WASHINGTON UNIVERSITY IN ST. LOUIS

Arts & Sciences
Department of Mathematics

Dissertation Examination Committee:
Xiang Tang, Chair
Aliakbar Daemi
Renato Feres
Jr-Shin Li
Alexander Seidel
Yanli Song

The Witten Deformation and Proper Cocompact Lie Group Actions

by
Hao Zhuang

A dissertation presented to
Washington University in St. Louis
in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

August 2025
St. Louis, Missouri

Table of Contents

Acknowledgments

First of all, I want to express my great appreciation to my PhD supervisor Prof. Xiang Tang for all his efforts in the past six years to cultivate my capacity and enthusiasm in mathematical research. I benefit a lot from all the inspiring discussions with him and really appreciate all his encouragement whenever I feel confused.

I want to thank Prof. Erkao Bao and Prof. Aliakbar Daemi for helpful discussions about the topological perspective of the Thom-Smale cohomology.

I want to thank Prof. Xianzhe Dai and Prof. Li-Sheng Tseng for helpful discussions about applying the Witten deformation technology to various situations.

I want to thank Prof. Rui Loja Fernandes, Prof. Nigel Higson, and Prof. Yanli Song for practical suggestions on approaching the analysis involving Lie group actions.

I want to thank Prof. Zhizhang Xie and Prof. Guoliang Yu for practical suggestions on generalizing the analysis from the abelian case to the nonabelian case.

I want to thank Prof. Renato Feres, Prof. Jr-Shin Li, and Prof. Alexander Seidel for agreeing to read my thesis and serve as my thesis defense committee members.

I want to thank Prof. Simone Cecchini, Prof. Xin Jin, Prof. Tian-Jun Li, Prof. Rudy Rodsphon, Dr. Minh Nguyen, Dr. Angel Roman, Dr. Jesus Sanchez, Dr. Joel Villatoro, and PhD students Clare Cheng, Ching-Chia Hsu, Sriharsha Indukuri, Pooja Joshi, Jiachen Lu, John Naughton, Hiroto Sasaki, Kuan Yang, and Hanzhang Yun for their helpful comments on my results and presentations.

I want to thank Prof. Jinmin Wang, Dr. Qiaochu Ma, Dr. Junrong Yan, and PhD students Wenxuan Guo and Shengzhen Ning for suggestions on relevant further research topics.

Especially, I really appreciate the McDonnell Scholarship, Lo Fellowship, and several other financial supports from our Department of Mathematics and McDonnell International Scholars Academy for my studies.

Finally, I am truly grateful to my parents Prof. Mingfang Wang and Prof. Lixin Zhuang for their boundless support and understanding throughout my journey in mathematical research, and for motivating me to pursue my passion in academia.

Hao Zhuang

Washington University in St. Louis

August 2025

Dedicated to my parents.

ABSTRACT OF THE DISSERTATION

The Witten Deformation and Proper Cocompact Lie Group Actions

by

Hao Zhuang

Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2025

Professor Xiang Tang, Chair

We study the interactions between the Witten deformation of the de Rham exterior differentiation and topological invariants in two scenarios of proper Lie group actions. In the first scenario, we work on a closed oriented manifold admitting an action by a compact connected Lie group. Using a special Morse-Bott function invariant under the group action, we deform the de Rham exterior derivative and get the associated Witten Laplacian. Applying asymptotic analysis, we localize the kernel of the Witten Laplacian around the critical components of the invariant Morse-Bott function. Finally, we build the chain isomorphism between the invariant Thom-Smale complex and the invariant Witten instanton complex.

In the second scenario, we work on an oriented noncompact manifold admitting a proper cocompact action by a Lie group. First, through the generalized mod 2 index map between real KK-groups, we deform the Dirac type operator associated with the de Rham exterior derivative and find the appropriate Witten deformation of the de Rham exterior derivative. Then, we show that the invariant cohomology associated with this Witten deformation can be used to compute the semi-characteristic of the manifold. Finally, we prove that the semi-characteristic vanishes when there are two independent invariant vector fields on the manifold.

Chapter 1

Introduction

In 1982, Witten proposed an analytic approach to Morse theory. In his highly influential paper [55], he proposed a deformation of the de Rham differentiation d using a Morse function and a positive parameter. Although the deformation does not change the cohomology group of the manifold, it localizes the cohomological information of the manifold to a finite set. This observation and its further developments suggest many analytic approaches to study topology.

In this thesis, we study the Witten deformation in two scenarios where manifolds admit proper Lie group actions. In the first scenarios, both the manifold and the Lie group are compact. We apply the Witten deformation to study the invariant Thom-Smale complex associated with a special Morse-Bott function. In the second scenario, the group action is proper and cocompact. By clarifying the image of a KKO-class under a generalized mod 2 index map on KKO-groups, we find that the Witten deformation appears in the invariant chain complex and is given by the average of a bump function.

In this chapter, we first introduce Witten's original proposal. Then, we briefly review the historical developments and our works in the analytic approach to Morse-Bott functions and Thom-Smale cohomology. Finally, we introduce the recent achievements and our works in the KKO-theoretic approach to the cohomology invariant under proper cocompact actions.

1.1 Witten's proposal

In Witten's settings [55], the de Rham exterior differentiation d is replaced by $d_T \coloneqq e^{-Tf} d e^{Tf}$, where f is a Morse function, and $T > 0$ is a parameter. Given a Riemannian metric on M, d_T admits a formal adjoint d_T^*. When $T \to +\infty$, the eigenspaces of the Witten Laplacian $(d_T + d_T^*)^2$ associated with small eigenvalues localize around the critical set of f, giving the well-known Morse inequalities an analytic proof.

As Smale presented in [50], let

$$C^k(M, f) = \mathrm{span}_{\mathbb{R}}\{p : (df)_p = 0 \text{ and the Morse index of } p \text{ is } k\}.$$

Then, under appropriate transversality assumptions, there is a chain complex

$$\partial : C^k(M, f) \to C^{k+1}(M, f)$$

called the Thom-Smale complex. The boundary map ∂ is defined by counting the number (with $\pm$ signs) of negative gradient flow lines from index k critical points to index $k+1$ critical points. Smale's original construction is topological. However, according to the localization phenomenon, Witten conjectured that it might be possible to realize the whole Thom-Smale complex associated to f in an analytic way.

1.2 Localization and asymptotic analysis

In 1985, Helffer and Sjöstrand first confirmed Witten's conjecture via semi-classical analysis tools in [27]. Later, in 1994, Bismut and Zhang provided an asymptotic analysis approach in [12], simplifying Helffer and Sjöstrand's work.

Based on their works, it is natural to ask for an analytic approach when f is a Morse-Bott function. In 1986, Bismut provided a heat kernel and probabilistic proof of the Morse-Bott inequalities in [10]. In 1988, using the semi-classical analysis tools again together with a perturbation on the function, Helffer and Sjöstrand also gave an analytic proof of the Morse-Bott inequalities in [26]. In 1991, Bismut and Lebeau developed an asymptotic analysis approach in [11] to study the kernel of the general Witten Laplacian deformed by a vector field. In 2014, following [11], Lu proved the Morse-Bott inequalities under the action of a compact Lie group in [41], along with the Morse-Bott inequalities for compact manifolds with boundaries as corollaries.

The above analytic results give thorough studies of the Morse-Bott inequalities. As a further topic, we hope to find an answer to the question on chain complexes:

Question 1.2.1. Given a Morse-Bott function f with transversality conditions on a closed oriented manifold, what is the analytic realization of its associated Thom-Smale complex?

From Chapter 2 to Chapter 4, we give an answer to the case in which f is G-invariant and satisfies some orbital and transversality assumptions. We will introduce not only a new chain complex to G-manifolds, but also give this complex an analytic realization.

More precisely, let G be a connected compact Lie group acting on an oriented closed m-dimensional manifold M. Then, we assume that M carries a G-invariant metric $\langle \cdot, \cdot \rangle$ and a smooth G-invariant function $f : M \to \mathbb{R}$ satisfying (a1) and (a2):

(a1) The critical submanifold $\mathrm{crit}(f)$ of f is a disjoint union of G-orbits, and along the normal direction of each critical orbit, the Hessian of f does not have zero eigenvalues;

(a2) For any submanifold $Y \subseteq \mathrm{crit}(f)$, we denote its unstable manifold (resp. stable manifold) by $W^u(Y)$ (resp. by $W^s(Y)$). With these notations, we assume $W^u(p)$ intersects $W^s(\mathcal{O})$ transversely for any critical point p and any critical orbit $\mathcal{O}$.

Remark 1.2.2. By [54, Lemma 4.8], the functions satisfying (a1) form a dense subset of the collection of smooth functions. For (a2), though it is not generic, it is often true according to [4, Section 5.1], e.g., if $M \backslash \mathrm{crit}(f)$ only has one type of orbits. See Section 2.3 for examples.

We first construct the topological side, the G-invariant Thom-Smale complex associated to f satisfying (a1) and (a2). It is adapted from Austin and Braam's model [4, Section 3].

By (a1), if on the normal direction of a critical orbit $\mathcal{O}$, there are i negative eigenvalues of the Hessian of f, then we say the Morse index of $\mathcal{O}$ is i. Let $\mathcal{O}_i$ be the union of all critical orbits $\mathcal{O} \subseteq \mathrm{crit}(f)$ with Morse index $= i$. By (a2), the endpoint map

$$\pi_i : W^u(\mathcal{O}_i) \to \mathcal{O}_i$$

gives a fiber bundle over $\mathcal{O}_i$. We let $\mathcal{E}_i$ be the orientation bundle of the fiber bundle $W^u(\mathcal{O}_i)$.

Let $\Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$ be the space of all G-invariant smooth $\mathcal{E}_i$-valued j-forms on $\mathcal{O}_i$. We define

$$C^k(M, f)^G := \bigoplus_{i+j=k} \Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G.$$

Then, we let $C^*(M,f)^G \coloneqq \oplus_{k=0}^m C^k(M,f)^G$. Furthermore, we obtain a boundary map

$$\partial : C^k(M,f)^G \to C^{k+1}(M,f)^G \tag{1.1}$$

based on the assumption (a2). Since (a2) ensures the fiber bundle structure of the moduli spaces between critical orbits, the map ∂ is mainly given by integrating densities along the fibers of these moduli spaces. We give a detailed description of ∂ in (2.4) and (2.5).

Definition 1.2.3. The complex given by $C^*(M,f)^G$ $(k = 0, 1, \cdots, m)$ and ∂ is called the G-invariant Thom-Smale complex of M associated to f.

Our topological main result shows that the G-invariant Thom-Smale complex is well-defined and computes the Betti numbers of M. See Examples 2.3.1 − 2.3.4 for the computations.

Theorem 1.2.4. *Under the assumptions (a1) and (a2), the G-invariant Thom-Smale complex of M associated to f is well-defined. It computes the de Rham cohomology of M.*

The G-invariant Thom-Smale complex provides us a simplified way to compute the cohomology of M using Morse-Bott functions whose critical sets have certain symmetry. There are much fewer G-invariant forms than all smooth forms, bringing us a lot of convenience.

The main idea to prove Theorem 1.2.4 is to apply the spectral sequence method as in [4, Section 3.3]. We adapt necessary steps to our situation for the proof in later chapters.

Next, we present the analytic side, which is given by G-invariant eigenforms of the Witten Laplacian and admits a strong correspondence to the topological side.

By adapting [56, (5.15)], we begin with the G-invariant Witten instanton complex of M associated to f. We let $\Omega^k(M)^G$ be the collection of all G-invariant smooth k-forms on M.

For any $T > 0$ and $\alpha > 0$, we let $d_T := e^{-Tf} d e^{Tf}$, $D_T := d_T + d_T^*$ and

$$F_T^k(M, f, \alpha)^G := \operatorname{span}_{\mathbb{R}} \left\{ \omega \in \Omega^k(M)^G : D_T^2 \omega = \delta \cdot \omega \text{ for some } 0 \leqslant \delta \leqslant \alpha \right\},$$

where d_T^* is the formal adjoint of d_T, and D_T^2 is called the Witten Laplacian. Similar to [56, (5.15)], we have $D_T^2 d_T = d_T D_T^2$, making

$$d_T : F_T^k(M, f, \alpha)^G \to F_T^{k+1}(M, f, \alpha)^G$$

a well defined complex. We then let $F_T^*(M, f, \alpha)^G := \oplus_{k=0}^m F_T^k(M, f, \alpha)^G$.

Definition 1.2.5. The complex given by $F_T^*(M, f, \alpha)^G$ $(k = 0, 1, \cdots, m)$ and d_T is called the G-invariant Witten instanton complex of M associated to f.

By the G-invariant version of Hodge theory in Section 3.1, the G-invariant Witten instanton complex computes the de Rham cohomology of M as well.

As we see from [11, (8.49)], it is possible for us to analyze the operator D_T and the Witten instanton complex associated with any G-invariant metric $\langle \cdot, \cdot \rangle$ under (a1) and (a2). However, to avoid the technical issues of non-isometric exponential maps (See [11, (8.21), (8.57)]), as in [56, (5.11)], we adjust the G-invariant metric $\langle \cdot, \cdot \rangle$ on M (See Section 4.1 for the adjustment) according to the G-action and the G-equivariant Morse-Bott lemma [54, Lemma 4.1] around each critical orbit.

Moreover, by restricting the adjusted $\langle \cdot, \cdot \rangle$ to each critical orbit, we get a (twisted) Dirac type operator (See (4.6) and Proposition 4.2.12)

$$d + d^* : C^*(M, f)^G \to C^{*\pm 1}(M, f)^G.$$

An inner product on $C^*(M,f)^G$ is given by letting the basis induced by the G-action be orthonormal (See (4.7) for details). Under this inner product, $d+d^*$ is self-adjoint with a finite spectral radius . We let α_0 be the spectral radius of the twisted $(d+d^*)^2$ on $C^*(M,f)^G$.

Our analytic main result realizes the G-invariant Thom-Smale complex by G-invariant Witten instanton complex. For some applications and extensions, see Corollaries 4.4.1 $-$ 4.4.4.

Theorem 1.2.6. *We equip M with the adjusted $\langle\cdot,\cdot\rangle$. For any $\alpha > \alpha_0$, the map*

$$\begin{aligned} \Phi_T : F_T^k(M,f,\alpha)^G &\to C^k(M,f)^G \\ \omega &\mapsto \sum_{i=0}^{k} (\pi_i)_* \left(e^{Tf} \cdot \omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \quad (k=0,1,\cdots,m) \end{aligned} \tag{1.2}$$

is a chain isomorphism when T is sufficiently large.

Remark 1.2.7. According to [15, Theorem 7.10], the map $(\pi_i)_*$ integrates forms along fibers to get an $\mathcal{E}_i$-valued form.

The most interesting part of Theorem 1.2.6 is "$\alpha > \alpha_0$". It shows that on the chain complex level, the nonzero eigenvalues coming from the horizontal direction around crit(f) has a nontrivial impact, which may be hard to find if we only look at the Morse-Bott inequalities. The phenomenon is, when $T \to +\infty$, at crit(f), the information along the vertical direction concentrates on each critical orbit, while that along the horizontal direction does not.

Following [11], the main idea to prove Theorem 1.2.6 is to write D_T into the summation

"horizontal operator" + "vertical operator" + "tail term".

The horizontal part contributes to the spectral radius α_0. The square of the vertical part along the normal direction around crit(f) is a harmonic oscillator, contributing to the necessity of a sufficiently large T. The tail term is controlled by choosing a sufficiently small tubular neighborhood of $\mathrm{crit}(f)$.

Remark 1.2.8. When G is a torus, we have $\alpha_0 = 0$. In particular, in the torus case, if $\mathcal{O}$ is a critical orbit admitting nonorientable $W^u(\mathcal{O})$, the only G-invariant Thom-Smale chain on $\mathcal{O}$ is 0. Moreover, by Examples 4.4.6 and 4.4.7, α_0 relies on both M and G.

The Thom-Smale part of this thesis is presented in Chapter 2, Chapter 3, and Chapter 4. It is organized in the following order. First, in Sections 2.1 and 2.2, we clarify Definition 1.2.3 and prove Theorem 1.2.4. Then, in Sections 3.1 and 4.1, we present prerequisites that are necessary for the subsequent analysis. Afterwards, in Sections 4.2 and 4.3, by asymptotic analysis on D_T, we prove Theorem 1.2.6. Finally, in Section 2.3, we give some examples and corollaries of Theorem 1.2.4 and Theorem 1.2.6.

1.3 Assembly map and KKO-theoretic perspective

In certain situations, it is hard to notice the appropriate deformation of d directly to reduce the complexity on the cohomological level. One example is related to the proper cocompact Hodge theory. Before that, we review our motivation to study the proper cocompact case.

In 1956, Kervaire defined a semi-characteristic $k(X)$ for any closed $(4n+1)$-dimensional manifold X in [35]. Following [56, (7.1)],

$$k(X) = \sum_{i \text{ is even}} \dim(\text{the } i\text{-th de Rham cohomology group of } X) \mod 2.$$

This $k(X)$ is related to cobordism [42] and admits characteristic class formulae [21]. These results have aroused the interest in finding a more accessible counting formula of $k(X)$. Like the Euler characteristic, we hope the counting formula of $k(X)$ is relevant to the number of zero points of certain vector fields. The very first result [1, Theorem 4.1] is the vanishing theorem proved by Atiyah:

Theorem 1.3.1 (Atiyah [1], 1970)**.** *When X is oriented, if there exist two linearly independent vector fields on X, then $k(X)$ vanishes.*

Atiyah's proof is index-theoretic. Let d be the exterior derivative, and d^* be its formal adjoint. Then, $k(X)$ is identified with the parity of the kernel of the skew-adjoint operator

$$D_{\mathrm{sig}} \coloneqq \hat{c}(\mathrm{dvol}) \circ (d + d^*)(1 + (d + d^*)^2)^{-1/2}$$

on the space of even-degree forms. Here, we call D_{sig} the signature operator on X, and $\hat{c}(\mathrm{dvol})$ is induced by the volume form of X. The parity is exactly the $\mathbb{Z}_2$-valued Atiyah-Singer mod 2 index (See [2, Theorem A] and [56, (7.5)]) of D_{sig}. The mod 2 index is defined for any real skew-adjoint Fredholm operator and invariant under certain perturbations. After applying to D_{sig} such a perturbation induced by the two vector fields, Atiyah noticed that the dimension of the kernel of the perturbed operator is always even. Thus, $k(X) = 0$.

In this thesis, motivated by Atiyah and Singer's work, we first construct a generalized mod 2 index map between real KK-groups (denoted by KKO). Then, we use this map to define the G-invariant Kervaire semi-characteristic $k(M, G)$ for M and G satisfying:

(b1) M is a noncompact oriented $(4n + 1)$-dimensional smooth manifold without boundary;

(b2) G is a Lie group acting smoothly, properly, and cocompactly on M without changing the orientation of M. Here, "cocompactly" means M/G is compact.

Afterwards, we prove that $k(M, G)$ can be computed via G-invariant forms twisted by the square root of the modular character of G and has a vanishing property like Theorem 1.3.1.

Our work follows the index-theoretic studies of the elliptic operators equivariant under group actions. Actually, for any G-equivariant Dirac type operator on M, its kernel and cokernel are unitary G-representations. Then, given a G-equivariant Dirac type operator, as in [38, Definition 9.3], we can use representation theory and K-theory to define and study its G-equivariant index. With this approach, we have seen remarkable success in the studies of the Connes-Kasparov conjecture (See [33, Section 7.6] and [7, (4.20)]) and the more general Baum-Connes conjecture (See [6, Section 2] and [7, (0.3), (0.4)]).

For our M and G under assumptions (b1) and (b2), we focus on the trivial representation component. One important work of this idea was given by Mathai and Zhang in 2010. They defined the $\mathbb{Z}$-valued G-invariant index [43, Definition 2.4] for Dirac type operators. Unlike the G-equivariant index, Mathai and Zhang's definition only involves forms invariant under the pullbacks of elements in G. Moreover, to deal with the case in which M is noncompact, they also used the deformation via a bump function. Meanwhile, in [43, Appendix], Bunke provided a complex KK-theoretic interpretation of the G-invariant index.

Following [43], by proving the proper cocompact Hodge theorem [51, Proposition 3.8], Tang, Yao, and Zhang revealed the relations between the G-invariant index and the G-invariant cohomology groups associated with forms invariant under the pullbacks of elements in G. By [51, Theorem 4.1(ii), (iii)], for M and G satisfying (b1) and (b2), the Euler characteristic associated with the G-invariant cohomology groups of M vanishes. This result motivates us to find a semi-characteristic to replace the G-invariant Euler characteristic. More precisely, we continue the study by finding an index-theoretic obstruction to the existence of two independent G-invariant vector fields on M under (b1) and (b2).

Inspired by [43] and [51], in this thesis, we construct a generalized mod 2 index map on KKO-groups. Let $C_0(M)$ be the space of continuous functions on M vanishing at infinity, and $Cl_{0,1}$ be the real Clifford C^*-algebra generated by 1 and v subject to

$$v^2 = -1, \ v^* = -v.$$

By adapting Bunke's construction [43, Definition C.1] of the G-invariant index into the mod 2 situation, we obtain the map

$$\mathrm{ind}_2^G : KKO^G(C_0(M), Cl_{0,1}) \to KKO(\mathbb{R}, Cl_{0,1})$$

in Definition 5.3.6. The natural identification $KKO(\mathbb{R}, Cl_{0,1}) \cong \mathbb{Z}_2$ in Theorem 5.2.20 makes ind_2^G a reasonable generalization of the Atiyah-Singer mod 2 index.

Definition 1.3.2. We call ind_2^G the generalized mod 2 index map.

Next, we study a KKO-class associated with a self-adjoint operator induced by the signature operator on M and apply ind_2^G to this class to get the definition of the G-invariant Kervaire semi-characteristic. By assigning M a G-invariant metric, we get the formal adjoint d^* of the de Rham d and let $D = d + d^*$. Given any oriented orthonormal local frame $e_1, \cdots, e_{4n+1}$, we let $\hat{c}(e_i) \coloneqq e_i^* \wedge + e_i \lrcorner$ and then

$$\hat{c}(\mathrm{dvol}) \coloneqq \hat{c}(e_1) \cdots \hat{c}(e_{4n+1}).$$

This $\hat{c}(\mathrm{dvol})$ is independent of the choices of oriented orthonormal local frames. Since M is complete due to the compactness of M/G, the operator

$$D_{\mathrm{sig}} = \hat{c}(\mathrm{dvol}) \circ D(1 + D^2)^{-1/2} : L^2(\Lambda^{\mathrm{even}} T^* M) \to L^2(\Lambda^{\mathrm{even}} T^* M) \tag{1.3}$$

is Fredholm on the real space spanned by all even-degree real L^2-forms on M. Later in Section 6.2, we will see that D_{sig} is skew-adjoint. Also, in (6.4), D_{sig} induces a self-adjoint $\mathscr{D}_{\text{sig}}$. We then have a class $[\mathscr{D}_{\text{sig}}] \in KKO^G(C_0(M), Cl_{0,1})$. Applying ind_2^G to $[\mathscr{D}_{\text{sig}}]$, the definition of the G-invariant semi-characteristic $k(M,G)$ is as follows.

Definition 1.3.3. We let $k(M,G) \coloneqq \operatorname{ind}_2^G([\mathscr{D}_{\text{sig}}])$ and call it the G-invariant Kervaire semi-characteristic of M.

Let χ be the modular character such that $dg^{-1} = \chi(g)dg$ for any left-invariant Haar measure dg. Then, we let A equal A_0 which is defined by letting $t = 0$ in (5.1). Also, we denote the space of all smooth G-invariant i-forms by $\Omega^i(M)^G$, and the i-th cohomology group of the chain complex

$$A^{-1}dA : \Omega^i(M)^G \to \Omega^{i+1}(M)^G \tag{1.4}$$

by $H^i_A(M)^G$. The following result shows that $k(M,G)$ can be computed as follows.

Theorem 1.3.4. *The G-invariant Kervaire semi-characteristic $k(M,G)$ is identified with*

$$\sum_{i \text{ is even}} \dim H^i_A(M)^G \mod 2 \tag{1.5}$$

through the isomorphism $KKO(\mathbb{R}, Cl_{0,1}) \cong \mathbb{Z}_2$.

The other result is an Atiyah type vanishing theorem, generalizing Theorem 1.3.1:

Theorem 1.3.5. *If there exist two linearly independent G-invariant vector fields on M, then $k(M,G)$ vanishes.*

The readers may think it is more convenient to define $k(M,G)$ using cohomology groups instead of using KKO-theory. Also, the readers may feel the deformation by A redundant. In fact, by Example 6.4.2, the G-invariant cohomology without the deformation does not satisfy the vanishing property like Theorem 1.3.1. According to [51, Section 4], we must deform either G-invariant forms or the de Rham d if we use chain complexes and cohomology groups to define $k(M, G)$. The difficulty is, on the chain complex side, when G is non-unimodular, it is hard to directly guess which power of A to deform d. Thus, we approach from the KKO-theoretic side. Once we obtain the index-theoretic definition of $k(M, G)$, we identify it with (1.5) based on the proper cocompact Hodge theorem [51, Proposition 3.8].

To prove Theorem 1.3.5, we apply to D a perturbation which does not change $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$. Similar to [1, Section 4], the perturbation is given by the two independent vector fields.

The KKO part of this thesis is presented in Chapter 5 and Chapter 6. It is organized in the following order. In Section 5.1, we introduce the background in Atiyah and Singer's mod 2 index for real skew-adjoint Fredholm operators. In Section 5.2, we review some basic definitions and facts about real KK-groups and then rephrase Atiyah and Singer's mod 2 index using real KK-groups. In Section 5.3, we construct the generalized mod 2 index. In Section 6.1, we present the proper cocompact Gårding's inequality and the associated Hodge theorem provided by [43] and [51]. In Sections 6.2, we prove Theorem 1.3.4. In Section 6.3, we prove Theorem 1.3.5. In Section 6.4, we discuss the effect of the modular character χ and the condition in Theorem 1.3.5.

Chapter 2

Invariant Thom-Smale complex

In this chapter, we introduce the G-invariant Thom-Smale complex of a closed oriented manifold admitting an action by a compact connected Lie group. This complex is associated with a special type of Morse-Bott functions. These functions satisfy the orbital and transversality conditions, allowing us to define the boundary map in the chain complex. In particular, the invariant Thom-Smale complex computes the de Rham cohomology of the manifold, and thus we will also provide some examples on such computations.

Throughout this chapter, we assume that M is a closed oriented m-dimensional manifold, G is a compact connected Lie group acting (from the left side) on M smoothly, and the G-invariant function f and the G-invariant metric $\langle \cdot, \cdot \rangle$ on M satisfy (a1) and (a2).

2.1 Invariant Thom-Smale complex

In this section, we explain the construction of the G-invariant Thom-Smale complex and clarify all the details needed for 1.1. Here, we do not adjust the Riemannian metric $\langle\cdot,\cdot\rangle$. We will only use the adjusted one in Sections 4.1, 4.2, and 4.3.

We recall the notations given in the introduction: $\mathcal{O}_i$ is the disjoint union of all critical orbits of f with Morse index i, and $\mathcal{E}_i$ is the orientation bundle of $\mathcal{O}_i$. As in [9, (1.1)], we have a left G-action on $\mathcal{E}_i$-valued j-forms. It allows us to define $\Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$, the space of all G-invariant $\mathcal{E}_i$-valued smooth j-forms on $\mathcal{O}_i$. Since $\mathcal{E}_i$ is a flat bundle, the de Rham d is still well-defined on $\Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$ (See [15, §7]).

Let $\mathbb{B}^i \subseteq \mathbb{R}^i$ be the open unit ball of dimension i. Then, given an open cover $\{U_a\}$ of $\mathcal{O}_i$ and the associated local trivializations $\varphi_a : U_a \times \mathbb{B}^i \to W^u(\mathcal{O}_i)$ of the fiber bundle $\pi_i : W^u(\mathcal{O}_i) \to \mathcal{O}_i$, we have transition maps

$$\begin{aligned} \varphi_{ab} : U_a \cap U_b &\to \mathrm{Diff}(\mathbb{B}^i) \\ p &\mapsto \varphi_a^{-1}(p) \circ \varphi_b(p). \end{aligned}$$

Let sgn be the function on $\mathrm{Diff}(\mathbb{B}^i)$ mapping orientation-preserving (resp. orientation-reversing) diffeomorphisms to 1 (resp. -1). The following transitions

$$\mathrm{sgn} \circ \varphi_{ab} : U_a \cap U_b \to \{\pm 1\}$$

define the orientation bundle $\mathcal{E}_i$. This $\mathcal{E}_i$ admits a G-action given as follows. For any $g \in G$, and any (p, t) in a local trivialization $U_a \times \mathbb{R}$ of $\mathcal{E}_i$, we find another $U_b \times \mathbb{R}$ such that $gp \in U_b$.

Then, $g(p,t)$ is defined to be

$$\big(gp, t\cdot \mathrm{sgn}(\varphi_b^{-1}(gp)\circ g\circ\varphi_a(p))\big)\in U_b\times\mathbb{R}$$

if written in the local trivialization $U_b\times\mathbb{R}$.

Lemma 2.1.1. *This action of G on $\mathcal{E}_i$ is well-defined.*

Proof. Suppose that $(p,t)\in U_a\times\mathbb{R}$ is identified with $(p,t')\in U_{a'}\times\mathbb{R}$, i.e., $t'\cdot\mathrm{sgn}(\varphi_{aa'}(p))=t$. Then, let $U_{b'}\times\mathbb{R}$ be another local trivialization of $\mathcal{E}_i$ such that $gp\in U_{b'}$, we find

$$\big(gp, t'\cdot\mathrm{sgn}(\varphi_{b'}^{-1}(gp)\circ g\circ\varphi_{a'}(p))\big)=(gp, t\cdot\mathrm{sgn}(\varphi_{b'b}(gp))\cdot\mathrm{sgn}(\varphi_b^{-1}(gp)\circ g\circ\varphi_a(p))).$$

Thus, when we change the local trivializations, $\big(gp, t\cdot\mathrm{sgn}(\varphi_b^{-1}(gp)\circ g\circ\varphi_a(p))\big)\in U_b\times\mathbb{R}$ is identified with $\big(gp, t'\cdot\mathrm{sgn}(\varphi_{b'}^{-1}(gp)\circ g\circ\varphi_{a'}(p))\big)\in U_{b'}\times\mathbb{R}$, showing that the action of G is well-defined. $\square$

For any $r\in\mathbb{Z}_{\geqslant 0}$, we have a moduli space

$$\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)\coloneqq\left(W^u(\mathcal{O}_{i+r})\cap W^s(\mathcal{O}_i)\right)/\mathbb{R}$$

consisting of the flow lines of $-\nabla f$ from $\mathcal{O}_{i+r}$ to $\mathcal{O}_i$. This "quotient $\mathbb{R}$" means quotient the time parameter of each flow line. Then, we have the following two natural maps

$$\ell_i^{i+r}:\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)\to\mathcal{O}_i\quad\text{and}\quad u_i^{i+r}:\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)\to\mathcal{O}_{i+r}.$$

By assumption (a2), u_i^{i+r} makes $\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)$ a fiber bundle over $\mathcal{O}_{i+r}$. In addition, each fiber of this bundle is of dimension $r-1$.

Similar to [3, Section VI.4.c], for any $\omega \in \Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$, we construct a form

$$(u_i^{i+r})_*(\ell_i^{i+r})^*\omega \in \Omega^{j-r+1}(\mathcal{O}_{i+r}, \mathcal{E}_{i+r})^G,$$

where $1 \leqslant r \leqslant j+1$. In fact, $(\ell_i^{i+r})^*\omega \in \Omega^j\left(\mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i), (\ell_i^{i+r})^*\mathcal{E}_i\right)$ is given by

$$\left((\ell_i^{i+r})^*\omega\right)_\gamma(\nu_1, \cdots, \nu_j) = \omega_{\ell_i^{i+r}(\gamma)}\left((d\ell_i^{i+r})\nu_1, \cdots, (d\ell_i^{i+r})\nu_j\right) \in \text{the fiber of } \mathcal{E}_i \text{ at } \ell_i^{i+r}(\gamma)$$

for any $\gamma \in \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i)$ and any $\nu_1, \cdots, \nu_j \in T_\gamma\mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i)$.

