Background Material: Cox Default Processes, CVA, and Expected Exposure

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Abstract

In this note we provide background material for a range of subjects related to the quantification of default risk in securities portfolios. Amongst other topics, we provide background on on Cox default processes; introduce the concepts of unilateral and bilateral CVA/DVA; and discuss the topic of expected exposure. Special emphasis is put on the numerical computation of the various quantities, both for credit valuation adjustment and for capital calculations (IMM).

1 Cox Process for Default[1]

1.1 Basic Setup

For a given company C, let its random default time be denoted τ. A classical method to model τ is to set it equal to the first jump time in a Poisson process with intensity $\lambda > 0$. If the probability measure is $\mathbb{Q}$, we then have, informally,

$$\mathbb{Q}\left(\tau \in [t, t + dt] | \tau > t\right) = \lambda \, dt.$$

Equivalently, for $T > t$, we have the *survival probability*

$$\mathbb{Q}\left(\tau > T | \tau > t\right) = e^{-\lambda(T-t)}. \tag{1}$$

As a constant intensity is insufficient to calibrate the model to observed term structures of bond and CDS spreads, it is natural to extend the model to allow for time-dependent intensity, in which case (1) becomes

$$\mathbb{Q}\left(\tau > T | \tau > t\right) = e^{-\int_t^T \lambda(u)\, du}, \tag{2}$$

where $\lambda(t)$ is a deterministic function of time.

While time-dependent default intensity often is enough to model vanilla instruments, more complicated securities or modeling frameworks will require the usage of stochastic intensities. In this case, we may rely on a *Cox process* formalism where default events can be considered originated in a two-step simulation process:

[1]For more details on Cox processes for default modelling, see [4].

1. Assuming that no default has taken place up to (current) time t, simulate[2] a stochastic path of $\lambda(u)$ for all $u > t$.

2. Freeze the path of $\lambda(\cdot)$, and use the result (2) to generate the default time τ. Specifically, for Step 2 we may draw a uniform $U(0,1)$ random number u, independent of the intensity process, and find τ from the inverse CDF method. That is, we generate τ by inverting the expression

$$e^{-\int_t^\tau \lambda(s)\,ds} = u,$$

where it is understood that the integral is applied to the random path of λ created in Step 1.

The two-step procedure of the Cox process setup may be formalized as

$$\mathbb{Q}\left(\tau > T | \tau > t\right) = \mathrm{E}_t\left(e^{-\int_t^T \lambda(u)\,du}\right), \tag{3}$$

where $\mathrm{E}_t(x) = \mathrm{E}(x|\mathcal{F}_t)$ denotes the $\mathbb{Q}$-expectation of x conditional on the information available at time t (as represented by the filtration $\mathcal{F}_t$) . It is understood that the expectation is over all paths of λ.

We note that the expression (3) is essentially identical to the pricing formula for a discount bond, except the short rate $r(t)$ has been replaced by an intensity $\lambda(t)$. This similarity is very convenient as it often allows us to adapt interest rate models and techniques to the default setting.

1.2 Risky Security Pricing in Cox Process Setting

We observe that, by definition,

$$\mathbb{Q}\left(\tau > T | \mathcal{F}_t\right) = \mathrm{E}_t\left(1_{\tau>T}\right)$$

where 1_A is an indicator function for the set A. From (3), we may write

$$\mathrm{E}_t\left(1_{\tau>T}\right) = 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^T \lambda(u)\,du}\right)$$

so, in a sense, the Cox process setting allows us to replace default indicator functions with stochastic survival probabilities. This analogy is especially useful for the pricing of contingent claims where the payout is subject to default risk.

For instance, consider a derivative where company C promises to pay a (possibly stochastic) non-negative payout $X(T) \geq 0$ at time T. In the case C defaults before time T, however, assume that the payment will not be made. From the perspective of C's trade counterparty, the net payout on the derivative is therefore

$$Y(T) = X(T)1_{\tau>T}.$$

[2] Some type of dynamic model for $\lambda(t)$ is required for this, typically an SDE.

