SEEM 5580 Midterm Exam Solutions **Name:**
9:30–11:15 am, Oct 23, 2017

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 5 problems.

1. (25 points; 5 points each) Suppose the customer arrivals to a service center can be modeled by $\{N(t) : t \geq 0\}$, a non-homogeneous Poisson process with rate function $\lambda(t)$, where $\lambda(t) = t$ for $0 \leq t \leq 4$ and $\lambda(t) = 4$ for $t > 4$. Answer the following questions. For (a) and (b), just circle True or False, no explanation needed. Answer the other parts in the space below.

   (a) $N(5) - N(2)$ has a Poisson distribution with mean 8. True or False?

   (b) $N(5) - N(2)$ is independent of $N(1)$. True or False?

   (c) Find the average number of customer arrivals during the time interval $[0, 10]$.

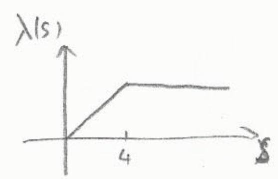
   (d) Find the moment generating function of $N(t)$ for any fixed $t > 0$.

   (e) Given that the 5-th customer arrives at time $t = 2$, find the (conditional) distribution of S_6, the arrival time of the 6-th customer.

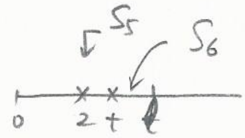
(c) $E[N(10)] = \int_0^{10} \lambda(t)\,dt = \int_0^4 t\,dt + \int_4^{10} 4\,dt = 8 + 24 = 32$

(d) $E[e^{\theta N(t)}] = \exp(m(t)\cdot(e^{\theta}-1))$ where

$$m(t) = \int_0^t \lambda(s)\,ds = \begin{cases} \frac{t^2}{2} & t \leq 4 \\ 8 + 4(t-4); & t > 4 \end{cases}$$



(e) $P(S_6 > t \mid S_5 = 2)$ for $t > 2$



$= P(\text{0 events in } (2, t] \mid S_5 = 2)$

$= P(\text{0 events in } (2, t])$

$= P(N(t) - N(2) = 0)$

$= e^{-\int_2^t \lambda(s)\,ds}$ as $N(t) - N(2) \sim \text{Poisson}(\int_2^t \lambda(s)\,ds)$

2. (10 points) Taxis are waiting in a queue for passengers to come. Passengers for those taxis arrive according to a Poisson process with an average of 60 passengers per hour. A taxi departs as soon as two passengers have been collected or 5 minutes have expired since the first passenger has got in the taxi. Suppose you get in the taxi as first passenger. What is your average waiting time for the departure?

Let ξ_1 be the arrival time of first passenger.

Then $\xi_1 \sim \exp(1)$ by the property of poisson process.

Here $E[\xi_1] = 1$ minute.

Let your waiting time be W.

Then
$$W = \begin{cases} \xi_1 & \text{if } \xi_1 < 5 \\ 5 & \text{if } \xi_1 \geq 5 \end{cases}$$

So
$$E[W] = \cancel{E[\xi_1]} \; E[\min\{\xi_1, 5\}].$$

$$= \int_0^5 x \cdot e^{-x} dx + \int_5^\infty 5 \cdot e^{-x} dx$$

$$= \int_0^5 x e^{-x} dx + 5 \cdot e^{-5}$$

$$= 1 - e^{-5} - 5e^{-5} + 5e^{-5}$$

$$= 1 - e^{-5} \quad \text{(minute)}$$

3. (15 points) Suppose that $U_1, U_2, \ldots$ are independent random variables with common probability function $g(k) = P(U_1 = k)$ where k belongs to the set of integers $\mathbb{Z}$. Let S be a countable set. Let X_0 be another random variable, independent of the sequence $(U_n)_{n\geq 1}$, taking values in S and let $f\colon S \times \mathbb{Z} \to S$ be a certain function. Define

$$X_{n+1} = f(X_n, U_{n+1}), \quad \text{for } n \geq 0. \tag{1}$$

Prove that $\{X_n : n \geq 0\}$ is a discrete time Markov chain and specify its transition probabilities using functions f and g.

Given any $i, j, i_{n-1}, \ldots, i_0 \in S$

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P(f(X_n, U_{n+1}) = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$$= P(f(i, U_{n+1}) = j \mid X_n = i, \ldots, X_0 = i_0) \qquad (*)$$

It is clear that X_n only depends on $X_0, U_1, \ldots, U_n$ by (1)

Hence, U_{n+1} is independent of $X_n, X_{n-1}, \ldots, X_0$.

