SEEM 5580 Midterm Exam Name:

9:30–11:00 am, Oct 22, 2018

This is a closed book, closed notes test. **CUHK student honor code applies to this test.**

1. (20 points) Assume that customers arrive at a service center between 10am-12noon following a non-homogeneous Poisson process with rate function $\lambda(t)$ given as follows. From 10am-11am, the arrival rate increases linearly from 0 to 60 customers per hour. From 11am to noon, the arrival rate is a constant, 60 customers per hour.

   (a) (6 points) What is the probability that there is at least 1 customer arrival in the first 5 minutes (after 10am)?

   (b) (7 points) Find the probability of the following event: there is 1 arrival between 10:00am to 10:10am and 2 arrivals between 10:05am to 10:15am.

   (c) (7 points) The total number of customer arrivals between 10:30am and 11:30am is a random variable. Find its distribution. Specify the mean and the variance.

$$\lambda(t) = \begin{cases} t/60; & 0 \le t \le 60 \\ 1; & 60 < t \le 120 \end{cases}$$

per minute

(a) $P(N(5) \ge 1) = 1 - P(N(5)=0) = 1 - e^{-\int_0^5 \lambda(s)ds} = 1 - e^{-\frac{5}{24}}$

(b) $P(N(10)=1,\ N(15)-N(5)=2)$

$= P(N(5)=1,\ N(10)-N(5)=0,\ N(15)-N(10)=2) +$
$P(N(5)=0,\ N(10)-N(5)=1,\ N(15)-N(10)=1)$

$= P(N(5)=1) \cdot P(N(10)-N(5)=0) \cdot P(N(15)-N(10)=2)$
$+ P(N(5)=0) \cdot P(N(10)-N(5)=1) \cdot P(N(15)-N(10)=1)$

$= m(5) \cdot e^{-m(5)} \cdot e^{-[m(10)-m(5)]} \cdot \frac{(m(15)-m(10))^2}{2} \cdot e^{-[m(15)-m(10)]}$
$+ e^{-m(5)} \cdot (m(10)-m(5)) \cdot e^{-[m(10)-m(5)]} \cdot [m(15)-m(10)] \cdot e^{-[m(15)-m(10)]}$

$= e^{-\frac{15}{8}} \cdot \left[\frac{25^2 \times 5}{24^3 \times 2} + \frac{25 \times 5}{24 \times 8} \right]$

write $m(t) = \int_0^t \lambda(s)ds$

$m(5) = \frac{5}{24}$

$m(10) - m(5) = \int_5^{10} \lambda(t)dt = \frac{5}{8}$

$m(15) - m(10) = \int_{10}^{15} \lambda(t)dt = \frac{25}{24}$

(c) Poisson distribution with mean $= \int_{30}^{90} \lambda(s)ds = 52.5$

Variance $= 52.5$

$E[\xi_i] = \frac{1}{2}$ $\Leftarrow$ $\xi_i = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$

$E[\xi_i^2] = \frac{1}{2}$

(in distribution it is the same)

2. (30 points) Suppose that $\{\xi_i, i = 1, 2, \ldots\}$ are independent and identically distributed random variables with moment generating function $\mathbb{E}[e^{s\xi_i}] = \frac{1}{2} + \frac{1}{2}e^s$. Suppose that $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λ, and that the process $\{N(t) : t \geq 0\}$ is independent of $\{\xi_i, i = 1, 2, \ldots\}$.

   (a) (7 points) What is the expected value of $Z = \xi_1 \cdot N(1) + \xi_2 \cdot N(2)$?

   (b) (7 points) What is the variance of the random variable Z defined above?

   (c) (8 points) Prove that $\{N(k) : k = 0, 1, 2, \ldots\}$ is a discrete-time Markov chain and specify the transition probabilities.

   (d) (8 points) Is $\{\sum_{i=1}^{N(t)} \xi_i : t \geq 0\}$ a homogeneous Poisson process? Explain and justify your answer.

