ENGG 5501: Foundations of Optimization	2021–22 First Term
Homework Set 2	
Instructor: Anthony Man–Cho So	Due: October 11, 2021

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Given any $A, B \in \mathcal{S}^n$, we define $A \bullet B = \mathrm{tr}(AB)$ to be the inner product between A and B.

(a) **(10pts).** Show that for any $A, B \in \mathcal{S}^n_+$, we have $A \bullet B \geq 0$.

(b) **(10pts).** The result in (a) implies that $\mathcal{S}^n_+ \subseteq \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$ for any $A \in \mathcal{S}^n_+$. Show that in fact

$$\mathcal{S}^n_+ = \bigcap_{A \in \mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}.$$

Problem 2 (25pts). We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ possesses Property C if for any sequence $\{x^k\}_{k \geq 0} \subset \mathbb{R}^n$ satisfying $\|x^k\|_2 \to +\infty$, we have $f(x^k) \to +\infty$.

(a) **(15pts).** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that f possesses Property C if and only if $L_t = \{x \in \mathbb{R}^n : f(x) \leq t\}$ is compact for any $t \in \mathbb{R}$. Hence, show that if the function f is continuous and possesses Property C, then the optimization problem

$$\inf_{x \in \mathbb{R}^n} f(x)$$

always has an optimal solution. (*Hint: Since the function f is continuous, the set L_t is closed for any $t \in \mathbb{R}$; see Handout C, Section 3.1.*)

(b) **(10pts).** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function that possesses Property C and $A \in \mathbb{R}^{m \times n}$ be a matrix. Does the function $x \mapsto f(Ax)$ necessarily possess Property C? Justify your answer.

Problem 3 (35pts). Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Define

$$K^\circ = \{w \in \mathbb{R}^n : w^T x \leq 0 \text{ for all } x \in K\}$$

to be the *polar cone* of K.

(a) **(5pts).** Show that K° is a convex cone.

(b) **(15pts).** Show that for any $x \in \mathbb{R}^n$, we have $z^* = \Pi_K(x)$ if and only if

$$z^* \in K, \quad x - z^* \in K^\circ, \quad (x - z^*)^T z^* = 0.$$

(c) **(15pts).** Using the result in (a), or otherwise, show that for any $x \in \mathbb{R}^n$, we have

$$x = \Pi_K(x) + \Pi_{K^\circ}(x).$$

Remark: The above identity shows that a closed convex cone K can be used to decompose any vector x into the orthogonal components $\Pi_K(x)$ and $\Pi_{K^\circ}(x)$.

Problem 4 (20pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. We say that f is ρ-convex (where $\rho \in \mathbb{R}$) if for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$, we have

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) + \frac{\rho\alpha(1-\alpha)}{2}\|x-y\|_2^2.$$

Note that the usual notion of convexity is the same as 0-convexity. Show that the following statements are equivalent:

1. The function f is ρ-convex for some $\rho \in \mathbb{R}$.
2. The function $x \mapsto f(x) + \frac{\rho}{2}\|x\|_2^2$ is convex.
3. For any $x, y \in \mathbb{R}^n$, we have

$$f(y) \ge f(x) + \nabla f(x)^T (y-x) - \frac{\rho}{2}\|y-x\|_2^2.$$