Numerical Methods for the Point Kinetics Equation

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Contents

Scope of the Thesis

- Implement the main algorithms in the literature for solving the Point Kinetics Equations (PKE).
- analyse their performances and limitations.
- Perform a real-life conditions benchmark on all schemes, not present in literature.

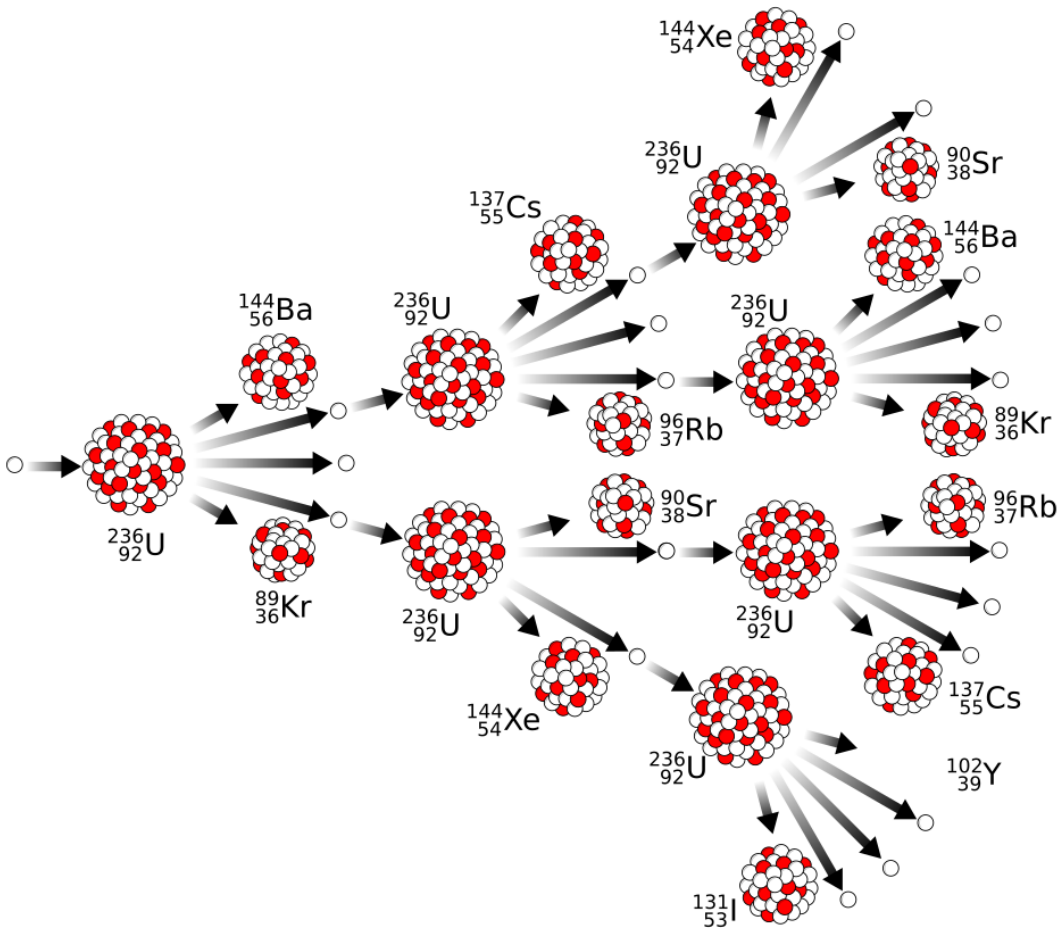
The Physical Model

During a fission reaction, neutrons are generated, by the decay of:

- **Fuel**, usually enriched Uranium.
- **Precursors**, intermediate stages of the reaction.

The standard model for a nuclear reactor in the literature contains 6 precursors and the neutrons generated by them are called *delayed*.

The Physical Model



Credit: *MikeRun, Wikimedia Commons*

Figure: An example of Nuclear fission with precursors.

