ENGG 5501: Foundations of Optimization	**2021–22 First Term**
Homework Set 5	
Instructor: Anthony Man–Cho So	Due: December 16, 2021

SOLVE THE FOLLOWING PROBLEMS. THE PARTS LABELED "EXTRA CREDIT" ARE OPTIONAL.

Problem 1 (20pts). Let $A \in \mathcal{S}^n$ be given. Let λ_1 be the largest eigenvalue of A and v_1 be a unit-length eigenvector associated with λ_1. Consider the following problem:

$$\begin{array}{ll} \max & x^T A x \\ \text{subject to} & \|x\|_2^2 = 1, \\ & v_1^T x = 0. \end{array} \tag{1}$$

(a) **(10pts).** Write down the first-order optimality conditions of Problem (1) and explain why they are necessary for optimality.

(b) **(10pts).** Let λ_2 be the optimal value of and v_2 be an optimal solution to Problem (1). Using the result in (a), show that λ_2 is the second largest eigenvalue of A and v_2 is an eigenvector associated with λ_2.

Problem 2 (15pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function and $C \subseteq \mathbb{R}^n$ be a non–empty closed convex set. Show that x^* is an optimal solution to the convex optimization problem

$$\min_{x \in C} f(x)$$

if and only if

$$x^* = \Pi_C(x^* - \nabla f(x^*)),$$

where $\Pi_C(\cdot)$ is the projection operator onto C.

Problem 3 (15pts). Let $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable convex functions. Consider the problem

$$\min_{x \in \mathbb{R}^n} \ \max\{g_1(x), \ldots, g_m(x)\} \tag{2}$$

Show that $x^* \in \mathbb{R}^n$ is an optimal solution to Problem (2) if and only if there exists a vector $u^* \in \mathbb{R}^m$ such that

$$\begin{array}{l} \displaystyle\sum_{j=1}^m u_j^* \nabla g_j(x^*) = \mathbf{0}, \quad u^* \geq \mathbf{0}, \quad \sum_{j=1}^m u_j^* = 1, \\ u_j^* = 0 \ \text{ if } \ g_j(x^*) < \max\{g_1(x^*), \ldots, g_m(x^*)\}, \ \text{for } j = 1, \ldots, m. \end{array}$$

Problem 4 (30pts). Let $A \in \mathcal{S}^n$ be given. Consider the following QCQP:

$$\begin{array}{ll} \inf & x^T A x \\ \text{subject to} & x_i^2 = 1 \quad \text{for } i = 1, \ldots, n. \end{array} \tag{3}$$

(a) **(5pts).** Derive the semidefinite relaxation of Problem (3) using the techniques introduced in class.

(b) **(10pts).** Write down the dual of the semidefinite relaxation you found in (a). Does the primal-dual pair of SDPs you obtained have zero duality gap? Justify your answer.

(c) **(15pts).** The Lagrangian dual of Problem (3) is given by

$$\sup_{w \in \mathbb{R}^n} \theta(w), \tag{4}$$

where

$$\theta(w) = \inf_{x \in \mathbb{R}^n} \left\{ x^T A x + \sum_{i=1}^{n} w_i (1 - x_i^2) \right\}.$$

Find an explicit expression for $\theta(w)$. Hence, or otherwise, show that Problem (4) is equivalent to the *dual* of the semidefinite relaxation you found in (b).

Problem 5 (20pts). Consider the following problem:

$$\begin{array}{ll} \inf & x_1^2 + 4x_2^2 + 16x_3^2 \\ \text{subject to} & x_1 x_2 x_3 = 1. \end{array} \tag{5}$$

(a) [**Extra Credit (15pts).**] Show that Problem (5) has an optimal solution.

(b) **(10pts).** Write down the first-order optimality conditions of Problem (5) and explain why they are necessary for optimality.

(c) **(10pts).** Using the result in (b), determine the set of optimal solutions to Problem (5).

Problem 6 [**Extra Credit (15pts).**] Let $A, X_0 \in \mathcal{S}_+^n$ and $r > 0$ be given with $A \neq \mathbf{0}$. Consider the following problem:

$$\begin{array}{ll} \sup & \operatorname{tr}(AX) \\ \text{subject to} & \|X - X_0\|_F^2 \leq r^2, \\ & X \in \mathcal{S}_+^n. \end{array} \tag{6}$$

Here, as usual, we have $\|M\|_F^2 = \operatorname{tr}(M^2)$ for any $M \in \mathcal{S}^n$. Determine the optimal solution to Problem (6) by considering its first-order optimality conditions. Justify your arguments carefully.