SEEM 5580 Final Exam **Name:**

9:30–11:15am, Dec 5, 2018

This is a closed book, closed notes test. **CUHK student honor code applies to this test.**

1. (24 points; 8 points each) A continuous time Markov chain $\{X_t : t \geq 0\}$ has state space $\{1, 2, 3\}$ and generator matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{pmatrix}.$$

Matlab gives (to two decimal places)

$$\exp(Q) = \begin{pmatrix} 0.52 & 0.33 & 0.15 \\ 0.33 & 0.44 & 0.23 \\ 0.29 & 0.46 & 0.25 \end{pmatrix} \quad \text{and} \quad \exp(100 \cdot Q) = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}.$$

(1) If $X_0 = 2$, find the probability that the chain will jump to state 3 in the first transition.

(2) Find $\mathbb{P}(X_2 = j | X_0 = 1)$ for $j = 1, 2, 3$.

(3) Find the stationary distribution of this Markov chain.

(1) $2/3$

(2) $e^{2Q} = \begin{bmatrix} 0.43 & 0.38 & 0.19 \\ 0.38 & 0.41 & 0.21 \\ 0.38 & 0.41 & 0.21 \end{bmatrix}$

$P(X_2 = j | X_0 = 1) = \begin{cases} 0.43 & j = 1 \\ 0.38 & j = 2 \\ 0.19 & j = 3 \end{cases}$

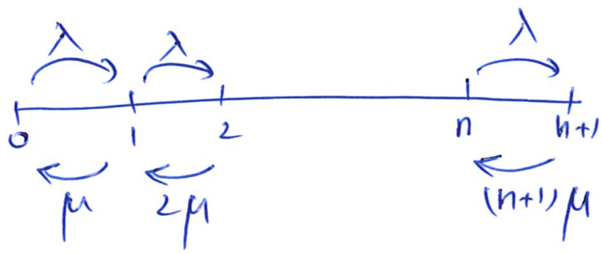
(3) $(\pi_1, \pi_2, \pi_3) = (0.4, 0.4, 0.2)$

2. (16 points; 8 points each) Consider an $M/M/\infty$ queueing system. Customers arrive according to a Poisson process with rate $\lambda > 0$. There are infinity number of servers. Each server processes customers at rate $\mu > 0$. Let $X(t)$ be the number of people in the system at time t and assume $X(0) = 0$.

(a) Consider the number of busy servers when the system is in steady state. Find its distribution and justify your answer.

(b) Suppose that there are 5 people in service at time $t = 8$, so $X(8) = 5$. If $\lambda = 2$ and $\mu = 3$, what is the expected time until the number of people in service changes, i.e. goes up or goes down.

(a) rate diagram



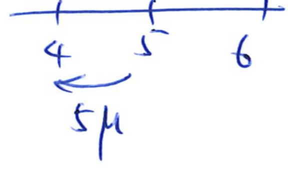
$\lambda \cdot \pi_{n-1} = n \cdot \mu \cdot \pi_n$, for $n \geq 1$.

$$\pi_n = \pi_0 \cdot \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}$$

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!}} = e^{-\frac{\lambda}{\mu}}$$

so number of busy servers $\sim$ Poisson $\left(\frac{\lambda}{\mu}\right)$.

(b) At state 5,



$v_5 = \lambda + 5\mu$

$= 2 + 5 \times 3 = \cancel{18}\ 17$

So $\frac{1}{v_5} = \cancel{\frac{1}{18}}\ \frac{1}{17}$.

3. (20 points; 10 points each)

There are two students Alice and Bob working at a fund raising car wash. Alice attracts passing cars at the rate of 3 per hour and she completes a car at a rate of 4 per hour. Bob does not do so well. He attracts cars a the rate of 2 per hour and completes a car at the rate of 2 per hour. An advisor wants to know the amount of time when Alice is working, when Bob is working, when no one is working, and when both are working. In other words, the advisor is interested in a certain Markov chain that has the four states just described.

(a) If we formulate the advisor's problem as a continuous time Markov model, what is the generator matrix?

(b) Use the Markov chain model to answer the following question: in the long run, what is the probability that no one is working? (Give an expression to get partiail credits if you can not solve (a).)

(a) (1, 1) : Both are working

(1, 0) Alice is working

(0, 1) Bob is working

(0, 0) no one is working

$$Q = \begin{array}{c} \\ (1,1) \\ (1,0) \\ (0,1) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (1,0) & (0,1) & (0,0) \end{array} \\ \begin{bmatrix} -6 & 2 & 4 & 0 \\ 2 & -6 & 0 & 4 \\ 3 & 0 & -5 & 2 \\ 0 & 3 & 2 & -5 \end{bmatrix} \end{array}$$

(b) $\pi Q = 0$

long-run probability no one is working : $\pi(0,0)$.

