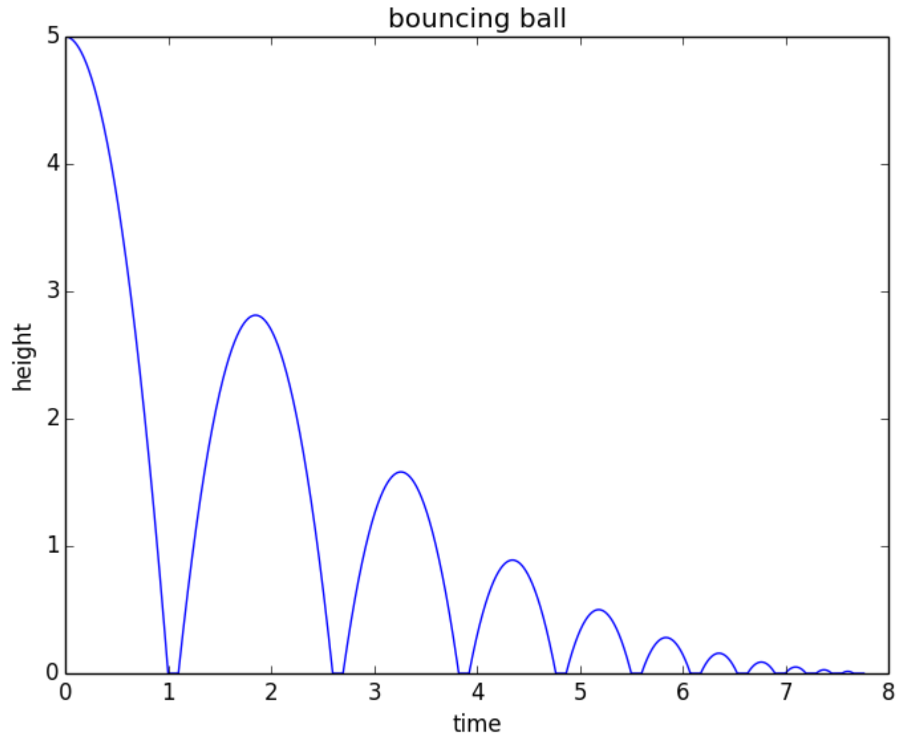

ASEN 2003: Lab 2
Bouncing Ball Lab

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Date Submitted: 02/11/2019

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Contents

The purpose of this lab was to experimentally determine the coefficient of restitution of a ball based on data obtained from experimental trials, performing analyses for sources of error in different calculation methods and experimental set-ups, and using the appropriate analyses to determine how the experimental methodology could be improved. The least effective method was found to be that which used the height of two adjacent bounces, while the most accurate was found to be that which measured the total time for the bouncing ball to come to rest. The measurements for the time-to-stop method were improved using an audio recorder and analysis of the sound profile of the ball impacting the ground. Following this improvement, the final coefficients of restitution were found to be 0.903 ± 0.002 for a ping pong ball and 0.928 ± 0.001 for a golf ball. Despite notable improvements in the accuracy of measurements, at the end of the lab there were still sources of error present that compromised the result and could still stand to be improved upon in follow-up experiments.

Nomenclature

e	=	coefficient of restitution
v_B	=	velocity of ball before collision
$v_B{}'$	=	velocity of ball after collision
v_A	=	velocity of floor before collision
$v_A{}'$	=	velocity of floor after collision
h_n	=	height of bounce
h_{n-1}	=	height of preceding bounce
t_n	=	time of bounce
t_{n-1}	=	time of preceding bounce
t_s	=	time for ball to stop bouncing
h_0	=	initial height of ball
g	=	acceleration due to gravity
$\frac{\partial e}{\partial x}$	=	Partial derivative of e with respect to variable x
σ_x	=	Uncertainty associated with term x.

I. Theory

For analysis of a direct central impact such as a ball hitting the ground, a useful value to know is the coefficient of restitution between the two objects. The coefficient of restitution is a measure of how much energy is lost between two objects during the collision. The general formula for coefficient of restitution between two objects is shown in Eq. (1) below:

$$e = \frac{v_B{}' - v_A{}'}{v_A - v_B} \tag{1}$$

However, the velocities of the two objects before and after the collision can be difficult quantities to experimentally measure in practice. Therefore, using the fundamental principles for direct central impacts such as momentum and energy, the coefficient of restitution can be calculated in several different ways in terms of variables that are easier to measure. In a simple experiment in which a ball is dropped from some initial height, quantities such as time between bounces and the change in height between bounces can be measured, as is represented by Fig. 1 below:

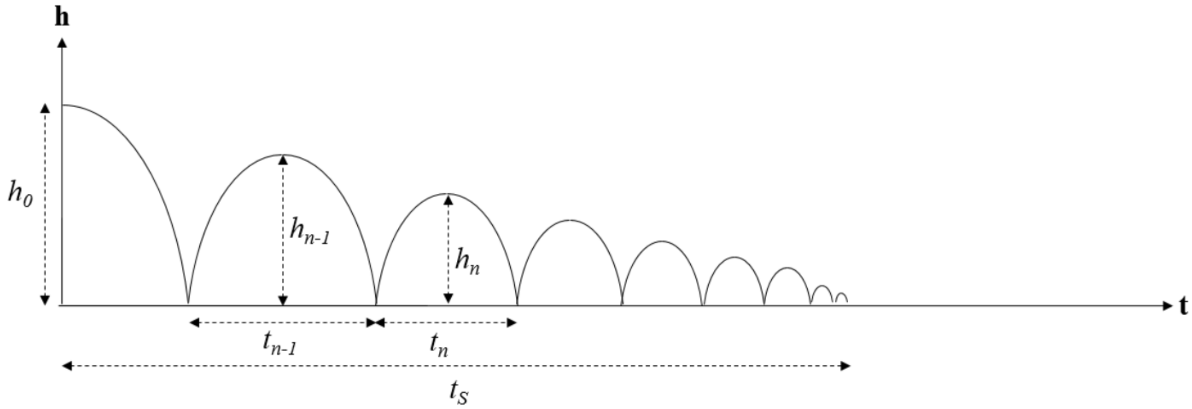

Fig. 1 Schematic of Variables Used in Derived Methods

From Equation 1, a formula for the coefficient of restitution can be derived in terms of the height of adjacent bounces of the ball.* This is shown below in Eq. (2) below:

$$e_{height} = \left(\frac{h_n}{h_{n-1}}\right)^{\frac{1}{2}} \tag{2}$$

Similarly, a formula for the coefficient of restitution can be derived in terms of the times of adjacent bounces, t_n and t_{n-1}, and also the total time the ball takes to stop bouncing once dropped, t_s. These are shown in Eqs. (3) and (4) respectively:

$$e_{bounces} = \left(\frac{t_n}{t_{n-1}}\right) \tag{3}$$

$$e_{stop} = \frac{t_s - \sqrt{\frac{2h_0}{g}}}{t_s + \sqrt{\frac{2h_0}{g}}} \tag{4}$$

