SEEM 5580 Final Exam **Name**:

9:30–11:15am, Dec 12, 2016

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 4 problems.

1. (30 points) A manufacturing setup consists of two distinct machines, each producing one component per hour. Each component is tested instantly and is identified as defective or non-defective. Let $0 < \alpha_i < 1$ be the probability that a component produced by machine i is non-defective, $i = 1, 2$. The defective components are discarded and the non-defective components are stored in two separate bins, one for each machine. When a component is present in each bin, the two are instantly assembled together and shipped out. Bin i can hold at most B_i components, $i = 1, 2$. When a bin is full the corresponding machine is turned off. It is turned on again when the bin has space for at least one component. Assume that successive components are independent.

   (a) (15 points) Let X_n be the number of items in bin 1 plus the number of items in bin 2 at the end of hour n. Is $X = \{X_n : n = 0, 1, \ldots,\}$ a DTMC? If not, briefly explain why and construct a one-dimensional DTMC (to answer the following questions below). (By one-dimensional I mean that the state space $S \subset \mathbb{Z}$.) What is the state space S of the Markov chain? What is the (one-step) transition probabilities? Show that your DTMC has a unique stationary distribution $\pi = (\pi_i : i \in S)$. **Even if you cannot get this part completely right, you should proceed to the other parts of this problem.**

   Solution. No. X is not DTMC. Let Y_n be the number of items in bin 1 minus the number of items in bin 2 at the end of hour n. When $Y_n = x > 0$, bin 1 has x items and bin 2 is empty. When $Y_n = x < 0$, bin 2 has $|x|$ items and bin 1 is empty. The state space

   $$S = \{-B_2, -B_2 + 1, \ldots, -1, 0, 1, \ldots, B_1\}.$$

   When $Y_n = B_1$, machine 1 is turned off the next period. When $Y_n = -B_2$, machine 2 is turned off. The transition matrix is

   $$P_{i,i+1} = \alpha_1(1 - \alpha_2) \quad P_{i,i-1} = \alpha_2(1 - \alpha_1), \quad P_{ii} = 1 - P_{i,i+1} - P_{i,i-1}$$

   for $-B_2 < i < B_1$, and

   $$P_{i,i-1} = \alpha_2, \quad P_{ii} = 1 - \alpha_2$$

   for $i = B_1$, and

   $$P_{i,i+1} = \alpha_1, \quad P_{ii} = 1 - \alpha_1$$

for $i = -B_2$. Since the DTMC is irreducible on a finite state space, it has a unique stationary distribution.

(b) (7 points) Derive an expression for the long-run fraction of the time that both machines are working.

Solution. The long-run fraction of the time both machines is

$$\sum_{-B_2<i<B_1} \pi_i.$$

(c) (8 points) Derive an expression for the long-run average number of assemblies shipped per hour.

Solution. The average number of assemblies shipped per period is

$$\sum_{-B_2\le i\le -1} \pi_i\alpha_1 + \sum_{1\le i\le B_1} \pi_i\alpha_2 + \pi_0\alpha_1\alpha_2.$$

2. (20 points; 10 points each) In an election, suppose we have two candidates A and B, such that A receives more votes than B in total (let's say A receives a votes, B receives b votes, and $a > b$). Suppose the total $n = a + b$ votes are counted in random order. Let S_k be the number of votes A is *leading by* after k votes counted. Define

$$X_k = \frac{S_{n-k}}{n-k} \quad \text{for } 0 \le k \le n-1.$$

(a) Is $\{X_k : 0 \le k \le n-1\}$ a martingale? Prove your assertion.

(b) Find the probability that A remains ahead of B throughout the counting process. Justify your answer. (For A to be "ahead", A's votes have to be strictly more than B's votes.)

1(a) yes. First, it is clear that X_k is bounded and hence integrable, $\forall k$.

