Homework 4

Problem 1

(a)

standard form of (2):

$$\begin{aligned} &\text{minimize} && (\mathbf{0}, e)^T (x_1, \ldots, x_n, y_1, \ldots, y_m) \\ &\text{subject to} && [A \mid I] \begin{bmatrix} x \\ y \end{bmatrix} = b \\ & && (x, y) \geq \mathbf{0}. \end{aligned}$$

dual of (2):

$$\begin{aligned} &\text{maximize} && b^T s \\ &\text{subject to} && \begin{bmatrix} A \\ \hline I \end{bmatrix} s \leq \begin{bmatrix} \mathbf{0} \\ e \end{bmatrix}. \end{aligned}$$

(b)

By the limitation of $(x, y) \geq \mathbf{0}$, there exist $n + m$ linearly independent vectors. So (2) has at least one vertex, then either the optimal values is $-\infty$, or there exists a vertex that is optimal via **Handout-3 Theorem 3 & 4**.

Besides, because $e, y \geq \mathbf{0}$, so $e^T y \geq \mathbf{0}$ which means the optimal values is impossible $-\infty$, so (2) always has an optimal solution.

(c)

proof of $\Rightarrow$:

Because (1) has solution so there $\exists \bar{x}$ satisfy (1). Then $(x, y) = (\bar{x}, \mathbf{0})$ is a feasible solution in (2) with value is zero.

Besides, $e^T y \geq 0$, so that is a optimal solution.

proof of $\Leftarrow$:

Because (2) has the optimal value is zero, denote one optimal solution as $(\bar{x}, \bar{y})$, we have $e^T \bar{y} = 0$ and $\bar{y} \geq \mathbf{0}$ which can refer that $\bar{y} = \mathbf{0}$.

So we can find $A\bar{x} + I\bar{y} = b \Rightarrow A\bar{x} = b - I\bar{y} = b$ and $\bar{x} \geq \mathbf{0}$

Hence $\bar{x}$ is a solution of (1)

In summary, system (1) has a solution $\Leftrightarrow$ opt value of (2) is 0.

Problem 2

We can construct dual LP as follow:

$$\begin{aligned} v_p^* = &\text{minimize} && -e^T x \\ &\text{subject to} && P^T x = x \\ & x \geq \mathbf{0} \end{aligned}$$

and dual:

$$\begin{aligned} v_d^* = \text{maximize} \quad & 0^T y \\ \text{subject to} \quad & (P - I)y \leq -e \end{aligned}$$

because P is a stochastic matrix with $P_{ij} \geq 0$ and $Pe = e$, so we have $0 \leq P_{ij} \leq 1, \sum_{j=1}^{n} P_{ij} = 1$.

We only need to show that $(P - I)y \leq -e$ is infeasible, i.e. $\forall y, \exists y_i, s.t. \sum_{j=1}^{n} P_{ij} y_j > y_i - 1$

Assume $\exists y, s.t (P - I)y \leq -e$, let $min(y_i) = y_0$.

we have $\forall i, (1 - P_{ii}) y_i \geq \sum_{j \neq i} P_{ij} y_j + 1$ and $P_{ii} \neq 1$ ($P_{ii} = 1$, then $0 \geq 1$)

$\Rightarrow y_i \geq \frac{\sum_{j \neq i} P_{ij} + 1}{1 - P_{ii}} \geq \frac{(1 - P_{ii}) y_0 + 1}{1 - P_{ii}} = y_0 + \frac{1}{1 - P_{ii}} > y_0$

but the inequality is not true when $y_i = y_0$, so $(P - I)y \leq -e$ is infeasible.

Hence, due to LP duality theory, primal problem is infeasible or unbounded.

And $x = \mathbf{0}$ is one solution of primal problem, so primal problem is unbounded, which means $P^T x = x, x \geq 0, x \neq \mathbf{0}$ is solvable.

