Incremental learning, game theory, and applications
Lecture 1: Finite zero-sum games

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Definition

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- The payoff of player 1 is then $g(i,j)$ and that of player 2 is $-g(i,j)$.
- Player 1 wants to maximize g and is called the maximizing player. Player 2 is the minimizing player.
- With the notations of the introduction, the strategy sets are $S^1 = I$ and $S^2 = J$ and the payoff functions are $g^1 = g = -g^2$.

- $G = (I, J, g)$ is a *finite* zero-sum game when I and J are finite.
- The game is then represented by an $I \times J$ matrix A, where player 1 chooses the row $i \in I$, player 2 chooses the column $j \in J$ and the entry A_{ij} of the matrix corresponds to the payoff $g(i, j)$.

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‘Matching Pennies”

	L	R
T	1	-1
B	-1	1

Bimatrix representation

- Formulation 1 : A zero-sum game

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$$\begin{array}{c|c|c|} & L & R \\ \hline T & 1 & -1 \\ \hline B & -1 & 1 \\ \hline \end{array}$$

- Formulation 2 : bi-matrix game

$$\begin{array}{c|c|c|} & L & R \\ \hline T & 1,-1 & -1,1 \\ \hline B & -1,1 & 1,-1 \\ \hline \end{array}$$

- Formulation 1 : A zero-sum game

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B	-1	1

- Formulation 2 : bi-matrix game

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

Saying that the game is zero-sum avoid to specify the payoffs of players 2.

Let $w \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition

P1 guarantees w if : $\exists i \in I, \quad \forall j \in J, \quad g(i,j) \geq w.$

Guaranteeing a payoff

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Definition

P2 can guarantee w if : $\exists j \in J, \quad \forall i \in I, \quad g(i,j) \leq w.$

Because player 2 is minimizing, the inequalities are reversed.

Let $G = (I, J, g)$ be a zero-sum game.

Definition

The *maxmin* of G is the supremum of quantities that P1 can guarantee. We denote it $\max\min(G)$, or $\underline{v}$. We have : $\underline{v} = \sup_{i \in I} \inf_{j \in J} g(i, j)$.

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Interpretation : if P1 plays before P2 and if P2 observes what P1 did before choosing his action, then the rational outcome of the game if $\underline{v}$ for player 1 and $-\underline{v}$ for player 2.

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The *minmax* of G is the infimum of the quantities that P2 can guarantee. It is denoted $\min\max(G)$, or $\overline{v}$. we have $\overline{v} = \inf_{i \in I} \sup_{j \in J} g(i, j)$.

Interpretation : if P2 plays before P1, the rational outcome must be $\overline{v}$ for p1 and $-\overline{v}$ for P2.

Duality gap and the value

Proposition

$\underline{v} \leq \overline{v}$.

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Proposition

If w can be guaranteed by both players, then w is unique and it is the value.

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T	1	-1
B	-1	1

$\underline{v} = -1 < 1 = \overline{v}$: No value.

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$\underline{v} = -1 < 1 = \overline{v}$: No value.

	L	R
T	1	2
B	-1	1

$\underline{v} = 1 = \overline{v}$: the value exists and is $v = 1$.

Optimal strategies

Suppose that the game has a value v.

Definition

- A strategy of player 1 is ε-optimal if it guarantees $v - \varepsilon$.

  A strategy of player 2 is ε-optimal if it guarantees $v + \varepsilon$.
- The 0-optimal strategies are called optimal. Let I^* be the optimal strategies for player 1, J^* be the optimal for player 2.

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- A strategy of player 1 is ε-optimal if it guarantees $v - \varepsilon$.

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- The 0-optimal strategies are called optimal. Let I^* be the optimal strategies for player 1, J^* be the optimal for player 2.

- If $(i^*, j^*) \in I^* \times J^*$, then $v = g(i^*, j^*)$.
- If a finite game has a value, each player has an optimal strategy.
- Not true when the game is infinite : If $G = (\boldsymbol{N}, \boldsymbol{N}, g)$, where $g(i,j) = 1/(i+j+1)$, what are the ε-optimal strategies of player 1 ? player 2 ?

Let $G = (g, I, J)$ be a zero-sum game which has a value. Then (i^*, j^*) belongs to $I^* \times J^*$ if and only if it is a saddle point of g, that is :

$$\forall (i,j) \in I \times J, \quad g(i,j^*) \leq g(i^*,j^*) \leq g(i^*,j),$$

Meaning that i^* is the best response against j^* et j^* is the best response against i^*, e.g. (i^*, j^*) is a Nash equilibrium.

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Careful !

