Exercises for Convexity

M2 IASD/MASH
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Ex. 1 — Existence of a minimizer of an extended-valued function
Prove that the following optimization problem

$$\min_{x\in\mathbb{R}^2} \sqrt{1+\left\|x-\begin{pmatrix}1\\0\end{pmatrix}\right\|_2^2} \qquad \text{s.t.} \quad \left\langle x, \begin{pmatrix}\cos(\|x\|_2)\\ \sin(\|x\|_2)\end{pmatrix}\right\rangle \geq 0$$

has a solution.

Ex. 2 — Meeting the ℓ^1 ball
We consider the ℓ^1 ball,

$$C = \left\{ x \in \mathbb{R}^p \mid \sum_{i=1}^{p} |x_i| \leq 1 \right\}.$$

1. Prove that it is a compact convex set.
2. What are the extreme points of C?
3. What is its (relative) interior?

Ex. 3 — On a set of positive semi-definite matrices
We denote by $\mathbb{S}_n(\mathbb{R})$ (resp. $\mathbb{S}_n^+(\mathbb{R})$) the set of real symmetric (resp. positive semi-definite) matrices of size $n \times n$. Consider the set

$$C = \left\{ M \in \mathbb{S}_n^+(\mathbb{R}) \mid \operatorname{Tr} M = 1 \right\}.$$

1. Prove that C is a compact convex set.
2. What are the extreme points of C?
3. a) What is the affine hull of C?
   b) Deduce the dimension of C. What can you conclude about the Minkowski-Carathéodory theorem for this convex?
4. What is the relative interior of C?

Ex. 4 — Some projections
Let $x \in \mathbb{R}^p$. Compute the projection of x onto

1. $C = \{ y \in \mathbb{R}^p \mid \|y\|_2 \leq 1 \}$ (ℓ^2 unit ball).
2. $C = \{ y \in \mathbb{R}^p \mid \|y\|_\infty \leq 1 \}$ (ℓ^∞ unit cube).
3. $C = \left\{ (y,t) \in \mathbb{R}^{p-1} \times \mathbb{R} \mid \sum_{i=1}^{p-1} (y_i)^2 \leq 1 \text{ and } 0 \leq t \leq 1 \right\}$, for $p \geq 2$ (cylinder).

Ex. 5 — Recognizing convex functions
Are the following functions convex?

1. $f(x) = \|Ax - b\|$, for $x \in \mathbb{R}^p$, where $A \in \mathbb{R}^{m \times N}$, $b \in \mathbb{R}^m$.
2. (ReLU) $f(x) = \max\{x, 0\}$, for all $x \in \mathbb{R}$.
3. (Quadratic over linear function) $f(x,y) = x^2/y$ for all $x, y \in \mathbb{R}$ such that $y > 0$.
4. (Log-sum-exp) $f(x) = \log(e^{x_1} + \cdots + e^{x_p})$, for all $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$.

5. (Maximal eigenvalue) $f(M) = \lambda_n(M)$ for all $M \in S_n^+(\mathbb{R})$.
Hint: *Observe that* $\lambda_n(M) = \sup \{ y^\top M y \mid y \in \mathbb{R}^n, \|y\| = 1 \}$.
6. (Sum of the k largest components) $f(x) = x_{[1]} + \cdots + x_{[k]}$ where $1 \leq k \leq p$ and $x_{[1]} \geq \cdots \geq x_{[p]}$ are the ordered components of $x \in \mathbb{R}^p$.
Hint: *Write f as the supremum of affine functions.*

Ex. 6 — Subdifferential of separable functions

1. Let $f_1, \ldots, f_p : \mathbb{R} \to \{+\infty\}$ be convex proper lower semi-continuous functions. Consider the function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x) = \sum_{i=1}^p f_i(x_i)$ for all $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$. Prove that
$$\forall x = (x_1, \ldots, x_p) \in \mathbb{R}^p, \quad \partial f(x) = (\partial f_1(x_1)) \times \ldots \times (\partial f_p(x_p))$$

