Projection

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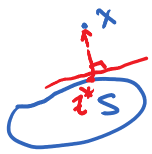
Projection: Given a set $S \subseteq \mathbb{R}^n$, $S \neq \phi$, and a point $x \notin S$,

we want to find a point in S that is closest to x.

wrt Euclidean distance

e.g.



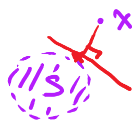
$z^* \in S$ and is closest to x.

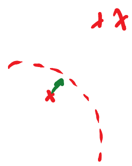
Formally, $z^* = \underset{z \in S}{\operatorname{argmin}} \|z - x\|_2$ is the projection of

x onto S and denoted by $z^* = \Pi_S(x)$.

Q: Existence?

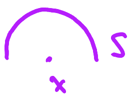
A: Not always. e.g.:





Q: Uniqueness?

A: Not always. e.g.



every point on S is a projection of x onto S.

Q: Conditions that guarantee existence and uniqueness?

Topological Preparations

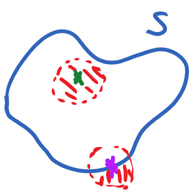
Let $S \subseteq \mathbb{R}^n$ be a set.

Definitions:

① We say that x is an interior point of S if $\exists \varepsilon > 0$

s.t. $B^\circ(x, \varepsilon) \triangleq \{ y \in \mathbb{R}^n : \|x - y\|_2 < \varepsilon \} \subseteq S$.

e.g.:

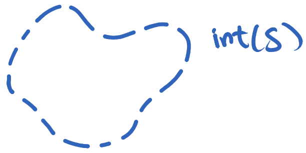


$\times$: interior point

$\times$: not an interior point

The collection of all interior points of S is called the interior of S, denoted by $\operatorname{int}(S)$.

e.g.:

e.g.: 

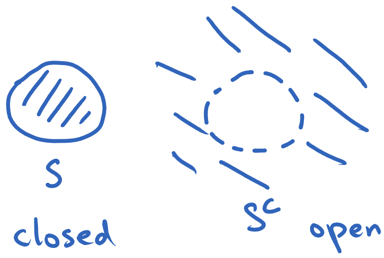
② We say that S is **open** if $S = \text{int}(S)$

e.g. 

not open

③ We say that S is **closed** if $\mathbb{R}^n \setminus S$ is open

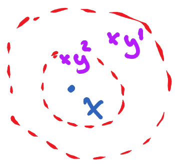
S^c : complement of S
everything in $\mathbb{R}^n$ but not in S

e.g. 

not closed
not open

④ We say that x is a **limit point** of S if every open ball centered at x contains $y \in S$ s.t. $y \neq x$.

In particular, $\exists$ sequence $y^i \in S$ s.t.
$y^i \neq x$ $\forall i$ and $y^i \to x$.



Note: x need not be in S.

e.g. $S = (0,1]$ 

Take $y^i = 1/i$, $i \geq 1$. Note: $y^i \in S$ $\forall i$, $y^i \to 0$

$\Rightarrow$ $x = 0$ is a limit point of S but $x \notin S$.

Exercise: Every $x \in \underset{\overset{||}{S}}{(0,1]}$ is a limit point of S.

Note: A finite set has no limit point.

Fact: S is closed iff every limit point of S belongs to S

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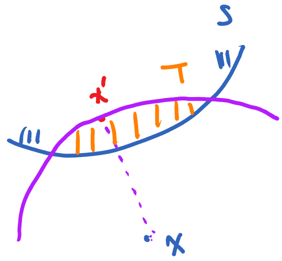
To prove S is closed, take any sequence $y^i \in S$ s.t. $y^i \to x$, $y^i \neq x$, then show $x \in S$.

Theorem: Let $S \subseteq \mathbb{R}^n$ be a non-empty, closed, and convex set.

Then, for every $x \in \mathbb{R}^n$, $\exists$ unique $z^* \in S$ s.t. $z^* = \Pi_S(x)$.

Proof: (Existence)



We may assume that $x \notin S$.

Consider any $x' \in S$ and define

$$T = S \cap B(x, \|x - x'\|_2)$$

Observe:

① $$\min_{z \in S} \|z - x\|_2 = \min_{z \in T} \underbrace{\|z - x\|_2}_{f(z)}$$

② T is closed (it is the intersection of 2 closed sets)

and bounded ($\exists R > 0$ s.t. $T \subseteq B(0, R)$)

In other words, T is compact (= closed and bounded)

Fact: (Weierstrass Theorem)

If f is continuous over a compact set T, then

then it attains its maximum and minimum on T.

(optimal solution always exists)

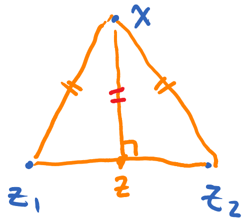
This proves the existence of the projection.

Note: The above argument does not need convexity.

(Uniqueness)

Suppose that $z_1, z_2 \in S$ s.t. both are projections

of x onto S.



① $\|x - z_1\|_2 = \|x - z_2\|_2$

② $z \in S$ because S is convex and z lies on the line segment $[z_1, z_2]$

③ (Exercise)

$\rightarrow \|x - z\|_2 < \|x - z_1\|_2 \Rightarrow$ contradiction