10. SS Lab 10

The 10th Signals and Systems lab covers topics such as representation and convergence of continuous time Fourier Series in the MATLAB environment.

Suggestions for improvement or correction of the manuscript would be appreciated.

10.1 Lab Objectives

In this lab, the following topics would be addressed:

- Fourier Series representation of continuous time period signals
- Convergence of continuous time Fourier Series

10.2 Fourier Series Representation of Continuous Time Signals

Consider a periodic signal having period T:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(2\pi/T)t}$$

This signal is the linear combination of harmonically related complex exponentials that are all periodic with T. Any usual periodic function can be expressed as a linear combination of harmonically related complex exponentials. The representation of a periodic signal in this manner is known as Fourier Series representation and the weights a_k are referred to as Fourier Series coefficients. Given a periodic signal $x(t)$, it is possible to determine its Fourier Series coefficients through the following integral:

$$a_k = \int_{-T}^{T} x(t)\, e^{-jk\omega_0 t} dt = \int_{-T}^{T} x(t)\, e^{-jk(2\pi/T)t} dt$$

This integral can be done over any time interval of length T, i.e., the period of the signal $x(t)$.

10.2.1 Synthesis of a Simple Periodic Signal

The following example demonstrates that the linear combination of harmonically related complex exponentials leads to a periodic function. The signal used in example is:

$$x(t) = \sum_{k=-3}^{3} a_k\, e^{jk\omega_0 t}, \text{where } a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}$$

Example: Fourier Series of continuous time periodic signal

```
clc
clear all
close all
t = -3:0.01:3; % duration of signal
% dc component for k=0 x0 = 1;
% first harmonic components for k=-1 and k=1
x1 = (1/4)*exp(j*(-1)*2*pi*t)+(1/4)*exp(j*(1)*2*pi*t);
```

```
y1 = x0 + x1; % sum of dc component and first harmonic
% second harmonic components for k=-2 and k=2
x2 = (1/2)*exp(j*(-2)*2*pi*t)+(1/2)*exp(j*(2)*2*pi*t);
y2 = y1 + x2; % sum of all components until second harmonic
% third harmonic components for k=-3 and k=3
x3 = (1/3)*exp(j*(-3)*2*pi*t)+(1/3)*exp(j*(3)*2*pi*t);
x = x0 + x1 + x2 + x3;
% sum of all components until third harmonic
figure;
subplot(3,2,1);
plot(t,x1); axis([-3 3 -2 2]); title('x1(t)');
subplot(3,2,2);
plot(t,y1); axis([-3 3 -0.2 2]); title('x0(t)+x1(t)');
subplot(3,2,3);
plot(t,x2); axis([-3 3 -2 2]); title('x2(t)');
subplot(3,2,4);
plot(t,y2); axis([-3 3 -1 3]);
title('x0(t)+x1(t)+x2(t)');
subplot(3,2,5);
plot(t,x3);
xlabel('t'); axis([-3 3 -1 1]); title('x3(t)');
subplot(3,2,6);
plot(t,x);
xlabel('t'); axis([-3 3 -1 4]);
title('x(t)=x0(t)+x1(t)+x2(t)+x3(t)')
```