Because these derivations are still in terms of experimentally obtained quantities, the coefficient of restitution in each calculation is subject to error due to measurement uncertainty of times and heights. Using the general method for error propagation, the team derived equations to estimate the error in the calculated coefficients of restitution due to the measurement errors. These are expressed in Eqs. (5)-(7).

$$\sigma_{e_{height}} = \sqrt{\left(\frac{1}{2(h_n h_{n-1})^{1/2}}\sigma_{h_n}\right)^2 + \left(-\frac{h_n^{1/2}}{2h_{n-1}^{3/2}}\sigma_{h_{n-l}}\right)^2} \tag{5}$$

It can be seen from Eq. (5) that the coefficient of restitution is sensitive to errors in both the height of the first bounce and also the subsequent bounce.

$$\sigma_{e_{bounces}} = \sqrt{\left(\frac{1}{t_{n-1}}\sigma_{t_n}\right)^2 + \left(-\frac{t_n}{t_{n-1}^2}\sigma_{t_{n-l}}\right)^2} \tag{6}$$

It can be seen from Eq. (6) above that this estimation of the coefficient of restitution is sensitive to errors in the time of the first bounce and also the time of the subsequent bounce.

$$\sigma_{e_{stop}} = \sqrt{\left(\frac{2g\sqrt{\frac{2h_0}{g}}}{(t_s\sqrt{g} + \sqrt{2h_0})^2}\sigma_{t_s}\right)^2 + \left(-\frac{t_s\sqrt{2g}}{\sqrt{h_0}(t_s\sqrt{g} + \sqrt{2h_0})^2}\sigma_{h_0}\right)^2} \tag{7}$$

It can be seen from Eq. (7) above that this estimation is sensitive to errors in the measurement of the initial height and the time taken to stop bouncing.

*Derivations in Appendix B

II. Procedures

A. Experiment 1

Applying the previously-detailed fundamental understanding of the coefficient of restitution and its calculation, three different experiments were conducted in order to obtain the desired coefficient of restitution with three different methods. These experiments used three data sets each from a total of 10 physical trials, effectively allowing for the collection of 30 trials of data in a third of the time and with fewer measurements. In each physical trial, measurements were taken for the heights of adjacent bounces, the times of adjacent bounces, and the total time for the ball to stop bouncing. One group member dropped the ball, another recorded video and obtained results for the height between bounces, and the last two measured the time between bounces and the total time for the ball to stop bouncing, respectively. This allowed efficient gathering of results in preparation for data analysis and experimental improvements. A step-by-step process of this method follows:

1) Tape a yardstick vertically along a wall (avoid covering any numbers with the tape).
2) Begin a video recording of the trial, level with the measuring stick. The two time recorders prepare their stopwatches.
3) Position the ping pong ball at the top of the measuring stick and release.
4) Record video of the bounces, remaining still for the whole video even if the ball goes out of frame. One stop watch is started as soon as the ball hits the floor the first time; a lap is recorded when the ball hits the floor a second floor, and the stopwatch is stopped when the ball hits the floor a third time (this provides the data for t_n and t_{n-1}). The other stopwatch is started as soon as the ball is released, and is stopped once the ball stops bouncing. Record results.
5) Perform 10 trials. Obtain the height between bounces from the videos. After this, take the three best videos and use these on an image tracking software to obtain more accurate results for bounce heights.

B. Experiment 2

Notable errors were found in the experimental setup and during data collection. When the ball was released, it rarely bounced perfectly straight back up, generally moving laterally as it bounced and sometimes moving out of frame in the video. Another big source of uncertainty was the impact of recorder reaction time when measuring the time between bounces. Due to these sources of error, the team decided to improve upon the original setup. Rather than listening for the bounces, the team used an audio recorder to record the bounces based on the sound of the impact of the ball with the floor. The resultant audio files were then transferred to the video-/audio-editing program iMovie, where the times between the spikes of noise were read off and recorded. Notable sources of error for this method included interference from environmental ambient noise, which necessitated the use of a team member's judgment in determining when sound profile fluctuations became small enough for the ball to be considered finished bouncing. A step-by-step process for the new experimental method follows

1) Set up the yardstick as before.
2) Hold the audio recorder near the floor where the ball is expected to impact and begin recording.
3) Release the ball from the same initial height as before in the initial data collection, and move the audio recorder with the ball to ensure consistently accurate and clear audio collection.
4) Stop the audio recorder and save the resulting file.
5) Perform 10 trials in like fashion.
6) Upload the voice memos into iMovie and read of the spikes in noise from when the ball hits the floor. Measure the time between these spikes and record results.

III. Results

A. Experiment 1

Compiled data was recorded following the conclusion of Experiment 1. The data collected included height and time of initial and subsequent bounces along with the total time to stop, as summarized in Table 1.

Trial	t_1 (s)	t_2 (s)	t_{stop} (s)	h_1 (in)	h_2 (in)
1	0.73	0.60	6.01	24.5	18.0
2	0.65	0.59	7.56	25.7	19.1
3	0.63	0.56	6.56	27.2	20.0
4	0.68	0.55	5.71	26.3	19.8
5	0.65	0.60	6.39	27.0	20.8
6	0.73	0.60	7.16	27.0	20.7
7	0.73	0.53	6.39	25.3	19.4
8	0.70	0.59	6.56	26.0	19.3
9	0.63	0.60	6.70	26.5	20.2
10	0.71	0.56	6.77	26.4	19.2

Table 1 Data collected for Experiment 1.

The means and standard deviations for each of the collected quantities are shown in Table 2.

Quantity	Mean	Standard Deviation
t_1 (s)	0.684	0.414
t_2 (s)	0.578	0.026
t_{stop} (s)	6.581	0.527
h_1 (in)	26.19	0.844
h_2 (in)	19.65	0.836

Table 2 Data statistics for Experiment 1.

The ITLL-provided Image Tracking software was also used to analyze three of the trial videos from Experiment 1, in order to obtain bounce height measurements for use in calculating the coefficient of restitution using the height of bounces method. The results from the Image tracking software are shown in Table 3, along with the means and standard deviations of the data.

Trial	h_1 (yd)	h_2 (yd)
1	1.01	0.690
7	1.02	0.683
8	0.99	0.667
Mean	1.01	0.680
Standard Deviation	0.015	0.012

Table 3 Data and statistics for image tracking software.

B. Experiment 2

In light of rather large deviations in the data recorded in Experiment 1, the team created Experiment 2 to consider a potentially better way for obtaining data for calculating the coefficient of restitution. The data from the 10 trials performed in Experiment 2 are shown in Table 4.

Trial	t_{stop} (s)
1	8.53
2	8.43
3	8.73
4	8.63
5	8.43
6	8.33
7	8.03
8	8.63
9	8.23
10	8.63
Mean	8.46
Standard Deviation	0.2163

Table 4 Experiment 2 data and statistics.

