Planning, Control, and Inference under Stochastic Uncertainties in Robotic Systems

Thesis proposal by

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ABSTRACT

This thesis proposal presents a framework that includes several valuable tools for solving robotics decision-making problems under uncertainties. Specifically, the problem of motion planning and trajectory optimization under actuation and observation noise in the robotic systems is considered. At the core of our formulation is a stochastic optimal control problem for general nonlinear smooth and hybrid dynamical systems with nonlinear cost functions, which describes a vast range of robotic systems and decision-making tasks, and also highlights the main challenges encountered in these tasks under uncertainties, including efficacy and efficiency of the solutions. This thesis presents several novel ideas, tools, and techniques for overcoming the above challenges: Efficient iterative methods based on iterative linearization and proximal gradient in the space of distributions to steer the state covariances, variational inference methods for robust trajectory distribution optimization and motion planning, stochastic control methods for linear and nonlinear dynamical systems with hybrid transitions, and control algorithms for stochastic systems. Overall, this thesis provides a principled framework for solving decision-making variables for robotic systems under uncertainties. Numerical experiments have been conducted to demonstrate the efficacy of the proposed paradigm. Future research proposals include receding horizon optimal control for hybrid systems under uncertainties, multi-modal variational motion planning, and stochastic optimal control for stochastic systems with partial and stochastic observations.

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Chapter 1

INTRODUCTION

Uncertainties arise from different aspects and components of a robotic system and its operating environments (Thrun, 2002; Thrun, 2000), such as imperfect modeling of the dynamics and the environments (Horak, 1988; Swevers, Verdonck, and De Schutter, 2007), actuation noises due to imperfect actuator parameters (Mesbah, 2018; Van Damme et al., 2011; Johannink et al., 2019), the noise and perceptual limitations in a robot's sensing systems (Mallick, Das, and Majumdar, 2014; Nguyen, Izadi, and Lovell, 2012; Durrant-Whyte, 1988), partial observations (Lovejoy, 1991a; Lovejoy, 1991b; Krishnamurthy, 2016), and external disturbances (W.-H. Chen et al., 2000; Mohammadi et al., 2013). Understanding and modeling these uncertainties and designing decision-making strategies against them is crucial for robots to perform efficiently and safely in real-world scenarios.

A deterministic model of a robot's dynamical systems and surrounding environment does not adequately account for uncertainties in decision-making processes. Instead, a probabilistic world model is more insightful (Thrun, 2002). Within this framework, the robot's state, observations, and actions are all uncertain variables characterized by probability distributions. This thesis focuses on the following questions: modeling probabilistic beliefs within the framework of systems, dynamics, and optimal control; measuring the quality or optimality of a robot's decision under probabilistic assumptions; understanding how uncertainties impact the performance of robotic systems; deriving solutions under the probabilistic assumptions; and evaluating the performances of the obtained solutions.

Motion Planning is one core decision-making component in robotic systems (LaValle, 2006; González et al., 2015). Given an environment, a start configuration, and a goal configuration, a motion planner computes a trajectory connecting the two configurations. The trajectory optimization paradigm (Ratliff et al., 2009; Schulman et al., 2014) formulates the motion planning problem as an optimization over all admissible trajectories with dynamics and environment constraints. An 'optimal' plan can be obtained by solving the optimization program by minimizing certain optimality indexes, such as time consumption, control energy, and safety.

Motion planning under uncertainties takes into account the uncertainties in the planning phase in their formulations (Kalakrishnan et al., 2011). Stochastic optimal control (Bertsekas and Shreve, 1996) and probabilistic robotics (Åström, 2012; Thrun, 2002) provide a principled framework to address this problem, where noises are explicitly modeled in the robots' dynamics as a stochastic differential equation, and the optimality index is converted into a statistical one over the trajectory distribution space. Motion planning in an obstacle-cluttered environment is a multi-modal problem, and evaluating and choosing motion planning among different candidate plans is an important task when facing uncertainties. Our method gives a measurement index related to the entropy of the proposal distribution. This index balances the performance and robustness of a motion plan under uncertainties.

The stochastic optimal control problem for linear Gaussian systems enjoys a probabilistic inference problem formulation over trajectory distributions, which stems from the duality between probability graph inference and optimal control (Kappen, Gómez, and Opper, 2012; Botvinick and Toussaint, 2012; Y. Chen, Tryphon T Georgiou, and Michele Pavon, 2016a; Y. Chen, Tryphon T Georgiou, and Michele Pavon, 2016b). We start from a stochastic control formulation to obtain the target posterior distribution in the inference problem for the motion planning problems for Gaussian processes with nonlinear collision costs. This posterior is the distribution induced by the optimal solution controller. Then, the inference problem can be solved through Variational Inference (VI), which formulates an optimization problem in the distribution space. We solve this motion planning inference problem within the Gaussian distribution family. The resulting paradigm is termed Gaussian Variational Inference Motion Planning (GVIMP).

For stochastic dynamical systems, the state covariance represents the uncertainty levels of a given controller. Controlling the covariance from an initial value to a terminal one is an uncertainty control task (Chen., T. Georgiou, and M. Pavon, 2016; Y. Chen, Tryphon T. Georgiou, and Michele Pavon, 2016d; Y. Chen, Tryphon T Georgiou, and Michele Pavon, 2018; Okamoto, Goldshtein, and Tsiotras, 2018; Okamoto and Tsiotras, 2019; Ridderhof, Okamoto, and Tsiotras, 2019). One paradigm to control uncertainties is iterative linearization, which approximately solves the approximated linear covariance control problems in closed form. In this chapter, we developed an iterative method for a class of stochastic control problems for nonlinear control-affine systems with additional initial and terminal time state covariance constraints. Each iteration results in a linear covariance control problem that has closed-loop solutions. We choose a proximal gradient paradigm in the outer loop iteration that enjoys a linear convergence rate. We demonstrate our methods in robot collision-avoiding problems.

The hybrid of continuous and discrete dynamics represents a wide range of scenarios in robotics. Rigid-body dynamics with contact, such as walking (Collins and Ruina, 2005; Laszlo, Panne, and Fiume, 1996; Todd, 2013; Kuindersma et al., 2016), running (Clark et al., 2001; Hutter et al., 2011; Westervelt et al., 2018), and manipulation (Johnson, Burden, and Koditschek, 2016; Billard and Kragic, 2019) are typical hybrid systems.

Uncertainties affect the stability of hybrid systems through both the smooth flows and the discrete jumps. The studies of the region of attraction (ROA) analysis and control design for limit cycles (Manchester, Tobenkin, et al., 2011; Manchester, Mettin, et al., 2011) revealed that hybrid systems are intrinsically unstable. Uncertainty control is, therefore, critical for hybrid systems (Tassa, 2011; Drnach and Zhao, 2021).

In the following two chapters, we solve stochastic control problems for hybrid systems with optimality by formulating the problem into a stochastic optimal control problem with hybrid guard and reset constraints. We consider the case of both smooth linear and nonlinear stochastic dynamics connected by deterministic hybrid transitions. The stochastic smooth flows with hybrid transitions form a hybrid path distribution under a given controller. We then convert the stochastic control problem into a path distribution control problem. We proposed two control strategies for linear and non-linear stochastic flows.

