CHAPTER # 4

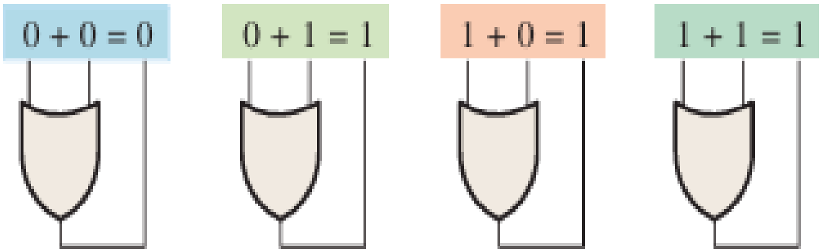
Boolean Algebra and Logic Simplification

Why Do we study this chapter????

In a microprocessor, the arithmetic logic unit (ALU) performs arithmetic and Boolean logic operations on digital data as directed by program instructions. Logical operations are equivalent to the basic gate operations that you are familiar with but deal with a minimum of 8 bits at a time. Examples of Boolean logic instructions are AND, OR, NOT, and XOR, which are called mnemonics. An assembly language program uses the mnemonics to specify an operation. Another program called an assembler translates the mnemonics into a binary code that can be understood by the microprocessor.

Boolean Addition

Boolean addition is equivalent to the OR operation.



Some examples of sum terms

$$A + B, A + \overline{B}, A + B + \overline{C}, \text{ and } \overline{A} + B + C + \overline{D}.$$

Determine the values of A, B, C, and D that make the sum term $A + \overline{B} + C + \overline{D}$ equal to 0.

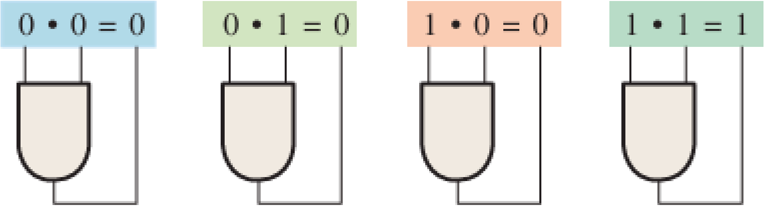
Solution

For the sum term to be 0, each of the literals in the term must be 0. Therefore, $A = \mathbf{0}$, $B = \mathbf{1}$ so that $\overline{B} = 0$, $C = \mathbf{0}$, and $D = \mathbf{1}$ so that $\overline{D} = 0$.

$$A + \overline{B} + C + \overline{D} = 0 + \overline{1} + 0 + \overline{1} = 0 + 0 + 0 + 0 = 0$$

Boolean Multiplication

Boolean multiplication is equivalent to the AND operation.



A product term is equal to 1 only if each of the literals in the term is 1. A product term is equal to 0 when one or more of the literals are 0.

Determine the values of A, B, C, and D that make the product term $A\overline{B}C\overline{D}$ equal to 1.

Solution

For the product term to be 1, each of the literals in the term must be 1. Therefore, $A = \mathbf{1}$, $B = \mathbf{0}$ so that $\overline{B} = 1$, $C = \mathbf{1}$, and $D = \mathbf{0}$ so that $\overline{D} = 1$.

$$A\overline{B}C\overline{D} = 1 \cdot \overline{0} \cdot 1 \cdot \overline{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

Related Problem

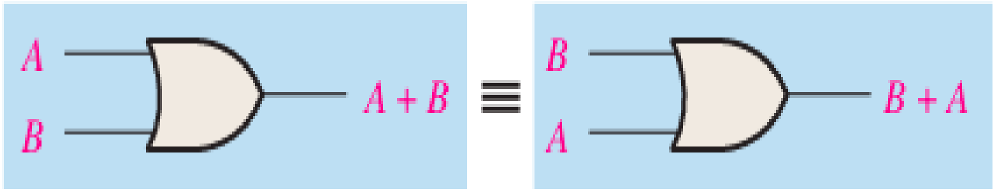
Determine the values of A and B that make the product term $\overline{A}\,\overline{B}$ equal to 1.

1. If $A = 0$, what does $\overline{A}$ equal?
2. Determine the values of A, B, and C that make the sum term $\overline{A} + \overline{B} + C$ equal to 0.
3. Determine the values of A, B, and C that make the product term $A\overline{B}C$ equal to 1.