In addition, for any $q \in \mathcal{O}_{i+r}$ and $\gamma \in \mathcal{M}(q, \mathcal{O}_i)$, there is the following isomorphism

$$T_qW^u(q) \cong T_\gamma\mathcal{M}(q, \mathcal{O}_i) \oplus T\gamma \oplus T_{\ell_i^{i+r}(\gamma)}W^u(\ell_i^{i+r}(\gamma)) \tag{2.1}$$

according to [31, Section 6.2.3(18)]. Thus, given an orientation on $W^u(q)$ and $j-r+1$ tangent vectors $w_1, \cdots, w_{j-r+1} \in T_q\mathcal{O}_{i+r}$, we get a density ξ on $\mathcal{M}(q, \mathcal{O}_i)$ in this way: Let $e_1, \cdots, e_{j-r+1}$ be any vectors in $\mathfrak{g}$ satisfying

$$w_1 = \left.\frac{d}{dt}\right|_{t=0} \exp(te_1)q, \ \cdots, \ w_{j-r+1} = \left.\frac{d}{dt}\right|_{t=0} \exp(te_{j-r+1})q. \tag{2.2}$$

Then, for any $\gamma \in \mathcal{M}(q, \mathcal{O}_i)$ and any $z_1, \cdots, z_{r-1} \in T_\gamma\mathcal{M}(q, \mathcal{O}_i)$, ξ at γ is given by

$$\begin{aligned} & \xi_\gamma(z_1, \cdots, z_{r-1}) \\ = \ & \left((\ell_i^{i+r})^*\omega\right)_\gamma\left(\left.\frac{d}{dt}\right|_{t=0}\exp(te_1)\gamma, \cdots, \left.\frac{d}{dt}\right|_{t=0}\exp(te_{j-r+1})\gamma, z_1, \cdots, z_{r-1}\right) \\ \in \ & \left((\ell_i^{i+r})^*\mathcal{E}_i\right)|_{\mathcal{M}(q,\mathcal{O}_i)} \cong \text{the orientation bundle of } \mathcal{M}(q, \mathcal{O}_i) \\ & \text{(the isomorphism is by (2.1))}. \end{aligned} \tag{2.3}$$

The construction of ξ is independent of the choices of $e_1, \cdots, e_{j-r+1}$. The integration of ξ on $\mathcal{M}(q, \mathcal{O}_i)$ gives us a real number. However, if we choose another orientation on $W^u(q)$, we get the opposite value. With an abuse of notation, we say that

$$\int_{\mathcal{M}(q,\mathcal{O}_i)} \xi \in \mathcal{E}_{i+r}.$$

Then, $(u_i^{i+r})_*(\ell_i^{i+r})^*\omega$ at each point $q \in \mathcal{O}_{i+r}$ is defined by

$$\left((u_i^{i+r})_*(\ell_i^{i+r})^*\omega\right)_q (w_1, \cdots, w_{j-r+1}) = \int_{\mathcal{M}(q,\mathcal{O}_i)} \xi$$

for any $w_1, \cdots, w_{j-r+1} \in T_q\mathcal{O}_{i+r}$.

Lemma 2.1.2. *This $(u_i^{i+r})_*(\ell_i^{i+r})^*\omega$ is G-invariant.*

Proof. For $q \in \mathcal{O}_{i+r}$, we let $\varphi_a : U_a \times \mathbb{B}^{i+r} \to W^u(\mathcal{O}_{i+r})$ be the local trivialization around q. Then, for any $g \in G$, we let

$$g^{-1} \circ \varphi_a \circ g : (g^{-1}U_a) \times \mathbb{B}^{i+r} \to W^u(\mathcal{O}_{i+r})$$

be the local trivialization around $g^{-1}q$. Under these two trivializations and their induced ones on $\mathcal{E}_{i+r}$, we find that

$$\operatorname{sgn}\left(\varphi_a^{-1}(q) \circ g \circ (g^{-1} \circ \varphi_a \circ g)(g^{-1}q)\right) = 1.$$

Thus, for any $w_1, \cdots, w_{j-r+1} \in T_q\mathcal{O}_{i+r}$,

$$\left(g \cdot (u_i^{i+r})_*(\ell_i^{i+r})^*\omega\right)_q (w_1, \cdots, w_{j-r+1}) = \left((u_i^{i+r})_*(\ell_i^{i+r})^*\omega\right)_{g^{-1}q} (g_*^{-1}w_1, \cdots, g_*^{-1}w_{j-r+1}).$$

With the same notations $e_1, \cdots, e_{j-r+1}$ as (2.2), for any $\tau \in \mathcal{M}(g^{-1}q, \mathcal{O}_i)$ and $\nu_1, \cdots, \nu_{r-1} \in T_\tau \mathcal{M}(g^{-1}q, \mathcal{O}_i)$, since ω is G-invariant, we define a density β on $\mathcal{M}(g^{-1}q, \mathcal{O}_i)$ and find

$$\begin{aligned}
&\beta_\tau(\nu_1, \cdots, \nu_r) \\
&= \left((\ell_i^{i+r})^*\omega\right)_\tau \left(\left.\frac{d}{dt}\right|_{t=0} \exp(t\mathrm{Ad}_{g^{-1}} e_1)\tau, \cdots, \left.\frac{d}{dt}\right|_{t=0} \exp(t\mathrm{Ad}_{g^{-1}} e_{j-r+1})\tau, \nu_1, \cdots, \nu_{r-1} \right) \\
&= \omega_{g^{-1}\ell_i^{i+r}(g\tau)} \left(\left.\frac{d}{dt}\right|_{t=0} g^{-1}\exp(te_1)\ell_i^{i+r}(g\tau), \cdots, \left.\frac{d}{dt}\right|_{t=0} g^{-1}\exp(te_{j-r+1})\ell_i^{i+r}(g\tau)\, , \right. \\
&\qquad\qquad \left. g_*^{-1}(d\ell_i^{i+r})g_*\nu_1, \cdots, g_*^{-1}(d\ell_i^{i+r})g_*\nu_{r-1} \right) \\
&= g^{-1}\cdot \left[\omega_{\ell_i^{i+r}(g\tau)} \left(\left.\frac{d}{dt}\right|_{t=0} \exp(te_1)\ell_i^{i+r}(g\tau), \cdots, \left.\frac{d}{dt}\right|_{t=0} \exp(te_{j-r+1})\ell_i^{i+r}(g\tau), \right.\right. \\
&\qquad\qquad \left.\left. (d\ell_i^{i+r})g_*\nu_1, \cdots, (d\ell_i^{i+r})g_*\nu_{r-1} \right) \right] \\
&= g^{-1}\cdot \xi_{g\tau}(g_*\nu_1, \cdots, g_*\nu_{r-1}).
\end{aligned}$$

According to [40, Proposition 16.41],

$$\left((u_i^{i+r})_*(\ell_i^{i+r})^*\omega\right)_{g^{-1}q}(g_*^{-1}w_1, \cdots, g_*^{-1}w_{j-r+1}) = \int_{\mathcal{M}(g^{-1}q,\mathcal{O}_i)} \beta = \int_{\mathcal{M}(q,\mathcal{O}_i)} \xi,$$

which is exactly $\left((u_i^{i+r})_*(\ell_i^{i+r})^*\omega\right)_q (w_1, \cdots, w_{j-r+1})$. □

We now precisely define ∂ in (1.1). For each $\Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$ and $r = 0, 1, \cdots, j+1$, we define

$$\begin{aligned}
\partial_r : \Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G &\to \Omega^{j-r+1}(\mathcal{O}_{i+r}, \mathcal{E}_{i+r})^G \\
\omega &\mapsto \begin{cases} d\omega & \text{if } r = 0, \\ (-1)^j \left(u_i^{i+r}\right)_* \left(\ell_i^{i+r}\right)^* \omega & \text{if } 1 \leqslant r \leqslant j+1. \end{cases}
\end{aligned} \tag{2.4}$$

Here, $\left(u_i^{i+r}\right)_*$ is given by integrating densities along fibers. Then, we define

$$\begin{aligned} \partial : C^k(M,f)^G &\to C^{k+1}(M,f)^G \\ \omega \in \Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G &\mapsto \partial_0\omega + \partial_1\omega + \cdots + \partial_{j+1}\omega. \end{aligned} \tag{2.5}$$

To verify that $\partial^2 = 0$, we need to clarify the manifold with corner structure of $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$, the local orientation of the codimension-1 stratum of $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$, and a generalized Stokes' theorem for the integration on

$$u_i^{i+r} : \overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)} \to \mathcal{O}_{i+r}$$

along fibers.

First, following the notations in [4, Section 3.2], for integers $0 < r_1 < \cdots < r_a < r$, we let

$$X_{r_1\cdots r_a} \coloneqq \mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_{i+r_a}) \times_{\mathcal{O}_{i+r_a}} \cdots \times_{\mathcal{O}_{i+r_2}} \mathcal{M}(\mathcal{O}_{i+r_2},\mathcal{O}_{i+r_1}) \times_{\mathcal{O}_{i+r_1}} \mathcal{M}(\mathcal{O}_{i+r_1},\mathcal{O}_i).$$

According to [4, Lemma 2.6, Lemma 3.3, Theorem A.11], $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$ is a smooth manifold with corner:

Theorem 2.1.3 (Austin-Braam [4], 1995)**.** *The compactification $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$ is a smooth manifold with corner:*

(i) *The codimension-1 stratum of $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$ is equal to $\bigcup_{0<s<r} X_s$. More precisely, there is an embedding*

$$\Psi_s : (-1,0] \times X_s \to \overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)} \tag{2.6}$$

mapping $(-1,0] \times X_s$ *to an open subset of* $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$. *In particular,* $\{0\} \times X_s$ *is mapped into the codimension-*1 *stratum of* $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$.

(ii) *Moreover,* Ψ_s *induces an embedding*

$$\Psi_{r_1\cdots r_a} : \underbrace{(-1,0] \times \cdots \times (-1,0]}_{a \text{ copies}} \times X_{r_1\cdots r_a} \to \overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}.$$

The codimension-a stratum of $\overline{\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)}$ *is* $\bigcup_{0<r_1<\cdots<r_a<r} X_{r_1\cdots r_a}$.

Second, we describe the local orientation of the codimension-1 stratum of $\mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i)$ induced by (2.6). Using the G-action, for any $q \in \mathcal{O}_{i+r}$, we can find an open subset $\mathcal{U}_{i+s}$ (resp. $\mathcal{U}_i$) of $\mathcal{O}_{i+s}$ (resp. $\mathcal{O}_i$) and some $q' \in \mathcal{U}_{i+s}$ (resp. $q'' \in \mathcal{U}_i$) such that $W^u(\mathcal{U}_{i+s})$ (resp. $W^u(\mathcal{U}_i)$) is diffeomorphic to $\mathcal{U}_{i+s} \times W^u(q')$ (resp. $\mathcal{U}_i \times W^u(q'')$), and the fiber bundle

$$\mathcal{M}(q,\mathcal{U}_{i+s}) \times_{\mathcal{U}_{i+s}} \mathcal{M}(\mathcal{U}_{i+s},\mathcal{O}_i) \to \mathcal{M}(q,\mathcal{U}_{i+s})$$

over $\mathcal{M}(q,\mathcal{U}_{i+s})$ is trivialized as the fiber bundle

$$\mathcal{M}(q,\mathcal{U}_{i+s}) \times \mathcal{M}(q',\mathcal{O}_i) \to \mathcal{M}(q,\mathcal{U}_{i+s})$$

with fiber $\mathcal{M}(q',\mathcal{O}_i)$. By the isomorphism (2.1), once we select the orientations of unstable manifolds $W^u(q)$, $W^u(q')$, and $W^u(q'')$, we automatically obtain the orientations of $\mathcal{M}(q,\mathcal{U}_i)$, $\mathcal{M}(q,\mathcal{U}_{i+s})$, and $\mathcal{M}(q',\mathcal{U}_i)$. We denote these three orientations by $[\mathcal{M}(q,\mathcal{U}_i)]$, $[\mathcal{M}(q,\mathcal{U}_{i+s})]$, and $[\mathcal{M}(q',\mathcal{U}_i)]$ respectively. Then, we have the following Lemma [4, Lemma 3.4].

Lemma 2.1.4 (Austin-Braam [4], 1995)**.** *The orientation of the manifold*

$$\mathcal{M}(q,\mathcal{U}_{i+s}) \times_{\mathcal{U}_{i+s}} \mathcal{M}(\mathcal{U}_{i+s},\mathcal{U}_i)$$

induced by $[\mathcal{M}(q,\mathcal{U}_i)]$ *under* (2.6) *is equal to* $(-1)^{r-s-1}[\mathcal{M}(q,\mathcal{U}_{i+s})][\mathcal{M}(q',\mathcal{U}_i)]$.

Third, we give the generalized Stokes' theorem. Continuing with the results and the notations in Theorem 2.1.3, for any $q \in \mathcal{O}_{i+r}$, we let

$$X_s(q) := \mathcal{M}(q,\mathcal{O}_{i+s}) \times_{\mathcal{O}_{i+s}} \mathcal{M}(\mathcal{O}_{i+s},\mathcal{O}_i),$$

$\Psi'_s = \Psi_s|_{\{0\}\times X_s(q)}$, and $\mathrm{Or}(q,\mathcal{O}_i)$ be the orientation bundle of $\overline{\mathcal{M}(q,\mathcal{O}_i)}$. We state the generalized Stokes' theorem on each fiber of $u_i^{i+r} : \mathcal{M}(\mathcal{O}_{i+r},\mathcal{O}_i) \to \mathcal{O}_i$.

Lemma 2.1.5. *For any* $\omega \in \Omega^{r-2}\left(\overline{\mathcal{M}(q,\mathcal{O}_i)}, \mathrm{Or}(q,\mathcal{O}_i)\right)$, *we have*

$$\int_{\overline{\mathcal{M}(q,\mathcal{O}_i)}} d\omega = \sum_{0<s<r} \int_{\overline{X_s(q)}} (\Psi'_s)^*\omega.$$

Proof. Let t be the coordinate on $(-1,0]$, $U = (x_1,\cdots,x_{\delta_i})$ be a local coordinate of $X_s(q)$, and e be the section 1 of $\mathrm{Or}(q,\mathcal{O}_i)|_{\Psi_s((-1,0]\times U)}$ induced by $dt \wedge dx_1 \wedge \cdots \wedge dx_{\delta_i}$. Meanwhile, we recall that $\dim \mathcal{M}(q,\mathcal{O}_i) = r-1$.

Suppose that $\mathrm{supp}(\omega) \subseteq \Psi_s((-1,0]\times U)$, we write

$$\omega = \varphi_0 dx_1 \wedge \cdots \wedge dx_{\delta_i} \otimes e + \sum_{j=1}^{r-2} \varphi_j dt \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{r-2} \otimes e$$

and then find

$$\begin{aligned}
\int_{\overline{\mathcal{M}(q,\mathcal{O}_i)}} d\omega = & \int_{(-1,0]\times U} d(\varphi_0 dx_1 \wedge \cdots \wedge dx_{r-2} \otimes e) \\
& + \sum_{j=1}^{r-2} \int_{(-1,0]\times U} d(\varphi_j dt \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{r-2} \otimes e) \\
= & \int_{(-1,0]\times U} \frac{\partial \varphi_0}{\partial t} |dt dx_1 \cdots dx_{r-2}| + \sum_{j=1}^{r-2} \int_{(-1,0]\times U} (-1)^j \frac{\partial \varphi_j}{\partial x_j} |dt dx_1 \cdots dx_{r-2}|
\end{aligned}$$

$$
\begin{aligned}
&= \int_U \varphi_0(0, x_1, \cdots, x_{r-2}) |dx_1 \cdots dx_{r-2}| \\
&= \int_{\overline{X_s(q)}} (\Psi'_s)^* \omega.
\end{aligned}
$$

The same proof applies to the situation where ω is supported in other types of local charts given by $\Psi_{r_1 \cdots r_a}$. The general case is obtained by partition of unity. $\square$

Remark 2.1.6. Similar to [40, Theorem 16.48], we need a nonvanishing outward-pointing direction induced by $(-1, 0]$ along $X_s(q)$. This is to ensure that $(\Psi'_s)^*\omega$ is a density.

Notice the fiber bundle structure

$$
u^{i+r}_{i,\partial} : \bigcup_{0<s<r} \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_{i+s}) \times_{\mathcal{O}_{i+s}} \mathcal{M}(\mathcal{O}_{i+s}, \mathcal{O}_i) \to \mathcal{O}_{i+r}
$$

induced by u^{i+r}_i on the codimension-1 stratum. We let

$$
\Psi' : \bigcup_{0<s<r} \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_{i+s}) \times_{\mathcal{O}_{i+s}} \mathcal{M}(\mathcal{O}_{i+s}, \mathcal{O}_i) \to \overline{\mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i)}
$$

be the map obtained by restricting each Ψ_s $(0 < s < r)$ in (2.6) to $\{0\} \times X_s$. We immediately have the following corollary of Lemma 2.1.5 (compare with [4, (3.3)]).

Corollary 2.1.7. *For any $\omega \in \Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$, we have the formula*

$$
(u^{i+r}_i)_* \left(d(\ell^{i+r}_i)^* \omega \right) = d(u^{i+r}_i)_* (\ell^{i+r}_i)^* \omega + (-1)^{j-r+2} (u^{i+r}_{i,\partial})_* (\Psi')^* (\ell^{i+r}_i)^* \omega.
$$

Proof. Around $q \in \mathcal{O}_{i+r}$, by [29, Theorem 9.3.7(iii)], we find $e_1, \cdots, e_n \in \mathfrak{g}$ such that there is a small $\delta > 0$ making

$$
\mathbb{B}^n(\delta) \coloneqq \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < \delta\} \to \mathcal{O}_{i+r}
$$

$$(x_1, \cdots, x_n) \mapsto \exp(x_1 e_1 + \cdots + x_n e_n) q$$

an embedding. Then, we obtain a local trivialization

$$\begin{aligned} \mathbb{B}^n(\delta) \times \mathcal{M}(q, \mathcal{O}_i) &\to \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i) \\ (x_1, \cdots, x_n, \gamma) &\mapsto \exp(x_1 e_1 + \cdots + x_n e_n)\gamma \end{aligned} \tag{2.7}$$

of $u_i^{i+r} : \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_i) \to \mathcal{O}_{i+r}$. On the local trivialization (2.7), we write

$$(\ell_i^{i+r})^* \omega = \sum_{a=0}^{n} \sum_{1 \leqslant k_1 < \cdots < k_a \leqslant n} \varphi_{k_1 \cdots k_a} dx_{k_1} \wedge \cdots \wedge dx_{k_a} \wedge \eta_{k_1 \cdots k_a},$$

where $\varphi_{k_1 \cdots k_a}$ is a smooth function, and η is an $\mathrm{Or}(q, \mathcal{O}_i)$-valued form on $\mathcal{M}(q, \mathcal{O}_i)$ with degree $j - a$. Recall that $r - 1 = \dim \mathcal{M}(q, \mathcal{O}_i)$, and let d_q be the de Rham exterior derivative on $\mathcal{M}(q, \mathcal{O}_i)$. Then, we find

$$\begin{aligned} & (u_i^{i+r})_* \left(d(\ell_i^{i+r})^* \omega) \right) \\ = {} & (u_i^{i+r})_* d \left(\sum_{a=0}^{n} \sum_{1 \leqslant k_1 < \cdots < k_a \leqslant n} \varphi_{k_1 \cdots k_{j-r+2}} dx_{k_1} \wedge \cdots \wedge dx_{k_a} \wedge \eta_{k_1 \cdots k_a} \right) \\ = {} & (u_i^{i+r})_* \left(\sum_{\substack{1 \leqslant k_1 < \cdots < k_{j-r+1} \leqslant n \\ \alpha \notin \{k_1, \cdots, k_{j-r+1}\}}} \frac{\partial}{\partial x_\alpha} \varphi_{k_1 \cdots k_{j-r+1}} dx_\alpha \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_{j-r+1}} \wedge \eta_{k_1 \cdots k_{j-r+1}} \right) \\ & + (-1)^{j-r+2} (u_i^{i+r})_* \left(\sum_{1 \leqslant k_1 < \cdots < k_a \leqslant n} dx_{k_1} \wedge \cdots \wedge dx_{k_{j-r+2}} \wedge d_q(\varphi_{k_1 \cdots k_{j-r+2}} \eta_{k_1 \cdots k_{j-r+2}}) \right) \\ = {} & d(u_i^{i+r})_* (\ell_i^{i+r})^* \omega + (-1)^{j-r+2} (u_{i,\partial}^{i+r})_* (\Psi')^* (\ell_i^{i+r})^* \omega. \end{aligned}$$

The second term in the last row is by Lemma 2.1.5. □

Remark 2.1.8. The definition of $(u_{i,\partial}^{i+r})_*$ follows the same pattern as defining $(u_i^{i+r})_*$. Given any $q \in \mathcal{O}_{i+r}$, once we choose an orientation of $W^u(q)$ and any $w_1, \cdots, w_{j-r+1} \in T_q \mathcal{O}_{i+r}$, similar to (2.3), $(\Psi')^*(\ell_i^{i+r})^*\omega$ induces a density $(\Psi')^*\xi$ on the boundary of $\mathcal{M}(q, \mathcal{O}_i)$. Then, applying Lemma 2.1.5, we integrate the density $(\Psi')^*\xi$ on the codimension-1 stratum

$$\bigcup_{0<s<r} \big(\mathcal{M}(q, \mathcal{O}_{i+s}) \times_{\mathcal{O}_{i+s}} \mathcal{M}(\mathcal{O}_{i+s}, \mathcal{O}_i)\big)$$

and obtain a number, which then gives us the form $(u_{i,\partial}^{i+r})_*(\Psi')^*(\ell_i^{i+r})^*\omega \in \Omega^{j-r+2}(\mathcal{O}_{i+r}, \mathcal{E}_{i+r})^G$.

Finally, we are able to check $\partial^2 = 0$. Similar to [4, Proposition 3.5], we only need to show:

Proposition 2.1.9. *For each $r \in \mathbb{Z}_{\geqslant 0}$, $\partial_0\partial_r + \partial_1\partial_{r-1} + \cdots + \partial_{r-1}\partial_1 + \partial_r\partial_0 = 0$.*

Proof. We follow the steps as in [4, Proposition 3.5]. Let Ψ' be the same map as that in Corollary 2.1.7. For any $\omega \in \Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G$, we find

$$\begin{aligned}
&\partial_r\partial_0\omega \\
=\ & (-1)^{j+1}(u_i^{i+r})_*\big(d(\ell_i^{i+r})^*\omega)\big) \\
=\ & (-1)^{j+1}d(u_i^{i+r})_*(\ell_i^{i+r})^*\omega + (-1)^{j+1+j-r+2}(u_{i,\partial}^{i+r})_*(\Psi')^*(\ell_i^{i+r})^*\omega \text{ (by Corollary 2.1.7)} \\
=\ & -\partial_0\partial_r\omega + (-1)^{-r+1}(u_{i,\partial}^{i+r})_*(\Psi')^*(\ell_i^{i+r})^*\omega.
\end{aligned}$$

By Lemma 2.1.4 and the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_{i+s}) \times_{\mathcal{O}_{i+s}} \mathcal{M}(\mathcal{O}_{i+s}, \mathcal{O}_i) \arrow[rr, "q_s^r"] \arrow[d, "p_s^r"] \arrow[rru] \arrow[dll, "u_{i,\partial}^{i+r}"'] & & \mathcal{M}(\mathcal{O}_{i+s}, \mathcal{O}_i) \arrow[u, "\ell_i^{i+s}"'] \arrow[d, "u_i^{i+s}"] & \mathcal{O}_i \\
\mathcal{O}_{i+r} & & \mathcal{M}(\mathcal{O}_{i+r}, \mathcal{O}_{i+s}) \arrow[ll, "u_{i+s}^{i+r}"] \arrow[rr, "\ell_{i+s}^{i+r}"] & & \mathcal{O}_{i+s} &
\end{tikzcd},$$

we write the second summand into

$$
\begin{aligned}
&(-1)^{-r+1}(u_{i,\partial}^{i+r})_*(\Psi')^*(\ell_i^{i+r})^*\omega \\
&= (-1)^{-r+1}\sum_{0<s<r}(-1)^{r-s-1}(u_{i+s}^{i+r})_*(p_s^r)_*(\Psi')^*(\ell_i^{i+r})^*\omega \text{ (by Lemma 2.1.4)} \\
&= (-1)^{-r+1}\sum_{0<s<r}(-1)^{r-s-1}(u_{i+s}^{i+r})_*(p_s^r)_*(q_s^r)^*(\ell_i^{i+s})^*\omega \\
&= (-1)^{-r+1}\sum_{0<s<r}(-1)^{r-s-1}(u_{i+s}^{i+r})_*(\ell_{i+s}^{i+r})^*(u_i^{i+s})_*(\ell_i^{i+s})^*\omega \text{ (by the diagram)} \\
&= -\sum_{0<s<r}\partial_{r-s}\partial_s\omega.
\end{aligned}
$$

Thus, we have $\partial_0\partial_r + \partial_1\partial_{r-1} + \cdots + \partial_{r-1}\partial_1 + \partial_r\partial_0 = 0$ and then $\partial^2 = 0$. $\square$

The first half of Theorem 1.2.4 is proved.

2.2 Topological realization

In this section, we prove the second half of Theorem 1.2.4 following the steps similar to [4, Theorem 3.8] and using the nonorientable Thom isomorphism [15, Theorem 7.10].

Definition 2.2.1. The G-invariant de Rham complex of M is formed by

$$\Omega^k(M)^G \coloneqq \{\omega \in \Omega^k(M) : g^*\omega = \omega \text{ for all } g \in G\}$$

together with the de Rham d. We let $H^k(M)^G$ be the k-th cohomology group of this complex, and $\Omega^*(M)^G \coloneqq \oplus_{k=0}^m \Omega^k(M)^G$.

By [22, Theorem 1.28], $H^k(M)^G$ is isomorphic to the original de Rham cohomology group of M. Thus, to prove Theorem 1.2.4, we need a quasi-isomorphism between the G-invariant Thom-Smale complex and the G-invariant de Rham complex.

Using the fiber bundle structure $\pi_i : W^u(\mathcal{O}_i) \to \mathcal{O}_i$ and the associated integration along fibers $(\pi_i)_*$ defined in [15, Theorem 7.10], we get a map

$$\begin{aligned}\Phi : \Omega^k(M)^G &\to C^k(M, f)^G \\ \omega &\mapsto \sum_{i=0}^{k} (\pi_i)_* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \quad (k = 0, 1, \cdots, m).\end{aligned} \tag{2.8}$$

We claim that Φ is a chain map. To show this fact, we need to clarify the manifold with corner structure of $W^u(\mathcal{O}_i)$, the local orientation of the codimension-1 stratum of $\overline{W^u(\mathcal{O}_i)}$, and a generalized Stokes' theorem for the integration on

$$\pi_i : \overline{W^u(\mathcal{O}_i)} \to \mathcal{O}_i$$

along fibers. The results are similar to Theorem 2.1.3, Lemma 2.1.4, Lemma 2.1.5, and Corollary 2.1.7.

First, following the notations in [4, Section 3.2] again, we let

$$Y_{r_1 \cdots r_a} \coloneqq \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-r_1}) \times_{\mathcal{O}_{i-r_1}} \cdots \times_{\mathcal{O}_{i-r_{a-1}}} \mathcal{M}(\mathcal{O}_{i-r_{a-1}}, \mathcal{O}_{i-r_a}) \times_{\mathcal{O}_{i-r_a}} W^u(\mathcal{O}_{i-r_a})$$

for integers $0 < r_1 < \cdots < r_a < i$. According to [4, Lemma 2.6, Lemma 3.3, Theorem A.11], $\overline{W^u(\mathcal{O}_i)}$ is a smooth manifold with corner:

Theorem 2.2.2 (Austin-Braam [4], 1995)**.** *The compactification $\overline{W^u(\mathcal{O}_i)}$ is a smooth manifold with corner:*

(i) *The codimension-*1 *stratum of* $\overline{W^u(\mathcal{O}_i)}$ *is equal to* $\bigcup_{0<j<i} Y_j$. *More precisely, there is an embedding*

$$\mathcal{P}_j : (-1, 0] \times Y_j \to \overline{W^u(\mathcal{O}_i)} \tag{2.9}$$

mapping $(-1, 0] \times Y_j$ *to an open subset of* $\overline{W^u(\mathcal{O}_i)}$. *In particular,* $\{0\} \times Y_j$ *is mapped into the codimension-*1 *stratum of* $\overline{W^u(\mathcal{O}_i)}$.

(ii) *Moreover,* $\mathcal{P}_j$ *induces an embedding*

$$\mathcal{P}_{r_1 \cdots r_a} : \underbrace{(-1, 0] \times \cdots \times (-1, 0]}_{a \text{ copies}} \times Y_{r_1 \cdots r_a} \to \overline{W^u(\mathcal{O}_i)}.$$

The codimension-a stratum of $\overline{W^u(\mathcal{O}_i)}$ *is* $\bigcup_{0<r_1<\cdots<r_a<i} Y_{r_1 \cdots r_a}$.

Second, we describe the local orientation of the codimension-1 stratum of $W^u(\mathcal{O}_i)$ induced by (2.9). Using the G-action, for any $q \in \mathcal{O}_i$, we can find an open subset $\mathcal{V}_{i-j}$ of $\mathcal{O}_{i-j}$ and some $q' \in \mathcal{V}_{i-j}$ such that $W^u(\mathcal{V}_{i-j})$ is diffeomorphic to $\mathcal{V}_{i-j} \times W^u(q')$, and the fiber bundle

$$\mathcal{M}(q, \mathcal{V}_{i-j}) \times_{\mathcal{V}_{i-j}} W^u(\mathcal{V}_{i-j}) \to \mathcal{M}(q, \mathcal{V}_{i-j})$$

over $\mathcal{M}(q, \mathcal{V}_{i-j})$ is trivialized as the fiber bundle

$$\mathcal{M}(q, \mathcal{V}_{i-j}) \times W^u(q') \to \mathcal{M}(q, \mathcal{V}_{i-j})$$

with fiber $W^u(q')$. Again, by (2.1), once we select the orientations of unstable manifolds $W^u(q)$ and $W^u(q')$, we automatically obtain the orientation of $\mathcal{M}(q', \mathcal{V}_{i-j})$. We denote the orientations of $W^u(q)$, $W^u(q')$, and $\mathcal{M}(q', \mathcal{V}_{i-j})$ by $[W^u(q)]$, $[W^u(q')]$, and $[\mathcal{M}(q', \mathcal{V}_{i-j})]$ respectively. Then, we have the following Lemma [4, Lemma 3.4].

Lemma 2.2.3 (Austin-Braam [4], 1995)**.** *The orientation of the manifold*

$$\mathcal{M}(q, \mathcal{V}_{i-j}) \times_{\mathcal{V}_{i-j}} W^u(\mathcal{V}_{i-j})$$

induced by $[W^u(q)]$ *under* (2.9) *is equal to* $(-1)^{j-1}[\mathcal{M}(q, \mathcal{V}_{i-j})][W^u(q')]$.

Third, we state the generalized Stokes' theorem. For simplicity, we only present it in the situation of this thesis. Notice the fiber bundle structure

$$\pi_{i,\partial} : \bigcup_{0<j<i} \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-j}) \times_{\mathcal{O}_{i-j}} W^u(\mathcal{O}_{i-j}) \to \mathcal{O}_i$$

induced by π_i on the codimension-1 stratum, we let

$$\mathcal{P}' : \bigcup_{0<j<i} \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-j}) \times_{\mathcal{O}_{i-j}} W^u(\mathcal{O}_{i-j}) \to \overline{W^u(\mathcal{O}_i)}$$

be the map obtained by restricting each $\mathcal{P}_j$ $(0 < j < i)$ in (2.9) to $\{0\} \times Y_j$. Then, we have the following result (compare with [4, (3.3)]):

Lemma 2.2.4. *For any* $\omega \in \Omega^k(M)^G$ *and any* $0 \leqslant i \leqslant k$*, we have the formula*

$$(\pi_i)_* \left(d\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) = d\left((\pi_i)_* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \right) + (-1)^{k-i+1} (\pi_{i,\partial})_* (\mathcal{P}')^* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right).$$

Proof. The proof is similar to that of Corollary 2.1.7. □

We are ready to show that Φ is a chain map.

