

Homework 2

Problem 1

(a)

$$\because A, B \in S_+^n$$

$$\therefore \exists U \in R^{k \times n}, k = \text{rank}(B), B = U^T U$$

$$\therefore \text{tr}(AB) = \text{tr}(AU^T U) = \text{tr}(UAU^T) \geq 0 \text{ with assumption } \text{tr}(AB) = \text{tr}(BA)$$

let us proof $\text{tr}(AB) = \text{tr}(BA), \forall A, B \in S^n$:

$$\text{let } A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$$

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$$

$$\text{tr}(BA) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} a_{ji}$$

we can rearrange the equation above and get $\text{tr}(AB) = \text{tr}(BA)$

(b)

First we proof $S_+^n \subseteq \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\}$

$$\because \text{the result in (a), we have } S_+^n \subseteq \{X \in S^n : A \bullet X \geq 0\}$$

$$\therefore \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\} \supseteq \bigcap_{A \in S_+^n} S_+^n = S_+^n$$

then we proof $S_+^n \supseteq \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\}$

$$\text{we only have to proof: } \forall X \notin S_+^n, \exists A \in S_+^n, s.t. A \bullet X < 0$$

$$\because X \notin S_+^n$$

$$\therefore \exists \mu \in R^n, s.t. \mu^T X \mu < 0$$

$$\text{let } A = \mu \mu^T, \text{ we can get } \text{tr}(AX) = \text{tr}(\mu \mu^T X) = \text{tr}(\mu^T X \mu) < 0$$

note that $A = \mu \mu^T$ because $\forall z \in R^n, z^T A z = z^T \mu \mu^T z = (\sum z_i u_i)^2 \geq 0$

$$\therefore \forall X \notin S_+^n, X \notin \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\}$$

$$\therefore S_+^n \supseteq \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\}$$

In summary, $S_+^n = \bigcap_{A \in S_+^n} \{X \in S^n : A \bullet X \geq 0\}$

Problem 2

(a)

First let us proof f possesses Property C $\Leftrightarrow L_t = \{x \in \mathbb{R}^n : f(x) \leq t\}$ is compact for any $t \in \mathbb{R}$

① proof of \Rightarrow :

first let us show L_t is bounded, which means $\forall x \in L_t, \exists T \in \mathbb{R}, s.t. \|x\|_2 < T$

Proof by contradiction:

assume L_t is not bounded, so there exists a sequence $\{x^k\}_{k \geq 0} \subseteq L_t, \|x^k\|_2 \rightarrow +\infty$

$\because f$ possesses Property C

$$\therefore f(x^k) \rightarrow +\infty$$

However, with the limitation of $f(x) \leq t$, we can find $\{x^k\}_{k \geq 0} \subsetneq L_t$

Contradicting the assumption, thus L_t is bounded

then let us show L_t is close, which means $\forall \{x^k\}_{k \geq 0} \subseteq L_t, \{x^k\} \rightarrow x, x \in L_t$

$\because f$ is a continuous function

$$\therefore \{f(x^k)\} \rightarrow f(x)$$

$$\because \{x^k\}_{k \geq 0} \subseteq L_t$$

$$\therefore \forall x^i \in \{x^k\}, f(x^i) \leq t$$

$$\therefore \{f(x^k)\} \rightarrow f(x) \leq t$$

$$\therefore x \in L_t, L_t \text{ is close}$$

In summary, L_t is compact

② **proof of** \Leftarrow :

Proof by contradiction:

assume f not possesses Property C, which means $\exists \{x^k\}_{k \geq 0} \subseteq \mathbb{R}^n, \|x^k\|_2 \rightarrow +\infty, f(x^k) \leq T, T \in \mathbb{R}$

let $t > T$, then $\{x^k\}_{k \geq 0} \subseteq L_t$. However $\|x^k\|_2 \rightarrow +\infty$, thus L_t is not bounded.

Contradicting the assumption, thus f possesses Property C

In summary, f possesses Property C $\Leftrightarrow L_t = \{x \in \mathbb{R}^n : f(x) \leq t\}$ is compact for any $t \in \mathbb{R}$

Then let us proof f is continuous and possesses Property C, $\inf_{x \in \mathbb{R}^n} f(x)$ always has an optimal solution.