Experiment 2 was also performed on a golf ball; the data gathered is provided in Table 5.

Trial	t_{stop} (s)
1	11.53
2	12.03
3	11.33
4	11.53
5	11.43
6	11.63
7	11.73
8	11.73
9	11.43
10	11.73
Mean	11.61
Standard Deviation	0.2044

Table 5 Experiment 2 with golf ball data and statistics.

IV. Performance Analysis

Once data was obtained from the experimental set-ups, the coefficient of restitution was calculated using the three calculation methods. The results from Experiment 1 for the three methods, along with the associated error, are shown in Table 6.

Method	e	Uncertainty
Time to stop	0.876	0.003
Height of Bounce	0.853	0.012
Time of Bounce	0.848	0.334

Table 6 Experiment 1 coefficients of restitution.

Coefficient of restitution calculations were also performed using the height of subsequent bounces generated from the image-tracking software. The results and associated error are shown in Table 7.

Trial	e	Uncertainty
1	0.818	0.022
7	0.823	0.021
8	0.818	0.023

Table 7 Coefficient of restitution calculations using image-tracking software data.

Finally, calculations were performed for the coefficients of restitution of the ping pong ball and golf ball using data obtained from Experiment 2. The results and associated error are shown in Table 8.

Ball type	e	Uncertainty
Golf ball	0.928	0.001
Ping pong ball	0.903	0.002

Table 8 Coefficients of restitution calculated for the ping pong ball and golf ball used in Experiment 2.

For all the error analysis on these coefficients of restitution, Eqs. (5)-(7) were used to propagate the error from the measurements to the final estimation of the coefficient of restitution. In general, each term calculation method had two sources of error which could be accounted for: height measurement uncertainty and time measurement uncertainty. In each case, random error within the experiments was accounted for by performing 10 physical trials for each experimental set-up.

Each of the experiments were subject to several types of error. One of these was instrument error; in methods requiring the use of a ruler (for measuring starting height or bounce heights), error was inherent in the ruler's limited resolution. Each inch on the ruler used was split into 8 equal segments, so the team couldn't provide any higher accuracy in estimates than this resolution. There was also human error associated with consistently starting the ball from the same height, and error in visual estimation of ball height from trial videos in Experiment 1.

There was also error associated with time measurements in two of the calculation methods. These measurements were mostly subject to human error, because the observer had to start and stop the clock fairly quickly to try and obtain an accurate time for the bounce of a the ball, or had to use a rough estimate to determine exactly when the ball stopped bouncing. As a result, it is justified to assume the instrument error to be the human reaction time to visual and audio stimuli, as the observer had to depend on the sight or sound of the ball impacting with the floor for their measurement. According to experts in neuroscience field, the average human reaction time to sonic and visual stimuli are .17 seconds and .25 seconds, respectively [1]. This assumes average wakefulness, physical conditions, physiology, and environment on the part of the observer.

Following all of this analysis, time-based calculation methods were expected to be more accurate than height-based, simply due to the low resolution of the ruler. Of the two time-based methods, it was hypothesized that the method which measured the time to stop was the best, since the time-of-bounces method required the observer to record much shorter periods of time (thus making it more prone to uncertainty). The results and error bars for the data obtained using Experiment 1 are shown in Fig. 2.

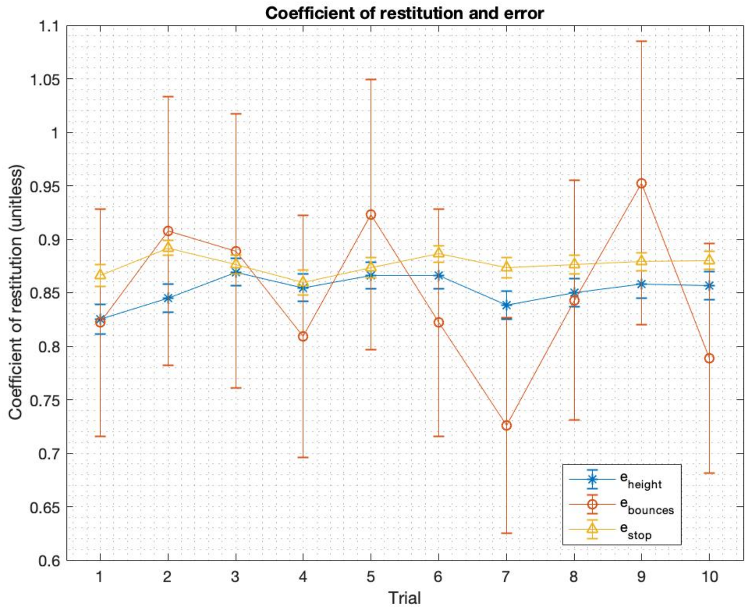

Fig. 2 Experiment 1 coefficients of restitution.

From Fig. 2 it is apparent that the time-to-stop calculation method was the most accurate of all of the methods used in Experiment 1. Following this analysis of Experiment 1, subsequent analysis of the data obtained from the image tracking software was performed to assess this method's accuracy. Results with associated uncertainty for data obtained using image-tracking software are graphed in Fig. 3.

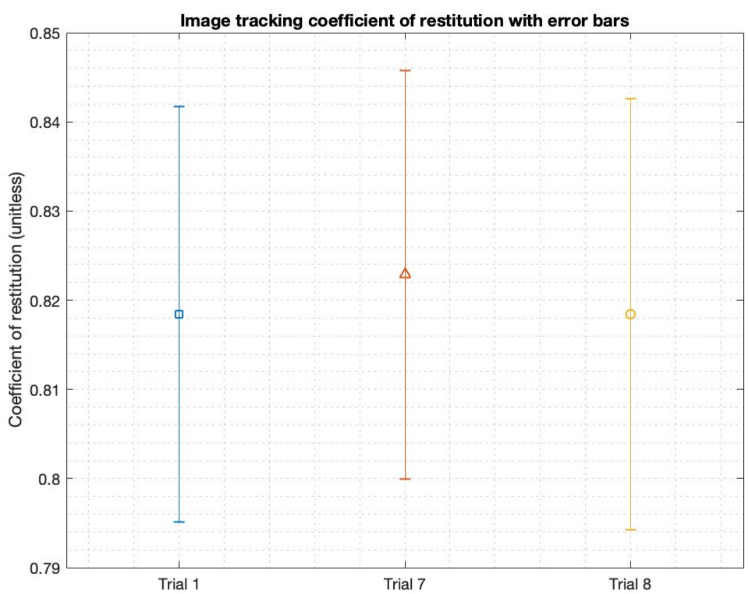

Fig. 3 Image tracking coefficients of restitution.

