Data-driven optimal control under safety constraints using sparse Koopman approximation

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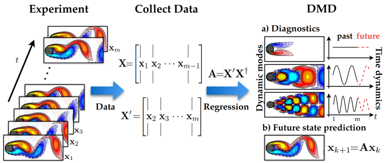
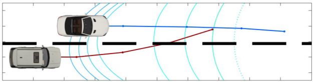
Motivation

- Koopman operator in system identification



- Koopman operator based dual optimal reach-safe control



1: Kutz, J. Nathan, et al. Dynamic mode decomposition: data-driven modeling of complex systems. Society for Industrial and Applied Mathematics, 2016.

2: B. Landry, M. Chen, S. Hemley and M. Pavone "Reach-Avoid Problems via Sum-of-Squares Optimization and Dynamic Programming", ArXiv 2018

Koopman and Perron-Frobenius Operator

- Koopman (composition) operator

$$[\mathbb{K}_t\varphi](x) = \varphi(s_t(x)), \quad \forall\varphi$$

- Perron-Frobenius (P-F, transfer) operator

$$\int_{s_{-t}(A)} \psi(x)dx = \int_A \mathbb{P}_t[\psi](x)dx, \quad \forall A \subset x, \quad \forall\psi$$

- Duality: $\forall\varphi, \psi$:

$$\langle[\mathbb{K}_t[\varphi](x), \psi(x)\rangle = \langle\mathbb{P}_t[\psi](x), \varphi(x)\rangle$$

$s_t(x)$ denotes the solution of $\dot{x} = f(x, u)$ starting from x

Koopman operator

- Definition

$$[\mathbb{K}_t\varphi](x) = \varphi(\mathbf{s}_t(x)), \quad \forall\varphi$$

- Physical meaning

Composite transitional operator in lifted space

- Finite dimension approximation

$$\mathrm{Basis}:\Psi(\mathrm{x}) \triangleq [\psi_1(\mathrm{x}), \ldots, \psi_\mathrm{N}(\mathrm{x})]^\mathrm{T}.$$

- Data $(x, y), y = f(x)$

$$\Psi_x = [\Psi(x_1)\ldots\Psi(x_M)], \Psi_y = [\Psi(y_1)\ldots\Psi(y_M)]$$

- Dynamic Mode Decomposition (DMD): $K \sim \mathbb{K}_t$

$$\min_K \|\Psi_y - K\Psi_x\|_F^2$$

Sparsity

- Arrangement of the basis (grid).
  2D case:
- Koopman transition operator
- Assumption: small sampling time
  $\mathrm{Data}\ (\mathrm{x}, \mathrm{y}), \mathrm{y}\ \mathrm{near}\ \mathrm{x}$
- Result

$$\Psi_y \approx K\Psi_x, K \text{ sparse } (\textbf{banded})$$

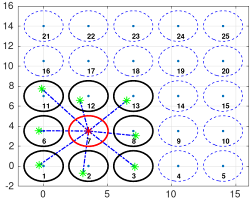


Figure 1: Basis function layout

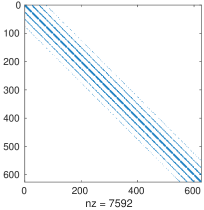


Figure 2: A typical sparsity pattern of K

Sparse Least Squares

- DMD: LS problem

$$\min_{K} \|\Psi_y - K\Psi_x\|_F^2 \,, K \text{ Sparse}$$

- Full matrix solution

$$K^\star = \left(\frac{1}{M}\sum_{k=1}^{M} \Psi(y_k)\Psi(x_k)^T\right)\left(\frac{1}{M}\sum_{k=1}^{M} \Psi(x_k)\Psi(x_k)^T\right)^\dagger$$

- Issue: computation efficiency
- Proposed: Formulating a sparse LS problem using only the nonzero elements

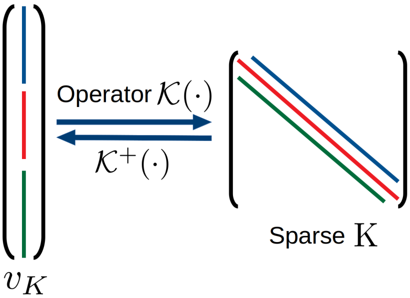
Sparse matrix construction

- Operator $\mathcal{K}(\cdot)$, dual operator $\mathcal{K}^{+}(\cdot)$

$$\langle \mathcal{K}_v, L \rangle = \langle v, \mathcal{K}_L^{+} \rangle, \forall L$$

- Derivations

$$\langle \mathcal{K}_v, L \rangle = \sum_{i,j,K_{i,j} \neq 0} K_{i,j} L_{i,j}$$
$$= \sum_{i=1}^{z} (v_K)_i (v_L)_i = \langle v, v_L \rangle$$



- Dual operator is the inverse operator!