Proposition 2.2.5. *The map* Φ *given by* (2.8) *is a chain map.*

Proof. Like in [4, Lemma 3.6], we check that for any $\omega \in \Omega^k(M)^G$ and any $0 \leqslant i \leqslant k$,

$$(\pi_i)_* \left(d\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) = \sum_{j=0}^{i} \partial_j (\pi_{i-j})_* \left(\omega|_{\overline{W^u(\mathcal{O}_{i-j})}} \right)$$

for all $\omega \in \Omega^k(M)^G$. By Lemma 2.2.4,

$$\begin{aligned} &(\pi_i)_* \left(d\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \\ &= d\left((\pi_i)_* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \right) + (-1)^{k-i+1} \left(\pi_{i,\partial}\right)_* (\mathcal{P}')^* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \\ &= \partial_0 (\pi_i)_* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) + (-1)^{k-i+1} \left(\pi_{i,\partial}\right)_* (\mathcal{P}')^* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right), \end{aligned}$$

where $(\pi_{i,\partial})_*$ is induced by the fiber bundle structure

$$\pi_{i,\partial} : \bigcup_{0<j<i} \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-j}) \times_{\mathcal{O}_{i-j}} W^u(\mathcal{O}_{i-j}) \to \mathcal{O}_i$$

of the codimension-1 stratum of $\overline{W^u(\mathcal{O}_i)}$. By Lemma 2.2.3 and the commutative diagram

$$\begin{tikzcd} & \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-j}) \times_{\mathcal{O}_{i-j}} W^u(\mathcal{O}_{i-j}) \arrow[r] \arrow[dl, "\pi_{i,\partial}"'] \arrow[d] & W^u(\mathcal{O}_{i-j}) \arrow[d, "\pi_{i-j}"] \\ \mathcal{O}_i & \mathcal{M}(\mathcal{O}_i, \mathcal{O}_{i-j}) \arrow[l, "u^i_{i-j}"] \arrow[r, "\ell^i_{i-j}"] & \mathcal{O}_{i-j} \end{tikzcd},$$

like in the proof Proposition 2.1.9, we obtain

$$\begin{aligned} &(-1)^{k-i+1} \left(\pi_{i,\partial}\right)_* \left(\omega|_{\overline{W^u(\mathcal{O}_i)}} \right) \\ &= (-1)^{k-i+1} \sum_{j=1}^{i} (-1)^{j-1} \left(u^i_{i-j}\right)_* (\ell^i_{i-j})^* (\pi_{i-j})_* \left(\omega|_{\overline{W^u(\mathcal{O}_{i-j})}} \right) \\ &= \sum_{j=0}^{i} \partial_j (\pi_{i-j})_* \left(\omega|_{\overline{W^u(\mathcal{O}_{i-j})}} \right). \end{aligned}$$

Thus, Φ is a chain map. $\square$

We then show that Φ is a quasi-isomorphism. Similar to [4, Section 3.3], without loss of generality, we assume that $f(\mathcal{O}_i) = i$ for all $0 \leqslant i \leqslant m$. Then, for any $r \in \mathbb{Z}_{\geqslant 0}$, we let

$$M_r = f^{-1}\left(r - \frac{1}{2}, +\infty\right), \quad M \backslash M_r = f^{-1}\left(-\infty, r - \frac{1}{2}\right]$$

so that for the G-invariant de Rham complex $(\Omega^*(M)^G, d)$, we get a filtration

$$\cdots \subseteq \Omega_c^k(M_{r+1})^G \subseteq \Omega_c^k(M_r)^G \subseteq \cdots \subseteq \Omega_c^k(M_0)^G = \Omega^k(M)^G$$

of $\Omega^k(M)^G$ for each k. Here, $\Omega_c^k(M_r)^G$ means the space of smooth G-invariant compactly supported k-forms on M_r. Furthermore, by defining

$$C_r^k(M, f)^G = \bigoplus_{i \geqslant r} \Omega^{k-i}(\mathcal{O}_i, \mathcal{E}_i)^G,$$

for the G-invariant Thom-Smale complex $(C^*(M, f)^G, \partial)$, we get a filtration

$$\cdots \subseteq C_{r+1}^k(M, f)^G \subseteq C_r^k(M, f)^G \subseteq \cdots \subseteq C_0^k(M, f)^G = C^k(M, f)^G$$

of $C^k(M, f)^G$ for each k as well. Let

$$\mathcal{G}\Omega_r^k = \Omega_c^k(M_r)^G / \Omega_c^k(M_{r+1})^G,$$

and

$$\mathcal{G}C_r^k = C_r^k(M, f)^G / C_{r+1}^k(M, f)^G = \Omega^{k-r}(\mathcal{O}_r, \mathcal{E}_r)^G.$$

Then, for each r, we get two chain complexes

$$(\mathcal{G}\Omega_r^*, d) : 0 \longrightarrow \mathcal{G}\Omega_r^0 \xrightarrow{d} \mathcal{G}\Omega_r^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{G}\Omega_r^m \longrightarrow 0$$

and

$$(\mathcal{G}C_r^*, d) : 0 \longrightarrow \mathcal{G}C_r^0 \xrightarrow{\partial=d} \mathcal{G}C_r^1 \xrightarrow{\partial=d} \cdots \xrightarrow{\partial=d} \mathcal{G}C_r^m \longrightarrow 0.$$

Let $H^k(\mathcal{G}\Omega_r^*)$ and $H^k(\mathcal{G}C_r^*)$ be their k-th cohomology groups respectively.

By [4, Lemma 3.7], if for any r and k, the chain map Φ induces an isomorphism between $H^k(\mathcal{G}\Omega_r^*)$ and $H^k(\mathcal{G}C_r^*)$, then Φ must be a quasi-isomorphism between the G-invariant Thom-Smale complex and the G-invariant de Rham complex. In addition, we notice that

$$H^k(\mathcal{G}C_r^*) = H^{k-r}(\mathcal{O}_r, \mathcal{E}_r)^G, \tag{2.10}$$

where the latter is the $(k-r)$-th G-invariant de Rham cohomology group of $\mathcal{O}_r$ with local coefficient $\mathcal{E}_r$. Thus, to check the quasi-isomorphism, we need the following proposition.

Proposition 2.2.6. *For any r and k,*

$$\Phi : H^k(\mathcal{G}\Omega_r^*) \to H^{k-r}(\mathcal{O}_r, \mathcal{E}_r)^G$$

is an isomorphism.

Proof. We let $V_r = M_r \cap W^u(\mathcal{O}_r)$ and $H_c^k(V_r)^G$ be its k-th G-invariant compactly supported de Rham cohomology group. By the nonorientable Thom isomorphism [15, Theorem 7.10], we find that Φ factors into

$$H^k(\mathcal{G}\Omega_r^*) \xrightarrow{\text{by restriction}} H_c^k(V_r)^G \xrightarrow{\text{Thom isomorphism}} H^{k-r}(\mathcal{O}_r, \mathcal{E}_r)^G,$$

where the Thom isomorphism is given by the integration along each fiber. Now, to finish the proof, we must show that the restriction

$$H^k(\mathcal{G}\Omega_r^*) \to H_c^k(V_r)^G$$

is an isomorphism.

We recall that for any embedded submanifold $S \subseteq M$, there is a relative de Rham cohomology [15, Section I.6] for the pair (M, S). If S is G-invariant, we get the G-invariant version of the relative de Rham cohomology. For $0 \leqslant k \leqslant m$, we let

$$\Omega^k(M, S)^G = \Omega^k(M)^G \oplus \Omega^{k-1}(S)^G$$

and extend the de Rham differentiation to

$$\begin{aligned} d : \Omega^k(M, S)^G &\to \Omega^{k+1}(M, S)^G \\ (\alpha, \beta) &\mapsto (d\alpha, \alpha|_S - d\beta) \end{aligned}$$

for the relative case. As in [15, Section 6.7], this defines the G-invariant de Rham complex of (M, S), and we let $H^k(M, S)^G$ be the k-th cohomology group of this complex.

We consider the following two short exact sequences

$$0 \to \Omega_c^k(M_{r+1})^G \to \Omega_c^k(M_r)^G \to \mathcal{G}\Omega_r^k \to 0$$

and

$$0 \to \Omega^k(M, M\backslash M_{r+1})^G \to \Omega^k(M, M\backslash M_r)^G \to \Omega^k(M\backslash M_{r+1}, M\backslash M_r)^G \to 0.$$

By [25, Theorem 2.16], they induce long exact sequences of cohomology groups $H^k_c(\cdot)^G$ and $H^k(\cdot)^G$ respectively. Then, we have the commutative diagram

$$\begin{array}{ccccc}
\cdots \to H^k_c(M_{r+1})^G & \longrightarrow & H^k_c(M_r)^G & \longrightarrow & H^k(\mathcal{G}\Omega^*_r) \to \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \to H^k(M, M\backslash M_{r+1})^G & \longrightarrow & H^k(M, M\backslash M_r)^G & \longrightarrow & H^k(M\backslash M_{r+1}, M\backslash M_r)^G \to \cdots
\end{array}$$

between two long exact sequences. Since for any k and r,

$$H^k_c(M_r)^G \to H^k(M, M\backslash M_r)^G$$

is an isomorphism by the definition of the G-invariant relative de Rham complex, the map

$$H^k(\mathcal{G}\Omega^*_r) \to H^k(M\backslash M_{r+1}, M\backslash M_r)^G$$

is an isomorphism according to the five lemma (See the proof of [25, Theorem 2.27]). Now, we obtain

$$H^k(M\backslash M_{r+1}, M\backslash M_r)^G \cong H^k((M\backslash M_r) \cup V_r, M\backslash M_r)^G$$

using the deformation retraction along the flow lines of ∇f. Then, by excision, we find

$$H^k((M\backslash M_r) \cup V_r, M\backslash M_r)^G \cong H^k(V_r, \partial V_r)^G \cong H^k_c(V_r)^G.$$

Thus, the restriction map $H^k(\mathcal{G}\Omega^*_r) \to H^k_c(V_r)^G$ is an isomorphism. □

The second half of Theorem 1.2.4 is thus a corollary of Proposition 2.2.6.

Remark 2.2.7. In the construction [4, Section 3] of the Thom-Smale complex associated with a more general Morse-Bott function f, there is no group action on the manifold, and it is required that all components of the critical submanifold of f and their unstable manifolds are orientable. In our study, due to the assumptions on the G-action, for each critical orbit $\mathcal{O}$, the unstable fiber bundle

$$W^u(\mathcal{O}) \to \mathcal{O}$$

is allowed to be nonorientable. See Example 2.3.3 for such a function admitting nonorientable unstable fiber bundles.

2.3 Examples and computations

In this section, we give some examples of computing the de Rham cohomology of M using Theorem 1.2.4. In each of the following examples, we first construct a G-invariant function satisfying assumptions (a1) and (a2). Then, we use the G-invariant Thom-Smale complex associated with this function to compute the de Rham cohomology of M.

Example 2.3.1. If $M = G$ acts by the left multiplication, and f is a constant, we have

$$C^k(M, f)^G = \Omega^k(G)^G, \ \forall\, k \in \mathbb{Z}_{\geqslant 0},$$

and the boundary map $\partial = d$. This repeats the G-invariant de Rham complex of G.

Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere centered at the origin, and $\mathbf{i} = \sqrt{-1}$ be the imaginary unit. In particular, we write $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and $\mathbb{S}^1 = \{e^{\mathbf{i}\theta} : \theta \in \mathbb{R}\}$.

Example 2.3.2. Let $M = \mathbb{S}^2 \times \mathbb{S}^1$ and $G = \mathbb{S}^1$ with the action

$$e^{\mathbf{i}\theta} \cdot (x, y, z, e^{\mathbf{i}t}) = \left(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z, e^{\mathbf{i}(2\theta + t)}\right).$$

Then, we see that

$$f(x, y, z, e^{\mathbf{i}t}) = (x^2 - y^2)\cos t + 2xy \sin t$$

is an $\mathbb{S}^1$-invariant function on $\mathbb{S}^2 \times \mathbb{S}^1$, and $\mathrm{crit}(f)$ consists of the following $\mathcal{O}_0, \mathcal{O}_1$ and $\mathcal{O}_2$:

(1) $\mathcal{O}_0 = \left\{(-\sin t, \cos t, 0, e^{\mathbf{i}2t}) \in \mathbb{S}^2 \times \mathbb{S}^1 : t \in \mathbb{R}\right\}$. It is diffeomorphic to the unit circle.

(2) $\mathcal{O}_1 = \mathcal{O}_{1,1} \cup \mathcal{O}_{1,2}$, where the two orbits are

$$\mathcal{O}_{1,1} = \left\{(0, 0, 1, e^{\mathbf{i}t}) \in \mathbb{S}^2 \times \mathbb{S}^1 : t \in \mathbb{R}\right\},$$
$$\mathcal{O}_{1,2} = \left\{(0, 0, -1, e^{\mathbf{i}t}) \in \mathbb{S}^2 \times \mathbb{S}^1 : t \in \mathbb{R}\right\}.$$

Both $\mathcal{O}_{1,1}$ and $\mathcal{O}_{1,2}$ are diffeomorphic to the unit circle.

(3) $\mathcal{O}_2 = \left\{(\cos t, \sin t, 0, e^{\mathbf{i}2t}) \in \mathbb{S}^2 \times \mathbb{S}^1 : t \in \mathbb{R}\right\}$. It is diffeomorphic to the unit circle.

Observe that $W^u(\mathcal{O}_0)$ and $W^u(\mathcal{O}_2)$ are orientable, while $W^u(\mathcal{O}_{1,1})$ and $W^u(\mathcal{O}_{1,2})$ are nonorientable, the G-invariant Thom-Smale complex is given by

$$0 \to \Omega^0(\mathcal{O}_0)^G \xrightarrow{0} \Omega^1(\mathcal{O}_0)^G \xrightarrow{0} \Omega^0(\mathcal{O}_2)^G \xrightarrow{0} \Omega^1(\mathcal{O}_2)^G \to 0.$$

Thus, we find that for $k = 0, 1, 2, 3$, the k-th cohomology group of M is $\mathbb{R}$. This gives the same result as applying the Künneth formula [45, Exercise 4.8].

Example 2.3.3. Let $G = \mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus, and $M = \mathbb{S}^2 \times \mathbb{T}^2$ equipped with the $\mathbb{T}^2$-action

$$\begin{aligned} & (e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}) \cdot (x, y, z, e^{\mathbf{i}t}, e^{\mathbf{i}s}) \\ = \ & \left(x\cos(\theta_1 + 2\theta_2) - y\sin(\theta_1 + 2\theta_2), x\sin(\theta_1 + 2\theta_2) + y\cos(\theta_1 + 2\theta_2), z, e^{\mathbf{i}(t+2\theta_1)}, e^{\mathbf{i}(s+\theta_2)}\right). \end{aligned}$$

Then, we find that

$$f(x, y, z, e^{\mathbf{i}t}, e^{\mathbf{i}s}) := (x^2 - y^2)\cos(t + 4s) + 2xy\sin(t + 4s)$$

is a $\mathbb{T}^2$-invariant function on $\mathbb{S}^2 \times \mathbb{T}^2$. Its critical set consists of the following $\mathcal{O}_0, \mathcal{O}_1$ and $\mathcal{O}_2$:

(1) $\mathcal{O}_0 = \left\{(-\sin t, \cos t, 0, e^{\mathbf{i}(2t-4s)}, e^{\mathbf{i}s}) \in \mathbb{S}^2 \times \mathbb{T}^2 : t, s \in \mathbb{R}\right\}$, which is diffeomorphic to $\mathbb{T}^2$.

(2) $\mathcal{O}_1 = \mathcal{O}_{1,1} \cup \mathcal{O}_{1,2}$, where the two critical orbits are

$$\mathcal{O}_{1,1} = \left\{(0, 0, 1, e^{\mathbf{i}t}, e^{\mathbf{i}s}) \in \mathbb{S}^2 \times \mathbb{T}^2 : t, s \in \mathbb{R}\right\},$$
$$\mathcal{O}_{1,2} = \left\{(0, 0, -1, e^{\mathbf{i}t}, e^{\mathbf{i}s}) \in \mathbb{S}^2 \times \mathbb{T}^2 : t, s \in \mathbb{R}\right\}.$$

Both $\mathcal{O}_{1,1}$ and $\mathcal{O}_{1,2}$ are diffeomorphic to $\mathbb{T}^2$.

(3) $\mathcal{O}_2 = \left\{(\cos t, \sin t, 0, e^{\mathbf{i}(2t-4s)}, e^{\mathbf{i}s}) \in \mathbb{S}^2 \times \mathbb{T}^2 : t, s \in \mathbb{R}\right\}$, which is diffeomorphic to $\mathbb{T}^2$.

Since $W^u(\mathcal{O}_0)$ and $W^u(\mathcal{O}_2)$ are orientable, while the unstable manifolds of $W^u(\mathcal{O}_{1,1})$ and $W^u(\mathcal{O}_{1,2})$ are nonorientable, we get the G-invariant Thom-Smale complex

$$0 \to \Omega^0(\mathcal{O}_0)^G \xrightarrow{0} \Omega^1(\mathcal{O}_0)^G \xrightarrow{0} \Omega^0(\mathcal{O}_2)^G \oplus \Omega^2(\mathcal{O}_0)^G \xrightarrow{0} \Omega^1(\mathcal{O}_2)^G \xrightarrow{0} \Omega^2(\mathcal{O}_2)^G \to 0.$$

Thus, we find that for $k = 0, 4$, the k-th cohomology group is $\mathbb{R}$, and for $k = 1, 2, 3$, the group is $\mathbb{R} \oplus \mathbb{R}$. This gives the same result as applying the Künneth formula.

Example 2.3.4. Let $M = \mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$ and $G = \mathbb{S}^1$ with the action

$$e^{\mathbf{i}\theta} \cdot (x, y, z, w) = (x\cos(2\theta) - y\sin(2\theta), x\sin(2\theta) + y\cos(2\theta), z\cos\theta - w\sin\theta, z\sin\theta + w\cos\theta).$$

Then, we find the function

$$f(x,y,z,w) = (z^2 - w^2)x + 2zwy$$

is an $\mathbb{S}^1$-invariant function on $\mathbb{S}^3$. The critical set of f consists of orbits

$$\begin{aligned}
\mathcal{O}_0 &= \left\{ \left(-\frac{\sqrt{3}}{3}\cos(2t), \frac{\sqrt{3}}{3}\sin(2t), \frac{2\sqrt{3}}{3}\cos t, \frac{2\sqrt{3}}{3}\sin t \right) \in \mathbb{S}^3 : t \in \mathbb{R} \right\}, \\
\mathcal{O}_1 &= \left\{ (\cos t, \sin t, 0, 0) \in \mathbb{S}^3 : t \in \mathbb{R} \right\}, \\
\mathcal{O}_2 &= \left\{ \left(\frac{\sqrt{3}}{3}\cos(2t), \frac{\sqrt{3}}{3}\sin(2t), \frac{2\sqrt{3}}{3}\cos t, \frac{2\sqrt{3}}{3}\sin t \right) \in \mathbb{S}^3 : t \in \mathbb{R} \right\}.
\end{aligned}$$

All of them are diffeomorphic to $\mathbb{S}^1$. Since $W^u(\mathcal{O}_1)$ is nonorientable, the G-invariant Thom-Smale complex is

$$0 \to \Omega^0(\mathcal{O}_0)^G \xrightarrow{0} \Omega^1(\mathcal{O}_0)^G \xrightarrow{\omega \mapsto \int_0^{2\pi}\omega} \Omega^0(\mathcal{O}_2)^G \xrightarrow{0} \Omega^1(\mathcal{O}_2)^G \to 0.$$

Therefore, we find for $k = 0, 3$, the k-th cohomology group is $\mathbb{R}$, and for $k = 1, 2$, the group is 0. This is exactly the de Rham cohomology of $\mathbb{S}^3$.

Example 2.3.5. Let M be the product $G \times X$, where X is a closed oriented smooth manifold. If G acts on $G \times X$ by

$$g \cdot (g', x) = (gg', x),$$

we use the natural projection $G \times X \to X$ to pull any generic Morse function to a Morse-Bott function satisfying (a1) and (a2) on $G \times X$. Then, according to the Thom-Smale complex associated with the generic Morse function on X, in the Thom-Smale complex of $G \times X$ associated with the pullback Morse-Bott function, the map $\partial_\alpha = 0$ when $\alpha \geqslant 2$. This fact gives the Künneth formula for the de Rham cohomology of $G \times X$.

Chapter 3

G-invariant Hodge theory

In this chapter, we present the G-invariant Hodge theory associated with the deformed de Rham complex and the Witten instanton complex. In fact, because our differential operators like the de Rham d and the Witten Laplacian D_T^2 are G-equivariant, the G-invariant Hodge theory is actually not far away from the classical one in [53, Chapter 6].

Throughout this chapter, we assume that G is a compact connected Lie group, M is a closed oriented manifold equipped with a G-invariant metric $\langle \cdot, \cdot \rangle$, and f is a G-invariant function. In fact, for G-invariant Hodge theory, we do not need (a1) or (a2).

3.1 Operators and forms

Associated to the G-invariant metric $\langle \cdot, \cdot \rangle$ on M, there is a unique form dvol_M satisfying the property that for any $p \in M$ and any oriented orthonormal basis $v_1, \cdots, v_m$ of T_pM,

$$(\mathrm{dvol}_M)_p(v_1, \cdots, v_m) = 1.$$

This dvol_M is called the volume form of M associated with $\langle\cdot,\cdot\rangle$ (See [40, Proposition 15.29]).

Proposition 3.1.1. *The volume form* dvol_M *is* G*-invariant.*

Proof. At any $p \in M$, we choose an oriented orthonormal basis $v_1, \cdots, v_m$ of T_pM. Then,

$$g \cdot v_1, \cdots, g \cdot v_m$$

give an oriented orthonormal basis of $T_{gp}M$ for any $g \in G$ since G is connected. Therefore,

$$(g^*\mathrm{dvol}_M)_p(v_1, \cdots, v_m) = (\mathrm{dvol}_M)_{gp}\,(g \cdot v_1, \cdots, g \cdot v_m) = 1.$$

Since dvol_M is unique, we must have $g^*\mathrm{dvol}_M = \mathrm{dvol}_M$. $\square$

We then impose the Riemannian metric $\langle\cdot,\cdot\rangle$ onto $\Omega^k(M)$. In fact, for any $\alpha, \beta \in \Omega^k(M)$, at each point $p \in M$, we choose any oriented orthonormal basis $v_1, \cdots, v_m$ of T_pM and define

$$\langle \alpha, \beta\rangle_p \coloneqq \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m} \alpha_p(v_{i_1}, \cdots, v_{i_k}) \cdot \beta_p(v_{i_1}, \cdots, v_{i_k}).$$

It can be verified directly that this definition is independent of the choice of oriented orthonormal basis of T_pM. Thus, on the collection

$$\Omega^*(M) \coloneqq \bigoplus_{k=1}^{m} \Omega^k(M)$$

of all smooth forms on M, we impose the metric by requiring that $\langle\alpha, \beta\rangle = 0$ when α and β are both homogeneous but with different degrees. Then, we get a smooth function $\langle\alpha, \beta\rangle$ on M for any $\alpha, \beta \in \Omega^*(M)$.

Proposition 3.1.2. *For any* $\alpha, \beta \in \Omega^*(M)$ *and any* $g \in G$*, we have* $g^*\langle\alpha, \beta\rangle = \langle g^*\alpha, g^*\beta\rangle$.

Proof. Without loss of generality, we assume $\alpha, \beta \in \Omega^k(M)$. At any point p, we choose an oriented orthonormal basis $v_1, \cdots, v_m$ of T_pM and get

$$\begin{aligned}
(g^*\langle\alpha,\beta\rangle)_p &= \langle\alpha,\beta\rangle_{gp} \\
&= \sum_{1\le i_1<i_2<\cdots<i_k\le m} \alpha_{gp}\left(gv_{i_1},\cdots,gv_{i_k}\right)\cdot\beta_{gp}\left(gv_{i_1},\cdots,gv_{i_k}\right) \\
&= \sum_{1\le i_1<i_2<\cdots<i_k\le m} (g^*\alpha)_p\left(v_{i_1},\cdots,v_{i_k}\right)\cdot(g^*\beta)_p\left(v_{i_1},\cdots,v_{i_k}\right) \\
&= \langle g^*\alpha, g^*\beta\rangle_p.
\end{aligned}$$

Thus, $g^*\langle\alpha,\beta\rangle = \langle g^*\alpha, g^*\beta\rangle$. $\square$

Recall that for each $0 \le k \le m$, we have the Hodge star

$$\star : \Omega^k(M) \to \Omega^{m-k}(M)$$

associated to $\langle\cdot,\cdot\rangle$. According to [46, equation (4.1.15)], this $\star$ is uniquely determined by

$$\alpha \wedge \star\beta = \langle\alpha,\beta\rangle \cdot \mathrm{dvol}_M$$

for all $\alpha, \beta \in \Omega^k(M)$. In particular, $\star 1 = \mathrm{dvol}_M$, and we can extend $\star$ from each $\Omega^k(M)$ to the whole $\Omega^*(M)$ linearly.

Proposition 3.1.3. *The Hodge star is G-equivariant.*

Proof. Without loss of generality, we let α and β be two k-forms. Then, we find

$$\beta \wedge \left((g^{-1})^* \star g^*\alpha\right) = (g^{-1})^*\left(g^*\beta \wedge \star g^*\alpha\right)) = (g^{-1})^*\Big(\langle g^*\beta, g^*\alpha\rangle \cdot \mathrm{dvol}_M\Big).$$

According to Proposition 3.1.1 and Proposition 3.1.2, the last term should be

$$\langle \beta, \alpha \rangle \cdot \mathrm{dvol}_M = \beta \wedge \star\alpha.$$

Since the Hodge star is uniquely defined, we must have $(g^{-1})^* \star g^*\alpha = \star\alpha$. $\square$

3.2 Witten instanton complex

Let $f \in \Omega^0(M)^G$. In addition, given any parameter $T > 0$, we deform d as [56, (5.5)].

Definition 3.2.1. We call the following operator

$$\begin{aligned} d_T : \Omega^*(M) &\to \Omega^{*+1}(M) \\ \omega &\mapsto e^{-Tf} d(e^{Tf}\omega) \end{aligned}$$

the Witten deformation of d associated with f and $T > 0$.

It is straightforward to see that $d_T^2 = 0$. Then, we get a chain complex

$$0 \to \Omega^0(M) \xrightarrow{d_T} \cdots \xrightarrow{d_T} \Omega^m(M) \to 0.$$

There is an L^2-norm on $\Omega^*(M)$. In fact, for any $\alpha \in \Omega^*(M)$, we define

$$\|\alpha\|^2 := \int_M \langle \alpha, \alpha \rangle \mathrm{dvol}_M = \int_M \alpha \wedge \star\alpha.$$

The completion of $\Omega^*(M)$ with respect to $\|\cdot\|$ is called the space of L^2-norms and denoted by $L^2(\Lambda^* T^* M)$. In particular, the norm $\|\|$ is G-invariant:

Proposition 3.2.2. *For any $g \in G$ and any $\alpha \in \Omega^*(M)$, $\|g^*\alpha\| = \|\alpha\|$.*

Proof. By Proposition 3.1.3, we get

$$\|g^*\alpha\|^2 = \int_M g^*\alpha \wedge \star g^*\alpha = \int_{gM} \alpha \wedge \star\alpha = \int_M \alpha \wedge \star\alpha = \|\alpha\|^2.$$

Thus, $\|\cdot\|$ is G-invariant. $\square$

As we know, the following operator

$$\begin{aligned} d^* : \Omega^k(M) &\to \Omega^{k-1}(M) \\ \omega &\mapsto (-1)^{nk+n+1} \star d \star \omega \end{aligned}$$

is the formal adjoint of d with respect to the L^2-norm $\|\cdot\|$. Immediately, we see the formal adjoint of d_T with respect to $\langle \cdot, \cdot \rangle_{L^{0,2}}$ is

$$d_T^* := e^{Tf} d^* e^{-Tf}.$$

Furthermore, we let $D_T := d_T + d_T^*$ be the deformed Dirac type operator and $D_T^2 := (d_T + d_T^*)^2$ be the Witten Laplacian.

Let c and $\hat{c}$ be the Clifford actions defined as in [56, (4.12)]. Then, we have

$$D_T = d + d^* + T\hat{c}(df).$$

A quick observation similar to [53, Propositions 6.2-6.3] is given as follows.

Proposition 3.2.3. *With respect to the inner product induced by* $\|\cdot\|$, D_T *is a formally self-adjoint operator. In addition, for any* $\omega \in \Omega^k(M)$, $D_T^2\omega = 0$ *if and only if* $d_T\omega = d_T^*\omega = 0$.

We find that D_T^2 is an elliptic operator because of the following Proposition 3.2.4.

Proposition 3.2.4. *D_T^2 has the same principal symbol as the Hodge Laplacian $(d+d^*)^2$.*

Proof. For any $h \in \Omega^0(M)$ and any $\omega \in \Omega^*(M)$, we have

$$[d_T, h]\omega = e^{-Tf}d(e^{Tf}h\omega) - he^{-Tf}d(e^{Tf}\omega) = dh \wedge \omega.$$

Then, by [46, Corollary 10.1.27], we find

$$[d_T^*, h]\omega = -(dh)\lrcorner\omega,$$

where $\lrcorner$ means the interior multiplication. Using [46, Proposition 10.1.10], we find

$$[[D_T^2, h], h]\omega = -2\langle dh, dh\rangle \cdot \omega.$$

This gives the same principal symbol as that of $(d+d^*)^2$ (See [46, Examples 10.1.18-10.1.22]). $\square$

By Proposition 3.2.4, we see that

$$D_T^2 : \Omega^k(M) \to \Omega^k(M)$$

is a second order self-adjoint elliptic operator on the k-th exterior bundle $\bigwedge^k T^*M$. Applying [46, Theorem 10.4.7, Corollary 10.4.10], we get an orthogonal decomposition of $\Omega^k(M)$:

Proposition 3.2.5. *For each $0 \le k \le m$, the dimension of $\ker(D_T^2 : \Omega^k(M) \to \Omega^k(M))$ is finite, and $\Omega^k(M)$ can be written into an orthogonal decomposition*

$$\ker(D_T^2 : \Omega^k(M) \to \Omega^k(M)) \oplus \mathrm{Im}(D_T^2 : \Omega^k(M) \to \Omega^k(M))$$

with respect to the inner product $\langle \cdot, \cdot \rangle_{L^{0,2}}$,

We denote the orthogonal projection from $\Omega^k(M)$ to $\ker(D_T^2 : \Omega^k(M) \to \Omega^k(M))$ by $\mathfrak{H}_k$. Then, for each $\omega \in \Omega^k(M)$, there is a unique $\eta \in \mathrm{Im}(D_T^2 : \Omega^k(M) \to \Omega^k(M))$ satisfying

$$\omega - \mathfrak{H}_k(\omega) = D_T^2 \eta. \tag{3.1}$$

Similar to [53, Definition 6.9], we define Green's operator

$$\begin{aligned} \mathfrak{G}_k : \Omega^k(M) &\to \Omega^k(M) \\ \omega &\mapsto \eta \text{ given in (3.1)}. \end{aligned}$$

It is clear that $\mathfrak{G}_k$ is a linear operator.

Proposition 3.2.6. *The Green's operator commutes with* d_T*. More precisely,*

$$\mathfrak{G}_{k+1} \circ d_T = d_T \circ \mathfrak{G}_k$$

on the collection $\Omega^k(M)$ *of smooth* k*-forms.*

Proof. For any $\omega \in \Omega^k(M)$, we have

$$\omega = \mathfrak{H}_k(\omega) + D_T^2 \mathfrak{G}_k(\omega).$$

Applying d_T to each side, by Proposition 3.2.3, we get

$$d_T\omega = d_T(\mathfrak{H}_k(\omega)) + d_T D_T^2 \mathfrak{G}_k(\omega) = 0 + D_T^2 d_T \mathfrak{G}_k(\omega).$$

Also, we have

$$d_T\omega = \mathfrak{H}_{k+1}(d_T\omega) + D_T^2\mathfrak{G}_{k+1}(d_T\omega).$$

Since the orthogonal decomposition of $d_T\omega$ is unique, we get

$$\mathfrak{H}_{k+1}(d_T\omega) = 0 \quad \text{and} \quad D_T^2 d_T\mathfrak{G}_k(\omega) = D_T^2\mathfrak{G}_{k+1}(d_T\omega).$$

Then, since d_T maps $\mathrm{Im}(D_T^2 : \Omega^k(M) \to \Omega^k(M))$ into $\mathrm{Im}(D_T^2 : \Omega^{k+1}(M) \to \Omega^{k+1}(M))$, by the definition of Green's operator, we obtain $d_T\mathfrak{G}_k(\omega) = \mathfrak{G}_{k+1}(d_T\omega)$. □

Recall that f is a G-invariant function. By Proposition 3.1.3, we find d_T, d_T^*, D_T and D_T^2 are all G-equivariant. Moreover, the operators $\mathfrak{H}_k$ and $\mathfrak{G}_k$ are also G-equivariant.

Proposition 3.2.7. *For any $g \in G$ and $\omega \in \Omega^k(M)$, we have $g^*\mathfrak{H}_k(\omega) = \mathfrak{H}_k(g^*\omega)$, and $g^*\mathfrak{G}_k(\omega) = \mathfrak{G}_k(g^*\omega)$.*

Proof. Using the decomposition $\omega = \mathfrak{H}_k(\omega) + D_T^2\mathfrak{G}_k(\omega)$, we get

$$g^*\omega = g^*\mathfrak{H}_k(\omega) + g^*D_T^2\mathfrak{G}_k(\omega) = g^*\mathfrak{H}_k(\omega) + D_T^2 g^*\mathfrak{G}_k(\omega).$$

In addition, we have another decomposition $g^*\omega = \mathfrak{H}_k(g^*\omega) + D_T^2\mathfrak{G}_k(g^*\omega)$. Since $D_T^2 g^*\mathfrak{H}_k(\omega)$ and $g^*D_T^2\mathfrak{H}_k(\omega)$ are both 0, by the uniqueness of the decomposition, we find

$$\mathfrak{H}_k(g^*\omega) = g^*\mathfrak{H}_k(\omega) \quad \text{and} \quad D_T^2 g^*\mathfrak{G}_k(\omega) = D_T^2\mathfrak{G}_k(g^*\omega).$$

Since g^* maps $\mathrm{Im}(D_T^2 : \Omega^k(M) \to \Omega^k(M))$ into itself, $g^*\mathfrak{G}_k(\omega)$ must be in $\mathrm{Im}(D_T^2 : \Omega^k(M) \to \Omega^k(M))$. By the definition of $\mathfrak{G}_k$, we find $\mathfrak{G}_k(g^*\omega) = g^*\mathfrak{G}_k(\omega)$. Therefore, $\mathfrak{H}_k$ and $\mathfrak{G}_k$ are both G-equivariant. □

3.3 G-invariant Hodge theory

For any $0 \leqslant k \leqslant m$, we let $\Omega^k(M)^G$ be the space of G-invariant smooth k-forms on M. Since d_T and d are both G-equivariant, we have the following two chain complexes

$$0 \to \Omega^0(M)^G \xrightarrow{d_T} \cdots \xrightarrow{d_T} \Omega^m(M)^G \to 0 \tag{3.2}$$

and

$$0 \to \Omega^0(M)^G \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M)^G \to 0. \tag{3.3}$$

We let $H^k_T(M)^G$ and $H^k(M)^G$ be the k-th cohomology groups of (3.2) and (3.3) respectively. In fact, the two cohomology groups are naturally isomorphic to each other.

Lemma 3.3.1. *For $0 \leq k \leq m$, $H^k_T(M)^G$ is naturally isomorphic to $H^k(M)^G$.*

Proof. As in the proof of [56, Proposition 5.3], the natural map

$$\Omega^k(M)^G \to \Omega^k(M)^G$$
$$\omega \mapsto e^{Tf}\omega$$

induces an isomorphism between $H^k_T(M)^G$ and $H^k(M)^G$. $\square$

By Lemma 3.3.1, in the G-invariant situation, we will use (3.2) more than (3.3).

