- This is an online open-book exam. Calculators are allowed.
- Total points: 100. Give sufficient details when you answer questions to get partial credits.
- Submit you answer scripts in a single PDF to blackboard by the **deadline 5:15 pm, March 1, 2021**. If there is internet connection issue and blackboard system does not work, please submit the scripts to xfgao@se.cuhk.edu.hk by the deadline. Late submission will be assigned a mark of zero.
- **You must work on the exam independently. Any form of Plagiarism will not be tolerated by the University.**

1. (20 points; 10 points each) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and $\{N(t) : t \geq 0\}$ models customer arrivals to a service center. Let ξ_i be the inter–arrival time between $(i-1)-$th and $i-$th customers, and let $T_n = \sum_{i=1}^{n} \xi_i$ be the arrival time of $n-$th customer where $T_0 := 0$. Answer the following questions. Assume throughout that $t > 0$ is fixed but arbitrary.
   (a) Find the distribution of $\xi_{N(t)+3}$. Justify your answer.
   (b) Is $\{N(k) : k = 0, 1, 2, \ldots\}$ a discrete-time Markov chain? If yes, prove your assertion and specify the state space and transition probabilities. If not, explain why.

2. (30 points; 10 points each) Assume that passengers arrive at a subway station between 10am -12noon following a non-homogeneous Poisson process with rate function $\lambda(t)$ given as follows. From 10am-11am, the arrival rate is constant, 1 passenger per minute. From 11am to noon, it increases linearly from 1 passenger per minute to 2 passengers per minute.

   (a) What is the probability that there are 2 arrivals between 10:00am to 10:10am *and* 1 arrival between 10:05am to 10:15am?

   (b) What is the expected number of arrivals between 10:30am and 11:30am?

   (c) What is the probability that the 1st passenger after 11am will take at least 5 minutes to arrival?

3. (30 points) Let $X = \{X_n : n = 0, 1, \ldots,\}$ be a Markov chain with state space $\{a, b\}$. The transition probabilities are given by

$$P_{aa} = 0.4, P_{ab} = 0.6, P_{ba} = 0.2, P_{bb} = 0.8.$$

When the Markov chain transits to state $x \in \{a, b\}$, a reward $f(x)$ is collected where

$$f(x) = \begin{cases} \$2 & \text{if } x = a, \\ \$1 & \text{if } x = b. \end{cases}$$

(a) (10 points) Find the finite-horizon expected total reward

$$\mathbb{E}\Big[f(X_0) + f(X_1) + f(X_2)\Big| X_0 = a\Big].$$

(b) (10 points) Find the infinite-horizon expected discounted total reward

$$\mathbb{E}\Big[\sum_{n=0}^{\infty} (0.8)^n f(X_n)\Big| X_0 = a\Big].$$

(c) (10 points) Find the expected long-run average reward

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\Big[\sum_{n=1}^{N} f(X_n)\Big| X_0 = a\Big].$$

4. (20 points, 5 points each) Consider a discrete time Markov chain $\{X_n : n = 0, 1, \ldots\}$ with the state space $S = \{1, 2, 3, 4\}$ and the (one-step) transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$$

Answer the following questions and *explain your answers.*

(1) This chain is irreducible. True or False?

(2) Specify the period of every state of this chain.

(3) Classify each state: which one is transient, which one is null recurrent and which one is positive recurrent?

(4) Is there an unique stationary distribution of this chain? If yes, provide the stationary distribution; If not, explain why.