Setting

$$\beta(t) = e^{\int_0^t r(u)\,du},$$

the usual risk-neutral pricing machinery shows that the time t price of this derivative is, for $t < T$,

$$\begin{aligned} V_{risky}(t) &= \mathrm{E}_t\left(\frac{\beta(t)}{\beta(T)} Y(T)\right) \\ &= \mathrm{E}_t\left(e^{-\int_t^T r(u)\,du} X(T) 1_{\tau > T}\right) \\ &= 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^T r(u)\,du} X(T) e^{-\int_t^T \lambda(u)\,du}\right) \\ &= 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^T [r(u)+\lambda(u)]\,du} X(T)\right), \end{aligned} \tag{4}$$

where we assume that the expectations are in the risk-neutral probability measure, i.e., the equivalent martingale measure induced by β. Notice how we in the third equality used the Cox process property to effectively replace the indicator function with a stochastic survival probability. As a result, the final pricing result (4) captures the credit risk embedded in the derivative through an increase of the discount rate, from r to $r + \lambda$.

We observe in passing that if $X(T) = 1$, the resulting expression

$$V_{risky}(t) = 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^T [r(u)+\lambda(u)]\,du}\right) \tag{5}$$

is the price formula for a so-called *risky discount bond*.

1.3 Recovery

Above, we assumed that a default of C would result in C's counterparty receiving nothing. In reality, a bankruptcy court will step in and use whatever assets C possess at the time of default to settle counterparty claims. As a result, some non-negative recovery amount $R(\tau)$ is normally paid to the counterparty at the time of default τ. The value of this payment may be written

$$V_R(t) = 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^\tau r(u)\,du} R(\tau) 1_{\tau \le T}\right). \tag{6}$$

Notice here the random upper limit on the discount factor, as well as the indicator function $1_{\tau \le T}$ which reflects the fact that the recovery payment will only take place if C defaults prior to the maturity T of the derivative in question.

The expression (6) is not particularly convenient, so consider rewriting it by integrating over the (pathwise) density of the default time:

$$\begin{aligned} V_R(t) &= 1_{\tau>t}\mathrm{E}_t\left(e^{-\int_t^\tau r(u)\,du} R(\tau) 1_{\tau \le T}\right) \\ &= 1_{\tau>t}\mathrm{E}_t\left(\int_t^T e^{-\int_t^s r(u)\,du} R(s)\lambda(s) e^{-\int_t^s \lambda(u)\,du}\,ds\right) \\ &= 1_{\tau>t}\mathrm{E}_t\left(\int_t^T e^{-\int_t^s [r(u)+\lambda(u)]\,du} R(s)\lambda(s)\,ds\right). \end{aligned} \tag{7}$$

We invite the reader to justify the second equality (hint: start by considering the well-known density of the first jump time of a Poisson process).

The total price $V(t)$ of a credit-risky contingent claim with promised payment $X(T) \geq 0$ and recovery $R \geq 0$ equals the sum of the pricing expressions (4) and (7). That is,

$$V(t) = V_{risky}(t) + V_R(t). \tag{8}$$

In practice, the actual amount of recovery paid depends on a variety of factors, including the characteristics of the security in question, the remainder of the trades done with the same counterparty, the jurisdiction, the detailed trade terms[3], the amount of collateral (if any), and more. Roughly speaking[4], however, one often models the recovery rate as either an exogenously specified constant (typically a certain fraction of the deal notional) or as a constant fraction of the risk-free value of the security (or portfolio) in question. For the latter, we evidently have, if $\tau < T$,

$$R(\tau) = \alpha \mathrm{E}_\tau \left(e^{-\int_\tau^T r(u)\,du} X(T) \right), \tag{9}$$

where α is a constant between 0 and 1. While this case may appear rather intractable, plugging (9) into (6) shows that we simply have

