Data-driven optimal control of nonlinear dynamics under safety constraints

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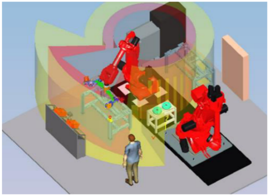
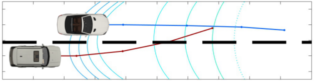
Motivation

- Safety in human-robot interaction[1]:



- Safety in autonomous car planning[2]:



1: G. Michalos, S. Makris, P. Tsarouchi, T. Guasch, D. Kontovrakis, G. Chryssolouris "Design Considerations for Safe Human-robot Collaborative Workplaces", Procedia CIRP 2015

2: B. Landry, M. Chen, S. Hemley and M. Pavone "Reach-Avoid Problems via Sum-of-Squares Optimization and Dynamic Programming", ArXiv 2018

Task: primal form

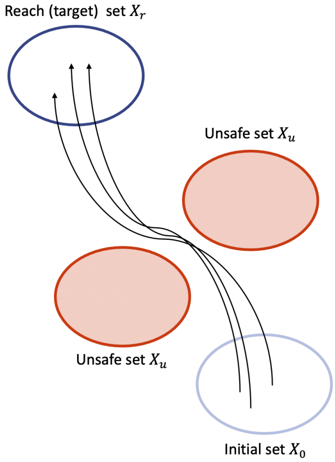
- Optimal reach-safe control problem

$$\min_{\mathbf{u}} \mathbb{E}_{\mathbf{x}_0}[\int_0^\infty q(\mathbf{x}(t)) + \mathbf{u}(\mathbf{x}(t))\mathbf{R}\mathbf{u}(\mathbf{x}(t))dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}\mathbf{u}$$

$$\mathbf{x}_0 \sim h_0$$

$$\mathbb{E}_{\mathbf{x}_0}[\int_0^\infty \mathbb{1}_{\mathbf{X}_u}(\mathbf{x}(t))dt] = 0$$



- control-affine dynamics
- initial distribution
- safety constraints
- Primal formulation

Dual formulation: Koopman operator

- Koopman (composition) operator $\mathbb{K}_t$

$$[\mathbb{K}_t y](\mathbf{x}) = y(\mathbf{x}(t)), \quad \forall y$$

- Example equality

$$\mathbb{E}_{\mathbf{x}_0}[q(\mathbf{x}(t))]$$
$$= \int_{\mathbf{X}} q(\mathbf{x}(t)) h_0(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{X}} [\mathbb{K}_t q] h_0 d\mathbf{x} = \langle [\mathbb{K}_t q], h_0 \rangle$$

- Dual: Perron-Frobenius (transfer) operator $\mathbb{P}_t$

$$\int_A y(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{s}_t(A)} \mathbb{P}_t[y](\mathbf{x}) d\mathbf{x}, \quad \forall A \subset \mathbf{X}, \quad \forall y$$

$\mathbf{x}(t)$ denotes the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ starting from $\mathbf{x}$

Dual formulation

- Duality: $\forall y_1, y_2$:

$$\langle [\mathbb{K}_t y_1](\mathbf{x}), y_2(\mathbf{x}) \rangle = \langle y_1(\mathbf{x}), [\mathbb{P}_t y_2](\mathbf{x}) \rangle$$

- Example equality

$$\begin{aligned} &\mathbb{E}_{\mathbf{x}_0}[q(\mathbf{x}(t))] \ s.t. \ \dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}\mathbf{u}, \mathbf{x}_0 \sim h_0 \\ &= \langle [\mathbb{K}_t q], h_0 \rangle \end{aligned}$$

- Equality: duality

$$\langle [\mathbb{K}_t q], h_0 \rangle = \langle q, [\mathbb{P}_t h_0] \rangle \triangleq \langle q, h_t \rangle$$

Occupation measure

- Duality equality

$$\mathbb{E}_{\mathbf{x}_0}[q(\mathbf{x}(t))] = \langle q, [\mathbb{P}_t h_0] \rangle \triangleq \langle q, h_t \rangle$$