Sparse LS reformulation

- LS problem in full vector $v, K = \mathcal{K}_v$

$$\begin{aligned} J_E(v) &= \|\mathcal{K}_v \Psi_x - \Psi_y\|_F^2 \\ &= \langle \mathcal{K}_v \Psi_x, \mathcal{K}_v \Psi_x \rangle - 2\langle \mathcal{K}_v \Psi_x, \Psi_y \rangle + \langle \Psi_y, \Psi_y \rangle \\ &= \mathrm{Tr}[\mathcal{K}_v^T \mathcal{K}_v \Psi_x \Psi_x^T] - 2\mathrm{Tr}[\mathcal{K}_v \Psi_x \Psi_y^T] \end{aligned}$$

- Compute the first order Taylor approximation

$$\begin{aligned} & J_E(v + \delta v) \\ &= \mathrm{Tr}[\mathcal{K}_{v+\delta v}^T \mathcal{K}_{v+\delta v} \Psi_x \Psi_x^T] - 2\mathrm{Tr}[\mathcal{K}_{v+\delta v} \Psi_x \Psi_y^T] \\ &= \dots \\ &\approx J_E(v) + \langle \mathcal{K}_{\delta v}, 2(\mathcal{K}_v \Psi_x \Psi_x^T - \Psi_y \Psi_x^T) \rangle \\ &= J_E(v) + \langle \delta v, \mathbf{2}(\boldsymbol{\mathcal{K}}^+(\boldsymbol{\mathcal{K}}_\mathbf{v} \boldsymbol{\Psi}_\mathbf{x} \boldsymbol{\Psi}_\mathbf{x}^\mathbf{T}) - \boldsymbol{\mathcal{K}}^+(\boldsymbol{\Psi}_\mathbf{y} \boldsymbol{\Psi}_\mathbf{x}^\mathbf{T})) \rangle \\ &= J_E(v) + \langle \delta v, \frac{\partial \mathbf{J_E}}{\partial \mathbf{v}} \rangle \end{aligned}$$

Sparse LS reformulation

- First-order approximation

$$\frac{\partial \mathbf{J_E}}{\partial \mathbf{v}} = \mathbf{2}(\mathcal{K}^+(\mathcal{K}_\mathbf{v}\mathbf{\Psi_x}\mathbf{\Psi_x^T}) - \mathcal{K}^+(\mathbf{\Psi_y}\mathbf{\Psi_x^T}))$$

- Minimizer v^* solves

$$\mathcal{K}^+(\mathcal{K}_{v^*}\Psi_x\Psi_x^T) - \mathcal{K}^+(\Psi_y\Psi_x^T) = 0 \tag{1}$$

- Linear function in v, seeks to solve instead

$$Sv = \mathcal{K}^+(\Psi_y\Psi_x^T)$$

- Construct matrix S using (1)

$$S_i = \mathcal{K}^+(\mathcal{K}_{e_i}\Psi_x\Psi_x^T)$$

Dual formulation: Koopman operator

- Koopman (composition) operator $\mathbb{K}_t$

$$[\mathbb{K}_t y](x) = y(x(t)), \quad \forall y$$

- Example equality

$$\begin{aligned}
&\mathbb{E}_{x_0}[q(x(t))] \\
&= \int_x q(x(t)) h_0(x) dx = \int_x [\mathbb{K}_t q] h_0 dx = \langle [\mathbb{K}_t q], h_0 \rangle
\end{aligned}$$

- Duality equality

$$\mathbb{E}_{x_0}[q(x(t))] = \langle q, [\mathbb{P}_t h_0] \rangle \triangleq \langle q, h_t \rangle$$

Dual optimal control using operator theory

- Optimal safe-reach control

$$\inf_{u(\cdot)} \int_X \int_0^\infty q(x_t) + \|u(x_t)\|_1 dt h_0(x_0) dx_0$$

$$\text{s.t.} \quad \dot{x} = f(x) + g(x)u(x), \int_0^\infty \mathbb{1}_{X_u}(\mathbf{s}_t(x_0))dt = 0, \ \forall x_0 \in X_0$$

- Dual formulation [1]

$$\inf_{\rho, u} \int_X (q(x) + \|u(x)\|_1)\rho dx$$

$$\text{s.t.} \quad \nabla \cdot [(f + gu)\rho](x) = h_0(x), \int_X \mathbb{1}_{X_u}(x)\rho(x)dx = 0$$

- $\rho(x)$: occupation measure

$$\rho(x) \triangleq \int_0^\infty [\mathbb{P}_t h_0](x)dt$$

[1] H. Yu, J. Moyalan, U. Vaidya and Y. Chen, "Data-Driven Optimal Control of Nonlinear Dynamics Under Safety Constraints," in IEEE Control Systems Letters, vol. 6, pp. 2240-2245, 2022, doi: 10.1109/LCSYS.2022.3140652.

