# Homework 2 (draft)

## Problem 1

**(a)**

$\because A, B \in S_+^n$

$\therefore \exists U \in R^{k\times n}, k = rank(B), B = U^TU$

$\therefore tr(AB) = tr(AU^TU) = tr(UAU^T) \geq 0$ with assumption $tr(AB) = tr(BA)$

let us proof $tr(AB) = tr(BA), \forall A, B \in S^n$:

let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$

$tr(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji}$

$tr(BA) = \sum_{i=1}^n \sum_{j=1}^n b_{ij}a_{ji}$

we can rearrange the equation above and get $tr(AB) = tr(BA)$

**(b)**

First we proof $\mathcal{S}_+^n \subseteq \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\because$ the result in (a), we have $\mathcal{S}_+^n \subseteq \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\therefore \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\} \supseteq \bigcap_{A\in\mathcal{S}_+^n} S_+^n = S_+^n$

then we proof $\mathcal{S}_+^n \supseteq \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

we only have to proof : $\forall X \notin S_+^n, \exists A \in S_+^n, s.t. A \bullet X < 0$

$\because X \notin S_+^n$

$\therefore \exists \mu \in R^n, s.t. \mu^TX\mu < 0$

let $A = \mu\mu^T$ , we can get $tr(AX) = tr(\mu\mu^TX) = tr(\mu^TX\mu) < 0$

note that $A = \mu\mu^T$ because $\forall z \in R^n, z^TAz = z^T\mu\mu^Tz = (\sum z_iu_i)^2 >= 0$

$\therefore \forall X \notin S_+^n, X \notin \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\therefore \mathcal{S}_+^n \supseteq \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

In summary, $\mathcal{S}_+^n = \bigcap_{A\in\mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

## Problem 2

**(a)**

First let us proof $f$ possesses Property C $\Leftrightarrow L_t = \{x \in \mathbb{R}^n : f(x) \leq t\}$ is compact for any $t \in \mathbb{R}$

① **proof of** $\Rightarrow$:

first let us show $L_t$ is bounded, which means $\forall x \in L_t, \exists T \in \mathbb{R}, s.t. ||x||_2 < T$

Proof by contradiction:

assume $L_t$ is not bounded, so there exists a sequence $\{x^k\}_{k\geq0} \subseteq L_t, ||x^k||_2 \to +\infty$

$\because f$ possesses Property C

$\therefore f(x^k) \to +\infty$

However, with the limitation of $f(x) \leq t,$ we can find $\{x^k\}_{k\geq0} \subsetneq L_t$

Contradicting the assumption, thus $L_t$ is bounded then let us show $L_t$ is close, which means $\forall \{x^k\}_{k\geq 0} \subseteq L_t, \{x^k\} \to x, x \in L_t$

$\because$ $f$ is a continuous function

$\therefore \{f(x^k)\} \to f(x)$

$\because \{x^k\}_{k\geq 0} \subseteq L_t$

$\therefore \forall x^i \in \{x^k\}, f(x^i) \leq t$

$\therefore \{f(x^k)\} \to f(x) \leq t$

$\therefore x \in L_t, L_t$ is close

In summary, $L_t$ is compact

② **proof of** $\Leftarrow$:

Proof by contradiction:

assume $f$ not possesses Property C, which means $\exists \{x^k\}_{k\geq 0} \subseteq \mathbb{R}, ||x^k||_2 \to +\infty, f(x^k) \leq T, T \in \mathbb{R}$

let $t > T$, then $\{x^k\}_{k\geq 0} \subseteq L_t$. However $||x^k||_2 \to +\infty$, thus $L_t$ is not bounded.

Contradicting the assumption, thus $f$ possesses Property C

In summary, $f$ possesses Property C $\Leftrightarrow L_t = \{x \in \mathbb{R}^n : f(x) \leq t\}$ is compact for any $t \in \mathbb{R}$

Then let us proof $f$ is continuous and possesses Property C, $\inf_{x\in\mathbb{R}^n} f(x)$ always has an optimal solution.

