APPM 2360 Project 1

A Mathematical Investigation of Populations and Predator-Prey Dynamics

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0 Introduction

Differential equations are a tool commonly used to predict things such as weather patterns, the growth of species populations, and the process of two objects reaching temperature equilibrium,etc. This type of math allows the behavior to be known with simply the derivative of the system.

Systems of differential equations are powerful tool to model real life complicated interactions between subjects under study. In praticularl, the tools provided by the study of differential equations can give great insights about the behavior of the subject differential equations describe. During the course of this project, systems of differential equations, and in particular the logistic growth equation were studied to describe and study the populations of predators and prey, in which both populations rely on each other as well as outside factors. The species under study were were in the Sierra-Nevada region, mule deer, the prey, and mountain lions, the predator.

1 Background

As the knowledge of differential equations and the behaviors and interactions of populations became more vast, the models for population evolved as well.

The first model was the logistic growth model, which modeled the population of mountain lions described by the logistic growth equation:

$$\frac{dy}{dx} = r(1 - \frac{x}{L})x \tag{1}$$

This equation includes two constants: r, which is the rate of growth of the species, and L, which represents the carrying capacity that the environment provides for the species. To fit this equation better with the behavior of the deer, a hunting equation, $H(x) = \frac{px^2}{q+x^2}$ is subtracted from the logistic growth equation, which gives:

$$\frac{dy}{dx} = r(1 - \frac{x}{L})x - \frac{px^2}{q + x^2} \tag{2}$$

The hunting equation also includes two constants, p and q, which change with the efficiency of the lions' hunting.

To analyze the behavior of two populations at once, a system of differential equations is necessary. The work of evolutionary biologist inspired great mathematicians like Mathematicians Vito Volterra and Alfred Lotka to develop models to study prey and predators relationship. The first system used is the Lotka-Volterra System.

$$\frac{dx_1}{dt} = -ax_1 + bx_1x_2 \tag{3}$$

$$\frac{dx_2}{dt} = cx_2 - dx_1x_2 \tag{4}$$

This system is derived from the Balance Law, which involves the net rate of change of the population of each population. More specifically, each constant a, b, c, d involves the birth and death rates of one of the two species studied. The final two terms in both $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ involve the populations relying on each other. However, this isn't the most accurate model, since the populations of each species depend on each other at a larger degree than is present in the Lotka-Volterra system.

The second system of equations used to represent the populations of mule deer and mountain lions over time is the Logistic Predator-Prey system:

$$\frac{dx_1}{dt} = -ax_1 + bx_1x_2 \tag{5}$$

$$\frac{dx_2}{dt} = c(1 - k)x_2 - dx_1x_2 \tag{6}$$

This system has both populations growing in a logistic fashion instead of in an exponential fashion, increasing the accuracy. Furthermore, there is now a limit on the number of lions that the environment can support, due to the $(1-k)$ included in $\frac{dx^2}{dt}$.

2 Modeling Individual Populations: the Logistic Equations

2.1 Mountain Lion Population Analysis

The units of the parameter r in the logistic growth equation (1) of the mule deer is going to be in dozens of lions/time, since this represents the growth rate of the deer over time. L is the carrying capacity, which should be in units of dozens of lions. Dozens of lions is what x is measured in, allowing for simpler calculations than in simply the number of deer. The analytical solution to equation 8 is given by:

$$x(t) = \frac{1}{1 + (\frac{L}{x_0} - 1)e^{-rt}} \tag{7}$$

And it can be obtained via separation of variables. Appendix ..... shows the derivations related.

Euler's method is then used to analyze the population of the mountain lions: equation (1) is used with $r = 0.65$ and $L_m = 5.4$. The step sizes for these Euler calculations, or h_1, h_2, h_3, are 0.5, 0.1, and 0.005, respectively. These three solutions are then represented on Figure 1.

