ASEN 2003: Lab 4
Unbalanced Wheel

Abdulla Al Ameri*, Adam Elsayed†, Samuel Firth‡, Ian Thomas§

University of Colorado Boulder, Boulder, CO, 80303

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[redacted]

Contents

The purpose of this lab was to use energy and work methods to model the motion of a cylinder rolling down a ramp. Four trials were run, two with a balanced cylinder and two with an extra mass a short distance off center; data was taken on the motion of the cylinder by way of a rotary encoder. Additionally, four Models were derived, two for each case. For the balanced wheel, Model 1 overestimated the angular velocity slightly, with an average overestimation of 0.61 rad/s; Model 2 (which incorporated a constant moment due to friction) matched the motion of the cylinder almost perfectly, with an average error in angular velocity of 0.025 rad/s. For the balanced wheel, both Models 3 & 4 also matched the motion of the cylinder very well, with an average error in angular velocity of 0.08 and 0.05 rad/s. It is believed that these discrepancies were mostly caused by irregularities in the surface of the ramp and the surface of the wheel.

Nomenclature

θ = angular position (i.e. how many degrees the blue wheel turned by from the beginning)
κ = radius of gyration of wheel
β = ramp angle
ω = angular velocity
m_w = mass of cylinder
m_s = mass of trailing supports
m_p = mass of extra mass
R = radius of cylinder
I = moment of inertia
r = radius to extra mass
r_m = radius of the extra mass
g = acceleration due to gravity
M_f = Moment due to friction
U = Potential Energy
T = Kinetic Energy
W = Work done

I. Model

To model the system under study [1] , four different models were derived form the kinematics relationships and energy methods to predict the angular velocity behavior as a function of θ. It is worth noting that all the models used a coordinate system with its origin at the center of the wheel at the initial position.

The first model considered the wheel without the extra added mass. In model 1 (MATLAB Code shown in Appendix B:2), the wheel was modeled assuming the wheel rolled without slipping, and supports translate but do not rotate. The derivations related to Model 1 are shown in Appendix C:1 .

Model 2 (MATLAB Code shown in Appendix B:3) was modeled after Model 1, however the only difference is that Model 2 accounts for a moment that has an unknown value applied at the center of mass of the wheel. This moments comes from the friction between the attachments of the supports and the big wheel. The moment does a non-conservative work because it is due friction, and hence it was added to the energy balance expressions. Model 2 derivations and expressions are shown in Appendix C:2.

Model 3, and Model 4 (MATLAB Code shown in Appendix B:4 , B:5) included the additional mass added at a distance away from the center of mass of the wheel. The unknown moment was included in both models, however the only difference is how the additional mass is treated. In Model 3, the additional mass was treated as a point mass. In Model 4, the additional mass was treated as a rigid body. This difference only changed the expression for the moment of inertia of the additional mass. Model 3 and 4 derivations and expressions are shown in figure Appendix C:3 and Appendix C:4 respectively.

II. Experiment

The experimental setup consisted of a large, blue cylinder rolling down a carpeted ramp. The cylinder is held in place at the top of the ramp by a doorstop until data is ready to be taken. A rotary encoder (similar to the encoder used in the Locomotive Crank Shaft Lab) is affixed to the central axis of the cylinder and is held in place by a trialing apparatus, which serves to keep the encoder at a fixed angle relative to the rotating cylinder. This means that the encoder can then determine the angular velocity and total angle rotated by the cylinder.

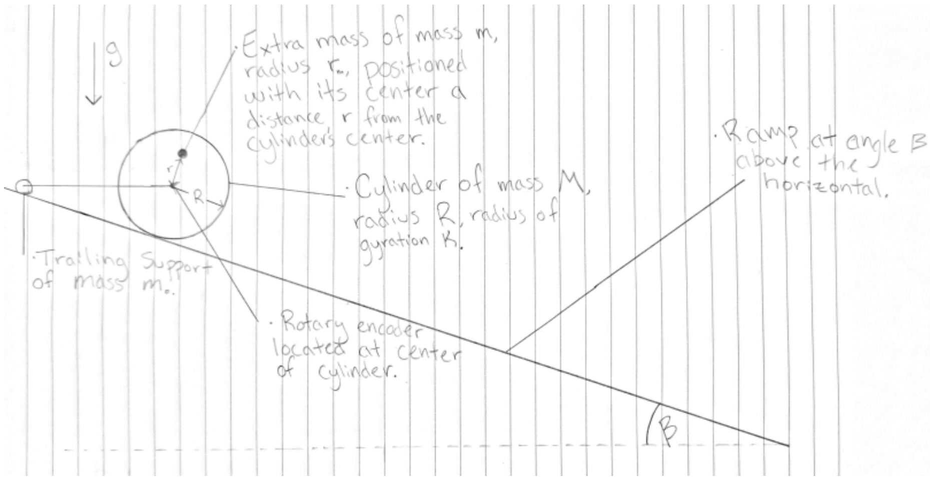

Fig. 1 Sketch of the experimental setup.

Four trials were run: two trials with an extra mass inserted off-center of the cylinder and two trials without. The cylinder was placed at a specific point at the top of the carpeted ramp and secured by two students. For the case of the unbalanced cylinder, the extra mass was positioned normal to the ramp, on the far side of the cylinder's central axis. Once data collection had been initiated, the doorstop was removed and the cylinder allowed to roll down the ramp. Once it reached the bottom, the cylinder was caught by another student to safely bring the mass to a halt.

During the balanced trials, the cylinder roller regularly down the hill, speeding gradually up as it rolled. Although Model 1 shared the same general shape, the lack of friction caused the model to speed up much more quickly than the actual cylinder. Model 2 incorporated friction and matched the experimental data almost perfectly.

During the unbalanced trials, the rolling motion was much more irregular. Once released from its initial position, the cylinder initially sped up much more quickly than the balanced case, but then slowed almost to a stop on the ramp upon completing one revolution. After the slow start, the unbalanced wheel kept rolling down the ramp, speeding up when the extra mass was on the downswing and slowing down when the extra mass was on the upswing. Both Models 3 & 4 predicted this motion and, after some tinkering, matched the actual motion of the cylinder impeccably for the first two revolutions. See Section 3 for more performance details.

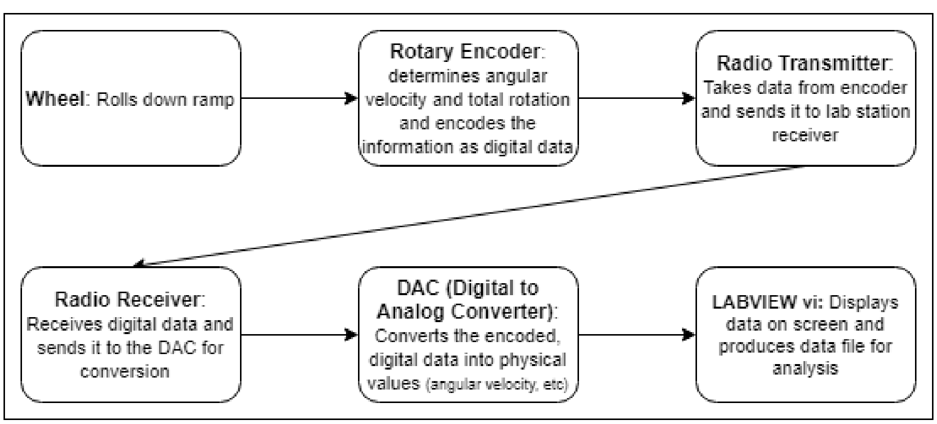

Fig. 2 Functional block diagram of the system.