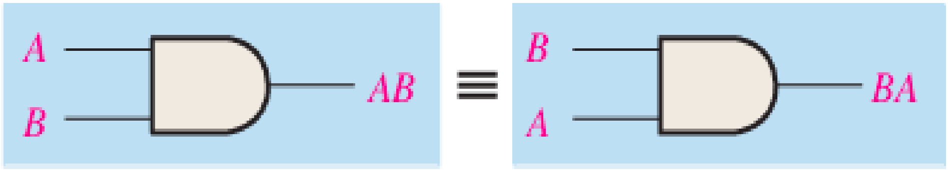
Laws and Rules of Boolean Algebra

Commutative Laws

$$A + B = B + A$$

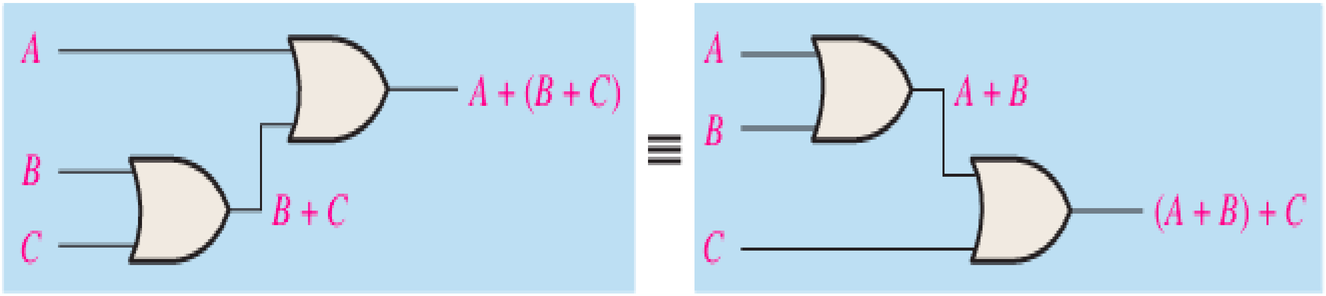


$$AB = BA$$

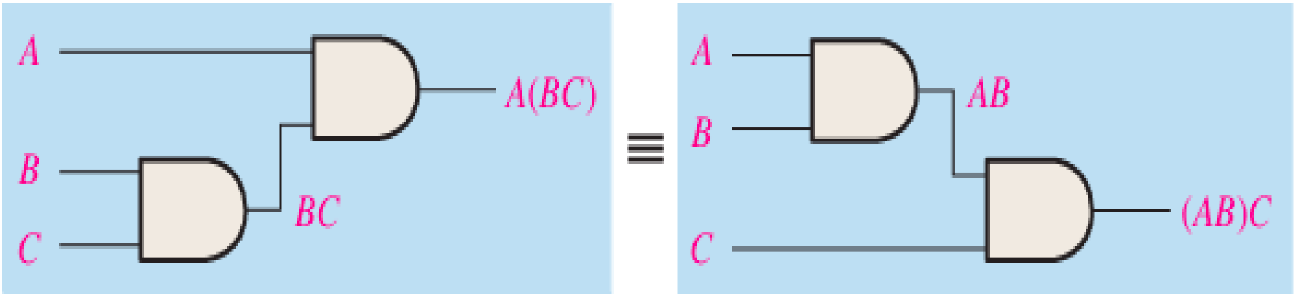


Associative Laws

$$A + (B + C) = (A + B) + C$$

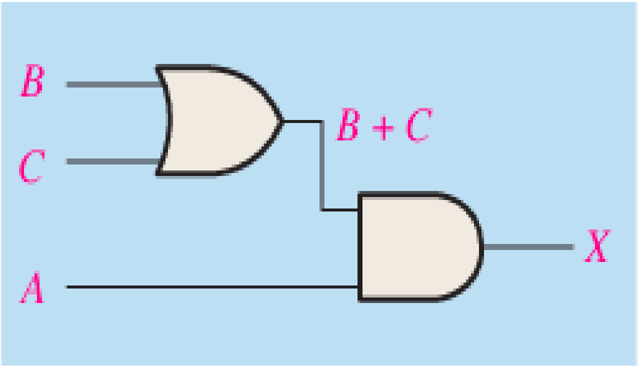


$$A(BC) = (AB)C$$



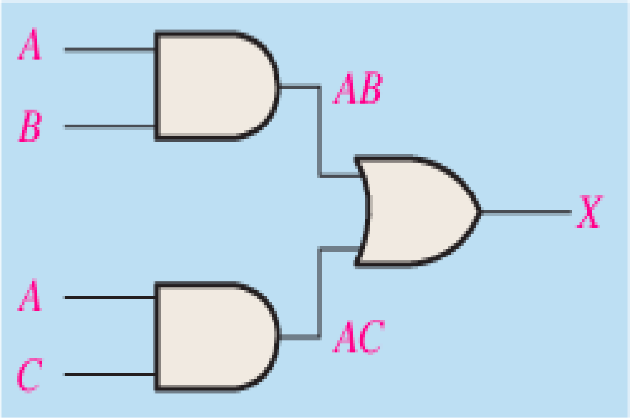
Distributive Law

$$A(B + C) = AB + AC$$



$X = A(B + C)$

≡

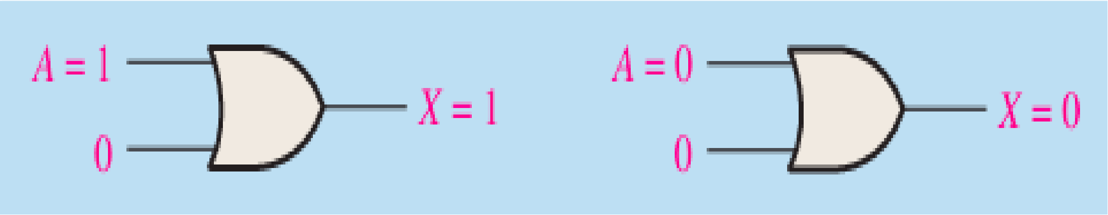


