Separation (cont'd)

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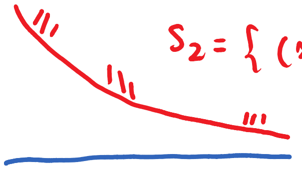
Set-set separation

Suppose that S_1, S_2 are non-empty, closed, convex. Consider

e.g.: $n=2$



$S_2 = \{(x_1, x_2) : x_2 \geq 1/x_1\}$

$S_1 = \{(x_1, 0) : x_1 \geq 0\}$

Observe: Any hyperplane separating S_1, S_2 must have normal in the direction $y=(0,1)$. But then

$$\max_{z \in S_1} y^T z \; \boxed{=} \; \inf_{z \in S_2} y^T z \qquad \text{(verify)}$$

Theorem (Set-Set Separation)

Let S_1, S_2 be non-empty, closed, convex sets with $S_1 \cap S_2 = \phi$.

Suppose further that S_2 is bounded (hence, S_2 is compact).

Then, $\exists y$ s.t.

$$\max_{z \in S_1} y^T z < \min_{z \in S_2} y^T z$$

Proof: Define $S_1 - S_2 \triangleq \{ z - u : z \in S_1, u \in S_2 \}$.

Observe:

① $0 \notin S_1 - S_2$; ② $S_1 - S_2$ is non-empty and convex (verify)

Claim: $S_1 - S_2$ is closed

Assuming the claim, then by point-set separation theorem,

$\exists y$ s.t.

$$\max_{v \in S_1 - S_2} y^T v < y^T 0 = 0.$$

Since $v \in S_1 - S_2 \iff \exists z \in S_1, u \in S_2$ s.t. $v = z - u$

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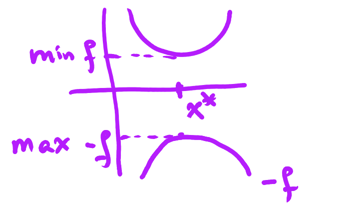
Hence,

$$\max_{v \in S_1 - S_2} y^T v = \max_{\substack{z \in S_1 \\ u \in S_2}} \underbrace{y^T(z-u)}_{y^T z \,+\, y^T(-u)}$$

$$= \max_{z \in S_1} y^T z + \max_{u \in S_2} y^T(-u)$$

$$= \max_{z \in S_1} y^T z - \min_{u \in S_2} y^T u$$

$\min f = -\max -f$



Rearrange to get desired result.

Proof of Claim:

Goal: Show that $S_1 - S_2$ is closed.

Let $x_1, x_2, \cdots$ be a sequence in $S_1 - S_2$ s.t. $x_k \to x$.

Want: Show $x \in S_1 - S_2$.

Since $x_k \in S_1 - S_2$, $\exists\, z_k \in S_1$, $u_k \in S_2$ s.t. $x_k = z_k - u_k$.

$\downarrow$
x

Note: Although $x_k \to x$, this does not imply z_k or u_k Converges. (e.g., $z_k = k$, $u_k = k$)

$\rightarrow$ Find subsequences of $\{z_k\}$ and $\{u_k\}$ that converge; i.e., find indices $k_1, k_2, \cdots$ s.t.

$$z_{k_i} \xrightarrow{i \to \infty} z, \quad u_{k_i} \xrightarrow{i \to \infty} u$$

Then, by closedness of S_1, S_2,

$$z \in S_1, \quad u \in S_2$$

$$\begin{array}{llll}
x_1 = z_1 - u_1 & & & k_1 = 1 \\
x_{13} = z_{13} - u_{13} & & & k_2 = 13 \\
x_{47} = z_{47} - u_{47} & & & k_3 = 47 \\
\downarrow \quad\ \ \downarrow \quad\ \ \downarrow & & & \vdots \\
x = z - u & & & \\
\cap\ \ \ \ \ \cap \ \ \ \ \cap & & & \\
S_1 - S_2 \quad S_1 \quad S_2 & \leftarrow \text{closedness of } S_1, S_2 & &
\end{array}$$

Note:

$$x_{k_i} = z_{k_i} - u_{k_i} \to z - u$$

And since $x_k \to x$ and the limit

is unique, we must have $x_{k_i} \to x = z - u \in S_1 - S_2$.