Ensured by the G-equivariance, the operators $\star$, d_T, d_T^*, D_T, $\mathfrak{H}_k$, and $\mathfrak{G}_k$ all map $\Omega^*(M)^G$ into itself. Moreover, by Proposition 3.2.5, we get an orthogonal decomposition of $\Omega^k(M)^G$ under the inner product $(\cdot,\cdot)$ induced by the L^2-norm $\|\cdot\|$.

Proposition 3.3.2. *For $0 \leqslant k \leqslant m$, we have*

$$\dim \ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G) < \infty,$$

and there is an orthogonal decomposition of $\Omega^k(M)^G$ into

$$\ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G) \oplus \operatorname{Im}(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G).$$

More precisely,

$$\omega = \mathfrak{H}_k(\omega) + D_{Tf}^2 \mathfrak{G}_k(\omega)$$

for any $\omega \in \Omega^k(M)^G$.

A direct corollary of Proposition 3.3.2 shows that each cohomology class $H_T^k(M)^G$ is represented by a unique element in $\ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G)$.

Corollary 3.3.3. *For $0 \leqslant k \leqslant m$,* $\ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G) \cong H_T^k(M)^G$.

Proof. As [53, Theorem 6.11], we map each $\xi \in \ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G)$ to its cohomology class in $H_T^k(M)^G$. Recall the inner product $(\cdot, \cdot)$ induced by the L^2-norm $\|\cdot\|$. If this ξ equals $d_T\eta$ for some $\eta \in \Omega^{k-1}(M)^G$, then we get

$$(\xi, \xi) = (\eta, d_T^* \xi) = 0$$

because $d_T^*\xi = 0$. Immediately, we find $\xi = 0$, and the map is injective.

Conversely, if $\omega \in \Omega^k(M)^G$ satisfies $d_T\omega = 0$, by Proposition 3.2.6, we find

$$\omega = \mathfrak{H}_k(\omega) + d_T d_T^* \mathfrak{G}_k(\omega) + d_T^* d_T \mathfrak{G}_k(\omega)$$

$$= \mathfrak{H}_k(\omega) + d_T d_T^* \mathfrak{G}_k(\omega) + d_T^* \mathfrak{G}_{k+1}(d_T\omega)$$
$$= \mathfrak{H}_k(\omega) + d_T d_T^* \mathfrak{G}_k(\omega).$$

Thus, the representative can be chosen as $\mathfrak{H}_k(\omega)$, and the map is surjective. □

According to [46, Theorem 10.4.20], the Witten Laplacian

$$D_T^2 : \Omega^k(M) \to \Omega^k(M)$$

has finitely many eigenvalues (counting multiplicity) in any finite-length interval. Following the idea in [56, Section 5.5], for any $\alpha > 0$, we let $F_T^k(M, f, \alpha)^G$ be the real vector space

$$\mathrm{span}_{\mathbb{R}}\{\omega \in \Omega^k(M)^G : D_T^2\omega = \lambda\omega \text{ for some } \lambda \in [0, \alpha]\}.$$

In other words, it is the finite dimensional space generated by G-invariant eigenforms associated with the eigenvalues of D_T^2 in $[0, \alpha]$. Since $D_T^2 d_T = d_T D_T^2$, we get a chain complex

$$0 \to F_T^0(M, f, \alpha)^G \xrightarrow{d_T} \cdots \xrightarrow{d_T} F_T^m(M, f, \alpha)^G \to 0 \tag{3.4}$$

and call it the Witten instanton complex associated with M, f, and the metric $\langle\cdot,\cdot\rangle$.

Theorem 3.3.4. *The complex* (3.4) *computes the same cohomology as* (3.2) *and* (3.3)*.*

Proof. By Corollary 3.3.3, we only need to prove that $\ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G)$ is isomorphic to the k-th cohomology of (3.4).

First, we map each $\xi \in \ker(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G)$ to its class in the k-th cohomology group of (3.4). Similar to the first part of the proof of Corollary 3.3.3, this map is injective.

Next, we show that this map is surjective. If $\omega \in F_T^k(M,f,\alpha)^G$ satisfies $d_T\omega = 0$, we have

$$\omega = \mathfrak{H}_k(\omega) + d_T d_T^* \mathfrak{G}_k(\omega).$$

To show that $\mathfrak{H}_k(\omega)$ represents the same cohomology class as ω, we must verify that $d_T^*\mathfrak{G}_k(\omega)$ is a form in $F_T^{k-1}(M,f,\alpha)^G$. In fact, let $\lambda_1, \cdots, \lambda_\ell$ be all eigenvalues of $D_T^2|_{\Omega^k(M)^G}$ in $(0,\alpha]$, we write ω into

$$\omega = \mathfrak{H}_k(\omega) + \omega_1 + \cdots + \omega_\ell,$$

where $D_T^2\omega_i = \lambda_i\omega_i$ $(1 \leqslant i \leqslant \ell)$. Then,

$$\frac{1}{\lambda_1}\omega_1 + \frac{1}{\lambda_2}\omega_2 + \cdots + \frac{1}{\lambda_\ell}\omega_\ell \in F_T^k(M,f,\alpha) \cap \mathrm{Im}(D_T^2 : \Omega^k(M)^G \to \Omega^k(M)^G)$$

satisfies that

$$\mathfrak{H}_k(\omega) + D_T^2\left(\frac{1}{\lambda_1}\omega_1 + \frac{1}{\lambda_2}\omega_2 + \cdots + \frac{1}{\lambda_\ell}\omega_\ell\right) = \omega.$$

By the definition of $\mathfrak{G}_k$, we find

$$\mathfrak{G}_k(\omega) = \frac{1}{\lambda_1}\omega_1 + \frac{1}{\lambda_2}\omega_2 + \cdots + \frac{1}{\lambda_\ell}\omega_\ell$$

and thus

$$d_T^*\mathfrak{G}_k(\omega) = \frac{1}{\lambda_1}d_T^*\omega_1 + \frac{1}{\lambda_2}d_T^*\omega_2 + \cdots + \frac{1}{\lambda_\ell}d_T^*\omega_\ell.$$

Since $D_T^2 d_T^*\omega_i = d_T^* D_T^2\omega_i = \lambda_i d_T^*\omega_i$, we immediately see that $d_T^*\mathfrak{G}_k(\omega) \in F_T^{k-1}(M,f,\alpha)^G$. $\square$

We have obtained several G-invariant complexes computing the de Rham cohomology of M. However, only (3.4) gives the analytic realization that we want in Theorem 1.2.6.

Chapter 4

Analytic Morse-Bott theory

In this chapter, we prove Theorem 1.2.6, showing that under a well-adjusted metric, the Witten instanton complex (3.4) analytically realizes the G-invariant Thom-Smale complex in Definition 1.2.3. The adjustment of the metric uses the G-equivariant Morse-Bott Lemma.

Throughout this chapter, we adopt the same assumptions on M and f as Chapter 2. The behavior of the eigenvalues of $D_T^2 = (d + d^* + T\hat{c}(df))^2$ ensures the application of Bismut and Lebeau's analysis. When T is large, we have two types of eigenvalues, determined by the horizontal direction and the vertical direction respectively around $\mathrm{crit}(f)$. The former type of eigenvalues have a finite upper bound, while the latter type of eigenvalues have a lower bound of order $O(T)$. This behavior means we can find a constant to separate these two types of eigenvalues and then apply the ideas from [11, Chapter VIII-X].

4.1 Metrics and Connections

In this section, we complete the following two preparations for the estimates of D_T and D_T^2:

(1) Adjust the original metric $\langle\cdot,\cdot\rangle$ according to the G-action and the G-equivariant Morse-Bott lemma [54, Lemma 4.1] around each critical orbit.

(2) Figure out how the Levi-Civita connection associated to the adjusted $\langle\cdot,\cdot\rangle$ acts on local frames around each critical orbit.

Given any critical orbit $\mathcal{O}$ with dimension $= n$ and Morse index $= i$, we let N (resp. $N(\varepsilon)$) be the normal bundle of $\mathcal{O}$ defined with respect to $\langle\cdot,\cdot\rangle$ (resp. collection of all $v \in N$ satisfying $\langle v, v\rangle < \varepsilon^2$). Applying [54, Lemma 4.1], we get:

Lemma 4.1.1. *There is a sufficiently small $\varepsilon > 0$, a G-equivariant bundle map*

$$P : N \to N$$

being an orthogonal projection with respect to $\langle\cdot,\cdot\rangle$ on each fiber, and a G-equivariant embedding

$$\varrho : N(8\varepsilon) \to M$$

satisfying that $\varrho|_{\mathcal{O}} = \mathrm{id}$, such that for all vectors $v \in N(8\varepsilon)$,

$$f \circ \varrho(v) = f(\mathcal{O}) - \frac{1}{2}\langle Pv, Pv\rangle + \frac{1}{2}\langle (1-P)v, (1-P)v\rangle . \tag{4.1}$$

In addition, given a point $p \in \mathcal{O}$, we let G_p be its stabilizer, and N_p be the fiber of N at p. If we view G as a principal G_p-bundle over G/G_p, we get an associated vector bundle $G \times_{G_p} N_p$ over G/G_p. Since $G/G_p \cong \mathcal{O}$, we have [44, Theorem 1.25]:

Lemma 4.1.2. *There is a G-equivariant bundle isomorphism $G \times_{G_p} N_p \cong N$.*

With Lemma 4.1.1 and Lemma 4.1.2, we identify N with $G \times_{G_p} N_p$. Now, we construct a Riemannian metric on $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $G \times_{G_p} N_p$ as in [44, Corollary 1.27] in the following steps:

(1) We let $\mathbb{1}$ be the identity of G. With respect to the adjoint representation of G on $\mathfrak{g}$, we assign $\mathfrak{g}$ a G-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

(2) Under $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, we let U be the orthogonal complement of the Lie algebra of G_p and choose an orthonormal basis $e_1, \cdots, e_n$ of U.

(3) We let $v_1, \cdots, v_i, v_{i+1}, \cdots, v_{m-n}$ be an orthonormal basis of N_p with respect to the original bundle metric $\langle \cdot, \cdot \rangle$ such that $v_1, \cdots, v_i$ is the image of $P|_{N_p}$.

(4) At each equivalence class $[g, v] \in G \times_{G_p} N_p$, we let the following tangent vectors

$$w_k[g, v] = \frac{d}{dt}\Big|_{t=0} [g \exp(te_k), v], \;\; z_\ell[g, v] = \frac{d}{dt}\Big|_{t=0} [g, v + tv_\ell] \;\; (1 \leqslant k \leqslant n, 1 \leqslant \ell \leqslant m - n).$$

be an orthonormal basis of the tangent space $T_{[g,v]}(G \times_{G_p} N_p)$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Moreover, we extend this $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ to the whole manifold M by using G-invariant bump functions so that the Riemannian metric equals $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ inside $N(4\varepsilon)$, while it is still $\langle \cdot, \cdot \rangle$ outside $N(6\varepsilon)$.

Proposition 4.1.3. *This $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is well-defined and G-invariant.*

Proof. We know that $[gh, h^{-1}v] = [g, v]$ when $h \in G_p$. Then, we have

$$w_k[g, v] = \frac{d}{dt}\Big|_{t=0} [g \exp(t\mathrm{Ad}_h e_k), v], \;\; z_\ell[g, v] = \frac{d}{dt}\Big|_{t=0} [g, v + thv_\ell] \;\; (1 \leqslant k \leqslant n, 1 \leqslant \ell \leqslant m - n).$$

Since both $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ are G-invariant, the choice of representatives for $[g, v]$ does not affect the metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. $\square$

Remark 4.1.4. As mentioned in [5, Section 3], in the Morse-Bott case, the transversality is not as generic as in the Morse case. Examples can be found in [37, Section 2]. However, since from the very beginning, we already assume that the original $\langle\cdot,\cdot\rangle$ satisfies the transversality conditions in (a2), by [4, Appendix B] and the continuous dependence on parameters of ODEs, the new metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ still satisfies (a2) when $\varepsilon > 0$ is sufficiently small.

Next, we describe the local frame of the adjusted metric.

Notation 4.1.5. We will use the new metric on M in the rest of this paper. For simplicity, we will still use the notation $\langle\cdot,\cdot\rangle$.

For any $g \in G$, we construct a local trivialization of N around $[g, 0] \in G \times_{G_p} \{0\} = \mathcal{O}$. By [29, Theorem 9.3.7(iii)], there is an open disk $\widetilde{U} \subset U$ centering at 0 such that

$$\begin{aligned} \phi : \widetilde{U} \times G_p &\to G \\ (e, h) &\to \exp(e)h \end{aligned}$$

is a diffeomorphism onto an open submanifold of G. Given any $g \in G$, we get a local trivialization

$$\begin{aligned} \varphi_g : \widetilde{U} \times N_p &\to N \\ (x_1e_1 + \cdots + x_ne_n, y_1v_1 \cdots + y_{m-n}v_{m-n}) &\mapsto [g\exp(x_1e_1 + \cdots + x_ne_n), y_1v_1 \cdots + y_{m-n}v_{m-n}] \end{aligned}$$

and also a coordinate system $(\mathbf{x}, \mathbf{y}) = (x_1, \cdots, x_n, y_1, \cdots, y_{m-n})$ of N. By the compactness of G, finitely many φ_g's cover the whole N.

For $\mathbb{1} \in G$ the identity element, we write $\varphi_{\mathbb{1}}$ as φ for convenience.

Since G can be nonabelian, w_k $(1 \leqslant k \leqslant n)$ and z_ℓ $(1 \leqslant \ell \leqslant m-n)$ do not define a global frame of TN. However, we can still define a local frame. For example, on the local chart φ of N around $[\mathbb{1}, 0]$, with the two projections

$$\pi_{\widetilde{U}} : \widetilde{U} \times G_p \to \widetilde{U} \quad \text{and} \quad \pi_{G_p} : \widetilde{U} \times G_p \to G_p,$$

we write $u_k(t) = \exp(x_1 e_1 + \cdots + x_n e_n)\exp(te_k)$ and get a smooth local orthonormal frame

$$\begin{aligned} w_k(\mathbf{x}, \mathbf{y}) &= \frac{d}{dt}\Big|_{t=0} \Big(\pi_{\widetilde{U}} \circ \phi^{-1}(u_k(t)), \big(\pi_{G_p} \circ \phi^{-1}(u_k(t))\big) \cdot (y_1 v_1 + \cdots y_{m-n} v_{m-n})\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqslant k \leqslant n) \\ z_\ell(\mathbf{x}, \mathbf{y}) &= \frac{d}{dt}\Big|_{t=0} (x_1 e_1 + \cdots + x_n e_n, y_1 v_1 + \cdots + (y_\ell + t) v_l + \cdots + y_{m-n} v_{m-n}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqslant \ell \leqslant m-n). \end{aligned} \tag{4.2}$$

These w_k's and z_ℓ's are orthonormal with respect to the new metric. Without loss of generality, we assume that they are oriented on N. Things are similar on any other φ_g $(g \in G)$.

Now, on the chart φ, we use the local coordinate system $(\mathbf{x}, \mathbf{y})$ and write

$$w_k(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} a_{kj}(\mathbf{x}) \frac{\partial}{\partial x_j} + \sum_{s=1}^{m-n} \left(\sum_{r=1}^{m-n} b_{ks}^r(\mathbf{x}) y_r \right) \frac{\partial}{\partial y_s} \quad \text{and} \quad z_\ell(\mathbf{x}, \mathbf{y}) = \frac{\partial}{\partial y_\ell}.$$

Here, $a_{kj}(\mathbf{x})$ and $b_{ks}^r(\mathbf{x})$ are functions in terms of $\mathbf{x}$.

Lemma 4.1.6. *The functions $b_{ks}^r(\mathbf{x})$ satisfy that*

$$b_{ks}^r(\mathbf{x}) = -b_{kr}^s(\mathbf{x}),$$

and that $b_{ks}^r(\mathbf{x}) = 0$ when $1 \leqslant s \leqslant i$ and $i+1 \leqslant t \leqslant m-n$.

Proof. We know that the Lie algebra of the orthogonal group consists of skew-symmetric matrices. Since in (4.2), $\pi_{G_p} \circ \phi^{-1}(u_k(t))$ acts isometrically, we then have

$$b^r_{ks}(\mathbf{x}) = -b^s_{kr}(\mathbf{x}).$$

The second conclusion is because the space N_p^- (resp. N_p^+) generated by $v_1, \cdots, v_i$ (resp. by $v_{i+1}, \cdots, v_{m-n}$) is invariant under the G_p-action on N_p. $\square$

Finally, we study two connections induced by the adjusted metric.

Let ∇^{TN} be the associated Levi-Civita connection on TN. Using the pullback induced by the inclusion $\mathcal{O} \hookrightarrow N$, we get a connection $\nabla^{TN|_{\mathcal{O}}}$ on the vector bundle $TN|_{\mathcal{O}}$. Then, by the projection $\pi : N \to \mathcal{O}$ and the bundle isomorphism

$$\begin{aligned} \iota : TN &\to \pi^*(TN|_{\mathcal{O}}) \\ w_k[g, v] &\mapsto (w_k[g, 0], [g, v]) \\ z_\ell[g, v] &\mapsto (z_\ell[g, 0], [g, v]) \end{aligned}$$

we get a connection ∇ on TN defined by

$$\iota \circ \nabla_X Y = \left(\pi^* \nabla^{TN|_{\mathcal{O}}}\right)_X (\iota \circ Y)$$

for any $X, Y \in \mathfrak{X}(N)$. Here, the notations of elements in the pullback bundle follow

$$\pi^*(TN|_{\mathcal{O}}) \subseteq TN|_{\mathcal{O}} \times \mathcal{O}.$$

Let a^{rs} be the function such that the matrix $[a^{rs}]_{1 \leqslant r,s \leqslant n}$ is the inverse of $[a_{rs}]_{1 \leqslant r,s \leqslant n}$. As in [41, Lemma 2.2], we can compute the difference between ∇^{TN} and ∇. For simplicity, we define the following notations

$$
\begin{aligned}
\mathcal{A}_{jk\ell}(\mathbf{x}) &:= \frac{1}{2}\sum_{r,s=1}^{n}\left(a_{jr}(\mathbf{x})\frac{\partial a_{\ell s}(\mathbf{x})}{\partial x_r} - a_{\ell r}(\mathbf{x})\frac{\partial a_{js}(\mathbf{x})}{\partial x_r}\right)a^{sk}(\mathbf{x}),\\
\mathcal{B}^{\ell t}_{jk}(\mathbf{x}) &:= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{m-n}\left(a_{jr}(\mathbf{x})\frac{\partial b^t_{k\ell}(\mathbf{x})}{\partial x_r} - a_{kr}(\mathbf{x})\frac{\partial b^t_{j\ell}(\mathbf{x})}{\partial x_r} + b^t_{js}(\mathbf{x})b^s_{k\ell}(\mathbf{x}) - b^t_{ks}(\mathbf{x})b^s_{j\ell}(\mathbf{x})\right).
\end{aligned}
$$

The precise computation is as follows.

Lemma 4.1.7. *Using the local frame $w_1, \cdots, w_n, z_1, \cdots, z_{m-n}$, we find*

$$
\begin{aligned}
\nabla^{TN}_{w_j} w_k &= \sum_{\ell=1}^{n}\left(-\mathcal{A}_{jk\ell}(\mathbf{x}) - \mathcal{A}_{k\ell j}(\mathbf{x}) + \mathcal{A}_{\ell jk}(\mathbf{x})\right)w_\ell + \sum_{\ell,t=1}^{m-n}\left(\mathcal{B}^{\ell t}_{jk}(\mathbf{x}) + \sum_{q=1}^{n}\mathcal{A}_{kqj}(\mathbf{x})b^t_{q\ell}(\mathbf{x})\right)y_t z_\ell\\
&= \nabla_{w_j} w_k + \sum_{\ell,t=1}^{m-n}\left(\mathcal{B}^{\ell t}_{jk}(\mathbf{x}) + \sum_{q=1}^{n}\mathcal{A}_{kqj}(\mathbf{x})b^t_{q\ell}(\mathbf{x})\right)y_t z_\ell\ ,\\
\nabla^{TN}_{w_j} z_k &= \sum_{\ell=1}^{m-n} b^\ell_{jk}(\mathbf{x})z_\ell + \sum_{\ell=1}^{n}\sum_{t=1}^{m-n}\left(\mathcal{B}^{kt}_{\ell j}(\mathbf{x}) + \sum_{q=1}^{n}\mathcal{A}_{jq\ell}(\mathbf{x})b^t_{qk}(\mathbf{x})\right)y_t w_\ell\\
&= \nabla_{w_j} z_\ell + \sum_{\ell=1}^{n}\sum_{t=1}^{m-n}\left(\mathcal{B}^{kt}_{\ell j}(\mathbf{x}) + \sum_{q=1}^{n}\mathcal{A}_{jq\ell}(\mathbf{x})b^t_{qk}(\mathbf{x})\right)y_t w_\ell\ ,\\
\nabla^{TN}_{z_j} w_k &= 0 + \sum_{\ell=1}^{n}\sum_{t=1}^{m-n}\mathcal{B}^{jt}_{\ell k}(\mathbf{x})y_t w_\ell = \nabla_{z_j} w_k + \sum_{\ell=1}^{n}\sum_{t=1}^{m-n}\mathcal{B}^{jt}_{\ell k}(\mathbf{x})y_t w_\ell\ ,\\
\nabla^{TN}_{z_j} z_k &= \nabla_{z_j} z_k = 0
\end{aligned}
$$

on the local chart of N given by φ.

Proof. This is a direct computation using Koszul's formula [39, Corollary 5.11(a)]. $\square$

4.2 Spectral gap of the Witten Laplacian

In this section, we figure out the spectral gap of D_T when T is sufficiently large. This spectral gap distinguishes between horizontal and vertical eigenvalues.

First of all, we clarify the behavior of D_T on G-invariant forms around each critical orbit.

Definition 4.2.1. On $\Omega^*(N)$, we call the operators

$$D^H = \sum_{k=1}^{n} c(w_k)\nabla_{w_k}$$

and

$$D^V = \sum_{\ell=1}^{m-n} c(z_\ell)\nabla_{z_\ell}$$

the horizontal operator and the vertical operator respectively.

We see that D^H and D^V are independent of local orthonormal frames. Thus, they are globally defined on N. As in [11, (9.69)], their supercommutator vanishes on $\Omega^*(N)$.

Proposition 4.2.2. *For any $\eta \in \Omega^*(N)$, we have $\left[D^H, D^V\right]\eta = 0$.*

Proof. The proof follows from Lemma 4.1.6, Lemma 4.1.7, and the fact that

$$c(Y)\nabla_X = \nabla_X c(Y) + c(\nabla_X Y)$$

for any $X, Y \in \mathfrak{X}(N)$. $\square$

Moreover, we notice that ∇ is a metric connection. Let $\star$ be the Hodge star, and $dV = \star 1$ be the volume form on N. As in [11, VIII(h)] and [41, Section 2.4], D^H and D^V are formal self-adjoint operators.

Proposition 4.2.3. *The operators D^H and D^V are formally self-adjoint on $\Omega^*(N)$.*

Proof. For any $\omega, \eta \in \Omega^*(N)$ with one of them being compactly supported,

$$\begin{aligned}
&\int_N \langle D^H\omega, \eta\rangle dV \\
&= \int_N \sum_{k=1}^n \langle \omega, \nabla_{w_k}(c(w_k)\eta)\rangle dV - \int_N \sum_{k=1}^n w_k\langle \omega, c(w_k)\eta\rangle dV \\
&= \int_N \sum_{k=1}^n \langle \omega, c(w_k)\nabla_{w_k}\eta\rangle dV + \int_N \sum_{k=1}^n \langle \omega, c\left(\nabla_{w_k}w_k\right)\eta\rangle dV - \int_N \sum_{k=1}^n d\langle \omega, c(w_k)\eta\rangle \wedge \star w_k^*
\end{aligned}$$

(Here, w_k^* is the dual 1-form $\langle w_k, \cdot\rangle$)

$$\begin{aligned}
&= \int_N \sum_{k=1}^n \langle \omega, D^H\eta\rangle dV + \int_N \sum_{k=1}^n \langle \omega, c\left(\nabla_{w_k}w_k\right)\eta\rangle dV \\
&\quad - \int_N d\left(\sum_{k=1}^n \langle \omega, c(w_k)\eta\rangle \wedge \star w_k^*\right) - \int_N \sum_{k=1}^n \langle \omega, c(w_k)\eta\rangle \wedge \star d^* w_k^* \\
&= \int_N \sum_{k=1}^n \langle \omega, D^H\eta\rangle dV + \int_N \sum_{k=1}^n \langle \omega, c\left(\nabla_{w_k}w_k\right)\eta\rangle dV \\
&\quad + \int_N \sum_{k=1}^n \sum_{\ell=1}^n \left\langle \langle \omega, c(w_k)\eta\rangle, w_\ell \lrcorner \nabla^{TN}_{w_\ell} w_k^*\right\rangle dV + \int_N \sum_{k=1}^n \sum_{r=1}^{m-n} \left\langle \langle \omega, c(w_k)\eta\rangle, z_r \lrcorner \nabla^{TN}_{z_r} w_k^*\right\rangle dV \\
&= \int_N \sum_{k=1}^n \langle \omega, D^H\eta\rangle dV + \int_N \sum_{k=1}^n \langle \omega, c\left(\nabla_{w_k}w_k\right)\eta\rangle dV - \int_N \sum_{\ell=1}^n \sum_{k=1}^n \langle \langle \omega, c(w_k)\eta\rangle, w_k^*\left(\nabla_{w_\ell}w_\ell\right)\rangle dV \\
&= \int_N \sum_{k=1}^n \langle \omega, D^H\eta\rangle dV.
\end{aligned}$$

The second to last equal sign is by Lemma 4.1.7. The proof for D^V is similar. □

Since both ∇^{TN} and ∇ are G-invariant connections (See [9, (1.10)]), we can also view D_T, D^H, and D^V as operators on $\Omega^*(N)^G$. Recall that by [56, (4.16)], we have

$$D_T = \sum_{k=1}^n c(w_k)\nabla^{TN}_{w_k} + \sum_{\ell=1}^{m-n} c(z_\ell)\nabla^{TN}_{z_\ell} + T\hat{c}(df) \tag{4.3}$$

on the local chart φ. Now, we hope to see how far away D_T is from $D^H + D^V + T\hat{c}(df)$.

Before we study the difference between D_T and $D^H + D^V + T\hat{c}(df)$, we give two lemmata about the G-invariant forms on N. Recall that $n = \dim \mathcal{O}$, and i is the Morse index of $\mathcal{O}$.

Notation 4.2.4. We let $N_p^- \subseteq N_p$ (resp. $N^- \subseteq N$) be the subspace (resp. subbundle) generated by $v_1, \cdots, v_i$ (resp. equal to $G \times_{G_p} N_p^-$). Meanwhile, we denote by $o(N^-)$ the orientation bundle of N^-. It identifies with the orientation bundle $\mathcal{E}_i|_{\mathcal{O}}$.

Lemma 4.2.5. *For any $\omega \in \Omega^j(\mathcal{O}, o(N^-))^G$, on the local chart $\varphi : \widetilde{V} \times \{0\} \to \mathcal{O}$, we have*

$$\omega = \sum_{k_1 < \cdots < k_j} c_{k_1 \cdots k_j} w_{k_1}^* \wedge \cdots \wedge w_{k_j}^* \otimes e^-,$$

where each $c_{k_1 \cdots k_j}$ is a constant, $w_1^*, \cdots, w_n^*$ are restricted on $\mathcal{O}$, and e^- is the "one-section" of $o(N^-)$ with respect to the local trivialization $\varphi : \widetilde{V} \times N_p^- \to N^-$.

Proof. Locally, we always have

$$\omega = \sum_{k_1 < \cdots < k_j} c_{k_1 \cdots k_j} w_{k_1}^* \wedge \cdots \wedge w_{k_j}^* \otimes e^-$$

and only need to show $c_{k_1 \cdots k_j}(\mathbf{x}) = c_{k_1 \cdots k_j}(0)$.

In fact, for any $u = x_1 e_1 + \cdots + x_n e_n \in \widetilde{V}$,

$$\begin{aligned}
&\sum_{k_1 < \cdots < k_j} c_{k_1 \cdots k_j}\left([\exp(u), 0]\right) w_{k_1 [\exp(u),0]}^* \wedge \cdots \wedge w_{k_j [\exp(u),0]}^* \otimes (e^-)_{[\exp(u),0]} \\
&= \omega_{[\exp(u),0]} \\
&= (\exp(u) \cdot \omega)_{[\exp(u),0]} \quad \text{(This is because } \omega \text{ is } G\text{-invariant)} \\
&= \exp(u) \cdot \left(\omega_{[\mathbb{1},0]}\right)
\end{aligned}$$

$$= \sum_{k_1<\cdots<k_j} c_{k_1\cdots k_j}\left([\mathbb{1},0]\right)\exp(u)\cdot\Big(w^*_{k_1[\mathbb{1},0]}\wedge\cdots\wedge w^*_{k_j[\mathbb{1},0]}\otimes(e^-)_{[\mathbb{1},0]}\Big)$$

$$= \sum_{k_1<\cdots<k_j} c_{k_1\cdots k_j}\left([\mathbb{1},0]\right)w^*_{k_1[\exp(u),0]}\wedge\cdots\wedge w^*_{k_j[\exp(u),0]}\otimes(e^-)_{[\exp(u),0]}.$$

Thus, each $c_{k_1\cdots k_j}$ is a constant. $\square$

Lemma 4.2.5 shows that $\dim\Omega^j(\mathcal{O},o(N^-))^G\leqslant\binom{n}{j}$.

Corollary 4.2.6. *The space $C^*(M,f)^G$ is finite dimensional.*

By a similar procedure, we get a similar lemma on N.

Lemma 4.2.7. *For any $\eta\in\Omega^*(N)^G$, on the local chart $\varphi:\widetilde{V}\times N_p\to N$, it has the form*

$$\eta=\sum_{\substack{k_1<\cdots<k_r\\ \ell_1<\cdots<\ell_t}} C^{\ell_1\cdots\ell_t}_{k_1\cdots k_r}(\mathbf{y})w^*_{k_1}\wedge\cdots\wedge w^*_{k_r}\wedge z^*_{\ell_1}\wedge\cdots\wedge z^*_{\ell_t},$$

where each $C^{\ell_1\cdots\ell_t}_{k_1\cdots k_r}(\mathbf{y})$ is a function in terms of only the vertical coordinate $\mathbf{y}$.

Notation 4.2.8. For any $\eta,\eta'\in\Omega^*(N)$ (resp. on $\Omega^*(M)$), we let $|\eta|\coloneqq\langle\eta,\eta\rangle^{1/2}$, $\|\eta\|$ be the L^2-norm of η, and (η,η') be the integral of $\eta\wedge\star\eta'$ on N (resp. on M). In particular, we write $|\mathbf{y}|^2=y_1^2+\cdots+y_{m-n}^2$.

In addition, for the calculation on N, we need an extension of f to resolve the issue that Lemma 4.1.1 holds true only in a bounded neighborhood.

Notation 4.2.9. Without loss of generality, when studying D_T on $N(4\varepsilon)$, we view f as a function given by (4.1) on the whole N if necessary. In this way, we extend D_T onto $\Omega^*(N)$.

As in [11, Theorem 8.18] and [41, Theorem 2.5], on $\Omega^*(N)^G$, we can now write D_T into the sum "horizontal" + "vertical" + "tail" and describe how the "tail" acts on forms.

Proposition 4.2.10. *We let R be the operator*

$$D_T - D^H - D^V - T\hat{c}(df).$$

Then, we have a constant $\Gamma_N > 0$ such that

$$|R\eta| \leqslant \Gamma_N \cdot |\mathbf{y}| \cdot |\eta| \tag{4.4}$$

for all $\eta \in \Omega^*(N)^G$. Also, R can be viewed as a matrix of order $O(|\mathbf{y}|)$ on the local chart φ.

Proof. By (4.3), we find

$$R = \sum_{k=1}^{n} c(w_k)\left(\nabla^{TN}_{w_k} - \nabla_{w_k}\right) + \sum_{\ell=1}^{m-n} c(z_\ell)\left(\nabla^{TN}_{z_\ell} - \nabla_{z_\ell}\right).$$

The estimate (4.4) follows from Lemma 4.1.7 and Lemma 4.2.7. The matrix form of R on the chart φ is by letting $\mathbf{x} = 0$ in Lemma 4.1.7 according to Lemma 4.2.7. □

Now, we construct a space generated by "approximate" eigenforms of D_T^2.

Let ρ be a G-invariant bump function on M such that $\rho \equiv 1$ on each $N(\varepsilon)$, and $\rho \equiv 0$ outside the union of all $N(2\varepsilon)$. With an unambiguous abuse of notations, we define

$$\begin{aligned} J_T : \Omega^*(\mathcal{O}, o(N^-))^G &\to \Omega^{*+i}(M)^G \\ \omega &\mapsto \rho \cdot \omega \wedge \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) z_1^* \wedge \cdots \wedge z_i^*. \end{aligned} \tag{4.5}$$

The image of J_T is globally defined because of the local coefficient $o(N^-)$.

Notation 4.2.11. As in [56, (5.18)], we let E_T be the space spanned by all such $J_T\omega$ $\big(\omega \in C^*(M,f)^G\big)$, and $E_T^\perp$ be the L^2-orthogonal complement of E_T in $\Omega^*(M)^G$. Meanwhile, we let p_T and $p_T^\perp$ be the orthogonal projection from $\Omega^*(M)^G$ to E_T and $E_T^\perp$ respectively.

