Convex Optimal Control Synthesis Under Safety Constraints

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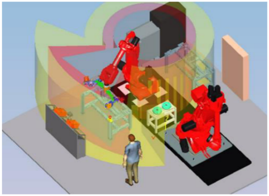
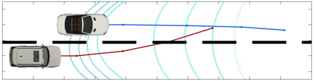
Motivation

- Safety in human-robot interaction[1]:



- Safety in autonomous car planning[2]:



- Unsafe set: the collection of states that we do not want our system to be at.

1: G. Michalos, S. Makris, P. Tsarouchi, T. Guasch, D. Kontovrakis, G. Chryssolouris "Design Considerations for Safe Human-robot Collaborative Workplaces", Procedia CIRP 2015

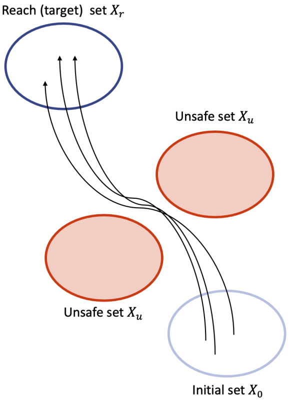
2: B. Landry, M. Chen, S. Hemley and M. Pavone "Reach-Avoid Problems via Sum-of-Squares Optimization and Dynamic Programming", ArXiv 2018

Task

Reach-avoid problem for dynamical system
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$

Initial sets, unsafe sets, and reach sets
$\mathbf{X}_0, \mathbf{X}_u, \mathbf{X}_r$

Optimal control problem, cost function
$\mathbb{E}[\int_0^\infty \left(q(\mathbf{x}(t)) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t)\right) dt]$



Koopman and Perron-Frobenius Operator

- Koopman (composition) operator

$$[\mathbb{K}_t\varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x})), \quad \forall\varphi$$

- Perron-Frobenius (transfer) operator

$$\int_{\mathbf{s}_{-t}(A)} \psi(\mathbf{x})d\mathbf{x} = \int_A \mathbb{P}_t[\psi](\mathbf{x})d\mathbf{x}, \quad \forall A \subset \mathbf{X}, \quad \forall\psi$$

Duality: $\forall\varphi,\psi$:

$$\langle[\mathbb{K}_t\varphi](\mathbf{x}), \psi(\mathbf{x})\rangle = \langle\mathbb{P}_t\psi](\mathbf{x}), \varphi(\mathbf{x})\rangle$$

$\mathsf{K}_t : \mathcal{L}_\infty(\mathbf{X}) \to \mathcal{L}_\infty(\mathbf{X}), \mathbb{P}_t : \mathcal{L}_1(\mathbf{X}) \to \mathcal{L}_1(\mathbf{X})$.

$\mathbf{s}_t(\mathbf{x})$ denotes the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ starting from $\mathbf{x}$

Intuition and occupation measure

- Evolution of distribution $h_0(\mathbf{x})$ under dynamics

$$\int_{\phi_t(A)} h_t(\mathbf{x})d\mathbf{x} = \int_A h_0(\mathbf{x})d\mathbf{x} \triangleq \int_{\phi_t(A)} \mathbb{P}_t[\psi](\mathbf{x})d\mathbf{x}, \forall A$$

- Occupation measure $\rho(\mathbf{x})$ - accumulated 'mass'

$$\rho(\mathbf{x}) := \int_0^{\infty} h_t(\mathbf{x})dt$$

$\phi_t(A)$ represents the evolution of set A under dynamics $\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}\mathbf{u}$

Problem reformulation

- Cost function reformulated:

$$J(\mu_0) = \int_{\mathbf{X}} (q + \mathbf{k}^\top \mathbf{R}\mathbf{k})(\mathbf{x})\rho(\mathbf{x})d\mathbf{x}$$