2. Consider the ℓ^1-norm, i.e. the function $f\colon x \mapsto \|x\|_1 \overset{\text{def.}}{=} \sum_{i=1}^p |x_i|$ and let $q \in \mathbb{R}^p$. Prove that $q \in \partial f(x)$ if and only if
$$\begin{cases} q_i = \operatorname{sign}(x_i) & \text{for all } i \in \{1, \ldots, p\} \text{ such that } x_i \neq 0, \\ q_i \in [-1, 1] & \text{for all } i \in \{1, \ldots, p\} \text{ such that } x_i = 0 \end{cases}$$

Ex. 7 — ℓ^1-regularization
Consider the minimization problem, for fixed $y \in \mathbb{R}^p$, and $\lambda > 0$,
$$\min_{x \in \mathbb{R}^p} \lambda \|x\|_1 + \frac{1}{2} \|x - y\|_2^2 .$$

Such a problem arises when considering the denoising of signals using the ℓ^1 norm. As it amounts to computing the proximity operator of the ℓ^1-norm, so it also appears in proximal algorithms.

1. Prove that there is a unique minimizer.

2. Prove that the solution is given by the *soft thresholding* of y,

$$\forall i \in \{1,\ldots,p\}, \quad x_i = \begin{cases} y_i + \lambda & \text{if } y_i < -\lambda, \\ 0 & \text{if } -\lambda \leq y_i \leq \lambda, \\ y_i - \lambda & \text{if } y_i > \lambda. \end{cases} \tag{1}$$

(you may use the result of the previous exercise).

Ex. 8 — Projection onto a convex set Let $C \subseteq \mathbb{R}^p$ be a nonempty closed convex set, and $\chi_C(x) \overset{\text{def.}}{=} 0$ if $x \in C$, $+\infty$ otherwise. Let $y \in \mathbb{R}^p$ and consider the problem

$$\min_{x\in\mathbb{R}^p} \frac{1}{2}\|x-y\|^2 + \chi_C(x).$$

1. Prove that there is a unique minimizer to that problem.
2. Let $q \in \mathbb{R}^p$, what does $q \in \partial\chi_C(x)$ mean?
3. Using the subdifferential, provide a characterization of the projection onto C.
4. Let $f \in \mathscr{C}^1(X)$, convex and coercive (w.r.t. the set C).

Ex. 9 — The Moreau-Yosida regularization and the proximal point Let f be a proper convex lower semi-continuous function, and $\lambda > 0$. Define the *Moreau-Yosida regularization* of f,

$$\forall x \in \mathbb{R}^p, \quad f_\lambda(x) \overset{\text{def.}}{=} \inf_{y\in\mathbb{R}^p} \left(f(y) + \frac{1}{2\lambda}\|x-y\|^2 \right) \tag{2}$$

1. Draw f_λ for $f(x) = \chi_{[-1,1]}(x)$, $x \in \mathbb{R}$.
2. Prove that there is a unique minimizer y in (2). It is called the proximal point of f at x. It is often denoted by $\mathrm{prox}_{\lambda f}(x)$.
   Hint: *To prove that f is coercive, justify that there exists some affine function which is below f.*
3. Prove that f_λ is convex proper, and that $\mathrm{dom}\, f_\lambda = \mathbb{R}^p$ (hence f_λ is continuous on $\mathbb{R}^p$).
4. Prove the following properties
   1. $f_\lambda(x) \leq f(x)$ for all $x \in \mathbb{R}^p$, $\lambda > 0$.
   2. $\lim_{\lambda\to0^+} \mathrm{prox}_{\lambda f}(x) = x$ for all $x \in \mathrm{dom}\, f$.
   3. $\lim_{\lambda\to0^+} f_\lambda(x) = f(x)$ for all $x \in \mathbb{R}^p$.

Ex. 10 — Convex conjugate.

1. Let $f\colon \mathbb{R}^p \to \mathbb{R}$ defined by $f(x) = \frac{1}{2}\|x\|_2^2$. Compute its Legendre-Fenchel transform f^*.

2. Compute the conjugate function of the ℓ^q norm: $x \mapsto \|x\|_q$.
Hint: *Remember the Hölder inequality:* $|\langle x, y \rangle| \leq \|x\|_q \|y\|_{q'}$ *for* $q, q' \in [1, +\infty]$ *such that* $1/q + 1/q' = 1$. *What is the equality case?*