$\inf_{x \in \mathbb{R}^n} f(x)$ always has an optimal solution, which means we can find a $x_0 \in \mathbb{R}^n$ s.t. $\forall x \in \mathbb{R}^n, f(x) \geq f(x_0)$

$\therefore f$ is continuous and possesses Property C, thus we can find a infimum c , which is the max c_i satisfy $f(x) \geq c_i$

then we only need to show that exist $x_0 \in \mathbb{R}^n$, s.t. $f(x_0) = c$

let $t = c + 1$, L_t is compact and c is also infimum of L_t

with the definition of infimum and continuous, $\exists \{x^k\}_{k \geq 0} \subseteq L_t, \{f(x^k)\} \rightarrow c, \{x^k\} \rightarrow x_0$

$\therefore L_t$ is compact

$\therefore x_0 \in L_t \subseteq \mathbb{R}^n$

Thus we can always find an optimal solution x_0

(b)

No.

assume $f(0) < +\infty$, let $A = 0^{m \times n}$, thus $\forall x \in \mathbb{R}^n, g(x) \equiv f(0) < \infty, g$ not possesses Property C.

Problem 3

(a)

$\forall x, y \in K^\circ, \alpha, \beta \in \mathbb{R}^+$, let $z = \alpha x + \beta y$

$$\forall u \in K, z^T u = (\alpha x + \beta y)^T u = \alpha x^T u + \beta y^T u \leq 0$$

$\therefore z \in K^\circ, K^\circ$ is a convex cone

(b)

First we know that, $z^* = \prod_K(x) \Leftrightarrow z^* \in K, (z - z^*)^T(x - z^*) \leq 0, \forall z \in K$

① **proof of** \Rightarrow :

With the property above, we have $z^* \in K$, then let us show that $(x - z^*)^T z^* = 0$

assume $(x - z^*)^T z^* \neq 0$, let $t = x - z^*, z^* = x - t$

$$\therefore t^T z^* \neq 0$$

$$\therefore t = \alpha z^* + z, \text{ where } \alpha \neq 0, z^* \neq 0, z^T z^* = 0 \text{ and } t^T z^* = \alpha(z^*)^T z^*$$

$$\therefore \|x - z^*\|_2^2 = \|x - (x - t)\|_2^2 = \|t\|_2^2 = \|\alpha z^*\|_2^2 + \|z\|_2^2 = |\alpha|^2 \|z^*\|_2^2 + \|z\|_2^2$$

$$\text{besides, let } z = 0, (-z^*)^T(x - z^*) = (-z^*)^T t = -\alpha(z^*)^T z^* \leq 0$$

$$\therefore \alpha \geq 0$$

by the definition of closed convex cone and $z^* \neq 0, \alpha \geq 0$, we can find a

$$z^o = (1 + \alpha)z^* \in K, \|x - z^o\|_2^2 = \|z\|_2^2 < \|x - z^*\|_2^2$$

so that $z^* \neq \prod_K(x)$, contradicting the assumption, thus $(x - z^*)^T z^* = 0$

last let us show that $x - z^* \in K^o$

$$\because (x - z^*)^T z^* = 0 \text{ and } (z - z^*)^T (x - z^*) \leq 0, \forall z \in K$$

$$\therefore z^T (x - z^*) = (x - z^*)^T z \leq 0, \forall z \in K$$

by the definition of K^o , we get $x - z^* \in K^o$

② proof of \Leftarrow :

$$\because x - z^* \in K^o \text{ and } (x - z^*)^T z^* = 0$$

$$\therefore (z - z^*)^T (x - z^*) \leq 0, \forall z \in K$$

$$\text{so that, we get } z^* \in K, (z - z^*)^T (x - z^*) \leq 0, \forall z \in K \Rightarrow z^* = \Pi_K(x)$$

In summary, $z^* = \Pi_K(x) \Leftrightarrow z^* \in K, x - z^* \in K^o, (x - z^*)^T z^* = 0$

(c)

Using the result in (b), we only need to show that $z^* = \Pi_K(x), x - z^* = \Pi_{K^o}(x)$

what's more, we only need to show that

$K = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K^o\}, K^o = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K\}, K \subseteq \mathbb{R}^n$ be a closed convex cone

$$\because x^T \omega = \omega^T x, \text{ so that } K = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K^o\} \text{ is true obviously by definition}$$

Using the result in (b):

$$x - z^* \in K^o, x - (x - z^*) = z^* \in K, (x - (x - z^*))^T (x - z^*) = x^T (x - z^*) = (x - z^*)^T x = 0 \Rightarrow x - z^* = \Pi_{K^o}(x)$$