Although the technologically-superior image-tracking software should have provided a better estimate for the height of the ball, this method also contained a notable amount of error. The software could only identify the location of the ball once per frame, and the poor quality of the video made it difficult to identify the exact location of the ball center. The software also let the user calibrate the distance measurements in the videos, but not all videos were completely level and some of the bench marked distances were slightly off. Due to these factors, the image-tracking software, despite using fewer trials than Experiment 1, had a similar amount of error to the data obtained from Experiment 1.

Following this error analysis, the most accurate calculation method was determined to be the time-to-stop method. Appropriately, Experiment 2 was designed with a focus on obtaining time-to-stop data. In this experiment, sound waves from the ball-ground impacts were used to estimate the time it took for the ball to stop bouncing. One of the sound files obtained from these trials is provided in Fig. 4 as an example.

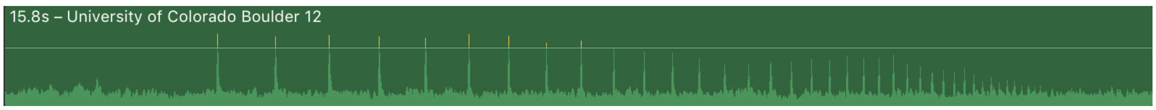

Fig. 4 Example sound file obtained in Experiment 2 trials.

This improved method was also subject to error, as each recording contained a certain amount of ambient noise from the surrounding environment that added to uncertainty in determining precisely when the ball hit the ground. This source of uncertainty was fairly insignificant for the first bounces because the ball made a loud, crisp sound upon impact; as the ball bounced to lower heights and the impact sound became quieter, however, this uncertainty began to notably influence the readability of the sound file, as seen in the tail end (right side) of Fig. 4. Taking this into account, the data obtained from Experiment 2 for both a ping pong ball and a golf ball is shown in Fig. 5.

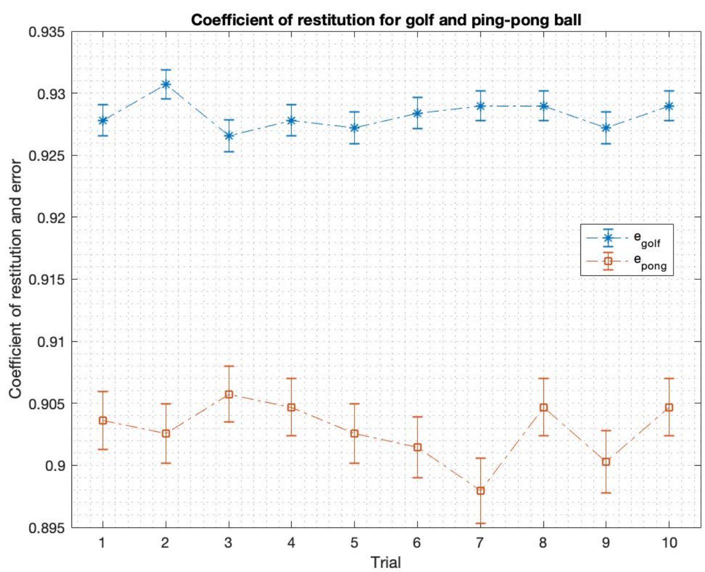

Fig. 5 Experiment 2 coefficients of restitution with associated error bars.

Experiment 2 still produced some amount of error, but from analysis its clear that the error in the time-to-stop calculations was significantly less when data from Experiment 2 was used as opposed to data from Experiment 1. This is most likely because it was much easier to obtain the total stop time for the bouncing ball using the visual cue of spikes in the sound profile as opposed to simply listening. However, it was also observed that the ping pong ball resulted in more uncertainty than the golf ball in Experiment 2. This was because the ping pong ball, being lighter, made quieter sounds upon impact in the ground, which corresponded to smaller spikes in the recorder audio files that made it more difficult to determine the time it took for the ping pong ball to stop bouncing.

Following the complete results and error analysis, Experiment 2 was found to be the most reliable source of data for time-to-stop calculations, and so this data was used to determine the final estimate for the coefficient of restitution. The final coefficient of restitution estimates for each of the balls are:

$$e_{\text{ping pong ball}} = 0.903 \pm 0.002$$

$$e_{\text{golf ball}} = 0.928 \pm 0.001$$

V. Conclusions and Recommendations

Following the performance of numerous experiments, trials, and analyses, an understanding was attained of the significance of various factors in influencing the coefficient of restitution of bouncing balls. While the process of recording results and calculating coefficients of restitution using derived equations was fairly simple, it was quickly clear that the results were notably inaccurate and that numerous sources of uncertainty and error were present in the experimental set-up. Following appropriate reflection and analysis, it was decided that the time-to-stop method was the most accurate and reliable method for determining the coefficient of restitution, as this contained the smallest uncertainty in collected data. Furthermore, the source of data for this method could be improved by designing a better experimental set-up in Experiment 2, which included the use of precise digital methods to significantly enhance the accuracy of results. The least accurate method was found to be that which used the height of ball bounces. This was largely due to the imperfection of the measuring stick that was used to measure the height, as well as the fairly poor quality of the videos from which the height could be measured off of. To possibly improve this experiment, the measuring stick used could have been taped to a piece of paper with grid lines on it, which would allow the determination of bounce height even when the bounce was not perfectly straight. The surface on which the ball bounced on could also have been more uniform, as roughness and dents caused the ball to bounce in different directions and sometimes out of the video frame. It is expected that those balls such as the ping pong and golf balls have high coefficients of restitution due to their applications, which is reflected on the results obtained.

References

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[4] Taylor, J., *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements*, 2nd ed., University Science Books, 1982.

Appendix

A. Team Member Participation Table

Team Member Participation
Lab #2
Date 2/11/2019

	Plan	Model	Experiment	Results	Report	Code
Name						
Hugo Stetz	2	1	1	1	2	1
Abdulla Al Ameri	1	1	1	1	1	2
Sarah Foley	1	1	1	2	1	1
Alexander Lowry	1	2	2	1	1	1

For each team member enter: 2 = Lead, 1 = Participate, 0 = Not involved, for each element.

Fig. 6 Team Participation Table for Lab 2.

B. Derivations

1. Coefficient of Restitution Equations

For each of our derivations, we begin with our standard coefficient of restitution equation, found in [2] and re-written in Eq. (1). Given that our experiments in this lab entail a ball hitting a solid surface (the floor, rather than a mobile), we can simplify this equation into Eq. (8), which is found in Section 16.2 of [3].

$$v'_{ball} = -ev_{ball} \tag{8}$$

Because we are looking for e, we can solve Eq. (8) for e to get Eq. (9).

$$e = -\frac{v'_{ball}}{v_{ball}} \tag{9}$$

For all of our derivations, positive velocities indicate movement in the upward direction and negative velocities indicate movement in the downward direction. We are assuming that our ball bounces exclusively along this up-down axis.

