Introduction

Monday, September 6, 2021 12:31 PM

infimum "minimize"

(P) $\quad v^* = \inf_{x \in X} f(x)$

- $f : \mathbb{R}^n \to \mathbb{R}$ objective function
- $X \subseteq \mathbb{R}^n$ feasible region; $x \in \mathbb{R}^n$: decision variable
- v^* : optimal value of (P)

(P)
- infeasible: $X = \emptyset$, $v^* = +\infty$ by convention
- feasible $X \neq \emptyset$
  - $v^* = -\infty$ $\quad$ e.g. $n=1$ $\quad \inf\ x$ s.t. $x \leq 0$
  - $v^* > -\infty$

Def: optimal solution

$x^* \in X$ s.t. $v^* = f(x^*)$

- opt. soln. exists

  e.g. $n=1$

  $\inf\ x$ s.t. $x \geq 0$

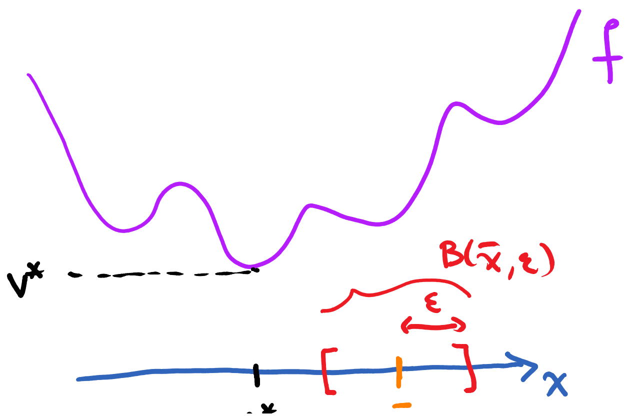
  $v^* = 0$, attained by $x^* = 0$

- opt. soln. does not exist

  e.g. $n=1$

  $\inf\ 1/x$ s.t. $x \geq 0$

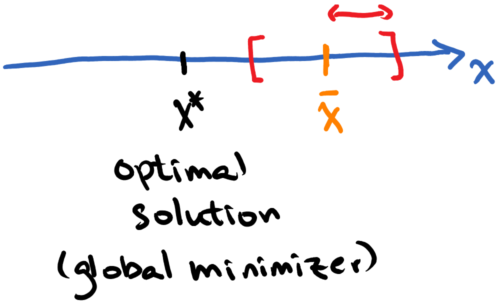
  $v^* = 0$, $\nexists x^* \geq 0$ s.t. $1/x^* = 0$.



Def: $\bar{x}$ is a local minimizer

if $\bar{x} \in X$ and $\exists \varepsilon > 0$ s.t.

$\forall x \in X \cap B(\bar{x}, \varepsilon)$, we have



$\forall x \in X \cap \underbrace{B(\bar{x},\varepsilon)}_{\text{neighborhood of } \bar{x}}$, we have

$f(\bar{x}) \le f(x)$

Here,

$B(\bar{x},\varepsilon) = \{ x \in \mathbb{R}^n : \|x - \bar{x}\|_2 \le \varepsilon \}$

center radius $\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$

Simple Examples of (P)

(1) Unconstrained : $X = \mathbb{R}^n$ $\quad \inf_{x \in \mathbb{R}^n} f(x)$

If f is differentiable, then $\nabla f(x) = 0$ is a necessary condition for optimality.

gradient of f: $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$

(2) Discrete : X is a discrete set; i.e.,

$\forall x \in X$, $\exists \varepsilon > 0$ s.t. $X \cap B(x,\varepsilon) = \{x\}$.

e.g. $X = \{0, 1, 2, 3\}$ is discrete

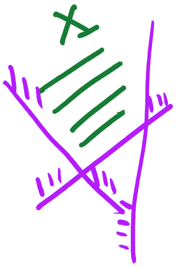
$X = [0,1]$ is not discrete

Note: For discrete optimization problems, local optimality is meaningless! This is because every feasible solution is a local minimizer.

(3) Linear Programming (LP)

$f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c^T x$ (linear function)

$c = (c_1, \cdots, c_n)$ }
$x = (x_1, \cdots, x_n)$ } Notation: column vectors



X: defined by a finite number of linear inequalities

$$X = \{ x \in \mathbb{R}^n : a_i^T x \leq b_i \quad i = 1, \cdots, m \} \qquad a_i \in \mathbb{R}^n, \; b_i \in \mathbb{R}$$

$$= \{ x \in \mathbb{R}^n : \underset{\uparrow \text{ componentwise}}{Ax \leq b} \} \qquad \underset{m \times n}{A} = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix}, \quad \underset{m}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Note: What if we want $a_i^T x = b_i$?

Simply consider

$$a_i^T x = b_i \iff \begin{cases} a_i^T x \leq b_i \\ -a_i^T x \leq -b_i \end{cases}$$