- Initial distribution:

$$\nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) = h_0$$

- Safety constraint: zero occupation in $\mathbf{X}_u$

$$\int_{X_u} \rho(\mathbf{x})d\mathbf{x} = \int_{\mathbf{X}} \mathbb{1}_{X_u}(\mathbf{x}) \int_0^\infty h_t(\mathbf{x})dtd\mathbf{x} = 0$$

Assume feedback control $\mathbf{u} = \mathbf{k}(\mathbf{x})$.

q is the state cost, and $\mathbf{R}$ is the penalty control magnitude, h_0 is the initial distribution

Convex formulation

- Is this a convex optimization problem?

$$\nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) = h_0$$

is bi-linearity in $(\rho, \mathbf{u})$.

- Define $\bar{\boldsymbol{\rho}} = \rho\mathbf{u}$, then convex in $(\rho, \bar{\boldsymbol{\rho}})$:

$$\begin{aligned} \inf_{\rho,\bar{\boldsymbol{\rho}}} \quad & \int_{\mathbf{X}} q(\mathbf{x})\rho(\mathbf{x}) + \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})^\top \mathbf{R}\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho(\mathbf{x})} d\mathbf{x} \\ \text{s.t.} \quad & \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = h_0 \geq 0 \\ & \int_{X_u} \rho(\mathbf{x}) d\mathbf{x} = 0 \end{aligned}$$

Solving: fixed Lagrangian multiplier

- Choose and fix a large Lagrangian multiplier $\bar{\lambda}$ and solve the problem

$$\inf_{\rho,\bar{\boldsymbol{\rho}}} \int_{\mathbf{X}} \left(q + \bar{\lambda}\mathbb{1}_{X_u}\right)\rho + \frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R}\bar{\boldsymbol{\rho}}}{\rho} d\mathbf{x}$$
$$\text{s.t.} \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) \geq 0$$

- How to solve?

  Using polynomials and SOS.

- How to deal with the term $\frac{\bar{\boldsymbol{\rho}}^\top \mathbf{R}\bar{\boldsymbol{\rho}}}{\rho}$?

  Construct an upper bound $w(\mathbf{x})$ and minimize w.

Polynomial parameterization

- Parameterize

$$\rho = \frac{a}{b^\alpha}, \overline{\boldsymbol{\rho}} = \frac{\mathbf{c}}{b^\alpha}$$

a, c are unknown polynomial variables, $b(\mathbf{x}) > 0$

- Construct $w(\mathbf{x})$. Define

$$\mathbf{M} \triangleq \begin{bmatrix} w & \mathbf{c}^T \\ \mathbf{c} & a\mathbf{R}^{-1} \end{bmatrix}$$

$$\mathbf{M} \succeq 0 \Leftrightarrow w \geq 0, w \geq \frac{\mathbf{c}^T\mathbf{R}\mathbf{c}}{a} = \mathbf{u}^T\mathbf{R}\mathbf{u}$$

Sum of squares (SOS)

- Polynomial $p(\mathbf{x})$ is SOS:

$$p(\mathbf{x}) \in \Sigma$$

- SDP formulation

$$\begin{aligned} \min_{C_a, C_{\mathbf{c}}, C_w} \quad & C_a^T(\mathbf{d}_1 + \bar{\lambda}\mathbf{d}_3) + C_w^T \mathbf{d}_2 \\ \text{s.t.} \quad & a \in \Sigma[\mathbf{x}], b^{\alpha+1} h_0 \in \Sigma[\mathbf{x}] \\ & \begin{bmatrix} w & \mathbf{c}^T \\ \mathbf{c} & a\mathbf{R}^{-1} \end{bmatrix} \succeq 0, \end{aligned}$$

which is a standard SDP.

C_a, $C_{\mathbf{c}}$, C_w are corresponding polynomial coefficients in a common monomial basis.

