A Gaussian Variational Inference Approach To Motion Planning

Hongzhe Yu Yongxin Chen

Georgia Institute of Technology, School of Aerospace Engineering

Background

Motion planning under uncertainty as stochastic optimal control

$$\min_{X(\cdot),u(\cdot)} \mathbb{E}\{\int_{t_0}^{t_N} \frac{1}{2}\|u_t\|^2 dt + \|\mathbf{h}(\mathbf{x})\|^2_{\Sigma_{\text{obs}}} + \frac{1}{2}\|X_0 - \mu_0\|_{K_0^{-1}} + \frac{1}{2}\|X_N - \mu_N\|_{K_N^{-1}}\} \tag{1a}$$

$$dX_t = A_t X_t dt + a_t dt + B_t(u_t dt + dW_t). \tag{1b}$$

Explanations and Applications

- Model the imperfect modeling and actuation as a **stochastic process**
- Applications: Safe navigation under uncertainty and Trajectory distribution control

Problem Formulation

Re-formulated as a **probabilistic inference** (Mukadam et al. (2018))

$$\mathbf{x}^\star = \arg\max_{\mathbf{x}} p(\mathbf{x}|\mathbf{z}) = \arg\max_{\mathbf{x}} p(\mathbf{z}|\mathbf{x})p(\mathbf{x}). \tag{2}$$

- $p(\mathbf{x}|\mathbf{z})$: Posterior probability modeling a feasible trajectory given the environment
- $p(\mathbf{x}) = \exp(-\frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|^2_{\mathbf{K}^{-1}})$: Prior induced by SDE (1b)
- $p(\mathbf{z}|\mathbf{x}) \propto \exp(-\|\mathbf{h}(\mathbf{x})\|^2_{\Sigma_{obs}^{-1}})$: Likelihood for collision avoidance.

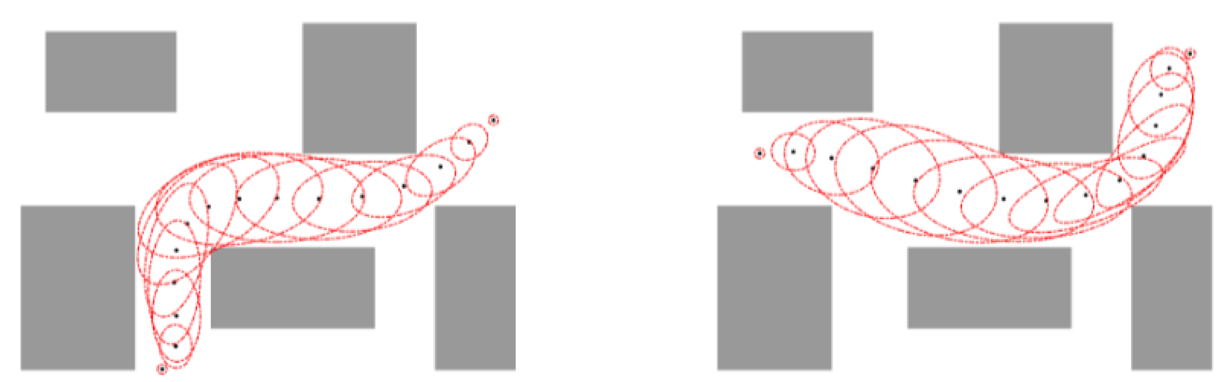

Figure 1. Feasible Trajectory Distribution Example ?

Gaussian Variational Inference

Variational Inference (VI): a standard method to solve probabilistic inference by optimization

$$\begin{aligned} q^\star &= \arg\min_{q\in\mathcal{Q}} \mathrm{KL}[q(\mathbf{x})||p(\mathbf{x}|\mathbf{z})] \\ &= \arg\min_{q\in\mathcal{Q}} \mathbb{E}_q[\log q(\mathbf{x}) - \log p(\mathbf{z}|\mathbf{x}) - \log p(\mathbf{x})] \\ &= \arg\max_{q\in\mathcal{Q}} \mathbb{E}_q \log p(\mathbf{z}|\mathbf{x}) - \mathrm{KL}[q(\mathbf{x})||p(\mathbf{x})] \end{aligned} \tag{3}$$

- $\mathcal{Q}$: Proposed distribution family
- $q(\mathbf{x})$: optimization variable
- KL: KL divergence, distance between distributions

VI optimizes the (parameterized) **proposal** distribution $\mathcal{Q}$ by (iteratively) minimizing the distance between the proposal and the target distribution. Gaussian Variational Inference (GVI) is when $\mathcal{Q}$ equals Gaussian distributions.

