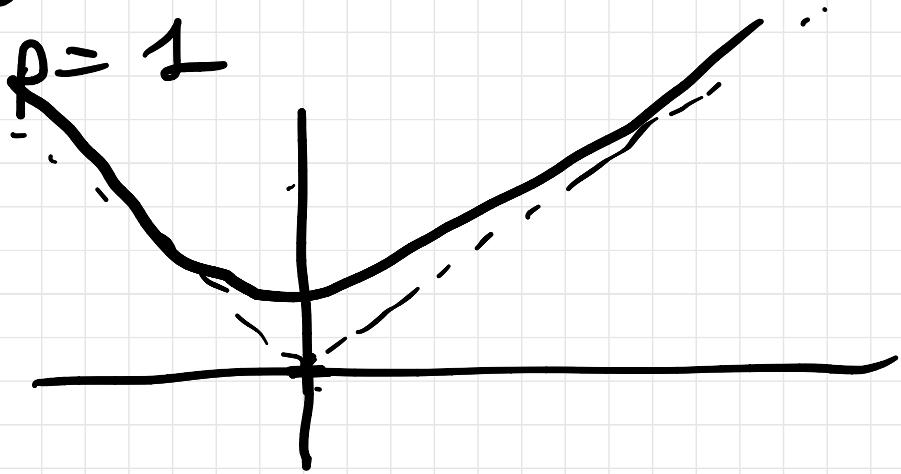
Def: The conjugate function of f is

$$\boxed{f^*(p) = \sup_{x \in X} \left(\langle p, x \rangle - f(x) \right).}$$

Remark: The "sup" is not always a «max»

$f(x) = \sqrt{x^2+1}$

$p = 1$



Prop: If f is proper then f^* is proper, convex l.s.c.

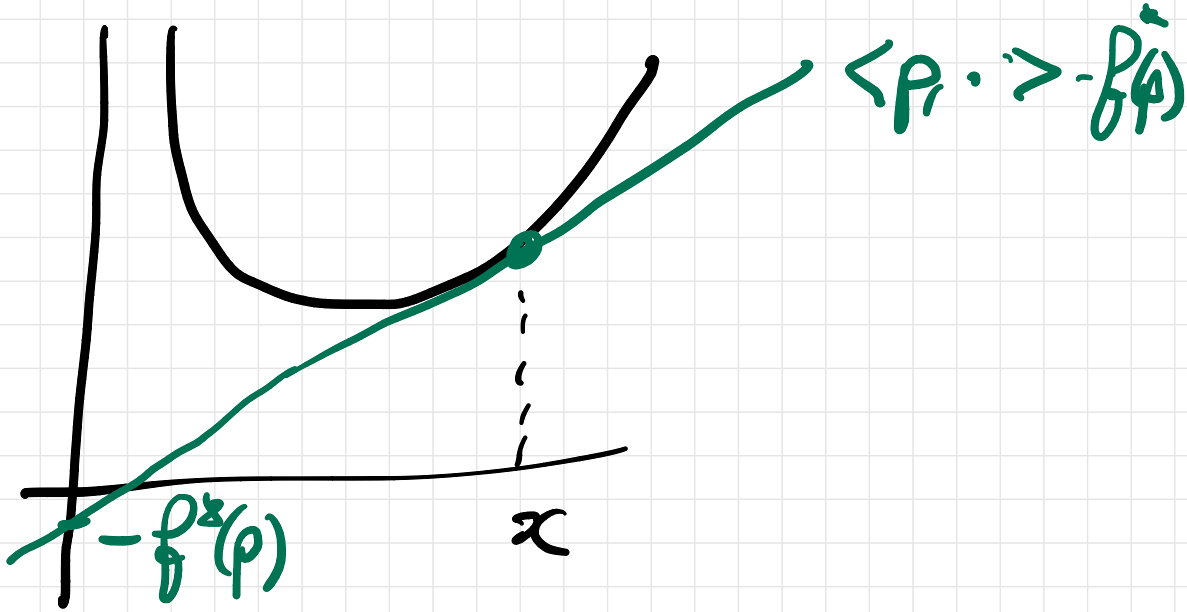
Prop: let f be convex proper.