Safety verification

- Safety: For almost all $\mathbf{x}_0 \in X_0$, the solution $\mathbf{s}_t(\mathbf{x})$ starting at $\mathbf{x}_0$ satisfies $\mathbf{s}_T(\mathbf{x}) \in X_r$, for some $T > 0$, $\mathbf{s}_t(\mathbf{x}) \notin X_u$, and $\mathbf{s}_t(\mathbf{x}) \in \mathbf{X}$ for all $t \in [0, T]$

- Criteria[1]: System safe if $\exists$ polynomial ρ_s:

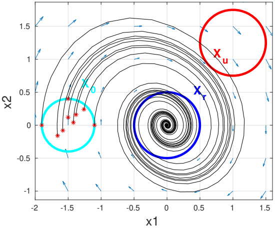
$$\begin{aligned} \rho_s(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \mathbf{X}_0 \\ \nabla \cdot (\rho_s(\mathbf{f} + \mathbf{g}\mathbf{u}))(\mathbf{x}) > 0, \forall \mathbf{x} \in cl(\mathbf{X} \setminus \mathbf{X}_r) \\ \rho_s < 0, \forall \mathbf{x} \in cl(\partial \mathbf{X} \setminus \partial \mathbf{X}_r) \cup \mathbf{X}_u \end{aligned}$$

- Find positive polynomial ρ_s inside compact sets defined by polynomials (semi-algebraic sets): an SDP feasibility problem.

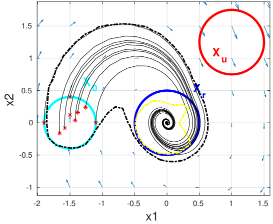
1. S. Prajna, and A. Rantzer "Convex programs for temporal verification of nonlinear dynamical systems." SIAM Journal on Control and Optimization, vol.46, no.3, pp.999–1021, 2007

Simulation results

Dynamical system: Van Der Pol

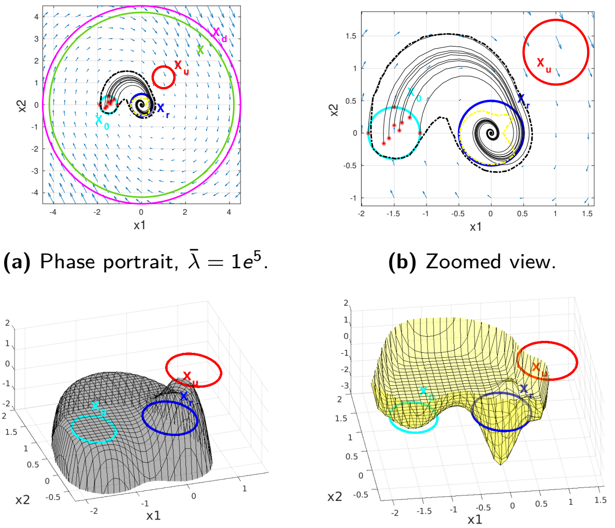


(a) Zoomed view, $\bar{\lambda} = 0$



(b) Zoomed view, $\bar{\lambda} = 1e^5$.

Simulation results

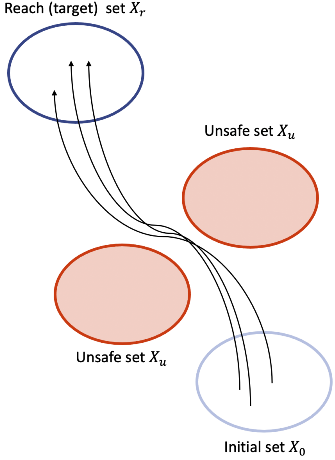


(a) Phase portrait, $\bar{\lambda} = 1e^5$.   **(b)** Zoomed view.

(c) Surface of $\rho_s(\mathbf{x})$ **(d)** Surface of $\nabla \cdot (\rho_s(\mathbf{f} + \mathbf{g}\hat{\mathbf{u}})$

Takeaway

- Introduced the notion of occupation measure
- Reformulated the optimal control problem under safety constraints
- Verified the solution in terms of safety constraints



Thank you!

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