Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 LP Basics

Linear Program. A *linear program* is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^d$ be the set of variables and $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. The canonical form of a linear program is

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Any linear program can be written in canonical form.

Let's check this is the case:

(i) What if the objective is maximization?

(ii) What if you have a constraint $Ax \leq b$?

(iii) What about $Ax = b$?

(iv) What if the constraint is $x \leq 0$?

(v) What about unconstrainted variables $x \in \mathbb{R}$?

Solution:

(i) Take the negative of the objective.

(ii) Negate both sides of the inequality.

(iii) Write both $Ax \leq$ and $Ax \geq b$ into the constraint set.

(iv) Change of variable: replace every x by $-z$, and add constraint $z \geq 0$.

(v) Replace every x by $x^+ - x^-$, add constraints $x^+, x^- \geq 0$. Note that for every solution to the original LP, there is a solution to the transformed LP (with the same objective value). Similarly, if there is a feasible solution for the transformed problem, then there is a feasible solution for the original problem with the same objective value.

Dual. The dual of the canonical LP is

$$\begin{aligned} \text{maximize } & b^\top y \\ \text{subject to } & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Weak duality: The objective value of any feasible dual $\leq$ objective value of any feasible primal
Strong duality: The *optimal* objective values of these two are equal.
Both are solvable in polynomial time by the Ellipsoid or Interior Point Method.

2 Job Assignment

There are I people available to work J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person i working c portion of a day on job j is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day such that the total value created by everyone working is optimized. No additional value comes from working on a job after it has been completed.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is x?

3.5cm

Solution: An assignment x is a choice of numbers x_{ij} where x_{ij} is the portion of person i's time spent on job j.

(b) What are the constraints we need to consider? Hint: there are three major types.

3.5cm

Solution: First, no person i can work more than 1 day's worth of time.

$$\sum_{j=1}^{J} x_{ij} \leq 1 \qquad \text{for } i = 1, \ldots, I.$$

Second, no job j can be worked past completion:

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \text{for } j = 1, \ldots, J.$$

Third, we require positivity.

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, I, j = 1, \ldots, J.$$

(c) What is the maximization function we are seeking?

3.5cm

Solution: By person i working job j for x_{ij}, they contribute value $a_{ij}x_{ij}$. Therefore, the net value is

$$\sum_{i=1,j=1}^{I,J} a_{ij}x_{ij} = A \bullet x.$$

3 Linear regression

In this problem, we show that linear programming can handle linear regression. Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ be given, where n, d are not assumed to be constant. However, assume all input numbers have constant bits.

(a) Recall that the ℓ_1 norm of a vector v is given by $\|v\|_1 = \sum_{i=1}^{d} |v_i|$. The L1 regression problem asks you to find $x \in \mathbb{R}^d$ that minimizes $\|Ax - b\|_1$.

(i) Provide a linear program that finds the opimtal x, given A, b.

(ii) Argue that it can be solved in polynomial time (in n, d).

Solution: (i). Let a_i be the ith row of A. The LP is given by

$$\begin{aligned} \text{minimize } & \sum_i t_i \\ \text{subject to } & a_i \cdot x - b_i \leq t_i \\ & a_i \cdot x - b_i \geq -t_i \\ & t_i \geq 0 \end{aligned}$$

(ii). The LP has $d+n$ variables and $3n$ constraints. Assuming all input numbers have constant bit complexity, the LP can be solved in polynomial time (via Ellipsoid or inteiror point method).

(b) Recall that the ℓ_∞ norm of a vector v is given by $\|v\|_\infty = \max_i |v_i|$. The L_∞ regression problem asks you to find $x \in \mathbb{R}^d$ that minimizes $\|Ax - b\|_\infty$.

(i) Provide a linear program that finds the opimtal x, given A, b.

(ii) Argue that it can be solved in polynomial time (in n, d).

Solution: (i).

$$\begin{aligned} \text{minimize } & t \\ \text{subject to } & a_i \cdot x - b_i \leq t \\ & a_i \cdot x - b_i \geq -t \\ & t \geq 0 \end{aligned}$$

(ii). Similar to (a)(ii).

4 Provably Optimal

Consider the following linear program:

$$\begin{aligned} \max \ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

For the linear program above,

(a) First compute the dual of the above linear program

(b) show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual. (*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

Solution: The dual of the given LP is:

$$\begin{aligned} \min \quad & y_1 + y_2 \\ & y_1 \geq 1 \\ & -y_1 + 2y_2 \geq 0 \\ & -y_2 \geq -2 \\ & y_1, y_2 \geq 0 \end{aligned} \qquad \equiv \qquad \begin{aligned} \min \quad & y_1 + y_2 \\ & y_1 \geq 1 \\ & y_2 \geq \frac{y_1}{2} \\ & y_2 \leq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

The objective value at the claimed optimum is 3/2. By the duality theorem, this would be optimum if and only if there is a feasible solution to the dual LP with the same objective value. Greedily trying to make y_1, y_2 as small as possible results in finding that$y_1 = 1, y_2 = 1/2$ is a feasible dual solution, with the objective value 3/2. Thus, the claimed primal optimal is indeed an optimal solution.

5 Taking a Dual

Consider the following linear program:

$$\begin{aligned} &\max\ 4x_1 + 7x_2 \\ x_1 + 2x_2 &\le 10 \\ 3x_1 + x_2 &\le 14 \\ 2x_1 + 3x_2 &\le 11 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Construct the dual of the above linear program.

Solution: If we scale the first constraint by $y_1 \ge 0$, the second by $y_2 \ge 0$, the third by $y_3 \ge 0$, and we add them up, we get an upperbound of $(y_1 + 3y_2 + 2y_3)x_1 + (2y_1 + y_2 + 3y_3)x_2 \le (10y_1 + 14y_2 + 11y_3)$. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned} &\min\ 10y_1 + 14y_2 + 11y_3 \\ y_1 + 3y_2 + 2y_3 &\ge 4 \\ 2y_1 + y_2 + 3y_3 &\ge 7 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$