An optimal strategy is not necessarily a best reply against any strategy of the opponent !... An example ?

Domination

Definition

- A strategy a of Player 1 is strongly dominated by another strategy b if $g(a,j) < g(b,j)$ for every action j of Player 2.
- A strategy a of Player 1 is weakly dominated by another strategy b if $g(a,j) \leq g(b,j)$ for every action j of Player 2, and the inequality is strict for at least one j.

This is basically a way to express the fact that a strategy is worse than another one. It should not be a big surprise that it is not worthwile to play dominated strategies.

Proposition

- The value, infsup, supinf of a game don't change when one removes weakly dominated strategies.
- If the supinf and infsup are maxmin and minmax (for example if the game is finite) then a strongly dominated strategy cannot be optimal.

Careful !

A weakly dominated strategy can be optimal, even in a finite game.

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- To be umpredictable, players must play at random : instead of choosing deterministically an element in I or J, they choose a probability distribution on I or J, that we call a *mixed strategy*.

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- A good player must thus hide his strategy. He must be unpredictable.
- To be umpredictable, players must play at random : instead of choosing deterministically an element in I or J, they choose a probability distribution on I or J, that we call a *mixed strategy*.
- For example, if we are playing Matching Pennies, or describing an algorithm that will play it “online”, it is clearly interesting to select each strategy with probability 1/2.

If E is finite with cardinality n, lets denote :

$$\Delta(E) = \{x \in \mathbb{R}^n_+; \sum_{e \in E} x^e = 1\}.$$

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Definition

The mixed extension of a finite game $G = (I, J, g)$ is the game

$$\Gamma = (\Delta(I), \Delta(J), g),$$

Where

$$g(x, y) = \mathbb{E}_{x \otimes y}\, g = \sum_{i,j} x^i y^j g(i, j).$$

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- The *support* of the mixed strategy x is : $supp(x) = \{i \in I : x^i > 0\}$,
- Let $A = (g(i,j))_{(i,j)\in I\times J})$ and $(x, y) \in \Delta(I) \times \Delta(J)$. Then we have

$$g(x, y) = xAy := x^t Ay$$

- For any $y \in \Delta(J)$, one has : $\max_{x \in \Delta(I)} g(x, y) = \max_{i \in I} g(i, y)$

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Proposition

The duality gap is smaller in Γ. If a player can guarantee $w \in \mathbb{R}$ in G, he can guarantee w in Γ, using the same strategy. In particular, if G has a value, then Γ has the same value.

The converse is false !

Theorem

Let A be a real valued matrix indexed by $I \times J$. There exists (x^*, y^*, v) in $\Delta(I) \times \Delta(J) \times \mathbb{R}$ such that :

$$\forall y \in \Delta(J), \;\; x^* A y \geq v \;\; \text{et} \;\; \forall x \in \Delta(I), \;\; x A y^* \leq v.$$

Said differently, the mixed extension of a game has a value, and players have optimal strategies.

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Moreover we have :

$$\begin{aligned} v &= \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} xAy = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} xAy \\ &= \max_{x \in \Delta(I)} \min_{j \in J} xAj = \min_{y \in \Delta(J)} \max_{i \in I} iAy. \end{aligned}$$

Properties of the optimal strategies

Let $A \in \mathbb{R}^{I \times J}$ be a matrix game. Let $X(A) \subset \Delta(I)$ and $Y(A) \subset \Delta(J)$ be optimal for players 1 and 2 respectively.

Proposition

- $X(A)$ and $Y(A)$ are non empty polytopes, Let $A \in \mathbb{R}^{I\times J}$ be a matrix game. Let $X(A) \subset \Delta(I)$ and $Y(A) \subset \Delta(J)$ be optimal for players 1 and 2 respectively.

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- $X(A) \times Y(A)$ is the set of saddle-points of A, e.g. elements of (x^*, y^*) on $\Delta(I) \times \Delta(J)$ such that :

$$x\,A\,y^* \leq \; x^*\,A\,y^* \leq \; x^*\,A\,y \quad \forall (x, y) \in \Delta(I) \times \Delta(J).$$

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- Let $(x^*, y^*) \in X(A) \times Y(A)$. Then for all $i \in \mathrm{supp}\,(x^*)$ and $j \in \mathrm{supp}\,(y^*)$,

$$iAy^* = v = x^*Aj = x^*Ay^* \quad \text{(complementarity)}.$$

1	-2
-1	3

Here $v = 1/7$. Player 1 optimal strategy : play Top with probab $(4/7, 3/7)$ on (T, B). Player 2 optimal strategy : $(5/7, 2/7)$ on (L, R).