We have $\nabla(z_1^* \wedge \cdots \wedge z_i^*) = 0$ by Lemma 4.1.6 and Lemma 4.1.7. Then, similar to [41, (2.50)], we get a Dirac type operator (which is twisted by the orientation bundle $o(N^-)$)

$$d + d^* := J_T^{-1} \circ D^H \circ J_T : \Omega^j(\mathcal{O}, o(N^-))^G \to \Omega^{j\pm 1}(\mathcal{O}, o(N^-))^G \tag{4.6}$$

on each critical orbit $\mathcal{O}$. In addition, using the local chart φ and the notations in 4.2.5, we get an inner product on $C^*(M,f)$ given by the magnitude

$$|\omega| := \sum_{k_1 < \cdots < k_j} c_{k_1 \cdots k_j}^2, \ \forall\, \omega \in \Omega^j(\mathcal{O}, o(N^-)). \tag{4.7}$$

Proposition 4.2.12. *On $C^*(M,f)^G$, $d + d^*$ is self-adjoint under the inner product (4.7).*

Proof. Notice that D^H maps E_T into E_T. Thus, by Proposition 4.2.3, D^H is self-adjoint on E_T. Then, by (4.6), $d + d^*$ is self-adjoint on $C^*(M,f)^G$. $\square$

Recall that α_0 is the spectral radius of $(d + d^*)^2$ on $C^*(M,f)^G$.

Proposition 4.2.13. *There exist $C_0, C_1, T_0 > 0$ and $\Gamma = \max_N\{\Gamma_N\}$ such that*

$$\|D_T\eta\| \leqslant \left(\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon\right)\|\eta\|$$

for all $T > T_0$ and all $\eta \in E_T$.

Proof. For any $\omega \in \Omega^j(\mathcal{O}, o(N^-))^G$, applying $D_T = D^H + D^V + T\hat{c}(df) + R$ to it, we find constants $C_0, C_1, T_0 > 0$ such that when $T > T_0$,

$$\begin{aligned}
&\|D_T J_T \omega\| \\
\leqslant\ & \left\|((d+d^*)\omega) \wedge \rho \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) z_1^* \wedge \cdots \wedge z_i^*\right\| + \left\|c(d\rho)\omega \wedge \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) z_1^* \wedge \cdots \wedge z_i^*\right\| \\
&+ \|R J_T \omega\| \\
\leqslant\ & \sqrt{\alpha_0}\|J_T\omega\| + C_0 e^{-C_1 T}\|J_T\omega\| + \Gamma_N \varepsilon \|J_T\omega\|.
\end{aligned}$$

The last inequality is by Proposition 4.2.12 and (4.4). □

Remark 4.2.14. The Γ in Proposition 4.2.13 is independent of ε, meaning that whenever $\alpha > \alpha_0$, we can choose small ε and assume that $\sqrt{\alpha_0} < \sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon < \sqrt{\alpha}$.

With the local expression given by Lemma 4.2.7 for every $\eta \in \Omega^*(N)^G$, we define

$$\mathcal{H}_T \coloneqq -\sum_{j=1}^{m-n} \frac{\partial^2}{\partial y_j^2} - (m-n)T + T^2|\mathbf{y}|^2 \quad \text{and} \quad \mathcal{L}_T \coloneqq 2T\sum_{l=1}^{i} z_l \lrcorner z_l^* \wedge + 2T \sum_{l=i+1}^{m-n} z_l^* \wedge z_l \lrcorner.$$

According to [56, Proposition 4.6] and Lemma 4.2.7, we find on the local chart φ that

$$\begin{aligned}
&(D^V + T\hat{c}(df))^2\eta \\
=\ & \sum_{\substack{k_1<\cdots<k_r \\ \ell_1<\cdots<\ell_t}} \mathcal{H}_T\left(C_{k_1\cdots k_r}^{\ell_1\cdots\ell_t}(\mathbf{y})\right) w_{k_1}^* \wedge \cdots \wedge w_{k_r}^* \wedge z_{\ell_1}^* \wedge \cdots \wedge z_{\ell_t}^* \\
&+ \sum_{\substack{k_1<\cdots<k_r \\ \ell_1<\cdots<\ell_t}} C_{k_1\cdots k_r}^{\ell_1\cdots\ell_t}(\mathbf{y}) \mathcal{L}_T\left(w_{k_1}^* \wedge \cdots \wedge w_{k_r}^* \wedge z_{\ell_1}^* \wedge \cdots \wedge z_{\ell_t}^*\right). \qquad (4.8)
\end{aligned}$$

Lemma 4.2.15. *When restricted to the space of L^2-sections of $\Lambda^* T^* N|_{N_p}$, the kernel of the positive operator $\mathcal{H}_T + \mathcal{L}_T$ is spanned by*

$$\exp\left(-\frac{T}{2}\left|\mathbf{y}\right|^2\right) w_{k_1}^* \wedge \cdots \wedge w_{k_j}^* \wedge z_1^* \wedge \cdots \wedge z_i^* \quad (1 \leqslant k_1 < \cdots < k_j \leqslant n),$$

where each w_k^* is restricted to N_p. Moreover, its first nonzero eigenvalue is $\geqslant 2T$.

Proof. The $\mathcal{H}_T$ is the harmonic oscillator on the space of L^2-functions. By [52, Section 8.6 (6.12)], we see that on L^2-functions, the kernel of $\mathcal{H}_T$ is generated by $\exp\left(-\frac{T}{2}\left|\mathbf{y}\right|^2\right)$, and all nonzero eigenvalues are $\geqslant 2T$.

For $\mathcal{L}_T$, on the space of frames with constant coefficients, its kernel is generated by

$$w_{k_1}^* \wedge \cdots \wedge w_{k_j}^* \wedge z_1^* \wedge \cdots \wedge z_i^* \quad (1 \leqslant k_1 < \cdots < k_j \leqslant n),$$

while all nonzero eigenvalues are $\geqslant 2T$. □

Now, on the space $\Omega^*(N)^G$, we let p'_T be the orthogonal projection from $\Omega^*(N)^G$ to the subspace E'_T spanned by all

$$\omega \wedge \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) z_1^* \wedge \cdots \wedge z_i^* \,, \forall\, \omega \in \Omega^*(\mathcal{O}, o(N^-))^G.$$

This E'_T is exactly the kernel of $\mathcal{H}_T + \mathcal{L}_T$ on $\Omega^*(N)^G$. In addition, on the chart φ, with the notations from Lemma 4.2.7, we find

$$\begin{aligned} p'_T\eta = &\left(\frac{T}{\pi}\right)^{\frac{m-n}{2}} \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) \cdot \\ &\sum_{k_1 < \cdots < k_r} \left(\int_{\mathbb{R}^{m-n}} C_{k_1 \cdots k_r}^{1 \cdots i}(\mathbf{y}) \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) |d\mathbf{y}|\right) w_{k_1}^* \wedge \cdots \wedge w_{k_r}^* \wedge z_1^* \wedge \cdots \wedge z_i^*, \end{aligned} \tag{4.9}$$

meaning that p'_T is largely determined by the projection from $L^2(\mathbb{R}^{m-n})$ to the kernel of the harmonic oscillator (compare with [11, (8.91)]).

For the convenience of estimating D_T on $E_T^\perp$, we present an auxiliary estimate of p'_T.

Lemma 4.2.16. *There is a constant $C' > 0$ such that when T is sufficiently large,*

$$\|p'_T\eta\| \leqslant C'T^{-1/4}\|\eta\| \tag{4.10}$$

for all $\eta \in \Omega^*(N)^G \cap E_T^\perp$ with $\operatorname{supp}(\eta) \subseteq N(4\varepsilon)$.

Proof. For any such η, as in [11, (9.80)], we can rewrite (4.9) into

$$\begin{aligned} & p'_T\eta \\ = & \left(\frac{T}{\pi}\right)^{\frac{m-n}{2}} \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) \cdot \\ & \sum_{k_1<\cdots<k_r} \left(\int_{\mathbb{R}^{m-n}} (1-\rho(\mathbf{y}))C^{1\cdots i}_{k_1\cdots k_r}(\mathbf{y}) \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) |d\mathbf{y}|\right) w^*_{k_1} \wedge \cdots \wedge w^*_{k_r} \wedge z^*_1 \wedge \cdots \wedge z^*_i. \end{aligned}$$

Then, (4.10) is because $\left|(1-\rho(\mathbf{y}))C^{1\cdots i}_{k_1\cdots k_r}(\mathbf{y}) \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right)\right| \leqslant \left|C^{1\cdots i}_{k_1\cdots k_r}(\mathbf{y}) \exp\left(-T\varepsilon^2/2\right)\right|$. □

Immediately, we state the estimate of D_T on $E_T^\perp$ as follows.

Proposition 4.2.17. *There exist $C_2 > 0$ and $T_1 > 0$ such that*

$$\|D_T\eta\| \geqslant C_2\sqrt{T}\|\eta\|$$

for all $T > T_1$ and all $\eta \in E_T^\perp$.

Proof. We prove it in three cases as [11, Theorem 9.11] and [56, Proposition 4.12].

Case I. We first assume that η is supported in one $N(4\varepsilon)$. By (4.4) and Proposition 4.2.2,

$$\begin{aligned}\|D_T\eta\|^2 &\geqslant \frac{1}{2}\left\|D^V\eta + T\hat{c}(df)\eta + D^H\eta\right\|^2 - \|R\eta\|^2 \\ &\geqslant \frac{1}{2}\left\|D^V\eta + T\hat{c}(df)\eta\right\|^2 + \frac{1}{2}\left\|D^H\eta\right\|^2 - \Gamma_N^2\cdot(4\varepsilon)^2\cdot\|\eta\|^2. \end{aligned} \tag{4.11}$$

According to Lemma 4.2.7, we write η into an orthogonal decomposition $\eta_1+\eta_2$, with η_1 carrying $z_1^*\wedge\cdots\wedge z_i^*$, while η_2 carrying other combinations of z_j^*'s. Recall the inner product $(\cdot,\cdot)$ associated with the L^2-norm $\|\cdot\|$. By Lemma 4.2.15,

$$\begin{aligned}(\mathcal{H}_T\eta_2,\eta_1) = (\mathcal{H}_T\eta_1,\eta_2) = (\mathcal{L}_T\eta_2,\eta_1) = (\mathcal{L}_T\eta_1,\eta_2) = (\mathcal{L}_T\eta_1,\eta_1) = 0,\\ (\mathcal{H}_T\eta_2,\eta_2)\geqslant 0.\end{aligned} \tag{4.12}$$

Thus, when T is sufficiently large,

$$\begin{aligned} & \|D^V\eta + T\hat{c}(df)\eta\|^2 \\ \text{(by (4.8))}\quad & = (\mathcal{H}_T\eta,\eta) + (\mathcal{L}_T\eta,\eta) \\ \text{(by (4.12))}\quad & \geqslant (\mathcal{H}_T\eta_1,\eta_1) + (\mathcal{L}_T\eta_2,\eta_2) \\ \text{(by Lemma 4.2.15)}\quad & \geqslant 2T\|\eta_1 - p_T'\eta_1\|^2 + 2T\,\|\eta_2\|^2 \\ \text{(by (4.9) and (4.10))}\quad & \geqslant 2T\|\eta_1\|^2 - 2C'T^{1/2}\|\eta_1\|^2 + 2T\|\eta_2\|^2 \\ & \geqslant 2T\|\eta\|^2 - 2C'T^{1/2}\|\eta\|^2. \end{aligned} \tag{4.13}$$

The same estimate holds when η is supported in the union of all $N(4\varepsilon)$'s.

Case II. Next, we assume that η is supported outside the union of all $N(2\varepsilon)$'s. Let $D := d + d^*$ on M. Then, there is a constant $C'' > 0$ greater than the norm of the supercommutator

$[D, \hat{c}(df)]$ on $\Omega^*(M)^G$. In addition, since there is another constant $C''' > 0$ such that

$$|\nabla f|^2 \geqslant C'''$$

outside the union of all $N(2\varepsilon)$'s, we obtain

$$\|D_T \eta\|^2 \geqslant T\left([D, \hat{c}(df)]\eta, \eta\right) + C'''T^2\|\eta\|^2 \geqslant (C'''T^2 - C''T)\|\eta\|^2. \tag{4.14}$$

Case III. Finally, for general $\eta \in E_T^\perp$, we unify the above estimates in exactly the same way as [56, Proposition 4.12 Step 3]. Let $\tilde{\rho}$ be the function on M satisfying $\tilde{\rho}(\mathbf{y}) = \rho(\mathbf{y}/2)$ on each $N(4\varepsilon)$ and $\tilde{\rho} = 0$ outside the union of all $N(4\varepsilon)$'s. Then, we have

$$\begin{aligned}
\|D_T\eta\| &\geqslant \frac{1}{\sqrt{2}}\|(1-\tilde{\rho})D_T\eta\| + \frac{1}{\sqrt{2}}\|\tilde{\rho}D_T\eta\| \\
&= \frac{1}{\sqrt{2}}\|D_T((1-\tilde{\rho})\eta) + [D, \tilde{\rho}]\eta\| + \frac{1}{\sqrt{2}}\|D_T(\tilde{\rho}\eta) + [\tilde{\rho}, D]\eta\| \\
&\geqslant \frac{1}{\sqrt{2}}\|D_T((1-\tilde{\rho})\eta)\| + \frac{1}{\sqrt{2}}\|D_T(\tilde{\rho}\eta)\| - \frac{1}{\sqrt{2}}\|[D, \tilde{\rho}]\eta\| - \frac{1}{\sqrt{2}}\|[\tilde{\rho}, D]\eta\|.
\end{aligned}$$

Since $\tilde{\rho}\eta$ and $(1-\tilde{\rho})\eta$ are both in $E_T^\perp$, we apply (4.11) and (4.13) to $\|D_T(\tilde{\rho}\eta)\|$, (4.14) to $\|D_T((1-\tilde{\rho})\eta)\|$, and bounded operator norms to $\|[D, \tilde{\rho}]\eta\|$ and $\|[\tilde{\rho}, D]\eta\|$. ◻

Finally, using Propositions 4.2.13 and 4.2.17, we present the spectral gap of D_T^2. The proof is adapted from the more general [57, Lemma 5.3], effective also for essential spectrum.

Proposition 4.2.18. *When $T > 0$ is sufficiently large, all eigenvalues of D_T^2 on $\Omega^*(M)^G$ are in $\left[0, (\sqrt{\alpha_0} + C_0e^{-C_1T} + \Gamma\varepsilon)^2\right] \cup [C_2^2T, +\infty)$.*

Proof. Suppose that $\omega \in \Omega^*(M)^G$ and

$$\left(\sqrt{\alpha_0} + C_0e^{-C_1T} + \Gamma\varepsilon\right)^2 < \lambda < C_2^2T$$

satisfy $D_T^2\omega = \lambda\omega$. Then, we write $\omega = \omega_1 + \omega_2$ with $\omega_1 \in E_T$ and $\omega_2 \in E_T^{\perp}$. Since D_T is self-adjoint, by the above two results, we get

$$\begin{aligned} 0 = & \left\langle (D_T^2 - \lambda)\omega, \omega_1 - \omega_2 \right\rangle \\ = & \left\langle (D_T^2 - \lambda)\omega_1, \omega_1 \right\rangle - \left\langle (D_T^2 - \lambda)\omega_2, \omega_2 \right\rangle \\ \leqslant & \left((\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon)^2 - \lambda \right) \|\omega_1\|^2 + (\lambda - C_2^2 T)\|\omega_2\|^2 \leqslant 0. \end{aligned}$$

This shows that $\omega = 0$. □

Let $P_T(\alpha)$ be the orthogonal projection from $\Omega^*(M)^G$ onto $F_T^*(M, f, \alpha)^G$. Then, we have $P_T(\alpha) \circ J_T$ mapping E_T into the Witten instanton complex. The following corollary shows that J_T is "almost" the one-to-one correspondence.

Corollary 4.2.19. *When T is sufficiently large, $P_T(\alpha) \circ J_T$ is an isomorphism.*

Proof. We notice that for any $\omega \in C^*(M, f)^G$, $P_T(\alpha)J_T\omega$ is in the sum of the eigenspaces of D_T^2 associated with eigenvalues $\leqslant \alpha$. Thus,

$$\|D_T P_T(\alpha) J_T \omega\| \leqslant \sqrt{\alpha}\|P_T(\alpha) J_T \omega\|. \tag{4.15}$$

Also, we know $J_T\omega - P_T(\alpha)J_T\omega$ is in the sum of the eigenspaces of D_T^2 associated with eigenvalues $> \alpha$. For sufficiently small ε and sufficiently large T, we have $(\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon)^2 < \alpha < C_2^2 T$. By Proposition 4.2.18, $J_T\omega - P_T(\alpha)J_T\omega$ is in the sum of the eigenspaces of D_T^2 associated with eigenvalues $\geqslant C_2^2 T$. Therefore, for any $\omega \in C^*(M, f)^G$,

$$\begin{aligned} C_2\sqrt{T}\|J_T\omega - P_T(\alpha)J_T\omega\| &\leqslant \|D_T(J_T\omega - P_T(\alpha)J_T\omega)\| \\ &\leqslant \|D_T J_T \omega\| + \|D_T P_T(\alpha) J_T \omega\| \end{aligned}$$

$$
\begin{aligned}
\text{(by Proposition 4.2.13 and (4.15))} \quad &\leqslant (\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon)\|J_T\omega\| + \sqrt{\alpha}\|P_T(\alpha)J_T\omega\| \\
&\leqslant (\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon + \sqrt{\alpha})\|J_T\omega\|. \qquad (4.16)
\end{aligned}
$$

Thus, $P_T(\alpha)J_T\omega = 0$ only when $\omega = 0$. Thus, $P_T(\alpha) \circ J_T$ is injective.

Next, if we have some $\eta \in F_T^*(M, f, \alpha)^G$ such that η is L^2-orthogonal to the image of $P_T(\alpha) \circ J_T$, we show that $\eta = 0$ and therefore $P_T(\alpha) \circ J_T$ is surjective. In fact, by (4.16),

$$
\begin{aligned}
\|p_T^{\perp}\eta\| = \|\eta - p_T\eta\| \quad &\text{(See Notation 4.2.11 for the definitions of } p_T \text{ and } p_T^{\perp}.) \\
&\geqslant \|\eta - P_T(\alpha)p_T\eta\| - \|P_T(\alpha)p_T\eta - p_T\eta\| \\
&\geqslant \|\eta\| - C_2^{-1}T^{-1/2}(\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon + \sqrt{\alpha})\|p_T\eta\| \\
&\geqslant \|\eta\| - C_2^{-1}T^{-1/2}(\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon + \sqrt{\alpha})\|\eta\|. \qquad (4.17)
\end{aligned}
$$

The second to last line in (4.17) is also because η is L^2-orthogonal to $P_T(\alpha)p_T\eta$. Thus,

$$
\begin{aligned}
\sqrt{\alpha}\|\eta\| &\geqslant \|D_T\eta\| \\
&\geqslant \|D_T p_T^{\perp}\eta\| - \|D_T p_T\eta\| \\
&\geqslant C_2\sqrt{T}\|p_T^{\perp}\eta\| - (\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon)\|p_T\eta\| \\
&\geqslant C_2\sqrt{T}\|\eta\| - 2(\sqrt{\alpha_0} + C_0 e^{-C_1 T} + \Gamma\varepsilon)\|\eta\| - \sqrt{\alpha}\|\eta\|. \qquad (4.18)
\end{aligned}
$$

This (4.18) is true only when $\eta = 0$. $\square$

4.3 Analytic realization

In this section, we prove the chain isomorphism between the G-invariant Thom-Smale complex and the G-invariant Witten instanton complex. We will use the map $P_T(\alpha) \circ J_T$ studied in Corollary 4.2.19 as an auxiliary map.

Before we carry out the analysis related to $P_T(\alpha) \circ J_T$, we follow [11, Section X] to find a formal series. For convenience, we define a transformation

$$\begin{aligned} Q_T : \Omega^*(N)^G &\to \Omega^*(N)^G \\ \omega_{[g,v]} &\mapsto \omega_{\left[g,v/\sqrt{T}\right]}, \end{aligned}$$

where $[g,v] \in N \cong G \times_{G_p} N_p$. Then, we find that on $\Omega^*(N)^G$,

$$Q_T D_T Q_T^{-1} = D^H + \sqrt{T}\left(D^V + \hat{c}(df)\right) + \frac{1}{\sqrt{T}} R.$$

Correspondingly, we define

$$\begin{aligned} K : \Omega^*(\mathcal{O}, o(N^-))^G &\to \Omega^*(N) \\ \omega &\mapsto \omega \wedge \exp\left(-\frac{1}{2}|\mathbf{y}|^2\right) z_1^* \wedge \cdots \wedge z_i^*. \end{aligned}$$

Recall that $\alpha > \alpha_0$ in $F_T^*(M,f,\alpha)^G$. For each $K\omega$, the formal series is as follows.

Proposition 4.3.1. *If $\delta \in \mathbb{R}$ and $\omega \in \Omega^*(\mathcal{O}, o(N^-))^G$ satisfy $(d+d^*)\omega = \delta \cdot \omega$ for some $\delta \in \mathbb{R}$, then for any $\lambda \in \mathbb{C}$ satisfying $|\lambda| = \sqrt{\alpha}$, there is a formal power series*

$$Z(\lambda, T) = \sum_{k=0}^{\infty} \sigma_k(\lambda) T^{-k/2}$$

such that $Q_T(\lambda - D_T)Q_T^{-1}Z(\lambda,T) = K\omega$.

Proof. We adapt the proof of [11, (10.3)]. Applying $Q_T(\lambda - D_T)Q_T^{-1}$ to $Z(\lambda,T)$, we find

$$-(D^V + \hat{c}(df))\sigma_0(\lambda) = 0, \tag{4.19}$$

$$(\lambda - D^H)\sigma_0(\lambda) - (D^V + \hat{c}(df))\sigma_1(\lambda) = K\omega, \tag{4.20}$$

$$(\lambda - D^H)\sigma_{k+1}(\lambda) - (D^V + \hat{c}(df))\sigma_{k+2}(\lambda) - R\sigma_k(\lambda) = 0,\ \forall\, k \in \mathbb{Z}_{\geqslant 0}. \tag{4.21}$$

Let q (resp. $q^{\perp}$) be the orthogonal projection from $\Omega^*(N)^G$ onto $\mathrm{Im}K$ (resp. the orthogonal complement of $\mathrm{Im}K$). Then, by (4.19) and (4.20), we choose

$$\sigma_0(\lambda) = \frac{K\omega}{\lambda - \delta} \ \text{ and } \ q^{\perp}\sigma_1(\lambda) = 0$$

as the initial values.

To find all $\sigma_k(\lambda)$ by (4.21), first, we see that $\lambda - D^H$ is invertible on $\mathrm{Im}K$ since $|\lambda| > \sqrt{\alpha_0}$. Next, by Lemma 4.2.15, $(D^V + \hat{c}(df))^{-2}$ is well-defined on the space of L^2-sections of $\Lambda^*T^*N|_{N_p}$:

(1) On the kernel of $(D^V + \hat{c}(df))^2$ restricted to $\Lambda^*T^*N|_{N_p}$, we let $(D^V + \hat{c}(df))^{-2}$ be 0.

(2) On the orthogonal complement of this kernel, $(D^V + \hat{c}(df))^{-2}$ is given by the functional calculus [48, Proposition 5.30].

In particular, since $D^V + \hat{c}(df)$ is G-equivariant, $(D^V + \hat{c}(df))^{-2}$ is well-defined on $\Omega^*(N)^G$.

Following [11, (10.15)], we write (4.21) into two equations:

$$(\lambda - D^H)\left(q\sigma_{k+1}(\lambda)\right) - q \circ R(\sigma_k(\lambda)) = 0, \tag{4.22}$$

$$-(D^V + \hat{c}(df))(q^{\perp}\sigma_{k+2}(\lambda)) + (\lambda - D^H)(q^{\perp}\sigma_{k+1}(\lambda)) - q^{\perp} \circ R(\sigma_k(\lambda)) = 0. \tag{4.23}$$

Similar to [11, (10.17)], we apply $(\lambda - D^H)^{-1}$ to (4.22) and

$$(D^V + \hat{c}(df))^{-2} \circ (D^V + \hat{c}(df))$$

to (4.23). Using $\sigma_0(\lambda)$ and $q^\perp \sigma_1(\lambda)$, we obtain the formal series $Z(\lambda, T)$ for ω. $\square$

We now use the formal series $Z(\lambda, T)$ to get the following Proposition 4.3.2 by adapting the proof of [11, Theorem 10.1]. Proposition 4.3.2 shows that $P_T(\alpha) \circ J_T$ is an isomorphism (but not a chain isomorphism) for large T, describing the size of $P_T(\alpha) \circ J_T - J_T$ inside and outside the radius-2ε tubular neighborhoods. Also, it is the key to prove Theorem 1.2.6.

Proposition 4.3.2. *For any $\mu \in \mathbb{N}$, there exist $C_3 > 0$ and $\Gamma' > 0$ (Γ' is irrelevant to ε) such that when T is sufficiently large,*

$$|P_T(\alpha) J_T \omega - J_T \omega| \leqslant C_3 T^{-\mu/2} |\omega| + \Gamma' \cdot \varepsilon \cdot |J_T \omega|$$

for all $\omega \in \Omega^*(\mathcal{O}, o(N^-))^G$.

Proof. Without loss of generality, we assume $(d + d^*)\omega = \delta \cdot \omega$ and $|\omega| = 1$, i.e.,

$$\sum_{k_1 < \cdots < k_j} c_{k_1 \cdots k_j}^2 = 1$$

in the notation from Lemma 4.2.5. We recall that $|\lambda|^2 = \alpha > \alpha_0$ and let

$$Z_\ell(\lambda, T) = \sum_{k=0}^{\ell+1} \sigma_k(\lambda) T^{-k/2}.$$

When ε is sufficiently small, and T is sufficiently large, by the spectral gap Proposition 4.2.18, we apply the functional calculus [11, (9.153)] to $P_T(\alpha)$ and get

$$\begin{aligned}
&P_T(\alpha)J_T\omega - J_T\omega \\
=&\frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|^2=\alpha}\frac{1}{\lambda-D_T}J_T\omega d\lambda - J_T\omega \\
=&\frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|^2=\alpha}\frac{1}{\lambda-D_T}J_T\omega d\lambda - \frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|^2=\alpha}\rho\cdot Q_T^{-1}Z_\ell(\lambda,T)d\lambda \\
&+\frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|^2=\alpha}\rho\cdot Q_T^{-1}Z_\ell(\lambda,T)d\lambda - J_T\omega.
\end{aligned} \tag{4.24}$$

Recall the bump function ρ on N. We now estimate (4.24) in two parts:

Part I: On the one hand, we find that

$$\begin{aligned}
&(\lambda-D_T)\left(\rho Q_T^{-1}Z_\ell(\lambda,T)\right) - J_T\omega \\
=&\rho(\lambda-D_T)Q_T^{-1}Z_\ell(\lambda,T) - c(d\rho)Q_T^{-1}Z_\ell(\lambda,T) - \rho Q_T^{-1}K\omega \\
=&Q_T^{-1}\Big[Q_T(\rho)\cdot\left((\lambda-D^H)\sigma_{\ell+1}(\lambda)T^{-(\ell+1)/2} - R\sigma_\ell(\lambda)T^{-(\ell+1)/2} - R\sigma_{\ell+1}(\lambda)T^{-(\ell+2)/2}\right) \\
&\qquad\qquad - c(Q_T(d\rho))Z_\ell(\lambda,T)\Big] \\
=&\rho\cdot\left[Q_T^{-1}((\lambda-D^H)\sigma_{\ell+1}(\lambda))T^{-(\ell+1)/2} - Q_T^{-1}(R\sigma_\ell(\lambda))T^{-(\ell+1)/2} - Q_T^{-1}(R\sigma_{\ell+1}(\lambda))T^{-(\ell+2)/2}\right] \\
&- c(d\rho)(Q_T^{-1}Z_\ell(\lambda,T)).
\end{aligned}$$

Following [11, (10.16)], we consider the Schwartz semi-norms [36, Part 4, Section 8] of $\sigma_\ell(\lambda)$ and $\sigma_{\ell+1}(\lambda)$. For any $\nu\in\mathbb{N}$ and $\nu_1+\cdots+\nu_{m-n}=\nu$, there are $\zeta_\nu,\zeta'_\nu>0$ such that

$$\left|\partial_{y_1}^{\nu_1}\cdots\partial_{y_{m-n}}^{\nu_{m-n}}\sigma_\ell(\lambda)\right| \leqslant \frac{\zeta_\nu}{(1+|\mathbf{y}|)^\nu} \quad\text{and}\quad \left|\partial_{y_1}^{\nu_1}\cdots\partial_{y_{m-n}}^{\nu_{m-n}}\sigma_{\ell+1}(\lambda)\right| \leqslant \frac{\zeta'_\nu}{(1+|\mathbf{y}|)^\nu}$$

uniformly for $|\lambda| = \sqrt{\alpha}$. Let $\|\cdot\|_\nu$ denote the ν-th Sobolev norm on M. Then, we have a constant $\widetilde{C}_\nu > 0$ such that

$$\|(\lambda - D_T)(\rho Q_T^{-1} s_\ell(\lambda, T)) - J_T\omega\|_\nu \leqslant \widetilde{C}_\nu \cdot T^{-(\ell+1)/2} \cdot T^{\nu/2} \cdot T^{-\mathrm{rank}(N)/2}$$

when T is sufficiently large.

On the other hand, by Proposition 4.2.18, we find that for all $\eta \in \Omega^*(M)^G$,

$$\|(\lambda - D_T)\eta\| \geqslant (\sqrt{\alpha} - \sqrt{\alpha_0} - C_0 e^{-C_1 T} - \Gamma\varepsilon)\|\eta\|. \tag{4.25}$$

In addition, by [56, (6.18)] or verifying inductively using the Gårding inequality of $D = d + d^*$ on M, we have a constant $C_4 > 0$ such that when T is sufficiently large,

$$\|\eta\|_{\nu+1} \leqslant C_4 T^{\nu+1} \left(\|(\lambda - D_T)\eta\|_\nu + \|\eta\|\right) \tag{4.26}$$

for all $\eta \in \Omega^*(M)^G$. By (4.25) and (4.26), there is a constant $C_5 > 0$ such that for any sufficiently large T and any unit eigenform ω of the twisted $d + d^*$ on $C^*(M, f)^G$,

$$\begin{aligned}
&\|\rho Q_T^{-1} Z_\ell(\lambda, T) - (\lambda - D_T)^{-1} J_T\omega\|_{\nu+1} \\
\leqslant\ & C_4 T^{\nu+1} \big(\|(\lambda - D_T)\rho Q_T^{-1} Z_\ell(\lambda, T) - J_T\omega\|_\nu + \|\rho Q_T^{-1} Z_\ell(\lambda, T) - (\lambda - D_T)^{-1} J_T\omega\|\big) \\
\leqslant\ & C_5 T^{\nu+1} \|(\lambda - D_T)\rho Q_T^{-1} Z_\ell(\lambda, T) - J_T\omega\|_\nu \\
\leqslant\ & C_5 T^{\nu+1} \cdot \widetilde{C}_\nu \cdot T^{-(\ell+1)/2} \cdot T^{\nu/2} \cdot T^{-\mathrm{rank}(N)/2}.
\end{aligned} \tag{4.27}$$

Part II: We look at the subtraction

$$\frac{1}{2\pi\sqrt{-1}} \int_{|\lambda|^2 = \alpha} \rho Q_T^{-1} Z_\ell(\lambda, T) d\lambda - J_T\omega$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{|\lambda|^2=\alpha} \rho Q_T^{-1} \left(\sigma_2(\lambda)T^{-1} + \sigma_3(\lambda)T^{-3/2} + \cdots + \sigma_{\ell+1}(\lambda)T^{-(\ell+1)/2}\right) d\lambda. \tag{4.28}$$

Recall that $\mathrm{supp}(\rho) \subseteq N(2\varepsilon)$. By checking the degree of Hermite polynomials [52, Section 8.6] arising from the eigenfunctions of the harmonic oscillator, we find $\Gamma_k > 0$ such that

$$\left| \frac{1}{2\pi\sqrt{-1}} \int_{|\lambda|^2=\alpha} \rho Q_T^{-1}(\sigma_k(\lambda))T^{-k/2} d\lambda \right| \leqslant \Gamma_k \cdot (2\varepsilon)^k \cdot \rho \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right) \tag{4.29}$$

for each $k = 1, \cdots, \ell+1$. Here, Γ_k is irrelevant to ε.

Combining (4.27), (4.28), and (4.29), by [53, Corollary 6.22], there is a $C_6 > 0$ such that

$$|P_T(\alpha)J_T\omega - J_T\omega| \leqslant C_6 T^{-(\ell-3\nu-1+\mathrm{rank}(N))/2} + \sum_{k=1}^{\ell+1} \Gamma_k \cdot (2\varepsilon)^k \cdot \rho \exp\left(-\frac{T}{2}|\mathbf{y}|^2\right).$$

The proposition is verified after reorganizing all the constants. $\square$

Remark 4.3.3. As in Remark 4.2.14, the constant Γ' is independent of the choice of ε. Thus, we can choose sufficiently small ε so that $\Gamma' \cdot \varepsilon < 1/2$.