Difference and connection to the MAP solution (2):

- (2) tries to find the probability maximizer of the target distribution.
- (3) tries to find the distribution that minimizes the distributional distance to the target.

Maximum Entropy Motion Planning

Re-write the objective function in (3) as

$$q^\star = \arg\max_{q\in\mathcal{Q}} \mathbb{E}_q[\log p(\mathbf{x}|\mathbf{z}) - \log q(\mathbf{x})] = \arg\max_{q\in\mathcal{Q}} \mathbb{E}_q[\log p(\mathbf{x}|\mathbf{z})] + H(q) \tag{4}$$

- $\mathbb{E}_q[\log p(\mathbf{x}|\mathbf{z})]$: Expected motion planning objective
- $H(q) = -\mathbb{E}_q[\log(q)]$: Entropy of the proposal q
- **Entropy regularized** motion planning

Natural Gradient Descent Algorithm

- Proposal Gaussian $q \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; Negative log-posterior $\psi(\mathbf{x}) = -\log p(\mathbf{x}|\mathbf{z})$
- Natural gradient descent iteration is (Opper and Archambeau (2009))

$$\boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\mu} = -\frac{\partial V}{\partial \boldsymbol{\mu}}, \quad \delta\boldsymbol{\Sigma}^{-1} = \frac{\partial^2 V(q)}{\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}^T} - \boldsymbol{\Sigma}^{-1} \tag{5}$$

- Important intermediate variables

$$\frac{\partial V(q)}{\partial \boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-1}\mathbb{E}[(\mathbf{x}-\boldsymbol{\mu})\psi(\mathbf{x})], \quad \frac{\partial^2 V(q)}{\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}^T} = \boldsymbol{\Sigma}^{-1}\mathbb{E}[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}^T)\psi(\mathbf{x})]\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbb{E}[\psi(\mathbf{x})] \tag{6}$$

- Methods used to compute the expectations
- Closed-forms for priors
- Gauss-Hermite quadratures for collision factors

Sparsity and Factorizations

- Sparse inverse covariance in prior $\mathbf{K}^{-1} = \mathbf{B}^T\mathbf{Q}^{-1}\mathbf{B}$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & & & \\ -\boldsymbol{\Phi}(t_1, t_0) & \mathbf{I} & & \\ & & \cdots & \\ & & -\boldsymbol{\Phi}(t_N, t_{N-1}) & \mathbf{I} \\ & & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{Q}^{-1} = \mathrm{diag}(\mathbf{K}_0^{-1}, \mathbf{Q}_{0,1}^{-1}, \dots, \mathbf{Q}_{N-1,N}^{-1}, \mathbf{K}_N^{-1})$$

and $\mathbf{Q}_{i,i+1} = \int_{t_i}^{t_{i+1}} \boldsymbol{\Phi}(t_{i+1}, s)\mathbf{F}(s)\mathbf{Q}_c\mathbf{F}(s)^T\boldsymbol{\Phi}(t_{i+1}, s)^T ds$

- Factorized objectives
  - Prior factors and Obstacle factors (Mukadam et al. (2018))

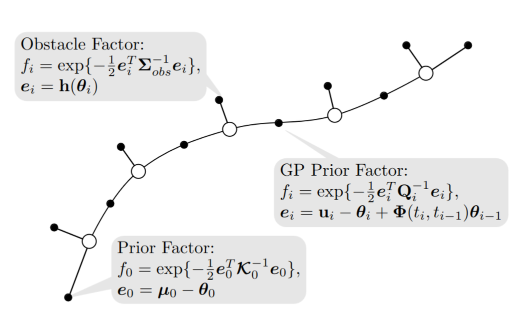

Figure 2. Factor graph for Motion planning

- Optimization on factor levels

$$\begin{aligned} V(q) &= \mathbb{E}_q[\log q(\mathbf{x})] - \sum_{k=1}^{K}\mathbb{E}_{q_k}[\psi_k(\mathbf{x}_k)] = \mathbb{E}_q[\log q(\mathbf{x})] - \sum_{k=1}^{K}\mathbb{E}_{q_k}[\log p(\mathbf{x}_k) + \log p(\mathbf{z}|\mathbf{x}_k)] \\ &\triangleq \frac{1}{2}\log(|\boldsymbol{\Sigma}^{-1}|) + \sum_{k=1}^{K} V_k(q_k) \end{aligned} \tag{7}$$