Thus we obatin

$(*) = P(f(i, U_{n+1}) = j)$

Similarly, $P(X_{n+1} = j \mid X_n = i) = P(f(i, U_{n+1}) = j)$

This proves that $\{X_n : n \geq 0\}$ is a DTMC.

$$P_{ij} = P(f(i, U_{n+1}) = j) = P(f(i, U_1) = j)$$

$$= \sum_{k\in\mathbb{Z}} P(f(i, U_1) = j \mid U_1 = k) \cdot P(U_1 = k)$$

$$= \sum_{k\in\mathbb{Z}} g(k) \cdot 1_{f(i,k)=j} \qquad \text{for } i, j \in S$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} & & & \\ \frac{25}{48} & \frac{13}{48} & \frac{7}{48} & \frac{4}{48} \\ & & & \\ & & & \end{bmatrix}$$

4. (25 points, 5 points each) Consider a discrete time Markov chain $\{X_n : n = 0, 1, \ldots\}$ with the state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

For the first three parts, just circle True or False. No explanation needed. Answer the other parts in the space below.

(1) This chain is irreducible. True or False?

(2) Every state of this chain has period 1. True or False?

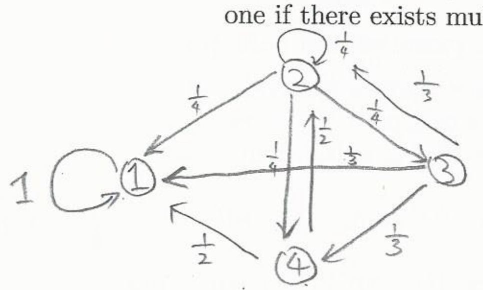
(3) Every state of this chain is positive recurrent. True or False?

(4) Suppose when the markov chain enters each state $x \in S$, there is a reward $h(x) = x^2$ dollars collected. Find $\mathbb{E}\left[h(X_0) + h(X_1) + h(X_2)\middle|X_0 = 2\right]$.

(5) Find one stationary distribution for this Markov chain (you only need to give one if there exists multiple stationary distributions).

$\pi = (1, 0, 0, 0)$

states : 1 2 3 4



(4): $E[h(X_0) \mid X_0 = 2] = E[2^2] = 4$

$E[h(X_1) \mid X_0 = 2] = \sum_{j=1}^{4} h(j) \cdot P_{2j}$

$= 1 \times \frac{1}{4} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{4} + 4^2 \times \frac{1}{4} = \frac{15}{2}$

$E[h(X_2) \mid X_0 = 2] = \sum_{j=1}^{4} h(j) \cdot P_{2j}^{(2)}$

$= 1 \times P_{21}^{(2)} + 4 \times P_{22}^{(2)} + 9 \times P_{23}^{(2)} + 16 \times P_{24}^{(2)}$

use P^2 ?

$\overset{?}{=} \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2}\right) + \frac{9}{4} \times \left(\frac{1}{4} + \frac{1}{3}\right) + 16 \times \frac{3}{48}$

$= \frac{25}{48} + \frac{52}{48} + \frac{63}{48} + \frac{48}{48} = \frac{47}{12}$

So $E[h(X_0) + h(X_1) + h(X_2)] = 2^2 + \frac{15}{2} + \frac{47}{12} = \frac{185}{12}$

5. (25 points, 5 points each) A customer arrives at a bank (with one clerk) at discrete time n with probability α. He or she waits in a queue (if any) which is served by one bank clerk in a first-come-first-served fashion. When at the head of the queue, the person requires a service which is distributed like a random variable S with values in $\mathbb{N}$: $P(S = k) = p_k$, for $k = 1, 2, \ldots$. Different people require services which are independent and identically distributed random variables. Consider the quantity W_n which is the total waiting time at time n: if I take a look at the queue at time n, then W_n represents the time I have to wait in line till I finish my service.

(a) Explain or show that W_n obeys the recursion $W_{n+1} = (W_n + S_n \cdot \xi_n - 1)^+$, where the S_n are i.i.d. random variables distributed like S, independent of the ξ_n. The latter are also i.i.d. with $P(\xi_n = 1) = \alpha$, $P(\xi_n = 0) = 1 - \alpha$. Thus $\xi_n = 1$ indicates that there is an arrival at time n.