__Fact:__ Every sequence in a compact set has a convergent subsequence.

e.g.: Consider $S = [-1, 1]$. Take $u_k = (-1)^k$. Note that u_k does not converge. However,

$$v_k \triangleq u_{2k} = 1 \quad , \quad w_k \triangleq u_{2k+1} = -1$$

Since S_2 is compact and $\{u_k\}_{k\geq 0}$ is a sequence in S_2, by the fact, it has a convergent subsequence $\{u_{k_i}\}_{i\geq 0}$, say $u_{k_i} \to u$. Since S_2 is closed, we know that $u \in S_2$.

Consider

$$x_{k_i} = z_{k_i} - u_{k_i}$$

$$\Rightarrow \quad z_{k_i} = x_{k_i} + u_{k_i}$$

$$\lim_{i\to\infty} z_{k_i} = \lim_{i\to\infty} x_{k_i} + \lim_{i\to\infty} u_{k_i} = x + u$$

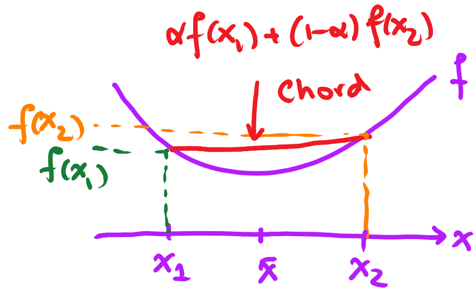
Since S_1 is closed, we have $x + u \in S_1$. But then

$$x = \underbrace{(x+u)}_{\in S_1} - \underbrace{u}_{\in S_2} \in S_1 - S_2 .$$

Definitions: Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function that is not identically $+\infty$.

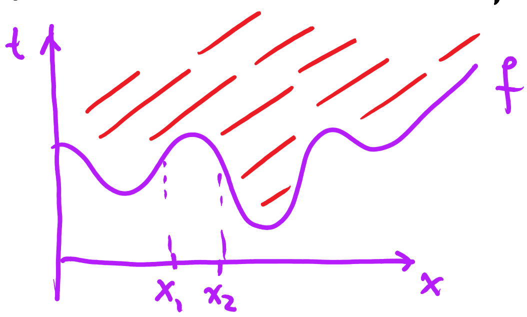
① We say that f is convex if $\forall x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$,

$$f(\underbrace{\alpha x_1 + (1-\alpha)x_2}_{\bar{x}}) \le \alpha f(x_1) + (1-\alpha) f(x_2)$$



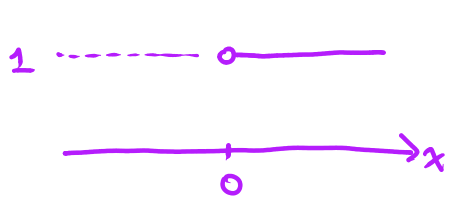
② The epigraph of f is the set

$$\text{epi}(f) \triangleq \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le t \} \subseteq \mathbb{R}^{n+1}$$



③ The effective domain of f is the set

$$\text{dom}(f) \triangleq \{ x \in \mathbb{R}^n : f(x) < +\infty \} \subseteq \mathbb{R}^n$$



$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ +\infty & \text{o/w} \end{cases}$$

$\text{dom}(f) = \mathbb{R}_{++}$

④ Let $S \subseteq \mathbb{R}^n$ be a set. The indicator of S is the function

$$\mathbb{1}_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{o/w} \end{cases}$$

e.g. Using the indicator,

$$\inf_{x \in S} f(x) \iff \inf_{x \in \mathbb{R}^n} \{ f(x) + \mathbb{1}_S(x) \}$$

Constrained Unconstrained