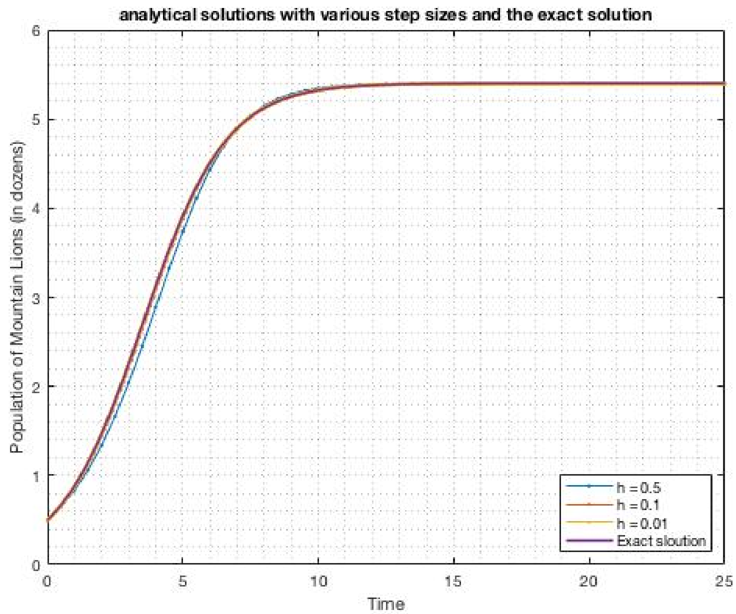

Figure 1: Euler Solutions for Mountain Lion Population

The Euler solution involving $h = 0.005$ is considered the most accurate of these three solutions, even though their graphs look very similar. In general decreasing the step size yields better results, however it takes more time to do the computations. Hence, It was then necessary to calculate the error using the exact solutions at the points in time that were calculated using Euler's method. This error was calculated by taking the absloute difference between the Euler's method results and the exact sloution given by equation 7.

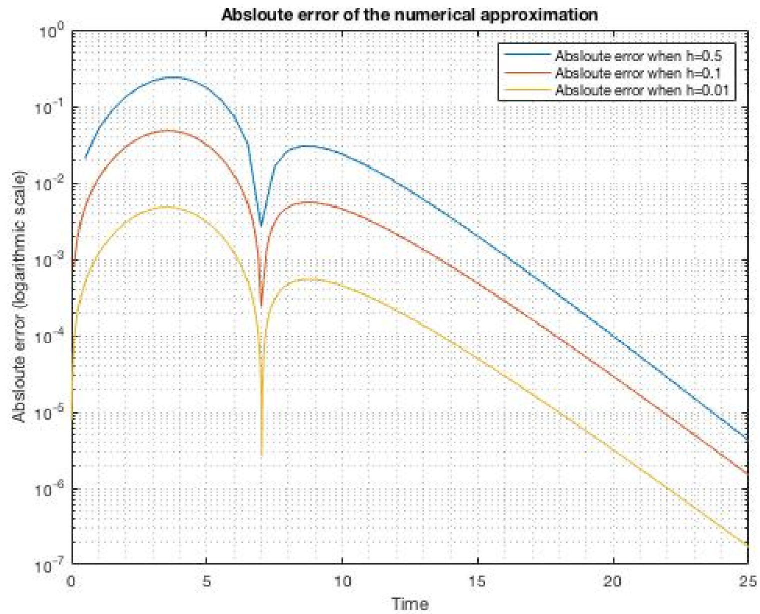

Figure 2: Absolute Error of Euler's Method in Deer Population Growth Equation

Looking at the graph of error (Figure 2), there is a dip in error at $t = 7$, which is represented in Figure 1, where the Euler solutions are considerably closer to the exact solution. This occurs because the population starts to level out around this point in time, since the population is approaching the carrying capacity. The best h size to use in terms of efficiency would be $h = 0.01$: the tic toc function of Matlab was used in the code, and no matter how many iterations were ran, the fastest of the three h values tested was $h = 0.01$. $h = 0.01$ still provides a small error compared to that of $h = 0.1$, which took longer to run. In general, the type of application dictates the preferable step size to have a balance between accuracy and computation time. In this case, it seems that a step size of 0.1 is more convenient since it does not take much time, and at the same time it has an error that is relatively close to a step size of 0.01.

2.2 Mule Deer Population Analysis

The use of the harvesting function (equation 6) allows the logistic growth equation of the mule deer to be more realistic, adding in the factor that deer are preyed on by the mountain lions. This function is considered to be first-order, nonlinear, and autonomous. This function is considered autonomous because x does not depend on t at all in this equation. Put it differently, the population of the deer depends only on themselves, and that is the physical meaning of autonomy.

