Convex Sets (cont'd)

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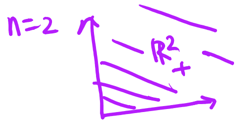
Examples (Convex Sets)

① Non-negative orthant

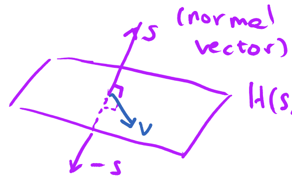
$\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \ \forall i \}$



Take $x, y \in \mathbb{R}^n_+$; $\alpha \in [0,1]$

$\Rightarrow \underbrace{\alpha x}_{\geq 0} + \underbrace{(1-\alpha) y}_{\geq 0} \in \mathbb{R}^n_+$

② Hyperplane



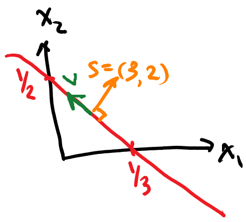
$H(s,c) = \{ x \in \mathbb{R}^n : s^T x = c \}$

(set of solutions to the linear equation $s^T x = c$)

e.g.: $n = 2$:

$H((3,2),1) = \{ (x_1, x_2) : 3x_1 + 2x_2 = 1 \}$

$= \{ (x_1, x_2) : \underbrace{(3,2)}_{s}^T (x_1, x_2) = \underbrace{1}_{c} \}$



Note: s is perpendicular to everything on $H((3,2),1)$

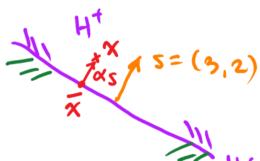
v is proportional to $v_0 = (0, \frac{1}{2}) - (\frac{1}{3}, 0) = (-\frac{1}{3}, \frac{1}{2})$

$s^T v_0 = (3,2)^T (-\frac{1}{3}, \frac{1}{2}) = 0.$

③ Halfspace

$H^+(s,c) = \{ x \in \mathbb{R}^n : s^T x \geq c \}$ Obviously,

$H^-(s,c) = \{ x \in \mathbb{R}^n : s^T x \leq c \}$ $H(s,c) = H^+(s,c) \cap H^-(s,c)$



$H^+(s,c)$ is the side where the normal vector s points into.

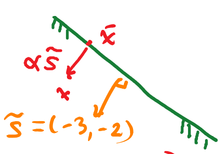
$H((3,2),1) = \{ (x_1, x_2) : 3x_1 + 2x_2 = 1 \}$

Let $x = \bar{x} + \alpha s$, $\alpha > 0$.

Then, $s^T x = s^T \bar{x} + \alpha s^T s$

$= c + \underbrace{\alpha \|s\|_2^2}_{>0} \geq c.$

$H((-3,-2),-1) = \{ (x_1, x_2) : -3x_1 - 2x_2 = -1 \}$

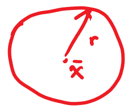


$H^+((-3,-2),-1)$ $\tilde{s}^T x = \underbrace{c + \alpha \|\tilde{s}\|_2^2}_{\geq c}$

④ Euclidean Ball

$$B(\bar{x}, r) = \{ x \in \mathbb{R}^n : \|x - \bar{x}\|_2 \le r \}$$

center radius



Verify its convexity:

Take $x, y \in B(\bar{x}, r)$; $\alpha \in [0,1]$

Want: $\alpha x + (1-\alpha) y \in B(\bar{x}, r)$

$$\|\alpha x + (1-\alpha)y - \bar{x}\|_2 = \|\alpha(x-\bar{x}) + (1-\alpha)(y-\bar{x})\|_2$$

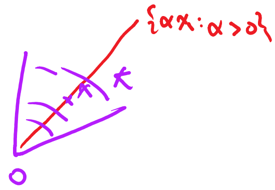
$$\le \|\alpha(x-\bar{x})\|_2 + \|(1-\alpha)(y-\bar{x})\|_2 \quad \text{(by } \Delta\text{-inequality)}$$

$$\le \alpha r + (1-\alpha) r = r.$$

⑤ Convex Cone

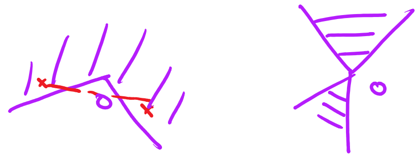
Definition: A set $K \subseteq \mathbb{R}^n$ is called a cone if

$\forall x \in K, \alpha > 0, \alpha x \in K$.



Q: Must a cone be convex?

A: No, e.g.:



Definition: A convex cone is a cone that is convex.

Examples: $\mathbb{R}^n_+$ (easy) ;

$S^n_+ = \{ X \in S^n : X \succeq 0 \}$ (exercise)

Convexity-Preserving Operations

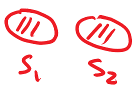
Motivation: So far we only know how to check convexity from first principles.

Q: Suppose that we apply some transformation to convex sets. Is the result convex?

Examples

① Let S_1, S_2 be convex sets.

- $S_1 \cup S_2$ convex? No:



$S_1 \cap S_2$ convex? Yes (check by first principles)

② Definition: We say that $A : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map if

$\forall x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$A(\alpha x + (1-\alpha)y) = \alpha A(x) + (1-\alpha) A(y)$$

e.g. $A: \mathbb{R}^n \to \mathbb{R}$, $A(x) = c^T x + d$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$

Proposition: Let $A: \mathbb{R}^n \to \mathbb{R}^m$ be an affine map, $S \subseteq \mathbb{R}^n$ be a convex set. Then,

$$A(S) \triangleq \{ A(x) : x \in S \} \text{ is convex.}$$

e.g:

① Translation: $A: \mathbb{R}^n \to \mathbb{R}^n$, $A(x) = x + d$, d is given

② Rotation: $A: \mathbb{R}^n \to \mathbb{R}^n$, $A(x) = Ux$, U orthogonal matrix ($U^T U = U U^T = I$)

e.g: $n=2$

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ (clockwise rotation by θ)

③ Projection: $A: \mathbb{R}^n \to \mathbb{R}^m$; $A(x) = Px$, P projection matrix ($P^2 = P$)

e.g.:

$$P = \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} m \\ n-m \end{array}$$

$$A(x) = P \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

④ Ellipsoid

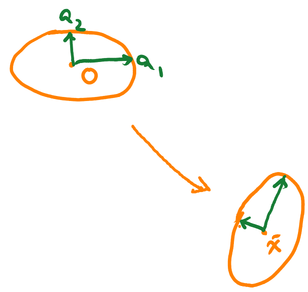
$$E(\bar{x}, Q) = \{ x \in \mathbb{R}^n : (x - \bar{x})^T Q (x - \bar{x}) \le 1 \}, \quad Q \succ 0$$

e.g. a) $Q = I \rightarrow E(\bar{x}, I) = B(\bar{x}, 1)$

b) $n=2$

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \le 1$$

$$\Longleftrightarrow \underbrace{[x_1 \ \ x_2]}_{x^T} \underbrace{\begin{bmatrix} 1/a_1^2 & \\ & 1/a_2^2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \le 1$$



Claim: There exists an affine map A s.t.

$$A(B(0,1)) = E(\bar{x}, Q).$$