Finally, with Proposition 4.3.2, we can prove Theorem 1.2.6 and establish the analytic realization. As in [12, Definition 6.10] and [56, Definition 6.8], we define two automorphisms on the space $C^*(M,f)^G$. Let $\mathcal{F}_T$ be the linear map

$$\begin{aligned} \mathcal{F}_T : C^*(M,f)^G &\to C^*(M,f)^G \\ \omega \in \Omega^j(\mathcal{O}, o(N^-))^G &\mapsto e^{Tf(\mathcal{O})} \cdot \omega, \end{aligned} \tag{4.30}$$

where $\mathcal{O}$ is a critical orbit. We immediately see $\mathcal{F}_T$ is an automorphism.

Meanwhile, we recall the coordinate $\mathbf{y} = (y_1, \cdots, y_{m-n})$ around each critical orbit $\mathcal{O}$, where the Morse index of $\mathcal{O}$ is i, and $\dim \mathcal{O} = n$. Using the same bump function ρ as that in (4.5), the map

$$
\begin{aligned}
\mathcal{N}_T : C^*(M, f)^G &\to C^*(M, f)^G \\
\omega \in \Omega^j(\mathcal{O}, o(N^-))^G &\mapsto \omega \cdot \int_{\mathbb{R}^i} \rho(y_1, \cdots, y_i, 0, \cdots, 0) e^{-T\left(y_1^2 + \cdots + y_i^2\right)} dy_1 \cdots dy_i
\end{aligned}
\tag{4.31}
$$

is also an automorphism.

Proof of Theorem 1.2.6. Recall the map

$$
\begin{aligned}
\Phi_T : F_T^k(M, f, \alpha)^G &\to C^k(M, f)^G \\
\eta &\mapsto \sum_{i=0}^{k} (\pi_i)_* \left(e^{Tf} \cdot \eta|_{\overline{W^u(\mathcal{O}_i)}} \right) \quad (k = 0, 1, \cdots, m)
\end{aligned}
$$

between two chain complexes. By Proposition 2.2.5, this Φ_T is a chain map. Also, for any $0 \leqslant r \leqslant k$ and any $\omega \in \Omega^{k-r}(\mathcal{O}_r, \mathcal{E}_r)^G$ satisfying $|\omega| = 1$, by Proposition 4.3.2, we find

$$
\Phi_T P_T(\alpha) J_T \omega = \Phi_T(J_T \omega + \tau) + \sum_{i=0}^{k} \sum_{\mathcal{O} \subseteq \mathcal{O}_i} e^{Tf(\mathcal{O})} \cdot (\pi_i)_* \left(e^{T(f - f(\mathcal{O}))} \cdot \tau'|_{\overline{W^u(\mathcal{O})}} \right), \tag{4.32}
$$

where $\mathcal{O}$ is an orbit in the union $\mathcal{O}_i$ of all critical orbits having the same Morse index i, and $\tau, \tau' \in \Omega^*(M)^G$ satisfies that

$$
|\tau| \leqslant \Gamma' \cdot \varepsilon \cdot |J_T \omega|, \text{ and } |\tau'| \leqslant C_3 T^{-\mu}.
$$

We first look at the tail term in (4.32) given by τ'. Since for every critical orbit $\mathcal{O} \subseteq \mathcal{O}_i$ $(0 \leqslant i \leqslant k)$, there is $f - f(\mathcal{O}) \leqslant 0$ on $\overline{W^u(\mathcal{O})}$, we then find a constant $\xi_{\mathcal{O}} > 0$ such that

$$
\left| (\pi_i)_* \left(e^{T(f - f(\mathcal{O}))} \cdot \tau'|_{\overline{W^u(\mathcal{O})}} \right) \right| \leqslant \xi_{\mathcal{O}} \cdot C_3 T^{-\mu} \quad (\mathcal{O} \subseteq \mathcal{O}_i, 0 \leqslant i \leqslant k) \tag{4.33}
$$

when T is sufficiently large.

Next, we look at the main term

$$\Phi_T(J_T\omega+\tau)=\sum_{i=0}^{k}\sum_{\mathcal{O}\subseteq\mathcal{O}_i} e^{Tf(\mathcal{O})}\cdot(\pi_i)_*\left(e^{T(f-f(\mathcal{O}))}\cdot(J_T\omega+\tau)\big|_{\overline{W^u(\mathcal{O})}}\right) \tag{4.34}$$

in (4.32). It separates into three parts:

Part I: When $i=r$, we write ω uniquely into the sum $\sum_{\mathcal{O}\subseteq\mathcal{O}_r}\omega_{\mathcal{O}}$, where $\omega_{\mathcal{O}}\in\Omega^{k-r}(\mathcal{O},o(N^-))^G$ for each critical orbit $\mathcal{O}\subseteq\mathcal{O}_r$. Applying (4.31), we obtain

$$\begin{aligned}
&\sum_{\mathcal{O}\subseteq\mathcal{O}_r} e^{Tf(\mathcal{O})}\cdot(\pi_r)_*\left(e^{T(f-f(\mathcal{O}))}\cdot(J_T\omega+\tau)\big|_{\overline{W^u(\mathcal{O})}}\right)\\
&=\sum_{\mathcal{O}\subseteq\mathcal{O}_r} e^{Tf(\mathcal{O})}\cdot\mathcal{N}_T(\omega_{\mathcal{O}})+\sum_{\mathcal{O}\subseteq\mathcal{O}_r} e^{Tf(\mathcal{O})}\cdot(\pi_r)_*\left(e^{T(f-f(\mathcal{O}))}\cdot\tau\big|_{\overline{W^u(\mathcal{O})}}\right)\\
&\quad+\sum_{\substack{\mathcal{O}'\subseteq\mathcal{O}_r\\ \mathcal{O}'\neq\mathcal{O}}}\sum_{\mathcal{O}\subseteq\mathcal{O}_r} e^{Tf(\mathcal{O}')}\cdot(\pi_r)_*\left(e^{T(f-f(\mathcal{O}'))}\cdot(J_T\omega_{\mathcal{O}}+\tau)\Big|_{\overline{W^u(\mathcal{O}')}}\right).
\end{aligned} \tag{4.35}$$

By Theorem 2.2.3, the boundary of $\overline{W^u(\mathcal{O})}$ is given by

$$\bigcup_{\nu=1}^{r}\bigcup_{i_0<i_1<\cdots<i_{\nu-1}<r}\mathcal{M}(\mathcal{O},\mathcal{O}_{i_{\nu-1}})\times_{\mathcal{O}_{i_{\nu-1}}}\cdots\times_{\mathcal{O}_{i_1}}\mathcal{M}(\mathcal{O}_{i_1},\mathcal{O}_{i_0})\times_{\mathcal{O}_{i_0}}W^u(\mathcal{O}_{i_0}).$$

Thus, in (4.35), $\mathcal{O}$ is disjoint from $\overline{W^u(\mathcal{O}')}$. Therefore, there is a constant $\xi_{\mathcal{O}'}>0$ such that

$$\left|(\pi_r)_*\left(e^{T(f-f(\mathcal{O}'))}\cdot(J_T\omega_{\mathcal{O}}+\tau)|_{\overline{W^u(\mathcal{O}')}}\right)\right|\leqslant e^{-T\xi_{\mathcal{O}'}} \tag{4.36}$$

for all $\mathcal{O}'\neq\mathcal{O}$ in (4.35). In addition, we have a constant $\Gamma''>0$ (irrelevant to ε) such that

$$\left|(\pi_r)_*\left(e^{T(f-f(\mathcal{O}))}\cdot\tau|_{\overline{W^u(\mathcal{O})}}\right)\right|\leqslant\Gamma''\cdot\varepsilon\cdot T^{-r/2} \tag{4.37}$$

in (4.35) when T is sufficiently large.

Part II: When $i < r$, again by Theorem 2.2.3, the boundary of $\overline{W^u(\mathcal{O}_i)}$ is equal to

$$\bigcup_{\nu=1}^{i} \bigcup_{i_0 < i_1 < \cdots < i_\nu = i} \mathcal{M}(\mathcal{O}_{i_\nu}, \mathcal{O}_{i_{\nu-1}}) \times_{\mathcal{O}_{i_{\nu-1}}} \cdots \times_{\mathcal{O}_{i_1}} \mathcal{M}(\mathcal{O}_{i_1}, \mathcal{O}_{i_0}) \times_{\mathcal{O}_{i_0}} W^u(\mathcal{O}_{i_0}).$$

Therefore, $\mathcal{O}_r$ is disjoint from $\overline{W^u(\mathcal{O}_i)}$. Thus, there exists a constant $\xi_{ir} > 0$ such that

$$\left| (\pi_i)_* \left(e^{T(f - f(\mathcal{O}))} \cdot J_T \omega|_{\overline{W^u(\mathcal{O})}} \right) \right| \leqslant e^{-T\xi_{ir}} \tag{4.38}$$

for all $\mathcal{O} \subseteq \mathcal{O}_i$ when $i > r$.

Part III: When $i > r$, we have a constant $\xi_{ir} > 0$ such that

$$\left| (\pi_i)_* \left(e^{T(f - f(\mathcal{O}))} \cdot J_T \omega|_{\overline{W^u(\mathcal{O})}} \right) \right| \leqslant \xi_{ir} \tag{4.39}$$

for all $\mathcal{O} \subseteq \mathcal{O}_i$.

Finally, we let $\kappa = \dim C^*(M, f)^G$ and recall that $m = \dim M$. The Gaussian integral (adjusted by ρ) in (4.31) satisfies

$$\int_{\mathbb{R}^i} \rho(y_1, \cdots, y_i, 0, \cdots, 0) e^{-T\left(y_1^2 + \cdots + y_i^2\right)} dy_1 \cdots dy_i = O(T^{-i/2}) \tag{4.40}$$

for all $0 \leqslant i \leqslant m$ when $T \to +\infty$. Combining (4.30) – (4.40), after selecting and arranging an orthonormal basis of $C^*(M, f)^G$, we find that when T is sufficiently large,

$$\Phi_T P_T(\alpha) J_T = \mathcal{F}_T \circ \mathcal{N}_T \circ (X + Y),$$

where X and Y are $\kappa \times \kappa$ matrices whose (s,t)-th entries X_{st} and Y_{st} satisfy that

$$\begin{aligned} X_{st} &= 1 + \Gamma''\varepsilon && \text{when } s = t, \\ |X_{st}| &\leqslant O(T^{m/2}) && \text{when } s > t, \\ X_{st} &= 0 && \text{when } s < t, \end{aligned}$$

and $|Y_{st}| \leqslant O(T^{-\mu+m/2})$ for any (s,t). According to the Leibniz formula [23, (4.16)] for determinants, we find when $T \to +\infty$,

$$\begin{aligned} \det(X+Y) &= \sum_{\substack{\sigma \text{ is a permutation} \\ \text{of } \{1,2\cdots,\kappa\}}} \operatorname{sgn}(\sigma) \cdot \prod_{s=1}^{\kappa} \left(X_{s\sigma(s)} + Y_{s\sigma(s)} \right) \\ &= (1+\Gamma''\varepsilon)^{\kappa} + O(T^{-\mu+\kappa m/2}). \end{aligned}$$

Notice that Γ'' is irrelevant to ε, we can let $\Gamma''\varepsilon \ll 1$. By choosing a sufficiently large μ in Proposition 4.3.2, the map $\Phi_T P_T(\alpha) J_T$ is an isomorphism when $T > 0$ is sufficiently large.

In addition, by Corollary 4.2.19, $P_T(\alpha) J_T$ is an isomorphism between vector spaces when T is sufficiently large. Therefore, Φ_T is a chain isomorphism for sufficiently large T. $\square$

4.4 Inequalities and eigenvalues

We now give some corollaries of the analytic result Theorem 1.2.6. A straightforward one counts the number of eigenvalues of D_T^2 on $\Omega^*(M)^G$.

Corollary 4.4.1. *For any constant $\alpha > \alpha_0$, when T is sufficiently large, the number of eigenvalues $\leqslant \alpha$ of $D_T^2|_{\Omega^*(M)^G}$ is equal to $\dim C^*(M,f)^G$.*

Proof. This is given by the definition of $F_T^*(M,f,\alpha)^G$. $\square$

In addition, since both $C^*(M,f)^G$ and $F_T^*(M,f,\alpha)^G$ are finite dimensional, we obtain simplified proofs of the weak and strong G-invariant Morse-Bott inequalities associated to our f. We recall the following notations:

(1) $m = \dim M$;

(2) $H^k(M)^G$ is the k-th G-invariant de Rham cohomology group of M;

(3) $H^j(\mathcal{O}_i, \mathcal{E}_i)^G$ is the j-th G-invariant de Rham cohomology group of $\mathcal{O}_i$ with local coefficients $\mathcal{E}_i$ (See (2.10)).

The weak version is as follows.

Corollary 4.4.2. *For any $0 \leqslant k \leqslant m$, $\dim C^k(M,f)^G \geqslant \dim H^k(M)^G$.*

Proof. Since $H^k(M)^G$ is isomorphic to the kernel of D_T^2 restricted on $\Omega^k(M)^G$, we find

$$\dim C^k(M,f)^G = \dim F_T^k(M,f,\alpha)^G \geqslant \dim\ker\left(D_T^2|_{\Omega^k(M)^G}\right) = \dim H^k(M)^G$$

and get the weak inequalities. $\square$

We then prove the following strong version. Actually, as the proof of [4, Corollary 3.9], the strong version can be obtained only using Theorem 1.2.4, or as [14, (3.6)], using the CW-complex structure determined by the Morse-Bott function. However, here we give a proof using Theorem 1.2.6 together with finite dimensional linear algebra.

Corollary 4.4.3. *For any $0 \leqslant k \leqslant m$, we have*

$$\sum_{r=0}^{k}\sum_{i+j=r}(-1)^{k-r}\dim H^j(\mathcal{O}_i,\mathcal{E}_i)^G \geqslant \sum_{r=0}^{k}(-1)^{k-r}\dim H^r(M)^G \tag{4.41}$$

and call them the G-invariant Morse-Bott inequalities associated to f.

Proof. Applying the dimension formula to the Witten instanton complex, we have

$$\begin{aligned}
&\dim F_T^k(M,f,\alpha)\\
=\ &\dim\ker\left(d_T|_{F_T^k(M,f,\alpha)^G}\right)+\dim\operatorname{Im}\left(d_T|_{F_T^k(M,f,\alpha)^G}\right)\\
=\ &\dim H^k(M)^G+\dim\operatorname{Im}\left(d_T|_{F_T^k(M,f,\alpha)^G}\right)+\dim\operatorname{Im}\left(d_T|_{F_T^{k-1}(M,f,\alpha)^G}\right). \qquad (4.42)
\end{aligned}$$

For the same reason on the Thom-Smale complex, we find

$$\begin{aligned}
&\dim C^k(M,f)^G\\
=\ &\sum_{i+j=k}\dim\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G\\
=\ &\sum_{i+j=k}\dim\ker\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right)+\dim\operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right)\\
=\ &\sum_{i+j=k}\dim H^j(\mathcal{O}_i,\mathcal{E}_i)^G+\dim\operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right)+\dim\operatorname{Im}\left(d|_{\Omega^{j-1}(\mathcal{O}_i,\mathcal{E}_i)^G}\right). \qquad (4.43)
\end{aligned}$$

By (4.42) and (4.43), we obtain

$$\begin{aligned}
&\sum_{r=0}^k(-1)^{k-r}\dim F_T^r(M,f,\alpha)^G\\
=\ &\sum_{r=0}^k(-1)^{k-r}\left(\dim H^r(M)^G+\dim\operatorname{Im}\left(d_T|_{F_T^r(M,f,\alpha)^G}\right)+\dim\operatorname{Im}\left(d_T|_{F_T^{r-1}(M,f,\alpha)^G}\right)\right)\\
=\ &\dim\operatorname{Im}\left(d_T|_{F_T^k(M,f,\alpha)^G}\right)+\sum_{r=0}^k(-1)^{k-r}\dim H^r(M)^G \qquad (4.44)
\end{aligned}$$

and then

$$\sum_{r=0}^k(-1)^{k-r}\dim C^r(M,f)^G$$

$$= \sum_{r=0}^{k} \sum_{i+j=r} (-1)^{k-r} \left(\dim\ker H^j(\mathcal{O}_i,\mathcal{E}_i)^G + \dim \operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right) + \dim \operatorname{Im}\left(d|_{\Omega^{j-1}(\mathcal{O}_i,\mathcal{E}_i)^G}\right)\right)$$

$$= \sum_{i+j=k} \dim \operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right) + \sum_{r=0}^{k}\sum_{i+j=r} (-1)^{k-r} \dim H^j(\mathcal{O}_i,\mathcal{E}_i)^G \tag{4.45}$$

By Theorem 1.2.6, (4.44), and (4.45), we get

$$\begin{aligned} &\sum_{r=0}^{k}\sum_{i+j=r} (-1)^{k-r} \dim H^j(\mathcal{O}_i,\mathcal{E}_i)^G - \sum_{r=0}^{k}(-1)^{k-r}\dim H^r(M)^G \\ =\ & \dim \operatorname{Im}\left(d_T|_{F_T^k(M,f,\alpha)^G}\right) - \sum_{i+j=k} \dim \operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right) \\ =\ & \dim \operatorname{Im}\left(\partial|_{C^k(M,f)^G}\right) - \sum_{i+j=k} \dim \operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right). \end{aligned} \tag{4.46}$$

Since $C^*(M,f)^G$ is finite dimensional, for each $0 \leqslant j \leqslant k$, we choose an independent subset

$$\{\omega_1^j, \cdots, \omega_s^j\} \subseteq \Omega^j(\mathcal{O}_{k-j},\mathcal{E}_i)^G$$

such that $\{d\omega_1^j, \cdots, d\omega_s^j\}$ is a basis of $\operatorname{Im}\left(d|_{\Omega^j(\mathcal{O}_i,\mathcal{E}_i)^G}\right)$. By the definition of ∂, the linear map

$$\begin{aligned} \operatorname{Im}\left(d|_{\Omega^0(\mathcal{O}_k,\mathcal{E}_k)^G}\right) \oplus \operatorname{Im}\left(d|_{\Omega^0(\mathcal{O}_k,\mathcal{E}_k)^G}\right) \cdots \oplus \operatorname{Im}\left(d|_{\Omega^k(\mathcal{O}_0,\mathcal{E}_0)^G}\right) &\to \operatorname{Im}\left(\partial|_{C^k(M,f)^G}\right) \\ (d\omega_{s_0}^0, d\omega_{s_1}^1, \cdots, d\omega_{s_k}^k) &\mapsto \partial\omega_{s_0}^0 + \partial\omega_{s_1}^1 + \cdots + \partial\omega_{s_k}^k \end{aligned}$$

is injective (but not canonical). We finish the prove by noticing that (4.46) is nonnegative. □

The proof of Corollary 4.4.3 is purely algebraic. However, it does not intrinsically explain why we involve the cohomology groups of the critical set. As in [10], [11], and [41], the intrinsic reason is, the kernel of the twisted $(d+d^*)^2|_{C^*(M,f)^G}$ corresponds to the sum of eigenspaces of D_T^2 associated to sufficiently small eigenvalues. More precisely, following the notations in

Proposition 4.2.18, we let $\xi(\varepsilon, T) = \left(C_0 e^{-C_1 T} + \Gamma\varepsilon\right)^2$ and give Corollary 4.4.1 a refinement which is similar to [41, (2.68)]:

Corollary 4.4.4. *For each $0 \leqslant k \leqslant m$, when T is sufficiently large, the map $P_T(\xi(\varepsilon, T)) \circ J_T$ is an isomorphism between $\ker\left((d + d^*)^2|_{C^k(M,f)^G}\right)$ and $F_T^k(M, f, \xi(\varepsilon, T))$.*

Proof. We replace the space E_T in (4.5) by the image of $\ker\left((d + d^*)^2|_{C^k(M,f)^G}\right)$ under J_T. Then, we follow the same analysis as either Sections $4.2 - 4.3$ or [11, Chapters VIII-X]. □

Now, using Corollary 4.4.4 and the fact that

$$\ker\left((d + d^*)^2|_{\Omega^j(\mathcal{O}_i, \mathcal{E}_i)^G}\right) \cong H^j(\mathcal{O}_i, \mathcal{E}_i)^G,$$

we prove Corollary 4.4.3 again and reveal a more intrinsic relation between the cohomology groups in the inequalities (4.41). In fact, this is exactly the spirit of the analysis on the Morse-Bott inequalities associated to more general Morse-Bott functions in [10] and [41].

Remark 4.4.5. We can even refine Corollary 4.4.4 more to correspond each eigenvalue of the twisted $(d + d^*)^2|_{C^k(M,f)^G}$ with the associated subspace of $F_T^k(M, f, \alpha)$. This refined correspondence is the asymptotic behavior of eigenvalues of D_T^2 first studied by Helffer and Sjöstrand using semi-classical analysis tools in [28].

We end this chapter by explaining how we notice the number α_0, which is the infimum of α and also the spectral radius of $(d + d^*)^2|_{C^*(M,f)^G}$ in Theorem 1.2.6.

Example 4.4.6. Let $M = G = SU(2)$ and use the left multiplication action. The basis

$$\mathbf{v}_1 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$$

of $\mathfrak{su}(2)$ (See [24, Example 3.27]) generates a left invariant frame X_1, X_2, X_3 on $SU(2)$, and we let X_1, X_2, X_3 be orthonormal. When $f = 0$ on $SU(2)$, each dual 1-form X_j^* satisfies

$$D_T^2(X_j^*) = 4X_j^*,\ j = 1, 2, 3.$$

Therefore, compared with the Witten instanton complex in [12] and [56] where the number α in $F_T^k(M, f, \alpha)^G$ can be arbitrarily small, we need a nontrivial lower bound (which is α_0) of α because of those eigenvalues of D_T^2 coming from the horizontal direction.

The next example shows that α_0 relies on both M and G instead of only on G.

Example 4.4.7. We let $G = SO(3)$, $M = \mathbb{S}^2$, and $f = 0$ on M. Then, G acts on M transitively. Equipping M with a G-invariant metric induced by G as in Section 4.1, we let dvol_M be the unit volume form with respect to this metric and find:

(1) $\Omega^0(M)^G = \mathbb{R}$.

(2) $\Omega^1(M)^G = \{0\}$. This is because the dual of a G-invariant 1-form is either 0 or a nonvanishing vector field. The latter is impossible on $\mathbb{S}^2$.

(3) $\Omega^2(M)^G = \mathbb{R} \cdot \mathrm{dvol}_M$.

Thus, in this case, α_0 should be 0. However, noticing the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ between Lie algebras, if we only consider $SO(3)$ instead of considering $SO(3)$ and $\mathbb{S}^2$ together when determining α_0, we will get $\alpha_0 \geqslant 4$ by Example 4.4.6, which is not the minimal choice.

Chapter 5

Generalized mod 2 index

In this chapter, we aim at giving the definition of the generalized mod 2 index. To achieve this goal, we first introduce Atiyah and Singer's mod 2 index for skew-adjoint Fredholm operators on real Hilbert spaces. Then, for the aim of generalizing the Atiyah-Singer mod 2 index, we review necessary prerequisites on Kasparov's KKO-theory and rephrase Atiyah and Singer's result using KKO-groups. Finally, we define the generalized mod 2 index map on KKO-groups using the Kasparov descent map, the Baum-Connes assembly map, and the KKO-theoretic rephrasing of the Atiyah-Singer mod 2 index.

Throughout this chapter and the next chapter, we let M be an oriented $(4n+1)$-dimensional smooth manifold without boundary, and G be a Lie group acting on M smoothly, properly, and cocompactly without changing the orientation of M.

5.1 Atiyah-Singer mod 2 index

In this section, we review the Atiyah-Singer mod 2 index presented in [2]. We begin by reviewing the definition of Fredholm operators and skew-adjoint operators. Then, we state the invariance of the mod 2 index under certain perturbations.

Within this section, we let H be a real Hilbert space and $\langle \cdot, \cdot \rangle$ be the inner product on the Hilbert space H. One special type of bounded linear operators on H are Fredholm operators.

Definition 5.1.1. We call a bounded linear operator

$$F : H \to H$$

Fredholm if there is another bounded operator $S : H \to H$ such that

$$FS - \mathrm{id}_H \text{ and } SF - \mathrm{id}_H$$

are both compact on H.

Let $F^* : H \to H$ be the adjoint of F. We have an important property of Fredholm operators:

Proposition 5.1.2. *For any Fredholm $F : H \to H$, $\ker F$ and $\ker F^*$ are finite dimensional.*

The other special type of bounded linear operators are skew-adjoint operators.

Definition 5.1.3. *For any bounded linear operator $F : H \to H$, if*

$$\langle Fv, w \rangle = -\langle v, Fw \rangle$$

for all $v, w \in H$, we call F a skew-adjoint operator.

When $F : H \to F$ is Fredholm and skew-adjoint, we see that $\dim \ker F = \dim \ker F^*$ are both finite dimensional. This observation means that we only need one of them to connect to topological information, motivating the following Atiyah-Singer mod 2 index.

Definition 5.1.4. When $F : H \to H$ is a skew-adjoint Fredholm operator, we call

$$\mathrm{ind}_2 F \coloneqq \dim \ker F \mod 2$$

the mod 2 index of F.

Like the Fredholm index [46, Definition 10.4.5] for general Fredholm operators, Atiyah and Singer showed in [2, Theorem A] that the mod 2 index for real skew-adjoint Fredholm operators is invariant under certain perturbations:

Theorem 5.1.5 (Atiyah-Singer [2], 1969)**.** *Let $F : H \to H$ be a skew-adjoint Fredholm operator. Then, we have*

$$\mathrm{ind}_2 F = \mathrm{ind}_2(F + F_0)$$

for any compact skew-adjoint operator $F_0 : H \to H$.

Remark 5.1.6. The mod 2 index is a substitute of the Fredholm index for skew-adjoint operators. In fact, when F is skew-adjoint, the Fredholm index of its complexification is always 0, meaning that $\ker F$ is more worthy of studying than $\ker F - \ker F^*$.

5.2 Real C^*-algebras and KKO-groups

In this section, we recall real C^*-algebras [49] and Kasparov's KKO-theory [32, 34]. All the definitions and results have their complex counterparts, but for simplicity, we stay in the real situation. We will also rephrase Theorem 5.1.5 using KKO-groups.

We recall the following definition [49, Remark 1.2] of real C^*-algebras and the associated concepts of homomorphisms and representations.

Definition 5.2.1. (i) A real C^*-algebra A is a real Banach $*$-algebra equipped with a norm $\|\cdot\|$ and an involution $*: A \to A$, such that for all $a \in A$, $\|a^*a\| = \|a\|^2$, and the spectrum of a^*a is a subset of $[0, +\infty)$.

(ii) Between two real C^*-algebras A and B, a $*$-homomorphism is a linear map $\varphi : A \to B$ such that $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ and $\varphi(a^*) = (\varphi(a))^*$ for all $a_1, a_2, a \in A$.

(iii) A representation of a real C^*-algebra A is a $*$-homomorphism from A to the space of bounded operators on a real Hilbert space.

Example 5.2.2. The real field $\mathbb{R}$ is itself a real C^*-algebra. The norm is given by $\|r\| = |r|$ for all $r \in \mathbb{R}$. The involution is given by $r^* = r$ for all $r \in \mathbb{R}$.

Example 5.2.3. Let $C_0(M, \mathbb{R})$ be the algebra of continuous real-valued functions vanishing at infinity on M. Then, equipped with the norm

$$\|\varphi\| = \sup_{x \in M} |\varphi(x)|$$

and the involution

$$\varphi^* = \varphi$$

for all $\varphi \in C_0(M, \mathbb{R})$, we see that $C_0(M, \mathbb{R})$ becomes a real C^*-algebra.

Over a real C^*-algebra, we have the concept of Hilbert modules [34, Section 1.10], which is a generalization of Hilbert spaces:

Definition 5.2.4. Let A be a real C^*-algebra. We call a real vector space H a right Hilbert A-module if H is a right A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : H \times H \to A$ such that for all $r \in \mathbb{R}, x, y, z \in H, a \in A$, we have (1)-(7):

(1) $r(xa) = (rx)a = x(ra)$,

(2) $\langle x, y \rangle = \langle y, x \rangle^*$,

(3) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(4) $\langle x, ry \rangle = r\langle x, y \rangle$,

(5) $\langle x, ya \rangle = \langle x, y \rangle a$,

(6) $\langle x, x \rangle$ is a positive element in A. In particular, $\langle x, x \rangle = 0$ if and only if $x = 0$.

(7) Let $\| \cdot \|_A$ be the norm on the C^*-algebra A, and let $\|x\|_H = \|\langle x, x \rangle\|_A^{1/2}$ for any $x \in H$. Then, H is complete in the norm $\| \cdot \|_H$.

For our further usage, we need the concept of $\mathbb{Z}_2$-grading (See [32, Sections 1.1 and 1.2] and [13, Definition 14.1.1]):

Definition 5.2.5. (i) A $\mathbb{Z}_2$-grading on a real C^*-algebra A is a C^*-automorphism $\epsilon : A \to A$ such that $\epsilon^2 = \mathrm{id}_A$. We say $a \in A$ has degree i ($i = 0$ or 1) if $\epsilon(a) = (-1)^i a$.

(ii) Using the A in the above condition (i), a $\mathbb{Z}_2$-grading on a right Hilbert A-module H (with inner product $\langle \cdot, \cdot \rangle$) is a linear map $\epsilon' : H \to H$ such that $(\epsilon')^2 = \mathrm{id}_H$, $\epsilon'(xa) = \epsilon'(x)\epsilon(a)$, $\epsilon(\langle x, y \rangle) = \langle \epsilon'(x), \epsilon'(y) \rangle$ for all $x, y \in H, a \in A$. Again, we say $x \in H$ has degree i if $\epsilon'(x) = (-1)^i x$.

Let $\mathbb{L}(H)$ be the space of all bounded adjointable operators on a $\mathbb{Z}_2$-graded right Hilbert A-module H. We write $H = H^0 \oplus H^1$ according to the degree of elements in H. Then, for any $\Phi \in \mathbb{L}(H)$, if $\Phi(H^i) \subseteq H^i$ $(i = 0, 1)$, we let $\deg \Phi = 0$. Else, if $\Phi(H^i) \subseteq H^{1-i}$ $(i = 0, 1)$, we let $\deg \Phi = 1$. This defines a $\mathbb{Z}_2$-grading on $\mathbb{L}(H)$.

Remark 5.2.6. On any $\mathbb{Z}_2$-graded real C^*-algebra A, we have the commutator $[a_1, a_2] = a_1 a_2 - (-1)^{\deg a_1 \deg a_2} a_2 a_1$, with $\deg a_i$ $(i = 1, 2)$ being the degree of $a_i \in A$.

In the graded situation, we will also require the tensor products to carry the $\mathbb{Z}_2$-grading information (See [13, Sections 13.5 and 14.4]).

Definition 5.2.7. Given a $\mathbb{Z}_2$-graded right Hilbert A_i-module H_i (equipped with inner product $\langle \cdot, \cdot \rangle_{H_i}$, where $i = 1, 2$), if there is a $*$-homomorphism $\varphi : A_1 \to \mathbb{L}(H_2)$, then we denote by $H_1 \hat{\otimes}_{A_1} H_2$ (or $H_1 \hat{\otimes}_{\varphi} H_2$) the completion of the algebraic tensor product $H_1 \otimes_{A_1} H_2$ in the norm induced by the A_2-valued inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_2, \varphi(\langle x_1, y_1 \rangle_{H_1}) y_2 \rangle_{H_2}.$$

This $H_1 \hat{\otimes}_{A_1} H_2$ is a $\mathbb{Z}_2$-graded right Hilbert A_2-module. Its $\mathbb{Z}_2$-grading is given by $\deg(x_1 \otimes x_2) = \deg x_1 + \deg x_2$.

From now on, before the end of this section, we let A, B, and C be $\mathbb{Z}_2$-graded separable real C^*-algebras. Also, for the rest of the thesis, we assume that all C^*-algebras and Hilbert modules are separable.

Let $\mathbb{K}(H) \subseteq \mathbb{L}(H)$ be the closure of all finite-rank operators on H. Then, using the description in [8, Definition 27] of the homotopy relation, we state Kasparov's definition [34, Section 4: Definition 3] of the KKO-group of A and B.

Definition 5.2.8. The KKO-group $KKO(A, B)$ consists of the equivalence classes of triples (H, ρ, Φ) over (A, B) satisfying (1)-(4):

(1) H is a $\mathbb{Z}_2$-graded right Hilbert B-module.

(2) $\rho : A \to \mathbb{L}(H)$ is a $*$-homomorphism.

(3) $\Phi \in \mathbb{L}(H)$ is of degree 1.

(4) $[F, \rho(a)],\ (F^2 - 1)\rho(a),\ (F - F^*)\rho(a)$ are in $\mathbb{K}(H)$ for all $a \in A$.

The equivalence relation consists of unitary equivalence (i) and homotopy equivalence (ii):

(i) (H_0, ρ_0, Φ_0) and (H_1, ρ_1, Φ_1) are unitarily equivalent if there is a degree-preserving isometry $u : H_1 \to H_2$ such that for all $a \in A$,

$$u \circ \Phi_0 = \Phi_1 \circ u \text{ and } u \circ \rho_0(a) = \rho_1(a) \circ u.$$

(ii) Let $B[0,1]$ be the collection of all continuous maps from $[0,1]$ to B. For any triple (H, ρ, Φ) over $(A, B[0,1])$ and each one of $i = 0$ and 1, we use the evaluation map

$$\begin{aligned} \mathrm{ev}_i : B[0,1] &\to B \\ \varphi &\mapsto \varphi(i) \end{aligned}$$

to obtain the triple

$$(\mathrm{ev}_i)_*(H, \rho, \Phi) \coloneqq (H \hat{\otimes}_{\mathrm{ev}_i} B, \rho \hat{\otimes}_{\mathrm{ev}_i} \mathrm{id}_B, \Phi \hat{\otimes}_{\mathrm{ev}_i} \mathrm{id}_B).$$

Then, for two triples (H_0, ρ_0, Φ_0) and (H_1, ρ_1, Φ_1) over (A, B), we call them homotopy equivalent if there is a triple (H, ρ, Φ) over $(A, B[0,1])$ such that for $i = 0$ and 1:

$$(H_i, \rho_i, \Phi_i) = (\mathrm{ev}_i)_*(H, \rho, \Phi).$$

The above $KKO(A,B)$ has a natural group structure induced by direct sum.