$$\begin{aligned}
V_R(t) &= \alpha 1_{\tau>t} \mathrm{E}_t \left(e^{-\int_t^\tau r(u)\,du} \mathrm{E}_\tau \left(e^{-\int_\tau^T r(u)\,du} X(T) \right) 1_{\tau \leq T} \right) \\
&= \alpha 1_{\tau>t} \mathrm{E}_t \left(e^{-\int_t^\tau r(u)\,du} e^{-\int_\tau^T r(u)\,du} X(T) 1_{\tau \leq T} \right) \\
&= \alpha 1_{\tau>t} \mathrm{E}_t \left(e^{-\int_t^T r(u)\,du} X(T) \left(1 - 1_{\tau>T}\right) \right) \\
&= \alpha 1_{\tau>t} \left\{ V_0(t) - V_{risky}(t) \right\},
\end{aligned}$$

where we have used the tower property of conditional expectations in the second equality, and where

$$V_0(t) = \mathrm{E}_t \left(e^{-\int_t^T r(u)\,du} X(T) \right)$$

is the default-free time t price of the derivative.

2 Unilateral CVA in the Cox Process Setting

2.1 Portfolio Exposure

Consider a situation where a firm B has traded a portfolio of N securities with firm C; let $v_i(t)$ be the time t default-free value of the ith security, $i = 1, \ldots, N$, as seen from the perspective of firm B. The default-free value to B of the portfolio is denoted[5] $\pi(t)$.

[3] In the swaps world, trades between counterparties is normally governed by a legally binding document called an *ISDA Master Agreement*. On top of this, there might be a CSA (Credit Support Annex) that details any collateral agreement.

[4] There is significant disagreement in the literature, and in industry, about how to best model recovery. As ISDA agreements have changed language over time as pertains to recovery, it might also be the case that different recovery models must be applied to different counterparties (depending on ISDA version).

[5] From the perspective of firm C, the value is $-\pi(t)$.

We assume that the time t values of all trades can be computed from knowledge of a set of M Markov financial market variables[6] $z(t) = (z_1(t), z_2(t), \ldots, z_M(t))^\top$, allowing us to write $v_i(t) = v_i(t, z(t))$ and $\pi(t) = \pi(t, z(t))$. Clearly we have

$$\pi(t, z(t)) = \sum_{i=1}^{N} v_i(t, z(t)). \tag{10}$$

Assume, for now, that firm B has no risk of bankruptcy, but that the default of firm C is characterized by a Cox process with random intensity $\lambda_C(t)$. If C defaults on its obligations, firm B may suffer a loss in the case that it is owed money by C at the time of default, i.e., if $\pi > 0$. In the absence of any credit mitigants, we may measure the potential loss through the *exposure* of B to C, defined as

$$E_C(t, z(t)) = \pi(t, z(t))^+, \tag{11}$$

where $x^+ = \max(x, 0)$. We use subscript C to indicate that the exposure is to C. Note that the exposure is zero if the value of the portfolio (to B) is negative, since C is then a creditor to B at the time of default. The exposure is also zero if default takes place after all securities in the portfolio have matured, as in this case $\pi(t) = 0$.

2.2 Credit Mitigants

It is rare that derivative portfolios are traded completely "naked", without any contractual features aiming to mitigate credit exposure. We list some common mitigation strategies here, all of which aim to reduce exposure.

2.2.1 Collateral

Collateral clauses are portfolio-level agreements under which a counterparty is required to post collateral (typically in the form of cash) if the exposure on the portfolio is large. There are a variety of collateral agreements, but a common clause involves a threshold value $c \geq 0$. To the extent that the portfolio exposure rises above c, counterparty C will be asked to post collateral to B in the amount of $\pi - c$ to bring back the exposure to c. As a result, B's effective exposure becomes[7]

$$E_C(t) = (\min(\pi(t), c))^+, \tag{12}$$

where $\pi(t)$ is computed from (10).

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[6] For swaps portfolios, $z(t)$ will consist of state variables that drive the state of the forward curve. See [1] for examples.