$\inf_{x\in\mathbb{R}^n} f(x)$ always has an optimal solution, which means we can find a $x_0 \in \mathbb{R}^n s.t. \forall x \in R^n, f(x) \geq f(x_0)$

$\because$ $f$ is continuous and possesses Property C, thus we can find a infimum $c$, which is the max $c_i$ satisfy $f(x) \geq c_i$

then we only need to show that exist $x_0 \in \mathbb{R}^n, s.t. f(x_0) = c$

let $t = c + 1$ , $L_t$ is compact and $c$ is also infimum of $L_t$

with the definition of infimum and continuous, $\exists \{x^k\}_{k\geq 0} \subseteq L_t, \{f(x^k)\} \to c, \{x^k\} \to x_0$

$\because L_t$ is compact

$\therefore x_0 \in L_t \subseteq R^n$

Thus we can always find an optimal solution $x_0$

**(b)**

No.

assume $f(0) < +\infty$, let $A = 0^{m\times n}$, thus $\forall x \in \mathbb{R}^n, g(x) \equiv f(0) < \infty, g$ not possesses Property C.

## Problem 3

**(a)**

$\forall x, y \in K^\circ, \alpha, \beta \in R^+$, let $z = \alpha x + \beta y$

$\forall u \in K, z^T u = (\alpha x + \beta y)^T x = \alpha x^T u + \beta y^T u \leq 0$

$\therefore z \in K^\circ, K^\circ$ is a convex cone

**(b)**

First we know that, $z^* = \prod_K(x) \Leftrightarrow z^* \in K, (z - z^*)^T(x - z^*) \leq 0, \forall z \in K$

① **proof of** $\Rightarrow$:

With the property above, we have $z^* \in K$, then let us show that $(x - z^*)^T z^* = 0$

assume $(x - z^*)^T z^* \neq 0$, let $t = x - z^*, z^* = x - t$

$\because t^T z^* \neq 0$

$\therefore t = \alpha z^* + z$, where $\alpha \neq 0, z^* \neq 0, z^T z^* = 0$ and $t^T z^* = \alpha (z^*)^T z^*$

$\therefore ||x - z^*||_2^2 = ||x - (x-t)||_2^2 = ||t||_2^2 = ||\alpha z^*||_2^2 + ||z||_2^2 = |\alpha|||z^*||_2^2 + ||z||_2^2$

by the definition of closed convex cone and $z^* \neq 0, \alpha \neq 0$, we can find a $z^o = (1 - sign(\alpha)min(\alpha, 1))z^* \in K, ||x - z^o|| = (1 - sign(\alpha)min(\alpha, 1))|\alpha|||z^*||_2^2 + ||z||_2^2 < ||x - z^*||_2^2$

so that $z^* \neq \prod_K(x)$, contradicting the assumption, thus $(x - z^*)^T z^* = 0$

last let us shat that $x - z^* \in K^o$

$\because (x - z^*)^T z^* = 0$ and $(z - z^*)^T (x - z^*) \leq 0, \forall z \in K$

$\therefore z^T (x - z^*) = (x - z^*)^T z \leq 0, \forall z \in K$

by the definition of $K^o$, we get $x - z^* \in K^o$

② **proof of** $\Leftarrow$:

$\because x - z^* \in K^o$ and $(x - z^*)^T z^* = 0$

$\therefore (z - z^*)^T (x - z^*) \leq 0, \forall z \in K$

so that, we get $z^* \in K, (z - z^*)^T (x - z^*) \leq 0, \forall z \in K \Rightarrow z^* = \prod_K(x)$

In summary, $z^* = \prod_K(x) \Leftrightarrow z^* \in K, x - z^* \in K^o, (x - z^*)^T z^* = 0$

**(c)**

Using the result in (b), we only need to show that $z^* = \prod_K(x), x - z^* = \prod_{K^o}(x)$

what's more, we only need to show that $K = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K^o\}, K^o = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K\}, K \subseteq \mathbb{R}^n$ be a closed convex cone

$\because x^T \omega = \omega^T x$, so that $K = \{\omega \in \mathbb{R}^n : \omega^T x \leq 0, \forall x \in K^o\}$ is true obviously by definition

Using the result in (b):

$$x - z^* \in K^o, x - (x - z^*) = x \in K, (x - (x - z^*))^T (x - z^*) = x^T (x - z^*) = (x - z^*)^T x = 0 \Rightarrow x - z^* = \prod_{K^o}(x)$$

In summary, $x = \prod_K(x) + \prod_{K^o}(x)$

## Problem 4

**(a)**