Theorem 5.2.9 (Kasparov [34], 1981)**.** *For any two real C^*-algebras A and B, $KKO(A,B)$ is a group.*

For the situation where we have a proper cocompact Lie group action, we need the concept [32, Definition 2.3] of G-equivariant KKO-groups.

Definition 5.2.10. Suppose that the Lie group G acts continuously on both A and B from the left. Then, the G-equivariant KKO-group $KKO^G(A,B)$ consists of the equivalence classes of G-equivariant triples (H,ρ,Φ) satisfying (1)-(4):

(1) H is a $\mathbb{Z}_2$-graded right Hilbert B-module such that there is a grading-preserving G-action on H satisfying that

$$g(xb) = (gx)(gb),\ \ g\langle x,y\rangle = \langle gx, gy\rangle,$$

and $G \times H \to H$ is continuous in the norm on H.

(2) $\rho : A \to \mathbb{L}(H)$ is a G-equivariant $*$-homomorphism.

(3) $\Phi \in \mathbb{L}(H)$ is G-equivariant with degree 1.

(4) $[F,\rho(a)],\ (F^2-1)\rho(a),\ (F-F^*)\rho(a)$ are in $\mathbb{K}(H)$ for all $a \in A$.

The equivalence relation is the same as that in Definition 5.2.8 except that we require the triple over $(A, B[0,1])$ to be G-equivariant.

As $KKO(A,B)$, the G-equivariant $KKO^G(A,B)$ is also a group [32, Proposition 2.4].

Theorem 5.2.11 (Kasparov [32], 1988)**.** *For any two real C^*-algebras A and B admitting continuous G-actions from the left, $KKO^G(A, B)$ is a group.*

When A admits a continuous G-action, following the notations in [18, II. Appendix C], we define covariant representations of (G, A).

Definition 5.2.12. A covariant representation π of the pair (G, A) on a real Hilbert space V consists of a unitary representation π_A of A on V, and a unitary representation π_G of G on V, such that

$$\pi_G(g)\pi_A(a)\pi_G(g^{-1}) = \pi_A(ga)$$

for all $g \in G$ and $a \in A$.

Let dg be a left invariant Haar measure on G, χ be the modular character on G satisfying

$$dg^{-1} = \chi(g)dg,$$

and $C_c(G, A)$ be the space of all real-valued continuous compactly supported maps from G to A. Continuing with the settings in Definition 5.2.12, for any $\varphi \in C_c(G, A)$ and any covariant representation π of (G, A) on the real Hilbert space V, we let

$$\tilde{\pi}(\varphi) = \int_G \pi_A(\varphi(g))\pi_G(g)dg \in V.$$

The following crossed product (See [32, Definition 3.7] and [18, II. Appendix C]) of G and A is important when we map KKO^G-groups to KKO-groups.

Definition 5.2.13. We equip $C_c(G, A)$ with the multiplication

$$(\varphi_1 \cdot \varphi_2)(h) = \int_G \varphi_1(g)g(\varphi_2(g^{-1}h))dg, \quad \forall \varphi_1, \varphi_2 \in C_c(G, A), h \in G,$$

and the involution

$$\varphi^*(g) = \chi(g)^{-1}g(\varphi(g^{-1})^*), \quad \forall \varphi \in C_c(G, A), g \in G.$$

Then, the completion of $C_c(G, A)$ with respect to the norm

$$\|\varphi\| = \sup \{\|\tilde{\pi}(\varphi)\| : \pi \text{ is a covariant representation of } (G, A)\}, \ \forall \varphi \in C_c(G, A)$$

is called the crossed product of G and A. This completion is a C^*-algebra and is denoted by $C^*(G, A)$.

Remark 5.2.14. In particular, when $A = \mathbb{R}$ is equipped with a trivial G-action, $C^*(G, \mathbb{R})$ is the (real version) group C^*-algebra of G.

Now, suppose that we have a triple (H, ρ, Φ) representing a class in $KKO^G(A, B)$, we construct an associated triple $(C^*(G, H), \tilde{\rho}, \tilde{\Phi})$ in the following way.

As in [32, Definition 3.8], we let $C_c(G, H)$ be the space of compactly supported continuous maps from G to H. Denote the inner product on H by $\langle \cdot, \cdot \rangle$, we get an element $\langle \omega, \omega \rangle \in C_c(G, B)$ by letting $\langle \omega, \omega \rangle(g) = \langle \omega(g), \omega(g) \rangle$. Denote the norm on $C^*(G, B)$ defined in Definition 5.2.13 by $\| \cdot \|_{C^*(G,B)}$, we let $C^*(G, H)$ be the completion of $C_c(G, H)$ with respect to the norm

$$\|\langle \omega, \omega \rangle\|_{C^*(G,B)}^{1/2}, \forall \omega \in C_c(G, H).$$

By [32, Lemma 3.10, Theorem 3.11], the right $C^*(G, B)$-module structure on $C^*(G, H)$ is determined by

$$(\omega\varphi)(h) = \int_G \omega(g)g(\varphi(g^{-1}h))dg, \forall \omega \in C_c(G, H), \varphi \in C_c(G, B).$$

In addition, $\tilde{\Phi}$ is determined by

$$\tilde{\Phi}(\omega)(g) = \Phi(\omega(g)), \forall \omega \in C_c(G, H), g \in G,$$

and $\tilde{\rho} : C^*(G, A) \to \mathbb{L}(C^*(G, H))$ is determined by

$$\left(\tilde{\rho}(\psi)(\omega)\right)(h) = \int_G \rho(\psi(g)) \left(g(\omega(g^{-1}h))\right) dg$$

for all $\psi \in C_c(G, A)$, $\omega \in C_c(G, H)$, and $h \in G$.

The following Kasparov descent map j^G [32, Theorem 3.11] maps KKO^G to KKO.

Theorem 5.2.15 (Kasparov [32], 1988)**.** *The following map*

$$j^G : KKO^G(A, B) \to KKO(C^*(G, A), C^*(G, B))$$
$$(H, \rho, \Phi) \mapsto (C^*(G, H), \tilde{\rho}, \tilde{\Phi})$$

is a homomorphism called the Kasparov descent map.

An operation between two KKO-groups is the Kasparov product. Connes and Skandalis gave it a concise description [19, Theorem A.5] using the concept of connections.

We now introduce connections. Suppose that $(H, \rho, \Phi) \in KKO(A, C)$ and $(H', \rho', \Phi') \in KKO(C, B)$. Let $H'' = H \hat{\otimes}_C H'$. For all $v \in H$, we let $T_v : H' \to H''$ be given by $T_v(w) = v \hat{\otimes}_C w$ for all $w \in H'$. The adjoint of T_v is given by

$$T_v^*(u \hat{\otimes}_C w) = \langle v, u \rangle w$$

for all $u \in H$ and $w \in H'$.

Definition 5.2.16. *Suppose that $\Psi \in \mathbb{L}(H')$ satisfies $[c, \Psi] \in \mathbb{K}(H')$ for all $c \in C$. Then, we call $\tilde{\Psi} \in \mathbb{L}(H'')$ a Ψ-connection on H if*

$$T_v \Psi - (-1)^{\deg v \cdot \deg \Psi} \tilde{\Psi} T_v \in \mathbb{K}(H', H'')$$

and

$$\Psi T_v^* - (-1)^{\deg v \cdot \deg \Psi} T_v^* \tilde{\Psi} \in \mathbb{K}(H'', H')$$

for any $v \in H$.

For $(H, \rho, \Phi) \in KKO(A, C)$ and $(H', \rho', \Phi') \in KKO(C, B)$, the following theorem [19, Theorem A.3, Theorem A.5] ensures the definition of the Kasparov product.

Theorem 5.2.17 (Connes-Skandalis [19], 1984)**.** *There exists a Φ'-connection Φ'' on H such that (H'', ρ, Φ'') is a triple over (A, B), and for all $a \in A$,*

$$\rho(a)[\Phi'', \Phi \hat{\otimes}_C \mathrm{id}_{H'}]\rho(a)^*$$

is positive after modulo $\mathbb{K}(H'')$. The triple (H'', ρ, Φ'') is unique up to homotopy equivalence.

Definition 5.2.18. We call

$$\begin{aligned} KKO(A, C) \times KKO(C, B) &\to KKO(A, B) \\ ((H, \rho, \Phi), (H', \rho', \Phi')) &\mapsto (H \hat{\otimes}_C H', \rho, \Phi'') \end{aligned}$$

the Kasparov product between $KKO(A, C)$ and $KKO(C, B)$.

In particular, we have the following Grassmann connection which brings us convenience when finding appropriate triples for KKO-classes.

Theorem 5.2.19 (Connes-Skandalis [19], 1984)**.** *Let V be a $\mathbb{Z}_2$-graded separable real Hilbert space, and C^+ be the unitalization of C. Suppose that $A = \mathbb{R}$, $H = V \hat{\otimes}_{\mathbb{R}} C^+$, $P \in \mathbb{L}(H)$ is a projection, and $\Psi \in \mathbb{L}(H')$ satisfies $[c, \Psi] \in \mathbb{K}(H')$ for all $c \in C$. Then,*

$$(P \hat{\otimes}_C \mathrm{id}_{H'})(\mathrm{id}_V \hat{\otimes}_{\mathbb{R}} \Psi)(P \hat{\otimes}_C \mathrm{id}_{H'})$$

is a Ψ-connection (which is also called the Grassmann connection) on PH.

Let $Cl_{0,1}$ be the real Clifford C^*-algebra generated by 1 and v subject to

$$v^2 = -1, \; v^* = -v.$$

We now rephrase Theorem 5.1.5 using KKO-groups.

Theorem 5.2.20 (Atiyah-Singer [2], 1969)**.** *Let $\mathcal{V}$ be a graded right Hilbert $Cl_{0,1}$-module, ε be a graded C^*-representation of $\mathbb{R}$ on $\mathcal{V}$, and Ψ be a bounded adjointable operator of degree 1 on $\mathcal{V}$ such that $(\mathcal{V}, \varepsilon, \Psi)$ represents a class in $KKO(\mathbb{R}, Cl_{0,1})$. Then, the map*

$$\begin{aligned} \delta : KKO(\mathbb{R}, Cl_{0,1}) &\to \mathbb{Z}_2 \\ (\mathcal{V}, \varepsilon, \Psi) &\mapsto \frac{1}{2} \dim \ker \left(\varepsilon(1)(\Psi + \Psi^*)\varepsilon(1) \right) \mod 2 \end{aligned}$$

is an isomorphism.

Remark 5.2.21. (1) As mentioned in [8, Part 1, Section 12], we can define both graded and ungraded KKO-groups. However, in this thesis, we always have the $\mathbb{Z}_2$-grading.

(2) When writing the triple representing a KKO-class, we always follow the order as (Hilbert module, C^*-representation, self-adjoint operator).

5.3 Generalized mod 2 index map

In this section, we define the generalized mod 2 index. We adapt Bunke's steps in [43, Appendix B, C] into our mod 2 index situation, using a more refined family of idempotents.

As [51, Section 3], we let Y be the compact subset of M such that $\bigcup_{g\in G} g\cdot Y = M$. Let U and U' be two open subsets of M such that $Y \subset \overline{U} \subset U' \subset M$, and both $\overline{U}$ and $\overline{U'}$ are compact. Then, we construct a bump function $f : M \to \mathbb{R}$ satisfying $f|_U = 1$ and $\mathrm{supp}(f) \subset U'$.

Let dg be a left-invariant Haar measure on G. With χ the modular character satisfying $dg^{-1} = \chi(g)dg$, for each $t \in \mathbb{R}$, we let

$$A_t(x) := \left(\int_G f(gx)^2 \chi(g)^t dg\right)^{1/2} \quad (\forall x \in M). \tag{5.1}$$

The function A_t has the following property under the pullback by $g \in G$:

Lemma 5.3.1. *The function $A_t : M \to \mathbb{R}$ satisfies*

$$A_t(gx) = \chi(g)^{\frac{1-t}{2}} A_t(x)$$

for all $g \in G$ and $x \in M$.

Proof. $A_t(gx)^2 = \int_G f(hgx)^2\chi(h)^t dh = \int_G f(hx)^2\chi(h)^t\chi(g)^{1-t}dh = \chi(g)^{1-t}A_t(x)^2.$ $\square$

With f and A_t, we then construct the generalized mod 2 index map step by step.

Using the bump function f, we define a family $p_t \in C^*(G, C_0(M,\mathbb{R}))$ $(t \in \mathbb{R})$ by

$$p_t(g,x) = \frac{f(x)}{(A_t(x))^2}\chi(g)^{1-\frac{t}{2}} f(g^{-1}x)$$

for all $g \in G$ and $x \in M$. We see that p_t is idempotent:

Proposition 5.3.2. *The family p_t $(t \in \mathbb{R})$ satisfies $p_t^2 = p_t^* = p_t$.*

Proof. This follows from the algebraic structure [18, II. Appendix C] on $C^*(G, C_0(M, \mathbb{R}))$:

$$\begin{aligned}
p_t^2(g,x) &= \int_G p_t(h,x)p_t(h^{-1}g, h^{-1}x)dh \\
&= \int_G \frac{f(x)}{(A_t(x))^2}\chi(h)^{1-\frac{t}{2}} f(h^{-1}x)\frac{f(h^{-1}x)}{(A_t(h^{-1}x))^2}\chi(h^{-1}g)^{1-\frac{t}{2}} f(g^{-1}x)dh \\
&= \frac{f(x)\chi(g)^{1-\frac{t}{2}} f(g^{-1}x)}{(A_t(x))^4}\int_G f(h^{-1}x)^2\chi(h)^{1-t}dh \quad \text{(by Lemma 5.3.1)} \\
&= p_t(g,x) \quad \text{(by the definition of } A_t\text{)},
\end{aligned}$$

and $p_t^*(g,x) = \chi(g)p_t(g^{-1}, g^{-1}x) = p_t(g,x)$. ☐

Remark 5.3.3. In Bunke's construction, there are more requirements on f. Thus, in [43, Appendix B], the idempotent is given only when $t = 1$. However, using our f, we make p_t self-adjoint for all t.

Similar to the construction of [43, Definition B.1], we need the KKO-class defined by the following three items:

(1) A projective right $C^*(G, C_0(M, \mathbb{R}))$-module $p_t C^*(G, C_0(M, \mathbb{R}))$ with the trivial grading.

(2) The representation ρ of $\mathbb{R}$ on $p_t C^*(G, C_0(M, \mathbb{R}))$ which is the scalar multiplication.

(3) The zero operator 0 on $p_t C^*(G, C_0(M, \mathbb{R}))$.

These items form a triple $(p_t C^*(G, C_0(M, \mathbb{R})), \rho, 0)$. This triple represents a KKO-class $[p_t] \in KKO(\mathbb{R}, C^*(G, C_0(M, \mathbb{R})))$ and acts as a projection (cf. [32, Remark 2.15(1)]).

Remark 5.3.4. The class $[p_t]$ is always independent of t. Also, it is independent of the compact subset Y, the neighborhoods U and U', and the bump function f. Similar to [43, Appendix B], for different (f, A_t) and $(\tilde{f}, \tilde{A}_t)$, we let

$$\gamma_{t,s}(x) = \sqrt{s(f(x))^2(A_t(x))^{-2} + (1-s)(\tilde{f}(x))^2(\tilde{A}_t(x))^{-2}} \ \ (\forall x \in M, 0 \leqslant s \leqslant 1, t \in \mathbb{R})$$

and then

$$q_{t,s}(g, x) = \gamma_{t,s}(x)\gamma_{t,s}(g^{-1}x)\chi(g)^{1/2} \ (\forall x \in M, g \in G, 0 \leqslant s \leqslant 1, t \in \mathbb{R}).$$

The path $q_{1,s}$ $(0 \leqslant s \leqslant 1)$ gives a homotopy which identifies $[p_1] = [p_t]$ and $[\tilde{p}_1] = [\tilde{p}_t]$.

Recall that $Cl_{0,1}$ is the real Clifford algebra generated by 1 and v subject to the relations $v^2 = -1$ and $v^* = -v$. For our usage, the $\mathbb{Z}_2$-grading on $Cl_{0,1}$ is given by letting $\deg(r) = 0$ and $\deg(sv) = 1$ for any $r, s \in \mathbb{R}$, while the $\mathbb{Z}_2$-gradings on $\mathbb{R}$, $C_0(M, \mathbb{R})$, and $C^*(G, C_0(M, \mathbb{R}))$ are all trivial.

According to [32, Section 3.11], we have the Kasparov descent map

$$j^G : KKO^G(C_0(M, \mathbb{R}), Cl_{0,1}) \to KKO(C^*(G, C_0(M, \mathbb{R})), C^*(G, Cl_{0,1})).$$

Furthermore, using the Kasparov product in Definition 5.2.18, we state the Baum-Connes assembly map for our skew-adjoint situation as follows.

Definition 5.3.5. Given the Kasparov descent map

$$j^G : KKO^G(C_0(M, \mathbb{R}), Cl_{0,1}) \to KKO(C^*(G, C_0(M, \mathbb{R})), C^*(G, Cl_{0,1}))$$

and the map

$$q : KKO(C^*(G, C_0(M, \mathbb{R})), C^*(G, Cl_{0,1})) \to KKO(\mathbb{R}, C^*(G, Cl_{0,1}))$$
$$\text{any class } K \mapsto \text{Kasparov product of } [p_t] \text{ and } K,$$

the Baum-Connes assembly map

$$\mu : KKO^G(C_0(M, \mathbb{R}), Cl_{0,1}) \to KKO(\mathbb{R}, C^*(G, Cl_{0,1}))$$

is their composition $q \circ j^G$.

With the Baum-Connes assembly map μ, we are one step away from the generalized mod 2 index map. We let $[\mathbb{1}] \in KKO(C^*(G, \mathbb{R}), \mathbb{R})$ be the class defined by these three items:

(1) The right $\mathbb{R}$-module $\mathbb{R}$ with the trivial grading.

(2) The representation ϱ of $C^*(G, \mathbb{R})$ on $\mathbb{R}$ which is determined by (See [20, Section VII.1])

$$\forall \psi \in C_c(G, \mathbb{R}) \text{ and } r \in \mathbb{R},\ \varrho(\psi)(r) := \left(\int_G \psi(g) dg \right) r.$$

(3) The zero operator $0 : \mathbb{R} \to \mathbb{R}$.

The definition of the generalized mod 2 index map follows immediately.

Definition 5.3.6. The generalized mod 2 index is

$$\mathrm{ind}_2^G : KKO^G(C_0(M, \mathbb{R}), Cl_{0,1}) \to KKO(\mathbb{R}, Cl_{0,1}) \cong \mathbb{Z}_2$$
$$\text{any class } \kappa \mapsto \mu(\kappa) \hat{\otimes}_{C^*(G,\mathbb{R})} [\mathbb{1}].$$

The identification $KKO(\mathbb{R}, Cl_{0,1}) \cong \mathbb{Z}_2$ is given by Theorem 5.2.20.

Remark 5.3.7. Instead of using the function A_t given in (5.1), there is another KKO-theoretic construction using a maximal compact subgroup $K < G$ and the space $C_0(G/K, \mathbb{R})$ of continuous real-valued functions vanishing at infinity on G/K to generalize the Atiyah-Singer mod 2 index. Following [30, (4.1)], in this way, the Baum-Connes assembly map

$$\mu : KKO^G(C_0(M, \mathbb{R}), Cl_{0,1}) \to KKO(\mathbb{R}, C^*(G, Cl_{0,1}))$$

is replaced by the composition of the descent map j^G, the map

$$KKO^G(C^*(G, C_0(M, \mathbb{R})), C^*(G, Cl_{0,1})) \to KKO(C^*(G, C_0(G/K, \mathbb{R})), C^*(G, Cl_{0,1}))$$

induced by a smooth equivariant map from M to G/K, and the map

$$KKO(C^*(G, C_0(G/K, \mathbb{R})), C^*(G, Cl_{0,1})) \to KKO(\mathbb{R}, C^*(G, Cl_{0,1}))$$

defined by left multiplying the KKO-class induced by the trivial representation of K. We will not use this construction in the proof of our main results, but we expect this construction to reveal the relations between the $k(M, G)$ in (1.5) and the semi-characteristic of a slice.

When M is closed and G is a single point, ind_2^G computes the Atiyah-Singer mod 2 index of the operator D_{sig} given by (1.3). Actually, if M is closed, then $C_0(M, \mathbb{R})$ is the space of all continuous real-valued functions on M, and D_{sig} is a real skew-adjoint Fredholm operator on the space $L^2(\Lambda^{\mathrm{even}} T^* M)$ of even-degree real L^2-forms on M. After extending the skew-adjoint D_{sig} to the self-adjoint $\mathscr{D}_{\mathrm{sig}}$ on $L^2(\Lambda^{\mathrm{even}} T^* M) \otimes Cl_{0,1}$ by

$$\mathscr{D}_{\mathrm{sig}}(\omega + \eta \cdot v) = -(D_{\mathrm{sig}}(\omega) + D_{\mathrm{sig}}(\eta) v) v,$$

we get a KKO-class in $KKO(C_0(M,\mathbb{R}), Cl_{0,1})$. By Theorem 5.2.20 and the definition of ind_2^G, the image of this KKO-class under ind_2^G is identified with

$$\dim\ker D_{\mathrm{sig}} \mod 2.$$

Thus, the map ind_2^G is a reasonable generalization of the Atiyah-Singer mod 2 index.

Remark 5.3.8. According to [47], we can define a generalization of the Fredholm index by coarse algebra. For the Atiyah-Singer mod 2 index, we also expect to generalize it from the perspective of coarse algebras. For this direction, see [16].

Chapter 6

Cohomology and vanishing theorem

Once we determine what the generalized mod 2 index is, we need to go from the image of the generalized mod 2 index map to the cohomological information of M under the action by G. The bridge to connect index theory and cohomology groups is the Hodge theory. Here, since our manifold is noncompact, and our group action is proper and cocompact, we need the proper cocompact Hodge theory in terms of the G-invariant cohomology twisted by A_t and $\chi^{t/2}$.

In this chapter, we prove our main results Theorem 1.3.4 and Theorem 1.3.5. First, we review the prerequisites in the proper cocompact Hodge theory, which identifies the G-invariant cohomology twisted by the modular character of G with the kernel of the deformed Laplacian on a deformed Sobolev space. Second, using the deformed Dirac type operator and the deformed Sobolev-1 space, we find an appropriate deformed triple representing $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$ and prove Theorem 1.3.4. Third, we prove Theorem 1.3.5 using a perturbation similar to that in [1, Section 4]. Finally, we explain why we need to twist the cohomology using A_t and $\chi^{t/2}$.

6.1 Proper cocompact Hodge theory

In this section, we present Mathai and Zhang's proper cocompact version of Gårding's inequality and the associated Tang, Yao, and Zhang's Hodge theorem.

According to [17, Theorem 2.1], we assign M a G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then, we obtain the formal adjoint d^* of the de Rham exterior differentiation d, the Levi-Civita connection ∇ on M, and the volume form dvol of M. Let $T = D + B$, where $D = d + d^*$, and B is a G-equivariant order 0 differential operator exchanging odd and even forms.

Let $\Omega^i(M)$ be the collection of all smooth i-forms on M. Using the Levi-Civita connection ∇, for any $s \in \mathbb{Z}_{\geqslant 0}$, we define the Sobolev s-norm [36, Definition 4.6]

$$\|\alpha\|_s \coloneqq \left(\sum_{j=0}^{s} \int_M \langle \nabla^j \alpha, \nabla^j \alpha \rangle \mathrm{dvol} \right)^{1/2}, \ \forall \alpha \in \bigoplus_{i=0}^{4n+1} \Omega^i(M).$$

In particular, $\| \cdot \|_0$ is the L^2-norm.

Next, we define the space

$$\Omega^i_{\chi^{t/2}}(M) \coloneqq \{\omega \in \Omega^i(M) : g^*\omega = \chi(g)^{t/2}\omega \text{ for all } g \in G\}.$$

With the bump function f from Section 5.3, we furthermore let

$$f \cdot \Omega^i_{\chi^{t/2}}(M) \coloneqq \left\{ f \cdot \omega : \omega \in \Omega^i_{\chi^{t/2}}(M) \right\}.$$

Remark 6.1.1. Our convention of pullback is $(g^*\alpha)(x) = \alpha(gx)$ for any form α on M, any $g \in G$, and any $x \in M$. It is the same as [40, Chapter 14] and [51, Section 2.2] but different from [43, Appendix].

Then, we let $\mathbf{H}^{s,i}_{\chi^{t/2}}(M,f)$ be the $\|\cdot\|_s$-completion of $f\cdot\Omega^i_{\chi^{t/2}}(M)$. Also, we let

$$\mathbf{H}^{s,\mathrm{even}}_{\chi^{t/2}}(M,f) \coloneqq \bigoplus_{i \text{ is even}} \mathbf{H}^{s,i}_{\chi^{t/2}}(M,f)$$

and

$$\mathbf{H}^{s}_{\chi^{t/2}}(M,f) \coloneqq \bigoplus_{i=0}^{4n+1} \mathbf{H}^{s,i}_{\chi^{t/2}}(M,f).$$

Let $L^2(\Lambda^*T^*M)$ be the real space spanned by all real L^2-forms on M, and P_t be the orthogonal projection from $L^2(\Lambda^*T^*M)$ onto $\mathbf{H}^{s}_{\chi^{t/2}}(M,f)$. We then state [43, Proposition 2.1], the proper cocompact Gårding's inequality:

Theorem 6.1.2 (Mathai-Zhang [43], 2010)**.** *For each t, the operator $P_tT : \mathbf{H}^{0}_{\chi^{t/2}}(M,f) \to \mathbf{H}^{0}_{\chi^{t/2}}(M,f)$ is densely defined on domain $\mathbf{H}^{1}_{\chi^{t/2}}(M,f)$, and there exist constants $C_1 > 0$ and $C_2 > 0$ satisfying*

$$\|P_tT(f\eta)\|_0 \geqslant C_1\|f\eta\|_1 - C_2\|f\eta\|_0$$

for all $f\eta \in \mathbf{H}^{1}_{\chi^{t/2}}(M,f)$. In addition, the kernel of P_tT is finite dimensional.

Remark 6.1.3. According to Bunke's construction in [43, Appendix D], the space $\mathbf{H}^{s,i}_{\chi^{t/2}}(M,f)$ is actually the image of all i-forms η satisfying $g^*\eta = \chi^{t/2}(g)\eta$ and

$$\sum_{j=0}^{s}\int_S \langle\nabla^j\eta,\nabla^j\eta\rangle \mathrm{dvol} < +\infty \ (\forall \text{ compact subset } S\subset M)$$

under the map $\eta \mapsto f\eta$. This is an injective map with a closed image.

We hope to apply these operators to study the G-invariant cohomology (with a deformation). Thus, we need a precise expression of P_t. The expression has been given in [43, Appendix D] and [51, Proposition 3.1] for $t = 1$ and 0 respectively. We adjust them and get

$$P_t(\omega)(x) = \frac{f(x)}{A_t(x)^2}\int_G \chi(g)^{t/2} f(gx)\omega(gx)dg, \ \forall \omega \in L^2(\Lambda^* T^* M), x \in M.$$

This P_t is exactly the orthogonal projection from $L^2(\Lambda^* T^* M)$ onto $\mathbf{H}^0_{\chi^{t/2}}(M, f)$:

Lemma 6.1.4. *The map $P_t : L^2(\Lambda^* T^* M) \to L^2(\Lambda^* T^* M)$ is an identity map when restricted to $\mathbf{H}^0_{\chi^{t/2}}(M, f)$. Also, P_t satisfies $P_t^* = P_t = P_t^2$. In addition,*

$$g^*\left(\frac{1}{f}P_t(\omega)\right) = \chi(g)^{t/2}\omega \text{ for all } g \in G \text{ and } \omega \in L^2(\Lambda^* T^* M),$$

meaning that the image of P_t is exactly $\mathbf{H}^0_{\chi^{t/2}}(M, f)$.

Moreover, writing T into $T = \sum_{i=1}^{4n+1} c(e_i)\nabla_{e_i} + B$, where $c(e_i) = e_i^* \wedge - e_i \lrcorner$, we find:

Lemma 6.1.5. *For any $\omega \in \Omega^i_{\chi^{t/2}}(M)$,*

$$P_t T(f\omega) = f \cdot (d + d^* + A_t^{-1} dA_t \wedge - A_t^{-1}\mathrm{grad} A_t \lrcorner)\omega + f \cdot B\omega.$$

The proofs of Lemma 6.1.4 and Lemma 6.1.5 are omitted because they are almost the same as the proof of [51, Proposition 3.1, Theorem 4.1].

We then look at the twisted de Rham differentiation

$$\begin{aligned} d_{A_t} : \Omega^i_{\chi^{t/2}}(M) &\to \Omega^{i+1}_{\chi^{t/2}}(M) \\ \omega &\mapsto A_t^{-1}(d(A_t\omega)). \end{aligned} \tag{6.1}$$

It defines a chain complex. We denote by $H^i_{\chi^{t/2}}(M, d_{A_t})$ the i-th cohomology group of the chain complex (6.1). Then, following [51, Section 3], we look at the operator

$$\begin{aligned} d_{t,f} : f \cdot \Omega^i_{\chi^{t/2}}(M) &\to f \cdot \Omega^{i+1}_{\chi^{t/2}}(M) \\ f \cdot \omega &\mapsto f \cdot d_{A_t}\omega = f \cdot (d\omega + A_t^{-1} dA_t \wedge \omega). \end{aligned}$$

Being a subspace of $\Omega^i(M)$, $f \cdot \Omega^i_{\chi^{t/2}}(M)$ inherites the L^2-norm $\|\cdot\|_0$ for each $0 \leqslant i \leqslant 4n+1$. As in the classical Hodge theory [53, Definition 6.1], we find the formal adjoint

$$\begin{aligned} d^*_{t,f} : f \cdot \Omega^{i+1}_{\chi^{t/2}}(M) &\to f \cdot \Omega^i_{\chi^{t/2}}(M) \\ f\omega &\mapsto f \cdot (d^*\omega - A_t^{-1} \mathrm{grad} A_t \lrcorner\, \omega) \end{aligned}$$

of $d_{t,f}$ with respect to the inner product induced by $\|\cdot\|_0$ on $f \cdot \Omega^{i+1}_{\chi^{t/2}}(M)$. Thus,

$$P_t D(f\omega) = d_{t,f}(f\omega) + d^*_{t,f}(f\omega) \tag{6.2}$$

by letting $B = 0$ in Lemma 6.1.5. By [51, Proposition 3.8], we state the proper cocompact Hodge theorem in which cohomology classes are identified with $\ker(P_t D)^2$:

Theorem 6.1.6 (Tang-Yao-Zhang [51], 2013)**.** *The kernel of the unbounded operator*

$$(P_t D)^2 : \mathbf{H}^{0,i}_{\chi^{t/2}}(M, f) \to \mathbf{H}^{0,i}_{\chi^{t/2}}(M, f)$$

is finite dimensional. In addition, the map $f\eta \mapsto \eta$ induces the canonical isomorphism between the kernel of $(P_t D)^2$ on $\mathbf{H}^{0,i}_{\chi^{t/2}}(M, f)$ and the cohomology group $H^i_{\chi^{t/2}}(M, d_{A_t})$.

Remark 6.1.7. In [51], Theorem 6.1.2 and Theorem 6.1.6 are proved for $t = 0$. However, the proof for any $t \in \mathbb{R}$ follows the same approach (See [51, Theorem 4.1(i)]).

6.2 Classes by skew-adjoint operators

In this section, we find an appropriate representative for the KKO-class $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$. Our steps are similar to [43, Appendix E]. Then, we prove Theorem 1.3.4.

Recall that $L^2(\Lambda^{\mathrm{even}}T^*M)$ is the real space spanned by all real L^2-forms with even degree on M. On $L^2(\Lambda^{\mathrm{even}}T^*M)$, D is self-adjoint and anti-commutes with the self-adjoint $\hat{c}(\mathrm{dvol})$. Thus, the operator

$$D_{\mathrm{sig}} \coloneqq \hat{c}(\mathrm{dvol}) \circ D(1+D^2)^{-1/2} : L^2(\Lambda^{\mathrm{even}}T^*M) \to L^2(\Lambda^{\mathrm{even}}T^*M) \tag{6.3}$$

is skew-adjoint and Fredholm. The parametrix of D_{sig} is $-D_{\mathrm{sig}}$. Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ be the inner product induced by $\|\cdot\|_0$ on $L^2(\Lambda^{\mathrm{even}}T^*M)$, and recall that $Cl_{0,1}$ is the real Clifford algebra generated by 1 and v subject to the relations $v^2 = -1$ and $v^* = -v$. We consider:

(1) The right $Cl_{0,1}$-module $L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1}$. The inner product is

$$\langle\!\langle \omega + \eta v, \omega' + \eta' v \rangle\!\rangle = \langle\!\langle \omega, \omega' \rangle\!\rangle + \langle\!\langle \eta, \eta' \rangle\!\rangle + \langle\!\langle \eta, \omega' \rangle\!\rangle v - \langle\!\langle \omega, \eta' \rangle\!\rangle v$$

for any $\omega, \omega', \eta, \eta' \in L^2(\Lambda^{\mathrm{even}}T^*M)$. The $\mathbb{Z}_2$-grading is by letting $|\omega| = 0$ and $|\eta v| = 1$.