Figure 2 shows the process by which the data from the rotary encoder mounted on the cylinder is turned into a data file on the ITLL servers. The angular velocity and angular position as functions of time are measured by the rotary encoder. The encoder consists of a plate with holes punched in it at specific locations, a laser, and a light sensor. As the encoder rotates, the laser either shines through a hole or is blocked by the plate. The light sensor sends out a low voltage if there is not light received and a high voltage if the light shines through, thus digitizing the information of the angular rotation and position as functions of time.

The newly digitized data is then sent to a radio transmitter located on the rotary encoder, which sends it to a radio receiver located at a nearby lab station. The digital data needs to be converted back into meaningful information before it can be parsed. This is done by a DAC (Digital to Analog Converter), which processes the data and sends it through to be read by the LABVIEW vi. The cylinder data is then displayed on-screen in real time and can be saved into a data file, ready to by analyzed by the group.

III. Results & Analysis

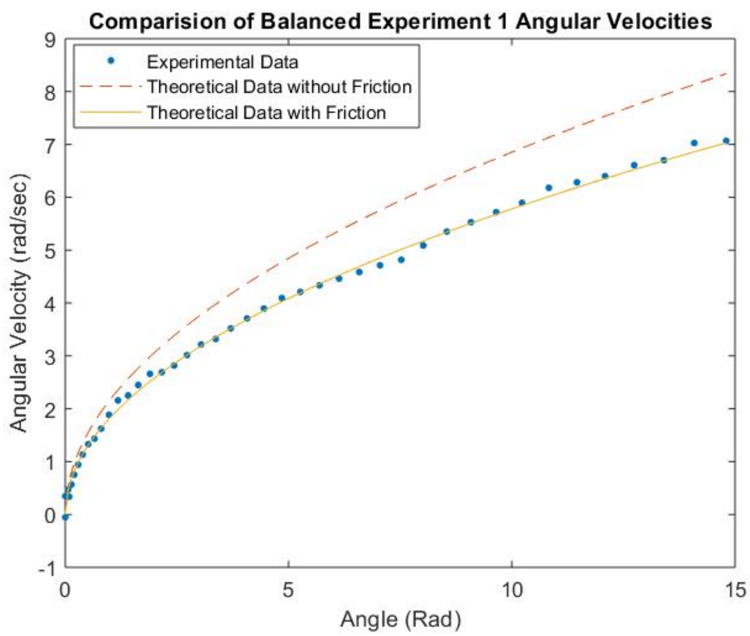

Fig. 3 Balanced wheel experiment 1 results, model 1, and model 2.

In general, the most detailed models showed no noticeable differences from the experimental data. In other words, the models did a very good job predicting the actual performance of the wheel. The general trend was a linear increase in the angular velocity of the wheel when the mass is not inserted, compared to a cyclic behavior when the extra mass is included.

While the balanced wheel with no friction model deviated from actual values, the inclusion of the frictional moment created a very accurate representation of the data. Of course, the actual data was not perfectly smooth due to imperfections in the experimental setup. However, the theoretical angular velocities compare to the data quite well.

When it came to the unbalanced wheel, the frictional moment was always included. Both modeling the extra mass as a point and a cylinder seemed to capture the data very well.

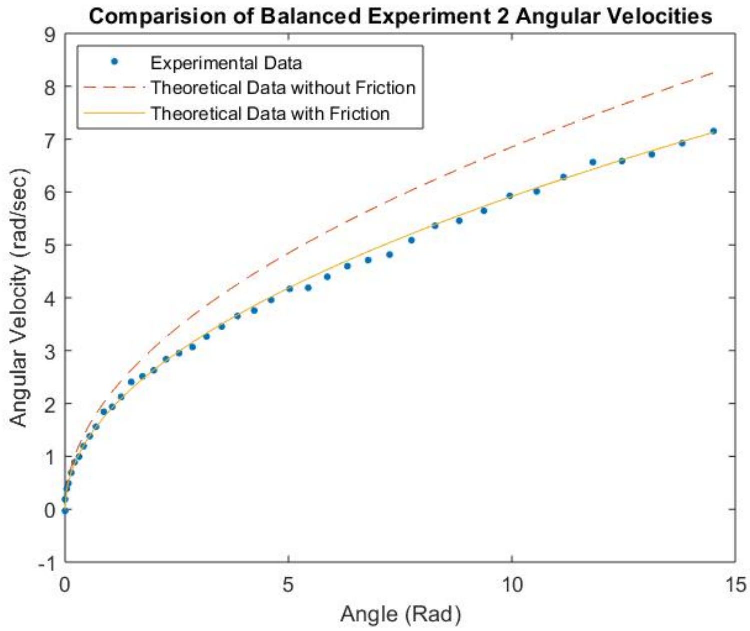

Fig. 4 Balanced wheel experiment 2 results, model 1, and model 2.

One difference that was observed is the amount of noise in the actual experimental data compared to the model that was smooth. Those differences might be due to the assumptions made when the model was derived. One assumption was that the wheel attached to the supports translates but do not rotate at all, where in reality this may not be entirely true. Another source of noise can be the surface of the ramp itself, that is to say that the imperfections in the surface of the ramp might be something that the sensors are sensitive to. In addition, it is also possible that the sensors themselves had some amount of error, especially since the sensors used radio waves to communicate the data and any obstacles in the path of the wave might affect the data.

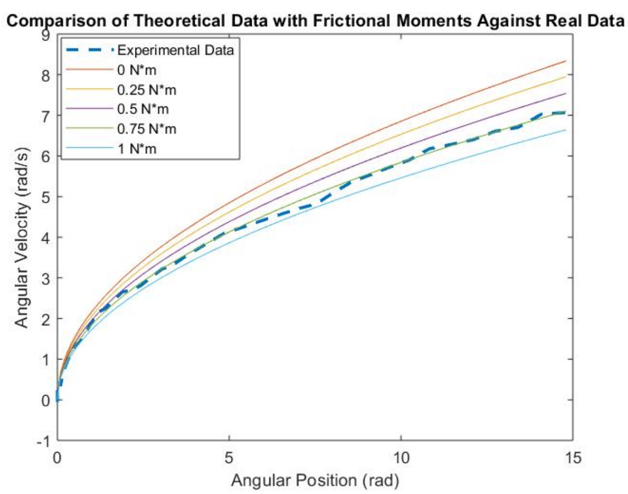

Fig. 5 Frictional Moment Comparisons

In order to accurately predict the angular velocity of any wheel rolling down a ramp, moments caused by friction must be taken into account. Figure 5 illustrates how altering the estimated moment due to friction can affect the validity of the theoretical model. Clearly, small changes to this value impact the predictions drastically. From this image, it is obvious that a frictional moment of approximately 0.75 N-m leads to the most precise representation of a wheel rolling down a ramp.