Second, we verify $E[X_{k+1} | X_0, X_1, \dots, X_k] = X_k$

Note that $E[X_{k+1} | X_0, \dots, X_k] = E\left[\frac{S_{n-k-1}}{n-k-1} \,\middle|\, X_0, \dots X_k\right]$

$$= E\left[\frac{S_{n-k-1}}{n-k-1} \,\middle|\, S_n, S_{n-1}, \dots, S_{n-k}\right]$$

let $\begin{cases} a_i = \text{\# of votes for A after first } i \text{ votes counted} \\ b_i = \text{\# of votes for B after first } i \text{ votes counted.} \end{cases}$

Then conditioned on S_{n-k}, we have

$$a_{n-k} = (S_{n-k} + (n-k))/2$$

$$b_{n-k} = \frac{-S_{n-k} + (n-k)}{2} \qquad \Leftarrow \left(\begin{array}{l} a_{n-k} - b_{n-k} = S_{n-k} \\ a_{n-k} + b_{n-k} = n-k \end{array} \right)$$

Now.

$$S_{n-k-1} = \begin{cases} S_{n-k} + 1 & \text{if } (n-k)\text{-th vote for B} \\ S_{n-k} - 1 & \text{if } (n-k)\text{-th vote for A.} \end{cases}$$

It follows that

$$E[S_{n-k-1} | S_{n-k}] = (S_{n-k} + 1) \cdot \frac{b_{n-k}}{n-k} + (S_{n-k} - 1) \cdot \frac{a_{n-k}}{n-k}$$

$$= S_{n-k} \cdot \frac{n-k-1}{n-k}$$

Hence $E[S_{n-k-1} | S_{n-k}, \dots, S_n] = E[S_{n-k-1} | S_{n-k}] = S_{n-k} \cdot \frac{n-k-1}{}$

(b). Define $T = \begin{cases} \min\{k : 0 \le k \le n-1, X_k = 0\} & \text{if such } k \text{ exists} \\ n-1 & \text{otherwise.} \end{cases}$

T is a bounded stopping time

Martingale stopping theorem applies and yields

$$E[X_T] = E[X_0] = \frac{E[S_n]}{n} = \frac{a-b}{a+b} \quad (*)$$

If A doesn't lead through the counting process,

then at some time k, $S_k = 0$ and $X_k = 0$, i.e. $T \le n-1$.

and $X_T = 0$.

If A leads B through the counting process, then $S_k > 0 \ \forall k$

and hence $X_k > 0 \quad \forall k$ from 0, to $n-1$.

Then $T = n-1$. by definition and $X_T = X_{n-1} = S_1 = 1$
(first vote A)

Now $(*)$ suggests that

$$\frac{a-b}{a+b} = E[X_T] = 1 \cdot P(\text{A leads through the process}) + 0 \cdot P(\text{A doesn't lead})$$

$$= P(\text{A leads B through the counting process}).$$

hence the probability is $\frac{a-b}{a+b}$

3. (20 points)

(a) (6 pts) Suppose $\{B_t : t \geq 0\}$ and $\{W_t : t \geq 0\}$ are two independent standard Brownian motions. For $a, b \in \mathbb{R}$, we define $Y_t = aB_t + bW_t$ for each $t \geq 0$. Find the sufficient and necessary condition on a, b for $\{Y_t : t \geq 0\}$ to be a standard Brownian motion.

(b) (6 pts) Suppose $\{X_t : t \geq 0\}$ is a Gaussian process with $\mathbb{E}[X_t] = 0$, and $Cov(X_s, X_t) = s$ for all $0 \leq s \leq t < \infty$. Suppose $X_0 = 0$ and $\{X_t : t \geq 0\}$ has continuous sample paths. Prove that $\{X_t : t \geq 0\}$ is a standard Brownian motion.

(c) (8 pts) Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion. Define $Z_t = B_{2t} - B_t$. Is $\{Z_t : t \geq 0\}$ a standard Brownian motion? Is $\{Z_t : t \geq 0\}$ a Gaussian process? Prove your assertion.

Solution.

(a) $a^2 + b^2 = 1$.

Easy to verify from definition that under this condition, $\{Y_t : t \geq 0\}$ is a standard Brownian Motion.