For hybrid systems with linear stochastic flows, we present a hybrid covariance steering (H-CS) method, which steers the state covariance around a nominal mean trajectory from an initial value to a target one with a guarantee. For hybrid systems with nonlinear stochastic smooth flows, we propose the hybrid path integral control (H-PI) algorithm, which can obtain the optimal controller via forward sampling of stochastic trajectories subject to hybrid transitions.

Chapter 2

GAUSSIAN VARIATIONAL INFERENCE MOTION PLANNING

Trajectory optimization leverages optimal control theory and formulates motion planning problems as an optimization in the time window $t \in [0, T]$ as follows

$$\begin{aligned} \min_{X_t, u_t} \; & J_0(X_t, u_t) \\ \text{s.t.} \; & g_i^c(X_t, u_t) \leq 0, \; i = 1, \dots, N_g \\ & f_i^c(X_t, u_t) = 0, \; i = 1, \dots, N_f. \end{aligned} \tag{2.1}$$

Problem (2.1) is an optimization over the continuous-time variables (X_t, u_t), where $X_t : [0, T] \to \mathbb{R}^n$ is the state function of time of dimension n, and $u_t : [0, T] \to \mathbb{R}^m$ represents the control signal function of dimension m. J_0 is a cost function that often integrates a running cost function of (X_t, u_t) over $[0, T]$. $\{g_i^c\}_{i=1}^{N_g}$, $\{f_i^c\}_{i=1}^{N_f}$ are the functions defining the inequality and equality constraints that the motion planning problem is subject to, respectively. We define the time discretization

$$\mathbf{t} \triangleq [t_0, \dots, t_N], \; t_0 = 0, \; t_N = T \tag{2.2}$$

that generates a vector of discretized states and control inputs of length $N + 1$. We denote the *support states* and *support controls* as the discretized variables

$$\mathbf{X} \triangleq [X_0, \dots, X_N]^T, \; \mathbf{U} \triangleq [u_0, \dots, u_N]^T, \tag{2.3}$$

where $X_i = X_{t_i}, \; u_i = u_{t_i}, i = 1, \dots, N$. When $\mathbf{X}$ is a random variable, the covariance matrix of $\mathbf{X}$ is denoted as $\mathbf{K}$. The discretized variables have the dimensions $\mathbf{X} \in \mathbb{R}^{(N+1)\times n}$, $\mathbf{U} \in \mathbb{R}^{(N+1)\times m}$, and $\mathbf{K} \in \mathbb{R}^{(N+1)n\times(N+1)n}$.

Motion Planning problem admits a probabilistic inference dual formulation (Toussaint, 2009; Mukadam, Yan, and Boots, 2016; Mukadam, Dong, et al., 2018). Gaussian Process Motion Planning (GPMP) formulated the motion planning problem (2.1) as a posterior probability over $\mathbf{X}$

$$p(\mathbf{X}|Z) \propto p(\mathbf{X})p(Z|\mathbf{X}), \tag{2.4}$$

where Z encodes the environment. Consider the state, control, and deviated states over the time discretization as in (2.2), (2.3).

Our formulation builds on the posterior probability (2.4) of an optimal motion plan. GVI-MP seeks to find the optimal distribution solution by solving

$$q^{\star} = \arg\min_{q \in \mathcal{Q}} \mathrm{KL}\left(q(\mathbf{X}) \parallel p(\mathbf{X}|Z)\right) \tag{2.5}$$

where $\mathcal{Q} \triangleq \{q_\theta : q_\theta \sim \mathcal{N}(\mu_\theta, \Sigma_\theta)\}$ consists of the parameterized proposal Gaussian distributions. (2.5) formulates the probability inference for an optimal robot trajectory as an optimization within the Gaussian distribution space, where the object function is a distributional distance measure, the Kullback–Leibler (KL) divergence (Van Erven and Harremos, 2014), between the proposal Gaussian and the target posterior.

(a) Method.

We propose a natural gradient numerical scheme for solving (2.5). Define the negative log probability for the posterior as $\psi(\mathbf{X}) \triangleq -\log p(\mathbf{X}|Z)$. The objective in (2.5) reads

$$\mathcal{J}(q) = \mathrm{KL}\left(q(\mathbf{X}) \parallel p(\mathbf{X}|Z)\right) = \mathbb{E}[\psi(\mathbf{X})] - \mathcal{H}(q),$$

where $\mathcal{H}(q)$ denotes the entropy of the distribution q. The sparsity of the Gaussian prior leads to a sparse structure in the inverse of the covariance matrix. We parameterize the proposed Gaussian using its mean μ_θ and inverse of covariance Σ_θ^{-1}. The derivatives with respect to μ_θ and Σ_θ^{-1} are

$$\frac{\partial \mathcal{J}(q)}{\partial \mu_\theta} = \Sigma_\theta^{-1}\mathbb{E}\left[(\mathbf{X} - \mu_\theta)\psi\right] \tag{2.6a}$$

$$\frac{\partial^2 \mathcal{J}(q)}{\partial \mu_\theta \partial \mu_\theta^T} = \Sigma_\theta^{-1}\mathbb{E}\left[(\mathbf{X} - \mu_\theta)(\mathbf{X} - \mu_\theta)^T\psi\right]\Sigma_\theta^{-1} - \Sigma_\theta^{-1}\mathbb{E}\left[\psi\right]. \tag{2.6b}$$

Natural gradient descent update w.r.t. objective $\mathcal{J}$ reads

$$\begin{bmatrix} \delta\mu_\theta \\ vec(\delta\Sigma_\theta^{-1}) \end{bmatrix} = -\begin{bmatrix} \Sigma_\theta & 0 \\ 0 & 2(\Sigma_\theta^{-1} \otimes \Sigma_\theta^{-1}) \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{J}(q)}{\partial \mu_\theta} \\ vec(\frac{\partial \mathcal{J}(q)}{\partial \Sigma_\theta^{-1}}) \end{bmatrix},$$

which in matrix form is written as

$$\Sigma_\theta^{-1}\delta\mu_\theta = -\frac{\partial \mathcal{J}(q)}{\partial \mu_\theta}, \quad \delta\Sigma_\theta^{-1} = -2\Sigma_\theta^{-1}\frac{\partial \mathcal{J}(q)}{\partial \Sigma_\theta^{-1}}\Sigma_\theta^{-1}. \tag{2.7}$$

Comparing (2.6) and (2.7), we have

$$\delta\Sigma_\theta^{-1} = \frac{\partial^2 \mathcal{J}(q)}{\partial \mu_\theta \partial \mu_\theta^T} - \Sigma_\theta^{-1}. \tag{2.8}$$

Equation (2.7) and (2.8) tells that, to calculate the update $\delta\mu_\theta, \delta\Sigma_\theta^{-1}$, we only need to compute the expectations (2.6a) and (2.6b). The updates use a step size $\eta < 1$

$$\mu_\theta \leftarrow \mu_\theta + \eta^R \, \delta\mu_\theta, \quad \Sigma_\theta^{-1} \leftarrow \Sigma_\theta^{-1} + \eta^R \, \delta\Sigma_\theta^{-1}, \tag{2.9}$$

with an increasing $R = 1, 2, \ldots$ to shrink the step size for backtracking until the cost decreases. To evaluate these expectations, we leveraged the sparsity of the factor graph for motion planning problems and updated the proposal Gaussian parameters in the factorized levels. The factor graph is shown in Fig. 2.1.

