SEEM 5580: Homework 2

October 5, 2021

PROBLEM 2.4

Answer:
First, according to the properties of Poisson process, we have that $E[N(t)] = \lambda t, \mathrm{Var}[N(t)] = \lambda t$. Thus,

$$\begin{aligned}
E[N(t)\cdot N(t+s)] &= E[N(t)\cdot(N(t)+(N(t+s)-N(t))] \\
&= E\left[N(t)^2\right] + E[N(t)]E[N(t+s)-N(t)] \; \big(\text{by independence of } N(t) \text{ and } N(t+s)-N(t)\big) \\
&= (\lambda t)^2 + \lambda t + \lambda t \lambda s \\
&= \lambda t(1+\lambda t+\lambda s)
\end{aligned}$$

PROBLEM 2.5

Answer:
We show that $\{N_1(t)+N_2(t), t \geq 0\}$ is a Poisson process by definition. First, it can be easily verified that $N_1(0)+N_2(0) = 0$ since $\{N_1(t), t \geq 0\}$ and $\{N_2(t)\}, t \geq 0$ are Poisson processes. Second, due to the facts that $N_1(s)$ is independent of $N_1(t+s)-N_1(s)$, $N_2(s)$ is independent of $N_2(t+s)-N_2(s)$, and $\{N_1(t), t \geq 0\}$ and $\{N_2(t)\}, t \geq 0\}$ are independent processes, it follows that $N_1(s)+N_2(s)$ is independent of $N_1(t+s)+N_2(t+s)-(N_1(s)+N_2(s))$, which implies that $N_1(t)+N_2(t)$ has independent increments.

Third,

$$\begin{aligned}
P\{N_1(t+s)+N_2(t+s)-N_1(s)-N_2(s)=n\} &= \sum_{i=0}^{n} P\{N_1(t+s)-N_1(s)=i, N_2(t+s)-N_2(s)=n-i\} \\
&= \sum_{i=0}^{n} P\{N_1(t+s)-N_1(s)=i\}P\{N_2(t+s)-N_2(s)=n-i\} \\
&= \sum_{i=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^i}{i!}\frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-i}}{(n-i)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!}\sum_{i=0}^{n}\frac{(\lambda_1 t)^i(\lambda_2 t)^{n-i}n!}{i!(n-i)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!}(\lambda_1 t+\lambda_2 t)^n.
\end{aligned}$$

Thus, we can conclude that $\{N_1(t)+N_2(t), t\geq 0\}$ is a Poisson process with rate $\lambda_1+\lambda_2$.
Assume that the first event of the combined process occurred at time t, thus

$$\begin{aligned}
P\{\text{the event comes from } \{N_1(t), t\geq 0\} | N_1(t)+N_2(t)=1\} &= \frac{P\{N_1(t)=1, N_2(t)=0\}}{P\{N_1(t)+N_2(t)=1\}} \\
&= \frac{e^{-\lambda_1 t}\lambda_1 t e^{-\lambda_2 t}}{e^{-(\lambda_1+\lambda_2)t}(\lambda_1+\lambda_2)t} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

This completes the proof.

PROBLEM 2.11

Answer:
Let X_1 denote the interval time from when the person arrived at that location until the first car arrived, and $X_i, i=2,3\cdots$ denote the inter arrival time between the i-th car and the $(i-1)$-th car, and we define $X_0=0$. Since cars arrive according to a Poisson process with rate λ, then $X_i, i=1,2,\cdots$ are i.i.d exponential random variable with mean $1/\lambda$.
Let random variable N denote the number of car arrivals it takes until the interarrival time first larger than T from the time when the person arrived at that location. Note that N includes the car with interarrival time larger than T, and $N=1,2,\cdots$. If $N=n$, it implies that $X_i\leq T$ for $i=0,1,\cdots,n-1$, and $X_n>T$. Let $p=P\{X_n>T\}=e^{-\lambda T}$, thus, N follows a geometric distribution with parameter p, and $E[N]=1/p$.
Let Y denote the time the person waits before starting to cross. Then we have $Y=\sum_{i=0}^{N-1}X_i$.