Problem 3

(a)

Because $p, q > 1$, $||x||_p$ is continuous and convex on R^n

- non-empty and closed under addition:

  $\mathbf{0} \in C_p \neq \emptyset$,
  $\forall (t_u, u), (t_v, v) \in C_p, ||u + v||_p \leq ||u||_p + ||v||_p \leq t_u + t_v \Rightarrow (t_u + t_v, u + v) \in C_p$
- cone:

  $\forall (t, u) \in C_p, \alpha > 0, ||\alpha u||_p = \alpha ||u||_p \leq \alpha t \Rightarrow (\alpha t, \alpha u) \in C_p$
- pointed:

  let $(t, u), (-t, -u) \in C_p$, we have $t = 0$ because $||x||_p \geq 0$, thus $u = 0$
- closed:

  We only need to show that, for all **limit point** of C_p in C_p. By definition, for any limit point (t, u) and correspond sequence $\{(t_i, u_i) \in C_p\}, lim\ (t_i, u_i) = (t, u)$, consider function $f(t, x) = ||x||_p - t$ is continuous and $f(t, x) \leq 0, (t, x) \in C_p$, thus $f(t, u) = lim\ f(t_i, u_i) \leq 0$ which indicate that $(t, u) \in C_p$.

In summary, C_p is a closed pointed cone.

(b)

$C_p^* = \{(t', x') \in R \times R^n : t * t' + x \bullet x' \geq 0, \forall (t, x) \in C_p\}$

$1^o : C_p^* \supseteq C_q :$

$\forall (t_p, x_p) \in C_p, (t_q, x_q) \in C_q, ||x_p||_p \leq t_p, ||x_q||_q \leq t_q$

By Holder's inequality: $||fg||_1 \leq ||f||_p ||g||_q$

Thus $t_p * t_q + x_p \bullet x_q \geq ||x_p||_p ||x_q||_q - ||x_p x_q||_1 \geq 0, (t_q, x_q) \in C_p^*$

which indicate that $C_p^* \supseteq C_q$

$2^o : C_p^* \subseteq C_q :$

$\forall (t', x') \in C_p^*, t * t' + x \bullet x' \geq 0, \forall (t, x) \in C_p$

let $x = -x', t = ||x'||_p$, thus we have $x' \bullet x' = ||x'x'||_1 = ||x'||_p||x'||_q$

$\Rightarrow ||x'||_p * t' - x' \bullet x' = ||x'||_p * t' - ||x'||_p||x'||_q \geq 0$

$\Rightarrow ||x'||_q \leq t'$

$\Rightarrow (t', x') \in C_q$

thus $C_p^* \subseteq C_q$

In summary, $C_p^* = C_q$

(c)

$int(C_p) = \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}$

proof:

$1^o : bound(C_p) = C_p \setminus int(C_p) \supseteq int(C_p) \setminus \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}$

$\Rightarrow int(C_p) \subseteq \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}:$

$\forall (t, x) \in \{||x||_p = t\}, \exists r \in R^+, (t - r, x) \in B((t, x), r), (t - r, x) \notin C_p$

therefore, $C_p \setminus int(C_p) \supseteq \{(t, x) \in R \times R^n : t \geq 0, ||x||_p = t\}$

$2^o : int(C_p) \supseteq \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}$

By definition, $\forall (||x||_p+r, x) \in \{(t, x) \in R \times R^n:t\ge0, ||x||_p < t\}, r \in R^+, \exist B((||x||_p+r, x), r'), r' = r * sin \theta, $ where $cos\theta = \frac{v_1 \bullet v_2}{||v_1||||v_2||}, v_1 = (1, 0, \ldots, 0), v_2 = det(t, ||x||_p)$

It's easy to verify that $r' < dis(||x||_p + r, bound(C_p))$, so $int(C_p) \supseteq \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}$

In summary, $int(C_p) = \{(t, x) \in R \times R^n : t \geq 0, ||x||_p < t\}$

Problem 4

(a)

let $y \in R^{n_j+1}, y = A^j \begin{bmatrix} x \\ u \end{bmatrix} - b^j,$

$y \in Q^{n_j+1} \Leftrightarrow ||y_{2 \sim n_j+1}||_2 \leq y_1$

We need to construct $A_j, b_j, s.t. z = y_{2 \sim n_j+1}, ||z||_2 \leq y_1 \Leftrightarrow x^T Q x = x^T Q^{1/2} Q^{1/2} x \leq t \Leftrightarrow ||Q^{1/2}x||_2^2 \leq t \Leftrightarrow ||Q^{1/2}x||_2 \leq \sqrt{t}$

which indicate that $z = Q^{1/2}x, y_1 = \sqrt{t}$ (for $t < 0$, let $y_1 = -1$)

Then we let $A^j = \begin{bmatrix} 1 & \mathbf{0} & -1 \\ \mathbf{0} & Q^{1/2} & \mathbf{0} \end{bmatrix}, \mu = (t - \sqrt{t}), b = \mathbf{0} \Rightarrow y = A^j \begin{bmatrix} x \\ \mu \end{bmatrix} - b^j = \begin{bmatrix} \sqrt{t} \\ Q^{1/2}x \end{bmatrix}$

Thus X is SOC-representable.

