SEEM 5580 Final Exam **Name**:

December 11, 2014

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 6 problems.

1. (10 points) Recall that a discrete time stochastic process $X = \{X_n : n = 0, 1, 2, \ldots\}$ in state space S satisfies the Markov property if

$$\mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i\} = \mathbb{P}\{X_{n+1} = j | X_n = i\}$$

for each $n \geq 0$ and $i_0, i_1, \ldots i_{n-1}, i, j \in S$. Toss a fair coin sequentially. Let $\xi_i = 1$ if the ith toss lands a head and $\xi_i = 0$ otherwise. Let $Y_0 = 0$, $Y_1 = \xi_1$ and $Y_n = \xi_n + \xi_{n-1}$ for $n \geq 2$. Does $Y = \{Y_n : n = 0, 1, 2, \ldots,\}$ satisfy the Markov property? Prove your assertion.

Solution.$Y = \{Y_n : n = 0, 1, 2, \ldots,\}$ doesn't satisfy the Markov property.

$$\begin{aligned}
\mathbb{P}\{Y_3 = 0 | Y_2 = 1, Y_1 = 0\} &= \mathbb{P}\{\xi_3 + \xi_2 = 0 | \xi_2 + \xi = 1, \xi_1 = 0\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0 | \xi_2 = 1, \xi_1 = 0\} \\
&= 0
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{P}\{Y_3 = 0 | Y_2 = 1\} &= \mathbb{P}\{\xi_3 + \xi_2 = 0 | \xi_2 + \xi_1 = 1\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0, \xi_2 + \xi_1 = 1\} / \mathbb{P}\{\xi_2 + \xi_1 = 1\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0, \xi_1 = 1\} / \mathbb{P}\{\xi_2 + \xi_1 = 1\} \\
&= (1/8)/(1/2) = 1/4
\end{aligned}$$

2. (20 points) Consider an inventory system. The weekly demands $\{D_i : i = 1, \ldots\}$ are iid following distribution

d	0	1	2
$\mathbb{P}\{D_i = d\}$	.2	.5	.3

Suppose the inventory policy is (s, S) with $s = 0$ and $S = 2$. Namely, by the end of Friday, if inventory becomes empty, order 2 item. Otherwise, do not order. Unsatisfied demand during a week is lost. Find the long run fraction of weeks when demand is not satisfied. Express your answer in terms the stationary distribution of a DTMC (Discrete-Time Markov Chain). You do not need to compute the stationary distribution. But you need to define the DTMC precisely and explain why the stationary distribution exists and is unique.

Solution. Let X_n be the inventory level at the end of week n after replenishment. Let $Y_{n+1} = X_n - D_{n+1}$. Then,

$$Y_{n+1} = \begin{cases} Y_n - D_{n+1}, & \text{if } s < Y_n \le S, \\ S - D_{n+1}, & \text{if } Y_n \le s, \end{cases}$$

where $s = 0, S = 2$. Therefore, $\{Y_n\}$ is a DTMC on $\{-1, 0, 1, 2\}$ and its transition matrix is

$$P = \begin{pmatrix} 0 & .3 & .5 & .2 \\ 0 & .3 & .5 & .2 \\ .3 & .5 & .2 & 0 \\ 0 & .3 & .5 & .2 \end{pmatrix}.$$

It is irreducible and the unique stationary distribution is $\pi = (0.1154, 0.3769, 0.3846, 0.1231)$. Define a function $f : \{-1, 0, 1, 2\} \to \mathbb{R}$ by $f(Y) = 1_{\{Y=-1\}}$. Then, the cumulative unsatisfied demand until time T_N is $\sum_{n=1}^{N} f(Y_n)$. By Strong Law of Large Numbers of DTMC, for large enough N,

$$\frac{1}{N} \sum_{n=1}^{N} f(Y_n) \approx \pi(f) = 0.1154.$$

The long run fraction of periods when demand if not satisfied is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(Y_n) = \pi(f) = 0.1154.$$

An alternative solution method: $\{X_n\}$ is a DTMC on $\{1, 2\}$ and its transition matrix is

$$Q = \begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix}.$$

It is irreducible and the unique stationary distribution is $\pi = (\frac{5}{13}, \frac{8}{13})$. Then, the long run fraction of periods when demand if not satisfied is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{\{X_n=1, D_{n+1}=2\}} = \pi(1) \times 0.3 = 0.1154.$$

3. (20 points; 5 points each question) A call center has four phone lines and two agents. Call arrival to the call center follows a Poisson process with rate 2 calls per minute. Calls that receive a busy signal are lost. The processing times are i.i.d exponentially distributed with mean 1 minute.

