SEEM 5580: Homework 1

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PROBLEM 1.1

Since N is a nonnegative integer-valued random variable, then we can obtain that

$$
\begin{aligned}
E[N] &= \sum_{k=0}^{\infty} k \cdot P\{N = k\} \\
&= \sum_{k=0}^{\infty} k \cdot [P\{N \geqslant k\} - P\{N \geqslant k+1\}] \\
&= \sum_{k=1}^{\infty} P\{N \geqslant k\};
\end{aligned}
$$

and also,

$$
\begin{aligned}
E[N] &= \sum_{k=0}^{\infty} k \cdot P\{N = k\} \\
&= \sum_{k=1}^{\infty} k \cdot [P\{N > k-1\} - P\{N > k\}] \\
&= \sum_{k=0}^{\infty} P\{N > k\}.
\end{aligned}
$$

Now consider the case where X is nonnegative with distribution F. In this case

$$
\begin{aligned}
E[X^n] &= \int_0^{\infty} x^n dF(x) \\
&= \int_0^{\infty} \int_0^{x} n y^{n-1} dy dF(x) \\
&= \int_0^{\infty} \int_y^{\infty} dF(x) n y^{n-1} dy \\
&= \int_0^{\infty} n x^{n-1} \overline{F}(x) dx
\end{aligned}
$$

where in the third equality we have used change of variables and Fubini–Tonelli theorem.

PROBLEM 1.2

(a) First,

$$P\{F(X) \leqslant F(x)\} = P\{X \leqslant x\} = F(x),$$

where the first equality is established because distribution function F is non-decreasing and right-continuous. Now let $Z = F(X), z = F(x)$, then $0 \leqslant z \leqslant 1$ and we have

$$P\{Z \leqslant z\} = z.$$

Then it can be easily obtained that the probability density function of random variable Z is

$$f(z) = 1, 0 \leqslant z \leqslant 1,$$

which means that the random variable $F(X)$ is uniformly distributed over (0,1).
(b) First, since distribution function F is non-decreasing and right-continuous, then its inverse function F^{-1} is also non-decreasing and right-continuous. Thus, we have

$$P\{F^{-1}(U) \leqslant F^{-1}(u)\} = P\{U \leqslant u\} = u = F[F^{-1}(u)], \tag{1}$$

where the second last equality holds because U is a uniform (0,1) random variable, and the last equality holds because the property of inverse function. Equation (1) proves that $F^{-1}(U)$ has distribution F.

PROBLEM 1.3

Since X_n is a binomial random variable with parameters (n, p_n), then

$$P\{X_n = i\} = \frac{n!}{(n-i)!i!} p_n^i (1-p_n)^{n-i}.$$

Since $np_n \to \lambda$ as $n \to \infty$, then it is implied that $p_n \to 0$ as $n \to \infty$. Thus, we obtain that as $n \to \infty$,

$$(1-p_n)^{n-i} = (1-p_n)^n (1-p_n)^{-i} \sim (1-p_n)^n \sim e^{-np_n} \to e^{-\lambda},$$

Also

$$\frac{n!}{(n-i)!} p_n^i = \frac{n(n-1)\cdots(n-i+1)(n-i)!}{(n-i)!} p_n^i \sim (np_n)^i \to \lambda^i, \quad as \quad n \to \infty,$$

Therefore, we can conclude that if $np_n \to \lambda$ as $n \to \infty$, then

$$P\{X_n = i\} \to e^{-\lambda} \lambda^i / i! \qquad as \quad n \to \infty.$$

PROBLEM 1.8

Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2.
(a) We apply moment generating function to derive the distribution of $X_1 + X_2$. The moment generating function of $X_1 + X_2$ is

$$\begin{aligned}
\psi_{X_1+X_2}(t) &= E[e^{t(X_1+X_2)}] \\
&= E[e^{tX_1}]E[e^{tX_2}] \\
&= \psi_{X_1}(t)\psi_{X_2}(t) \\
&= exp\{(\lambda_1+\lambda_2)(e^t-1)\}.
\end{aligned}$$