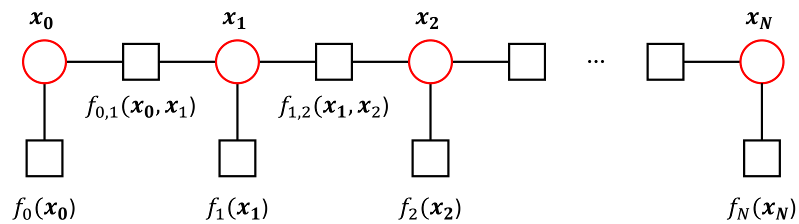


Figure 2.1: GVIMP Factor Graph.

On the factor levels, closed-form expressions when the posterior is quadratic and leveraged sparse Gauss-Hermite (G-H) quadrature methods (Heiss and Winschel, 2008) when the posterior is an arbitrary nonlinear function. The overall complexity of the GVIMP algorithm is linear to the time discretization and polynomial in the state dimensions (Yu and Y. Chen, 2024).

(b) Numerical Experiments.

Experiment results for a 2-D robot navigating through a narrow gap are shown in Fig. 2.2. Our formulation enjoys a robust motion planning equivalent interpretation. Fig. 2.3 shows an example of taking the robustness of motion plans into the decision-making for robot motion planning tasks. Tab. 2.1 recorded the corresponding costs.

	Prior	**Collision**	**MP**	**Entropy**	**Total**
Left	**34.4583**	9.1584	**43.6168**	44.1752	87.7920
Right	42.9730	**2.0464**	45.0193	**39.9193**	**84.9387**

Table 2.1: Comparing costs for plans in Fig. 2.3. The regularized entropy cost changed the order of the total costs and promoted a less risky decision.

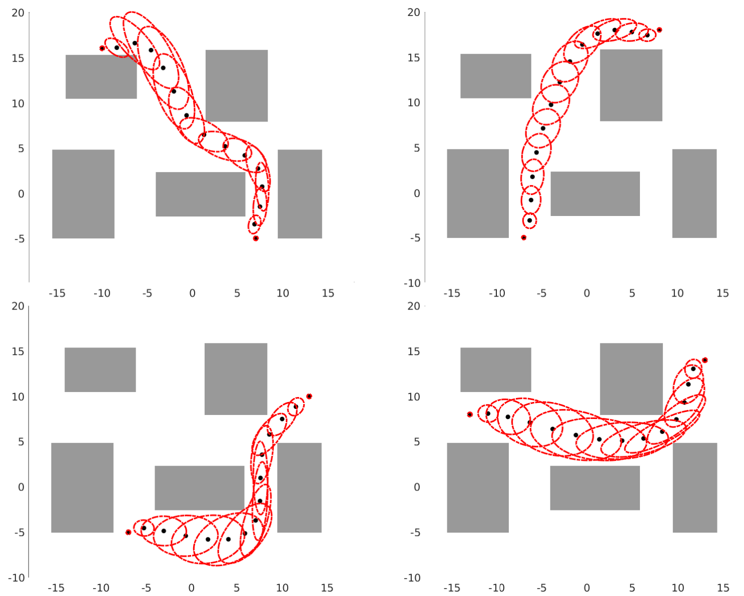


Figure 2.2: Planning in cluttered environments. 15 support states are used.

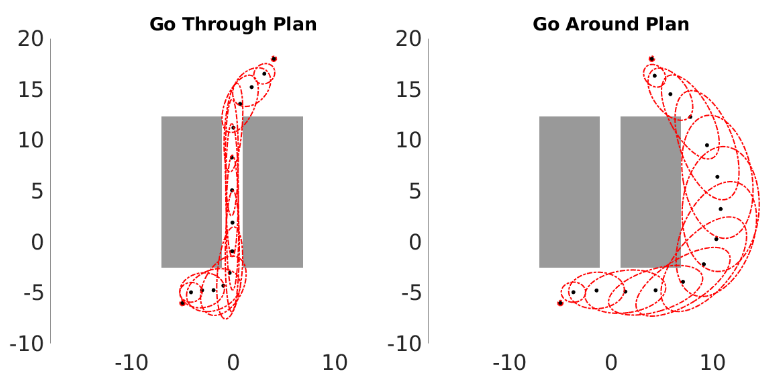


Figure 2.3: An example of comparing different motion plans. Entropy regularizes the motion planning objective. Costs are shown in Tab. 2.1.

Chapter 3

PROXIMAL COVARIANCE STEERING FOR NONLINEAR CONTROL-AFFINE SYSTEMS

(a) Problem formulation We propose an iterative covariance steering algorithm to solve stochastic control problems with nonlinear state costs, nonlinear dynamics, and initial and terminal state covariance constraints. The covariance steering problem we consider is

$$\min_{u} \quad \mathbb{E}\left\{\int_0^1 [\frac{1}{2}\|u_t\|^2 + V(X_t)]dt\right\} \tag{3.1a}$$
$$dX_t = f(t, X_t)dt + B(t)(u_t dt + \sqrt{\epsilon}dW_t) \tag{3.1b}$$
$$X_0 \sim \rho_0, \quad X_1 \sim \rho_1, \tag{3.1c}$$

where ρ_0 (ρ_1) is a probability distribution with mean m_0 (m_1) and covariance Σ_0 (Σ_1). The cost function has two decoupled terms: $\frac{1}{2}\|u_t\|^2$ that only depends on the control and $V(X_t)$ that only depends on the state. The problem (3.1) has been investigated in the study of distribution control (Y. Chen, Tryphon T. Georgiou, and Michele Pavon, 2016c; Caluya and Halder, 2020) for general distributions ρ_0, ρ_1, which links control theory and optimal transport theory. The duality between the distributional control and the stochastic optimal control is shown in Fig. 3.1.