Analyzing $H(x)$ with different x_0 values produces a graph that ends up leveling off around 1.2 dozen as x_0 increases. At $x = 0$, the harvesting function is zero, due to there being no deer to harvest.

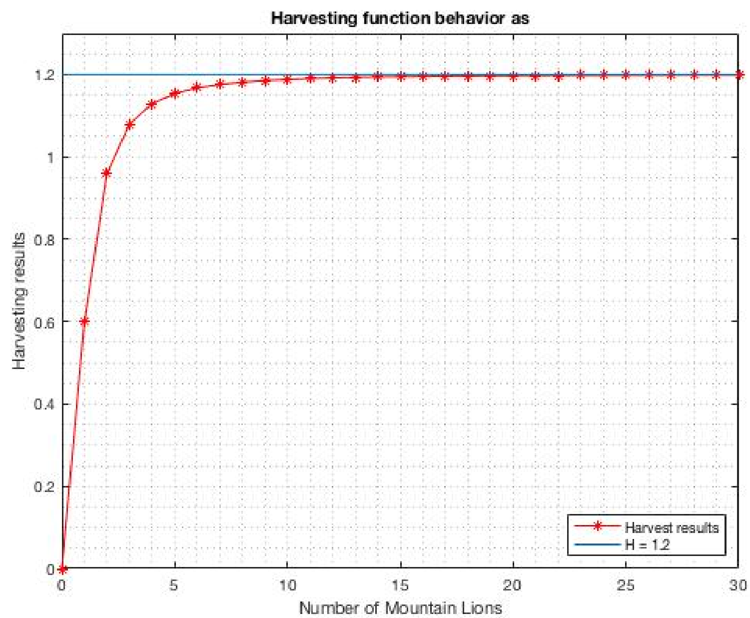

Figure 3: Behavior of Harvesting Function as Number of Lions Increases

When x_0 is very small, $H(x)$ is very small, since the less deer there exist in the ecosystem, the less deer that can be harvested. In other words, near x = 0 the harvesting goes to 0. However, the lions can only harvest so many deer no matter how large the population gets, which is why the function seems to level out at around 1.2 dozen deer.

First, the direction field of Equation (2) is analyzed. This is then compared to the Euler solutions with a step size of $h = 0.1$ over the interval $t = [0, 75]$. Two initial x values were chosen: $x = 2$ and $x = 8$.

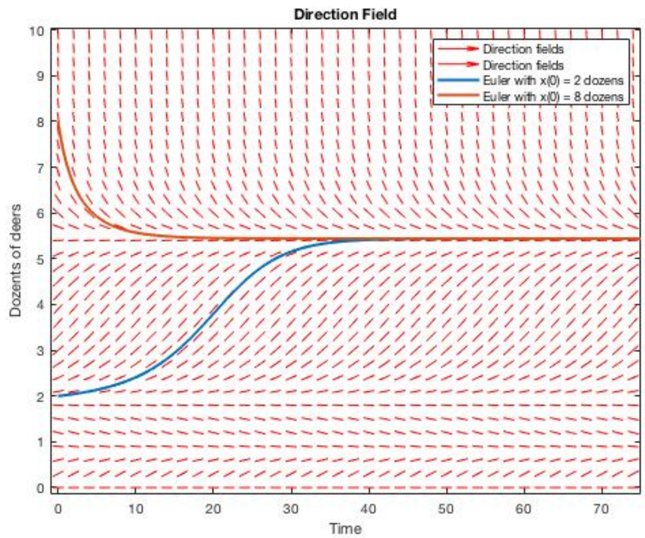

Figure 4: Direction field of the Deer Population Differential Equation

At $x = 2$, the population trends upwards towards the carrying capacity (minus the maximum of $H(x)$ and stays there when it is reached, while at $x = 8$, the population trends downwards towards 5.5 dozen deer, which is the amount that the ecosystem can sustain minus the amount of deer harvested by the lions. As time went on, the population of deer stayed around 5.5 dozen deer after it was initially reached. This is reasonable to expect, since any population in a system would converge towards the maximum carrying capacity.

