ENGG 5501: Foundations of Optimization	**2021–22 First Term**
Homework Set 3	
Instructor: Anthony Man–Cho So	Due: October 25, 2021

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Let $C \subseteq \mathbb{R}^n$ be a non-empty closed convex set and $\mathbb{I}_C$ denote its associated indicator function; i.e.,

$$\mathbb{I}_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

(a) **(5pts).** Argue directly using the definition of the subdifferential that for any $x \in C$,

$$\partial \mathbb{I}_C(x) = \{ s \in \mathbb{R}^n : s^T(y - x) \le 0 \text{ for all } y \in C \}.$$

(b) **(15pts).** Using the result in (a), show that for any $x \in \mathbb{R}^n_+$,

$$\partial \mathbb{I}_{\mathbb{R}^n_+}(x) = \{ s \in \mathbb{R}^n : s \le \mathbf{0},\ x^T s = 0 \}.$$

(*Hint: Consider the cases $x \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ separately.*)

$s \le 0,\ x^T s = 0 \iff x \ge 0.\ \forall y \ge 0$
$s^T(y - x) \le 0$
$\forall\ I_k = (0, 0, \ldots, \underset{i}{1}, 0, \ldots)$
$\therefore \forall\ x + I_k \ge 0$ k-th
and $s^T(x + I_k - x)$
$= s^T I_k \le 0$
$\therefore s \le 0$
assume $s^T x \ne 0$
$\therefore s^T x \ge 0$. when
let $y = 0$
$\therefore$ let $y = 2x \ge 0$
$s^T(y - x) = s^T x$
> 0
$\therefore s^T x = 0.$

Problem 2 (20pts). Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be the function given by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } \|(x_1, x_2)\|_2 < 1, \\ \in [0, +\infty] & \text{if } \|(x_1, x_2)\|_2 = 1, \\ +\infty & \text{if } \|(x_1, x_2)\|_2 > 1. \end{cases}$$

(a) **(10pts).** Show that f is convex.

(b) **(10pts).** By considering an appropriate instance of f, give an example of a convex function whose epigraph is not closed.

Problem 3 (15pts). Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -\dfrac{1}{2} & \text{if } |x| \le 1, \\ \dfrac{1}{2}x^2 - |x| & \text{if } |x| > 1 \end{cases}$$

is convex.

$f(\alpha x_1 + (1-\alpha) x_2)$
$= \frac{1}{2} \alpha^2 x_1^2 + \alpha(1-\alpha) x_1 x_2 + (1-\alpha) x_2^2 - |\alpha x_1 + (1-\alpha) x_2|$

Problem 4 (15pts). Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron. Suppose that for $i = 1, \ldots, n$, we either have the constraint $x_i \ge 0$ or the constraint $x_i \le 0$ in the description of P. Is it true that P has at least one vertex? Justify your answer.

Problem 5 (30pts). Let $A \in \mathbb{R}^{m \times n}$ be given. Show that exactly one system in each of the following pairs has a solution.

$$Ax < 0 \Rightarrow Ax \leq e$$
$$\Rightarrow Ax + Is = e$$

(a) **(15pts).**

$$\begin{array}{ll} \text{(I)} & Ax < \mathbf{0}, \;\; x \geq \mathbf{0}. \\ \text{(II)} & A^T y \geq \mathbf{0}, \;\; y \geq \mathbf{0}, \;\; y \neq \mathbf{0}. \end{array}$$

$$\tilde{A}^T y = \begin{pmatrix} A^T y \\ I^T y \end{pmatrix} \geq 0$$

(b) **(15pts).**

$$\begin{array}{ll} \text{(I)} & Ax \geq \mathbf{0}, \;\; Ax \neq \mathbf{0}. \\ \text{(II)} & A^T y = \mathbf{0}, \;\; y > \mathbf{0}. \end{array}$$

(*Hint: Follow the idea in the proof of Corollary 2 in Handout 3.*)

$$0 = y^T A x = y^T (Ax) \geq 0$$

$$y > 0 \Rightarrow y - \alpha \bar{I} \geq 0$$

$$y \geq \alpha \bar{I}$$

$$Ax \geq 0, \; Ax \neq 0$$

$$Ax = e + \bar{I}s, \quad e > 0$$

$$Ax + \alpha \bar{I} = e$$

$$[A, -A, -\bar{I}], \; (x^+, x^-, s)$$

$$Ax = \underset{\geq 0}{e} + \bar{I}\underset{> 0}{s}$$

$$s = \alpha \bar{I} + s_0$$