The System of ODEs

The system of ordinary differential equations (ODEs) describing point kinetics is as follows:

$$\begin{cases} \frac{dN(t)}{dt} = \frac{(\rho(t)-\beta)}{\Lambda} N(t) + \sum_{i=1}^{6} \lambda_i c_i(t) \\ \frac{dc_i(t)}{dt} = \frac{\beta_i}{\Lambda} N(t) - \lambda_i c_i(t) \quad i = 1, ..., 6 \end{cases} \tag{1}$$

The computational complexity lies in the reactive term $\rho(t)$; if we were limited to the constant case, we would be able to find an analytical solution.
The reactivity $\rho(t)$ is expressed in dollars (\$), a pure number given by the normalization of the number of free neutrons.
This is considered a stiff problem, in[1] this stiffness was decoupled from the precursors, giving us (1).

[1] A. Attard Y.A. Chao. "A resolution to the Stiffness Problem of Reactor Kinetics". In: *Nuclear Science and Engineering* 90 (1985), pp. 40–46.

The Considered Reactor

The reactor is the environment in which we solve the system and is defined by the following parameters:

- $\vec{\beta}$: vector of the fraction of neutrons produced by each precursor.
- $\vec{\lambda}$: vector of precursor lifetimes.
- Λ: neutron generation time of the reactor.
- α: reactor feedback term.

The initial tests will be without feedback, assuming that the reactive term does not generate additional reactivity.

PCA Scheme

The idea: treat the problem as a system of linear ODEs and then approximate the nonlinear term.[2]

- The eigenvalues are computed through the *inhour equation:*

$$\rho_i = \beta + \Lambda\omega - \sum_{j=1}^{m} \frac{\beta_j \lambda_j}{\omega + \lambda_j} \tag{2}$$

- Once computed the relative eigenvectors the next step is:

$$\vec{x}_{i+1} = P_i e^{D_i h} P_i^{-1} \vec{x}_i \tag{3}$$

With $\rho(t)$ =constant this method computes the exact solution for the system of ODEs.

[2] E.J. Allen Matthew Kinard. "Efficient numerical solution of the point kinetics equations in nuclear reactor dynamics". In: *Annals of Nuclear Energy* 31 (2004), pp. 1039–1051.

CORE Model

The idea: Solve the system of ODEs with a Laplace Transform.[3]

- The transforrmed system becomes:

$$\begin{cases} s\mathcal{N} = \frac{(\rho-\beta)}{\Lambda}\mathcal{N} + \sum_{i=1}^{6} \lambda_i \mathcal{C}_i \\ s\mathcal{C}_i = \frac{\beta_i}{\Lambda}\mathcal{N} - \lambda_i \mathcal{C}_i + c_i(0^-) \end{cases} \tag{4}$$

- Using the Heaviside expansion on $\mathcal{N}(s)$ one can obtain the solution for $N(t)$ as:

$$N(t) = \sum_{k=1}^{7} R_k e^{s_k t} \tag{5}$$

With $s_k \forall k = 1, .., 7$ solutions to the *inhour equation*.

[3] B. Quintero-Leyva. "CORE: A numerical algorithm to solve the point kinetics equations". In: *Annals of Nuclear Energy* 35 (2008).

TSE Model

- A simple first order expansion of the differential equations.
- A lightweight algorithm.

$$\begin{cases} N(t+h) = N(t) + h\frac{\rho(t)-\beta}{\Lambda}N(t) + h\sum_{i=1}^{6}\lambda_i C_i(t) \\ C_i(t+h) = C_i(t) + h\frac{\beta_i}{\Lambda}N(t) - h\lambda_i C_i(t) \end{cases} \tag{6}$$

BEFD Model

The idea: Create a more robust scheme using an implicit method, and add an equation to account for adiabatic feedback.[4]

- The equation for adiabatic feedback is:

$$\frac{d\rho(t)}{dt} = \frac{d\rho_0(t)}{dt} - BN(t) \tag{7}$$

- The scheme thus becomes (supposing no source or drain of neutron is present):

$$[I - hA(t_{j+1}, N(t_{j+1}))]Y(t_{j+1}) = Y(t_j) \tag{8}$$

- **Richardson Extrapolation** is performed to the result:

$$\varphi_s(h) = \frac{2^s\varphi(h/2) - \varphi_{s-1}(h)}{2^s - 1} \tag{9}$$

  This assures a more stable initial condition for the next time step.

[4] Ganapol B.D. "A highly accurate algorithm for the solution of the point kinetics equations". In: *Annals of Nuclear Energy* 62 (2012).

