# Incremental Learning

## Equilibrium selection in repeated games

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# Introduction

# Introduction

- **Equilibrium Selection in Repeated Games** : the strategic decision-making in infinite repeated games.
- Infinite repeated games offer a platform to analyze how individuals influence each other's choices through continuous interactions.
- Researchers have begun to pay attention to how individuals adjust their strategy choices through the learning and information updating process in historical interactions.
- But there is still a significant research gap in how individuals select among many possible equilibria through a dynamic learning process in infinite repeated games.

# Theoretical Background

# Kalai and Lehrer's Interactive Learning Problem

- 'Interactive Learning' Problem : how each player adjusts their strategy based on their opponent's behavior, especially when they have limited information about their opponent's strategy.
- They assumed that each player holds certain "beliefs" (i.e., expectations) about their opponent's possible strategies at the beginning of the game and chooses their best strategy based on these beliefs. As the game progresses, players update their beliefs about their opponent's strategy according to the actual behavior observed.
- The "grain-of-truth" assumption : initial beliefs must be compatible with eventual equilibrium states. This allows for accurate predictions of behavior and reaching a Nash equilibrium.

# Foster and Young's Near-Rational Learning

- Their learning model explores how players with limited information approximate game equilibrium through near-rational behavior.
- "near-rational" actions of players : in most instances, attempts are made to respond optimally, but non-optimal choices may occur due to reasons like misunderstandings or misjudgments.
- Their result indicates that even in the presence of errors and near-relational behavior, continuous learning and strategy adjusment can stabilize the system's dynamics at an equilibrium state.

# The Kalai–Smorodinsky Bargaining Solution

- Foundational Assumptions : The model is based on players forming beliefs from empirical evidence, demonstrating rational behavior by optimizing decisions according to these beliefs, and operates on uncoupled learning mechanisms.
- Dynamic Process of Belief Adjustment : Players engage in an iterative process of forming and revising bounded-memory beliefs based on the observed actions, leading to a cycle of uncertainty and exploration when beliefs conflict with observed behaviors.
- Emphasizes that principles of equitable and efficient outcomes in bargaining contexts are applicable to equilibrium selection in repeated games.

# Problem Setting

# Equilibrium Selection Problem

- In game theory, an "equilibrium" is a state where no player has the incentive to deviate from their strategy, believing it to be optimal given the strategies of the others.
- The Folk Theorem is a significant concept especially in the context of infinite repeated games. Under certain conditions, almost any individually rational payoff configuration can be sustained as an equilibrium through appropriate strategy combinations.
- A key question arises : if multiple equilibria are possible, how can we predict which one players will ultimately choose ?

# Learning Rules

# Certain/Uncertain Mode

- **Certain** : When a player is in this state, they have strong confidence in predicting their opponent's next move. This assurance allows them to react optimally according to the learning rule.
- **Uncertain** : In this state, the player lacks confidence in forecasting their opponent's next move. They attempt to guess, but with considerable uncertainty, which increases the likelihood of random actions.

# Definition of Player's Behavior

**Definition 1 ($\epsilon$-history) :** The $\epsilon$-history represents the sequence of action profiles played in the most recent $\epsilon$ periods. This historical segment is fundamental to the decision-making process, as it allows players to anticipate opponents' next moves and respond accordingly.

**Definition 2 (Memory $k$ strategy) :** This implies a player's strategy is contingent on the most recent $k$ rounds of action history.

**Definition 3 (Consistency with strategies) :** Players also evaluate the effectiveness of their strategies by noting deviations from expected strategy, which prompts reassessment and potential change in their strategic approach.

# Strategy Transitions in the Learning Process

- **Certain Mode :** When a player has a strong belief about their strategy, they act on this conviction and respond with strategies they deem best aligned with anticipated actions of their opponents.
- **Uncertain Mode :** This mode occurs when a player doubts their current understanding of the game. This leads to more experimental actions with varied strategies.

Transitions between these modes occur based on the player's perception of consistency with the recent $m$-history. When outcomes align with expectations, players stay in Certain Mode, otherwise, they shift to Uncertain Mode.

## Stability and Effectiveness in Repeated Games

**Key Challenge :** Selecting the most likely equilibrium from many possibilities where players' actions are coordinated and optimal.
**Stability :** Refers to the long-term consistency of strategies and payoffs, preventing arbitrary strategy changes. It involves :

- Robust learning that guides through trial and error.
- Probabilities of transitioning between strategic mindsets based on confidence in the strategy's effectiveness.

**Effectiveness :** Focuses on making strategic choices influenced by past interactions and current equilibria. It includes :

- Addressing deviations due to randomness or errors.
- Steering towards an optimal equilibrium, like the Kalai–Smorodinsky equilibrium.

# Main Result : Theorem 1

# Theorem 1

- Key discovery : Predicting how players will ultimately behave in repeated games.
- Theorem 1 addresses the conditions under which a learning rule in repeated games leads to a stochastically stable state.
- Conditions include :
  1. Accurate players' beliefs about opponents' strategies.
  2. Strategies form a subgame perfect equilibrium.
  3. Payoff vectors are close to the Kalai-Smorodinsky bargaining solution.
- Significance : Convergence to a state of fairness and efficiency.

# Lemma 1

Without uncertainty, the game cannot reach a stochastically stable state.

- Formal statement : If $\forall i, Pr(si|h) = 1$, then $s \notin Support(\mu^*)$.
- Highlights the importance of exploration and uncertainty.

# Lemmas 2 to 5

- Lemma 2 : Conditions for a strategy profile to be part of a stochastically stable state.
- Lemma 3 : Transition probabilities between states.
- Lemma 4 : Persistence of stochastically stable states.
- Lemma 5 : Efficiency of stochastically stable states.

Lemma 1-5 established a theoretical framework to support the conclusion of Theorem 1.

# Auxiliary Theorems : Theorem A and B

- Theorem A (Barlo, Carmona, and Sabourian 2016) : For all $v \in V^*$ and $\lambda > 0$, there exists a $k$-memory pure subgame-perfect equilibrium of $H$ such that each player $i$ receives continuation payoffs within $\lambda$ of $v_i$ at every subgame.
- This confirms the existence of a $k$-memory pure subgame-perfect equilibrium.
- Theorem B (Ellison 2000) : Let $\omega, \omega'$ be states within the process. If the resistance to leaving state $\omega$ exceeds the resistance to transitioning from $\omega$ to $\omega'$, denoted by $r(\omega) > r'(\omega', \omega)$, then state $\omega$ is not stochastically stable.
- This provides criteria for determining stochastically stable states.

Theorems A and B not only assert the possibility of reaching a stable equilibrium but also ensure that the equilibrium is resilient to fluctuations inherent in the learning dynamics of the game.

# Conclusion

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- Reveals the complex dynamics of strategic interactions and the importance of learning rules in repeated games.
- Introduction of a novel learning rule, P,
  1. Facilitates the convergence to equilibrium states corresponding to the Kalai–Smorodinsky bargaining solution.
  2. Illuminates the path towards achieving efficient and equitable outcomes.
- Provides a valuable framework for predicting and influencing outcomes in interactive decision-making scenarios.