- Similarly

$$\mathbb{E}_{\mathbf{x}_0}[\mathbb{1}_{\mathbf{X}_u}(\mathbf{x}(t))] = \langle \mathbb{1}_{\mathbf{X}_u}, [\mathbb{P}_t h_0] \rangle$$

- $h_t = [\mathbb{P}_t h_0]$: Evolution of density h_0 driven by dynamics

$$\int_{\phi_t(A)} h_t(\mathbf{x}) d\mathbf{x} = \int_A h_0(\mathbf{x}) d\mathbf{x}, \forall A$$

- Occupation measure $\rho(\mathbf{x})$ - accumulated 'mass'

$$\rho(\mathbf{x}) := \int_0^\infty h_t(\mathbf{x}) dt$$

$\phi_t(A)$ represents the evolution of set A under dynamics $\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}\mathbf{u}$

Problem reformulation

- Dual / Primal formulation

$$\min_{\mathbf{u},\rho} \int_{\mathbf{X}} (q + \mathbf{u}^\top \mathbf{R}\mathbf{u})(\mathbf{x})\rho(\mathbf{x})d\mathbf{x}$$
$$\text{s.t. } \nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) = h_0$$
$$\int_{\mathbf{X}} \mathbb{1}_{\mathbf{X}_u} \rho(\mathbf{x})d\mathbf{x} = 0$$

$$\min_{\mathbf{u}} \ \mathbb{E}_{\mathbf{x}_0}[\int_0^\infty q(\mathbf{x}(t)) + \mathbf{u}(t)\mathbf{R}\mathbf{u}(t)dt]$$
$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}\mathbf{u}$$
$$\mathbf{x}_0 \sim h_0$$
$$\mathbb{E}_{\mathbf{x}_0}[\int_0^\infty \mathbb{1}_{\mathbf{X}_u}(\mathbf{x}(t))dt] = 0$$

  - Objective: running cost inner product with $\rho(\mathbf{x})$
  - Constraints on dynamics and initial distribution
  - Constraints on safety: zero occupation in $\mathbf{X}_u$

Advantage: Convex formulation

- Is this a convex optimization problem?

$$\nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) = h_0$$

  is bi-linear in $(\rho, \mathbf{u})$.

- Define $\bar{\boldsymbol{\rho}} = \rho\mathbf{u}$, then convex in $(\rho, \bar{\boldsymbol{\rho}})$:

$$\begin{aligned} \min_{\rho,\bar{\boldsymbol{\rho}}} \quad & \int_{\mathbf{X}} q(\mathbf{x})\rho(\mathbf{x}) + \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})^\top \mathbf{R}\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho(\mathbf{x})} d\mathbf{x} \\ \text{s.t.} \quad & \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = h_0 \geq 0 \\ & \int_{\mathbf{X}} \rho(\mathbf{x}) \mathbb{1}_{X_u} d\mathbf{x} = 0 \end{aligned}$$

Solving: Penalty method, fixed Lagrangian multiplier

- Fix a Lagrangian multiplier $\bar{\lambda}$ (penalty method) for the equality constraint $\int_{\mathbf{X}} \mathbb{1}_{X_u} \rho d\mathbf{x} = 0$

$$\min_{\rho, \bar{\boldsymbol{\rho}}} \int_{\mathbf{X}} \left(q + \bar{\lambda} \mathbb{1}_{X_u} \right) \rho + \frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R} \bar{\boldsymbol{\rho}}}{\rho} d\mathbf{x}$$
$$\text{s.t.} \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) \geq 0 (= h_0)$$

- Inequality constraints? Parameterize

$$\rho = \frac{a}{b^\alpha}, \bar{\boldsymbol{\rho}} = \frac{\mathbf{c}}{b^\alpha}$$

a, c are unknown polynomial variables, $b(\mathbf{x}) > 0, \alpha > 0$

- solve using SOS

Advantage: Data driven approximation of constraints

- Koopman infinitesimal generator: time derivative

$$\mathcal{K}_{\mathbf{f}} y := \lim_{t \to 0} \frac{[\mathbb{K}_t y](\mathbf{x}) - y(\mathbf{x})}{t} = \mathbf{f}(\mathbf{x}) \cdot \nabla y(\mathbf{x})$$