Step and Ramp Reactivity Analysis[5]

Since there is no variable reactive term, we can solve the system analytically:

- The closed form of the solution is:

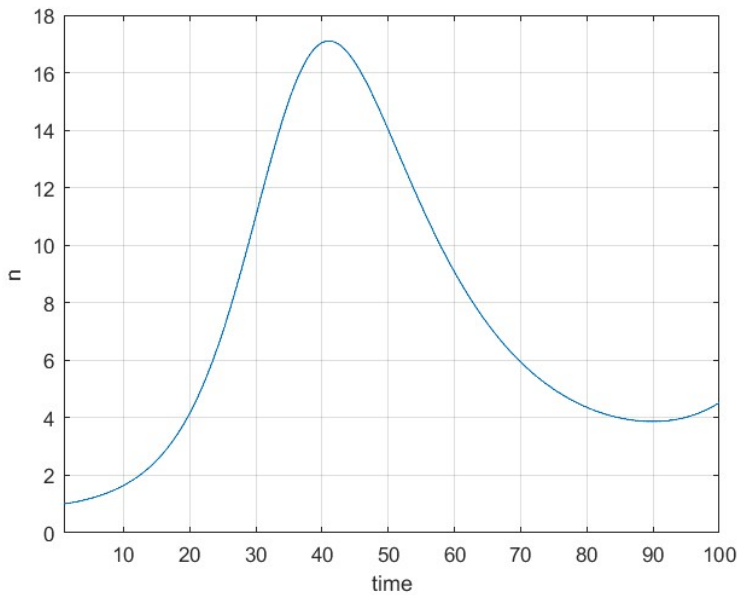
$$N(t) = \sum_{i=1}^{7} c_i e^{\lambda_i t} X_{1i} \tag{10}$$

  with λ_i eigenvalues and X matrix of eigenvectors.
- Three stages can be defined at constant $\rho(t)$:
  - *Subcritical Stage*: $\rho = 0.003$
  - *Critical Stage*: $\rho = 0.007$
  - *Supercritical Stage*: $\rho = 0.008$

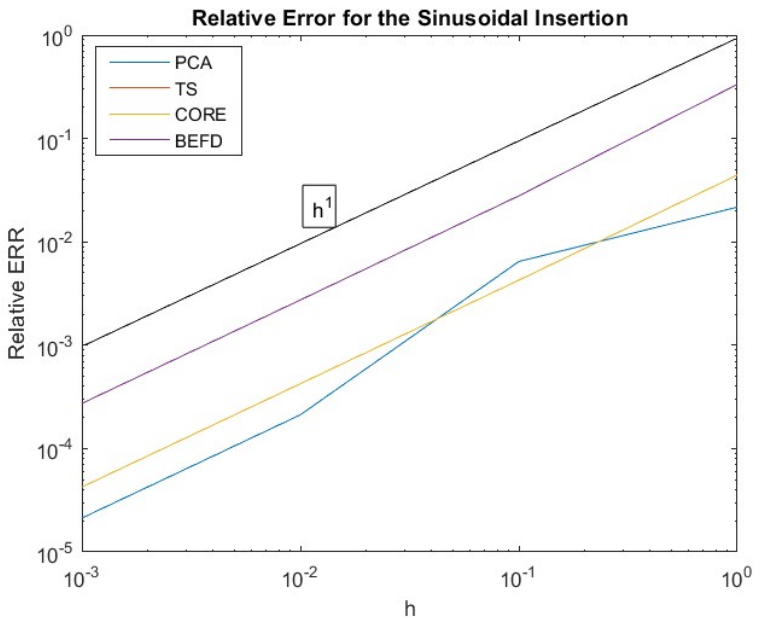
[5] E.J. Allen Matthew Kinard. "Efficient numerical solution of the point kinetics equations in nuclear reactor dynamics". In: *Annals of Nuclear Energy* 31 (2004), pp. 1039–1051.

Sinusoidal Reactivity Analysis[6]

- This is the only example where only one precursor is present, as it is common in literature.
- The sinusoidal case is important to validate the schemes with a nonlinear reactivity term.



(a) Neutron profile for sinusoidal insertion.