3 Modeling Population Interactions

3.1 Lotka-Volterra System

The Lotka-Volterra system is a non-linear, first order, autonomous set of differential equations. Neither of these equations rely on t, so these equations can both be classified as autonomous. In addition, the x_1x_2 portions of both equations present render the system non-linear, since these 2 variables do rely on each other. This system is a first-order system, since both population variables each have one solution at each time value, t.

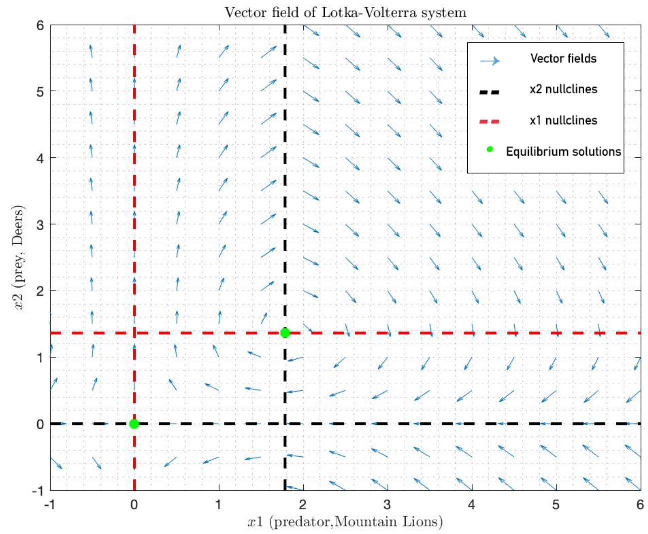

Figure 5: Direction field of the Lotka-Volterra Equation System

The nullclines of a system of differential equations occur where the the derivative of one variable with respect to time is zero, while equilibrium points occur at the intersection of two nullclines, or where $\frac{dx_2}{dt} = 0$. Those nullclines can be expressed as $\frac{dx_1}{dt} = 0$ and where $\frac{dx_2}{dt} = 0$ (individually). Equilibrium points occur at the intersection of two nullclines, or where $\frac{dx_2}{dt} = 0$ and $\frac{dx_1}{dt} = 0$. It can be seen from equations 3 and 4 that equation 3 that describes the variable x1 (predator) has two nullclines that occur when x2 = (a/b), and x1 = 0. Similarly, the variable x2 (the prey) have two nullclines, when x2 = 0, and x1 = c/d. The equilibrium points hence are at the points (0,0) and $(\frac{c}{d}, \frac{a}{b})$, where the points represent points on the x1-x2 plane, i.e. (x1,x2).

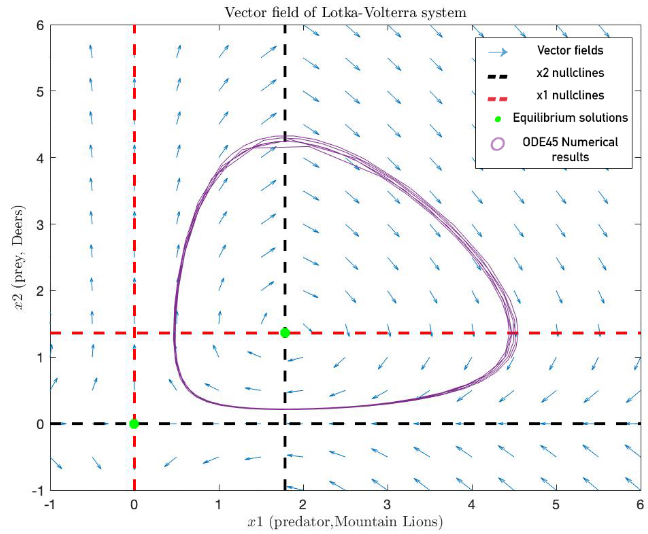

Figure 6: Solution Curve of the Lotka-Volterra System using Euler's method with $a = 1.5, b = 1.1, c = 2.5, d = 1.4$

The curves plotted on top of the direction field show that these curves are in phase with the rest of the solution curves. Asymptotic behavior is shown at $y = 0$; the equilibrium solution at (0,0) is semi-stable due to the direction field trending partly away from and partly towards this solution, and the equilibrium solution at $(\frac{c}{d}, \frac{a}{b})$ is also semi-stable, as the direction lines trend partly towards, and partly away from it. As the deer population increases, the mountain lion population is seen to increase as well; moreover, when the mountain lions eat too many of the deer, the deer population decreases, causing a decrease in the mountain lion population due to too little food. This decrease in the predators of the deer allows them to flourish, starting the cycle over again. The curves clearly depict a cyclic solution.