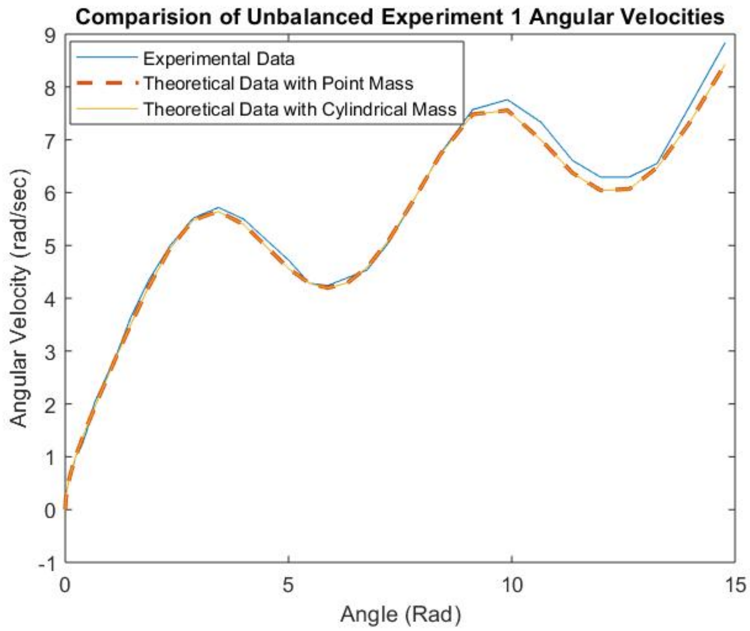

Fig. 6 Unbalanced wheel (experiment 1) results and Model 3.

Figure 3 and figure 4 shows the experimental results compared to model 1 (without added moment) and model 2 (with added moment). It is very clear from figure 3 and figure 4 that the additional moment added from friction modeled the actual data far better than the model without friction. The difference is not significant for lower angles, however the difference increases as the angle of rotation increases. Please note that the moments used here were derived from each experimental data set individually, not taken to be the approximated values discussed above. These values were 0.69 N-m for experiment 1 and 0.79 N-m for experiment 2, clearly showing that 0.75 N-m was an accurate estimation.

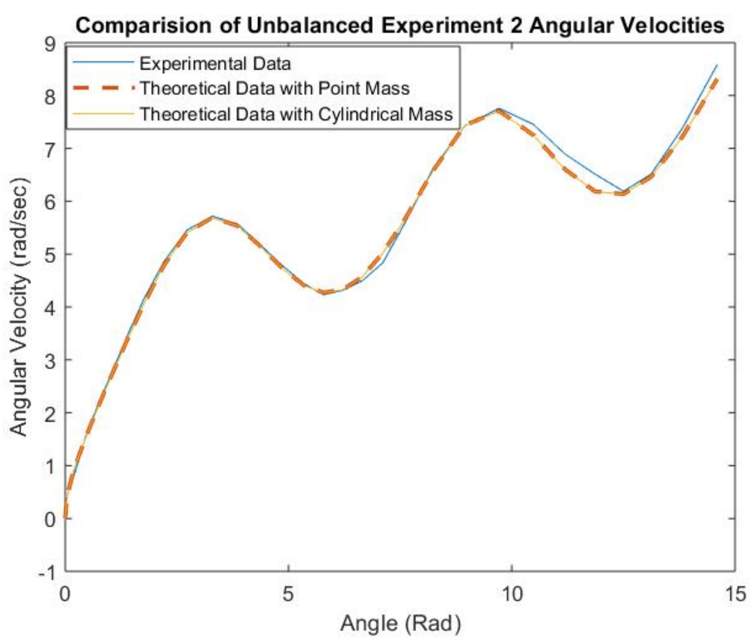

Fig. 7 Unbalanced wheel (experiment 1) results and Model 4.

Furthermore, the difference between modeling the extra mass as point mass vs a rigid body can be seen in figures 6 7. Both models, model 3 (consider the extra mass as a point mass) and model 4 (consider the extra mass as a rigid body), were almost identical. Hence, it can be observed that expressing the moment of inertia of the extra mass as rigid body or point mass does not results in any significant differences. Moreover, both models matched the trend of the experimental data. In figures 6 7, we notice a spike down at around $\theta = 15$ (rad). This happens because the wheel is forced to stop by a team-member on the other side of the ramp, which the models do not account for. Therefore, only data up to 15 radians is shown.

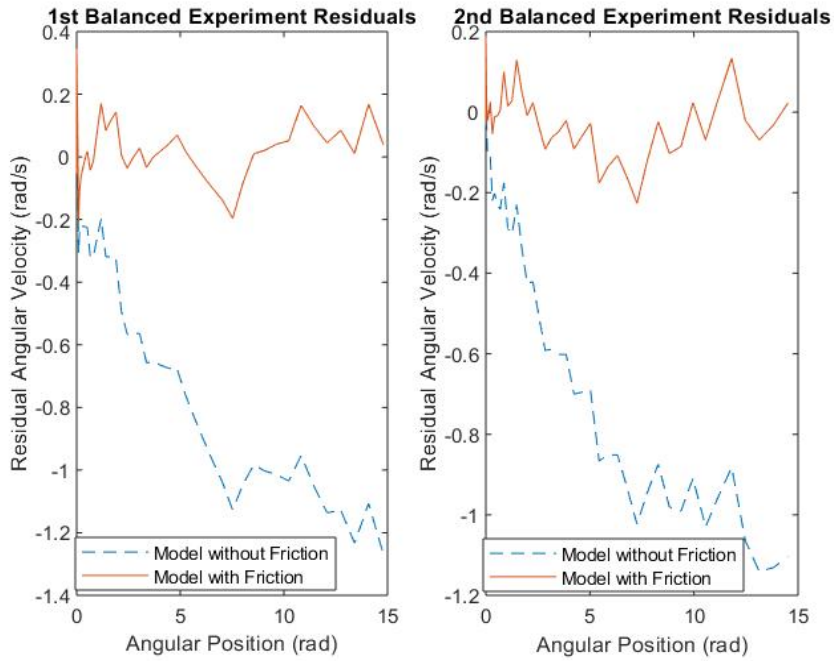

Fig. 8 Balanced Wheel Residuals for Both Experiments

The residuals between the theoretical calculations and collected data for the balanced wheel experiments is shown in Figure 8. The deviation between friction-less modeling and actual data is clear and distinct, while the differences between model 2 and the data is minuscule and random. This simply quantifies the observations made above from Figures 3 and 4. Note that the residuals for the friction-less model were negative. This is because the lack of non-conservative, negative work on the system for model 1 allowed those values to be higher. Since the expected values were higher than those observed, the residual $r = observed - expected$ gave a negative value.

The differences between models 3 and 4, however, are far less obvious. In fact, they are indistinguishable in Figure 9. Therefore, it is a fair point to say that modeling the extra mass as a point in this particular setup is not only acceptable but likely preferred. This goes to show that simplifications, when appropriate, are extremely useful.

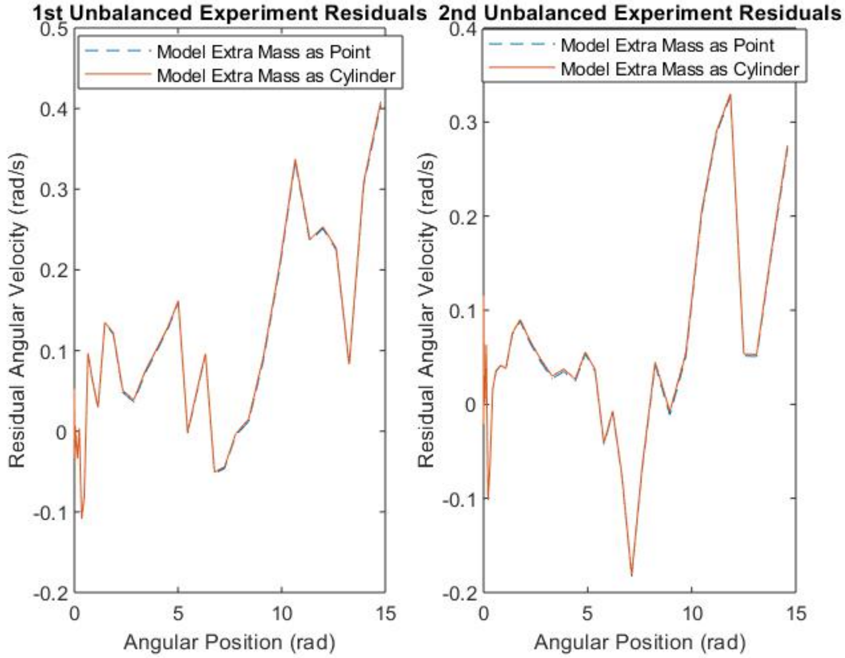

Fig. 9 Unbalanced Wheel Residuals for Both Experiments.