$X = AB + AC$

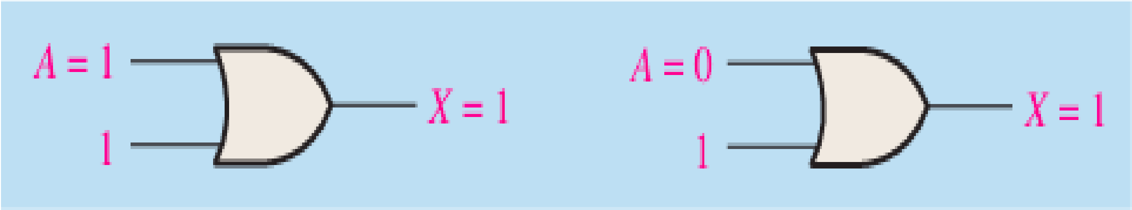
Rules of Boolean Algebra

Basic rules of Boolean algebra.

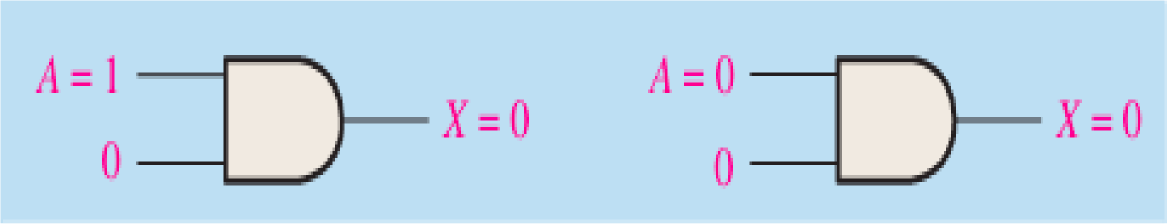
1. $A + 0 = A$	**7.** $A \cdot A = A$
2. $A + 1 = 1$	**8.** $A \cdot \overline{A} = 0$
3. $A \cdot 0 = 0$	**9.** $\overline{\overline{A}} = A$
4. $A \cdot 1 = A$	**10.** $A + AB = A$
5. $A + A = A$	**11.** $A + \overline{A}B = A + B$
6. $A + \overline{A} = 1$	**12.** $(A + B)(A + C) = A + BC$