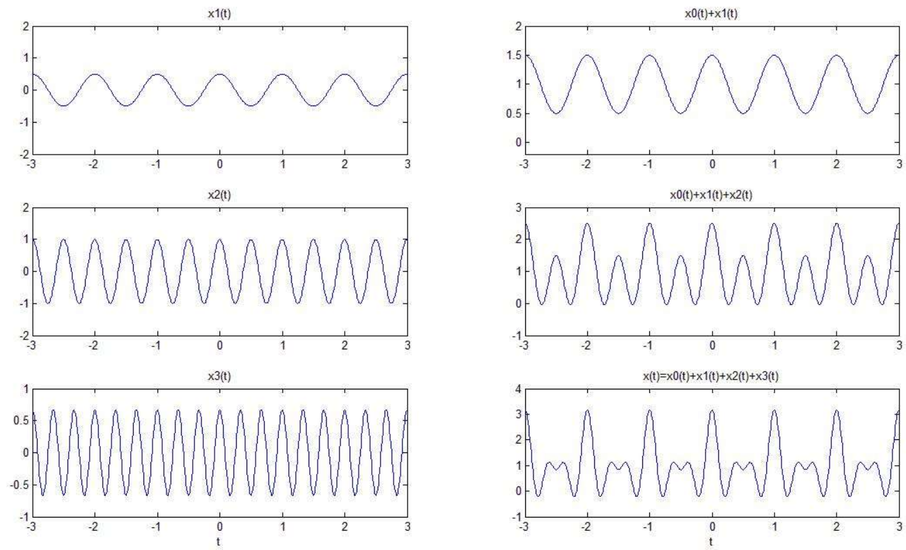

Figure 10.1: Fourier Series illustration of a continuous time periodic signal

10.2.2 Synthesis of a Simple Periodic Signal

Once the Fourier Series coefficient of a continuous time periodic signal is determined analytically using analysis equation, signal can be reconstructed using synthesis equation. Consider the periodic square wave signal defined as:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < {}^{T}/_{2} \end{cases}$$

where T is the time period and T_1 is the duty cycle with Fourier Series coefficients:

$$a_0 = \frac{2T_1}{T}, a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin\big(k2\pi(T_1/T)\big)}{k\pi} \text{ for } k \neq 0$$

In the following examples, first and ideal square wave is created, and then square wave is approximated from its harmonics using synthesis equation by letting k in the partial sum to go from $-M$ to M instead of $-\infty$ to $+\infty$, where M is 10, 20, and 100. In all examples, T is taken as 1 sec.

Example: Ideal square wave created by thresholding 1 Hz cosine wave

```
% generate perfect square wave
t = -1.5:0.005:1.5;    %duration of square wave
xcos = cos(2*pi*t);    %cosine wave of 1 Hz
xpsqw = xcos > 0;
%thresholding cosine wave using relational operator
figure(1);
set(gcf,'defaultaxesfontsize',9);
plot(t,xpsqw,'lineWidth',2);
xlabel('t');
```

```
ylabel('x(t)');
title('Periodic Square Wave (T=1)');
axis([-1.5 1.5 -0.1 1.1]);
grid;
```

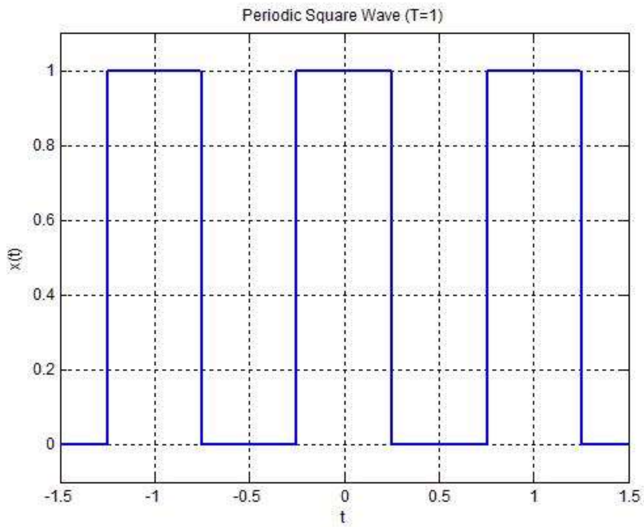

Figure 10.2: Fourier Series illustration

Example: Fourier Series coefficients for square wave with period 1 sec and variable duty cycle

```
k = -15:15; %number of square wave coefficients
T = 1; %time period of square wave
T1 = 1/4; %duty cycle of square wave
ak1 = sin(k*2*pi*(T1/T))./(k*pi);
%square wave Fourier series coefficients
% Ignore the "divide by zero" warning that happens
% because k in the denominator hits 0. We will now do
% a manual correction for a0 -> ak1(16)
ak1(16) = 2*T1/T;
figure;
set(gcf,'defaultaxesfontsize',8);
subplot(3,1,1);
stem(k,ak1,'filled');
ylabel('ak');
title('FS Coefficients for Periodic Square Wave (T=1, T1=1/4)');
T1 = 1/8;
ak2 = sin(k*2*pi*(T1/T))./(k*pi);
ak2(16) = 2*T1/T;  % Manual correction for a0 -> ak2(16)
```

```
subplot(3,1,2);
stem(k,ak2,'filled');
ylabel('ak');
title('FS Coefficients for Periodic Square Wave... (T=1, T1=1/8)');
T1 = 1/16;
ak3 = sin(k*2*pi*(T1/T))./(k*pi);
ak3(16) = 2*T1/T;  % Manual correction for a0 -> ak3(16)
subplot(3,1,3);
stem(k,ak3,'filled');
xlabel('k');
ylabel('ak');
title('FS Coefficients for Periodic Square Wave (T=1, T1=1/16)');
```