(a) $E[Z] = E[\xi_1] \cdot \cancel{E[\xi_2]}\, E[N(1)] + E[\xi_2] \cdot E[N(2)]$

$= \frac{1}{2} \cdot \lambda + \frac{1}{2} \cdot 2\lambda = 1.5\lambda$

(b) $E[Z^2] = E[\xi_1^2 N(1)^2 + \xi_2^2 \cdot N(2)^2 + 2\xi_1\xi_2 \cdot N(1) \cdot N(2)]$

$= \frac{1}{2}(\lambda+\lambda^2) + \frac{1}{2} \cdot E[N(2)]^2 + 2 \cdot (\frac{1}{2})^2 \cdot E[N(1) \cdot N(2)]$

$\downarrow$ $E[N(1) \cdot N(1)] + \lambda^2$ as $N(2)-N(1)$ independent of $N(1)$

$= \frac{1}{2} \cdot [(\lambda + \lambda^2) + [2\lambda + 4\lambda^2] + [\lambda + 2\lambda^2]]$

$= 2\lambda + 3.5\lambda^2$

$\Rightarrow Var(Z) = 2\lambda + \frac{5}{4}\lambda^2$

(c) $P(N(k+1)=j \mid N(k)=i, N(k-1)=i_{k-1}, \ldots, N(1)=i_1, N(0)=0)$

$= P(N(k+1)-N(k)=j-i \mid N(k)=i, N(k-1)=i_{k-1}, \ldots, N(0)=0)$

independent increments of N $\overset{\Downarrow}{=} P(N(k+1)-N(k)=j-i) = P(N(k+1)=j \mid N(k)=i)$

$= P_{ij} = \begin{cases} \frac{\lambda^{j-i}}{(j-i)!} e^{-\lambda} & j \geq i \\ 0 & j < i \end{cases}$

(d) Yes. It is clear that it[2] has stationary and independent increments.

In addition, $P(\sum_{i=1}^{N(t)} \xi_i = k) = \sum_{n=0}^{\infty} P(\sum_{i=1}^{N(t)} \xi_i = k \mid N(t)=n) \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

$= \sum_{n=0}^{\infty} \binom{n}{k} \cdot (\frac{1}{2})^n \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \sum_{n=k}^{\infty} \frac{1}{(n-k)!k!} (\frac{\lambda t}{2})^n \cdot e^{-\lambda t}$

$m = n-k$. $\overset{?}{=} \frac{(\frac{1}{2}\lambda t)^k}{k!} \cdot e^{-\frac{1}{2}\lambda t} \Rightarrow$ So $\{\sum_{i=1}^{N(t)} \xi_i, t \geq 0\}$ is Poisson process with rate $\lambda/2$.

3. (20 points) A machine produces two items per day. The probability that an item is non-defective is p. The quality of successive items are independent. Defective items are thrown away instantly, and the non-defective items are stored to satisfy the demand of one item per day which occurs at the end of a day. Any demand that cannot be satisfied immediately is lost. Let X_n be the number of items in storage at the beginning of the n-th day (before the demand and production for that day are taken into account), with $X_0 = 2$.

Prove that $\{X_n : n = 0, 1, 2, \ldots\}$ is a discrete time Markov chain. Provide the state space and transition probabilities.

If $X_n > 0$,

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p^2 \\ X_n & \text{w.p } 2p(1-p) \\ X_n - 1 & \text{w.p } (1-p)^2 \end{cases}$$

If $X_n = 0$,

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p^2 \\ X_n & \text{w.p. } 1-p^2 \end{cases}$$

So X_{n+1} only depends on X_n, and Markov property holds.