The problem (3.1) can be reformulated as the distributional control problem

$$\min_{\mathcal{P}^u} \quad \int d\mathcal{P}^u \left[\log\frac{d\mathcal{P}^u}{d\mathcal{P}^0} + \frac{1}{\epsilon}V\right] \tag{3.2a}$$
$$(X_0)_\sharp\mathcal{P}^u = \rho_0, \quad (X_1)_\sharp\mathcal{P}^u = \rho_1. \tag{3.2b}$$

Let $F(\mathcal{P}^u) = \int[\frac{1}{\epsilon}V - \log d\mathcal{P}^0]d\mathcal{P}^u$ and $G(\mathcal{P}^u) = \int d\mathcal{P}^u \log d\mathcal{P}^u$. Then, (3.2) becomes a composite optimization

$$\min_{\mathcal{P}^u\in\Pi(\rho_0,\rho_1)} F(\mathcal{P}^u) + G(\mathcal{P}^u). \tag{3.3}$$

The proposed algorithm starts from a proximal-gradient descent paradigm in the distribution space. Let $D(\cdot,\cdot)$ be a Bregman divergence, then the generalized non-Euclidean proximal gradient algorithm reads

$$y^{k+1} = arg\, min_{y\in\mathcal{Y}} G(y) + \frac{1}{\eta}D(y, y^k) + \langle\nabla F(y^k), y - y^k\rangle. \tag{3.4}$$

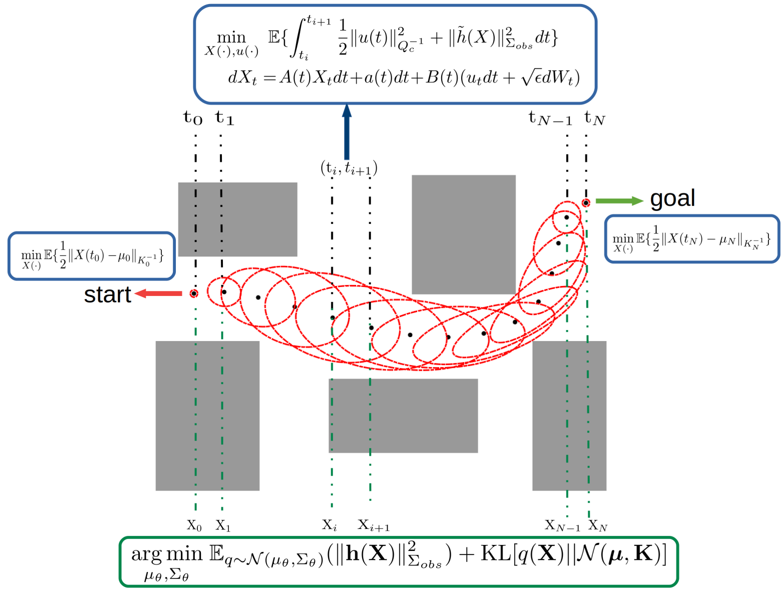
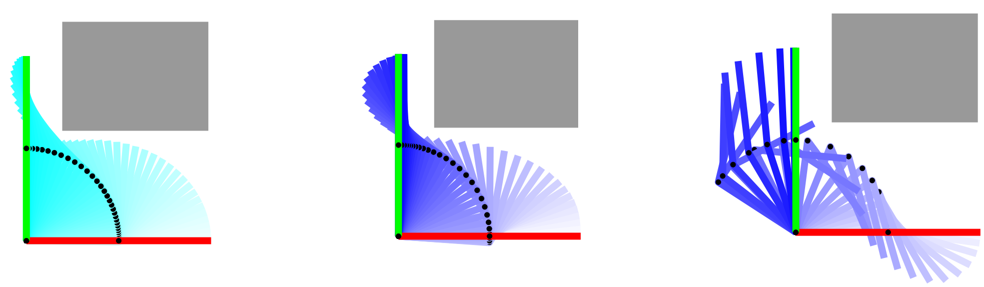


Figure 3.1: A visual demonstration of the connection between Gaussian variational inference motion planning (GVI-MP) and stochastic optimal control problems in a 2-DOF point robot planning example.

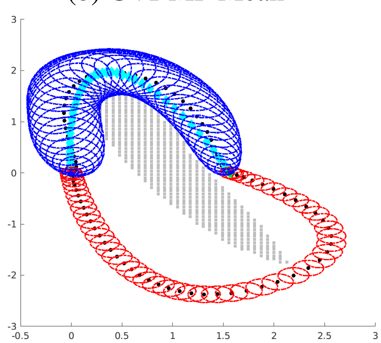
A popular choice of $D(\cdot,\cdot)$ is the Kullback-Leibler divergence $\mathrm{KL}(\cdot\|\cdot)$, which is suitable for optimization over probability vectors/distributions. Since (3.3) is an optimization over the space of probability measures, we use Kullback-Leibler divergence in (3.4). This leads to the following iteration

$$\mathcal{P}_{k+1} = arg\ min_{\mathcal{P}\in\hat{\Pi}(\rho_0,\rho_1)} G(\mathcal{P}) + \frac{1}{\eta}\mathrm{KL}(\mathcal{P}\|\mathcal{P}_k) + \langle \frac{\delta F}{\delta \mathcal{P}}(\mathcal{P}_k), \mathcal{P}\rangle. \tag{3.5}$$

At each step of the proximal gradient, by approximating the function F to the second order and approximating the dynamics to the first order, we show that the sub-problem is solving a linear covariance control problem with a closed-form solution (Yu, Z. Chen, and Y. Chen, 2023). Comparisons of the SOC and GVI methods on the same experiments for a 2-link bar motion planning are shown in Fig. 3.2 (Yu and Y. Chen, 2024).



(a) GPMP2 Result (b) GVI-MP Mean (c) PCS-MP Mean



(d) Configuration space

Figure 3.2: Motion planning for a 2-link arm. Result comparison between GVIMP, PCSMP, and baselines.

Chapter 4

COVARIANCE CONTROL FOR HYBRID DYNAMICAL SYSTEMS

A hybrid dynamical system contains different modes with different smooth flow and discrete jump dynamics between them. It is usually defined by a tuple (Grossman et al., 1993; Johnson, Burden, and Koditschek, 2016) $\mathcal{H} := \{\mathcal{I}, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}\}$. The set

$$\mathcal{I} := \{I_1, I_2, \ldots, I_{N_I}\} \subset \mathbb{N} \tag{4.1}$$

is a finite set of *modes*, $\mathcal{D}$ is the set of continuous *domains* containing the state spaces, with domain D_j for mode I_j, $\mathcal{F}$ is the set of *flows*, consisting of individual flow F_j that describes the smooth dynamics in mode I_j. $\mathcal{G} \subseteq \mathcal{D}$ denotes the set of *guards* triggering the resets.

