Convex Optimal control under safety constraints

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Abstract

- Optimal reach-safe control arises often in applications
- Introduced the notion of occupation measure as the time accumulation of the system on each state
- Optimal control for control-affine dynamics can be formulated as a convex optimization
- Safety and stability constraints can be introduced in a straightforward manner
  - Use penalty function method to deal with the equality constraints
  - Use SOS to relax the problem into an SDP

Preliminaries

For dynamical system $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$ with solution $\mathbf{s}_t(\mathbf{x})$, and arbitrary functions φ, ψ

Koopman (Composition) Operator $[\mathbb{K}_t\varphi](\mathbf{x})$

$$[\mathbb{K}_t\varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x}))$$

Perron-Frobenius (Transfer) Operator $[\mathbb{P}_t\psi](\mathbf{x})$

$$\int_{\mathbf{s}_{-t}(A)} \psi(\mathbf{x})d\mathbf{x} = \int_A [\mathbb{P}_t\psi](\mathbf{x})d\mathbf{x}, \quad \forall A \subset \mathbf{X}$$

Duality $< [\mathbb{K}_t\varphi](\mathbf{x}), \psi(\mathbf{x}) >=< [\mathbb{P}_t\psi](\mathbf{x}), \varphi(\mathbf{x}) >$

Occupancy measure: time accumulation of density

$$\rho(\mathbf{x}) \triangleq \int_0^\infty [\mathbb{P}_t h_0](\mathbf{x})dt$$

Convex formulation

Optimal control problem with safety constraints

- Formulation in **primal** space

$$\inf_{\rho,\mathbf{u}} \int_{\mathbf{X}}\int_0^\infty [\mathbb{K}_t l(\mathbf{x}, u(\mathbf{x}))](\mathbf{x})dt h_0(\mathbf{x})d\mathbf{x}$$
$$\text{s.t.} \qquad \int_0^\infty\int_{\mathbf{X}} [\mathbb{K}_t \mathbb{1}_{\mathbf{X}_u}](\mathbf{x})h_0(\mathbf{x})d\mathbf{x}dt$$

- Formulation in **dual** space and convexify the problem by change of variables $\rho\mathbf{u} \triangleq \bar{\boldsymbol{\rho}}$

$$\inf_{\rho,\bar{\boldsymbol{\rho}}} \int_{\mathbf{X}} \left(c(\mathbf{x}, \mathbf{u}) + \bar{\lambda}\mathbb{1}_{\mathbf{X}_u}(\mathbf{x})\right) \rho(\mathbf{x})\; d\mathbf{x}$$
$$\text{s.t.} \quad \nabla\cdot(\mathbf{f}\rho(\mathbf{x}) + \mathbf{g}\bar{\boldsymbol{\rho}}(\mathbf{x})) = h_0(\mathbf{x})$$

Two types of **cost functions** are considered

- L2 regularized: $c(\mathbf{x}) = q(\mathbf{x}) + \mathbf{u}^T\mathbf{R}\mathbf{u}$
- L1 regularized: $c(\mathbf{x}) = q(\mathbf{x}) + \beta\|\mathbf{u}\|_1$

Rational Polynomial parameterization [2]

$$\rho = \frac{a(\mathbf{x})}{b(\mathbf{x})^\alpha}, \quad \bar{\boldsymbol{\rho}} = \frac{\mathbf{c}(\mathbf{x})}{b(\mathbf{x})^\alpha}$$

- Relax the equality constraint to non-negative inequality

Sum-of-Squares (SOS) solution [1]

- Polynomial $p(\mathbf{x}) \geq 0$ if $p(\mathbf{x}) = \boldsymbol{\Psi}_d^T\mathbf{Q}\boldsymbol{\Psi}_d$ for monomial basis $\boldsymbol{\Psi}_d$ and some $\mathbf{Q} \succeq 0$
- The non-negative polynomial constraint becomes **LMI(Linear Matrix Inequality)** and linear equality
- The objective becomes **linear** in polynomial coefficient
- The whole problem becomes a **SDP**