where the second equality is derived by independence of X_1 and X_2, and the last equality is derived from the definition of moment generating function of Poisson random variables. Thus the moment generating function of $X_1 + X_2$ is that of a Poisson random variable with mean $\lambda_1 + \lambda_2$. By uniqueness, this is the distribution of $X_1 + X_2$.
(b) The probability mass function of X_1 given $X_1 + X_2 = n$ is

$$\begin{aligned}
P\{X_1 = x_1 | X_1 + X_2 = n\} &= \frac{P\{X_1 = x_1, X_2 = n - x_1\}}{P\{X_1 + X_2 = n\}} \\
&= \frac{P\{X_1 = x_1\}P\{X_2 = n - x_1\}}{P\{X_1 + X_2 = n\}} \\
&= \frac{\lambda_1^{x_1}\lambda_2^{n-x_1} n!}{(\lambda_1+\lambda_2)^n x_1!(n-x_1)!} \\
&= C_n^{x_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x_1}.
\end{aligned}$$

PROBLEM 1.16

(a)

$$\begin{aligned}
P(X \le x) &= P\left(Y \le x \middle| U \le \frac{f(Y)}{cg(Y)}\right) \\
&= \frac{P\left(Y \le x, U \le \frac{f(Y)}{cg(Y)}\right)}{P\left(U \le \frac{f(Y)}{cg(Y)}\right)}
\end{aligned} \tag{2}$$

Since U is uniformly distributed on $(0,1)$, the denominator in (2) is given by:

$$P\left(U \le \frac{f(Y)}{cg(Y)}\right) = \int_R \frac{f(t)}{cg(t)} g(t)dt = \frac{1}{c} \tag{3}$$

Making this substitution in (2), we find that:

$$P(X \le x) = cP\left(Y \le x, U \le \frac{f(Y)}{cg(Y)}\right) = c\int_{-\infty}^{x} \frac{f(t)}{cg(t)} g(t)dt = \int_{-\infty}^{x} f(t)dt \tag{4}$$

Since x is arbitrary, this verifies that X has density function f.

(b) In fact, Equation (3) shows that the probability of acceptance on each attempt is $1/c$. Because the attempts are mutually independent, the number of iterations of the algorithm needed to generate X is geometrically distributed with mean c.

PROBLEM 1.25

Clearly, we have $M_0 = M_k = 0$ and condition on the first gamble we have $M_n = \frac{1}{2}M_{n-1} + \frac{1}{2}M_{n+1} + 1$. So we have:

$$M_1 = M_2 - M_1 + 2 = M_3 - M_2 + 4 = M_4 - M_3 + 6 = ... = M_k - M_{k-1} + 2(k-1)$$

By symmetry, we know $M_1 = M_{k-1}$, so $M_1 = M_{k-1} = k-1$ and $M_2 = 2M_1 - 2 = 2(k-2)$. By induction, suppose $M_i, i = 1,2,..n-1$ satisfying $M_i = i(k-i)$. Then $M_n = 2M_{n-1} - M_{n-2} - 2 = 2(n-1)(k-n+1) - (n-2)(k-n+2) - 2 = n(k-n)$.

PROBLEM 1.29

First, we can easily see that X_i itself is a gamma distribution with parameters $(1,\lambda)$. Then by induction, it would suffice to show that a gamma $(1,\lambda)$ plus an independent gamma (s,λ) is a gamma distribution with parameters $(s+1,\lambda)$. Now let X is gamma $(1,\lambda)$, Y is gamma (s,λ), and X and Y are independent. So

$$f_X(x) = \lambda e^{-\lambda x}, f_Y(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{s-1}}{(s-1)!}, \quad t \geqslant 0.$$

Since X and Y are independent, therefore,

$$f_{X+Y}(t) = \int_0^t f_X(t-y) f_Y(y) dy = \int_0^t \lambda e^{-\lambda(t-y)} \frac{\lambda e^{-\lambda y}(\lambda y)^{s-1}}{(s-1)!} dy = \frac{\lambda e^{-\lambda t}(\lambda t)^s}{s!}.$$

Thus $X+Y$ follows a gamma distribution with parameters $(s+1,\lambda)$. Therefore, using this result, it is easy to show that $\sum_{i=1}^n X_i$ has a gamma distribution with parameters (n,λ).