Homework 3

Problem 1

(a)

By the definition of the subdifferential:

$$\begin{aligned}\partial \mathbb{I}_C(x) &= \{s \in \mathbb{R}^m : \mathbb{I}_C(y) - \mathbb{I}_C(x) \geq s^T(y-x), \forall y \in C\} \\ &= \{s \in \mathbb{R}^m : s^T(y-x) \leq 0, \forall y \in C\}\end{aligned}$$

(b)

By the result in (a), $\partial \mathbb{I}_{R_+^n}(x) = \{s \in \mathbb{R}^m : s^T(y-x) \leq 0, \forall y \in \mathbb{R}_+^n\}$

First, let us show that $\partial \mathbb{I}_{R_+^n}(x) \subseteq \left\{s \in \mathbb{R}^n : s \leq \mathbf{0}, x^T s = 0\right\}$

let $I_k = \{0, 0, \ldots, 1, 0, \ldots\} \in \mathbb{R}^n$ who have 1 at k-th dim and other dims are 0

$\therefore \forall I_k, x + I_k \geq 0$ and $s^T(x + I_k - x) = s^T I_k = s_k I_k \leq 0$

$\therefore s \leq \mathbf{0}$

assume $s^T x \neq 0$

$\therefore s^T x > 0$, when let $y = 0$

let $y = 2x \geq \mathbf{0}, s^T(y-x) = s^T x > 0$

By contradiction, $s^T x = 0$

Second, let us show that $\partial \mathbb{I}_{R_+^n}(x) \supseteq \left\{s \in \mathbb{R}^n : s \leq \mathbf{0}, x^T s = 0\right\}$

$\because s \leq \mathbf{0}, x^T s = 0$

$\therefore s^T(y-x) = s^T y \leq 0, \forall y \geq \mathbf{0}$

In conclusion, $\partial \mathbb{I}_{R_+^n}(x) = \left\{s \in \mathbb{R}^n : s \leq \mathbf{0}, x^T s = 0\right\}$

Problem 2

(a)

By the property of $||\cdot||_2$

$\forall p_1, p_2 \in \mathbb{R}^2, \alpha \in [0, 1]$,

$$\begin{aligned}||\alpha p_1 + (1-\alpha)p_2||_2 &\leq \alpha||p_1||_2 + (1-\alpha)||p_2||_2 \\ &\leq max(||p_1||_2, ||p_2||_2)\end{aligned}$$ (equal iff $\alpha = 0$ or 1)

assume $||p_1||_2 \leq ||p_2||_2$:

① $||p_2||_2 < 1$:

$f(\alpha p_1 + (1-\alpha)p_2) \leq \alpha f(p_1) + (1-\alpha) f(p_2) = 0$

② $||p_1||_2 < 1, ||p_2||_2 = 1$:

$f(\alpha p_1 + (1-\alpha)p_2) \leq \alpha f(p_1) + (1-\alpha) f(p_2) = (1-\alpha) f(p_2)$ (equal iff $\alpha = 0$ or 1)

③ $||p_1||_2 = ||p_2||_2 = 1$:

$f(\alpha p_1 + (1-\alpha)p_2) \leq \alpha f(p_1) + (1-\alpha) f(p_2)$ (equal iff $\alpha = 0$ or 1)

④ $||p_2||_2 > 1$:

$f(p_2) = +\infty$

$f(\alpha p_1 + (1-\alpha)p_2) \leq \alpha f(p_1) + (1-\alpha) f(p_2)$ (equal iff $\alpha = 0$ or 1)

In conclusion, f is convex.

(b)

let $f(1,0) = 1$ and $f(x_1, x_2) = 0, ||(x_1, x_2)||_2 = 1, (x_1, x_2) \neq (1,0)$

It's obviously that, $(1,0,1) \in epi(f)$ and is **isolated**

so $epi(f)$ is not closed.

Problem 3

$\forall x_1, x_2 \in \mathbb{R}, \alpha \in [0,1]$,

assume $x_1 \leq x_2$:

① $x_1, x_2 < -1$

$$\begin{aligned} f(\alpha p_1 + (1-\alpha)p_2) &= \frac{\alpha^2 x_1^2}{2} + \alpha(1-\alpha)x_1x_2 + \frac{(1-\alpha)^2 x_2^2}{2} - |\alpha p_1 + (1-\alpha)p_2| \\ &= \frac{\alpha^2 x_1^2}{2} + \alpha(1-\alpha)x_1x_2 + \frac{(1-\alpha)^2 x_2^2}{2} + \alpha p_1 + (1-\alpha)p_2) \\ &\leq \frac{\alpha x_1^2}{2} + \frac{(1-\alpha)x_2^2}{2} + \alpha p_1 + (1-\alpha)p_2) \\ &= \alpha f(x_1) + (1-\alpha) f(x_2) \end{aligned}$$