Operator Approximation

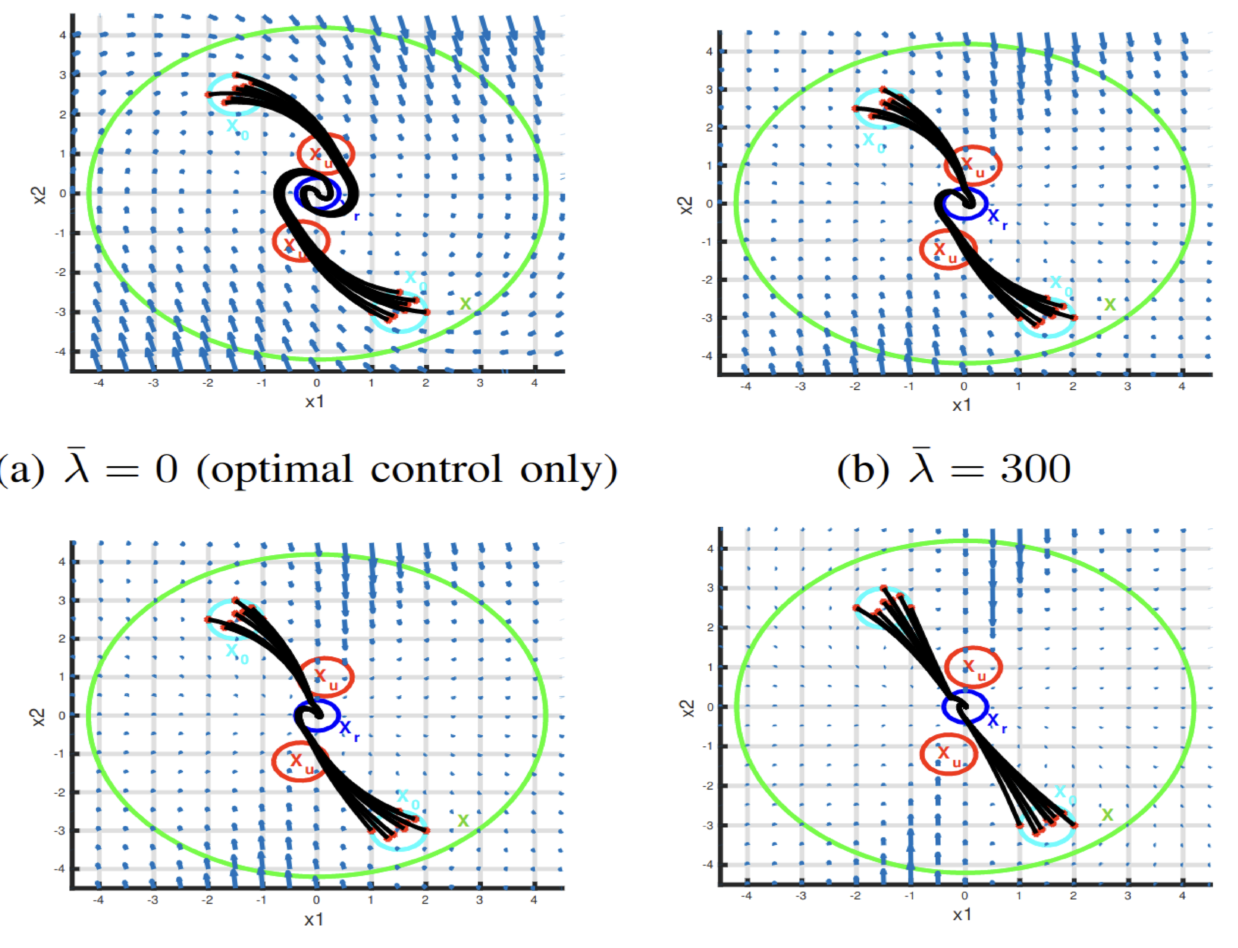
- Collect data $(\mathbf{x}_k, \mathbf{u}_k, \dot{\mathbf{x}}_k)$ and define
  $\dot{\boldsymbol{\Psi}}_d(\mathbf{x}_k, \dot{\mathbf{x}}_k) = [\nabla\psi_1(\mathbf{x}_k)\cdot\dot{\mathbf{x}}_k, \ldots, \nabla\psi_N(\mathbf{x}_k)\cdot\dot{\mathbf{x}}_k]^T$
- Solve least-squares problem

$$\min_{\mathbf{L}_0,\ldots,\mathbf{L}_m} \sum_{k=1}^M \left\| \dot{\boldsymbol{\Psi}}_d(\mathbf{x}_k, \dot{\mathbf{x}}_k) - [\mathbf{L}_0, \ldots, \mathbf{L}_m] \begin{bmatrix} \boldsymbol{\Psi}_d(\mathbf{x}_k) \\ \boldsymbol{\Psi}_d(\mathbf{x}_k)\mathbf{u}_k^1 \\ \cdots \\ \boldsymbol{\Psi}_d(\mathbf{x}_k)\mathbf{u}_k^m \end{bmatrix} \right\|_2^2$$

Simulations

Van Der Pol system: $\dot{x}_1 = x_2, \; \dot{x}_2 = (1 - x_0^2)x_2 - x_1 + u$

- Iteratively increase the penalty $\bar{\lambda}$



(a) $\bar{\lambda} = 0$ (optimal control only) (b) $\bar{\lambda} = 300$

(c) $\bar{\lambda} = 1000$ (d) $\bar{\lambda} = 10^4$

References

[1] Pablo A Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* California Institute of Technology, 2000.

[2] Stephen Prajna, Pablo A Parrilo, and Anders Rantzer. "Nonlinear control synthesis by convex optimization". In: *IEEE Transactions on Automatic Control* 49.2 (2004), pp. 310–314.