(2) The representation ϕ_{sig} of $C_0(M, \mathbb{R})$ on $L^2(\Lambda^{\mathrm{even}}T^*M\hat{\otimes}Cl_{0,1})$. It is given by left multiplication of functions.

(3) A self-adjoint bounded Fredholm operator

$$\begin{aligned} \mathscr{D}_{\mathrm{sig}} : L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1} &\to L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1} \\ \omega + \eta v &\mapsto -(D_{\mathrm{sig}}(\omega) + D_{\mathrm{sig}}(\eta)v)v. \end{aligned} \tag{6.4}$$

The operator $\mathscr{D}_{\mathrm{sig}}$ is G-equivariant. The self-adjoint property follows from the identity $v^2 = -1$.

Then, $\left(L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1}, \phi_{\mathrm{sig}}, \mathscr{D}_{\mathrm{sig}}\right)$ determines a class $[\mathscr{D}_{\mathrm{sig}}] \in KKO^G(C_0(M,\mathbb{R}), Cl_{0,1})$.

We now find a representative for $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$ step by step.

First, by Theorem 5.2.15, $j^G([\mathscr{D}_{\mathrm{sig}}]) \in KKO(C^*(G, C_0(M,\mathbb{R})), C^*(G, Cl_{0,1}))$ is represented by the triple

$$\left(C^*(G, L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1}), \tilde{\phi}_{\mathrm{sig}}, \tilde{\mathscr{D}}_{\mathrm{sig}}\right):$$

(1) $\tilde{\phi}_{\mathrm{sig}}$ is the representation of $C^*(G, C_0(M,\mathbb{R}))$ on $C^*(G, L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1})$ given by

$$\left(\tilde{\phi}_{\mathrm{sig}}(\psi)(\omega+\eta v)\right)(h) = \int_G \psi(g)(g^{-1})^*(\omega(g^{-1}h))dg + \left(\int_G \psi(g)(g^{-1})^*(\eta(g^{-1}h))dg\right)v \tag{6.5}$$

for all $\psi \in C_c(G, C_0(M,\mathbb{R}))$, $\omega + \eta v \in C_c(G, L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1})$, and $h \in G$.

(2) $\tilde{\mathscr{D}}_{\mathrm{sig}}$ is given by

$$\left(\tilde{\mathscr{D}}_{\mathrm{sig}}(\omega+\eta v)\right)(g) = \mathscr{D}_{\mathrm{sig}}\left(\omega(g)+\eta(g)v\right) = -\left(D_{\mathrm{sig}}(\omega(g)) + D_{\mathrm{sig}}(\eta(g))v\right)v \tag{6.6}$$

for all $\omega + \eta v \in C_c(G, L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1})$ and $g \in G$.

Second, we find a triple representing $j^G([\mathscr{D}_{\mathrm{sig}}])\hat{\otimes}_{C^*(G)}[\mathbb{1}]$:

Lemma 6.2.1. *The class $j^G([\mathscr{D}_{\mathrm{sig}}])\hat{\otimes}_{C^*(G)}[\mathbb{1}] \in KKO(C^*(G, C_0(M,\mathbb{R})), Cl_{0,1})$ is represented by the triple $(L^2(\Lambda^{\mathrm{even}}T^*M)\hat{\otimes}Cl_{0,1}, \tau_{\mathrm{sig}}, \mathscr{D}_{\mathrm{sig}})$. Here, τ_{sig} is the representation of*

$C^*(G, C_0(M, \mathbb{R}))$ on $L^2(\Lambda^{\text{even}} T^*M) \hat{\otimes} Cl_{0,1}$ determined by

$$\tau_{\text{sig}}(\psi)(\omega + \eta v) = \int_G \psi(g)(g^{-1})^* \omega dg + \left(\int_G \psi(g)(g^{-1})^* \eta dg \right) v \tag{6.7}$$

for all $\psi \in C_c(G, C_0(M, \mathbb{R}))$ and $\omega + \eta v \in L^2(\Lambda^{\text{even}} T^*M) \hat{\otimes} Cl_{0,1}$.

Proof. According to [19, Remark A.6(1)], the Hilbert module in the triple should be

$$C^*(G, L^2(\Lambda^{\text{even}} T^*M) \hat{\otimes} Cl_{0,1}) \hat{\otimes}_{Cl_{0,1} \hat{\otimes} C^*(G)} (Cl_{0,1} \hat{\otimes} \mathbb{R}).$$

As mentioned in [20, Section VII.1], the representation ϱ of $C^*(G, \mathbb{R})$ on $\mathbb{R}$ is given by

$$\forall \psi \in C_c(G, \mathbb{R}) \text{ and } r \in \mathbb{R}, \varrho(\psi)(r) = \left(\int_G \psi(g) dg \right) r.$$

Thus, as in the proof of [43, Lemma E.1], we have an isomorphism

$$\begin{aligned} \iota : C^*(G, L^2(\Lambda^{\text{even}} T^*M) \hat{\otimes} Cl_{0,1}) \hat{\otimes}_{Cl_{0,1} \hat{\otimes} C^*(G)} (Cl_{0,1} \hat{\otimes} \mathbb{R}) &\to L^2(\Lambda^{\text{even}} T^*M) \hat{\otimes} Cl_{0,1} \\ (\omega + \eta v) \cdot (a + bv) &\mapsto \left(\int_G \omega(g) dg + \int_G \eta(g) dg v \right) (a + bv). \end{aligned}$$

Therefore, to obtain τ_{sig}, we integrate both sides of (6.5) with respect to $h \in G$. More precisely, for any $\omega, \eta \in C_c(G, L^2(\Lambda^{\text{even}} T^*M))$ and $\psi \in C_c(G, C_0(M, \mathbb{R}))$, using the map ι,

$$\begin{aligned} &\tau_{\text{sig}}(\psi)(\iota(\omega + \eta v)) \\ &= \iota \left(\tilde{\phi}_{\text{sig}}(\psi)(\omega + \eta v) \right) \\ &= \int_G \left(\tilde{\phi}_{\text{sig}}(\psi) \omega \right)(h) dh + \left(\int_G \left(\tilde{\phi}_{\text{sig}}(\psi) \eta \right)(h) dh \right) v \\ &= \int_G \int_G \psi(g)(g^{-1})^* (\omega(g^{-1}h)) dg dh + \left(\int_G \int_G \psi(g)(g^{-1})^* (\eta(g^{-1}h)) dg dh \right) v \end{aligned}$$

$$= \int_G \int_G \psi(g)(g^{-1})^*(\omega(g^{-1}h))dhdg + \left(\int_G \int_G \psi(g)(g^{-1})^*(\eta(g^{-1}h))dhdg\right) v$$

(Since the Haar measure is left invariant:)

$$= \int_G \int_G \psi(g)(g^{-1})^*(\omega(g^{-1}h))d(g^{-1}h)dg + \left(\int_G \int_G \psi(g)(g^{-1})^*(\eta(g^{-1}h))d(g^{-1}h)dg\right) v$$

$$= \int_G \psi(g)(g^{-1})^* \left(\iota(\omega + \eta v)\right) dg.$$

Similarly, by integrating both sides of (6.6) with respect to $g \in G$, we obtain $\mathscr{D}_{\mathrm{sig}}$. □

Third, we apply $[p_t]$ to $j^G([\mathscr{D}_{\mathrm{sig}}])\hat{\otimes}_{C^*(G)}[\mathbb{1}]$ to find a representative of $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$.

Proposition 6.2.2. *Let* $F_{\mathrm{sig}} = \hat{c}(\mathrm{dvol}) \circ P_t D\left(1 + (P_t D)^2\right)^{-1/2}$ *and*

$$\begin{aligned} \mathcal{F}_{\mathrm{sig}} : \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1} &\to \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1} \\ \omega + \eta v &\mapsto -(F_{\mathrm{sig}}\omega + (F_{\mathrm{sig}}\eta)v)v. \end{aligned}$$

Let ρ_{sig} *be the scalar multiplication of* $\mathbb{R}$ *on* $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1}$. *Then, the triple*

$$\left(\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1}, \rho_{\mathrm{sig}}, \mathcal{F}_{\mathrm{sig}}\right) \tag{6.8}$$

represents $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$.

Proof. Since $[p_t]$ is an idempotent, we use the Grassmann connection from Theorem 5.2.19 to obtain the triple (6.8).

Recall that $(p_t C^*(G, C_0(M, \mathbb{R})), \rho, 0)$ represents $[p_t]$, where ρ is the scalar multiplication of $\mathbb{R}$ on $p_t C^*(G, C_0(M, \mathbb{R}))$. By (6.7), the idempotent $p_t \in C^*(G, C_0(M, \mathbb{R}))$ satisfies $\tau_{\mathrm{sig}}(p_t)(\omega + \eta v) = P_t\omega + (P_t\eta)v$ for all $\omega, \eta \in L^2(\Lambda^{\mathrm{even}} T^* M)$. Then, the Hilbert module in the triple representing $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$ is

$$\begin{aligned}
& p_t C^*(G, C_0(M,\mathbb{R}))\hat{\otimes}_{C^*(G,C_0(M,\mathbb{R}))} L^2(\Lambda^{\mathrm{even}} T^*M)\hat{\otimes} Cl_{0,1} \\
= \; & P_t L^2(\Lambda^{\mathrm{even}} T^*M)\hat{\otimes} Cl_{0,1} \\
= \; & \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\hat{\otimes} Cl_{0,1}.
\end{aligned}$$

The second "=" is because of Lemma 6.1.4. The representation ρ_{sig} is immediate to see.

For the operator part, let $\mathrm{id}_{Cl_{0,1}}$ be the identity map on $Cl_{0,1}$. According to the Grassmann connection, the operator in the triple should be

$$\left(P_t\hat{\otimes}\mathrm{id}_{Cl_{0,1}}\right)\mathscr{D}_{\mathrm{sig}}\left(P_t\hat{\otimes}\mathrm{id}_{Cl_{0,1}}\right)$$

on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\hat{\otimes} Cl_{0,1}$. To replace it by $\mathcal{F}_{\mathrm{sig}}$, we show that $\mathcal{F}_{\mathrm{sig}}$ is well-defined and self-adjoint, and the difference

$$\left(P_t\hat{\otimes}\mathrm{id}_{Cl_{0,1}}\right)\mathscr{D}_{\mathrm{sig}}\left(P_t\hat{\otimes}\mathrm{id}_{Cl_{0,1}}\right) - \mathcal{F}_{\mathrm{sig}}$$

is compact. Equivalently, we show that

$$F_{\mathrm{sig}} : \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f) \to \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)$$

is well-defined and skew-adjoint, and

$$P_t D(1+D^2)^{-1/2} P_t - P_t D(1+(P_t D)^2)^{-1/2}$$

is compact on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)$. We put the verification into Lemma 6.2.3 and Lemma 6.2.4. $\square$

Lemma 6.2.3. *The operator*

$$F_{\mathrm{sig}} : \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f) \to \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)$$

is well-defined and skew-adjoint.

Proof. By (6.2), $P_t D$ is formal self-adjoint on $\bigoplus_{i=0}^{4n+1} f \cdot \Omega^i_{\chi^{t/2}}(M)$. According to Theorem 6.1.2, the spectrum $\mathrm{spec}(P_t D)$ of $P_t D$ is an unbounded countable discrete closed subset of $\mathbb{R}$ (cf. [46, Theorem 10.4.20]) such that

$$\mathbf{H}^0_{\chi^{t/2}}(M, f) = \sum_{\lambda \in \mathrm{spec}(P_t D)} \ker(\lambda - P_t D).$$

We then apply functional calculus to obtain $P_t D \left(1 + (P_t D)^2\right)^{-1/2}$. In addition, $\hat{c}(\mathrm{dvol})$ commutes with P_t and anti-commutes with D, so $F_{\mathrm{sig}} = \hat{c}(\mathrm{dvol}) \circ P_t D \left(1 + (P_t D)^2\right)^{-1/2}$ is skew-adjoint on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)$. ☐

Lemma 6.2.4. *The operator*

$$P_t D(1 + D^2)^{-1/2} P_t - P_t D(1 + (P_t D)^2)^{-1/2}$$

is compact on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)$*.*

Proof. In fact, we have

$$\begin{aligned}
&P_t D(1 + (P_t D)^2)^{-1/2} - P_t D(1 + D^2)^{-1/2} \\
= &\int_0^\infty P_t D \left(((P_t D)^2 + 1 + s^2)^{-1} - (D^2 + 1 + s^2)^{-1}\right) ds \\
= &\int_0^\infty P_t D(D^2 + 1 + s^2)^{-1} \left(D^2 - (P_t D)^2\right) ((P_t D)^2 + 1 + s^2)^{-1} ds
\end{aligned}$$

$$= \int_0^\infty P_t D(D^2+1+s^2)^{-1}\left(D(1-P_t)D+(1-P_t)DP_tD\right)\left((P_tD)^2+1+s^2\right)^{-1}ds. \qquad (6.9)$$

Notice that for any $\omega \in \Omega^{\mathrm{even}}_{\chi^{t/2}}(M)$, we have $(1-P_t)D(f\omega) = c(df - fA^{-1}dA)\omega$. Therefore, $(1-P_t)D|_{\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)}$ is bounded. Also, since $P_tD((P_tD)^2+1+s^2)^{-1}|_{\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)}$ is compact, the integrand of (6.9) is compact on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)$. By the following four inequalities

$$\|P_tD((P_tD)^2+1+s^2)^{-1}\| \leqslant \frac{1}{2}(1+s^2)^{-1/2}, \quad \|D(D^2+1+s^2)^{-1}\| \leqslant \frac{1}{2}(1+s^2)^{-1/2},$$

$$\|D(D^2+1+s^2)^{-1}D\| \leqslant 1, \quad \|(P_tD)^2((P_tD)^2+1+s^2)^{-1}\| \leqslant 1,$$

the integral (6.9) is norm convergent. Therefore,

$$P_tD(1+D^2)^{-1/2}P_t - P_tD(1+(P_tD)^2)^{-1/2}$$

is compact on $\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)$. □

Now, we prove Theorem 1.3.4.

Proof of Theorem 1.3.4. By Theorem 5.2.20 and Proposition 6.2.2, $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$ is identified with

$$\dim\ker\left(F_{\mathrm{sig}} : \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f) \to \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\right) \mod 2 \qquad (6.10)$$

for all $t \in \mathbb{R}$. By Theorem 6.1.6, (6.10) equals

$$\sum_{i \text{ is even}} \dim H^i_{\chi^{t/2}}(M, d_{A_t}) \mod 2$$

for all $t \in \mathbb{R}$. A special case is $t = 0$, where we get rid of the twist by $\chi^{t/2}$ on differential forms. Following the same notation as the Witten deformation (1.4), we have

$$\operatorname{ind}_2^G([\mathscr{D}_{\mathrm{sig}}]) = \sum_{i \text{ is even}} \dim H^i_{\chi^{0/2}}(M, d_{A_0}) = \sum_{i \text{ is even}} \dim H^i_A(M)^G \mod 2.$$

Theorem 1.3.4 is now proved. □

Remark 6.2.5. The other special case is $t = 1$, By Lemma 5.3.1, when $t = 1$, $A(gx) = A(x)$ for any $x \in M$ and $g \in G$. Then, we can directly check the following commutative diagram which gives an isomorphism

$$\begin{tikzcd}
\Omega^i_{\chi^{1/2}}(M) \arrow[r, "d"] \arrow[d, "A_1^{-1}"] & \Omega^{i+1}_{\chi^{1/2}}(M) \arrow[d, "A_1^{-1}"] \\
\Omega^i_{\chi^{1/2}}(M) \arrow[r, "d_{A_1} = A_1^{-1} d A_1"] & \Omega^{i+1}_{\chi^{1/2}}(M)
\end{tikzcd}$$

between chain complexes. By the commutative diagram, we get rid of the deformation on d but still have the twists on differential forms. In general, by Lemma 5.3.1, we need to either twist differential forms using $\chi^{t/2}$, or deform d using A_t.

6.3 Vanishing theorem

In this section, we prove Theorem 1.3.5 using Atiyah's perturbation trick provided in the proof of Theorem 1.3.1 (See [1, Section 4] and [56, Chapter 7]).

Without loss of generality, we assume two G-invariant vector fields V_1 and V_2 on M such that $\langle V_1, V_2 \rangle = 0$ and $\langle V_1, V_1 \rangle = \langle V_2, V_2 \rangle = 1$.

As in [1, Section 4] and [56, Section 7.2], we perturb $D = d + d^*$ into

$$T = \frac{1}{2}\left(D - \hat{c}(V_1)\hat{c}(V_2)D\hat{c}(V_2)\hat{c}(V_1)\right).$$

Then, we have the same calculation as [56, (7.10)]:

Lemma 6.3.1. *The operator $\hat{c}(\mathrm{dvol})$ anti-commutes with T. Moreover, we have*

$$T = D + \frac{1}{2}\sum_{i=1}^{4n+1} c(e_i)\hat{c}(V_1)\hat{c}(\nabla_{e_i}V_1) + \frac{1}{2}\sum_{i=1}^{4n+1} c(e_i)\hat{c}(V_1)\hat{c}(V_2)\hat{c}(\nabla_{e_i}V_2)\hat{c}(V_1),$$

where $e_1, \cdots, e_{4n+1}$ is an oriented local orthonormal frame, and $c(e_i) = e_i^* \wedge - e_i \lrcorner$.

We notice that the tail term

$$\frac{1}{2}\sum_{i=1}^{4n+1} c(e_i)\hat{c}(V_1)\hat{c}(\nabla_{e_i}V_1) + \frac{1}{2}\sum_{i=1}^{4n+1} c(e_i)\hat{c}(V_1)\hat{c}(V_2)\hat{c}(\nabla_{e_i}V_2)\hat{c}(V_1)$$

is a G-equivariant order 0 differential operator exchanging odd and even forms. Let ρ_{sig} be the same as in Proposition 6.2.2, $F'_{\mathrm{sig}} = \hat{c}(\mathrm{dvol}) \circ P_t T(1 + (P_t T)^2)^{-1/2}$ and

$$\begin{aligned}\mathcal{F}'_{\mathrm{sig}} : \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1} &\to \mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1}\\ \omega + \eta v &\mapsto -(F'_{\mathrm{sig}}\omega + (F'_{\mathrm{sig}}\eta)v)v.\end{aligned}$$

Lemma 6.3.2. *The triple*

$$\left(\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M, f)\hat{\otimes} Cl_{0,1}, \rho_{\mathrm{sig}}, \mathcal{F}'_{\mathrm{sig}}\right)$$

is also a representative of $\mathrm{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$.

Proof. We only need to check that $P_tD\left(1+(P_tD)^2\right)^{-1/2} - P_tT\left(1+(P_tT)^2\right)^{-1/2}$ is compact. In fact, this difference equals

$$\frac{2}{\pi}\int_0^\infty ((P_tD)^2+s^2+1)^{-1}\left(P_tD\cdot P_tB\cdot P_tT-(s^2+1)P_tB\right)(T^2+s^2+1)^{-1}ds.$$

Since P_tB is bounded, we just need a procedure similar to the proof of Lemma 6.2.4. □

By Lemma 6.3.2 and Theorem 5.2.20, when we have V_1 and V_2, $\operatorname{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])$ is identified with

$$\dim\ker\left(F'_{\mathrm{sig}}:\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\to\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\right)\mod 2.$$

However, the operator $\hat{c}(V_1)\hat{c}(V_2)$ commutes with F'_{sig} and satisfies $(\hat{c}(V_1)\hat{c}(V_2))^2=-1$. Thus, it assigns a complex structure to the kernel of

$$F'_{\mathrm{sig}}:\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f)\to\mathbf{H}^{0,\mathrm{even}}_{\chi^{t/2}}(M,f),$$

making the kernel even-dimensional. Thus, $\operatorname{ind}_2^G([\mathscr{D}_{\mathrm{sig}}])=0$, and Theorem 1.3.5 is proved.

6.4 Modular characters and cohomology

In this section, we explain the influences of the modular character χ and the parameter t. Also, we illustrate that the condition in Theorem 1.3.5 makes sense.

Recall the i-th cohomology group $H^i_{\chi^{t/2}}(M,d_{A_t})$ associated with (6.1). By the proof of Theorem 1.3.4, if we define $k(M,G)$ using cohomology groups, it should be given as follows.

Definition 6.4.1. For each $t \in \mathbb{R}$, we call

$$k_t(M, G) \coloneqq \sum_{i \text{ is even}} \dim H^i_{\chi^{t/2}}(M, d_{A_t}) \mod 2$$

the t-th Kervaire semi-characteristic. In particular, $k_t(M, G) = k(M, G)$ for all t.

Here is the question. Is it possible that we define $k_t(M, G)$ without deforming d into $d_{A_t} = A_t^{-1} d A_t$? This seems to be confirmed by the "canonical chain isomorphism"

$$\begin{tikzcd}
\Omega^i_{\chi^{t/2}}(M) \arrow[r, "d"] \arrow[d, "A_t^{-1}"] & \Omega^{i+1}_{\chi^{t/2}}(M) \arrow[d, "A_t^{-1}"] \\
\Omega^i_{\chi^{t/2}}(M) \arrow[r, "A_t^{-1} d A_t"] & \Omega^{i+1}_{\chi^{t/2}}(M)
\end{tikzcd}.$$

Unfortunately, by Lemma 5.3.1, for any ω satisfying $g^*\omega = \chi(g)^{t/2}\omega$, we have $g^*(A_t^{-1}\omega) = \chi(g)^{t-\frac{1}{2}}\omega$. So, the commutative diagram is well-defined only when $t = 1$ (See Remark 6.2.5). Therefore, no matter $t = 0$ or 1, the formula for $k(M, G)$ always involves an adjustment by either $A = A_0$ on d or $\chi^{1/2}$ on forms.

We still wish to use the most intuitive G-invariant cohomology. Let $\Omega^i(M)^G$ be the space of all smooth i-forms ω satisfying $g^*\omega = \omega$ for all $g \in G$. Then, we let $H^i(M)^G$ be the i-th cohomology group of the chain complex

$$d : \Omega^i(M)^G \to \Omega^{i+1}(M)^G. \tag{6.11}$$

It would be nice if we could replace $k(M, G)$ by the following more intuitive

$$k'(M, G) \coloneqq \sum_{i \text{ is even}} \dim H^i(M)^G \mod 2.$$

However, since M is noncompact, and G can be non-unimodular, this $k'(M, G)$ does not fit well into the vanishing theorem as $k(M, G)$.

Example 6.4.2. Let Aff(1) be the affine group

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0, b \in \mathbb{R} \right\}.$$

Its Lie algebra is generated by two matrices $Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. They satisfy $[Z, W] = W$, so Aff(1) is non-unimodular (See [29, Example 7.5.25(c)]). Now, we let

$$G = \mathrm{Aff}(1) \times \mathrm{Aff}(1) \times \mathbb{R}$$

with the left action on itself. This action is free and proper, and G admits five everywhere independent G-invariant vector fields. However, by applying [40, Proposition 14.29] together with $[Z, W] = W$, we find

$$H^0(G)^G \cong \mathbb{R}, H^1(G)^G \cong \mathbb{R}^3, H^2(G)^G \cong \mathbb{R}^5, H^3(G)^G \cong \mathbb{R}^4, H^4(G)^G \cong \mathbb{R}, H^5(G)^G = 0.$$

Therefore, we see that

$$k'(G, G) = \sum_{i=0,2,4} \dim H^i(G)^G = 1 \neq 0 \mod 2,$$

which does not fit into the vanishing theorem.

The phenomenon in Example 6.4.2 is because $H^i(M)^G$ in general does not satisfy the Poincaré duality (See the proof of [51, Theorem 4.1]), which means $k'(M, G)$ cannot represent the even and odd degree parts at the same time, and is not a perfect choice to be the semi-characteristic. However, when G is unimodular, $k'(M,G) = k(M,G)$.

We end this thesis by an example about the condition in Theorem 1.3.5.

Example 6.4.3. We equip $\mathbb{Z}$ with the discrete topology and the addition operation, making it a noncompact 0-dimensional Lie group. Then, we equip the 5-dimensional $\mathbb{S}^5 \times \mathbb{Z}$ with the natural action by integers. By the de Rham cohomology of $\mathbb{S}^5$, we find

$$
\begin{aligned}
&H^0(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong H^5(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong \mathbb{R}, \\
&H^1(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong H^2(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong H^3(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong H^4(\mathbb{S}^5 \times \mathbb{Z})^{\mathbb{Z}} \cong 0.
\end{aligned}
$$

Therefore, $k(\mathbb{S}^5 \times \mathbb{Z}, \mathbb{Z}) = k'(\mathbb{S}^5 \times \mathbb{Z}, \mathbb{Z}) = 1$. In fact, $\mathbb{S}^5 \times \mathbb{Z}$ has only one everywhere independent $\mathbb{Z}$-invariant vector field on it. This vector field is induced by the one on $\mathbb{S}^5$.

Example 6.4.3 shows that the condition of two vector fields in Theorem 1.3.5 makes sense. Meanwhile, we see that the connectedness of M or G is not required.

References

[1] Michael F. Atiyah. *Vector Fields on Manifolds*. Springer, 1970.

[2] Michael F. Atiyah and Isadore M. Singer. "Index theory for skew-adjoint Fredholm operators." In: *Publications Mathématiques de l'IHÉS* 37 (1969), pp. 5–26.

[3] Michèle Audin. *Torus Actions on Symplectic Manifolds*. Birkhäuser, 2012.

[4] David M. Austin and Peter J. Braam. "Morse-Bott theory and equivariant cohomology." In: *The Floer Memorial Volume*. Birkhäuser, 1995, pp. 123–183.

[5] Augustin Banyaga and David E. Hurtubise. "Morse-Bott homology." In: *Trans. Amer. Math. Soc.* 362.8 (2010), pp. 3997–4043.

[6] Paul Baum and Alain Connes. "Geometric K-theory for Lie groups and foliations." In: *L'Enseignement Mathématique. IIe Série* 46 (2000), pp. 3–42.

[7] Paul Baum, Alain Connes, and Nigel Higson. "Classifying Space for Proper Actions and K-theory of group C^*-algebras." In: *Contemporary Mathematics* 167 (1994), pp. 241–291.

[8] Paul F. Baum, Guillermo Cortiñas, Ralf Meyer, Rubén Sánchez-García, Marco Schlichting, and Bertrand Toën. *Topics in Algebraic and Topological K-Theory*. Springer Berlin Heidelberg, 2010.

[9] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat Kernels and Dirac Operators*. Springer, 2003.

[10] Jean-Michel Bismut. "The Witten Complex and the Degenerate Morse Inequalities." In: *Journal of Differential Geometry* 23.3 (1986), pp. 207–240.

[11] Jean-Michel Bismut and Gilles Lebeau. "Complex immersions and Quillen metrics." In: *Publications mathématiques de l'I.H.É.S.* 74 (1991), pp. 1–298.

[12] Jean-Michel Bismut and Weiping Zhang. "Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle." In: *Geometric & Functional Analysis* 4.2 (1994), pp. 136–212.

[13] Bruce Blackadar. *K-Theory for Operator Algebras*. 2nd edition. Cambridge University Press, 1998.

[14] Raoul Bott. "Lectures on Morse theory, old and new." In: *Bulletin (New Series) of the American Mathematical Society* 7.2 (1982), pp. 331–358.

[15] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer, 2013.

[16] Chris Bourne. "Locally equivalent quasifree states and index theory." In: *Journal of Physics A: Mathematical and Theoretical* 55.10 (2022), p. 104004.

[17] Glen E. Bredon. *Introduction to Compact Transformation Groups*. Elsevier Science, 1972.

[18] Alain Connes. *Noncommutative Geometry*. Elsevier Science, 1995.

[19] Alain Connes and Georges Skandalis. "The Longitudinal Index Theorem for Foliations." In: *Publications of the Research Institute for Mathematical Sciences* 20 (1984).

[20] Kenneth R. Davidson. *C^*-Algebras by Example*. Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs. American Mathematical Society, 1996.

[21] James F. Davis and R. James Milgram. "Semicharacteristics, bordism, and free group actions." In: *Transactions of the American Mathematical Society* 312 (1989), pp. 55–83.

[22] Yves Félix, John Oprea, and Daniel Tanré. *Algebraic Models in Geometry*. Oxford University Press, 2008.

[23] Werner H. Greub. *Linear Algebra*. Springer, 1981.

[24] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer, 2015.

[25] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[26] Bernard Helffer and Johannes Sjöstrand. "A proof of the Bott inequalities." In: *Algebraic Analysis*. Academic Press, 1988, pp. 171–183.

[27] Bernard Helffer and Johannes Sjöstrand. "Puits multiples en mécanique semi-classique IV: Etude du complexe de Witten." In: *Commun. P. D. E.* 10.3 (1985), pp. 245–340.

[28] Bernard Helffer and Johannes Sjöstrand. "Puits multiples en mécanique semi-classique VI. (Cas des puits sous-variétés)." In: *Annales de l'I.H.P. Physique théorique* 46.4 (1987), pp. 353–372.

[29] Joachim Hilgert and Karl-Hermann Neeb. *Structure and Geometry of Lie Groups*. Springer, 2011.

[30] Peter Hochs and Yanli Song. "An equivariant index for proper actions II: Properties and applications." In: *Journal of Noncommutative Geometry* 12.1 (2018), pp. 157–193.

[31] Michael Hutchings. "Lectures notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)." In: Lecture notes at UC Berkeley (2002).

[32] Gennadi G. Kasparov. "Equivariant KK-theory and the Novikov conjecture." In: *Invent Math* 91 (1988).

[33] Gennadi G. Kasparov. “Operator K-theory and its applications: Elliptic operators, group representations, higher signatures, C^*-extensions.” In: *Proceedings of the International Congress of Mathematicians, Vol. 2 (Warsaw, 1983)*. PWN, Warsaw, 1983, pp. 987–1000.

[34] Gennadi G. Kasparov. “The operator K-functor and extensions of C^*-algebras.” In: *Mathematics of the USSR-Izvestiya* 16.3 (1981).

[35] Michel Kervaire. “Courbure intégrale généralisée et homotopie.” In: *Mathematische Annalen* 131 (1956), pp. 219–252.

[36] Demeter Krupka and David Saunders. *Handbook of Global Analysis*. Elsevier Science, 2011.

[37] Janko Latschev. “Gradient flows of Morse-Bott functions.” In: *Mathematische Annalen* 318 (2000), pp. 731–759.

[38] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1989.

[39] John M. Lee. *Introduction to Riemannian Manifolds*. 2nd. Springer, 2019.

[40] John M. Lee. *Introduction to Smooth Manifolds*. 2nd edition. Springer, 2012.

[41] Wen Lu. “Morse–Bott inequalities in the presence of a compact Lie group action and applications.” In: *Differential Geometry and its Applications* 32 (2014), pp. 68–87.

[42] George Lusztig, John Milnor, and Franklin P. Peterson. “Semi-characteristics and cobordism.” In: *Topology* 8.4 (1969), pp. 357–359.

[43] Varghese Mathai and Weiping Zhang. “Geometric quantization for proper actions (With an Appendix by U. Bunke).” In: *Advances in Mathematics* 225.3 (2010), pp. 1224–1247.

[44] Eckhard Meinrenken. “Group actions on manifolds.” In: Lecture notes at University of Toronto (2003).

[45] Shigeyuki Morita, Teruko Nagase, and Katsumi Nomizu. *Geometry of Differential Forms*. American Mathematical Society, 2001.

[46] Liviu I. Nicolaescu. *Lectures on the Geometry of Manifolds*. 3rd. World Scientific, 2020.

[47] John Roe. “Comparing analytic assembly maps.” In: *The Quarterly Journal of Mathematics* 53.2 (2002), pp. 241–248.

[48] John Roe. *Elliptic Operators, Topology, and Asymptotic Methods*. 2nd. Taylor & Francis, 1999.

[49] Jonathan Rosenberg. “Structure and applications of real C^*-algebras.” arXiv:1505.04091.

[50] Stephen Smale. “On Gradient Dynamical Systems.” In: *Annals of Mathematics* 74.1 (1961), pp. 199–206.

[51] Xiang Tang, Yi-Jun Yao, and Weiping Zhang. “Hopf cyclic cohomology and Hodge theory for proper actions.” In: *Journal of Noncommutative Geometry* 7 (2013), pp. 885–905.

[52] Michael E. Taylor. *Partial Differential Equations II: Qualitative Studies of Linear Equations*. 3rd. Springer, 2023.

[53] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, 2013.

[54] Arthur G. Wasserman. “Equivariant differential topology.” In: *Topology* 8.2 (1969), pp. 127–150.

[55] Edward Witten. “Supersymmetry and Morse theory.” In: *Journal of Differential Geometry* 17.4 (1982), pp. 661–692.

[56] Weiping Zhang. *Lectures on Chern-Weil Theory and Witten Deformations*. World Scientific, 2001.

[57] Weiping Zhang and Huitao Feng. *Geometry and Analysis on Manifolds*. Chinese edition. Higher Education Press, 2022. 张伟平, 冯惠涛: 流形上的几何与分析. 高等教育出版社, 2022.