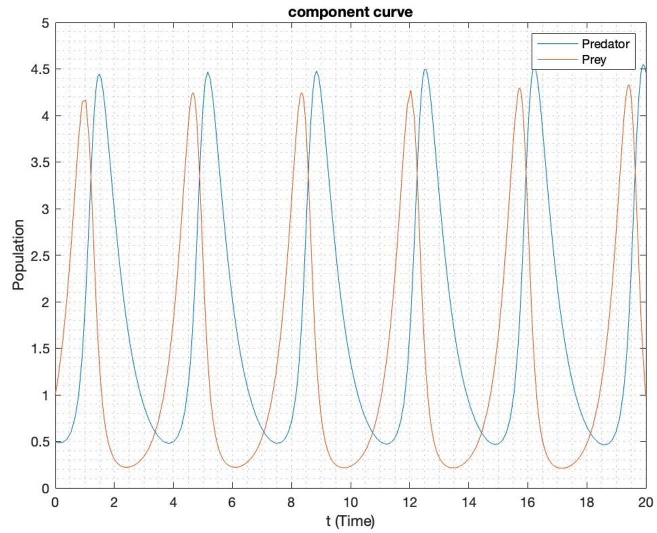

Figure 7: Solution Curve of the Lotka-Volterra System over time using Euler's method with $a = 1.5, b = 1.1, c = 2.5, d = 1.4$

In addition, the populations don't exactly peak and dip at the same time due to the time needed for populations to grow and decrease.

3.2 Logistic Predator-Prey System

As explained above, the nullclines of a system of differential equations happen when the derivatives are zero. For the system expressed by equations 5 and 6. For the variable x1 (the predator), $x_2 = \frac{a}{b}$, and $x_1 = 0$. In addition, for the prey x_2, one nullcline occurs when x2 = 0, and the other nullcline appears at line x2 = (1/k) * (1 - (d/c)*x1) . The equilibrium points here are at the points (0,0) and (0, $\frac{c}{d}$, and ($\frac{c}{d}$, $\frac{a}{b}$)), where the points represent points on the x1-x2 plane, i.e. (x1,x2).

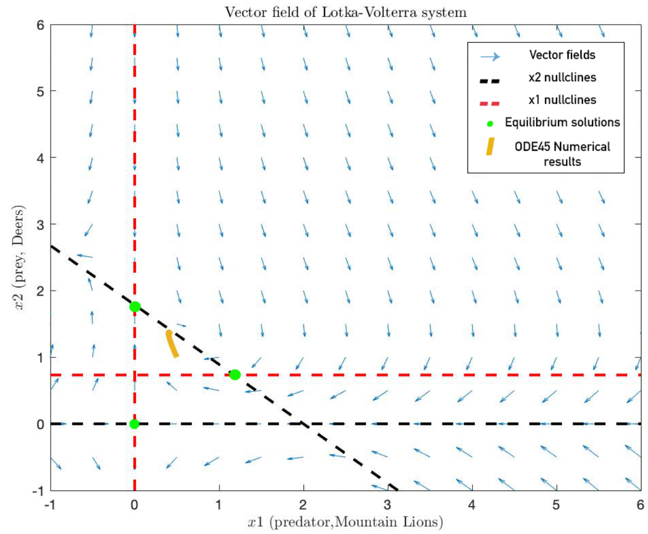

Figure 8: Plot of Direction Field of Logistic Predator-Prey System w/ Equilibrium Solutions and Nullclines

The plot of the direction field shows the direction trending towards the nullcline $x_1 = 0$, but then trending upwards around the nullcline $x_2 =$ a/b. Unlike in the Lotka-Volterra system, this system has an equilibrium point that is stable, $(\frac{c}{d}, \frac{a}{b}))$. The direction field seems to be trending the solution curves towards an equilibrium, at which they stay.