(b)

let $\pi \geq 0, \sqrt{x_1 x_2} \geq \pi \geq t$, it easy to show that π exist and $t \leq \sqrt{x_1 x_2} \Leftrightarrow \pi^2 \leq x_1 x_2 = \frac{1}{4}((x_1 + x_2)^2 - (x_1 - x_2)^2)$

$\Leftrightarrow ||(\pi, \frac{1}{2}(x_1 - x_2)||_2 \leq \frac{1}{2}(x_1 + x_2)$ which indicate that then final vector $v = (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 - x_2), \pi)^T \in Q^3$

The we let $A^j = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mu = (\pi), b^j = \mathbf{0} \Rightarrow v = A^j \begin{bmatrix} x \\ \mu \end{bmatrix} - b^j = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) \\ \frac{1}{2}(x_1 - x_2) \\ \pi \end{bmatrix}$

Thus X is SOC-representable.

Problem 5

(a)

$$\forall p = u + v \in K_1^* + K_2^*, u \in K_1^*, v \in K_2^*, \forall x \in K_1 \cap K_2, p \bullet x = (u + v) \bullet x = u \bullet x + v \bullet x \geq 0$$

Hence, $p \in (K_1 + K_2)^*, K_1^* + K_2^* \subseteq (K_1 \cap +K_2)^*$

(b)

$1^o : K_1^* + K_2^*$ is a cone:

$\forall p = u + v \in K_1^* + K_2^*, u \in K_1^*, v \in K_2^*, \forall \alpha > 0, \alpha p = \alpha u + \alpha v, \alpha u \in K_1^*, \alpha v \in K_2^*$

Hence, $\alpha p \in K_1^* + K_2^*, K_1^* + K_2^*$ is a cone

$2^o : K_1^* + K_2^*$ is convex:

$\forall p_1, p_2 \in K_1^* + K_2^*, p_1 = u_1 + v_1, p_2 = u_2 + v_2, u_1, u_2 \in K_1^*, v_1, v_2 \in K_2^*, \forall \alpha \in (0, 1),$

$\alpha p_1 + (1 - \alpha)p_2 = (\alpha u_1 + (1 - \alpha)u_2) + (\alpha v_1 + (1 - \alpha)v_2) = u + v$, where $u \in K_1^*, v \in K_2^*$ because they are convex

Hence, $\alpha p_1 + (1 - \alpha)p_2 \in K_1^* + K_2^*$ which indicates $K_1^* + K_2^*$ is convex

In summary, $K_1^* + K_2^*$ is a convex cone

(c)

By the definition of close, we need to show that $\forall p = u + v \in K_1^* + K_2^*, \exists \{p_i = u_i + v_i\} \in K_1^* + K_2^*, \{p_i\} \to p$

Because, $\exists x \in int(K_1) \cap int(K_2)$, thus for any given $t, S_t = \{w \in K_1^* : ||w||_2 = t\}$ is compact and $inf_{S_t} \langle x, w \rangle > 0$ and same as K_2

which indicate that $\{\langle x, u_i \rangle\}, \{\langle x, v_i \rangle\}$ is also bounded and $\{u_i\}, \{v_i\}$ is bounded.

Hence, $\{u_i\}, \{v_i\}$ must exist some convergent subsequences let them converge on u, v and belong to K_1^*, K_2^* with the closeness, thus $K_1^* + K_2^*$ is closed.

(d)

Proof by contradiction:

Assume $(K_1 \cap K_2)^* \subsetneq K_1^* + K_2^*$ which means there exist $x \in (K_1 \cap K_2)^*, x \notin K_1^* + K_2^*$. Thus exist a vector b, let $b^T w \leq 0 < b^T x, \forall w \in K_1^* + K_2^*$

Because $0 \in K_1^*, K_2^*$, thus $b^T w \leq 0, \forall K_1^* \cup K_2^*$

However, $x \in (K_1 \cap K_2)^*$ which means $\forall w \in K_1 \cap K_2, x^T w \geq 0$ thus $x \in K_1^* \cup K_2^*$ which means $b^T x \leq 0$, and it is a contradiction

Therefore, $(K_1 \cap K_2)^* \subseteq K_1^* + K_2^*$