- Dynamics $\mathbf{f}, \mathbf{g}$ is unknown while data is easy to collect
- Collect data $(\mathbf{x}_k, \dot{\mathbf{x}}_k)_{k=1}^{M}$, lift data to

$$\boldsymbol{\Psi}(\mathbf{x}_k) \text{ and } \dot{\boldsymbol{\Psi}}(\mathbf{x}_k, \dot{\mathbf{x}}_k) \triangleq [\nabla \psi_1(\mathbf{x}_k) \cdot \dot{\mathbf{x}}_k, \ldots, \nabla \psi_N(\mathbf{x}_k) \cdot \dot{\mathbf{x}}_k]^T$$

- Approximate Koopman generator

$$\mathbf{L}^* = \arg\min_{\mathbf{L}} \sum_{k=1}^{M} \left\| \dot{\boldsymbol{\Psi}}(\mathbf{x}_k, \dot{\mathbf{x}}_k) - \mathbf{L} \boldsymbol{\Psi}(\mathbf{x}_k) \right\|_2^2$$

Approximating the constraints

- Constraints $\nabla \cdot (\mathbf{f}a + \mathbf{g}\mathbf{c})$ and $\nabla \cdot (\mathbf{f}ab + b\mathbf{g}\mathbf{c})$
- Expand

$$\nabla \cdot (\mathbf{f}a + \mathbf{g}\mathbf{c}) = \nabla \cdot \mathbf{f}a + \mathbf{f}^T \nabla a + \sum_{i=1}^{m} (\nabla \cdot \mathbf{g}_i c_i + \mathbf{g}_i^T \nabla c_i)$$

- Approximate each $\nabla \cdot \mathbf{f}$ and $\nabla \cdot \mathbf{g}_i$

$$\nabla \cdot \mathbf{f} = \nabla \cdot [\mathcal{K}_{\mathbf{f}} \mathbf{x}_1, \ldots, \mathcal{K}_{\mathbf{f}} \mathbf{x}_n] \approx \nabla \cdot (C_{\mathbf{x}}^T \mathbf{L}_0 \mathbf{\Psi}_d)$$
$$\nabla \cdot \mathbf{g}_i = \nabla \cdot [\mathcal{K}_{\mathbf{g}_i} \mathbf{x}_1, \ldots, \mathcal{K}_{\mathbf{g}_i} \mathbf{x}_n] \approx \nabla \cdot (C_{\mathbf{x}}^T \mathbf{L}_i \mathbf{\Psi}_d)$$

- Approximate each $\mathbf{f}^T \nabla a$ and $\mathbf{g}_i^T \nabla c_i$

$$\mathbf{f}^T \nabla a \approx C_a^T \mathbf{L}_0 \mathbf{\Psi}_d$$
$$\mathbf{g}_i^T \nabla c_i \approx C_{c_i}^T \mathbf{L}_i \mathbf{\Psi}_d$$

Solving the problem

- Objective: quadratic over linear term: convex

$$\min_{\rho,\bar{\boldsymbol{\rho}}} \int_{\mathbf{X}} \left(q + \bar{\lambda}\mathbb{1}_{X_u}\right)\rho + \frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R}\bar{\boldsymbol{\rho}}}{\rho} d\mathbf{x}$$

- Construct upper bound $w(\mathbf{x}) \geq \frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R}\bar{\boldsymbol{\rho}}}{\rho}$: Schur complement

$$\mathbf{M} \triangleq \begin{bmatrix} w & \mathbf{c}^T \\ \mathbf{c} & a\mathbf{R}^{-1} \end{bmatrix}$$

$$\text{LMI: } \mathbf{M} \succeq 0 \Leftrightarrow w \geq 0, w \geq \frac{\mathbf{c}^T\mathbf{R}\mathbf{c}}{a} = \frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R}\bar{\boldsymbol{\rho}}}{\rho} b^\alpha$$