Dual optimal control: convex formulation

- Bi-linear constraint in (ρ, u): $\nabla \cdot (\rho(f + gu)) = h_0$
- Define $\bar{\rho} = \rho u$, then convex in $(\rho, \bar{\rho})$:

$$\nabla \cdot (f\rho + g\bar{\rho}) = h_0 \geq 0$$

- Approximation using P-F operator

$$\begin{aligned}\nabla \cdot (f\rho + g\bar{\rho}) &= \nabla \cdot (f\rho) + \sum_{i=1}^{m} \nabla \cdot (g_i \rho u_i) \\ &= -\mathcal{P}_f \rho - \sum_{i=1}^{m} \mathcal{P}_{g_i} \bar{\rho}_i \\ &\approx \Psi^T \left(\frac{I - P_0^T}{\Delta t} v\right) + \Psi^T \left(\frac{I - (P_i - P_0)^T}{\Delta t} w_i\right)\end{aligned}$$

Linear Programming Formulation

- LP formulation of the optimal reach-safe problem

$$
\begin{aligned}
\inf_{C_\rho, w_1,\ldots,w_m, s_1,\ldots,s_m} \quad & I_q^T C_\rho + \sum_{i=1}^{m} I_\Psi^T s_i + I_m^T C_\rho \\
\text{s.t.} \quad & \frac{I - P_0^T}{\Delta t} C_\rho + \sum_{i=1}^{m} \frac{I - (P_i - P_0)^T}{\Delta t} w_i = C_h \\
& I_{X_u}^T C_\rho = 0 \\
& s_i \geq |w_i|, i = 1, \ldots, m \\
& |w_i| \leq u_{\max} C_\rho, i = 1, \ldots, m
\end{aligned}
\tag{3}
$$

where $\rho = C_\rho^T \Psi, u_i = w_i^T \Psi, h = C_h^T \Psi$, and

$$
I_q = \int_X q \cdot \Psi, \quad I_\Psi = \int_X \Psi, \quad I_{X_u} = \int_X \Psi \cdot \mathbb{1}_{X_u} \tag{4}
$$

Results for sparse Koopman approximation

# basis	# nonzero elements		solving time (s)		norm diff.
	full	sparse	full	sparse	
25	390625	5329	9.86	**0.91**	$3.03e^{-8}$
30	810000	7744	40.23	**2.21**	$1.04e^{-8}$
40	2560000	13924	117.82	**3.39**	$1.88e^{-8}$
60	12960000	31684	765.81	**14.69**	$7.41e^{-8}$

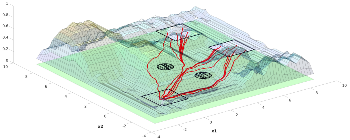
Table 1: 2D Integrator dynamics

# basis	# nonzero elements		solving time (s)		norm diff.
	full	sparse	full	sparse	
25	390625	43557	1.95	**0.8**	$2.12e^{-8}$
30	810000	62500	47.14	**3.4**	$2.09e^{-8}$
40	2560000	115600	131.3	**5.67**	$1.09e^{-8}$
60	12960000	270400	854.7	**21.65**	$4.78e^{-8}$

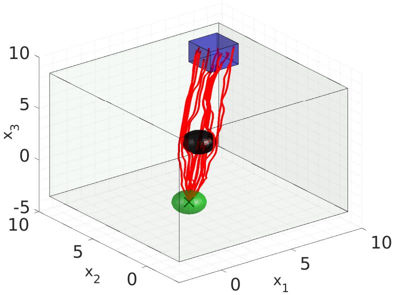
Table 2: Van der Pol dynamics

Optimal safe-reach results

- 2D navigation

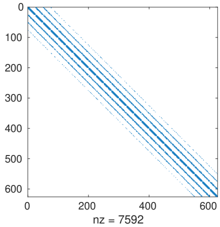


- 3D reach-safe for integrator dynamics

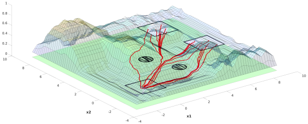


Summarize and Future work

- Observed the sparsity of Koopman operator approximation
- Formulated a sparse least-squares problem to solve DMD
- Used the solved Koopman operator approximation to solve an optimal reach-safe problem



- Future work: Explore different types of basis functions (in this work we only consider the RBF basis).



Thank you!

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