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For all $t \in \mathrm{R}$, the game has value $v = 1$, and each player has a unique optimal strategy, which is pure : Top for player 1, Left for player 2.

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a	b
c	d

In the case where each player has two actions, either there exists a pair of pure optimal strategies (and then the value is one of the numbers a, b, c, d) or the optimal strategies are completely mixed and the value is given by

$$v = \frac{ad - bc}{a + d - b - c}.$$

Theorem

Let A be an $n \times m$ matrix, b an $1 \times m$ vector and c a $n \times 1$ vector with real coefficients. The two dual linear programs

$$(\mathcal{P}_1) \quad \begin{array}{c} \min\langle c, x\rangle \\ xA \geq b \\ x \geq 0 \end{array} \qquad\qquad (\mathcal{P}_2) \quad \begin{array}{c} \max\langle y, b\rangle \\ Ay \leq c \\ y \geq 0 \end{array}$$

have the same value as soon as they are feasible, i.e. when the sets $\{xA \geq b; x \geq 0\}$ *and* $\{Ay \leq c; y \geq 0\}$ *are non-empty.*

Proof of minmax theorem from duality

By considering $A + tE$ with $t \geq 0$ and E being the matrix with $E_{ij} = 1, \forall (i,j) \in I \times J$, one can assume $A \gg 0$, of dimension $m \times n$.

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Let us consider the dual programs

$$(\mathcal{P}_1) \quad \begin{array}{c} \min\langle X, c\rangle \\ XA \geq b \\ X \geq 0 \end{array} \qquad\qquad (\mathcal{P}_2) \quad \begin{array}{c} \max\langle b, Y\rangle \\ AY \leq c \\ Y \geq 0 \end{array}$$

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where the variables satisfy $X \in \mathrm{R}^m$, $Y \in \mathrm{R}^n$ and the parameters are given by $c \in \mathrm{R}^m, c_i = 1, \forall i$ and $b \in \mathrm{R}^n, b_j = 1, \forall j$.

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Thus by the duality theorem there exists a triple (X^*, Y^*, w) with :

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$X^* \neq 0$ implies $w > 0$, hence dividing X^* and Y^* by w, one obtains the existence of $(x^*, y^*) \in \Delta(I) \times \Delta(J)$ with

$$x^* A e^j \geq 1/w, \forall j, \quad e^i A y^* \leq 1/w, \forall i.$$

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$(\mathcal{P}_2)$ is feasible with $Y = 0$, as is $(\mathcal{P}_1)$ by taking X large enough.

Thus by the duality theorem there exists a triple (X^*, Y^*, w) with :

$$X^* \geq 0,\; Y^* \geq 0,\; X^*A \geq b,\; AY^* \leq c, \qquad \sum_i X_i^* = \sum_j Y_j^* = w.$$

$X^* \neq 0$ implies $w > 0$, hence dividing X^* and Y^* by w, one obtains the existence of $(x^*, y^*) \in \Delta(I) \times \Delta(J)$ with

$$x^*Ae^j \geq 1/w, \forall j, \quad e^iAy^* \leq 1/w, \forall i.$$

Hence there is a value, namely $1/w$, and x^* and y^* are optimal strategies.

Fictitious play : Brown (1951)

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Principle

- We start from any (i_1, j_1) in $I \times J$,
- At each stage $n \geq 2$, each player will play a best response to the average past behavior of the opponent up to stage $n - 1$.

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Definition

A sequence $(i_n, j_n)_{n \geq 1}$ is a realisation of FP process if for each $n \geq 1$:

- i_{n+1} is a best response of player 1 against $y_n := \frac{1}{n} \sum_{t=1}^{n} j_t \in \Delta(J)$,
- j_{n+1} is a best response of player 2 against $x_n := \frac{1}{n} \sum_{t=1}^{n} i_t \in \Delta(I)$.

Theorem (Robinson, 1951)

Let $(i_n, j_n)_{n\geq 1}$ be the realization of a fictitious play process for the matrix A. Then :

1) The distance from (x_n, y_n) to the set of optimal strategies $X(A) \times Y(A)$ goes to 0 as $n \to \infty$. Explicitly :
$\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in \Delta(I), \forall y \in \Delta(J)$

$$x_n A y \geq val(A) - \varepsilon \;\text{ and }\; x A y_n \leq val(A) + \varepsilon.$$

2) The average payoff on the trajectory, namely $\frac{1}{n}\sum_{t=1}^{n} A_{i_t,j_t}$, converges to $val(A)$.

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This is an alternative (and constructive) proof of the minmax theorem.