To get this, note that $Var(Y_t) = t = Var(aB_t + bW_t)$

$$= a^2 \cdot t + b^2 \cdot t$$

(b). $Var(X_t) = Cov(X_t, X_t) = t$.

Only need to show X has stationary and independent increment.

$$Cov(X_{t+s} - X_t, X_t) = Cov(X_{t+s}, X_t) - Cov(X_t, X_t)$$
$$= t - t = 0$$

As $X_{t+s} - X_t$, X_t are both Gaussian r.v.s

Then we have $X_{t+s} - X_t$ independent of X_t.

~~then~~ Note that $Var(X_{t+s} - X_t) = t + s + t - 2t = s$.

So $X_{t+s} - X_t$ is also $N(0, s)$ $\Rightarrow$ stationary increment.

A rigorous proof used mathematical induction on the # of time intervals considered. (only 2 intervals above)

(c). $\{Z_t : t \geq 0\}$ is NOT standard Brownian Motion, e.g. $Var(Z_{2t} - Z_t) = 3t \neq t$.

$\{Z_t : t \geq 0\}$ is a Gaussian process

To find the long-run averge nuber of customers in system, we need to ~~sta~~ solve the stationary distribution for each CTMC with generator matrix Q. Suppose it is $\{\pi_{i,j}\}$ then

long-run average # of customers in system $= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(i+j)\cdot\pi_{ij}$

4. (30 points; 10 points each) Consider a two independent server queue. Two different queues are formed in front of the two servers (first-come-first-served). A common stream of customers arrive at the two different queues according to a Poisson process with rate λ. The service times at each server follow i.i.d exponential distributions with a mean $1/\mu$. Assume $\lambda < 2\mu$.

For each of the following three routing policies, construct an appropriate CTMC model and briefly explain how your model can be used to find the long-run average number of customers in the system (you do not need to give a numerical answer). Specify the state space and the transition rates for three CTMCs you construct.

(a) An arriving customer joins the server with the shorter queue. If the number of customers in the two queues are equal, the arriving customer joins either one with probability 0.5.

(b) An arriving customer is randomly assigned to either server with probability 0.5.

(c) An arriving customer is randomly assigned to either server with probability 0.5. In addition, when one queue is empty, the server will serve a customer (if any) in the other queue.

let $Q_1(t)$ = # of customers in the first queue at time $t \geq 0$

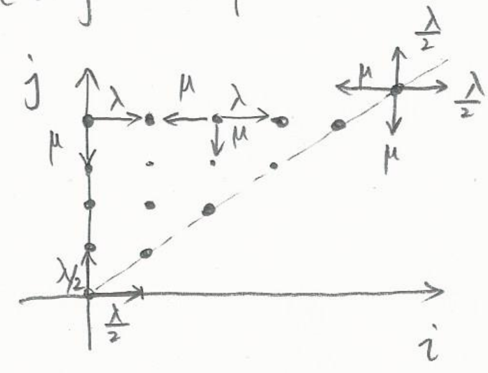
$Q_2(t)$ = # of customers in the second queue at time $t \geq 0$.

Then $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a CTMC for policy (a), (b), (c).

state space $S = \mathbb{Z}^+ \times \mathbb{Z}^+$ where $\mathbb{Z}^+ = \{z : z \text{ is a nonnegative integer}\}$.

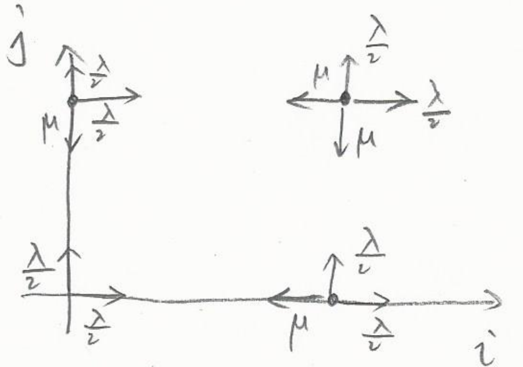
let (i, j) represent one state $Q_1(t) = i$, $Q_2(t) = j$.

(a)



Symmetric in the diagonal

(b)



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