(a) Model the system by a continuous time Markov chain. Specify the state space. Clearly describe the meaning of each state. Specify the generator matrix.

(b) What is the long-run fraction of time that there are two calls in the system?

(c) What is the long-run average number of waiting calls, excluding those in service, in the system?

(d) What is the throughput (the rate at which completed calls leaves the call center) of the call center?

Solution.

(a) Let $X(t)$ denote the total number of customers in the system, including those in service and waiting. The state space is $\{0, 1, 2, 3, 4\}$ as there are 4 phone lines in total.

State 0 means that there is no one in the system. State 1 and 2 mean that there is (are) 1 and 2 customer(s) in service, respectively. State 3 and 4 mean that there are 2 customers in service, with 1 and 2 customer(s) waiting, respectively.

Let $\lambda = 2$ denote the arrival rate, and $\mu = 1$ denote the service rate. The rate diagram is shown in Figure 1.

(b) Let π_i denote the long-run fraction of time that there are i customers in the system, $i = 0, 1, \ldots, 4$. The system equation is as follow:

$$\begin{aligned}
\lambda\pi_0 &= \mu\pi_1 \\
\lambda\pi_1 &= 2\mu\pi_2 \\
\lambda\pi_2 &= 2\mu\pi_3 \\
\lambda\pi_3 &= 2\mu\pi_4
\end{aligned}$$

and $\sum_{i=0}^{4} \pi_i = 1$. Solve the above equations, we have that

$$\begin{aligned}
(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) &= (\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}) \\
&= (0.1111, 0.2222, 0.2222, 0.2222, 0.2222).
\end{aligned}$$

The desired probability is $\pi_2 = \frac{2}{9} = 0.2222$.

(c) The long-run average number of waiting calls equals to

$$\begin{aligned}
L_q &= \pi_3 + 2\pi_4 \\
&= \frac{2}{3} = 0.6667.
\end{aligned}$$

(d) The throughput equals to the unblocked arrival rate

$$(1-\pi_4)\lambda = \frac{14}{9} = 1.5556.$$

It is also equal to the rate at which completed calls leaves the call center, i.e.,

$$\mu\pi_1 + 2\mu(\pi_2+\pi_3+\pi_4) = \frac{14}{9} = 1.5556.$$

4. (20 points; 5 points each question) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and $\{N(t) : t \geq 0\}$ models customer arrivals to a service center. Let ξ_i be the inter–arrival time between $(i-1)-$th and $i-$th customers. For each statement, say "true" or "false" and explain your answers. Assume throughout that $t > 0$ is fixed but arbitrary.

   (a) $N(t)$ has a Poisson distribution with mean λ.

   (b) ξ_i has an exponential distribution with mean λ for each i.

   (c) $\xi_{N(t)+1}$ has an exponential distribution.

   (d) $\xi_{N(t)+2}$ has an exponential distribution.

   Solution.

   (a) No. Mean is λt.

   (b) No. Mean is $1/\lambda$.

   (c) No. Note that $N(t) = k$ is equivalent to $S_k \leq t < S_{k+1}$. Direct computation yields

$$\begin{aligned}
P(\xi_{N(t)+1} > x) &= \sum_{k\geq 0} P(\xi_{N(t)+1} > x | N(t) = k) P(N(t) = k) \\
&= \sum_{k\geq 0} P(\xi_{k+1} > x | N(t) = k) P(N(t) = k) \\
&= \sum_{k\geq 0} P(\xi_{k+1} > x, S_k \leq t < S_{k=1}) P(N(t) = k) \\
&= \sum_{k\geq 0} \int_0^t P(\xi_{k+1} > x, \xi_{k+1} > t-y) dP(S_k \leq y) \cdot P(N(t) = k) \\
&\neq e^{-\lambda x}
\end{aligned}$$

