Homework 5

Problem 1

(a)

$f(x) = -x^T Ax = -\sum_{i,j} x_i A_{ij} x_j$

the first-order optimality conditions:

$$\nabla f(x) + w_1 \nabla g_1(x) + w_2 \nabla g_2(x) = \mathbf{0} \Rightarrow -2Ax - 2w_1 x + w_2 v_1 = \mathbf{0}, w_i \geq 0$$

By the **Handout 7-2 Theorem 3**, we convert Problem (1) to $min f(x) = -x^T Ax$ and it's easy to verify that $f(x)$ is continuously differentiable in R^n. Consider $\nabla g_1(x) = -2x, \nabla g_2(x) = v_1$ is independent because not $\exists k, s.t. -2x \equiv kv_1$。

(b)

Consider the result in (a), let it $\cdot v_1^T$ at left and right, the we get:

$$-2Axv_1^T - 2w_1 xv_1^T + w_2 v_1 v_1^T = \mathbf{0} = w_2 ||v_1||_2^2$$

which indicate that $w_2 = 0$, thus we have $Ax = w_1 x$, which means the optimal result must be eigenvector, so we have:

$f(x) = -x^T Ax = \lambda ||x||_2^2 = \lambda$

which is equal to some eigenvalue.

However, due to $v_1^T x = 0$, thus $x \neq v_1$, so the optimal value can only be the second largest eigenvalue of A. Thus, λ_2 is the second largest eigenvalue of A and v_2 is an eigenvector associated with λ_2

Problem 2

$1^o :\Rightarrow$

Because x^* is an optimal solution and $f(x)$ is continuous differentiable convex function. Thus, $\forall x \in C, f(x) \geq f(x^*), \text{and } \nabla f(x^*)(x - x^*) \geq 0$

By **Handouts 2 Theorem3** , we have $x^* = \Pi_C(x^* - \nabla f(x^*))$

$\Leftrightarrow x^* \in C \text{ and } (x - x^*)(x^* - \nabla f(x^*) - x^*) = -(x - x^*)\nabla f(x^*) \leq 0, \forall x \in C$

Thus, $x^* = \Pi_C(x^* - \nabla f(x^*))$

$2^o :\Leftarrow$

Because $x^* = \Pi_C(x^* - \nabla f(x^*))$, we have $\nabla f(x^*)(x - x^*) \geq 0$, and $f(x)$ is continuous differentiable convex function. Thus, $f(x) \geq f(x^*) + \nabla f(x^*)(x - x^*) \geq f(x^*), \forall x \in C$。

Thus, x^* is an optimal solution.

Problem 3

We can convert this problem to:

$$\begin{array}{ll} \min & z \\ \text{subject to} & g_1(x) - z \leq 0 \\ & g_2(x) - z \leq 0 \\ & \dots \\ & g_m(x) - z \leq 0 \end{array}$$

$1^o :\Rightarrow$

Because x^* is an optimal solution, by **KKT condition** and Slater condition, we have:

$\sum_{j=1}^m u_j^* \nabla g_j(x^*) = \mathbf{0}, u^* \geq \mathbf{0}$

and

$u_j^*(g_j(x^*) - z^*) = 0, \text{for } j = 1, \dots m$

Thus, for $g_j(x^*) < max\{g_1(x^*), \dots, g_m(x^*)\}, u_j^* = 0$

Besides, $g_j(x^*) = max\{g_1(x^*), \dots, g_m(x^*)\}$ must exists, so that $u^* \neq \mathbf{0}$ exists and can scale it to satisfy $\sum_{j=1}^m u_j^* = 1$

$2^o :\Leftarrow$

For given x^*, there exist a vector $u^* \in R^m, s.t.$

$\sum_{j=1}^m u_j^* \nabla g_j(x^*) = \mathbf{0}, u^* \geq \mathbf{0}, \sum_{j=1}^m u_j^* = 1$

$u_j^* = 0$, if $g_j(x^*) < max\{g_1(x^*), \dots, g_m(x^*)\}$

$z^* = max\{g_1(x^*), \dots, g_m(x^*)\}$

let $I = \{j | g_j(x^*) = max\{g_1(x^*), \dots, g_m(x^*)\}\}$,

1. $\forall i \in I, \nabla g_i(x^*) = \mathbf{0}$:

   $\forall x \in R^n, z = max\{g_i(x)\}_{i=1 \dots m} \geq max\{g_i(x)\}_I$

   g_i is convex function, thus $\forall i \in I, g_i(x) \geq g_i(x^*)$

   thus, $z^* \leq z, \forall x$

2. $\exists i, \nabla g_i(x^*) \neq \mathbf{0}$:

   because g_i is convex function, thus

   $\forall x \in R^n, \exists i \in I, \nabla g_i(x^*)(x - x^*) \geq 0 \Rightarrow g_i(x) \geq g_i(x^*)$

   $z = max\{g_i(x)\}_{i=1 \dots m} \geq max\{g_i(x)\}_I \geq z^*$

In summary, x^* is an optimal solution if and only if the condition above it true.

