ENGG 5501: Foundations of Optimization	2021–22 First Term

Homework Set 4

Instructor: Anthony Man–Cho So — Due: November 22, 2021

SOLVE THE FOLLOWING PROBLEMS. THE PARTS LABELED "EXTRA CREDIT" ARE OPTIONAL.

Problem 1 (20pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m_+$ be given. Consider the linear system

$$Ax = b, \ x \geq \mathbf{0} \tag{1}$$

and the related LP

$$\begin{array}{ll} \text{minimize} & e^T y \\ \text{subject to} & Ax + Iy = b, \\ & x \geq \mathbf{0}, \, y \geq \mathbf{0}. \end{array} \tag{2}$$

Here, $e = (1, \ldots, 1) \in \mathbb{R}^m$ is the vector of all ones and I is the $m \times m$ identity matrix. The LP (2) is commonly known as the *Phase One Problem.*

(a) **(5pts).** Write down the dual of (2).

(b) **(5pts).** Show that (2) always has an optimal solution.

(c) **(10pts).** Show that the system (1) has a solution iff the optimal value of (2) is zero.

Problem 2 (15pts). Let $P \in \mathbb{R}^{n \times n}$ be a *stochastic* matrix; i.e., $P_{ij} \geq 0$ for $i, j \in \{1, \ldots, n\}$ and $Pe = e$. Show that the system

$$P^T x = x, \quad x \geq \mathbf{0}, \quad x \neq \mathbf{0}$$

is solvable. (*Hint: Consider the duality between an appropriate pair of LPs.*)

Problem 3 (25pts). Let $p, q > 1$ be such that $1/p + 1/q = 1$. Define

$$C_p = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq 0, \, \|x\|_p \leq t\}.$$

(a) **(10pts).** Show that C_p is a closed pointed cone. (*Hint: To establish closedness of C_p, you may use the fact that the function $(t, x) \mapsto \|x\|_p - t$ is continuous.*)

(b) **(15pts).** Show that $C_p^* = C_q$ (recall that C_p^* is the dual cone of C_p). (*Hint: Hölder's inequality.*)

(c) **[Extra Credit (10pts).]** Give an explicit expression for $\text{int}(C_p)$. Justify your answer.

Problem 4 (15pts). We say that a set $X \subseteq \mathbb{R}^n$ is *SOC-representable* if there exist matrices $A^j \in \mathbb{R}^{(n_j+1) \times (n+\ell)}$ and vectors $b^j \in \mathbb{R}^{n_j+1}$ for $j = 1, \ldots, m$ such that

$$x \in X \quad \Longleftrightarrow \quad \exists u \in \mathbb{R}^\ell \text{ such that } A^j \begin{bmatrix} x \\ u \end{bmatrix} - b^j \in \mathcal{Q}^{n_j+1} \text{ for } j = 1, \ldots, m.$$

(a) **(15pts).** Let $Q \in \mathcal{S}^n_+$ be given and $X = \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : x^T Q x \le t\right\}$. Show that X is SOC-representable.

(b) **[Extra Credit (10pts).]** Show that $X = \left\{(t,x_1,x_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : x_1, x_2 \ge 0,\, t \le \sqrt{x_1 x_2}\right\}$ is SOC-representable.

Problem 5 (25pts). Let E be a finite-dimensional Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$. Let $K_1, K_2 \subseteq E$ be two closed convex cones. In the spirit of creating new cones from old ones, the purpose of this problem is study the closed convex cone $K_1 \cap K_2$ and its dual.

(a) **(5pts).** Show that $K_1^* + K_2^* \subseteq (K_1 \cap K_2)^*$ (recall that $K_1^* + K_2^* = \{u + v \in E : u \in K_1^*,\, v \in K_2^*\}$).

(b) **(10pts).** Show that $K_1^* + K_2^*$ is a convex cone.

(c) **[Extra Credit (15pts).]** Suppose that there exists a vector $u \in E$ satisfying $u \in \mathrm{int}(K_1) \cap \mathrm{int}(K_2)$. Show that $K_1^* + K_2^*$ is closed.

(d) **(10pts).** Using the results in (b) and (c) above, show that if there exists a vector $u \in E$ satisfying $u \in \mathrm{int}(K_1) \cap \mathrm{int}(K_2)$, then $(K_1 \cap K_2)^* \subseteq K_1^* + K_2^*$. (*Hint: Proposition 2(b) of Handout 5 and Problem 3 of the Midterm Examination.*)