In summary, $x = \Pi_K(x) + \Pi_{K^o}(x)$

Problem 4

① $1 \Leftrightarrow 2$

let $g(x) = f(x) + \frac{\rho}{2} \|x\|_2^2$, $g(x)$ is convex which means

$$g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y), \forall x, y \in \mathbb{R}^n, \alpha \in [0, 1]$$

which is

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) + \frac{\rho}{2} \|\alpha x + (1 - \alpha)y\|_2^2 &\leq \alpha(f(x) + \frac{\rho}{2} \|x\|_2^2) + (1 - \alpha)(f(y) + \frac{\rho}{2} \|y\|_2^2) \\ \Leftrightarrow f(\alpha x + (1 - \alpha)y) &\leq \alpha f(x) + (1 - \alpha)f(y) + \alpha \frac{\rho}{2} \|x\|_2^2 + (1 - \alpha) \frac{\rho}{2} \|y\|_2^2 - \frac{\rho}{2} \|\alpha x + (1 - \alpha)y\|_2^2 \\ \Leftrightarrow f(\alpha x + (1 - \alpha)y) &\leq \alpha f(x) + (1 - \alpha)f(y) + \frac{\rho}{2} (\alpha \|x\|_2^2 + (1 - \alpha) \|y\|_2^2 - \|\alpha x + (1 - \alpha)y\|_2^2) \\ \Leftrightarrow f(\alpha x + (1 - \alpha)y) &\leq \alpha f(x) + (1 - \alpha)f(y) + \frac{\rho}{2} (\alpha \|x\|_2^2 + (1 - \alpha) \|y\|_2^2 - (\alpha^2 \|x\|_2^2 + (1 - \alpha)^2 \|y\|_2^2 + 2\alpha(1 - \alpha) \sum_{i=1}^n x_i y_i)) \\ \Leftrightarrow f(\alpha x + (1 - \alpha)y) &\leq \alpha f(x) + (1 - \alpha)f(y) + \frac{\rho}{2} \alpha(1 - \alpha) (\|x\|_2^2 + \|y\|_2^2 - 2 \sum_{i=1}^n x_i y_i) \\ \Leftrightarrow f(\alpha x + (1 - \alpha)y) &\leq \alpha f(x) + (1 - \alpha)f(y) + \frac{\rho \alpha(1 - \alpha)}{2} \|x - y\|_2^2 \end{aligned}$$

$\because f$ is ρ -convex, so the inequality is true by the definition

$\therefore f$ is ρ -convex $\Leftrightarrow g(x)$ is convex

② $2 \Leftrightarrow 3$

$$g(x) = f(x) + \frac{\rho}{2} \|x\|_2^2, \nabla g(x) = \nabla f(x) + \rho x$$

$\because g$ is convex

$$\therefore \forall x, y \in \mathbb{R}^n, g(y) \geq g(x) + \nabla g(x)^T (y - x)$$

which is

$$\begin{aligned} f(y) + \frac{\rho}{2} \|y\|_2^2 &\geq f(x) + \frac{\rho}{2} \|x\|_2^2 + (\nabla f(x) + \rho x)^T (y - x) \\ \Leftrightarrow f(y) + \frac{\rho}{2} \|y\|_2^2 &\geq f(x) + \frac{\rho}{2} \|x\|_2^2 + \nabla f(x)^T (y - x) + \rho x^T (y - x) \\ \Leftrightarrow f(y) &\geq f(x) + \nabla f(x)^T (y - x) + \frac{\rho}{2} \|x\|_2^2 - \frac{\rho}{2} \|y\|_2^2 + \rho x^T (y - x) \\ \Leftrightarrow f(y) &\geq f(x) + \nabla f(x)^T (y - x) + \frac{\rho}{2} (\|x\|_2^2 - \|y\|_2^2 + 2x^T (y - x)) \\ \Leftrightarrow f(y) &\geq f(x) + \nabla f(x)^T (y - x) - \frac{\rho}{2} (\|y\|_2^2 + \|x\|_2^2 - 2x^T y) \\ \Leftrightarrow f(y) &\geq f(x) + \nabla f(x)^T (y - x) - \frac{\rho}{2} \|y - x\|_2^2 \end{aligned}$$

$$\therefore g \text{ is convex} \Leftrightarrow \forall x, y \in \mathbb{R}^n, f(y) \geq f(x) + \nabla f(x)^T(y - x) - \frac{\rho}{2} \|y - x\|_2^2$$

In conclusion, $1 \Leftrightarrow 2 \Leftrightarrow 3$, the statements are equivalent.