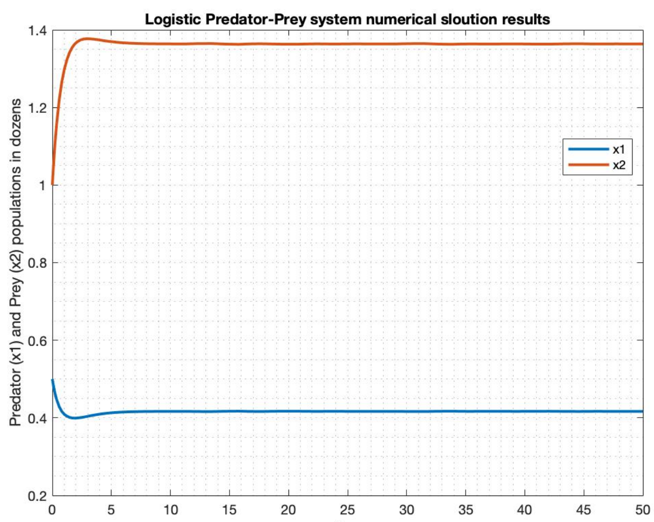

Figure 9: Logistic Predator-Prey Curves with $a = 1.5, b = 1.1, c = 2.5, d = 1.4, k = 0.5$.

This is proven while looking at the solution curves graph of the Logistic Predator-Prey

System. There is no periodic behavior; however, the solution curves for x_1 and x_2 do both trend towards asymptotes.

3.3 Comparing the Two Systems

The main strength of the Lotka-Volterra (3,4) model is that it easily shows the cyclic nature of the populations of these species over time in response to the population of one another; its' main downfall is that it doesn't show the populations interacting, and it also doesn't have a check on the availability of food for predators. The main strengths of the Logistic Predator Prey (5,6) model is that it takes into account the interactions of the populations and has them rely on each other; however, the solution graph looks almost nothing like that of an actual population graph of a predator and prey species over time, making it hard to figure out what is going on regarding each population. Each of these systems have their strengths and weaknesses; however, neither of these systems took into account the shared resources that both these species rely on. The addition of a limiting factor due to water consumption coming from both species could make the population modeling systems more accurate.

4 Conclusion

Through using the three models of populations of a predator-prey system, conclusions can be drawn about predator-prey relationships through finding the nullclines and equilibrium values. The three different models represented different aspects of population growth. The simple use of the logistic equation allows a single population to be plotted over time; however, this method doesn't show any predator-prey interaction. The Lotka-Volterra model provides information about population change in a very basic way, showing the cyclic nature of a predator-prey population relationship. The Logistical Predator-prey model brought in the factor of carrying capacities to the relationship, which showed a more realistic version of what happens to the populations throughout time.

Euler's method was extremely helpful in finding solution curves to these systems, allowing for a better visual of what was happening than simply looking at the direction fields. In addition, the nullclines showed where the populations' minimum values were over the course of the cycle. The equilibrium values showed points where the population wouldn't change, with the stable one being in the center oft he Euler solution curves. Overall, the use of differential equations in examining the relationship between predators and their prey shows the cyclic behavior due to population change or predation and the asymptotic trending of each population over time.

Appendices

A Solution of the Logistic Equation

$$\frac{dy}{dx} = r(1 - \frac{x}{L})x \tag{8}$$

$$\frac{dy}{y(1 - \frac{y}{L})} = rdt \tag{9}$$

$$\frac{1}{y} + \frac{\frac{1}{L}}{(1 - \frac{y}{L})}dy = rdt \tag{10}$$

$$\ln \frac{y}{1 - \frac{y}{L}} = rt + c \tag{11}$$

$$\frac{y}{1 - \frac{y}{L}} = Ce^{rt} \tag{12}$$

5 References

- APPM 2360 Project 1 A Mathematical Investigation of Populations and Predator-Prey Dynamics

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- Predators and Prey A Case of Imbalance Mountain Lions and the North Kings Deer Herd. Johnston Ridge