ENGG 5501: Foundations of Optimization	**2019–20 First Term**
Solution to Midterm Examination	
Time Limit: 2 Hours	October 24, 2019

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (15pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function satisfying $f(x) \in (0, \infty)$ for all $x \in \mathbb{R}^n$. Show that the function $x \mapsto \frac{1}{f(x)}$ is convex on $\mathbb{R}^n$.

ANSWER: Let $g : \mathbb{R}_{--} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(t) = -\frac{1}{t}$ and $h(x) = -f(x)$. Then, g is convex on $\mathbb{R}_{--}$ and h is convex on $\mathbb{R}^n$. Moreover, g is non-decreasing on $\mathbb{R}_{--}$. Since $g(h(x)) = \frac{1}{f(x)}$, the desired conclusion follows from Theorem 11(d) of Handout 2. $\square$

Problem 2 (10pts). Consider the function $f : \mathbb{R}^n \to \mathbb{R}_+$ given by $f(x) = \sum_{i=1}^n |x_i|$. Determine f^*, the conjugate of f. Show all your work.

ANSWER: By definition of f^*, we have

$$f^*(y) = \sup_{x\in\mathbb{R}^n} \left\{ y^T x - \sum_{i=1}^n |x_i| \right\} = \sup_{x\in\mathbb{R}^n} \left\{ \sum_{i=1}^n (y_i x_i - |x_i|) \right\} = \sum_{i=1}^n \sup_{x_i\in\mathbb{R}} (y_i x_i - |x_i|),$$

where the last equality is due to the separability of coordinates in the maximization. Now, using the fact that $|t| = \mathrm{sgn}(t)t$, we compute

$$\sup_{x_i\in\mathbb{R}} (y_i x_i - |x_i|) = \sup_{x_i\in\mathbb{R}} (y_i - \mathrm{sgn}(x_i))x_i = \begin{cases} 0 & \text{if } |y_i| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

It follows that $f^*(y) = i_B(y)$, where $i_B : \mathbb{R}^n \to \{0, +\infty\}$ is the indicator function of $B = \{y \in \mathbb{R}^n : |y_i| \le 1 \text{ for } i = 1, \ldots, n\}$. $\square$

Problem 3 (25pts). Let $P = \{x \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \le 1\}$. Show that $P = \mathrm{conv}(\{\pm e_1, \ldots, \pm e_n\})$.

ANSWER: Suppose that $x \in \mathrm{conv}(\{\pm e_1, \ldots, \pm e_n\})$. Then, we can write

$$x = \sum_{i=1}^n (\alpha_i^+ e_i + \alpha_i^-(-e_i)), \tag{1}$$

where $\alpha_1^+, \ldots, \alpha_n^+, \alpha_1^-, \ldots, \alpha_n^- \ge 0$ and

$$\sum_{i=1}^n (\alpha_i^+ + \alpha_i^-) = 1.$$

Now, using (1), we have $x_i = \alpha_i^+ - \alpha_i^-$ for $i = 1, \ldots, n$. It follows that

$$\sum_{i=1}^n |x_i| = \sum_{i=1}^n |\alpha_i^+ - \alpha_i^-| \le \sum_{i=1}^n (|\alpha_i^+| + |\alpha_i^-|) = 1;$$

i.e., $x \in P$.

Conversely, suppose that $x \in P$. Let

$$\Delta = \frac{1}{2n}\left(1 - \sum_{i=1}^{n} |x_i|\right) \in \left[0, \frac{1}{2n}\right],$$
$$I^+ = \{i : x_i \geq 0\}, \;\; I^- = \{i : x_i < 0\}.$$

Clearly, we have $I^+ \cup I^- = \{1, \ldots, n\}$. Moreover, we can express x as

$$x = \sum_{i \in I^+} (x_i + \Delta) e_i + \sum_{i \in I^+} \Delta(-e_i) + \sum_{i \in I^-} (|x_i| + \Delta)(-e_i) + \sum_{i \in I^-} \Delta e_i.$$

Now, observe that

$$\sum_{i \in I^+} (x_i + \Delta) + \sum_{i \in I^-} \Delta + \sum_{i \in I^+} \Delta + \sum_{i \in I^-} (|x_i| + \Delta) = \sum_{i=1}^{n} |x_i| + 2n\Delta = 1.$$

It follows that $P \in \operatorname{conv}(\{\pm e_1, \ldots, \pm e_n\})$, as desired. □

Problem 4 (15pts). Let $A \in \mathbb{R}^{m \times n}$ be given. Show that exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(I)} & Ax < \mathbf{0}, \;\; x \geq \mathbf{0}. \\ \text{(II)} & A^T y \geq \mathbf{0}, \;\; y \geq \mathbf{0}, \;\; y \neq \mathbf{0}. \end{array}$$

ANSWER: First, we show that (I) and (II) cannot be simultaneously solvable. Suppose to the contrary that $\bar{x}$ (resp. $\bar{y}$) solves (I) (resp. (II)). Then, on one hand, we have $\bar{y}^T A\bar{x} \geq 0$ because $\bar{x} \geq \mathbf{0}$ and $A^T\bar{y} \geq \mathbf{0}$. On the other hand, we have $\bar{y}^T A\bar{x} < 0$ because $A\bar{x} < \mathbf{0}$ and $\mathbf{0} \neq \bar{y} \geq \mathbf{0}$. This results in a contradiction.