Table 1 shows gives statistical data on the residuals taken from each experiment/model combination. The final column is particularly interesting and reassuring for this lab group. The fact that there was only a single outlier for all data sets indicates that the experimental setup and theoretical modeling were on par with each other. This leads to great confidence in the calculated results. The single outlier in the first trial with friction was likely due to a small "hiccup" that occurred at the beginning of the roll for that attempt. Again, the mean residual was much lower for model 2 than model 1 in both cases. This was as expected. Likewise, it was expected that the mean residual for models 3 and 4 would be almost identical, as they were.

Table 1 Statistical Data on Residuals Between Test Data and Theoretical Values

Trial/Model	Mean Residual [rad/s]	SD [rad/s]	SDOM [rad/s]	Sample Population	Number of Outliers
Trial 1/ No Friction	-0.64	0.40	0.06	43	0
Trial 1/ Friction	0.02	0.10	0.02	43	1
Trial 2/ No Friction	-0.59	0.37	0.06	42	0
Trial 2/ Friction	-0.03	0.08	0.01	42	0
Trial 3/ Point	0.08	0.12	0.02	36	0
Trial 3/ Cylinder	0.08	0.12	0.02	36	0
Trial 4/ Point	0.05	0.10	0.02	35	0
Trial 4/ Cylinder	0.05	0.11	0.02	35	0

IV. Conclusions and Recommendations

In conclusion, the development of models and collecting data helped to further enhance the knowledge about planar motion of objects. It is important to note the moments (i.e. friction here) were significant when it came to develop accurate models. On the other hand, different expressions for the moment of inertia of external mass were not as important to the model, as can bee seen in the similarities between Models 3 & 4. The first experiment did not perform as well as the others due to a small "hiccup" at the beginning of the roll. The setup could be improved by removing bumps and unnecessary tape from the ramp to allow for better data collection. However, with all issues considered, this lab was certainly a success.

References

[1] "ASEN 2003 Lab 4: Unbalanced Wheel REV2," , 2019.

V. Acknowledgements

The assistance provided by the course faculty and teaching assistants and their guidance is highly appreciated. Thank you.

Appendix

A. Team Member Participation Table

Name	Plan	Model	Experiment	Results	Report	Code
Abdulla Al Ameri	2	1	1	1	1	1
Adam Elsayed	1	2	1	2	1	1
Samuel Firth	1	1	1	1	2	1
Ian Thomas	1	1	2	1	1	1

B. MATLAB

1. main.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed, Ian Thomas
% Date written: 2/27/2019
% Date modified: 3/6/2019
%
% Purpose:
% To calculate dynamics of a wheel rolling
% down a ramp under various conditions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


%% housekeeping
clear; clc; close all;
%% Define given parameters
m_wheel=11.7; %kg
m_support=0.7; %kg
m=3.4; %kg Mass of extra mass
beta=0.096; %rad: equal to 5.5 deg
R=0.235; %m: Radius of wheel
k=0.203; %m
r_tomass=0.178; %m Distance to extra mass center
r_mass=0.019; %m Radius of extra mass (cylinder)
g=9.81;

%% Read data in
[angaccel_balanced1,angaccel_balanced2,angaccel_unbal1,angaccel_unbal2,...
    theta_balanced1,theta_balanced2,theta_unbal1,theta_unbal2,omega_expbalanced1,...
    omega_expbalanced2,omega_expunbal1,omega_expunbal2]=...
    readin('balanced_1.xlsx','balanced_2.xlsx','unbalanced_1.xlsx','unbalanced_2.xlsx');

%% Model Balanced wheel, no friction
omega_balanced1_nofric=MODEL1(m_wheel,m_support,beta,R,k,g,theta_balanced1);
omega_balanced2_nofric=MODEL1(m_wheel,m_support,beta,R,k,g,theta_balanced2);

%% Create a figure showing the use of 5 different moment values for model 2,
% for comparison purposes. Use just experiment 1
[omega_m1]=MODEL2_exp(m_wheel,m_support,beta,R,k,g,theta_balanced1,0);
[omega_m2]=MODEL2_exp(m_wheel,m_support,beta,R,k,g,theta_balanced1,.25);
```

```
[omega_m3]=MODEL2_exp(m_wheel,m_support,beta,R,k,g,theta_balanced1,.5);
[omega_m4]=MODEL2_exp(m_wheel,m_support,beta,R,k,g,theta_balanced1,.75);
[omega_m5]=MODEL2_exp(m_wheel,m_support,beta,R,k,g,theta_balanced1,1);
%Plot them against each other
figure(1)
plot(theta_balanced1,omega_expbalanced1,'--','LineWidth',2)
hold on
plot(theta_balanced1,omega_m1,theta_balanced1,omega_m2,theta_balanced1,...
    omega_m3,theta_balanced1,omega_m4,theta_balanced1,omega_m5)
xlabel('Angular Position (rad)'); ylabel('Angular Velocity (rad/s)');
title('Comparison of Theoretical Data with Frictional Moments Against Real Data');
legend('Experimental Data','0 N*m','0.25 N*m','0.5 N*m','0.75 N*m','1 N*m');

%% Model Balanced wheel, friction added
[omega_balanced1_fric,mom_fric1]=MODEL2(m_wheel,m_support,beta,R,k,g,...
    theta_balanced1,omega_expbalanced1);
[omega_balanced2_fric,mom_fric2]=MODEL2(m_wheel,m_support,beta,R,k,g,...
    theta_balanced2,omega_expbalanced2);

%% Plot the three types of data for balanced experiments against each other (Experimental, ...
    no friction theoratical, and friction-included theoretical)
figure(2)
plot(theta_balanced1,omega_expbalanced1,'.','MarkerSize',10)
hold on
plot(theta_balanced1,omega_balanced1_nofric,'--')
plot(theta_balanced1,omega_balanced1_fric)
title('Comparision of Balanced Experiment 1 Angular Velocities')
xlabel('Angle (Rad)'); ylabel('Angular Velocity (rad/sec)');
legend('Experimental Data','Theoretical Data without Friction','Theoretical Data with ...
    Friction');
figure(3)
plot(theta_balanced2,omega_expbalanced2,'.','MarkerSize',10)
hold on
plot(theta_balanced2,omega_balanced2_nofric,'--')
plot(theta_balanced2,omega_balanced2_fric)
title('Comparision of Balanced Experiment 2 Angular Velocities')
xlabel('Angle (Rad)'); ylabel('Angular Velocity (rad/sec)');
legend('Experimental Data','Theoretical Data without Friction','Theoretical Data with ...
    Friction');

