SEEM 5580 Final Exam **Name**:
December 11, 2014

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 6 problems.

1. (10 points) Recall that a discrete time stochastic process $X = \{X_n : n = 0, 1, 2, \ldots\}$ in state space S satisfies the Markov property if

$$\mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i\} = \mathbb{P}\{X_{n+1} = j | X_n = i\}$$

for each $n \geq 0$ and $i_0, i_1, \ldots i_{n-1}, i, j \in S$. Toss a fair coin sequentially. Let $\xi_i = 1$ if the ith toss lands a head and $\xi_i = 0$ otherwise. Let $Y_0 = 0$, $Y_1 = \xi_1$ and $Y_n = \xi_n + \xi_{n-1}$ for $n \geq 2$. Does $Y = \{Y_n : n = 0, 1, 2, \ldots,\}$ satisfy the Markov property? Prove your assertion.

2. (20 points) Consider an inventory system. The weekly demands $\{D_i : i = 1, \ldots\}$ are iid following distribution

d	0	1	2
$\mathbb{P}\{D_i = d\}$	.2	.5	.3

Suppose the inventory policy is (s, S) with $s = 0$ and $S = 2$. Namely, by the end of Friday, if inventory becomes empty, order 2 item. Otherwise, do not order. Unsatisfied demand during a week is lost. Find the long run fraction of weeks when demand is not satisfied. Express your answer in terms the stationary distribution of a DTMC (Discrete-Time Markov Chain). You do not need to compute the stationary distribution. But you need to define the DTMC precisely and explain why the stationary distribution exists and is unique.

3. (20 points; 5 points each question) A call center has four phone lines and two agents. Call arrival to the call center follows a Poisson process with rate 2 calls per minute. Calls that receive a busy signal are lost. The processing times are i.i.d exponentially distributed with mean 1 minute.

   (a) Model the system by a continuous time Markov chain. Specify the state space. Clearly describe the meaning of each state. Specify the generator matrix.

   (b) What is the long-run fraction of time that there are two calls in the system?

   (c) What is the long-run average number of waiting calls, excluding those in service, in the system?

   (d) What is the throughput (the rate at which completed calls leaves the call center) of the call center?

4. (20 points; 5 points each question) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and $\{N(t) : t \geq 0\}$ models customer arrivals to a service center. Let ξ_i be the inter–arrival time between $(i-1)$–th and i–th customers. For each statement, say "true" or "false" and explain your answers. Assume throughout that $t > 0$ is fixed but arbitrary.

   (a) $N(t)$ has a Poisson distribution with mean λ.

   (b) ξ_i has an exponential distribution with mean λ for each i.

   (c) $\xi_{N(t)+1}$ has an exponential distribution.

   (d) $\xi_{N(t)+2}$ has an exponential distribution.

5. (20 points; 5 points each question) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose $\mathbb{P}(\xi_1 = 1) = 1 - \mathbb{P}(\xi_1 = -1) = p \in (0,1)$. Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ for $n \geq 1$. For fixed integers $a > 0$ and $b > 0$, define

$$\tau = \inf\{k \geq 1 : \xi_k = 1\} \quad \text{and} \quad T = \inf\{k \geq 0 : S_k \in \{a, -b\}\}$$

(a) Is τ a stopping time for $\{S_n : n \geq 1\}$? Explain why or why not.

(b) Suppose $p = 0.5$. Calculate $\mathbb{E}[S_\tau]$.

(c) Suppose $p = 0.5$. Compute $\mathbb{P}(S_T = a)$.

(d) Suppose $p \neq 0.5$. Compute $\mathbb{P}(S_T = a)$.

6. (10 points) Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Suppose Z is a standard normal random variable with mean 0 and variance 1. Define $Y(t) = \sqrt{t} \cdot Z$ for $t \geq 0$. For each statement, say "true" or "false". No explanations are needed. Assume throughout that $t > 0$ is fixed but arbitrary.

   (a) (3 points) The process $\{Y(t) : t \geq 0\}$ is a standard Brownian motion.

   (b) (4 points) $\max_{0 \leq s \leq t} B(s)$ and $|B(t)|$ have the same distribution.

   (c) (3 points) $B(t) - \min_{0 \leq s \leq t} B(s)$ and $|B(t)|$ have the same distribution.

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