- Initial proof : by induction

Two proofs

- Initial proof : by induction
- Modern proof : go to continuous time (this lecture).

Continuous Fictitious Play

Take as variables the empirical frequencies x_n and y_n, so that the discrete dynamics for player 1 reads as

$$x_{n+1} = \frac{1}{n+1}[i_{n+1} + nx_n] \qquad \text{with} \qquad i_{n+1} \in \mathrm{BR}^1(y_n)$$

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Theorem (Harris (1998) ; Hofbauer and Sorin (2006))

For the CFP process, the duality gap converges to 0 at a speed $O(1/t)$ and $(x(t), y(t))$ to the set of optimal strategies $X(A) \times Y(A)$

Make the time change $z(t) = x(\exp(t))$, which leads to the autonomous differential inclusion

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Write the payoff as $g(x, y) = xAy$ and for $(x, y) \in \Delta(I) \times \Delta(J)$, let

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and write

$$\alpha(t) = x(t) + \dot{x}(t) \in \mathrm{BR}^1(y(t)) \quad \text{and} \quad \beta(t) = y(t) + \dot{y}(t) \in \mathrm{BR}^2(x(t)).$$

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We have $L(y(t)) = g(\alpha(t), y(t))$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t} L(y(t)) = \dot{\alpha}(t) A y(t) + \alpha(t) A \dot{y}(t).$$

Proof : Part 2

$$\alpha(t) = x(t) + \dot{x}(t) \in \mathrm{BR}^1(y(t)) \quad \text{and} \quad \beta(t) = y(t) + \dot{y}(t) \in \mathrm{BR}^2(x(t)).$$

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The envelope theorem shows that the first term collapses hence we obtain

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Thus $w(t) = w(0)\, e^{-t}$. There is convergence of $w(t)$ to 0 at exponential speed, hence convergence to 0 at a speed $O(1/t)$ in the original problem.

Let C be a non-empty closed convex subset of R^k (endowed with the Euclidean norm) and $\{x_n\}$ a bounded sequence in R^k.

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For $x \in \mathrm{R}^k$, $\Pi_C(x)$ stands for the projection of x on C and $\bar{x}_n$ is the Cesàro mean up to stage n of the sequence $\{x_i\}$:

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Say that $\{x_n\}$ is a Blackwell C-sequence if it satisfies :

$$\langle x_{n+1} - \Pi_C(\bar{x}_n), \bar{x}_n - \Pi_C(\bar{x}_n)\rangle \leq 0, \quad \forall n.$$

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Theorem

If $\{x_n\}$ is a Blackwell C-sequence then $d_n = d(\bar{x}_n, C)$ converges to 0.

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Decompose

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Using the hypothesis we obtain

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$$\|x_{n+1} - \overline{x}_n\|^2 \leq 2\|x_{n+1}\|^2 + 2\|\overline{x}_n\|^2 \leq 4M^2,$$

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$$= \left(\frac{1}{n+1} \right) (\langle x_{n+1} - y_n, \overline{x}_n - y_n \rangle - \|\overline{x}_n - y_n\|^2).$$

Using the hypothesis we obtain

$$d_{n+1}^2 \leq \left(1 - \frac{2}{n+1} \right) \; d_n^2 + \left(\frac{1}{n+1} \right)^2 \|x_{n+1} - \overline{x}_n\|^2.$$

From

$$\|x_{n+1} - \overline{x}_n\|^2 \leq 2\|x_{n+1}\|^2 + 2\|\overline{x}_n\|^2 \leq 4M^2,$$

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Let $y_n = \Pi_C(\bar{x}_n)$. Then

$$d_{n+1}^2 \leq \|\overline{x}_{n+1} - y_n\|^2 = \|\overline{x}_n - y_n\|^2 + \|\overline{x}_{n+1} - \overline{x}_n\|^2 + 2\langle \overline{x}_{n+1} - \overline{x}_n, \overline{x}_n - y_n \rangle.$$

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Thus by induction $d_n \leq \frac{2M}{\sqrt{n}}$.

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Theorem

$\{x_n\}$ is a Blackwell C-sequence with $C = \{x \in \mathrm{R}^k;\ x \geq 0\}$. Consequently, there is $s \in \Delta(I)$ with $sAt \geq 0$, for all $t \in \Delta(J)$.

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To conclude. Consider the empirical frequencies arising in $\bar{x}_n$ as a mixed strategy of player 1 and use compactness of $\Delta(I)$ to deduce that its limit provides a strategy $s \in \Delta(I)$ which satisfies $sAt \geq 0$, for all $t \in \Delta(J)$.