2. Measurements Sensitivity

As part of the analysis of this lab, equations for the error of the relevant variables were used. These equations were all based on the General Method for propagating error, as found in [4] and written out fully in Eq. (10).

$$\sigma_x = \sqrt{\left(\frac{\partial x_1}{\partial y_1}\sigma_{y_1}\right)^2 + \left(\frac{\partial x_2}{\partial y_2}\sigma_{y_2}\right)^2} \tag{10}$$

3. Coefficient of Restitution Equations: Height of Bounce

Given that our first equation for e (Eq. (2) in [2]) should be based on heights rather than velocities and we do not wish to consider time, we will use our standard equation for velocity in terms height and acceleration, as found in [3] and rewritten in Eq. (11).

$$v_f^2 = v_i^2 + 2a(h_f - h_i) \tag{11}$$

Given that our ball will start at zero velocity at the height of its bounce, we can set $v_i = 0$. Our sole acting acceleration is that due to gravity, so we set $a = -g$, $h_i = h_{n-1}$ to represent the height of the ball's $(n-1)$th bounce, and $h_f = 0$ to represent the ball hitting the floor (which we consider our zero-point). Because our v_f represents the velocity of the ball just before it hits the floor, this is our pre-impact velocity, $v_{ball,i}$. We thus arrive at Eq. (12).

$$v_{ball,i}^2 = 2gh_{n-1} \tag{12}$$

We solve for $v_{ball,i}$ to arrive at Eq. (13). Our final equation contains a negative sign because this velocity will be in the downward direction.

$$v_{ball,i} = -\sqrt{2gh_{n-1}} \tag{13}$$

We then adapt Eq. (11) to account for the ball's movement following impact. In this case, we set $v_f = 0$, representing that the ball will be momentarily at rest at the top of its nth bounce. Because the ball starts on the floor for this motion, $h_i = 0$. The height of the ball's nth bounce is represented by $h_f = h_n$. Acceleration is still $a = -g$. v_i represents our velocity just after impact with the floor, so that $v_i = v'_{ball}$.

$$0 = v_{ball,f}^2 - 2gh_n \tag{14}$$

We solve for v'_{ball} to arrive at Eq. (15).

$$v_{ball,f} = \sqrt{2gh_n} \tag{15}$$

We can plug these velocity equations into Eq. (9) to arrive at Eq. (16).

$$e = -\frac{\sqrt{2gh_n}}{-\sqrt{2gh_{n-1}}} \tag{16}$$

After simplifying, we obtain Eq. (17), thus having derived Eq. (2) from Method 1 in [2].

$$e = \left(\frac{h_n}{h_{n-1}}\right)^{\frac{1}{2}} \tag{17}$$

4. Measurements Sensitivity: Height of Bounce

This equation for the coefficient of restitution is subject to errors in the initial height of the ball and height of the first bounce. These variables, then, are applied with Eq. (10) to provide Eq. (18).

$$\sigma_{e_{height}} = \sqrt{\left(\frac{\partial e}{\partial h_n}\sigma_{h_n}\right)^2 + \left(\frac{\partial e}{\partial h_{n-1}}\sigma_{h_{n-l}}\right)^2} \tag{18}$$

The partial derivatives in Eq. (18) are calculated in Eqs. (19)-(20).

$$\frac{\partial e}{\partial h_n} = \frac{1}{2(h_n h_{n-1})^{1/2}} \tag{19}$$

$$\frac{\partial e}{\partial h_{n-1}} = -\frac{h_n^{1/2}}{2h_{n-1}^{3/2}} \tag{20}$$

Substituting these equations into Eq. (18), the error for the coefficient of restitution for this method yields the following final equation.

$$\sigma_{e_{height}} = \sqrt{\left(\frac{1}{2(h_n h_{n-1})^{1/2}}\sigma_{h_n}\right)^2 + \left(-\frac{h_n^{1/2}}{2h_{n-1}^{3/2}}\sigma_{h_{n-l}}\right)^2} \tag{21}$$

5. Coefficient of Restitution Equations: Time of Bounce

Next we take a look at our second equation for e (Eq. (3) in [2]), which should be based on times rather than velocities (without considering height). We can start our derivation from our previously-derived Eq. (17). Given that each bounce consists of parabolic (and therefore symmetrical) motion, we know that half of the time for each bounce will entail the ball rising from the ground to its peak bounce height, and the other half of the bounce time will entail the ball falling from its peak height back to the ground. We can use this information with our standard kinematics equation Eq. (22).

$$h_f = h_i + v_{y,i}t + \frac{1}{2}a_y t^2 \tag{22}$$

If we consider our starting point as the peak of bounce $n-1$ and our end point as when the ball hits the floor again, we can set $h_i = h_{n-1}$, $h_f = 0$, $v_{y,i} = 0$, and $t = \frac{t_{n-1}}{2}$. As usual, $a = -g$. With these considerations, we arrive at Eq. (23).

$$0 = h_{n-1} - \frac{1}{2}g\left(\frac{t_{n-1}}{2}\right)^2 \tag{23}$$

We can rearrange Eq. (23) to arrive at Eq. (24).

$$h_{n-1} = \frac{gt_{n-1}^2}{8} \tag{24}$$

Next, we consider our consecutive bounce, bounce n. We again consider our starting point to be the peak of the bounce and the end point to be the floor at which the bounce finishes. Thus, we set $h_i = h_n$, $h_f = 0$, $v_{y,i} = 0$, and $t = \frac{t_n}{2}$. Again, $a = -g$. We arrive at Eq. (25),

$$0 = h_n - \frac{1}{2}g\left(\frac{t_n}{2}\right)^2, \tag{25}$$

which we can rearrange Eq. (25) to arrive at Eq. (26).

$$h_n = \frac{gt_n^2}{8} \tag{26}$$

We can plug these expressions for bounce height into Eq. (17) (as shown in Eq. (27)) to arrive at Eq. (29), having thus derived Eq. (3) from Method 2 in [2].

$$e = \left(\frac{h_n}{h_{n-1}}\right)^{\frac{1}{2}} = \left(\frac{\frac{gt_n^2}{8}}{\frac{gt_{n-1}^2}{8}}\right)^{\frac{1}{2}} \tag{27}$$

$$e = \frac{\sqrt{t_n^2}}{\sqrt{t_{n-1}^2}} \tag{28}$$

$$e = \left(\frac{t_n}{t_{n-1}}\right) \tag{29}$$

6. Measurements Sensitivity: Time of Bounce

This equation for the coefficient of restitution is also subject to errors in the time of the first bounce of the ball and time of the second bounce. These variables, then, are applied with Eq. (10) to provide Eq. (30).

$$\sigma_{e_{bounces}} = \sqrt{\left(\frac{\partial e}{\partial t_n}\sigma_{t_n}\right)^2 + \left(\frac{\partial e}{\partial t_{n-1}}\sigma_{t_{n-l}}\right)^2} \tag{30}$$

The partial derivatives in Eq. (30) are calculated in Eqs. (31)-(32).