$\Rightarrow \{X_n : n \geq 0\}$ is a DTMC with $S = \{0, 1, 2, \cdots\}$

The transition probabilities are: for $i \geq 0$, $j \geq 0$

$$P_{ij} = P(X_{n+1} = j \mid X_n = i) = \begin{cases} p^2 & \text{if } j = i+1 \\ 2p(1-p) & \text{if } j = i > 0 \\ 1-p^2 & \text{if } j = i = 0 \\ (1-p)^2 & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

Specifically

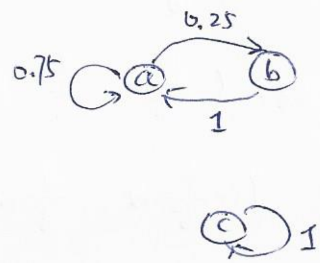
$X_{n+1} = (X_n + Y_n - 1)^+$ where Y_n = production # of non-defective items

$Y_n \sim \text{Binomial}(2, p)$. i.i.d

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = 2) = P((i + Y_{n+1} - 1)^+ \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = 2)$$
$$= P(X_{n+1} = j \mid X_n = i).$$

4. (30 points, 5 points each) Let $X = \{X_n : n = 0, 1, \ldots,\}$ be a discrete time Markov Chain with state space $S = \{a, b, c\}$. The transition probability matrix is given by

$$P = \begin{pmatrix} .75 & .25 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



Let

$$f(x) = \begin{cases} \$1 & \text{if } x = a, \\ \$2 & \text{if } x = b, \\ \$3 & \text{if } x = c. \end{cases}$$

(a) This Markov chain is irreducible. True or false? (No explanation needed)

(b) Every state of this Markov chain has period 1. True or False? (No explanation needed)

(c) Find $\mathbb{E}\Big[f(X_1) + f(X_2)\Big|X_0 = b\Big]$.

(d) Find $\mathbb{E}\Big[\sum_{n=0}^{\infty}(.5)^n f(X_n)\Big|X_0 = b\Big]$.

(e) Find ALL the stationary distributions of the Markov chain.

(f) Find $\lim_{n\to\infty} \mathbb{P}(X_n = a|X_0 = x)$ for $x = a, b$ and c. If the limit does not exist, explain why.

(c) $E[f(X_1)|X_0=b] = f(a)\cdot P_{ba} + f(b)\cdot P_{bb} = f(a) = 1$

$E[f(X_2)|X_0=b] = 0.75 \times f(a) + 0.25\, f(b) = 1.25$

so the answer is 2.25

Consider $A = \begin{bmatrix} 0.75 & 0.25 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow$ $I - 0.5A = \begin{bmatrix} \frac{5}{8} & -\frac{1}{8} \\ -\frac{1}{2} & 1 \end{bmatrix}$

(d) $E[\sum_{n=0}^{\infty}(0.5)^n f(X_n)|X_0=b] = \sum_{n=0}^{\infty}(0.5)^n A^n \underbrace{f}_{2\times 1 \text{ vector } \binom{1}{2}}(b)$

$= \left[\sum_{n=0}^{\infty}(0.5A)^n \cdot f\right](b)$

$= \left[(I-0.5A)^{-1} f\right](b)$

$= \frac{16}{9}\begin{bmatrix} 1 & \frac{1}{8} \\ \frac{1}{2} & \frac{5}{8} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}(b) = \frac{16}{9} \times \frac{7}{4} = \frac{28}{9}$

(b): 2-nd component of the 2x1 vector

(e) $\begin{cases} (\pi_a, \pi_b) \begin{bmatrix} 0.75 & 0.25 \\ 1 & 0 \end{bmatrix} = (\pi_a, \pi_b) \\ \pi_a, \pi_b \geq 0 \\ \pi_a + \pi_b = 1 \end{cases} \Rightarrow \begin{cases} \pi_a = \frac{4}{5} \\ \pi_b = \frac{1}{5} \end{cases}$

$\{a, b\}$ is closed communicating class.

All stationary distributions for the DTMC:

$$\alpha \cdot \left(\frac{4}{5}, \frac{1}{5}, 0\right) + (1-\alpha)(0, 0, 1)$$

$$= \left(\frac{4}{5}\alpha, \frac{1}{5}\alpha, 1-\alpha\right) \qquad 0 \leq \alpha \leq 1.$$

(f) $\lim_{n \to \infty} P(X_n = a \mid X_0 = x) = \begin{cases} \pi_a = 0.8 ; & x = a \\ \pi_a = 0.8 ; & x = b \\ 0 ; & x = c \end{cases}$