Homework 2 (draft)

Problem 1

(a)

$\because A,B \in S^n_+ $

$\therefore \exists U \in R^{k\times n}, k = rank(B), B = U^TU$

$\therefore tr(AB) = tr(AU^TU) = tr(UAU^T) \geq 0$ with assumption $tr(AB) = tr(BA)$

(b)

First we proof $\mathcal{S}^n_+ \subseteq \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\because$ the result in (a), we have $\mathcal{S}^n_+ \subseteq \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\therefore \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\} \supseteq \bigcap_{A\in\mathcal{S}^n_+} S^n_+ = S^n_+$

then we proof $\mathcal{S}^n_+ \supseteq \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

we only have to proof : $\forall X \notin S^n_+, \exists A \in S^n_+, s.t. A \bullet X < 0$

$\because X \notin S^n_+$

$\therefore \exists \mu \in R^n, s.t. \mu^T X \mu < 0$

let $A = \mu\mu^T$, we can get $tr(AX) = tr(\mu\mu^T X) = tr(\mu^T X \mu) < 0$

note that $A = \mu\mu^T$ because $\forall z \in R^n, z^T Az = z^T \mu\mu^T z = (\sum z_i u_i)^2 >= 0$

$\therefore \forall X \notin S^n_+, X \notin \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

$\therefore \mathcal{S}^n_+ \supseteq \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$

In summary, $\mathcal{S}^n_+ = \bigcap_{A\in\mathcal{S}^n_+} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$