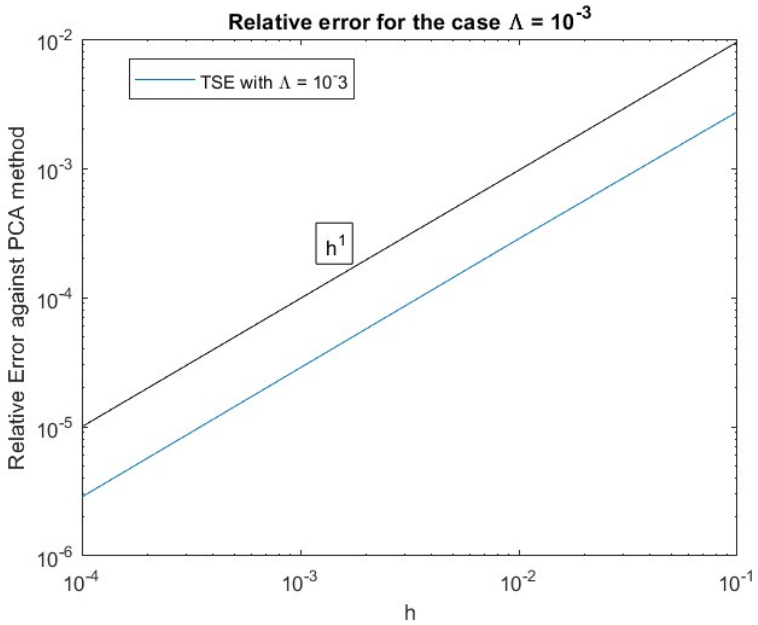


(b) Relative error of the schemes.

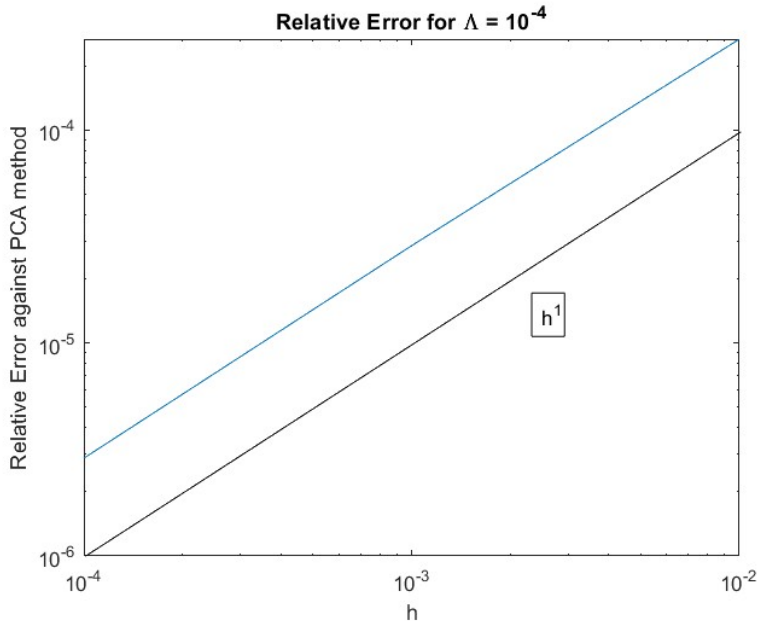
[6] Ganapol B.D. "A highly accurate algorithm for the solution of the point kinetics equations". In: *Annals of Nuclear Energy* 62 (2012).

TSE[7] Scheme Case

- For $\Lambda = 10^{-7}$ the TSE diverges.
- Λ and h, the step size, are not independent.
- There seems to be convergence only if $\frac{h}{\Lambda} \leq 100$.



(a) $\Lambda = 10^{-3}$.



(b) $\Lambda = 10^{-4}$.

[7] A. Pierson D. McMahon. "A Taylor series solution of the reactor point kinetics equations". In: *Paper of the Department of Nuclear Safety Analysis,Sandia National Laboratories, Albuquerque* (2010).