$$\frac{\partial e}{\partial t_n} = \frac{1}{t_{n-1}} \tag{31}$$

$$\frac{\partial e}{\partial t_{n-1}} = -\frac{t_n}{t_{n-1}^2} \tag{32}$$

Substituting these equations into Eq. (30), the error for the coefficient of restitution for this method yields the following final equation.

$$\sigma_{e_{bounces}} = \sqrt{\left(\frac{1}{t_{n-1}}\sigma_{t_n}\right)^2 + \left(-\frac{t_n}{t_{n-1}^2}\sigma_{t_{n-l}}\right)^2} \tag{33}$$

7. Coefficient of Restitution Equations: Time to Stop

We can describe t_s as the sum of the times of all of the ball bounces, where each bounce time is represented with t_n. Starting at bounce 1 (beginning the first time the ball hits the ground and ending the second time the ball hits the ground) and ending at $n = \infty$ (theoretically, without friction or energy loss, the ball bounces forever), we can describe this mathematically using Eq. (34).

$$t_s = \sum_{n=1}^{\infty} t_n \tag{34}$$

However, we must account for the time it takes our ball to fall from its initial height to the floor. We call this time value $\frac{t_0}{2}$ and add it to our above equation to obtain Eq. (35). Note that we divide t_0 by two because our ball starts from the peak of a "bounce" rather than from the floor (i.e. the time will be half of what it would be for a full bounce).

$$t_s = \sum_{n=1}^{\infty}(t_n) + \frac{t_0}{2} \tag{35}$$

$$t_s - \frac{t_0}{2} = \sum_{n=1}^{\infty} t_n \tag{36}$$

Note that we can redefine the summation in Eq. (36) to be in terms of t_0, because every bounce of the ball will require a time equal to the time of the previous bounce times the coefficient of restitution, which is equivalent to the expression $t_0 e^n$, the initial bounce time multiplied by e n times for n bounces.

$$t_s - \frac{t_0}{2} = \sum_{n=1}^{\infty} t_0 e^n \tag{37}$$

We can solve the infinite geometric series in Eq. (37) because we know that $e < 1$. In doing so, we arrive at Eq. (38).

$$t_s - \frac{t_0}{2} = \frac{e t_0}{1 - e} \tag{38}$$

We can perform some algebraic manipulation to turn Eq. (38) into the more useful form expressed in Eq. (43).

$$\left(t_s - \frac{t_0}{2}\right)(1-e) = et_0 \tag{39}$$

$$t_s - et_s - \frac{t_0}{2} + \frac{et_0}{2} = et_0 \tag{40}$$

$$t_s - \frac{t_0}{2} = et_0 - \frac{et_0}{2} + et_s \tag{41}$$

$$t_s - \frac{t_0}{2} = et_s + \frac{et_0}{2} \tag{42}$$

$$e = \frac{t_s - \frac{t_0}{2}}{t_s + \frac{t_0}{2}} \tag{43}$$

We solve Eq. (26) from our previous derivation for time and arrive at Eq. (44).

$$t_n = \sqrt{\frac{8h_n}{g}} \tag{44}$$

We plug the result into Eq. (43) and, after solving, arrive at Eq. (46). We have thus obtained the desired equation for coefficient of restitution with respect to time-to-stop and initial height, expressed in Eq. (4) in [2].

$$e = \frac{t_s - \frac{\sqrt{\frac{8h_n}{g}}}{2}}{t_s + \frac{\sqrt{\frac{8h_n}{g}}}{2}} \tag{45}$$

$$e = \frac{t_s - \sqrt{\frac{2h_n}{g}}}{t_s + \sqrt{\frac{2h_n}{g}}} \tag{46}$$

8. Measurements Sensitivity: Time to Stop

This equation for the coefficient of restitution is also subject to errors in the time of the first bounce of the ball and time of the second bounce. These variables, then, are applied with Eq. (10) to provide Eq. (47).

$$\sigma_{e_{stop}} = \sqrt{\left(\frac{\partial e}{\partial t_s}\sigma_{t_s}\right)^2 + \left(\frac{\partial e}{\partial h_0}\sigma_{h_0}\right)^2} \tag{47}$$

The partial derivatives in Eq. (47) are calculated in Eqs. (48)-(49).

$$\frac{\partial e}{\partial t_s} = \frac{2g\sqrt{\frac{2h_0}{g}}}{(t_s\sqrt{g} + \sqrt{2h_0})^2} \tag{48}$$

$$\frac{\partial e}{\partial h_0} = -\frac{t_s\sqrt{2g}}{\sqrt{h_0}(t_s\sqrt{g} + \sqrt{2h_0})^2} \tag{49}$$

Substituting these equations into Eq. (47), the error for the coefficient of restitution for this method yields the following final equation.

$$\sigma_{e_{stop}} = \sqrt{\left(\frac{2g\sqrt{\frac{2h_0}{g}}}{(t_s\sqrt{g} + \sqrt{2h_0})^2}\sigma_{t_s}\right)^2 + \left(-\frac{t_s\sqrt{2g}}{\sqrt{h_0}(t_s\sqrt{g} + \sqrt{2h_0})^2}\sigma_{h_0}\right)^2} \tag{50}$$