Now we have

$$\begin{aligned}
E[\sum_{i=0}^{N-1} X_i|N=n] &= E[\sum_{i=0}^{n-1} X_i|X_i \le T, i=1,\cdots,n-1; X_n > T] \\
&= (n-1)E[X_i|X_i \le T] \text{ (follows from i.i.d of } X_i, i=1,\cdots,n) \\
&= \frac{n-1}{1-e^{-\lambda T}}\left[\frac{1}{\lambda} - (\frac{1}{\lambda}+T)e^{-\lambda T}\right]
\end{aligned}$$

Hence,

$$E[Y] = \frac{E[N]-1}{1-e^{-\lambda T}}\left[\frac{1}{\lambda} - (\frac{1}{\lambda}+T)e^{-\lambda T}\right] = \frac{e^{\lambda T}-1}{1-e^{-\lambda T}}\left[\frac{1}{\lambda} - (\frac{1}{\lambda}+T)e^{-\lambda T}\right] = \frac{e^{\lambda T}}{\lambda} - \frac{1}{\lambda} - T$$

PROBLEM 2.22

Answer:
Call an entering car a type-I car if it is still in the interval (a,b) by time t. Now, if the car enters at time $s, s \le t$, then it will be a type-I car if its velocity is such that $a < (t-s)V < b$, and since the velocity distribution of a car is F, the probability of this will be

$$F\left(\frac{b}{t-s}\right) - F\left(\frac{a}{t-s}\right).$$

Hence, we obtain that the distribution of the number of cars located in the interval (a,b) at time t is Poisson with mean

$$\lambda\int_0^t \left(F\left(\frac{b}{t-s}\right) - F\left(\frac{a}{t-s}\right)\right) ds.$$

PROBLEM 2.30

Answer:
(a) We first show that

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-m(t)},$$

where

$$m(t) = \int_0^t \lambda(s) ds.$$

Hence, the probability density function of T_1 is

$$f_{T_1}(t) = \lambda(t)e^{-m(t)}.$$

Then, we compute the distribution of T_2 condition on T_1. This gives

$$\begin{aligned}
P\{T_2 > t|T_1 = s\} &= P\{0 \text{ events in } (s,s+t]|T_1 = s\} \\
&= P\{0 \text{ events in } (s,s+t]\} \quad \text{(by independent increments)} \\
&= e^{-[m(s+t)-m(s)]}.
\end{aligned}$$

From above it can be seen that the distribution of T_2 is not independent of the value of T_1. Thus, T_i are not independent.
(b)

$$\begin{aligned} P\{T_2 > t\} &= \int_0^\infty P\{T_2 > t | T_1 = s\} f_{T_1}(s) ds \\ &= \int_0^\infty e^{-[m(s+t)-m(s)]} \lambda(s) e^{-m(s)} ds \\ &= \int_0^\infty \lambda(s) e^{-m(s+t)} ds. \end{aligned}$$

Thus, the probability density function of T_2 is

$$f_{T_2}(t) = \int_0^\infty \lambda(s)\lambda(s+t) e^{-m(s+t)} ds.$$

Hence, it follows that T_i are not identically distributed.
(c) The distribution of T_1 is derived in part (a).
(d) The distribution of T_2 is derived in part (b).

PROBLEM 2.31

Answer:
We show that $\{N^*(t), t \geq 0\}$ is a Poisson process by definition. First, it can be easily verified that $N^*(0) = 0$ since $N(m^{-1}(0)) = N(0) = 0$.
Second, due to the facts that $N(t)$ has independent increments and $m(t)$ is an increasing function, one can also readily verify that $N^*(t)$ has independent increments.
Third,

$$\begin{aligned} P\{N^*(t+s) - N^*(s) = n\} &= P\{N(m^{-1}(t+s)) - N(m^{-1}(s)) = n\} \\ &= exp\Big(-\int_{m^{-1}(s)}^{m^{-1}(t+s)} \lambda(u) du\Big) \frac{\Big(\int_{m^{-1}(s)}^{m^{-1}(t+s)} \lambda(u) du\Big)^n}{n!} \\ &= e^{-t} t^n / n! \end{aligned}$$

Thus, we can conclude that $\{N^*(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 1$.

PROBLEM 2.41

Answer:
See answers at the back of the textbook.