Then $\forall x, p \in \mathbb{R}^N$

$$f(x) + f^*(p) \geq \langle p, x \rangle$$

Moreover

$$p \in \partial f(x) \iff f(x) + f^*(p) = \langle p, x \rangle$$



2) The double conjugate

Let f be convex l.s.c and proper.

$$f(x) = \sup\left\{ L(x) \;/\; L \text{ affine},\ L \leq f \right\}$$

$$= \sup\left\{ \langle p, x\rangle - \alpha \;/\; p \in \mathbb{R}^N,\ \alpha \in \mathbb{R},\ \forall y\ \langle p, y\rangle - \alpha \leq f(y) \right\}$$

$$= \sup\left\{ \langle p, x\rangle - f^*(p) \;/\; p \in \mathbb{R}^N \right\}$$

$$= f^{**}(x).$$

Exercise 6

Let $p \in \mathbb{R}^N$.

$$p \in \partial f(x) \iff \forall y \in \mathbb{R}^N \quad f(y) \geq f(x) + \langle p, y-x \rangle$$

$$\iff \forall y \in \mathbb{R}^N, \sum_{i=1}^{N} f_i(y_i) \geq \sum_{i=1}^{N} \left(f(x_i) + p_i(y_i - x_i) \right)$$

$\boxed{\supset}$ If $p \in \prod_{i=1}^{N} \partial f_i(y_i)$ then

$$\forall i \quad f_i(y_i) \geq f_i(x_i) + p_i(y_i - x_i)$$

$$\sum_i \qquad\qquad \sum$$

$$f(y) \geq f(x) + \langle p, y-x \rangle$$

Hence $\prod \partial f_i(x_i) \subset \partial f(x)$.

$\boxed{C}$ Let $p \in \partial f(x)$

Pick $y_i = \begin{cases} x_i & \text{if } i \neq i_0 \\ t & i = i_0 \end{cases}$

Then

$$f(y) \geq f(x) + \langle p, y - x \rangle$$

$$\Leftrightarrow f_{i_0}(t) \geq f_{i_0}(x_{i_0}) + p_{i_0}(t - x_{i_0})$$

hence $p_{i_0} \in \partial f_{i_0}(x_{i_0})$.

True for all $i_0 \in \{1, \dots, N\}$

hence $p \in \prod_{i=1}^{N} \partial f_i(x_i)$

2) $f(x) = \sum_{i=1}^{N} |x_i|$

$$q \in \partial f(x) \iff \forall i \quad q_i \in \partial f_i(x_i)$$

We have

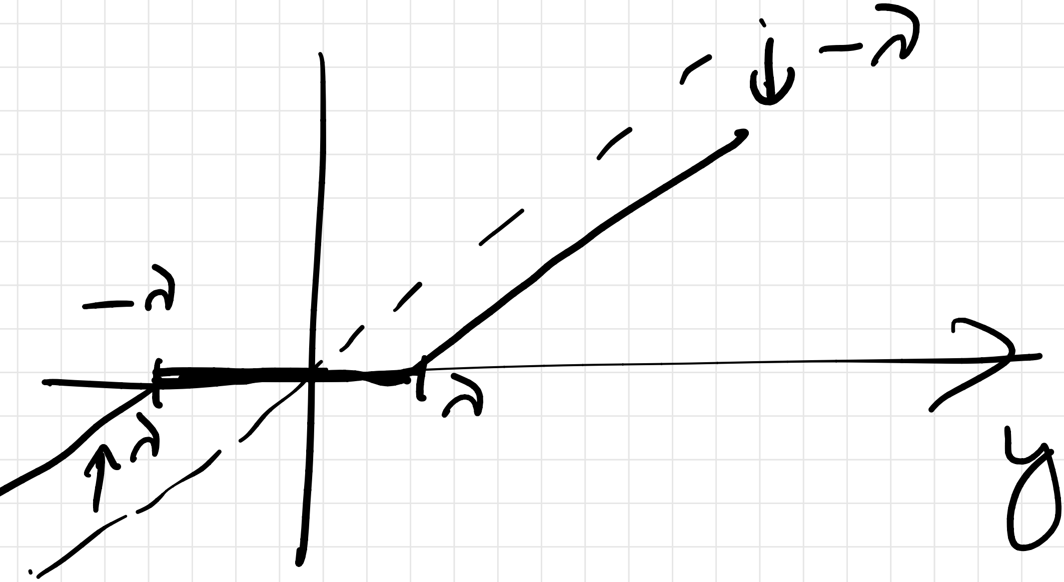
$$\partial f_i(x_i) = \begin{cases} \{-1\} & \text{if } x_i < 0 \\ [-1, 1] & x_i = 0 \\ \{+1\} & x_i > 0 \end{cases}$$

Typically

$$q = (\text{sign}(x_1), \dots, t_j, \dots, \text{sign}(x_N))$$

if $x = (x_1, \dots, 0, \dots x_N)$

2) $0 \in \partial(f+g)(x)$



Theorem: let f convex, proper
l.s.c.

Then $f^{**} = f$

Corollary: let f convex l.s.c
proper

Then
$\forall x, p \in \mathbb{R}^N \quad p \in \partial f(x) \iff x \in \partial f^*(p)$