We denote $X_j(t) \in \mathbb{R}^{n_j}$ as the continuous-time state in mode I_j, and $u_j(t) \in \mathbb{R}^{m_j}$ as the state-dependent control input. In mode I_j, the flow F_j is considered to be a linear stochastic process

$$dX_j(t) = (A_j(t)X_j(t) + B_j(t)u_j(t))dt + \sqrt{\epsilon}B_j(t)dW_j(t)), \tag{4.2}$$

where $A_j(t) \in \mathbb{R}^{n_j \times n_j}, B_j(t) \in \mathbb{R}^{n_j \times m_j}$ are the system matrices in mode I_j, and $dW_j(t) \in \mathbb{R}^{m_j}$ denotes a standard Wiener process with noise intensity ϵ. All the above variables are *mode-dependent*. A transition from mode I_j to mode I_k happens at state $X_j(t^-)$ and time t^- if the guard condition is satisfied, i.e.,

$$g_{jk}(t^-, X_j(t^-)) \leq 0, \; X_j(t^-) \in I_j. \tag{4.3}$$

The guard function triggers the time instances and states on which the jump dynamics events happen. At the event t^-, an instant jumping dynamics is applied to the system. From mode I_j to mode I_k, the reset map is defined as

$$t^+ = t^-, \quad X_k(t^+) = R_{jk}(X_j(t^-)). \tag{4.4}$$

The reset map R_{jk} is nonlinear and instantaneous in general. Saltation Matrices (Kong, Payne, Zhu, et al., 2024) provide precise linear approximations for the nonlinear reset functions. The Saltation Matrix $\Xi_{jk} \in \mathbb{R}^{n_k \times n_j}$ from mode I_j to I_k is defined as

$$\Xi_{jk} \triangleq \partial_x R_{jk} + \frac{(F_k - \partial_x R_{jk} \cdot F_j - \partial_t R_{jk})\partial_x g_{jk}}{\partial_t g_{jk} + \partial_x g_{jk} \cdot F_j}, \tag{4.5}$$

where $\partial_x(\cdot)$ and $\partial_t(\cdot)$ denote the partial derivatives in state and time, respectively. The dynamics of a perturbed state δX_t at hybrid transitions can be approximated as

$$\delta X(t^+) \approx \Xi_{jk}\delta X(t^-). \tag{4.6}$$

Controller design tasks based on iterative linearization of the dynamics have been applied to robot locomotion and estimation tasks for hybrid systems (Kong, Li, et al., 2023; Kong, Payne, Council, et al., 2021). Motivated by this linear approximation, this work considers *saltation* dynamics for the stochastic state variable $X(t)$ around its mean at the hybrid events

$$t^+ = t^-, \quad X_k(t^+) = \Xi_{jk}X_j(t^-), \tag{4.7}$$

where n_j, n_k are the dimensions of the state space before and after the jump.

Assumptions and Notations. We consider our optimal control problems in the time window $[0, T]$. Without loss of generality, we only consider *one* jump event at $t = t^-$, which separates the total time window into 2 pieces. Denote the initial and terminal time in the two windows as t_0^j and t_f^j respectively, with $t_0^1 = 0, t_f^1 = t^-$ and $t_0^2 = t^+, t_f^2 = T$. Denote the two modes before and after the jump event as I_1 and I_2, where the pre-event states $X_1(t) \in \mathbb{R}^{n_1}, \forall t \leq t^-$ and the post-event states $X_2(t) \in \mathbb{R}^{n_2}, \forall t \geq t^+$. We use

$$\Xi \triangleq \Xi_{12} \in \mathbb{R}^{n_2 \times n_1}$$

to represent the saltation transition from mode I_1 to I_2 at $t^- = t^+$.

The covariance steering for systems with saltation transitions is formulated as follows

$$\min_{u_j(t)} \mathcal{J}_H \triangleq \mathbb{E}\left\{\int_0^T [\|u_j(t)\|^2 + X_j'(t)Q_j(t)X_j(t)]dt\right\}$$

$$dX_1 = A_1(t)X_1dt + B_1(t)(u_1dt + \sqrt{\epsilon}dW_1), \tag{4.8a}$$

$$X_2(t^+) = \Xi X_1(t^-), \tag{4.8b}$$

$$dX_2 = A_2(t)X_2dt + B_2(t)(u_2dt + \sqrt{\epsilon}dW_2), \tag{4.8c}$$

$$X_1(0) \sim \mathcal{N}(m_0, \Sigma_0), \quad X_2(T) \sim \mathcal{N}(m_T, \Sigma_T), \tag{4.8d}$$

Let $\mathbb{P}_j^*$ be the measure induced by the solution to the LQG problem

$$\min_{\hat{u}_j(t)} \ \mathbb{E}\left\{\int_{t_0^j}^{t_f^j} \|\hat{u}_j(t)\|^2 + X_j(t)'Q_j(t)X_j(t)dt\right\} \tag{4.9}$$

for systems (4.2) without marginal constraints.

In mode I_j, denote the joint marginal distribution between the initial and terminal time state under path measure $\mathbb{P}_j$ by $\mathbb{P}_{(t_0^j, t_f^j)}$, and denote the same joint marginals for measure $\mathbb{P}_j^*$ as $\mathbb{P}^*_{(t_0^j, t_f^j)}$. The covariance steering's objective function in the mode I_j can be converted into a finite-dimensional optimization problem whose objective reads (Yu, Franco, et al., 2024a)

$$\mathcal{J}_j = \mathrm{KL}\left(\mathbb{P}_{(t_0^j, t_f^j)} \parallel \mathbb{P}^*_{(t_0^j, t_f^j)}\right). \tag{4.10}$$

In the distribution dual, the covariance propagation at the hybrid events becomes a constraint on the covariances

$$\Sigma^+ = \Xi \Sigma^- \Xi'. \tag{4.11}$$

The H-CS problem is thus equivalent to minimizing the KL-divergence in each mode, subject to the constraint (4.11). The overall formulation reads

$$\begin{aligned}
&\min_{W_1, W_2, \Sigma^-, \Sigma^+, Y_1, Y_2} \mathcal{J}_{\mathrm{SDP}} && \text{(4.12a)}\\
\text{s.t.}\quad & \Sigma^- \succ 0,\ Y_1 \succ 0,\ \Sigma^+ \succcurlyeq 0,\ Y_2 \succcurlyeq 0, && \text{(4.12b)}\\
& \Sigma^+ = \Xi \Sigma^- \Xi', && \text{(4.12c)}\\
& \begin{bmatrix} \Sigma^+ & W_2' \\ W_2 & \Sigma_T - Y_2 \end{bmatrix} \succcurlyeq 0, && \text{(4.12d)}
\end{aligned}$$

where $\mathcal{J}_{\mathrm{SDP}}$ equals (Yu, Franco, et al., 2024a)

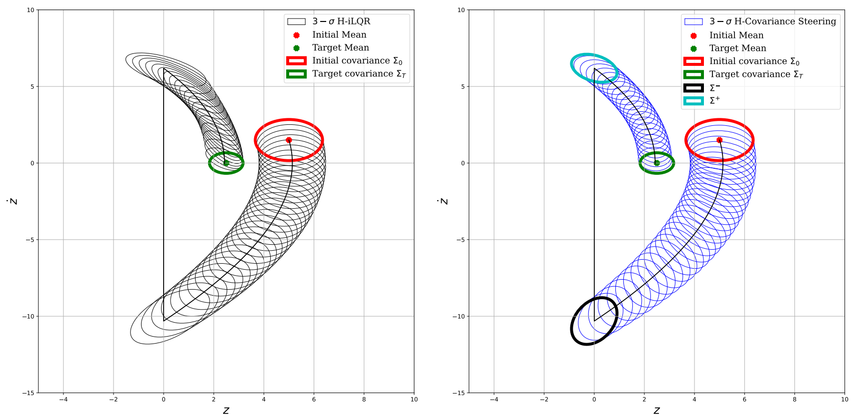
$$\begin{aligned}
&\frac{1}{\epsilon}\operatorname{Tr}(S_1^{-1}\Sigma^-) - \frac{2}{\epsilon}\operatorname{Tr}(\Phi_{A_2}' S_2^{-1} W_2) - \frac{2}{\epsilon}\operatorname{Tr}(\Phi_{A_1}' S_1^{-1} W_1)\\
&\quad + \frac{1}{\epsilon}\operatorname{Tr}(\Phi_{A_2}' S_2^{-1} \Phi_{A_2} \Sigma^+) - \log\det(Y_1) - \log\det(Y_2).
\end{aligned}$$

This formulation is convex in its variables $W_1, W_2, \Sigma^-, \Sigma_\eta^+$.