Experiment Results

- Convergence results

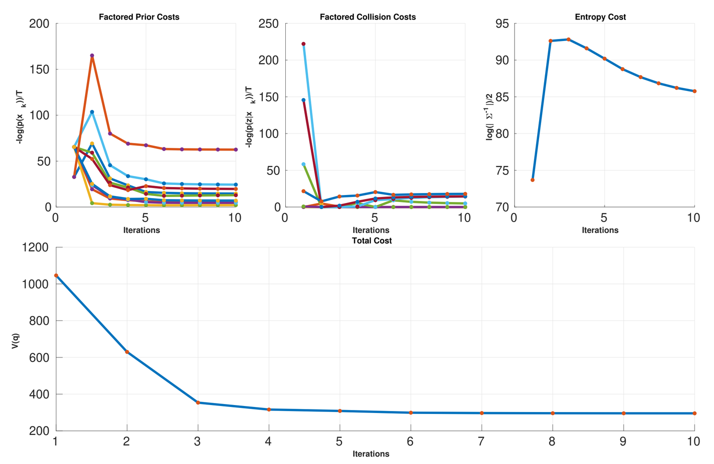

Figure 3. Factor and total costs.

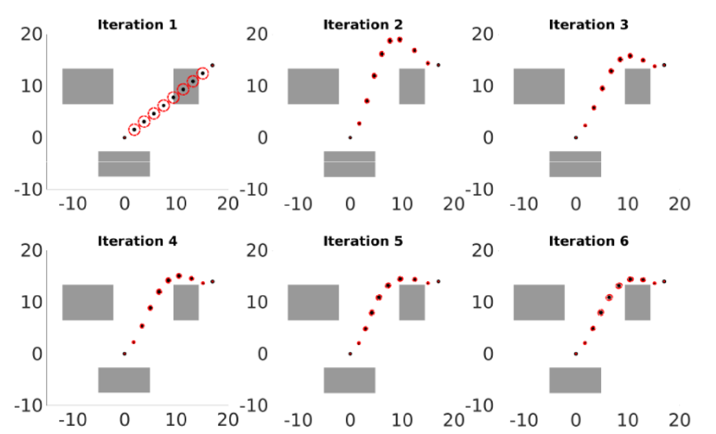

Figure 4. Convergence

Robust Planning

- **High temperature** planning T in (4) formulation: $q^\star = \arg\max_{q\in\mathcal{Q}} \mathbb{E}_q[\log p(\mathbf{x}|\mathbf{z})] + TH(q)$