%% Model Unbalanced wheel, extra mass as a point
omega_unbalpoint1=MODEL3(m_wheel,m_support,m,beta,R,k,r_tomass,g,theta_unbal1,mom_fric1);
omega_unbalpoint2=MODEL3(m_wheel,m_support,m,beta,R,k,r_tomass,g,theta_unbal2,mom_fric2);

%% Model Unbalanced wheel, extra mass as a cylinder
omega_unbalcylinder1=MODEL4(m_wheel,m_support,m,beta,R,k,r_tomass,r_mass,g,...
    theta_unbal1,mom_fric1);
omega_unbalcylinder2=MODEL4(m_wheel,m_support,m,beta,R,k,r_tomass,r_mass,g,...
    theta_unbal2,mom_fric2);

%% Plot the three types of data for unbalanced experiments against each other ...
    (Experimental, no friction theoratical, and friction-included theoretical)
figure(4)
plot(theta_unbal1,omega_expunbal1,'Linewidth',.5)
hold on;
plot(theta_unbal1,omega_unbalpoint1,'--','Linewidth',2)
plot(theta_unbal1,omega_unbalcylinder1,'Linewidth',.5)
title('Comparision of Unbalanced Experiment 1 Angular Velocities')
xlabel('Angle (Rad)'); ylabel('Angular Velocity (rad/sec)');
legend('Experimental Data','Theoretical Data with Point Mass',...
    'Theoretical Data with Cylindrical Mass');
figure(5)
plot(theta_unbal2,omega_expunbal2,'Linewidth',.5)
hold on;
plot(theta_unbal2,omega_unbalpoint2,'--','Linewidth',2)
plot(theta_unbal2,omega_unbalcylinder2,'Linewidth',.5)
title('Comparision of Unbalanced Experiment 2 Angular Velocities')
xlabel('Angle (Rad)'); ylabel('Angular Velocity (rad/sec)');
```

```
legend('Experimental Data','Theoretical Data with Point Mass',...
    'Theoretical Data with Cylindrical Mass');

%% Find residuals
[res_balanced1nofric, res_balanced1fric, res_balanced2nofric, res_balanced2fric,...
    res_unbal1point, res_unbal1cyl, res_unbal2point, res_unbal2cyl]=...
    residuals(omega_expbalanced1,omega_expbalanced2,omega_expunbal1,...
    omega_expunbal2,omega_balanced1_nofric,omega_balanced2_nofric,...
    omega_balanced1_fric,omega_balanced2_fric,omega_unbalpoint1,...
    omega_unbalpoint2,omega_unbalcylinder1,omega_unbalcylinder2);

%% Calculate statistical info on the residuals
[mean_balanced1nofric,mean_balanced1fric,mean_balanced2nofric,mean_balanced2fric,...
    mean_unbal1point,mean_unbal1cyl,mean_unbal2point,mean_unbal2cyl,...
    SD_balanced1nofric,SD_balanced1fric,SD_balanced2nofric,SD_balanced2fric,...
    SD_unbal1point,SD_unbal1cyl,SD_unbal2point,SD_unbal2cyl,SEM_balanced1nofric,...
    SEM_balanced1fric,SEM_balanced2nofric,SEM_balanced2fric,SEM_unbal1point,...
    SEM_unbal1cyl,SEM_unbal2point,SEM_unbal2cyl,N1,N2,N3,N4]=...
    statistics(res_balanced1nofric,res_balanced1fric, res_balanced2nofric, ...
    res_balanced2fric,res_unbal1point, res_unbal1cyl, res_unbal2point, res_unbal2cyl);

%% Plot residuals
figure(6)
subplot(1,2,1)
plot(theta_balanced1,res_balanced1nofric,'--')
hold on
plot(theta_balanced1,res_balanced1fric)
xlabel('Angular Position (rad)'); ylabel('Residual Angular Velocity (rad/s)');
legend('Model without Friction','Model with Friction')
title('1st Balanced Experiment Residuals')

subplot(1,2,2)
plot(theta_balanced2,res_balanced2nofric,'--')
hold on
plot(theta_balanced2,res_balanced2fric)
xlabel('Angular Position (rad)'); ylabel('Residual Angular Velocity (rad/s)');
legend('Model without Friction','Model with Friction')
title('2nd Balanced Experiment Residuals')

figure(7)
subplot(1,2,1)
plot(theta_unbal1,res_unbal1point,'--')
hold on
plot(theta_unbal1,res_unbal1cyl)
xlabel('Angular Position (rad)'); ylabel('Residual Angular Velocity (rad/s)');
legend('Model Extra Mass as Point','Model Extra Mass as Cylinder')
title('1st Unbalanced Experiment Residuals')

subplot(1,2,2)
plot(theta_unbal2,res_unbal2point,'--')
hold on
plot(theta_unbal2,res_unbal2cyl)
xlabel('Angular Position (rad)'); ylabel('Residual Angular Velocity (rad/s)');
legend('Model Extra Mass as Point','Model Extra Mass as Cylinder')
title('2nd Unbalanced Experiment Residuals')

%% Find out how many outliers there are in each data set

%Find where the outliers are
outliers_balanced1nofric=[find(res_balanced1nofric<mean_balanced1nofric-...
    3*SD_balanced1nofric) find(res_balanced1nofric>mean_balanced1nofric+...
    3*SD_balanced1nofric)];
outliers_balanced1fric=[find(res_balanced1fric<mean_balanced1fric-...
    3*SD_balanced1fric) find(res_balanced1fric>mean_balanced1fric+...
    3*SD_balanced1fric)];

outliers_balanced2nofric=[find(res_balanced2nofric<mean_balanced2nofric-...
    3*SD_balanced2nofric) find(res_balanced2nofric>mean_balanced2nofric+...
```

```
    3*SD_balanced2nofric)];
outliers_balanced2fric=[find(res_balanced2fric<mean_balanced2fric-...
    3*SD_balanced2fric) find(res_balanced2fric>mean_balanced2fric+...
    3*SD_balanced2fric)];

outliers_unbal1point=[find(res_unbal1point<mean_unbal1point-3*SD_unbal1point...
    ) find(res_unbal1point>mean_unbal1point+3*SD_unbal1point)];
outliers_unbal1cyl=[find(res_unbal1cyl<mean_unbal1cyl-3*SD_unbal1cyl)...
    find(res_unbal1cyl>mean_unbal1cyl+3*SD_unbal1cyl)];

outliers_unbal2point=[find(res_unbal2point<mean_unbal2point-3*SD_unbal2point)...
    find(res_unbal2point>mean_unbal2point+3*SD_unbal2point)];
outliers_unbal2cyl=[find(res_unbal2cyl<mean_unbal2cyl-3*SD_unbal2cyl)...
    find(res_unbal2cyl>mean_unbal2cyl+3*SD_unbal2cyl)];

