Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Basics

Flow. The *capacity* indicates how much flow can be allowed on an edge. Given a directed graph with edge capacity $c(u,v)$ and s,t, a flow is a mapping $f : E \to \mathbb{R}^+$ that satisfies

- Capacity constraint: $f(u,v) \leq c(u,v)$, the flow on an edge cannot exceed its capacity.
- Conservation of flows: $f^{\text{in}}(v) = f^{\text{out}}(v)$, flow in equals flow out for any $v \notin \{s,t\}$

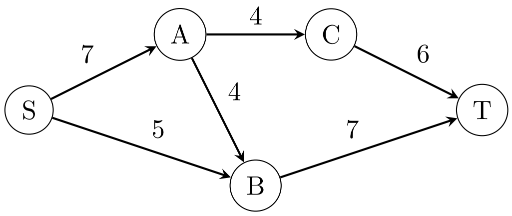
Here, we define $f^{\text{in}}(v) = \sum_{u:(u,v)\in E} f(u,v)$ and $f^{\text{out}}(v) = \sum_{u:(v,u)\in E} f(u,v)$. We also define $f(v,u) = -f(u,v)$, and this is called *skew-symmetry*.

Residual Graph. Given a flow network (G,s,t,c) and a flow f, the *residual capacity* (w.r.t. flow f) is denoted by $c_f(u,v) = c_{uv} - f_{uv}$. And the *residual network* $G_f = (V, E_f)$ where $E_f = \{(u,v) : c_f(u,v) > 0\}$.

Ford-Fulkerson. Keep pushing along s-t paths in the residual graph. Runs in time $O(mF)$.

2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

3 Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph G and the additional variant constraints, show how to construct a directed graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v)\in E} f_{uv} \le c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

4 Secret Santa

Imagine you are throwing a party and you want to play Secret Santa. Thus you would like to assign to every person at the party another partier to whom they must anonymously give a gift. However, there are some restrictions on who can give gifts to who. For instance, nobody should be assigned to give a gift to themselves or to their spouse. Since you are the host, you know all of these restrictions. Give an efficient algorithm that determines if you and your guests can play Secret Santa.