Simulation results.

We conducted covariance control experiments for bouncing ball dynamics and for Spring-Loaded Inverted Pendulum (SLIP) dynamics. The results for the bouncing ball are shown in Fig. 4.1. In Fig. 4.1, the mean is obtained from the H-iLQR controller, and the covariance control is achieved by integrating the hybrid Riccati equations starting from the same initial conditions. We compute the covariance propagation under the H-iLQR controller, plotted in black for comparison. H-CS controller steered the state covariance to the target terminal one.

For the SLIP system, we sample the initial states from the initial Gaussian distribution and apply the H-CS feedback controller to obtain the samples' state trajectories under uncertainties around the mean trajectory controlled by the H-iLQR controller in the last iteration of H-iLQR. Fig. 4.2 shows the nominal and the controlled stochastic body position trajectory samples.



(a) Controlled Covariance, H-iLQR. (b) Controlled Covariance, H-CS.

Figure 4.1: Covariance steering for a bouncing ball dynamics with elastic impacts. The H-CS controller guarantees the terminal covariance constraint.

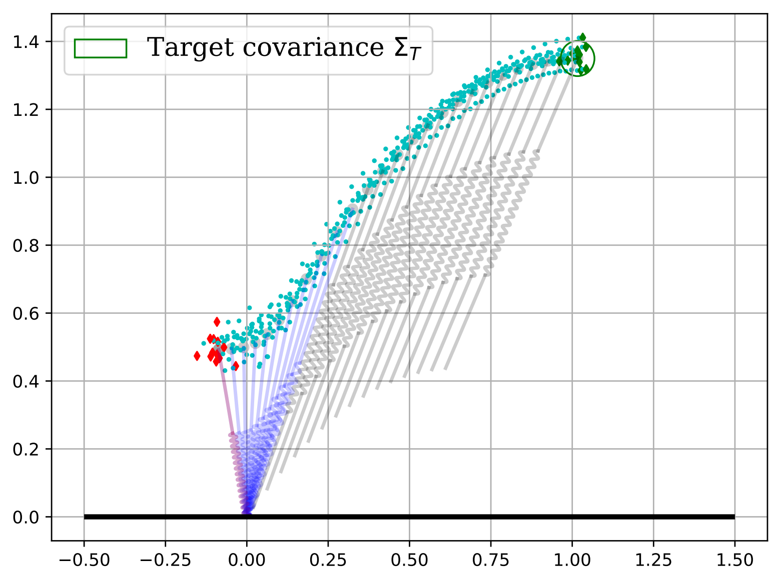


Figure 4.2: Deterministic nominal trajectory under H-iLQR controller and stochastic trajectories under the H-CS controller for the SLIP model.

Chapter 5

PATH INTEGRAL CONTROL FOR HYBRID DYNAMICAL SYSTEMS

In this chapter, we consider optimal control problems for nonlinear smooth flows with hybrid transitions. For the hybrid system (4.1), (4.3), (4.4), we consider mode-dependent state $X_t^j \in \mathbb{R}^{n_j}$ and control $u_t^j \in \mathbb{R}^{m_j}$ variables in mode I_j, where the *controlled* smooth flow is

$$dX_t^j = F_j(t, X_t^j)dt + \sigma_j(t, X_t^j)(u^j(t, X_t^j)dt + \sqrt{\epsilon}dW_t^j). \tag{5.1}$$

Define the *uncontrolled* smooth flow in mode I_j as

$$dX_t^j = F_j(t, X_t^j)dt + \sqrt{\epsilon}\sigma_j(t, X_t^j)dW_t^j, \tag{5.2}$$

with the jump dynamics defined by guards (4.3) and resets (4.4). For a given control sequence $u(t, X_t)$ and a realization of the random process dW_t, the state trajectory rollout $\{X_t | t \in [0, T]\}$ is assumed to have N_J hybrid transitions at time $\{t_j^-\}_{j=1,\dots,N_J}$. We extend the jump time set by letting $t_0^+ = 0$ and $t_{N_J+1}^- = T$. For simplicity, we assume that the system is in mode I_j in the time $[t_j^+, t_{j+1}^-]$, i.e.,

$$X_t^j \in I_j, \forall t \in [t_j^+, t_{j+1}^-];\;\; X^{j+1}(t_{j+1}^+) = R_{j,j+1}(X^j(t_j^-)).$$

The control problem we consider is defined in the time window $[0, T] = \cup_{j=0,\dots,N_J}[t_j^+, t_{j+1}^-]$ as

$$\begin{aligned} &\min_u \mathcal{J}_H \triangleq \mathbb{E}\left[\sum_{j=0}^{N_J}\int_{t_j^+}^{t_{j+1}^-}\left(V(t, X_t^j) + \frac{1}{2}\|u_t^j\|^2\right)dt + \Psi_T\right] \\ &\text{s.t. } (5.1),\ (4.3),\ (4.4),\ \forall j = 1, \dots, N_I. \end{aligned} \tag{5.3}$$

Our observation is that the probability measure induced by the controlled process (5.1), and the measure $d\mathbb{P}^0$ induced by the uncontrolled process (5.2) with the conditions of the same jump (4.4) has a similar form to the case without the hybrid events, which is given by (Yu, Franco, et al., 2024b)

$$
\begin{aligned}
\frac{d\mathbb{P}^u}{d\mathbb{P}^0} &= \exp\left(\sum_{j=0}^{N_J}\int_{t_j^+}^{t_{j+1}^-} -\frac{\|u_t^j\|^2}{2\epsilon}dt + \frac{1}{\sqrt{\epsilon}}(u_t^j)' dW_t^j\right) && \text{(5.4a)}\\
&= \exp\left(\sum_{j=0}^{N_J}\int_{t_j^+}^{t_{j+1}^-} \frac{1}{2\epsilon}\|u_t^j\|^2 dt + \frac{1}{\sqrt{\epsilon}}(u_t^j)' d\tilde{W}_t^j\right) && \text{(5.4b)}\\
&:= \exp\left(\frac{1}{\epsilon}\Lambda_H\right), && \text{(5.4c)}
\end{aligned}
$$

where dW_t is a Wiener process under $\mathbb{P}^0$, and $\tilde{W}_t$ is a Wiener process under $\mathbb{P}^u$.