Next, observe that (I) is solvable iff the system

$$\text{(I}') \quad Ax + s = -e, \;\; (x, s) \geq \mathbf{0}$$

is solvable. Indeed, if $(\bar{x}, \bar{s})$ is a solution to (I'), then $A\bar{x} \leq -e < \mathbf{0}$ and $\bar{x} \geq \mathbf{0}$, which implies that $\bar{x}$ is a solution to (I). Conversely, if $\bar{x}$ is a solution to (I), then there exists a $\theta > 0$ such that $A\bar{x} \leq -\theta e$. By letting $\tilde{x} = \bar{x}/\theta$ and $\tilde{s} = -e - A\tilde{x}$, we have $A\tilde{x} + \tilde{s} = -e$ and $(\tilde{x}, \tilde{s}) \geq \mathbf{0}$, which implies that $(\tilde{x}, \tilde{s})$ is a solution to (I').

Now, if (I') is not solvable, then by Farkas' theorem, the system

$$\text{(II}') \quad A^T w \leq \mathbf{0}, \;\; w \leq \mathbf{0}, \;\; -e^T w > 0$$

is solvable. However, it is clear that if $\bar{w}$ is a solution to (II'), then $\bar{y} = -\bar{w}$ is a solution to (II). This completes the proof. □

Problem 5 (15pts). Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron. Suppose that for $i = 1, \ldots, n$, we either have the constraint $x_i \geq 0$ or the constraint $x_i \leq 0$ in the description of P. Is it true that P has at least one vertex? Justify your answer.

ANSWER: Yes. By assumption, the polyhedron P contains the constraints

$$\left\{ \begin{array}{lll} x_i & \geq & 0 \quad \text{for } i \in I, \\ x_i & \leq & 0 \quad \text{for } i \notin I, \end{array} \right.$$

where $I \subseteq \{1, \ldots, n\}$. We claim that P does not contain a line, which would then imply the desired conclusion. Suppose that this is not the case. Then, there exist $x_0 \in P$ and $d \neq \mathbf{0}$ such that $x_0 + \alpha d \in P$ for all $\alpha \in \mathbb{R}$. Let $j \in \{1, \ldots, n\}$ be such that $d_j \neq 0$. If $j \in I$, then $(x_0 + \alpha d)_j < 0$ as $\alpha \searrow -\infty$, which contradicts the hypothesis that $(x_0 + \alpha d)_j \geq 0$ for all $\alpha \in \mathbb{R}$. One can derive a similar contradiction for the case where $j \notin I$. Hence, the claim is established. $\square$

Problem 6 (20pts). Let $B = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1 \text{ for } i = 1, \ldots, n\}$. Show that for any $y \in \mathbb{R}^n$, the projection $\Pi_B(y)$ of y onto B is given by

$$[\Pi_B(y)]_i = \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\} \quad \text{for } i = 1, \ldots, n,$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

ANSWER: Observe that for $i = 1, \ldots, n$, we have

$$y_i - \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\} = \begin{cases} 0 & \text{if } |y_i| \leq 1, \\ y_i - \operatorname{sgn}(y_i) & \text{otherwise.} \end{cases}$$

Hence, for any $x \in B$,

$$\sum_{i=1}^n (y_i - \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\})(x_i - \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\}) = \sum_{i:|y_i|>1} (y_i - \operatorname{sgn}(y_i))(x_i - \operatorname{sgn}(y_i)).$$

Now, if $y_i > 1$, then $y_i - \operatorname{sgn}(y_i) = y_i - 1 > 0$ and $x_i - \operatorname{sgn}(y_i) = x_i - 1 \leq 0$. On the other hand, if $y_i < -1$, then $y_i - \operatorname{sgn}(y_i) = y_i + 1 < 0$ and $x_i - \operatorname{sgn}(y_i) = x_i + 1 \geq 0$. In both cases, we have $(y_i - \operatorname{sgn}(y_i))(x_i - \operatorname{sgn}(y_i)) \leq 0$, which implies that

$$\sum_{i=1}^n (y_i - \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\})(x_i - \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\}) \leq 0$$

for any $x \in B$. It then follows from Theorem 5 of Handout 2 that

$$[\Pi_B(y)]_i = \operatorname{sgn}(y_i) \cdot \min\{1, |y_i|\} \quad \text{for } i = 1, \ldots, n,$$

as desired. $\square$