[7] A technical note: in reality the exposure may rise slightly above c, due to a time-lag between the date on which the collateral call is made and the date on which the collateral is actually posted. The time-lag is normally small (one or two weeks are common assumption), and often ignored. Ad-hoc adjustments for time-lags are, however, possible.

In many cases, the collateral trigger c is not a constant, but instead depends on the credit-worthiness of counterparty C. For instance, a contractually specified table may map c to the credit rating of firm C; the map will virtually always cause c to decrease (and tighten up the collateral protection level) as the rating of counterparty C deteriorates. To incorporate ratings-based thresholds into a model, we can assume that the ratings-based threshold maps can be translated into functional forms of C's default intensity processes $\lambda_C(t)$. Specifically, we assume that

$$E_C(t) = \left(\min\left(\pi(t), m\left(\lambda_C(t)\right)\right)\right)^+, \tag{13}$$

for a given time-independent mapping function m. Estimating m will require building a link between credit ratings and proxies for λ_C (often bond or CDS spreads[8]), an exercise that can be done using historical data.

2.2.2 Downgrade Provision

Some portfolios may be subject to *downgrade provision*, a more extreme form of ratings-based credit mitigation than the collateral agreements described above. Under the terms of unilateral downgrade provision (in B's favor), the entire portfolio will be settled at its market value the first time the counterparty credit rating falls below a predefined level.

In our setup, credit ratings of C can (as above) be mapped to levels of the credit processes $\lambda_C(t)$, so downgrade provisions will, effectively, involve the specification of a λ_C-threshold D that cause termination of the portfolio. We define

$$\Gamma = \inf\{t : \lambda_C(t) > D\},$$

and incorporation of a downgrade provision in the exposure computation then proceeds by adjusting (11) to

$$E_C(t) = 1_{t<\Gamma}\pi(t)^+. \tag{14}$$

Notice that downgrade provisions and collateral agreements rarely, if ever, co-exist in the same portfolio.

2.2.3 Termination Clauses

So far we have only described portfolio-level credit mitigants. In many cases, some individual trades may additionally be subject to *termination clauses*. Consider trade i and assume that the trade has a *unilateral termination option* (in B's favor) running on a trade-specific schedule $\{T_k\}$; typically the dates in this schedule are 5 years apart. On a date T_k in the schedule, B has the right to terminate the trade at the fair market value of the trade. Provided that such termination takes place, we then must have $v_i(t) = 0$ for all $t > T_k$.

[8] Intensities λ are associated with CDS spreads s through the approximate relation $\lambda = s/(1 - R_{CDS})$, where R_{CDS} is the recovery rate on CDSs (often 40%).

Exercising a termination option will negatively impact B's relationship with C and may hurt B's overall reputation in the marketplace, so the termination decision must be approached carefully. In practice, two conditions must be met for the termination option to be exercised: a) the exposure of the trade to B must be material; and b) the credit quality of C must be bad enough for significant doubt to exist whether the counterparty will survive until the next termination date (T_{k+1}). We can incorporate a) by assuming that some materiality threshold $H \geq 0$ exists, such that B termination option will only be exercised at time T_k provided that $v_i(T_k) > H$. To model b), we assume that the termination option will only be exercised if the default intensity process $\lambda_C(T_k)$ exceeds a certain threshold K.

To summarize, for a unilateral termination option in B's favor, we define an exercise time of

$$\mathcal{T}_i = \inf \{t \in \{T_k\} : v_i(t) > H, \lambda_C(t) > K\}, \tag{15}$$

where we understand that the $\{T_k\}$ schedule is specific to trade i. Note that the portfolio value in (10) must then be re-written as

$$\pi(t, z(t)) = \sum_{i=1}^{N} v_i(t, z(t)) 1_{t<\mathcal{T}_i}. \tag{16}$$

The various definitions of exposure in (11)-(14) carry over to the case of termination clauses, provided that the definition of $\pi(t)$ in (16) is used.

We point out that in addition to mutual and unilateral termination, some trades may have *mandatory termination*, where termination will be strictly enforced on the first possible date (T_1). We can accommodate this in (15) above by simply setting $K = H = -\infty$.