Benchmark with Adiabatic Feedback[8]

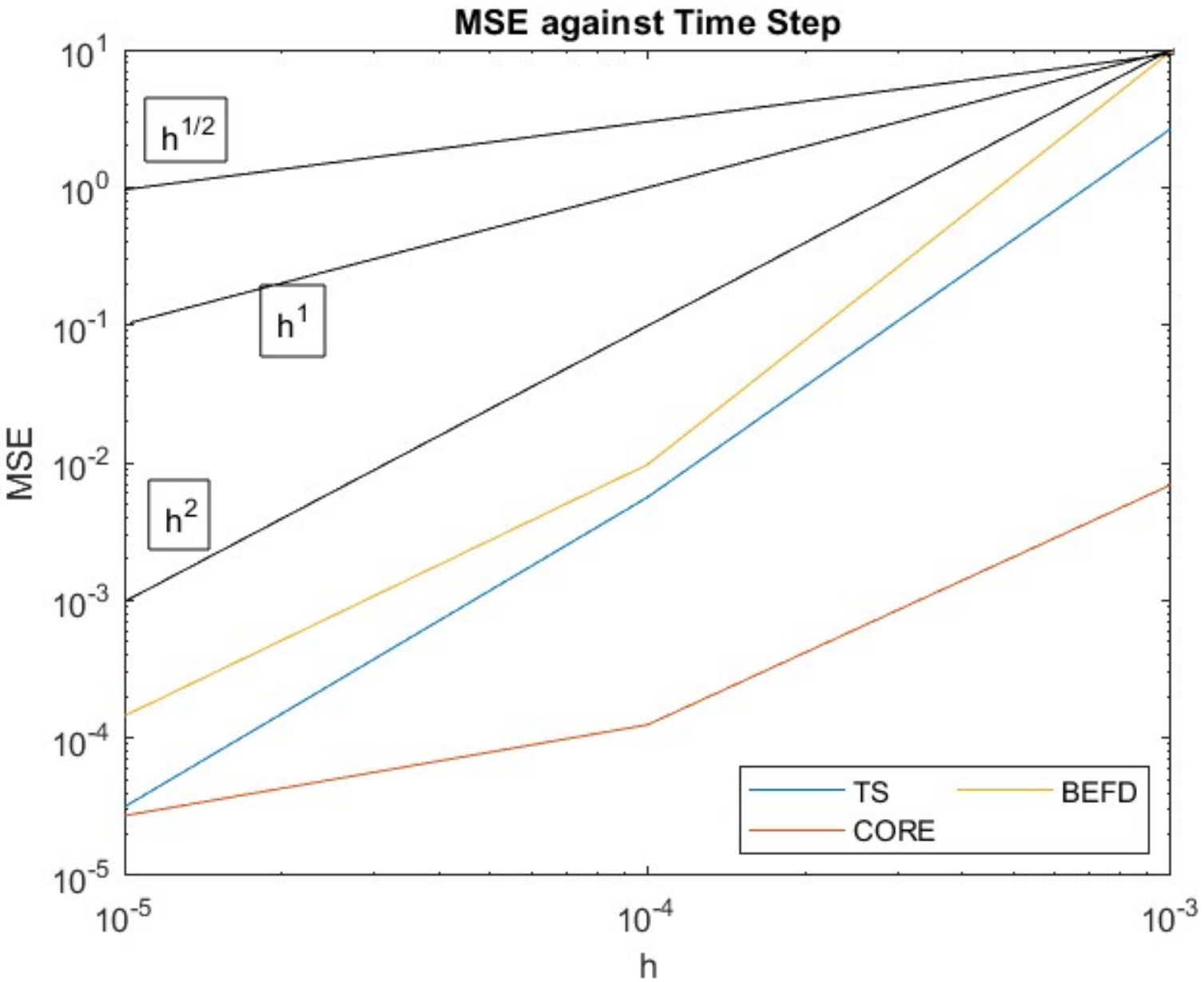
- With $\alpha = 10^{-11}$ the reactivity ρ is redefined:

$$\frac{d\rho}{dt}(t) = \frac{d\rho_0}{dt}(t) + \alpha N(t) \tag{11}$$

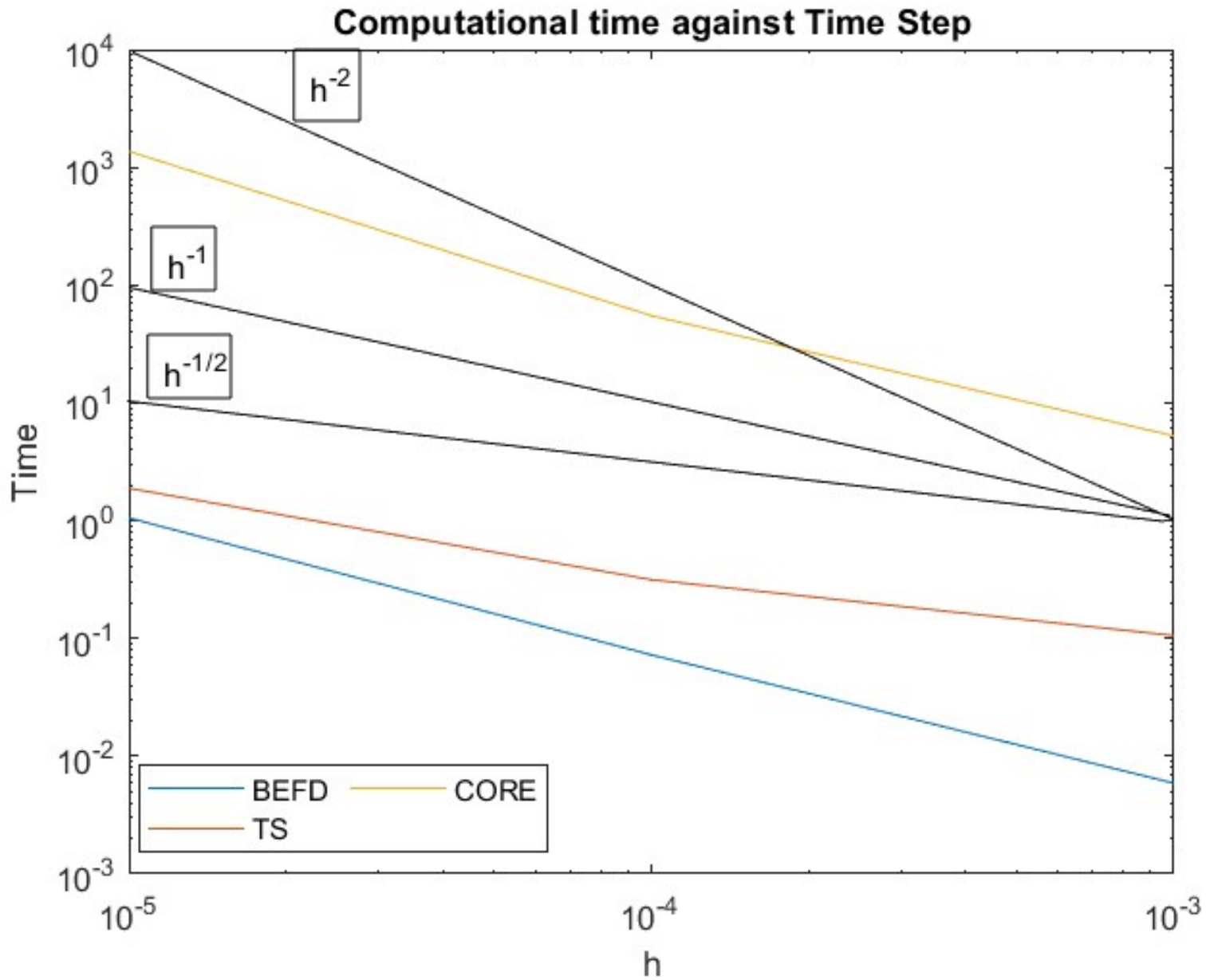
- The only scheme designed specifically for Adiabatic Feedback is BEFD.
- This experiment is run to compare the numerical schemes, and show their adaptability.

[8] R. Furfaro P. Picca B.D. Ganapol. "An accurate technique for the solution of the non-linear point kinetics equations". In: *International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering* (2011).

Benchmark MSE Plot



Benchmark Computational Time Plot



Conclusions

- This thesis aimed to implement in MATLAB the main algorithms for solving PKE.
- Simulations were carried out to ensure accuracy as in the literature.
- Real-life application conditions, like adiabatic feedback.
- The most performing algorithms on all simulations are BEFD and CORE.
- For the TSE scheme a dependence on the problem parameters has been suggested.

Future Developments

- Seek to find the limitations of the CORE algorithm compared to the implicit BEFD scheme.
- Prove with rigour the necessary condition for the convergence of the TSE scheme, with sinusoidal reactivity.