%Calculate how many outliers there are
numout_balanced1nofric=numel(outliers_balanced1nofric);
numout_balanced1fric=numel(outliers_balanced1fric);
numout_balanced2nofric=numel(outliers_balanced2nofric);
numout_balanced2fric=numel(outliers_balanced2fric);
numout_unbal1point=numel(outliers_unbal1point);
numout_unbal1cyl=numel(outliers_unbal1cyl);
numout_unbal2point=numel(outliers_unbal2point);
numout_unbal2cyl=numel(outliers_unbal2cyl);
```

2. MODEL1.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed
% Date written: 2/27/2019
% Date modified: 2/27/2019
%
% Purpose:
% To calculate dynamics of a balanced hollow cylindrical object rolling
% down a ramp, assuming no friction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function parameters:
%
% IN:
% m_wheel - Mass of wheel in kg
% m_support - Mass of support in kg
% beta  - angle of inclination in degrees
% R     - Radius of wheel in m
% k     - radius of gyration of wheel in kg*m^2
% g     - Gravitational constant
% theta - Vector of angular positions in radians
% OUT:
% omega - angular velocity vector in radians per second

function omega=MODEL1(m_wheel,m_support,beta,R,k,g,theta)
%Calculate angular velocity for balanced wheel using energy methods
%omega in rad/s
%Potential energy comes from mass of both wheel and support
PE_dif=(m_wheel+m_support)*g*R.*theta*sin(beta); %Potential Energy drop of system
KE_coef=.5*(m_wheel+m_support)*R^2+.5*m_wheel*k^2; %Kinetic energy without the omega term
omega=sqrt(PE_dif/KE_coef);
end
```

3. MODEL2.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed
% Date written: 2/27/2019
% Date modified: 2/27/2019
%
% Purpose:
% To calculate dynamics of a balanced hollow cylindrical object rolling
% down a ramp, assuming friction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function parameters:
%
% IN:
% m_wheel - Mass of wheel in kg
% m_support - Mass of support in kg
% beta  - angle of inclination in degrees
% R     - Radius of wheel in m
% k     - radius of gyration of wheel in kg*m^2
% g     - Gravitational constant
% theta - Vector of angular positions in radians
% omega_exp - Experimental angular velocity measurements in radians per
% second
% OUT:
% omega - angular velocity vector in radians per second
% mom_fric - Frictional moment causing work on system

function [omega,mom_fric]=MODEL2(m_wheel,m_support,beta,R,k,g,theta,omega_exp)
%Calculate angular velocity for balanced wheel with friction using energy methods
%omega in rad/s
%Potential energy comes from mass of both wheel and support

%Find moment caused by friction. Use a reasonable set of experimental theta
%and omega values to caluclate moment. Derived from work-energy equation
mom_fric=(-(m_wheel+m_support)*g*R.*theta(5:25)*sin(beta)+.5*(m_wheel+ ...
    m_support)*R^2.*omega_exp(5:25).^2+.5*m_wheel*k^2.*omega_exp(5:25).^2)...
    ./theta(5:25); %N*m

%NOTE THAT ALTHOUGH THIS MOMENT WON'T BE THE SAME FOR BOTH EXPERIMENTS, IT
%WILL CREATE THE BEST RESULTS AND WILL ACCOUNT FOR DIFFERENCES BETWEEN EACH
%TRIAL

%Create a single value from this vector.
mom_fric=mean(mom_fric);

%Create values for numerator and denominator
PE_dif=(m_wheel+m_support)*g*R.*theta*sin(beta);
KE_coef=(.5*(m_wheel+m_support)*R^2+.5*m_wheel*k^2);

%Find omega, this time with the frictional moment factored in.
omega=sqrt((mom_fric.*theta+PE_dif)/KE_coef);
end
```

4. MODEL3.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Ian Thomas,
% Date written: 2/27/2019
% Date modified: 2/27/2019
%
% Purpose:
% To calculate dynamics of an unbalanced hollow cylindrical object with an
%extra massrolling down a ramp, assuming point mass
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function parameters:
%
% IN:
% m_wheel - Mass of wheel in kg
% m_support - Mass of support in kg
% m      - Mass of extra mass in kg
% beta   - angle of inclination in degrees
% R      - Radius of wheel in m
% k      - radius of gyration of wheel in kg*m^2
% r_tomass - Distance from center of wheel to point mass in m
% g      - Gravitational constant
% theta - Vector of angular positions in radians
% mom_fric - Frictional Moment doing work on system
% OUT:
% omega - angular velocity vector in radians per second

function omega=MODEL3(m_wheel,m_support,m,beta,R,k,r_tomass,g,theta,mom_fric)
%Calculate angular velocity for unbalanced wheel using energy methods
%omega in rad/s. Treat extra mass as a point mass

r=r_tomass;

%Calculate potential energy changes for each subsystem
PE_wheel=m_wheel*g*R.*theta*sin(beta); %Difference between initial wheel PE and wheel PE ...
    at any point
PE_support=m_support*g*R.*theta*sin(beta); %Assume this support is at same level as center ...
    of wheel.
PE_mass=m*g*(R.*theta*sin(beta)+r.*(cos(beta)-cos(beta+theta)));

%Calculate kinetic energy coefficients (ie without omega) of each subsystem
KE_wheel=.5*m_wheel*(R^2+k^2);
KE_support=.5*m_support*R^2;
KE_mass=.5*m*R^2+m*r^2+m*R*r.*(cos(theta+beta)*cos(beta)+sin(theta+beta)*sin(beta));

%Calculate omega. Frictional moment also accounted for
omega=sqrt((PE_wheel+PE_support+PE_mass+mom_fric.*theta)./(KE_wheel+KE_support+KE_mass));
end
```

5. MODEL4.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Ian Thomas,
```

```
% Date written: 2/27/2019
% Date modified: 2/27/2019
%
% Purpose:
% To calculate dynamics of an unbalanced hollow cylindrical object with an
%extra massrolling down a ramp, assuming cylidrical, rigid body mass
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function parameters:
%
% IN:
% m_wheel - Mass of wheel in kg
% m_support - Mass of support in kg
% m      - Mass of extra mass in kg
% beta   - angle of inclination in degrees
% R      - Radius of wheel in m
% k      - radius of gyration of wheel in kg*m^2
% r_tomass - Distance from center of wheel to point mass in m
% r_mass - radius of cyldinrical mass in m
% g      - Gravitational constant
% theta - Vector of angular positions in radians
% mom_fric - Frictional Moment doing work on system
% OUT:
% omega - angular velocity vector in radians per second

function omega=MODEL4(m_wheel,m_support,m,beta,R,k,r_tomass,r_mass,g,theta,mom_fric)
%Calculate angular velocity for balanced wheel using energy methods
%omega in rad/s
r=r_tomass;

%Calculate potential energy changes for each subsystem
PE_wheel=m_wheel*g*R.*theta*sin(beta); %Difference between initial wheel PE and wheel PE ...
    at any point
PE_support=m_support*g*R.*theta*sin(beta); %Assume this support is at same level as center ...
    of wheel.
PE_mass=m*g*(R.*theta*sin(beta)+r.*(cos(beta)-cos(beta+theta)));

%Calculate kinetic energy coefficients (ie without omega) of each
%subsystem. Only difference from model 3 is the moment of inertia for the
%mass, which is now considered to be an actual cylinder and not just a
%point.
KE_wheel=.5*m_wheel*(R^2+k^2);
KE_support=.5*m_support*R^2;
KE_mass=.5*m*R^2+m*r^2+m*R*r.*(cos(theta+beta)*cos(beta)+sin(theta+beta)*...
    sin(beta))+.5*m*r_mass^2;