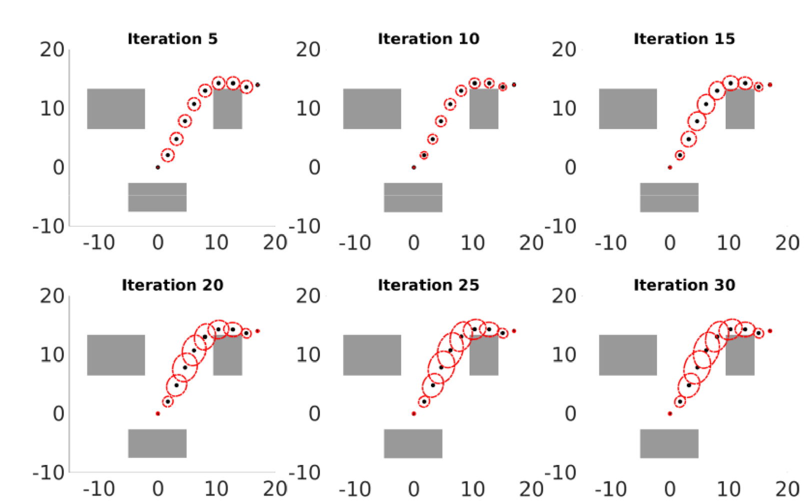

Figure 5. High temperature

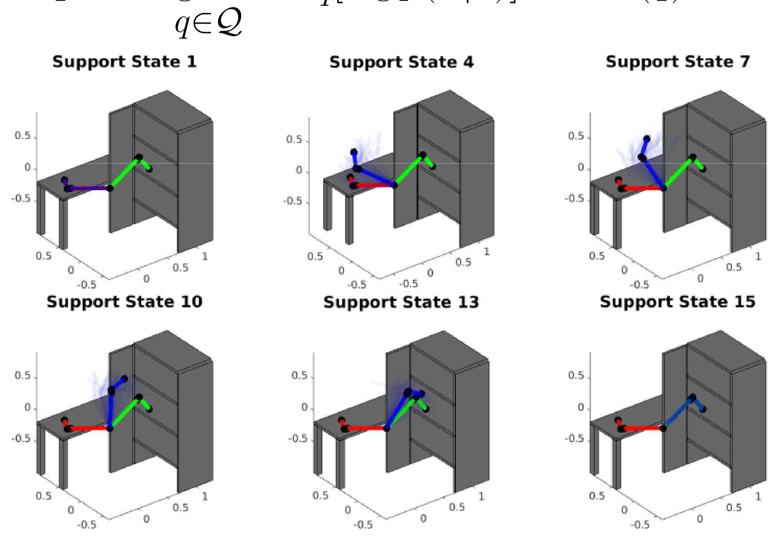

Figure 6. 7-DOF WAM Arm

- **Shorter** but more **risky** trajectory or **longer** but **safer** one?

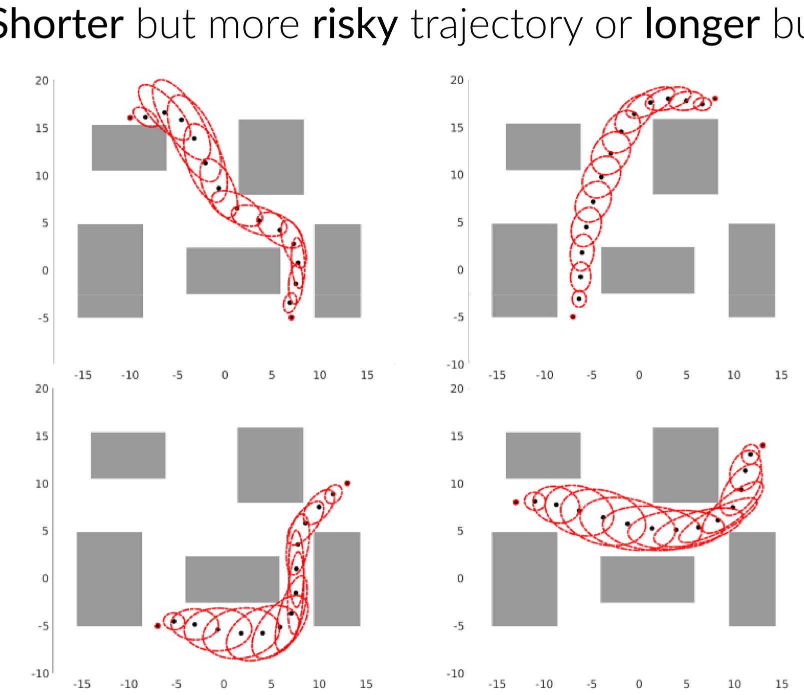

Figure 7. Planning in cluttered environment

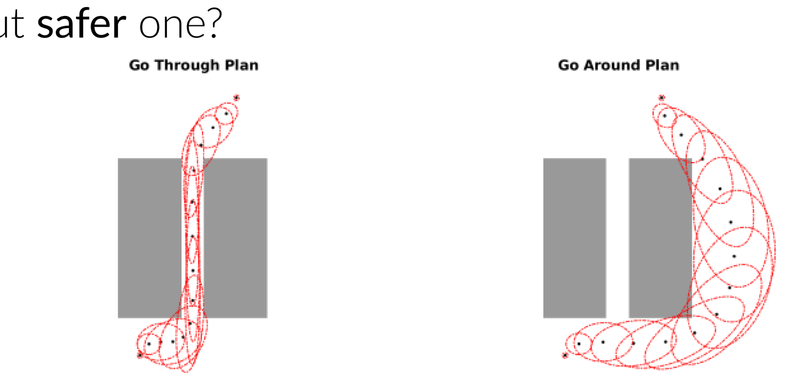

Fig. 3: Entropy regularized motion planning [35].

	Prior	Collision	MP	Entropy	Total
Left	**34.4583**	9.1584	**43.6168**	44.1752	87.7920
Right	42.9730	**2.0464**	45.0193	**39.9193**	**84.9387**

Figure 8. High-entropy plan indicates a more robust plan in the probabilistic sense.

References

Mukadam, M., Dong, J., Yan, X., Dellaert, F., and Boots, B. (2018). Continuous-time gaussian process motion planning via probabilistic inference. *The International Journal of Robotics Research*, 37(11):1319–1340.

Opper, M. and Archambeau, C. (2009). The variational gaussian approximation revisited. *Neural computation*, 21(3):786–792.