1 (a): $p(\xi_{N(t)+3} > x) = \sum_{n=0}^{\infty} p(\xi_{N(t)+3} > x \mid N(t)=n) \cdot p(N(t)=n)$

$$= \sum_{n=0}^{\infty} p(\xi_{n+3} > x \mid N(t)=n) \cdot p(N(t)=n)$$

$$\overset{①}{=} \sum_{n=0}^{\infty} p(\xi_{n+3} > x) \cdot p(N(t)=n)$$

$$= \sum_{n=0}^{\infty} e^{-\lambda x} \cdot p(N(t)=n)$$

$$= e^{-\lambda x}$$

where in Equation ① we used the fact that $\{\xi_{n+3} > x\}$ is independent of the event $\{N(t)=n\}$. The reason for this is that $\{N(t)=n\}$ is equivalent to $\{S_n \le t < S_{n+1}\}$, which only depends on $\xi_1, \cdots, \xi_{n+1}$.

1 (b): Yes. for any $i \le j$, we have

$\cancel{p(N(t)=j \mid N(}$

$$p(N(k+1)=j \mid N(k)=i, N(k-1)=i_{k-1}, \cdots, N(0)=i_0)$$

$$= p(N(k+1) - N(k) = j-i \mid N(k)=i, N(k-1)=i_{k-1}, \cdots, N(0)=i_0)$$

$$= p(N(k+1) - N(k) = j-i) \qquad \text{"independent increments"}$$

$$= p(N(k+1)=j \mid N(k)=i)$$

$$= P_{ij} = \frac{\lambda^{j-i}}{(j-i)!} \cdot e^{-\lambda} \qquad \text{for } i \le j$$

For $i > j$, it is clear that the Markov property also holds and $P_{ij} = 0$

2: Denote $\{N(t): t \geq 0\}$ the arrival process.

(a) $P(N(10)-N(0)=2,\ N(15)-N(5)=1)$

$= P(N(5)=2,\ N(10)-N(5)=0,\ N(15)-N(10)=1)$

$+ P(N(5)=1,\ N(10)-N(5)=1,\ N(15)-N(10)=0)$

$\overset{①}{=} P(N(5)=2) \cdot P(N(10)-N(5)=0) \cdot P(N(15)-N(10)=1)$

$+ P(N(5)=1) \cdot P(N(10)-N(5)=1) \cdot P(N(15)-N(10)=0)$

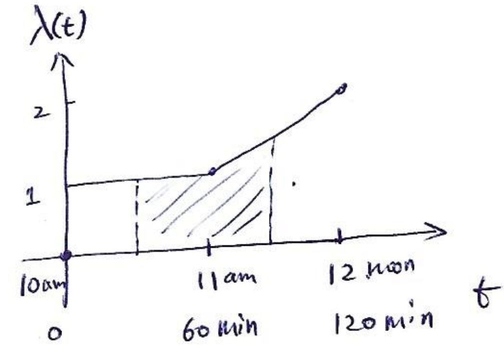
$\overset{②}{=} \frac{5^2}{2!} \cdot e^{-5} \cdot \frac{5^0}{0!} \cdot e^{-5} \cdot \frac{5}{1} \cdot e^{-5}$

$+ \frac{5}{1} \cdot e^{-5} \cdot \frac{5}{1} \cdot e^{-5} \cdot \frac{5^0}{0!} \cdot e^{-5}$

$= 87.5 \cdot e^{-15}$

where ① follows from the independent increment property of $\{N(t): t \geq 0\}$, and ② follows from the fact that $\lambda(t)=1$ for $0 \leq t \leq 60$ (min)

(b) $\lambda(t) = \begin{cases} 1 & 0 \leq t \leq 60 \\ \frac{t}{60} & 60 \leq t \leq 120 \end{cases}$




$E[N(90)-N(30)]$

$= \int_{30}^{90} \lambda(t)\, dt$

$= \int_{30}^{60} 1\, dt + \int_{60}^{90} \frac{t}{60}\, dt$

$= 30 + 37.5 = 67.5$

(c) $P(N(65)-N(60)=0) = e^{-\int_{60}^{65} \frac{t}{60} dt} = e^{-\frac{125}{24}}$

since $\{N(t): t \geq 0\}$ is a non-homogeneous Poisson process with rate function $\{\lambda(t): t \geq 0\}$.