Example: Reconstruction of square wave using 10 terms i.e. $M = 10$

```
clc
clear all
close all
t = -1.5:0.005:1.5; %square wave duration
T = 1; %time period of square wave
T1 = 1/4; %duty cycle of square wave
w0 = 2*pi/T;  %fundamental radian frequency of square wave
M = 10;%number of coefficients
k = -M:M; %2M+1 total coefficients to construct square wave
ak = sin(k*2*pi*(T1/T))./(k*pi);
ak(M+1) = 2*T1/T;  % Manual correction for a0 -> ak(M+1)
x = zeros(1,length(t));
for k = -M:M
   x = x + ak(k+M+1)*exp(j*k*w0*t);
end
figure;
set(gcf,'defaultaxesfontsize',9);
plot(t,real(x),'lineWidth',2);
grid;
xlabel('t'); ylabel('x(t)') ;
title('Reconstruction from Fourier Series with 21 terms');
```

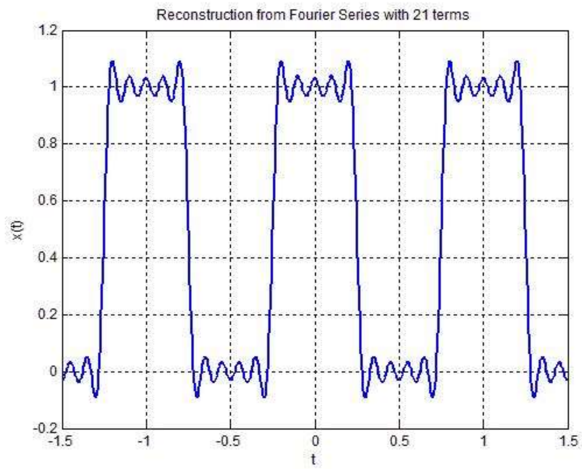

Figure 10.3: Reconstruction of square wave, M = 10

Example: Reconstruction of square wave using 20 terms i.e. $M = 20$

```
clc
clear all
close all
t = -1.5:0.005:1.5; T = 1;
T1 = 1/4;
w0 = 2*pi/T; M = 20;
k = -M:M;
ak = sin(k*2*pi*(T1/T))./(k*pi);
% Manual correction for a0 -> ak(M+1) ak(M+1) = 2*T1/T;
x = zeros(1,length(t)); for k = -M:M
x = x + ak(k+M+1)*exp(j*k*w0*t); end
figure;
set(gcf,'defaultaxesfontsize',9);
plot(t,real(x),'lineWidth',2);
grid;
xlabel('t');
ylabel('x(t)');
title('Reconstruction from Fourier Series with 41 terms');
```

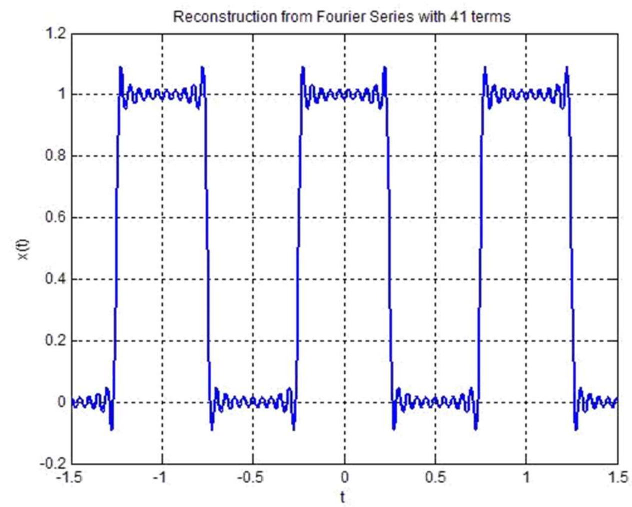

Figure 10.4: Reconstruction of square wave, M = 20

Example: Reconstruction of square wave using 100 terms i.e. $M = 100$

```
clc
clear all
close all
t = -1.5:0.005:1.5; T = 1;
T1 = 1/4;
w0 = 2*pi/T; M = 100;
k = -M:M;
ak = sin(k*2*pi*(T1/T))./(k*pi);
ak(M+1) = 2*T1/T;    % Manual correction for a0 -> ak(M+1)
x = zeros(1,length(t)); for k = -M:M
x = x + ak(k+M+1)*exp(j*k*w0*t); end
figure;
set(gcf,'defaultaxesfontsize',9)
plot(t,real(x),'lineWidth',2);
grid;
xlabel('t'); ylabel('x(t)') ;
title('Reconstruction from Fourier Series with 201 terms');
```

