SEEM 5580 — Midterm Exam — **Name:**

9:30–11:15 am, Oct 17, 2016

This is a closed book, closed notes test. **CUHK student honor code applies to this test.** There are a total of 5 problems. For a matrix A with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ we have } A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \text{ and } \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

1. (25 points; 5 points each) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and $\{N(t) : t \geq 0\}$ models customer arrivals to a service center. Let ξ_i be the inter–arrival time between $(i-1)$–th and i–th customers, and let $T_n = \sum_{i=1}^n \xi_i$ be the arrival time of n–th customer where $T_0 := 0$. For each statement, say "true" or "false" and *briefly explain your answers.* Assume throughout that $t > 0$ is fixed but arbitrary.

   (a) $N(t)$ has a Poisson distribution with mean λ.

   (b) ξ_i has an exponential distribution with mean λ for each i.

   (c) $\xi_{N(t)+1}$ has an exponential distribution.

   (d) Let $N_1(s) = \sum_{k=1}^{\infty} 1_{T_{2k-1} \leq s}$. Then $\{N_1(s) : s \geq 0\}$ is a Renewal process.

   (e) Let $N_2(s) = \sum_{k=1}^{\infty} 1_{T_{2k} \leq s}$. Then $\{N_2(s) : s \geq 0\}$ is a Renewal process.

(a) False. $E[N(t)] = \lambda t$

(b) False. $E[\xi_i] = \frac{1}{\lambda}$

(c) False. Consider $y > t$

$$P(\xi_{N(t)+1} > y) = \sum_{n=0}^{\infty} P(\xi_{n+1} > y;\ N(t) = n) \qquad (\xi_{n+1} \text{ not independent of } \{N(t)=n\})$$

$$= \sum_{n=0}^{\infty} P(\xi_{n+1} > y;\ T_n \leq t < T_{n+1})$$

$$= \sum_{n=0}^{\infty} P(\xi_{n+1} > y;\ T_n \leq t;\ \xi_{n+1} > t - T_n)$$

$$= \sum_{n=0}^{\infty} P(\xi_{n+1} > y,\ T_n \leq t) \qquad (\text{since } y > t \geq t - T_n)$$

$$= \sum_{n=0}^{\infty} P(\xi_{n+1} > y) \cdot P(T_n \leq t) \qquad (\xi_{n+1} \text{ independent of } T_n)$$

$$= e^{-\lambda y} \cdot \sum_{n=0}^{\infty} P(T_n \leq t) = e^{-\lambda y} \cdot (1 + \lambda t) \rightarrow \text{non-exponential}$$

(d) False. first inter-arrival $\rightarrow \xi_1$

2nd inter-arrival $\rightarrow \xi_2 + \xi_3$

$>$ not same distribution

(e) True: inter-arrival times are $\xi_1 + \xi_2$; $\xi_3 + \xi_4$; $\xi_5 + \xi_6$; ---.

2. (20 points; 5 points each) Assume that passengers arrive at a subway station between 10am -12noon following a non-homogeneous Poisson process with rate function $\lambda(t)$ given as follows. From 10am-11am, the arrival rate is constant, 60 passengers per hour. From 11am to noon, it increases linearly to 120 passengers per hour.

   (a) What is the probability that there is at least 1 passenger arrival in the first 2 minutes (after 10am)?

   (b) What is the probability that there are 2 arrivals between 10:00am to 10:10am *and* 1 arrival between 10:05am to 10:15am?

   (c) What is the expected number of arrivals between 10:50am and 11:10am?

   (d) What is the probability that the 1st passenger after 11am will take at least 2 minutes to arrival?

Solution. Let $N(t)$ be the non-homogeneous Poisson process. If we use the time units of mins, then the arrival rate is

$$\lambda(t) = \begin{cases} 1 & 0 \le t \le 60mins \\ \frac{t}{60} & 60mins < t \le 120mins \end{cases}$$

(a) $P(N(2) \ge 1) = 1 - P(N(2) = 0) = 1 - e^{-2}$.

(b) The probability that 2 arrivals between 10:00am to 10:10am *and* 1 arrival between 10:05am to 10:15am is

$$\begin{aligned} & P(N(5) - N(0) = 2, N(10) - N(5) = 0, N(15) - N(10) = 1) \\ + \; & P(N(5) - N(0) = 1, N(10) - N(5) = 1, N(15) - N(10) = 0) \\ = \; & \frac{5^2}{2!}e^{-5} \times \frac{5^0}{0!}e^{-5} \times \frac{5^1}{1!}e^{-5} + \frac{5^1}{1!}e^{-5} \times \frac{5^1}{1!}e^{-5} \times \frac{5^0}{0!}e^{-5} \\ = \; & \frac{175}{2}e^{-15}. \end{aligned}$$

(c) Expected number of arrivals between 10:50am and 11:10am is equal to

$$\int_{50}^{70} \lambda(t)dt = \int_{50}^{60} 1dt + \int_{60}^{70} \frac{t}{60}dt = \frac{125}{6}.$$

(d) Probability that the 1st passenger after 11am will take at least 2 minutes to arrival:

$$P(N(11am, 11:02am) = 0) = e^{-\int_{60}^{62} \lambda(t)dt} = \cancel{e^{-\frac{13}{6}}}. = e^{-\frac{61}{30}}$$

3. (10 points) Let $\{\xi_i\}$ be a sequence of i.i.d random variables with distribution

$$\mathbb{P}(\xi_i = 0) = \mathbb{P}(\xi_i = 1) = .5, \quad \text{for } i = 1, 2, \ldots.$$

Let $Y_0 = 0$, $Y_1 = \xi_1$ and $Y_n = \xi_n + \xi_{n-1}$ for $n \geq 2$. Is $Y = \{Y_n : n = 0, 1, 2, \ldots, \}$ a discrete time Markov chain? If yes, work out the one-step transition probabilities; if not, prove your assertion.