The search for the optimal controller is obtained by minimizing the KL-divergence between the optimal distribution $\mathbb{P}^*$ and a controlled distribution $\mathbb{P}^u$. This formulation is inspired by the cross-entropy methods for the optimal control for smooth stochastic systems (Zhang et al., 2014) and was first introduced to the robotics community for applications in autonomous driving (Williams, Wagener, et al., 2017; Williams, Drews, et al., 2018). The cross-entropy formulation is

$$
\begin{aligned}
u^* &= \arg\min_u \mathrm{KL}\left(\mathbb{P}^* \parallel \mathbb{P}^u\right) = \arg\min_u \mathbb{E}_{\mathbb{P}^*}\left[\log \frac{d\mathbb{P}^*}{d\mathbb{P}^0}\frac{d\mathbb{P}^0}{d\mathbb{P}^u}\right]\\
&= \arg\min_u \mathbb{E}_{\mathbb{P}^*}\left[-\frac{1}{\epsilon}\Lambda_H\right],
\end{aligned} \tag{5.5}
$$

since the term $\mathcal{L}_H$ is independent to the control. Λ_H has different forms (5.4a) and (5.4b) when considering Wiener processes under different measures. Minimizing the above over the control u_t gives the optimal controller.

The optimal controller for the problem (5.5) at time t can be expressed as an expectation

$$
u_t^* = \frac{\sqrt{\epsilon}\times \mathbb{E}_{\mathbb{P}^0}\left[\Delta W_t \exp\left(-\frac{1}{\epsilon}\mathcal{L}_H(t)\right)\right]}{\Delta t \times \mathbb{E}_{\mathbb{P}^0}\left[\exp\left(-\frac{1}{\epsilon}\mathcal{L}_H(t)\right)\right]}. \tag{5.6}
$$

Importance sampling in the hybrid path distribution space is leveraged to reduce the variance of (5.6) by replacing the underlying measure $\mathbb{P}^0$ with another path measure $\mathbb{P}^u$. In this work, we choose $\mathbb{P}^u$ as the one induced by the H-iLQR. Given the ratio (5.4b), we have

$$
\mathbb{E}_{\mathbb{P}^0}\left[\Delta W_t \exp\left(-\frac{\mathcal{L}_H}{\epsilon}\right)\right] = \mathbb{E}_{\mathbb{P}^u}\left[\Delta W_t \exp\left(-\frac{\mathcal{S}_H}{\epsilon}\right)\right],
$$

and

$$
\mathbb{E}_{\mathbb{P}^0}\left[\exp\left(-\frac{\mathcal{L}_H}{\epsilon}\right)\right] = \mathbb{E}_{\mathbb{P}^u}\left[\exp\left(-\frac{\mathcal{S}_H}{\epsilon}\right)\right],
$$

where $\mathcal{S}_H^u(t)$ is defined as the piece-wise smooth integration

$$\mathcal{S}_H^u(t) \triangleq \mathcal{I}_S[t, t_{j_m-1}^-] + \sum_{j=j_m}^{N_J} \mathcal{I}_S[t_j^+, t_{j+1}^-], \tag{5.7}$$

where j_m is the minimum j that $t_j^+ \geq t$, and each integral $\mathcal{I}_S$ is defined as

$$\mathcal{I}_S[t_j^+, t_{j+1}^-] \triangleq \int_{t_j^+}^{t_{j+1}^-} \left(\frac{1}{2} \|u_t^j\|^2 + V(t, X_t^j) dt + \sqrt{\epsilon} (u_t^j)' d\tilde{W}_t^j \right) + \Psi_T. \tag{5.8}$$

The optimal control (5.6) is then equivalently

$$u_t^* = \frac{\sqrt{\epsilon} \times \mathbb{E}_{\mathbb{P}^u} \left[\Delta W_t \exp\left(-\frac{1}{\epsilon} \mathcal{S}_H(t) \right) \right]}{\Delta t \times \mathbb{E}_{\mathbb{P}^u} \left[\exp\left(-\frac{1}{\epsilon} \mathcal{S}_H(t) \right) \right]}. \tag{5.9}$$

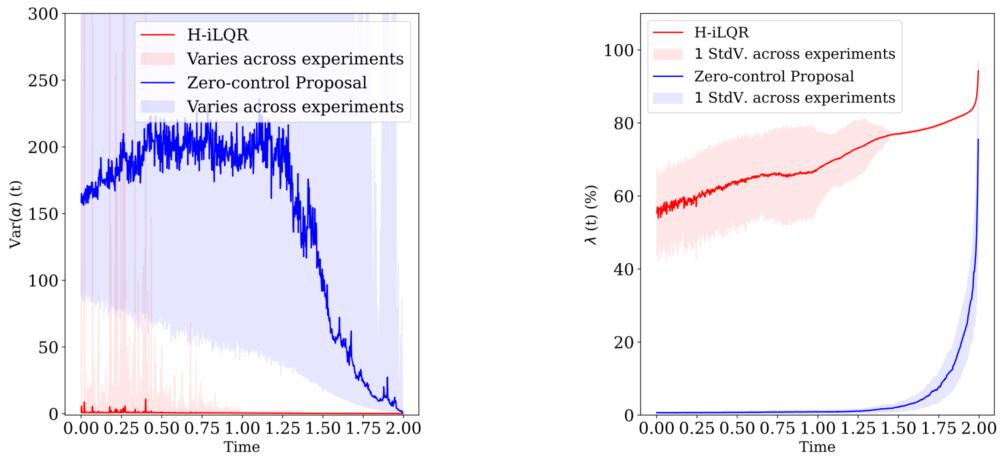
Experiment results.

We evaluate the expected cost $\mathcal{J}_H$ in (5.3) using a Monte Carlo estimation. We conduct 100 experiments under independent randomness and compute the empirical mean of the expected path costs. We use the same randomnesses to compare the expected path costs for the H-PI controller and the H-iLQR proposal controller. The improvements in the expected costs over H-iLQR are recorded in Tab. 5.1.

We test the impact on the H-iLQR feedback controller of the jump dynamics and the nonlinearities in the smooth flows. In Fig. 5.2, we observe stronger vibrations in the variance and effective sample portion before the bounce time than after, for different noise realizations, for both SLIP and bouncing ball. As all the samples before the bounce time will experience the bouncing event, this indicates that the hybrid transition affects the robustness of the feedback controller used by these samples.

Tab. 5.3 separately computes the averages before and after the hybrid event. For the linear reset map in the bouncing ball example, the λ increases from 65.83% to 79.85%, and for the nonlinear SLIP model, λ increases from 10.32% to 93.89%.

We use the H-iLQR controller as the proposal controller in the importance sampling process in H-PI because it can reduce the variance of this process and thus reduce the sampling efficiency of the H-PI controller. The comparison of variance and effective samples under H-iLQR and a zero-controller is conducted and shown in Fig. 5.1. H-iLQR significantly reduced the sampling variance.



(a) Variance.

(b) Effective Samples.

Figure 5.1: Comparison of the variance and effective samples are shown in subfigure 5.1a and 5.1b. H-iLQR provides a much lower variance proposal path distribution than $\mathbb{P}^0$.