4. (14 points) Let $\{Y_n : n \geq 1\}$ be an irreducible discrete time Markov chain on a countable state space S with transition probability matrix $\mathbb{P}$. A real–valued function $h : S \to \mathbb{R}$ is called harmonic for the transition probability matrix $\mathbb{P}$ if

$$\sum_{j \in S} \mathbb{P}_{ij} h(j) = h(i), \quad \text{for each } i \in S.$$

Prove that if h is bounded and h is harmonic for the transition probability matrix $\mathbb{P}$, then the process $\{h(Y_n) : n \geq 1\}$ is a martingale.

① $E[h(Y_{n+1}) \mid Y_1, \dots, Y_n]$

$= E[h(Y_{n+1}) \mid Y_n]$ (Markov property)

$= \sum_{j \in S} P_{Y_n, j} h(j)$

$= h(Y_n)$ (harmonic h).

② $E|h(Y_n)| < +\infty$ as h is bounded for all n.

5. (16points; 8 points each) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributied random variables. Suppose $\mathbb{P}(\xi_1 = 1) = 1 - \mathbb{P}(\xi_1 = -1) = p$. Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ for $n \geq 1$. For fixed integers $a > 0$ and $b > 0$, define

$$\tau = \inf\{k \geq 1 : \xi_k = 1\} \quad \text{and} \quad T = \inf\{k \geq 1 : S_k \notin (-a, b)\}.$$

(a) Is τ a stopping time for $\{S_n : n \geq 1\}$? Explain your answer.

(b) Suppose $p \neq 0.5$. Find the probability distribution of S_T. Justify your answer.

(a) $\{\tau = n\} \Leftrightarrow \{\xi_1 = -1, \cdots, \xi_{n-1} = -1, \xi_n = 1\}$

$\Leftrightarrow \{S_1 = -1, S_2 = -2, \cdots, S_{n-1} = -n+1, S_n = -n+2\}$

in addition,

$P(\tau = n) = P(\xi_1 = -1) \cdot \cdots P(\xi_{n-1} = -1) \cdot P(\xi_n = 1)$

$= p^{n-1} \cdot (1-p)$

so $P(\tau < \infty) = 1$.

so τ is a stopping time for $\{S_n : n \geq 1\}$

(b) T is a stopping time for $\{S_k : k \geq 1\}$.

$\left\{ \left(\frac{q}{p}\right)^{S_n} : n \geq 0 \right\}$ is a martingale $\quad (q = 1-p)$

$$\left\{ \begin{array}{l} E\left[\left(\frac{q}{p}\right)^{S_T}\right] = 1 = \left(\frac{q}{p}\right)^{-a} \cdot P(S_T = -a) + \left(\frac{q}{p}\right)^{b} \cdot P(S_T = b). \\ P(S_T = -a) + P(S_T = b) = 1. \end{array} \right.$$

Martingale stopping theorem applies as the stopped process $\left\{ \left(\frac{q}{p}\right)^{S_{n \wedge T}} \right\}$ are uniformly bounded.

6. (10 points) Let $\{X_n : n \geq 0\}$ be a discrete time martingale with $\mathbb{E}[X_n^2] < \infty$ for each n and $X_0 = 0$. Define $\xi_i = X_i - X_{i-1}$ for each $i \geq 1$. Prove that $Cov(\xi_i, \xi_j) = 0$ for $i \neq j$, that is, the random variables ξ_i and ξ_j are uncorrelated.

$$E[\xi_i] = E[X_i] - E[X_{i-1}] = 0$$

$$Cov(\xi_i, \xi_j) = E[\xi_i \xi_j) - E[\xi_i] \cdot E[\xi_j]$$
$$= E[\xi_i \xi_j]$$

Consider $i < j$

$$E[\xi_i \xi_j] = E[\ E[\xi_i \xi_j \mid X_1, --, X_{j-1}]]$$
$$= E[\ \xi_i \cdot E[\xi_j \mid X_1, --, X_{j-1})]$$
$$= E[\xi_i \cdot 0] = 0$$

$$E[\xi_j \mid X_1, -, X_{j-1}] = E[X_j \mid X_1 -, X_{j-1}]$$
$$- E[X_{j-1} \mid X_1 --, X_{j-1}]$$
$$= X_{j-1} - X_{j-1} = 0$$