2.3 Unilateral CVA

To characterize actual (rather than just potential) losses, we need to combine the exposure with the default process of C. For this, first notice that at the time τ of default, there are two opposing valuation effects. First, if the exposure is positive B will lose $E_C(\tau)$, the present value of all the deal cash flows that were promised, net of posted collateral[9]. Second, B gains whatever recovery $R_C(\tau)$ may be granted by the bankruptcy court. Applying (7), we may then compute the unilateral[10] *credit valuation adjustment*[11] (CVA)

[9]We assume that there is no delay in settling the portfolio.

[10]The term unilateral refers to the fact that we only consider C to have any default risk. We consider how to relax this assumption in Section 3 below.

[11]This number is here defined to be positive, so it needs to be *subtracted* from the default-free portfolio value to computed the fair (credit-adjusted) portfolio value.

as

$$
\begin{aligned}
CVA(t) &= 1_{\tau_C>t}\mathrm{E}_t\left(\int_t^{T_{\max}} e^{-\int_t^s [r(u,z(u))+\lambda_C(u)]\,du}\,(E_C(s,z(s))-R_C(s))\,\lambda_C(s)\,ds\right)\\
&= (1-\alpha_C)1_{\tau_C>t}\mathrm{E}_t\left(\int_t^{T_{\max}} e^{-\int_t^s [r(u,z(u))+\lambda_C(u)]\,du} E_C(s,z(s))\lambda_C(s)\,ds\right) \qquad (17)
\end{aligned}
$$

where $\alpha_C \in [0,1]$ and where we have assumed, as is standard for most derivatives portfolios, that the recovery amount is an α_C-fraction of the default-free exposure value (similar to (9)). Note that that we assume that the short-rate is a function of our state-vector $z(t)$, and that the upper limit of the main integral in (17) is set to be the longest maturity of any trade in the portfolio, here denoted $T_{\max}$. We emphasize that the exposure variable E depends on both the market state variable process z, but also (suppressed in the notation) potentially on the path of the credit intensity $\lambda_C(t)$, through its impact on termination and downgrade decisions.

2.4 Practical Computation of Unilateral CVA

In practice, computation of CVA at time 0 is done by Monte Carlo simulation on a time-grid $\{t_k\}_{k=1}^K$, where $t_K \geq T_{\max}$. Loosely speaking, an algorithm can proceed as follows:

1. Simulate in $\mathbb{Q}$ all relevant[12] market data $z(t_k)$, $k = 1, 2, \ldots, K$.

2. Simulate in $\mathbb{Q}$ the intensity process path $\lambda_C(t_k)$, $k = 1, 2, \ldots, K$.

3. Using the outcome of Steps 1 and 2 and the fact that $r(t) = r(t, z(t))$, construct for all t_k (numerically) the integrals

$$
\beta_\lambda(t_k) = e^{\int_0^{t_k} \lambda_C(u)\,du}, \quad \beta(t_k) = e^{\int_0^{t_k} r(u)\,du}.
$$

4. At each t_k, use the market state vector $z(t_k)$ to price all trades in the portfolio. Also construct the portfolio price $\pi(t_k)$, applying downgrade provisions as needed (see (16)).

5. Use (11)-(14) to compute $E_C(t_1), E_C(t_2), \ldots, E_C(t_K)$.

6. Set $Q(t_k) = E_C(t_k)\lambda_C(t_k)\beta(t_k)^{-1}\beta_\lambda(t_k)^{-1}$, $k = 1, 2, \ldots, K$.

7. Repeat M times, and form the sample averages

$$
\widehat{A}(t_k) = \frac{1}{M}\sum_{m=1}^{M} Q^{(m)}(t_k), \qquad (18)
$$

where $Q^{(m)}$ denotes the random sample value of Q in path number m.

[12]Only the particular risk factors that affect a particular portfolio should be included. Some pre-processing may be necessary before the simulation is launched.