3. $P = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$ $\vec{f} = \begin{bmatrix} f(a) \\ f(b) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(a) $E[f(X_0) \mid X_0 = a] = f(a) = 2$

$E[f(X_1) \mid X_0 = a] = f(a) \cdot P_{aa} + f(b) \cdot P_{ab} = 2 \cdot 0.4 + 1 \cdot 0.6 = 1.4$

Since $P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}$,

$E[f(X_2) \mid X_0 = a] = f(a) \cdot P_{aa}^{(2)} + f(b) \cdot P_{ab}^{(2)}$

$= 2 \times 0.28 + 1 \times 0.72 = 1.28$

$\Rightarrow E[f(X_0) + f(X_1) + f(X_2) \mid X_0 = a]$

$= 2 + 1.4 + 1.28 = 4.68$

(b) $E[f(X_n) \mid X_0 = a] = f(a) \cdot P_{aa}^{(n)} + f(b) \cdot P_{ab}^{(n)} = P^n_{a,\cdot} \vec{f}$

$\left( E[\sum_{n=0}^{\infty} (0.8)^n \cdot f(X_n) \mid X_0 = a],\ E[\sum_{n=0}^{\infty} (0.8)^n f(X_n) \mid X_0 = b] \right)^T$

$= \sum_{n=0}^{\infty} 0.8^n \cdot P^n \cdot \vec{f}$

$= (I - 0.8P)^{-1} \cdot \vec{f}$

$= \begin{bmatrix} 0.68 & -0.48 \\ -0.16 & 0.36 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2.14 & 2.86 \\ 0.95 & 4.05 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Hence $E[\sum_{n=0}^{\infty} (0.8)^n \cdot f(X_n) \mid X_0 = a]$

$= 2.14 \times 2 + 2.86 \times 1 = 7.14$

3 (c): We can find the stationary distribution $\pi = (\pi_a, \pi_b)$ as follows.

$$\begin{cases} \pi_a = 0.4\,\pi_a + 0.2\,\pi_b \\ \pi_b = 0.6\,\pi_a + 0.8\,\pi_b \\ \pi_a + \pi_b = 1, \quad \pi_a, \pi_b \geq 0 \end{cases} \Rightarrow \begin{cases} \pi_a = 1/4 \\ \pi_b = 3/4 \end{cases}$$

This Markov chain is irreducible, aperiodic, and positive recurrent.

Hence, with probability one we have as $N \to \infty$,

$$① \quad \frac{1}{N}\sum_{n=1}^{N} f(X_n) \longrightarrow \sum_{i\in S} \pi_i \cdot f(i) = \pi_a f(a) + \pi_b f(b)$$

$$= 1/4 \times 2 + 3/4 \times 1 = 5/4.$$

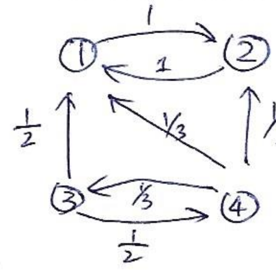
Since $f$ is bounded, by dominated convergence theorem,

$$\lim_{N\to\infty} \frac{1}{N} E\left[\sum_{n=1}^{N} f(X_n) \,\middle|\, X_0 = a\right] = 5/4. \quad ②$$

Here ~~$\frac{1}{N} E\left[\sum_{n=1}^{N} f(X_n) \middle| X_0=a\right] \leq 2$ for any $N$.~~

Dominate convergene theorem is needed because we need to obtain convergence in expectation (②) from convergence with probability one (①).

4. Transition Diagram




(1) False. it is clear that 1 and 3 doesn't communicate.

(2) $d(1) = d(2) = 2$, because $P_{ii}^{(2n)} = 1$ and $P_{ii}^{(2n+1)} = 0$ for $i = 1, 2$.

$d(3) = d(4) = 2$. because $P_{ii}^{(2n)} > 0$ and $P_{ii}^{(2n+1)} = 0$ for $i = 3, 4$.

(3) states 3 and 4 are transient

$P_{33}^{(2n)} = \left(\frac{1}{6}\right)^n$ for $n \geq 1$ $\Rightarrow \sum_{k=1}^{\infty} P_{33}^{(k)} = \sum_{n=0}^{\infty} P_{33}^{(2n)} < 1$

Hence state 3 is transient.

state 4 communicates with 3, hence it is also transient.

states 1 and 2 are positive recurrent.

If the Markov chain starts from state 1, the first time $T_1$ it returns is $T_1 \equiv 2$. Hence state 1 is positive recurrent.

state 2 $\leftrightarrow$ state 1 $\Rightarrow$ state 2 is also positive recurrent.

(4) We solve:

$$\begin{cases} \pi_1 = \pi_2 + \frac{1}{2}\pi_3 + \frac{1}{3}\pi_4 \\ \pi_2 = \pi_1 + \frac{1}{3}\pi_4 \\ \pi_3 = \frac{1}{3}\pi_4 \\ \pi_4 = \frac{1}{2}\pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \\ \pi_i \geq 0 \quad i = 1, 2, 3, 4 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{1}{2} \\ \pi_2 = \frac{1}{2} \\ \pi_3 = 0 \\ \pi_4 = 0 \end{cases}$$

Hence there is unique stationary distribution $\pi = (\frac{1}{2}, \frac{1}{2}, 0, 0)$.