		H-iLQR	H-PathIntegral	Improved (%)
Expected Cost	Bouncing Ball	60.11	57.86	3.74
	SLIP	0.2459	0.2164	**12.00**
H-iLQR 10% tail	Bouncing Ball	141.65	115.40	**11.46**
	SLIP	0.9451	0.2616	**63.29**

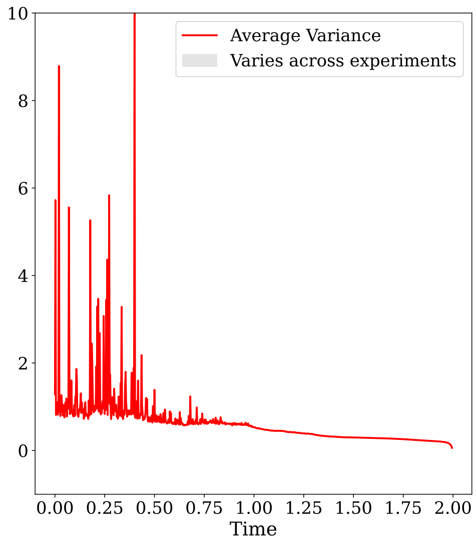
Table 5.1: Expected cost improvement and the statistics on the experiments where H-iLQR has the highest 10% costs.

	Confidence level	0.7	0.8	0.9
Improvement (%)	Bouncing Ball	15.05	21.34	32.44
	SLIP	37.86	47.48	59.32

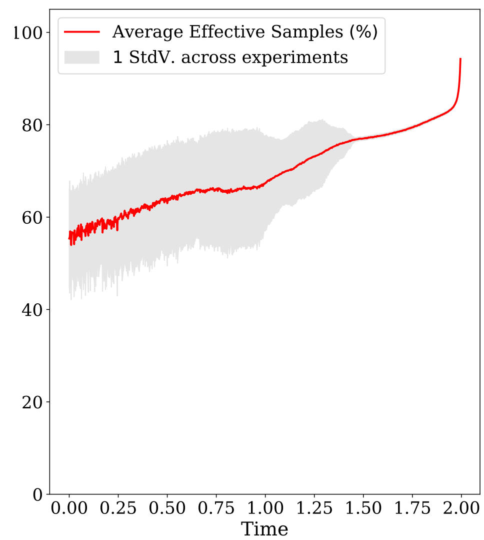
Table 5.2: Conditional Value at Risk (CVaR) statistics for the cost improvement in Tab. 5.1, conditioned on confidence levels 70%, 80%, and 90%.

		$[0, T]$	$[0, t^-]$	$[t^-, T]$	**Jump Influence**
Bouncing Ball	Avg. Var(α)	0.64	0.78	0.25	67.95%
	Avg. λ(%)	69.59	65.83	79.85	21.2%
SLIP	Avg. Var(α)	14.20	34.03	3.55	89.57%
	Avg. λ(%)	64.69	10.32	93.89	809.8%

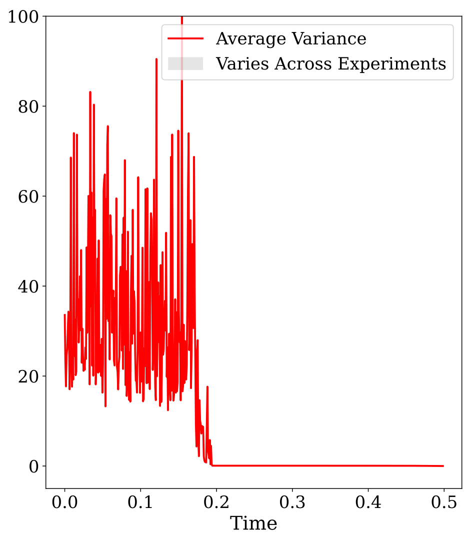
Table 5.3: Average weight distribution variance and effective sample portion along the time axis.



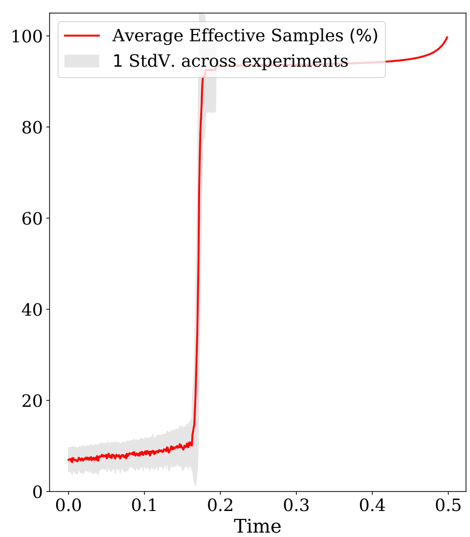
(a) Variance, bouncing ball.



(b) Effective samples, ball.



(c) Variance, SLIP.



(d) Effective samples, SLIP.

Figure 5.2: Sample variance and effective sample portions for the bouncing ball and the SLIP jumping tasks. Both indicators experience changes before and after the jump events, showing the impact of the jump dynamics on the feedback controller.

Chapter 6

FUTURE RESEARCH PROPOSAL

Several interesting future directions may follow the current works presented in this thesis proposal. One potential future direction is to develop and extend the above paradigm into a receding-horizon fashion. Secondly, in motion planning problems, obstacles in the environment naturally make the problem non-convex, and therefore, the cost functions will have multiple modes, which can be captured in the GVIMP paradigm using variational inference. Finally, optimal robot decision-making under partially observed environments is also one potential direction in the author's future research.

6.1 Variational Solutions to Non-convex Multi-modal Trajectory Optimization Problems

A trajectory optimization problem with obstacles is non-convex, since for example, two feasible trajectories go around one obstacle from the two sides, the convex combination of the two may be infeasible (it can hit the obstacle in the middle). The cost function for the problem is therefore also non-convex and has multiple picks. In the variational inference problem settings, the posterior related to the optimal trajectory distribution has multiple modes. Multi-modal proposal distributions and other variational techniques (Bishop and Nasrabadi, 2006; Blei, Kucukelbir, and McAuliffe, 2017) can capture this multi-modality in motion plans.

6.2 Optimal Control for Partially Observed Stochastic Systems

The optimal control and planning problem for stochastic systems with partial observations has been studied in the literature (Bensoussan, 1992; Kaelbling, Littman, and Cassandra, 1998). In discrete-time setting, it is modeled as the problem of Partially Observed Markov Decision Processes (POMDP). Solving POMDP with optimality faces computation scalability issues (Madani, Hanks, and Condon, 1999). Therefore, approximations have been proposed to find sub-optimal solutions to the original problem, such as MDP and QMDP approximations using fully observed future systems to approximate the partially observed ones (Hauskrecht, 2000; Littman, Cassandra, and Kaelbling, 1995), and point-based approximations (Lovejoy, 1991b).

For continuous-time systems, the partially observed control problems are modeled by dynamics observation SDEs. In the linear quadratic case, the optimal solution can be decomposed into a filtering problem and a control problem, which is known as the separation principle. For general cases, similar paradigm to DMP and QMDP approximations can be proposed in the continuous-time settings.

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