8. Finally compute the estimated CVA as

$$\widehat{CVA} = (1 - \alpha_C) \sum_{k=1}^{K} \widehat{A}(t_k)\,(t_k - t_{k-1})\,, \tag{19}$$

with $t_0 = 0$.

Several comments are in order here.

- In Step 1, to ensure consistency with observed market values of the underlying portfolio, a properly calibrated pricing model is required to perturb $z(t)$ through time. For interest rate portfolios, a good choice is a quasi-Gaussian model, see [1].
- In Step 2, the simulation of $\lambda_C(t)$ will require a model calibrated to quoted bond or CDS spreads (a quasi-Gaussian model is reasonable). It is important that $\lambda_C(t)$ be correlated in a realistic way with $z(t)$, to ensure that *wrong-way risk* is properly captured[13].
- While the algorithm has been outlined for a single counterparty portfolio only, there are significant computational benefits in running the algorithm on multiple portfolios simultaneously, as duplication of market data simulation will be avoided. Other computational tricks can be considered, e.g. the option of terminating simulations early if a downgrade provision is triggered.

3 Unilateral DVA and Bilateral CVA/DVA

3.1 DVA and Net CVA

In Section 2, when computing the CVA from B's perspective, we assumed that B itself could not default. This assumption is obviously not realistic, and we could equally well have defined a CVA charge from the perspective of C. This (if assumed unilateral) would involve the default intensity λ_B of firm B, and would basically reverse all portfolio values when computing exposures. For instance, if $E_C(t) = \pi(t)^+$ is the (naked) portfolio exposure from the perspective of B, then $E_B(t) = (-\pi(t))^+$ is the exposure from the perspective of C. Altering signs[14] and credit intensity in this manner, we would end up with an expression for C's CVA adjustment for B-exposure similar to (17), except that subscripts C are replaced with B everywhere. From the perspective of B, C's CVA charge

[13] If a counterparty is likely to default in market scenarios where the exposure is higher (lower) than average, we say that a portfolio has wrong-way (right-way) risk. Correlating the default intensity with the market state vector $z(t)$ will help capture right- and wrong-way risk.

[14] In addition, we need to ensure that all relevant credit mitigants are applied. Often, credit mitigation clauses are not symmetric, so just because B has a particular type of protection in place against a default of C it does not follow that C has the same rights as B. For instance, although collateral postings are often symmetric and bilateral in nature, there are many cases where only one firm (typically the weakest) is obligated to post collateral.

can be considered a "negative" CVA, a value *benefit* that offsets the CVA *liability* from (17). Indeed, from B's perspective, the option effectively owned by B to default its way out of its obligation should logically result in an increase in the portfolio value. This increase (as seen from B) is often called a *debt valuation adjustment* (DVA). The definition is

$$DVA(t) = (1-\alpha_B)1_{\tau_B>t}\mathrm{E}_t\left(\int_t^{T_{\max}} e^{-\int_t^s [r(u,z(u))+\lambda_B(u)]\,du} E_B(s,z(s))\lambda_B(s)\,ds\right). \tag{20}$$

Taking into account both CVA and DVA, the total credit adjusted portfolio value (to B) would be computed as

$$\pi_{risky}(t) = \pi(t) + DVA(t) - CVA(t) \triangleq \pi(t) + V(t).$$

The difference $V(t) = DVA(t) - CVA(t)$ is often known as the *net CVA charge*.

It is interesting to note that $\pi_{risky}(t)$ will increase as firm B's credit spreads increase, as the DVA benefit obviously grows in λ_B. In other words, when firm B's credit quality deteriorates, the value of its securities portfolio will increase, all things equal. This somewhat paradoxical result is a consequence of the fact that B's default option will increase in value as it becomes more likely that B defaults.

Many financial pundits consider DVA benefits counterintuitive and even misleading, and there are many in industry that would like to see it removed from accounting laws (where it is currently mandatory). This, however, will cause an breakdown in the law of one price, as in this case the reported value from B's perspective would equal $\pi(t) - CVA(t)$, and the price from C's perspective would[15] equal $-\pi(t) - DVA(t)$. As these prices generally do not net out to zero, the market does not clear.