(d) Yes. The key is to observe that $N(t) = k$ is equivalent to $S_k \le t < S_{k+1}$, so this event is independent of ξ_{k+2}. With this observation one can directly check

$$\begin{aligned} P(\xi_{N(t)+2} > x) &= \sum_{k\ge 0} P(\xi_{N(t)+2} > x | N(t) = k) P(N(t) = k) \\ &= \sum_{k\ge 0} P(\xi_{k+2} > x | N(t) = k) P(N(t) = k) \\ &= \sum_{k\ge 0} P(\xi_{k+2} > x) P(N(t) = k) \\ &= e^{-\lambda x} \sum_{k\ge 0} P(N(t) = k) = e^{-\lambda x} \end{aligned}$$

5. (20 points; 5 points each question) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose $\mathbb{P}(\xi_1 = 1) = 1 - \mathbb{P}(\xi_1 = -1) = p \in (0, 1)$. Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ for $n \ge 1$. For fixed integers $a > 0$ and $b > 0$, define

$$\tau = \inf\{k \ge 1 : \xi_k = 1\} \quad \text{and} \quad T = \inf\{k \ge 0 : S_k \in \{a, -b\}\}$$

(a) Is τ a stopping time for $\{S_n : n \ge 1\}$? Explain why or why not.

(b) Suppose $p = 0.5$. Calculate $\mathbb{E}[S_\tau]$.

(c) Suppose $p = 0.5$. Compute $\mathbb{P}(S_T = a)$.

(d) Suppose $p \ne 0.5$. Compute $\mathbb{P}(S_T = a)$.

Solution.

(a) Yes. The event $\{\tau = n\}$ occurs if and only if $\xi_1 = -1, \ldots, \xi_{n-1} = -1, \xi_n = 1$, which is equivalent to $S_1 = -1, S_2 = -2, \ldots, S_{n-1} = -(n-1), S_n = -n+2$. So $\{\tau = n\}$ only depends on $S_1, \ldots, S_n$. In addition, for each $n \ge 1$, we have

$$P(\tau = n) = P(\xi_1 = -1, \ldots, \xi_{n-1} = -1, \xi_n = 1) = (1-p)^{n-1} p$$

which implies $P(\tau < \infty) = 1$ (actually $E[\tau] < \infty$). Hence τ is a stopping time for $\{S_n : n \ge 1\}$.

(b) Since $\{S_n : n \ge 1\}$ is a martingale, $E[\tau] < \infty$, and $E[|S_{n+1} - S_n||S_1, \ldots, S_n] = E[|\xi_{n+1}||S_1, \ldots, S_n] = 1$, so we can apply the martingale stopping theorem (condition iii is satisfied), and conclude that $\mathbb{E}[S_\tau] = \mathbb{E}[S_1] = 0$.

(c) You can verify that T is a stopping time for $\{S_n : n \ge 1\}$. In addition, the stopped process $\{\bar{S}_n : n \ge 1\}$ is uniformly bounded by $\max\{|a|, |b|\}$. By martingale stopping theorem we have $E[S_T] = E[S_0] = 0$. Note that $E[S_T] = a \cdot P(S_T = a) - b \cdot P(S_T = -b)$ and $P(S_T = a) + P(S_T = -b) = 1$, we have

$$P(S_T = a) = \frac{-b}{|a| + |b|}$$

(d) Using the martingale $\{\left(\frac{q}{p}\right)^{S_n} : n \geq 1\}$ where $q = 1 - p$ and martingale stopping theorem, one finds that

$$1 = \left(\frac{q}{p}\right)^{S_T} = \left(\frac{q}{p}\right)^{a} P(S_T = a) + \left(\frac{q}{p}\right)^{-b} P(S_T = -b)$$

Hence we obtain when $p \neq 0.5$,

$$P(S_T = a) = \frac{1 - \left(\frac{q}{p}\right)^{-b}}{\left(\frac{q}{p}\right)^{a} - \left(\frac{q}{p}\right)^{-b}}$$

6. (10 points) Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Suppose Z is a standard normal random variable with mean 0 and variance 1. Define $Y(t) = \sqrt{t} \cdot Z$ for $t \geq 0$. For each statement, say "true" or "false". No explanations are needed. Assume throughout that $t > 0$ is fixed but arbitrary.

(a) (3 points) The process $\{Y(t) : t \geq 0\}$ is a standard Brownian motion.

(b) (4 points) $\max_{0 \leq s \leq t} B(s)$ and $|B(t)|$ have the same distribution.

(c) (3 points) $B(t) - \min_{0 \leq s \leq t} B(s)$ and $|B(t)|$ have the same distribution.

Solution.

(a) No.

(b) Yes.

(c) Yes.