(b) Show that $\{W_n : n \geq 0\}$ is a Markov chain and compute its transition probabilities P_{ij} for $i, j = 0, 1, 2, \ldots$, in terms of the parameters α and $(p_k)_{k \geq 1}$.

(c) Suppose that $p_1 = 1 - \beta$ and $p_2 = \beta > 0$. Find conditions on α and β so that the stationary distribution of $\{W_n : n \geq 0\}$ exists. Give a physical interpretation of this condition.

(d) Find the stationary distribution of $\{W_n : n \geq 0\}$ when it exists.

(e) Find the average waiting time of a customer in steady-state. (If you can not solve (d), then give an expression of the average waiting time using the stationary distribution of $\{W_n : n \geq 0\}$. Partial credits will be given to this expression.).

(a) If at time n, $W_n > 0$ and nobody arrives then $W_{n+1} = W_n - 1$ as the waiting time decrease by 1 unit in 1 unit of time

If at time n, someone arrives and has service time S_n, then $W_{n+1} = W_n + S_n - 1$; If $W_n = 0$ and nobody arrives, then $W_{n+1} = W_n = 0$. Putting all things together, we have

$$W_{n+1} = (W_n + S_n \cdot \xi_n - 1)^+ \quad ; \quad x^+ = \max\{x, 0\}$$

(b) $\{W_n : n \geq 0\}$ is a DTMC follows from Problem 3

The state space is $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$.

$$P(W_{n+1} = j \mid W_n = i) = P((i + S_n \cdot \xi_n - 1)^+ \mid W_n = i)$$

For $j < i - 1$, clearly $P_{ij} = 0$

For $j = i - 1$, $P_{ij} = P_{i,i-1} = 1 - \alpha$ = the probability no one arrives

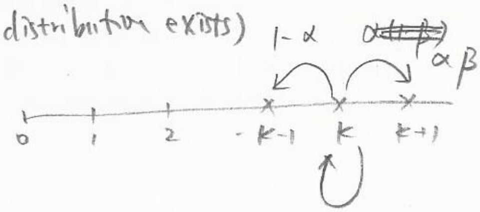
For $j \geq i$, $P_{ij} = P(S_n \cdot \xi_n = j - i + 1) = \alpha \cdot P(S_n = j - i + 1)$

(this page is left blank)

(c). In this case, we have $P_{00} = 1-\alpha\beta$, $P_{01} = \alpha\beta$

$$P_{k,k-1} = 1-\alpha \qquad P_{k,k+1} = \alpha\beta \qquad P_{k,k} = \alpha(1-\beta). \quad \text{for } k \ge 1$$

Using $\pi = \pi P$. (if the stationary distribution exists)



we can obtain

$$\pi_k \cdot (1-\alpha) = \pi_{k-1} \cdot \alpha\beta$$

which implies

$$\pi_k = \left(\frac{\alpha\beta}{1-\alpha}\right)^k \cdot \pi_0 \qquad k \ge 0.$$

Now $$\sum_{k=0}^{\infty} \pi_k = \sum_{k=0}^{\infty} \left(\frac{\alpha\beta}{1-\alpha}\right)^k \cdot \pi_0 = 1$$

So stationary distribution exists $\Longleftrightarrow$ $\sum_{k=0}^{\infty} \left(\frac{\alpha\beta}{1-\alpha}\right)^k < +\infty$

That is, $$\frac{\alpha\beta}{1-\alpha} < 1.$$

or $$1+\beta < \frac{1}{\alpha}$$

Interpretation: Average service time < average time between two consecutive arrivals.

(d) $$\sum_{k=0}^{\infty} \left(\frac{\alpha\beta}{1-\alpha}\right)^k = \frac{1}{1-\frac{\alpha\beta}{1-\alpha}} = \frac{1-\alpha}{1-\alpha(1+\beta)}$$

Hence $$\pi_0 = \frac{1-\alpha(1+\beta)}{1-\alpha} \quad \text{and} \quad \pi_k = \pi_0 \cdot \left(\frac{\alpha\beta}{1-\alpha}\right)^k.$$

(e): Average waiting time in steady-state is

$$\sum_{k=0}^{\infty} k \cdot \pi_k = \sum_{k=0}^{\infty} k \cdot \left(\frac{\alpha\beta}{1-\alpha}\right)^k \cdot \pi_0 = \frac{\alpha\beta}{1-\alpha(1+\beta)}$$