Solution:

$Y = \{Y_n : n = 0, 1, 2, \ldots, \}$ doesn't satisfy the Markov property.

$$\begin{aligned}
\mathbb{P}\{Y_3 = 0 | Y_2 = 1, Y_1 = 0\} &= \mathbb{P}\{\xi_3 + \xi_2 = 0 | \xi_2 + \xi_1 = 1, \xi_1 = 0\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0 | \xi_2 = 1, \xi_1 = 0\} \\
&= 0
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{P}\{Y_3 = 0 | Y_2 = 1\} &= \mathbb{P}\{\xi_3 + \xi_2 = 0 | \xi_2 + \xi_1 = 1\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0, \xi_2 + \xi_1 = 1\} / \mathbb{P}\{\xi_2 + \xi_1 = 1\} \\
&= \mathbb{P}\{\xi_3 = 0, \xi_2 = 0, \xi_1 = 1\} / \mathbb{P}\{\xi_2 + \xi_1 = 1\} \\
&= (1/8)/(1/2) = 1/4
\end{aligned}$$

4. (15 points) Let $X = \{X_n : n = 0, 1, \ldots,\}$ be a DTMC on two states a and b. The transition matrix is given by

$$P = \begin{pmatrix} .5 & .5 \\ 1 & 0 \end{pmatrix}.$$

Let

$$f(x) = \begin{cases} \$10 & \text{if } x = a, \\ \$20 & \text{if } x = b. \end{cases}$$

(a) (7 points) Find $\mathbb{E}\Big[f(X_0) + 2f(X_1) + f(X_2)\Big| X_0 = a\Big]$.

(b) (8 points) Find $\mathbb{E}\Big[\sum_{n=0}^{\infty}(.5)^n f(X_n)\Big| X_0 = a\Big]$.

Solution.

$$\begin{aligned} \mathbb{E}_a\Big[f(X_0) + 2f(X_1) + f(X_2)\Big] &= f(a) + 2\mathbb{E}_a f(X_1) + \mathbb{E}_a f(X_2) \\ &= f(a) + 2(Pf)(a) + (P^2 f)(a) \\ &= 10 + 2 \times 15 + 12.5 \\ &= 52.5 \end{aligned}$$

Solution.

$$\begin{aligned} \mathbb{E}_a\Big[\sum_{n=0}^{\infty}(.5)^n f(X_n)\Big] &= \sum_{n=0}^{\infty}(.5)^n \mathbb{E}_a f(X_n) \\ &= \sum_{n=0}^{\infty}(.5)^n (P^n f)(a) \\ &= \Big((\sum_{n=0}^{\infty}(.5)^n P^n) f\Big)(a) \\ &= \Big((I - .5P)^{-1} f\Big)(a) \\ &= \begin{bmatrix} 1.6 & 0.4 \\ 0.8 & 1.2 \end{bmatrix} * \begin{bmatrix} 10 \\ 20 \end{bmatrix} (a) \\ &= 24 \end{aligned}$$

5. (30 points, 5 points each) Consider a discrete time Markov chain $\{X_n : n = 0, 1, \ldots\}$ with the state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For the first three parts, just circle True or False. No explanation needed. Answer the other three parts in the space below.

(1) This chain is irreducible. True or False?

(2) Every state of this chain has period 2. True or False?

(3) Every state of this chain is positive recurrent. True or False?

(4) Is there an unique stationary distribution of this chain? If yes, provide the stationary distribution; If not, explain why.

(5) Does $\lim_{n\to\infty} \mathbb{P}(X_n = 2 | X_0 = 3)$ exist? If yes, provide the limit; if not, explain why.

(6) Find $\mathbb{E}[\tau_3 | X_0 = 3]$, where $\tau_3 = \inf\{n \geq 1 : X_n = 3\}$. (You do not need to provide a proof.)

(4) $$\begin{cases} (\pi_1\ \pi_2\ \pi_3\ \pi_4) = (\pi_1\ \pi_2\ \pi_3\ \pi_4) \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$\Rightarrow \pi_1 = 0.4 \quad \pi_2 = 0.3 \quad \pi_3 = 0.2 \quad \pi_4 = 0.1$

yes, unique stationary distribution

(5) yes. $\lim_{n\to\infty} P(X_n = 2 | X_0 = 3) = \pi_2 = 0.3$

Finite DTMC. ergodic

(6) $E[T_3 | X_0 = 3] = \frac{1}{\pi_3} = \frac{1}{0.2} = 5$