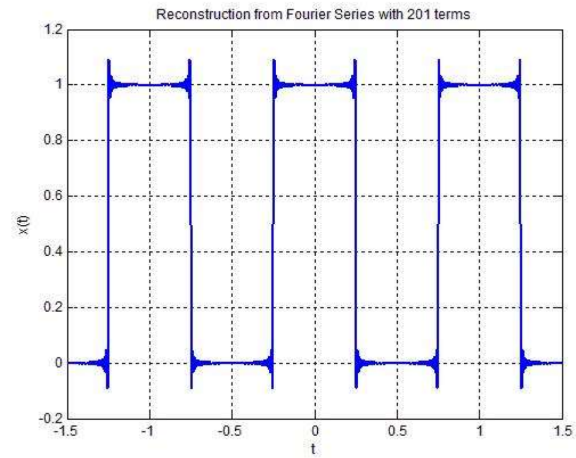

Figure 10.5: Reconstruction of square wave, M = 100

10.3 Tasks

Perform the following tasks:

10.3.1 Task 01

In the example in section 10.2.1, a_ks are chosen to be symmetric about the index $k = 0$, i.e., $a_k = a_{-k}$. Select new a_ks on your own to alter this symmetry and form a new signal. What do you observe? Is $x(t)$ a real signal when coefficients are not symmetric?

10.3.2 Task 02

A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period of $T = 8$. The non-zero Fourier Series coefficients for $x(t)$ are:

$$a_1 = a_{-1} = 2, a_3 = a^*{}_{-3} = 4j$$

Express $x(t)$ as the linear combination of these coefficients.

10.3.3 Task 03

A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period of $N = 5$. The non-zero Fourier Series coefficients for $x[n]$ are:

$$a_0 = 1, a_2 = a^*{}_{-2} = e^{j\frac{\pi}{4}}, a_4 = a^*{}_{-4} = 2e^{j\frac{\pi}{3}}$$

Express $x[n]$ as the linear combination of these coefficients.

10.3.4 Task 04

Considering the Fourier Series coefficients plot given below, what do you observe happens to the envelope of the coefficients when T_1 is reduced from 1/4 to 1/16 with constant time period T?

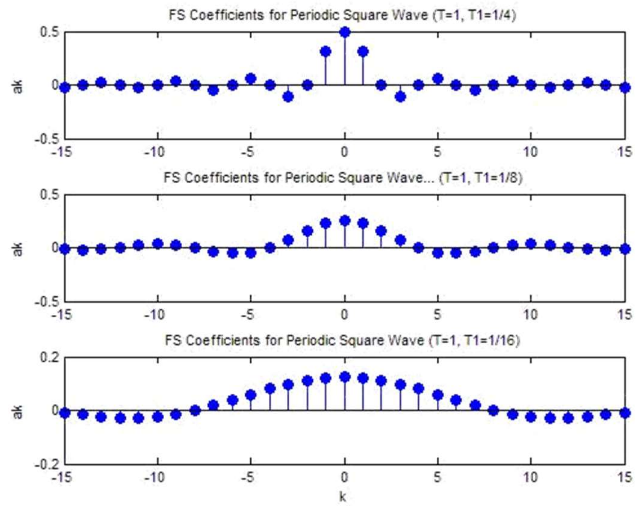

Figure 10.6: Fourier Series coefficients plot

10.3.5 Task 05

Considering the plots of square wave reconstructed using $M = 10$, 20, and 100 terms in section 10.2.2, what do you observe about Gibb's phenomena?

10.3.6 Task 06

Given the following Fourier Series coefficients:

$$a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$$

Plot the coefficients and reconstructed signal. Take the terms for reconstructed signal to be $M =$ 10, 20, and 50. What effect do you see when M is varied?

10.3.7 Task 07

Given the following Fourier Series coefficients:

$$a_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$$

Plot the coefficients and reconstructed signal. Take 10 terms ($M = 10$) for the reconstructed signal.