Problem 4

(a)

$x^T A x = A \cdot x x^T, x_i^2 = 1 \Leftrightarrow x^T E_i x = 1 = E_i \cdot x x^T$, where E_i is matrix $e_{ii} = 1$ and other is 0.

let $X = x x^T$, the semidefinite relaxation of Problem (3) is

$$\begin{array}{lll} inf & A \cdot X & \\ \text{subject to} & E_i \cdot X = 1 \quad \text{for } i = 1, \dots, n & (P) \\ & X \geq \mathbf{0} & \end{array}$$

(b)

dual of (P):

$$\begin{array}{ll} sup & e^T y \\ \text{subject to} & \begin{bmatrix} y_1 & 0 & 0,\dots,0 \\ 0 & y_2 & 0,\dots,0 \\ & \dots & \\ 0 & \dots & y_n \end{bmatrix} + S = A \quad (D) \\ & y \in R^n, S \in S_+^n \end{array}$$

It is easy to solve (D) that $e^T y = tr(A)$ with $y_i = A_{ii}$, if (D) is solvable, which means $A_{ij} \geq 0, i \neq j$

At this condition, (P) has solution $A \cdot X = tr(A) = e^T y$ with $X = E$。

Besides, if exist $A_{ij} < 0, i \neq j$, then $v_p^* = -\infty$ and (D) cannot be solved.

So, the duality gap is zero.

(c)

$\theta(w) = inf_{x \in R^n}\{x^T A x + \sum_{i=1}^n w_i(1 - x^2)\} = e^T w + inf_{x \in R^n} x^T (A - Diag(w))x$

consider, if exist $A_{ij} < 0, i \neq j$, then $inf_{x \in R^n} x^T (A - Diag(w))x = (A - Diag(w)) \cdot (xx^T)$ must be $-\infty$ and $\theta(w) = -\infty$

So the maximum target holds $A - Diag(w) \in S_+^n$ when $\theta(w) > -\infty$

Besides, when $A - Diag(w) \in S_+^n, inf_{x \in R^n} x^T (A - Diag(w))x = 0$, so $\theta(w) = e^T w$

So we can equivalent Problem(4) to:

$$\begin{array}{ll} sup & e^T w \\ \text{subject to} & Diag(w) + S = A \\ & w \in R^n, S \in S_+^n \end{array}$$

which is same as (D) in (b)

Problem 5

(a)

Let $f(x) = x_1^2 + 4x_2^2 + 16x_3^2$

It is easy to verify that $f(x)$ is a continuous function on R^3

let $x_1 = x_2 = x_3 = 1, f(x) = 21$, so $inf f(x) \leq 21, x_1 x_2 x_3 = 1$, so we can let $B = \{x | x_1^2, x_2^2, x_3^2 \leq 21\}$ as an add condition which will not effect the origin problem.

So that $\{x | x_1^2 + x_2^2 + x_3^2 = 1\} \cup B$ is a compact and f is continuous, $inf f$ must exists

(b)

By **KKT condition**:

$\nabla f(x) + w \nabla g(x) = (2x_1, 8x_2, 32x_3) + w(x_2 x_3, x_1 x_3, x_1 x_2) = \mathbf{0}$

there is only one constrain $g(x)$ so it's must be independent, so the KKT condition is necessary

(c)

$$
\begin{aligned}
2x_1 + wx_2x_3 = 0 \\
8x_2 + wx_1x_3 = 0 \\
32x_3 + wx_1x_2 = 0 \\
x_1x_2x_3 = 1
\end{aligned}
$$

we can get $x = (2, 1, 1/2)$ with $w = -8$ as the optimal solution set

Problem 6

The KKT condition of Problem (6):

$$\nabla tr(AX) + v\nabla tr((X - X_0)^2) = \mathbf{0}, v \geq 0 \quad (1)$$

with the complementary slackness:

$$v(tr((X - X_0)^2) - r^2) = 0 \quad (2)$$

we can solve equation (1) $\Leftrightarrow X = X_0 + \frac{A}{2v}$ so the $X_0 + \frac{A}{2v} \in S_+$ or the optimal may not exist

combine (1) and (2) :

$$tr((X_0 - \frac{A}{2v} - X_0)^2) = \frac{tr(A^2)}{4v^2} \leq r^2$$

$$\Rightarrow v \geq \frac{\sqrt{tr(A^2)}}{2r}$$

$$tr(AX) = tr(AX_0) + \frac{tr(A^2)}{2v}$$

we can find that there is a optimal solution when $v = \frac{\sqrt{tr(A^2)}}{2r}$ with $X^* = X_0 - \frac{rA}{\sqrt{tr(A^2)}}$