C. MATLAB

1. BouncingBall.m

```
%% info

% this script is part of CU's SP 18, 2003: Dynamics
% lab to study the coefficient of restitution for balls, mainly a Ping Pong
% ball. Data are stored in folder /Data in xcel file.
%
% Done by:
%
% - Sarah Foley
% - Hugo Stetz
% - Alexander Lowry
% - Abdulla Al Ameri


%% get data

clear
clc
close all

% read data
Data = xlsread('Data/ExperimentalData.xlsx');

Trial = Data(:,1); % trial number
Bounce1_Time = Data(:,2); % time between the first two bounces
Bounce2_Time = Data(:,3);
TotalTime = Data(:,4);
Error_time_values = Data(:,5);
Height_firstbounce = Data(:,7); % inches,
Height_secondboune = Data(:,6); % inches,


h0_inches = 36 ; %inches.
h0_SI = 0.9144 ; % meters
g = 9.81 ; % gravity
g = 386.09 ; % gravity in inches/s^2


%% error analysis : all error sources


% error from instrument
h0_error = 1/14 ; % the same for all trials
TotalTime_error = 0.5 ; % changes per trial
Height_error = 1/14 ; % 1/4 inche is our error from height, eyeballed from the video
Time_error_each_bounce = 0.05; %

% random error

Height_firstbounce_std = std(Height_firstbounce);
Height_secondbounce_std = std(Height_secondboune);
Error_time_values_std = std(Error_time_values); %std + 0.5 time to reaction?
Bounce1_Time_std = std(Bounce1_Time);
Bounce2_Time_std = std(Bounce2_Time);

% total error:

Height_firstbounce_total_error = sqrt ( (Height_firstbounce_std).^2 + (Height_error).^2 );
Height_secondbounce_total_error = sqrt ( (Height_secondbounce_std).^2 + (Height_error).^2);
time_values_total_error = sqrt ( (Error_time_values_std).^2 + (TotalTime_error).^2);
Bounce1_Time_total_error = sqrt ( (Time_error_each_bounce).^2 + (Bounce1_Time_std).^2);
Bounce2_Time_total_error = sqrt ( (Time_error_each_bounce).^2 + (Bounce2_Time_std).^2);
```

```
h0_total_error = sqrt ( h0_error^2 + Height_error^2 ) ;


%% error analysis : functions


syms h0 ts hn tn tn_1 hn_1 tn n
% hn_1 tn_1 is previous height, and time

e_stop_error(h0,ts) = (ts - sqrt((2*h0)/g))/(ts + sqrt((2*h0)/g));
e_height_error(hn,h0,n) = (hn/h0)^(1/(2*n)) ;
e_bonces_error(tn,tn_1) = ( tn / tn_1 ) ;


%% e : time to stop

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

% error analysis : derivatives: e_Stop

%partial of e_stop with respect to h0;
Partial_stop_h0 = diff(e_stop_error,h0);
Partial_stop_ts = diff(e_stop_error,ts);


% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -


% do the math!

for i=1:length(Trial)

e_stop(i) = (TotalTime(i) - sqrt((2*h0_inches)/g))/(TotalTime(i) + sqrt((2*h0_inches)/g));
Error_stop(i) = double(sqrt ( ((Partial_stop_h0(h0_inches,TotalTime(i))) * h0_total_error ...
    ).^2 + (((Partial_stop_ts(h0_inches,TotalTime(i))) * time_values_total_error).^2) ));

end

%% e : time of bounce

% error analysis : derivatives: e_bounces


% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

%partial of e_bouncs with respect to tn, tn_1;

Partial_bounces_tn = diff(e_bonces_error,tn);
Partial_bounces_tn_1 = diff(e_bonces_error,tn_1);


% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -


for i=1:length(Trial)

e_bounces(i) = Bounce2_Time(i) / Bounce1_Time(i) ;
Error_bounces(i) = double(sqrt ( ((Partial_bounces_tn(Bounce2_Time(i),Bounce1_Time(i)) * ...
    Bounce2_Time_total_error )).^2 + ...
    ((Partial_bounces_tn_1(Bounce2_Time(i),Bounce1_Time(i)) * ...
    Bounce1_Time_total_error)).^2) );

end

%% e : height of bounce

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

```

```
% error analysis : derivatives: e_height

%partial of e_height with respect to tn, tn_1;
Partial_height_hn = diff(e_height_error,hn);
Partial_height_h0 = diff(e_height_error,h0);

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

for i=1:length(Trial)

    % the equation says * n but we only using the first bounce of each
    % trial so n = 1 ;

e_height(i) = ( Height_firstbounce(i) / h0_inches ) ^ ( 1 / ( 2 )) ;
e_height_2(i) = ( Height_secondboune(i) / Height_firstbounce(i) ) ^ ( 1 / ( 4 )) ;

Error_height(i) = double(sqrt ( ((Partial_height_hn(Height_firstbounce(i),h0_inches,1) * ...
    Height_error ))^2 + ((Partial_height_h0(Height_firstbounce(i),h0_inches,1) * ...
    h0_total_error))^2 ));
Error_height_2(i) = double(sqrt ( ((Partial_height_hn(Height_secondboune(i), ...
    Height_firstbounce(i),2) * Height_secondbounce_total_error ))^2 + ...
    ((Partial_height_h0(Height_secondboune(i), Height_firstbounce(i),2) * ...
    Height_firstbounce_total_error))^2 ));


end


%% plot error


% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

figure(1)

errorbar([1:length(Trial)],e_height,Error_height_2,'-*')
hold on
errorbar([1:length(Trial)],e_bounces,Error_bounces,'-o')
hold on
errorbar([1:length(Trial)],e_stop,Error_stop,'-^')
grid minor
ylabel('Coefficient of restitution (unitless)')
xlabel('Trial')
title('Coefficient of restitution and error')
xlim([0.5 10.5])
legend('e_h_e_i_g_h_t','e_b_o_u_n_c_e_s','e_s_t_o_p','Location','SouthEast')

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

fprintf('e_stop: %0.3f',mean(e_stop) );
fprintf('    %0.3f \n', mean(Error_stop) );

fprintf('e_height: %0.3f',mean(e_height) );
fprintf('    %0.3f \n', mean(Error_height_2) );

fprintf('e_bounces: %0.3f',mean(e_bounces) );
fprintf('    %0.3f \n', mean(Error_bounces) );



% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
```