%Calculate omega. Frictional moment also accounted for
omega=sqrt((PE_wheel+PE_support+PE_mass+mom_fric.*theta)./(KE_wheel+KE_support+KE_mass));
end
```

6. $MODEL2_{exp}$.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed
% Date written: 2/27/2019
% Date modified: 2/27/2019
%
% Purpose:
```

```
% To estimate the frictional moment caused by the support connection.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function parameters:
%
% IN:
% m_wheel - Mass of wheel in kg
% m_support - Mass of support in kg
% beta  - angle of inclination in degrees
% R     - Radius of wheel in m
% k     - radius of gyration of wheel in kg*m^2
% g     - Gravitational constant
% theta - Vector of angular positions in radians
% mom_fric - Educated guess for magnitude of frictional moment in Newton-meters
% second
% OUT:
% omega_m - Omega with certain moment coefficient

function [omega_m]=MODEL2_exp(m_wheel,...
    m_support,beta,R,k,g,theta,mom_fric)
%Calculate angular velocity for balanced wheel with friction using energy
%methods and a given frictional moment guess
%omega in rad/s
%Potential energy comes from mass of both wheel and support

%Create values for numerator and denominator
PE_dif=(m_wheel+m_support)*g*R.*theta*sin(beta);
KE_coef=(.5*(m_wheel+m_support)*R^2+.5*m_wheel*k^2);

%Find omega, this time with the frictional moment factored in.
omega_m=sqrt((-mom_fric.*theta+PE_dif)/KE_coef);
end
```

7. residuals.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed,
% Date written: 3/11/2019
% Date modified: 3/11/2019
%
% Purpose:
% Calculate residuals of data sets for unbalanced wheel
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [res_balanced1nofric, res_balanced1fric, res_balanced2nofric, res_balanced2fric,...
    res_unbal1point, res_unbal1cyl, res_unbal2point, res_unbal2cyl]=...
    residuals(omega_expbalanced1,omega_expbalanced2,omega_expunbal1,...
    omega_expunbal2,omega_balanced1_nofric,omega_balanced2_nofric,...
    omega_balanced1_fric,omega_balanced2_fric,omega_unbalpoint1,...
    omega_unbalpoint2,omega_unbalcylinder1,omega_unbalcylinder2)

%Simply do observed - expected
res_balanced1nofric=omega_expbalanced1-omega_balanced1_nofric;
res_balanced1fric=omega_expbalanced1-omega_balanced1_fric;

res_balanced2nofric=omega_expbalanced2-omega_balanced2_nofric;
res_balanced2fric=omega_expbalanced2-omega_balanced2_fric;

```

```
res_unbal1point=omega_expunbal1-omega_unbalpoint1;
res_unbal1cyl=omega_expunbal1-omega_unbalcylinder1;

res_unbal2point=omega_expunbal2-omega_unbalpoint2;
res_unbal2cyl=omega_expunbal2-omega_unbalcylinder2;

end
```

8. statistics.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Group Names:
% 1. Abdulla Al Ameri
% 2. Adam Elsayed
% 3. Samuel Firth
% 4. Ian Thomas
%
% Group 11-14
% Author(s): Adam Elsayed,
% Date written: 3/11/2019
% Date modified: 3/11/2019
%
% Purpose:
% Calculate statistical data on residuals of data sets for unbalanced wheel
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [mean_balanced1nofric,mean_balanced1fric,mean_balanced2nofric,mean_balanced2fric...
    mean_unbal1point,mean_unbal1cyl,mean_unbal2point,mean_unbal2cyl,...
    SD_balanced1nofric,SD_balanced1fric,SD_balanced2nofric,SD_balanced2fric,...
    SD_unbal1point,SD_unbal1cyl,SD_unbal2point,SD_unbal2cyl,SEM_balanced1nofric,...
    SEM_balanced1fric,SEM_balanced2nofric,SEM_balanced2fric,SEM_unbal1point,...
    SEM_unbal1cyl,SEM_unbal2point,SEM_unbal2cyl,N1,N2,N3,N4]=...
    statistics(res_balanced1nofric,res_balanced1fric, res_balanced2nofric, ...
    res_balanced2fric,res_unbal1point, res_unbal1cyl, res_unbal2point, res_unbal2cyl)

%Calculate means, SDs and SEMs of each residual
[N1,¬]=size(res_balanced1nofric);
mean_balanced1nofric=mean(res_balanced1nofric);
SD_balanced1nofric=std(res_balanced1nofric);
SEM_balanced1nofric=SD_balanced1nofric/sqrt(N1);

mean_balanced1fric=mean(res_balanced1fric);
SD_balanced1fric=std(res_balanced1fric);
SEM_balanced1fric=SD_balanced1fric/sqrt(N1);

[N2,¬]=size(res_balanced2nofric);
mean_balanced2nofric=mean(res_balanced2nofric);
SD_balanced2nofric=std(res_balanced2nofric);
SEM_balanced2nofric=SD_balanced2nofric/sqrt(N2);

mean_balanced2fric=mean(res_balanced2fric);
SD_balanced2fric=std(res_balanced2fric);
SEM_balanced2fric=SD_balanced2fric/sqrt(N2);

[N3,¬]=size(res_unbal1point);
mean_unbal1point=mean(res_unbal1point);
SD_unbal1point=std(res_unbal1point);
SEM_unbal1point=SD_unbal1point/sqrt(N3);

mean_unbal1cyl=mean(res_unbal1cyl);
SD_unbal1cyl=std(res_unbal1cyl);
SEM_unbal1cyl=SD_unbal1cyl/sqrt(N3);

[N4,¬]=size(res_unbal2point);
mean_unbal2point=mean(res_unbal2point);
```

```
SD_unbal2point=std(res_unbal2point);
SEM_unbal2point=SD_unbal2point/sqrt(N4);

mean_unbal2cyl=mean(res_unbal2cyl);
SD_unbal2cyl=std(res_unbal2cyl);
SEM_unbal2cyl=SD_unbal2cyl/sqrt(N4);

end
```

C. Derivations

1. Frictionless Balanced Wheel

Assuming all energy is conserved and the origin the center of the wheel, we have:

$$U_1 + T_1 + W_{12} = U_2 + T_2 \tag{1}$$

Define the wheel to have zero potential energy at the start, without any work due to friction, and since the wheel is not moving we have:

$$-U_2 = T_2 \tag{2}$$

Where $U_2 = U_{wheel} + U_{support}$ and $T_2 = T_{wheel} + T_{support}$. We can express the height h of the wheel as a function of θ, where $h = -R\theta \sin\beta$. We can express the potential energy of the wheel, then, using the relationship $U = mgh$ as:

$$U_{wheel} = -m_w g R\theta \sin\beta \tag{3}$$

$$U_{support} = -m_s g R\theta \sin\beta \tag{4}$$

$$U = U_{wheel} + U_{support} = -(m_w + m_s) g R\theta \sin\beta \tag{5}$$

We can express the kinetic energy of the system in terms of the linear velocity of the center of mass of the system and the angular velocity of the rotating wheel as:

$$T_{wheel} = \frac{1}{2} m_w v_w^2 + \frac{1}{2} I_w \omega^2 \tag{6}$$

$$T_{support} = \frac{1}{2} m_s v_s^2 \tag{7}$$

We know since the wheel and the support are fixed that $v_w = v_s = v$, and that $v = R\omega$ by circular geometry. Substituting variables we have:

$$T = T_{wheel} + T_{support} = \frac{1}{2}(m_w + m_s)(R\omega)^2 + \frac{1}{2} I_w \omega^2 \tag{8}$$

Using (2) we can now write:

$$(m_w + m_s) g R\theta \sin\beta = \frac{1}{2}(m_w + m_s)(R\omega)^2 + \frac{1}{2} I_w \omega^2 \tag{9}$$

