Introduction (cont'd)

09/09/2021 12:33

Sep 23: Change to ELB LT1

Recall:

(P) $\inf_{x \in X} f(x)$

(4) Quadratic Programming (QP)

$$f(x) = \sum_{i,j=1}^{n} Q_{ij} x_i x_j = \underset{1\times n}{x^T} \underset{n\times n}{Q} \underset{n\times 1}{x} \quad , \quad Q = [Q_{ij}] \in \mathbb{R}^{n\times n}$$

homogeneous quadratic: $f(\alpha x) = \alpha^2 f(x)$

X: Same as LP

* Compare quadratic constrained QP (QCQP)

$$\inf \quad x^T Q x$$
$$\text{s.t.} \quad x^T A_i x \le b_i \quad , \quad i = 1, \cdots, m.$$

* Note: Q can be assumed WLOG to be symmetric when defining f. Indeed, observe

$$x^T Q x = x^T \underbrace{\left(\frac{Q+Q^T}{2}\right)}_{\text{Symmetric}} x$$

$$x^T \frac{Q}{2} x + x^T \frac{Q^T}{2} x = \frac{1}{2} x^T Q x + \frac{1}{2} (\underline{x^T Q^T x})^T = x^T Q x$$

(5) Semidefinite Programming (SDP)

Definition: Let $Q \in \underbrace{S^n}_{\text{Set of } n\times n \text{ real Sym. matrices}}$. Then, the following are equivalent:

(1) Q is positive semidefinite (psd)

(2) $\forall x \in \mathbb{R}^n$: $x^T Q x \ge 0$

(3) All eigenvalues of Q are non-negative.

Let $C, A_1, \cdots, A_m \in S^n$; $b_1, \cdots, b_m \in \mathbb{R}$ be given.

$$\inf \quad b^T y$$

(SDP) $$\text{s.t.} \quad C - \sum_{i=1}^{m} y_i A_i \underline{\succeq 0} \quad (*)$$ (linear matrix

(SDP) s.t. $C - \sum_{i=1}^{m} y_i A_i \underbrace{\succeq 0}_{\text{is psd}}$ (*) (linear matrix inequality (LMI))

$y \in \mathbb{R}^m$

Remarks

(1) (*) is equivalent to $\underbrace{-\sum_{i=1}^{m} y_i A_i}_{\stackrel{\Delta}{=} M(y)} \succeq -C$

$\mathbb{R}^m \ni y \mapsto M(y) \in S^n$

Note: $M(\cdot)$ is a linear map: $\forall \alpha, \beta \in \mathbb{R},\ y, z \in \mathbb{R}^m$,

$M(\alpha y + \beta z) = \alpha M(y) + \beta M(z)$ (exercise)

(2) Suppose that $C, A_1, \cdots, A_m$ are diagonal. Then, (*) becomes

$$\begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix} - \sum_{i=1}^{m} y_i \begin{bmatrix} a_{i1} & & \\ & \ddots & \\ & & a_{in} \end{bmatrix} \succeq 0$$

$$\Leftrightarrow \begin{bmatrix} c_1 - \sum_i y_i a_{i1} & & \\ & \ddots & \\ & & c_n - \sum_i y_i a_{in} \end{bmatrix} \succeq 0$$

$$\Leftrightarrow \quad c_j - \sum_i y_i a_{ij} \geq 0 \quad \text{for } j = 1, \cdots, n.$$ linear inequalities

(3) How about more LMIs in the constraint?

e.g.

$$\begin{aligned} C_1 - \sum_{i=1}^{m} y_i A_i &\succeq 0 \\ C_2 - \sum_{i=1}^{m} y_i B_i &\succeq 0 \end{aligned} \quad \overset{\text{(exercise)}}{\Longleftrightarrow} \quad \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} - \sum_{i=1}^{m} y_i \begin{bmatrix} A_i & \\ & B_i \end{bmatrix} \succeq 0$$

(Hint: Let $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in S^n$. Then, $A \succeq 0$ iff $A_1, A_2 \succeq 0$ (why?))

Reformulation Examples

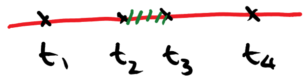
(1) Air Traffic Control

- n planes arriving
- i^{th} plane arrives within $[a_i, b_i]$
- assume that the planes land in order

- Let t_i be the assigned landing time of i^{th} plane

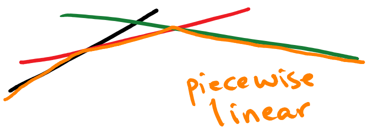
For Safety, define the shortest metering time as

$$\min_{1 \le i \le n-1} \{t_{i+1} - t_i\}$$



- <u>Goal</u>: maximize shortest metering time

$$\max \quad \min_{1 \le i \le n-1} \{t_{i+1} - t_i\}$$



$$\text{s.t.} \quad a_i \le t_i \le b_i \qquad i = 1, \cdots, n$$

$$t_i \le t_{i+1} \qquad i = 1, \cdots, n-1$$

} linear constraints

$$\Longleftrightarrow \quad \max \quad z$$

$$\text{s.t.} \quad a_i \le t_i \le b_i, \quad t_i \le t_{i+1}$$

$$z = \min_{1 \le i \le n-1} \{t_{i+1} - t_i\}$$

can be replaced by $\le$

At optimality, equality is achieved (why?)

$$\Longleftrightarrow \quad \max \quad z$$

$$\text{s.t.} \quad a_i \le t_i \le b_i, \quad t_i \le t_{i+1}$$

$$z \le t_{i+1} - t_i$$

} LP in variables $t_1, \cdots, t_n, z$

(2) Data Fitting

data points $(a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$; $i = 1, \cdots, m$.

$$\underset{\text{input}}{a_i} \longrightarrow \boxed{?} \longrightarrow \underset{\text{output}}{b_i}$$

Need to restrict the class of functions for the black box

Typical choice: affine function

$$f(y) = \underbrace{x^T y}_{\text{linear}} + \underbrace{t}_{\text{constant}} \qquad x, y \in \mathbb{R}^n, \; t \in \mathbb{R}$$

<u>Goal</u>: Find (x, t) s.t. $f(a_i) \simeq b_i \;\; \forall i$

e.g.: $n=1$