2. *ImageTracking.m*

```
%% info:
```

```
% this script is to estimate the coefficient of restitution of a ball using
% an image trackign software.

% Done by:
%
% - Sarah Foley
% - Hugo Stetz
% - Alexander Lowry
% - Abdulla Al Ameri


%% import data:

clear
clc
close all;


Data1 = xlsread('Data/Trial_1_Data_Image_Tracking.xlsx');
Data7 = xlsread('Data/Trial_7_Data_Image_Tracking.xlsx');
Data8 = xlsread('Data/Trial_8_Data_Image_Tracking.xlsx');

Data1(1,:) = [];

Data1(:,2) = Data1(:,2)*39.3701; % convert from inches to meters
Data7(:,2) = Data7(:,2)*39.3701; % convert from inches to meters
Data8(:,2) = Data8(:,2)*39.3701; % convert from inches to meters

% correct outlayer in all data set by making them hit the ground at height
% 0

% Data8(2:end,2) = Data8((2:end),2)  ;

Data8(:,2) = Data8(:,2) - min(Data8(:,2));
Data7(:,2) = Data7(:,2) - min(Data7(:,2));
Data1(:,2) = Data1(:,2) - min(Data1(:,2));

% zero the time

Data1(:,1) = Data1(:,1) - Data1(1,1) ;
Data7(:,1) = Data7(:,1) - Data7(1,1) ;
Data8(:,1) = Data8(:,1) - Data8(1,1) ;


% h0 is just the first value in each data

h0_trial_1 = Data1(1,2);
h0_trial_7 = Data7(1,2);
h0_trial_8 = Data8(1,2);


% height of second bounce, it happens after 0.4 seconds so we can start
% looking after that

% get time bigger than 0.4

time_index_1 = find(Data1(:,1)>0.4);
time_index_7 = find(Data7(:,1)>0.4);
time_index_8 = find(Data8(:,1)>0.6); %for this one it happens after 0.6 seconds


% hn is height of the second bounce.

hn_Data1 = max(Data1(time_index_1(1):end,2));
hn_Data7 = max(Data7(time_index_7(1):end,2));
hn_Data8 = max(Data8(time_index_8(1):end,2));

%% get e values:
```

```
e_height_1 = ( hn_Data1 / h0_trial_1 ) ^ ( 1 / ( 2 )) ;
e_height_7 = ( hn_Data7 / h0_trial_7 ) ^ ( 1 / ( 2 )) ;
e_height_8 = ( hn_Data8 / h0_trial_8 ) ^ ( 1 / ( 2 )) ;

%% error analysis : all error sources

% error from instrument
h0_error_insturment = 1/14 ; % the same for all trials
hn_error_insturment = 1/14 ; % 1/4 inche is our error from height, eyeballed from the video

% random error

% std of all the three h0's because in theory they should be the same
% this's the same concept for hn

h0_error_random = std([ h0_trial_1 h0_trial_7 h0_trial_8 ]);
hn_error_random = std([ hn_Data1 hn_Data7 hn_Data8 ]);

% total error:

h0_total_error = sqrt ( (h0_error_insturment).^2 + (h0_error_random).^2 );
hn_total_error = sqrt ( (hn_error_insturment).^2 + (hn_error_random).^2 );

% error:

syms hn h0

e_height_error(hn,h0) = (hn/h0)^(1/(2));
Partial_height_hn = diff(e_height_error,hn);
Partial_height_h0 = diff(e_height_error,h0);

total_error_Data1 = double(sqrt ( ((Partial_height_h0(h0_trial_1,hn_Data1)) * ...
    h0_total_error ).^2 + (((Partial_height_hn(h0_trial_1,hn_Data1)) * hn_total_error).^2) ));
total_error_Data7 = double(sqrt ( ((Partial_height_h0(h0_trial_7,hn_Data7)) * ...
    h0_total_error ).^2 + (((Partial_height_hn(h0_trial_7,hn_Data7)) * hn_total_error).^2) ));
total_error_Data8 = double(sqrt ( ((Partial_height_h0(h0_trial_8,hn_Data8)) * ...
    h0_total_error ).^2 + (((Partial_height_hn(h0_trial_8,hn_Data8)) * hn_total_error).^2) ));



%% plot results

figure(1)

plot(Data1(:,1),Data1(:,2),'-.*')
hold on
plot(Data7(:,1),Data7(:,2),'-.s')
hold on
plot(Data8(:,1),Data8(:,2),'-.o')
hold on
hold on
plot([0:0.1:1.3], zeros(1,length([0:0.1:1.3])),'LineWidth',1.3)
legend('Trial 1','Trial 8','Trial 8','Ground')
grid minor
xlabel('time (second)');
ylabel('Height (inches)');
ylim([-1 40])
xlim([0 1.05])
title('Trajectory of the ball from height h0')

figure(2)
errorbar(1,e_height_1,total_error_Data1,'s')
hold on
errorbar(2,e_height_7,total_error_Data7,'^')
hold on
errorbar(3,e_height_8,total_error_Data8,'o')
grid minor
```

```
title('Image tracking coefficient of restitution with error bars')
xticks([ 1 2 3 ])
xticklabels({'Trial 1','Trial 7','Trial 8'})
xlim([ 0.5 3.5])
ylabel('Coefficient of restitution (unitless)')


%% printout resutls

fprintf('e_height_1: %0.3f',e_height_1 );
fprintf('   %0.4f \n', total_error_Data1 );

fprintf('e_height_7: %0.3f',e_height_7 );
fprintf('   %0.4f \n', total_error_Data7 );

fprintf('e_height_8: %0.3f',e_height_8 );
fprintf('   %0.4f \n', total_error_Data8 );
```

3. *ImprovedMethod.m*

```
%% info

% this script is to re-estimate the Coefficient of restitution for the pong ball
% and also a tennis ball using an improved method, which depends on using audio waves to
% estimate the time between bounces and total time it took the ball to stop

% Done by:
%
% - Sarah Foley
% - Hugo Stetz
% - Alexander Lowry
% - Abdulla Al Ameri



%% read data

clear
clc
close all


Data = xlsread('Data/ImprovedMethodData.xlsx');

trial_pong = Data(:,1);
trial_golf = Data(:,3);
time_pong = Data(:,2); % time to stop.
time_golf = Data(:,4); % time to stop.

%% estimate errors:

% let the random error in each individual ball for time to stop be std because they
% in theory should have the same time.

random_error_pong = std(time_pong);
random_error_golf = std(time_golf);

% the insturment error really depends on when we decide to make the
% cut-off, this method is really really accurate, so we can take this error
% to be a very small fraction of a second!

inst_error_pong = 0.00001 ;
inst_error_golf = 0.00001 ;


```

```
time_total_error_pong = sqrt ( (random_error_pong)^2 + (inst_error_pong)^2 ) ;
time_total_error_golf = sqrt ( (random_error_golf)^2 + (inst_error_golf)^2 ) ;

h0_inches = 36 ; %inches.
g = 386.09 ; % gravity in inches/s^2

h0_error = 1/14 ; % the same for all trials


%% calculating e:


syms h0 ts

e_stop_error(h0,ts) = (ts - sqrt((2*h0)/g))/(ts + sqrt((2*h0)/g));
%partial of e_stop with respect to h0;
Partial_stop_h0 = diff(e_stop_error,h0);
Partial_stop_ts = diff(e_stop_error,ts);


% for ping-pong ball

% do the math!

for i=1:length(time_pong)

e_pong(i) = (time_pong(i) - sqrt((2*h0_inches)/g))/(time_pong(i) + sqrt((2*h0_inches)/g));
total_error_pong(i) = double( sqrt ( (Partial_stop_h0(h0_inches,time_pong(i)) * h0_error) ...
    .^2 + ((Partial_stop_ts(h0_inches,time_pong(i)) * time_total_error_pong).^2)));

end

% repeat for golf ball.

for i=1:length(time_golf)

e_golf(i) = (time_golf(i) - sqrt((2*h0_inches)/g))/(time_golf(i) + sqrt((2*h0_inches)/g));
total_error_golf(i) = double( sqrt ( (Partial_stop_h0(h0_inches,time_golf(i)) * h0_error) ...
    .^2 + ((Partial_stop_ts(h0_inches,time_golf(i)) * time_total_error_golf).^2)));

end


%% plot results

figure(1)

errorbar(trial_golf,e_golf,total_error_golf,'-.*')
hold on
errorbar(trial_pong,e_pong,total_error_pong,'-.s')
grid minor
ylabel('Coefficient of restitution and error')
xlabel('Trial')
title('Coefficient of restitution for golf and ping-pong ball')
legend('e_g_o_l_f','e_p_o_n_g','Location','SouthEast')
xlim([ 0.5 10.5])

%% printout results

fprintf('e_golf: %0.3f',mean(e_golf) );
fprintf('   %0.3f \n', mean(total_error_golf) );

fprintf('e_pong: %0.3f',mean(e_pong) );
fprintf('   %0.3f \n', mean(total_error_pong) );
```