- Solving using SOS (sums of squares) techniques

Sum of squares (SOS): SDP problem

- Polynomial $p(\mathbf{x})$ is SOS:

$$p(\mathbf{x}) \in \Sigma$$

- SDP formulation

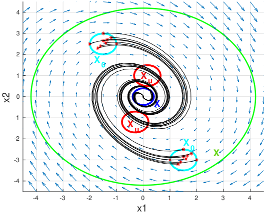
$$\begin{aligned} \min_{C_a, C_{\mathbf{c}}, C_w} \quad & C_a^T(\mathbf{d}_1 + \bar{\lambda}\mathbf{d}_3) + C_w^T \mathbf{d}_2 \\ \text{s.t.} \quad & a \in \Sigma[\mathbf{x}], b^{\alpha+1} h_0 \in \Sigma[\mathbf{x}] \\ & \begin{bmatrix} w & \mathbf{c}^T \\ \mathbf{c} & a\mathbf{R}^{-1} \end{bmatrix} \succeq 0, \end{aligned}$$

which is a standard SDP.

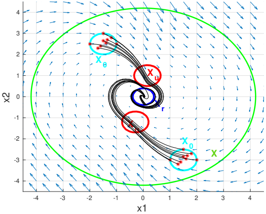
C_a, $C_{\mathbf{c}}$, C_w are corresponding polynomial coefficients in a common monomial basis.

Simulation results

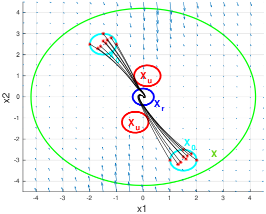
Dynamical system: Van Der Pol, L2 penalty



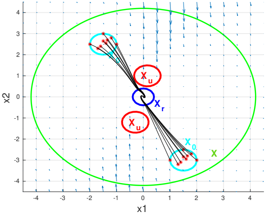
(a) $\bar{\lambda} = 0$



(b) $\bar{\lambda} = 10^4$



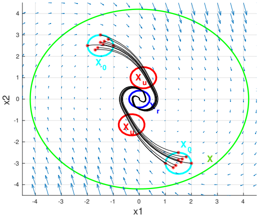
(c) $\bar{\lambda} = 10^6$



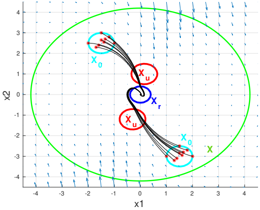
(d) $\bar{\lambda} = 5 \times 10^7$

Simulation results

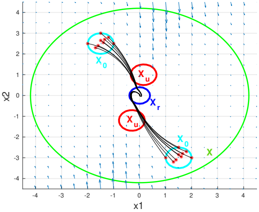
Dynamical system: Van Der Pol, L1 penalty



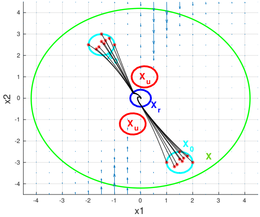
(a) $\bar{\lambda} = 0$



(b) $\bar{\lambda} = 300$



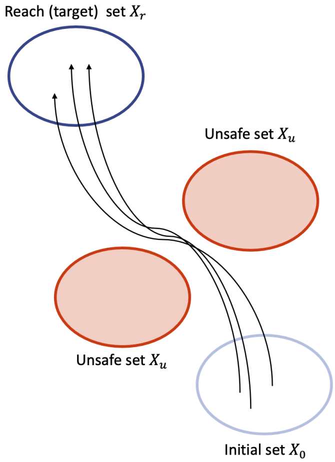
(c) $\bar{\lambda} = 1000$



(d) $\bar{\lambda} = 10^4$

Summarize and Future work

- Reformulated the optimal control problem under safety constraints
- Approximated the constraints using data
- Solved the problem using polynomials and SOS
- Future work: Use different basis functions, explore the structure of the problem (sparsity, etc.)



Thank you!

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