3.2 Bilateral CVA/DVA

Above, when computing CVA for B we assumed that B could not default. On the other hand, when computing DVA for B, we assumed that C could not default. It should be evident that neither calculation is fully coherent, as ideally we would want to incorporate *joint* default risk into both CVA and DVA calculations. To see how this impacts the analysis, let us revisit the computation of CVA from B's perspective. If C defaults at time τ_C, we earlier argued that the exposure to B is $\pi(\tau_C)^+$ (assuming no credit mitigants). In reality, when B itself carries credit risk, the exposure will only exist if B itself did not default prior to time τ_C. In a setting where both B and C can default, we will need to adjust the exposure to $\pi(\tau_C)^+1_{\tau_B\geq\tau_C}$. This, in turn, requires a modification of (17) to

$$\begin{aligned} CVA_{bi}(t) &= (1-\alpha_C)1_{\tau_C>t}\mathrm{E}_t\left(\int_t^{T_{\max}} e^{-\int_t^s [r(u,z(u))+\lambda_C(u)]\,du} 1_{\tau_B\geq s} E_C(s,z(s))\lambda_C(s)\,ds\right) \\ &= (1-\alpha_C)1_{\tau_B>t}1_{\tau_C>t}\mathrm{E}_t\left(\int_t^{T_{\max}} e^{-\int_t^s [r(u,z(u))+\lambda_C(u)+\lambda_B(u)]\,du} E_C(s,z(s))\lambda_C(s)\,ds\right) \end{aligned} \tag{21}$$

[15] B's DVA (as computed in (20)) is C's CVA.

where we in the second equality have used the Cox process property of τ_B. Notice that this expression depends jointly on λ_B and λ_C, so the simulation machinery in Section 2.4 needs to be updated accordingly. In particular, we need to be able to simulate correlated paths of both λ_B and λ_C.

A straightforward extension of (21) – basically just reverse subscripts B and C – leads to a definition of bilateral DVA. The net bilateral CVA charge (as seen from B) is then

$$V_{bi}(t) = CVA_{bi}(t) - DVA_{bi}(t). \tag{22}$$

While it is clear that $CVA(t) > CVA_{bi}(t)$ and $DVA(t) > DVA_{bi}(t)$, the net bilateral charge $V_{bi}(t)$ may be both larger or smaller than the unilateral charge $V(t)$. In practice, the two numbers are often fairly close. In part[16] because of this, most firms in the industry compute (and report) only unilateral CVA and DVA numbers.

4 Expected Exposure

As our final topic, let us consider the computation of portfolio *expected exposure*, as, for instance, needed by Basel III capital rules. Expected exposure is also useful for CVA purposes.

4.1 Expected Exposure in a CVA Setting

Set $t = 0$, freeze a particular future point in time $T \in [0, T_{\max}]$, and consider a reasonable way to characterize the contribution of the exposure $E_C(T)$ to the unilateral CVA formula (17). An obvious quantity of interest is the expectation

$$\Omega(T) = (1 - \alpha_C)\mathrm{E}\left(e^{-\int_0^T [r(u)+\lambda_C(u)]\,du} E_C(T)\lambda_C(T)\right)$$

which allows us to write

$$CVA(0) = \int_0^{T_{\max}} \Omega(u)\,du.$$

In other words, $\Omega(T)\,dT$ can be considered the contribution to CVA from the interval $[T, T + dT]$. As $\Omega(T)$ contains both interest rate discounting and a default risk element, it is not in itself a meaningful measure of the expected exposure at time $[T, T + dT]$. To

[16] Another reason is that Basel III capital standards around CVA exclusively work with the unilateral definition. Using different CVA definitions for valuation and capital leads to confusion in hedge objectives, something that most firms would like to avoid. In any case, accounting rules allow for both unilateral and bilateral numbers to be used in official accounting statements.

