SEEM 5580 Final Exam **Name**:

9:30–11:30am, Dec 14, 2015

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 6 problems.

1. (25 points, 5 points each) Consider a discrete time Markov chain $\{X_n : n = 0, 1, \ldots\}$ with the state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For the first three parts, just circle True or False. Answer the other two parts in the space below.

(1) This chain is irreducible. True or False?

(2) This chain is aperiodic. True or False?

(3) Every state of this chain is recurrent. True or False?

(4) Calculate $P(X_3 = 1 | X_0 = 4)$.

(5) Does $\lim_{n\to\infty} P(X_n = 1 | X_0 = 4)$ exist? If yes, explain why and provide the limit; if not, explain why.

2. (20 points) Taxis looking for customers arrive at a taxi station as a Poisson process (rate 1 per minute), while customers looking for taxis arrive as a Poisson process (rate 1.25 per minute). Suppose taxis will wait, no matter how many taxis are in line before them. But customers who arrive to find 2 other customers in line go away immediately. Over the long run, what is average number of customers waiting at the station?

3. (15 points) Suppose cars enter a one-way infinite highway according to a Poisson process with rate λ. The i-th car to enter chooses a velocity V_i and travels at this velocity. We assume that the V_i's are independent positive random variables having a common cumulative distribution function F. *Find the distribution of the number of cars that are located in the interval* $(0, b)$ *at time* t. Assume that no time is lost when one car overtakes another car, and at time zero there is no car on the highway.

4. (20 points, 10 points each) The following problem illustrates nice connections between martingales and Markov chains. Let $\{Y_n : n \geq 1\}$ be an irreducible discrete time Markov chain on a countable state space S with transition probability matrix $\mathbb{P}$. A real–valued function $h : S \to \mathbb{R}$ is called harmonic for the transition probability matrix $\mathbb{P}$ if

$$\sum_{j \in S} \mathbb{P}_{ij} h(j) = h(i), \quad \text{for each } i \in S.$$

(a) Show that if h is bounded and h is harmonic for the transition probability matrix $\mathbb{P}$, then the process $\{h(Y_n) : n \geq 1\}$ is a martingale.

(b) If $\{Y_n : n \geq 1\}$ is recurrent, prove that all the bounded harmonic functions are constants.

5. (10 points) Suppose a continuous time Markov chain $X = \{X_t : t \geq 0\}$ has a countable state space S with state $0 \in S$, and for each $t \geq 0$

$$P(X_t = 0|X_0 = 0) = \sum_{j=1}^{k} a_j e^{-b_j t}$$

for some finite k and positive numbers a_j, b_j for $j = 1, \ldots, k$. Let H_0 denote the holding time of the chain in state 0 before it jumps to some other state. Find the expected value of H_0.

6. (10 points) Kolmogrov's continuity criterion states that if for a real–valued stochastic process $X = \{X_t : t \geq 0\}$, there exist three *positive* constants $\alpha, \beta, C > 0$ such that

$$E[|X_{t+h} - X_t|^\alpha] \leq C \cdot h^{1+\beta}$$

for every t and $h \geq 0$, then X has a modification with continuous sample path. (A process Y is called a modification of X if for each t, $X_t = Y_t$ with probability one)

Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion. Prove that $\{B_t : t \geq 0\}$ has a modification with continuous sample path.

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