We can now factor ω^2 out of the right side of the equation and solve, making the substitution $I_w = m_w \kappa_w^2$:

$$\omega = \sqrt{\frac{2(m_w + m_s) g R\theta \sin\beta}{(m_w + m_s) R^2 + m_w \kappa_w^2}} \tag{10}$$

2. Balanced Wheel with Friction

The derivation of this scenario is the same as that of the first, except the W_{12} term is a negative value. We now have:

$$W_{12} - U_2 = T_2 \tag{11}$$

Where $W_{12} = -M\theta$, where M is the moment due to the frictional force between the wheel and the support. Substituting variables, we now have:

$$-M\theta + (m_w + m_s)gR\theta\sin\beta = \frac{1}{2}(m_w + m_s)(R\omega)^2 + \frac{1}{2}I_w\omega^2 \tag{12}$$

The moment can be solved if some experimental values of angular position and velocity are known:

$$M = \frac{-(m_w + m_s)gR\theta sin(\beta) + \frac{1}{2}(m_w + m_s)R^2\omega^2 + \frac{1}{2}m_w\kappa_w^2\omega^2}{\theta} \tag{13}$$

Factoring, substituting, and solving for ω yields:

$$\omega = \sqrt{\frac{2[(m_w + m_s)gR\theta\sin\beta - M\theta]}{(m_w + m_s)R^2 + m_w\kappa_w^2}} \tag{14}$$

3. Unbalanced Wheel, Point Mass Approximation

This derivation is the same as the previous section, except with an added point mass of mass m_p radius r away from the center of the rotating disk. The approach to this problem is to model this as two individual systems: the wheel/support system and the point mass. These systems are linked by the same angular velocity and angle displacement, and so it is possible to express the dynamics of the entire system as a function of ω and θ. Using conservation of energy:

$$U_1 + W_{12} = U_2 + T_2 \tag{15}$$

Where $W_{12} = -M\theta$, $U_2 = U_{wheel} + U_{support} + U_{point}$, and $T_2 = T_{wheel} + T_{support} + T_{point}$. Defining the zero of potential energy to be the center of the wheel at the start, U_1 is simply the initial potential energy of the point mass relative to the center of the wheel. It is given that the point mass is starting at an offset of β from vertical, so:

$$U_1 = m_p gr\cos\beta \tag{16}$$

We know U_2 is the sum of the potential energies from model 2, except with the extra potential energy of the point mass. It is useful to find the height of the point mass relative to the height of the center of the disk, and taking θ from the vertical:

$$h_{point} = r\cos\beta + \theta + h \tag{17}$$

$$h = -R\theta\sin\beta \tag{18}$$

$$U_{point} = m_p g h_{point} = m_p g[r\cos(\beta + \theta) - R\theta\sin\beta] \tag{19}$$

$$\tag{20}$$

Summing all the individual potential energies from model 2 gives:

$$U_2 = -(m_w + m_s)gR\theta\sin\beta + m_p g[r\cos(\beta + \theta) - R\theta\sin\beta] \tag{21}$$

The kinetic energies of the wheel and the support are the same functions of θ and ω as they were in model two, so it is only necessary to calculate T_{point}. The kinetic energy of a point mass is given as $\frac{1}{2}mv^2$, so it is important to solve for the velocity of the point in terms of ω to get a kinetic energy. From dynamics relationships:

$$\vec{v}_p = \vec{v}_w + \vec{\omega} \times \vec{r}_{p/w} \tag{22}$$

Where the subscript w refers to the center of the wheel. Rewriting this equation in terms of known constants yields:

$$\vec{v}_p = R\omega[\cos\beta\hat{i} - \sin\beta\hat{j}] + [-\omega\hat{k}] \times r[\sin(\beta + \theta)\hat{i} + \cos(\beta + \theta)\hat{j}] \tag{23}$$

Evaluating the cross product and simplifying yields:

$$\vec{v}_p = [r\omega\cos(\beta + \theta) + R\omega\cos\beta]\hat{i} - [r\omega\sin(\beta + \theta) + R\omega\sin\beta]\hat{j} \tag{24}$$

$$= \omega[r\cos(\beta + \theta) + R\cos\beta]\hat{i} - \omega[r\sin(\beta + \theta) + R\sin\beta]\hat{j} \tag{25}$$

Using vector identity $v^2 = \vec{v} \cdot \vec{v}$:

$$\begin{aligned} v_p^2 =& \omega^2[r^2 \cos^2(\beta + \theta) + 2Rr \cos(\beta) \cos(\beta + \theta) + R^2 \cos^2(\beta) + \\ & r^2 \sin^2(\beta + \theta) + 2Rr \sin(\beta) \sin(\beta + \theta) + R^2 \sin^2(\beta)] \qquad (26) \\ =& \omega^2[r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] \qquad (27) \end{aligned}$$

Using the kinetic energy relationship:

$$T_{point} = \frac{1}{2} m_p \omega^2 [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] \qquad (28)$$

Substituting into (15):

$$\begin{aligned} & m_p gr \cos\beta + (m_w + m_s + m_p) gR\theta \sin\beta - m_p gr \cos(\beta + \theta) - M\theta = \\ & \frac{1}{2}(m_w + m_s) R^2 \omega^2 + \frac{1}{2} I_w \omega^2 + \frac{1}{2} m_p \omega^2 [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] \qquad (29) \end{aligned}$$

Solving for ω:

$$\omega = \sqrt{\frac{2[m_p gr \cos\beta + (m_w + m_s + m_p) gR\theta \sin\beta - m_p gr \cos(\beta + \theta) - M\theta]}{(m_w + m_s) R^2 + m_w \kappa_w^2 + m_p [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))]}} \qquad (30)$$

4. Unbalanced Wheel, Rigid Body

The expression for the angular velocity in a situation where the extra mass is modeled as a rigid body is the same as that of model 3, except the rotational kinetic energy of the extra mass must be accounted for. In terms of the variables given, where r_m is the radius of the extra mass and m_m is the mass of the extra mass, the rotational kinetic energy is given by:

$$T_{mass,rot} = \frac{1}{2} \left(\frac{1}{2} m_m r_m^2 \right) \omega^2 \qquad (31)$$

Where $\frac{1}{2} m_m r_m^2$ is the moment of inertia of the extra mass. The linear kinetic energy of the extra mass is given by:

$$T_{mass,lin} = T_{point} = \frac{1}{2} m_m \omega^2 [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] \qquad (32)$$

Because it is simply the velocity of the center of mass used in the calculations, which is the same as the velocity of the point mass from model 3. The total kinetic energy of the extra mass is:

$$\begin{aligned} T_{mass} &= T_{mass,lin} + T_{mass,rot} \qquad (33) \\ &= \frac{1}{2} m_m \omega^2 [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] + \frac{1}{2} \left(\frac{1}{2} m_m r_m^2 \right) \omega^2 \qquad (34) \end{aligned}$$

All that's left to do is add this to the total kinetic energy term and solve for ω, which is given by:

$$\omega = \sqrt{\frac{2[m_m gr \cos\beta + (m_w + m_s + m_m) gR\theta \sin\beta - m_m gr \cos(\beta + \theta) - M\theta]}{(m_w + m_s) R^2 + m_w \kappa_w^2 + m_m [r^2 + R^2 + 2Rr(\cos(\beta + \theta) \cos(\beta) + \sin(\beta + \theta) \sin(\beta))] + \frac{1}{2} m_m r_m^2}} \qquad (35)$$