address this, consider first adjusting for default risk by writing

$$PVEE(T) = \frac{\mathrm{E}\left(e^{-\int_0^T [r(u)+\lambda_C(u)]\,du} E_C(T)\lambda_C(T)\right)}{\mathrm{E}\left(e^{-\int_0^T \lambda_C(u)\,du}\lambda_C(T)\right)} = \frac{\mathrm{E}\left(e^{-\int_0^T [r(u)+\lambda_C(u)]\,du} E_C(T)\lambda_C(T)\right)}{-\partial X_C(0,T)/\partial T}, \quad (23)$$

where we have defined the survival probability

$$X_C(t,T) = \mathrm{E}_t\left(e^{-\int_t^T \lambda_C(u)\,du}\right).$$

The notation $PVEE(T)$ stands for "present value of expected exposure", where the "present value" part refers to the existence of the interest rate discount factor $1/\beta(T)$ in the numerator of (23). To compensate for this discount effect, one can write

$$EE(T) = \frac{\mathrm{E}\left(e^{-\int_0^T [r(u)+\lambda_C(u)]\,du} E_C(T)\lambda_C(T)\right)}{-\partial X_C(0,T)/\partial T \cdot P(0,T)}, \quad (24)$$

where we have introduced the discount bond price

$$P(t,T) = \mathrm{E}_t\left(e^{-\int_t^T r(u)\,du}\right).$$

We define $EE(T)$ in (24) as the (CVA-style) *expected exposure* at the future time T.

Note that the expectation in (24) uses the risk-neutral measure $\mathbb{Q}$. We may, of course, rewrite this using other probability measures. For instance, if we move to the T-forward measure $\mathbb{Q}^T$ induced by $P(t,T)$, we get the simpler expression

$$EE(T) = \frac{\mathrm{E}^T\left(e^{-\int_0^T \lambda_C(u)\,du} E_C(T)\lambda_C(T)\right)}{-\partial X_C(0,T)/\partial T},$$

where $\mathrm{E}^T(\cdot)$ denotes time 0 expectation in measure $\mathbb{Q}^T$. Further simplifications are possible.

When computing CVA numbers, it is common to also return the expected exposure profile $EE(T)$ for all T in the simulation time line. Besides being a very useful diagnostic/debugging tool, the expected exposure profile reveals where the major exposure problems reside, and can aid substantially in the design of CVA hedges. We notice that

$$CVA(0) = (1-\alpha_C)\int_0^{T_{max}} EE(u)P(0,u)\left(-\partial X_C(0,u)/\partial u\right)\,du.$$

4.2 Expected Exposure in a Capital Setting

Under Basel III capital rules, expected exposures are a key input in the IMM framework for counterparty credit risk charges (see [2] for details). In a capital setting, the computation of $EE(T)$ is a little different from (24), as no rules of no-arbitrage pricing need be enforced. Instead, we simply redefine expected exposure for regulatory capital (RC) to be

$$EE_{RC}(T) = \tilde{\mathrm{E}}\left(E_C(T)\right) \tag{25}$$

where $\tilde{\mathrm{E}}(\cdot)$ is the expectation in some chosen probability measure.

Exactly which probability measure is to be used in computing $EE_{RC}(T)$ is not made precise in the regulatory language. Most commonly, the measure is interpreted loosely as a "historical" probability measure, in the sense that all diffusion volatilities are set equal to their historical average – for more discussion, see [3]. In any case, whenever the measure has been fixed, the numerical computation of (25) can be done by Monte Carlo simulation, using the algorithm outlined in Section 2.4.

References

[1] Andersen, L. (2013), "Background Material: Gaussian and Quasi-Gaussian Models," Lecture Notes.

[2] Andersen, L. (2013), "Capital and IMM – Part II," Lecture Notes.

[3] Andersen, L. (2013), "Capital and IMM – Part III," Lecture Notes.

[4] Lando, D. (1998), "On Cox Processes and Credit Risky Securities", *Derivatives Research*, Vol. 2, pp. 99-120.