Projection (cont'd), Separation

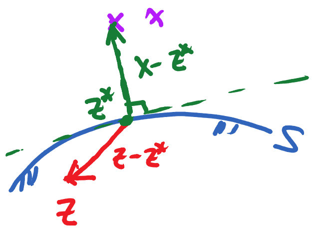
23/09/2021 12:35

Recap: Existence and uniqueness of projection is guaranteed for non-empty, closed, convex sets

Q: Consider such a set S. Let $x \notin S$. Suppose that $z \in S$ is given. How to verify $z = \Pi_S(x)$?

A: **Theorem**: Under the previous setting, given $x \in \mathbb{R}^n$,

$$z^* = \Pi_S(x) \iff z^* \in S, \ (z - z^*)^T(x - z^*) \le 0 \quad \forall z \in S.$$



Example: Consider $S = B(0,1)$. Let $x \in \mathbb{R}^n$ be s.t. $x \notin S$.

Note that $\|x\|_2 > 1$. Then,

$$\Pi_S(x) = \frac{x}{\|x\|_2}$$

Indeed, $\forall z \in B(0,1)$

$$\left(z - \frac{x}{\|x\|_2}\right)^T\left(x - \frac{x}{\|x\|_2}\right) = x^T z - \|x\|_2 - \frac{x^T z}{\|x\|_2} + 1$$

$$= \left(1 - \frac{1}{\|x\|_2}\right) x^T z - \|x\|_2 + 1$$

$\downarrow$ Cauchy-Schwarz

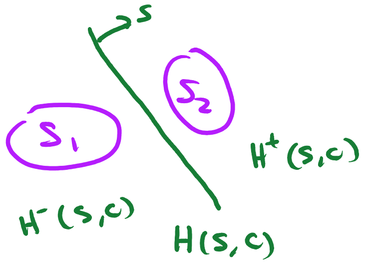
$$\le \left(1 - \frac{1}{\|x\|_2}\right) \|x\|_2 \cdot \underbrace{\|z\|_2}_{\le 1 \text{ since } z \in B(0,1)} - \|x\|_2 + 1$$

$$\le \left(1 - \frac{1}{\|x\|_2}\right) \|x\|_2 - \|x\|_2 + 1 = 0.$$

Separation

Motivation: Given sets $S_1, S_2 \subseteq \mathbb{R}^n$, how can we certify that $S_1 \cap S_2 = \emptyset$?

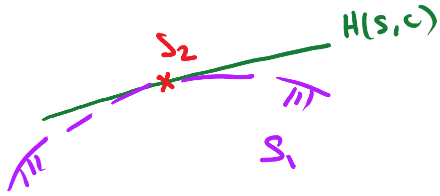
Idea:



Observe:

$S_1 \subseteq H^-(s,c)$
$S_2 \subseteq H^+(s,c)$ $\rightarrow$ use hyperplane $H(s,c)$ to certify $S_1 \cap S_2 = \emptyset$.

but



$S_1 \cap S_2 = \emptyset$

a hyperplane cannot "properly" separate S_1 and S_2

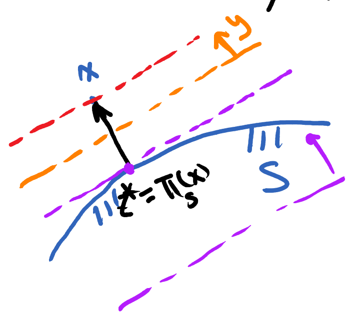
$S_1 \cap S_2 = \emptyset$, but no hyperplane can separate the two

Theorem: (Point-Set Separation)

Let $S \subseteq \mathbb{R}^n$ be a non-empty, closed, convex set and $x \notin S$.

Then, $\exists y \in \mathbb{R}^n$ s.t

$$\max_{z \in S} y^T z < y^T x$$



Proof: By projection theorem,

$z^* = \Pi_S(x)$ exists and is unique. Consider $y = x - z^*$.

Note that $y \neq 0$ because $x \notin S$. Then,

$$\forall z \in S: \quad (z - z^*)^T y \leq 0$$

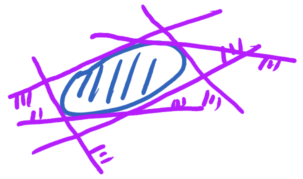
$$\Rightarrow \quad y^T z \leq y^T z^* = y^T x + y^T \underbrace{(z^* - x)}_{= -y}$$

$$= y^T x - \|y\|_2^2$$

$$\Rightarrow \quad \max_{z \in S} y^T z \leq y^T x - \|y\|_2^2 < y^T x$$

Theorem: A closed convex set $S \subseteq \mathbb{R}^n$ is the intersection of all halfspaces containing S; i.e., $S = \bigcap_{\substack{H \text{ halfspace} \\ H \supseteq S}} H$

e.g.



Proof: Special cases:

$\phi = H^-(0,-1) \cap \cdots$ (recall: $H^-(s,c) = \{x : s^T x \le c\}$)

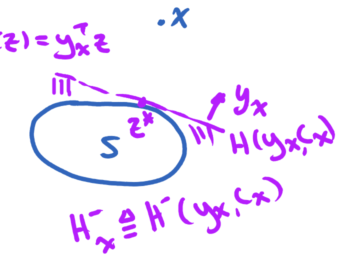
$\mathbb{R}^n = \bigcap_{c \geq 0} H^-(0,c)$

We may assume without loss of generality that $\phi \subsetneq S \subsetneq \mathbb{R}^n$.

Let $x \notin S$. Then, by Separation theorem,

$\exists\, y_x \in \mathbb{R}^n$ and $\underline{c_x = \max_{z \in S} y_x^T z}$ s.t. $f(z) = y_x^T z$

$$S \subseteq H^-(y_x, c_x) \triangleq H_x^- \qquad \text{"} f(z^*) \text{)}$$



Note: $x \notin H_x^-$.

Claim: $S = \bigcap_{x \notin S} H_x^-$

Proof of Claim: Note that $\forall x \notin S$, we have $S \subseteq H_x^-$ by construction $\Rightarrow S \subseteq \bigcap_{x \notin S} H_x^-$.

Conversely, suppose that $\bigcap_{x \notin S} H_x^- \not\subseteq S$. Then,

$\exists z$ s.t. $z \in \bigcap_{x \notin S} H_x^-$ but $z \notin S$

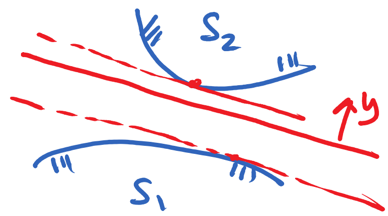
However, $z \in \bigcap_{x \notin S} H_x^- \subseteq H_z^- \not\ni z$.

a contradiction.

How about set-set separation?

S_1, S_2 : non-empty, closed, convex

and $S_1 \cap S_2 = \phi$



<u>Want</u>: $\exists y \in \mathbb{R}^n$ s.t.

$$\max_{z \in S_1} y^T z < \min_{z \in S_2} y^T z$$