Homework Set 1

Instructor: Anthony Man–Cho So Due: September 24, 2021

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (15pts). Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given. Reformulate the following optimization problem as a linear program. Justify your answer.

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_\infty \\ \text{subject to} & \|x\|_1 \le 1. \end{array}$$

(Here, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the ℓ_1- and ℓ_∞-norm, respectively. Their definitions can be found in Handout B, Section 1.2.)

Problem 2 (30pts).

(a) **(10pts).** Let $S \subseteq \mathbb{R}^n$ be arbitrary and $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map. Is it true that $A(\mathrm{conv}(S)) = \mathrm{conv}(A(S))$? Justify your answer.

(b) **(10pts).** Show that the set $S = \{X \in \mathcal{S}^n : \lambda_{\max}(X) \le 1,\ X \succeq \mathbf{0}\}$ is convex.

(c) **(10pts).** Is the set $S = \{X \in \mathcal{S}^n : \mathrm{rank}(X) \le 1\}$ convex? Justify your answer.

Problem 3 (25pts). Let $B(\mathbf{0}, 1) \subset \mathbb{R}^n$ be the unit Euclidean ball in $\mathbb{R}^n$ centered at the origin. For any $x \in B(\mathbf{0}, 1)$, consider the set

$$N(x) = \{u \in \mathbb{R}^n : u^T(y-x) \le 0 \text{ for all } y \in B(\mathbf{0}, 1)\}.$$

(a) **(10pts).** Show that $N(x)$ is a convex cone for any $x \in B(\mathbf{0}, 1)$.

(b) **(15pts).** Give an explicit description of $N(x)$. Simplify your answer as much as possible. Show all your work.

Problem 4 (30pts).

(a) **(15pts).** Consider the halfspace $H^-(s,c) = \{x \in \mathbb{R}^n : s^Tx \le c\}$, where $s \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $c \in \mathbb{R}$ are given. Let $x \in \mathbb{R}^n$ be arbitrary. Find a formula for $\Pi_{H^-(s,c)}(x)$ in terms of s, c, x and prove its correctness.

(b) **(15pts).** Let $\Delta = \{x \in \mathbb{R}^n : e^Tx = 1, x \ge \mathbf{0}\}$ be the standard simplex, where $e = (1, 1, \dots, 1)$ is the vector of all ones. Show that for any $v \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$, we have

$$\Pi_\Delta(v) = \Pi_\Delta(v + \delta e).$$