② $x_1 < -1, |x_2| \leq 1$

1' $\alpha p_1 + (1-\alpha)p_2 < -1$:

$$\begin{aligned} f(\alpha p_1 + (1-\alpha)p_2) &\leq \frac{\alpha x_1^2}{2} + \frac{(1-\alpha)x_2^2}{2} - |\alpha p_1 + (1-\alpha)p_2| \\ &\leq \frac{\alpha x_1^2}{2} + \frac{(1-\alpha)}{8} - |\alpha p_1| \\ &= \alpha f(x_1) + (1-\alpha) f(x_2) \end{aligned}$$

2' $\alpha p_1 + (1-\alpha)p_2 \geq -1$:

$f(\alpha p_1 + (1-\alpha)p_2) = \frac{1}{2}$ and $f(p_1), f(p_2) \geq \frac{1}{2}$

$\therefore f(\alpha p_1 + (1-\alpha)p_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

③ $x_1 < -1, x_2 > 1$

1' $|\alpha p_1 + (1-\alpha)p_2| > 1$

same as ①

2' $|\alpha p_1 + (1-\alpha)p_2| \leq 1$

same as ②-2'

④ $|x_1|, |x_2| \leq 1$

same as ②-2'

⑤ $|x_1| \leq 1, x_2 > 1$

same as ②

⑥ $x_1, x_2 > 1$

same as ①

Problem 4

Yes.

By the problem, we have constraints $\forall i = 1, \dots, n, x_i \geq 0$ or $x \leq 0$ which corresponding to n vectors $\{a_i\}_{i=1}^n$

let $I_k = \{0, 0, \dots, 1, 0, \dots\} \in \mathbb{R}^n$ who have 1 at k-th dim and other dims are 0

$a_i = I_k$ if $x_i \geq 0$ and $a_i = -I_k$ if $x_i \leq 0$

It's obvious that $\{a_i\}_{i=1}^n$ is linearly independent and $\{a_i\}_{i=1}^n \subseteq P$

By *Handout 3, Theorem 2(3)*, we get that P has at least one vertex

Problem 5

(a)

① assume both have solution:

$\exists x \in R^n, y \in R^m, s.t.$

$\mathbf{0} \leq (y^T A)x = y^T(Ax) \leq \mathbf{0} \Rightarrow y = \mathbf{0}$

Contradiction that $y \neq \mathbf{0}$

② exactly one system has a solution:

$Ax < \mathbf{0} \Leftrightarrow Ax \leq e$

which equivalent to $\widetilde{A}z = e, z \geq \mathbf{0}$, where $\widetilde{A} = [A, I], z = (x, s)$

By Farkas' lemma, if the system $\widetilde{A}z = e, z \geq \mathbf{0}$ has no solution, then there exists a $y \in \mathbb{R}^m s.t. \widetilde{A}^T y \leq \mathbf{0}$ and $e^T y > 0$.

From the definition of $\widetilde{A}$, we see that $A^T y \geq \mathbf{0}, y \geq \mathbf{0}$

Moreover, since $e^T y > 0$, we conclude that $y \neq 0$.

Thus (I) has solution (II) has no solution

(b)

① assume both have solution:

$\exists x \in R^n, y \in R^m, s.t.$

$\mathbf{0} = (y^T A)x = y^T(Ax) > \mathbf{0}$

Contradiction

② exactly one system has a solution:

$\because Ax \geq 0, Ax \neq 0$ is equivalent to $Ax \geq 0, b^T Ax = 1, b > \mathbf{0}$

let $\widetilde{A} = \begin{pmatrix} A & -A & -I \\ b^T A & -b^T A & 0 \end{pmatrix}$, $z = (x^+, x^-, s)$

(I) can be written as $\widetilde{A}z = e = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}, z \geq 0$

By Farkas' lemma, if the system $\widetilde{A}z = e, z \geq \mathbf{0}$ has no solution, then there exists a $y \in \mathbb{R}^m s.t. \widetilde{A}^T(y', \alpha) \leq \mathbf{0}$ and $e^T(y', \alpha) > 0$.

Let $y = y' + \alpha \cdot b, y' \geq 0, \alpha > \mathbf{0}, b > \mathbf{0}$

so we can get $A^T y = 0, y > \mathbf{0}$

Thus (I) has solution (II) has no solution