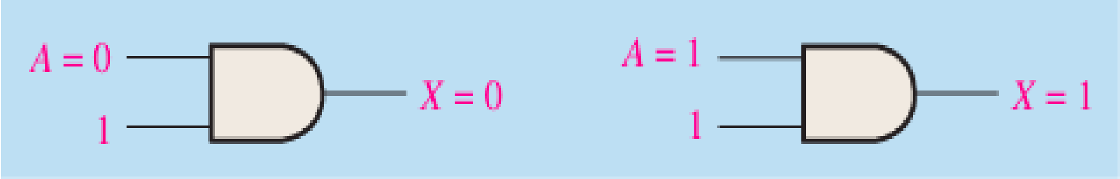
$X = A + 0 = A$



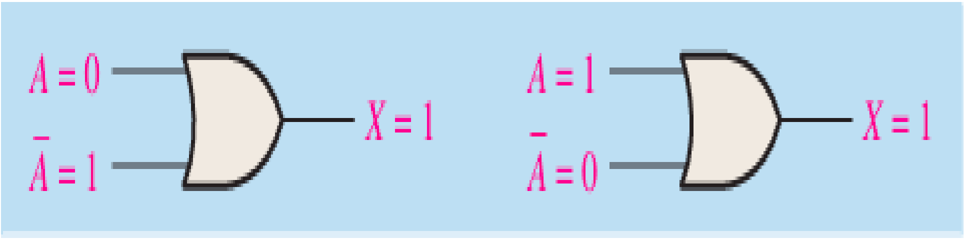
$X = A + 1 = 1$



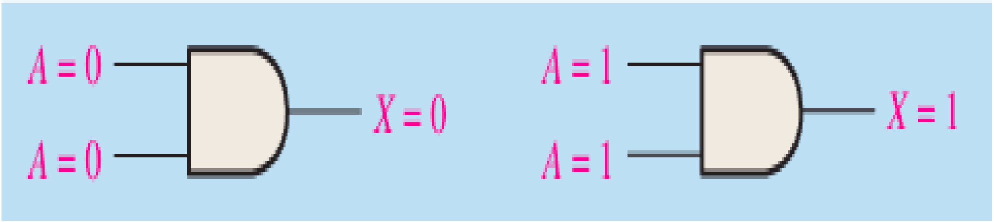
$X = A \cdot 0 = 0$



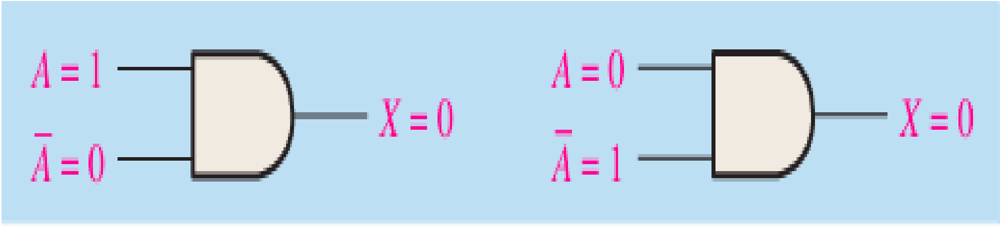
$X = A \cdot 1 = A$



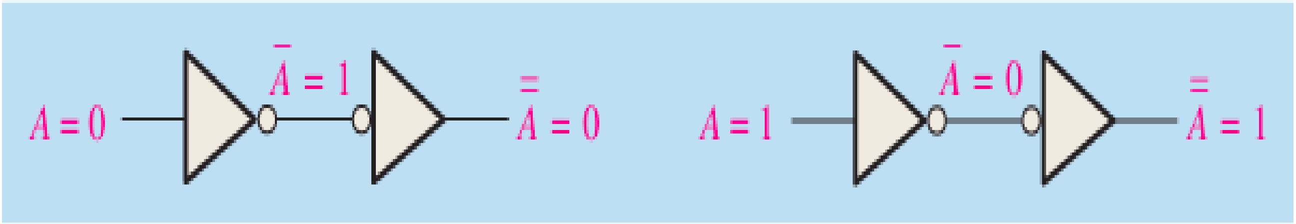
$X = A + \overline{A} = 1$



$X = A \cdot A = A$



$X = A \cdot \overline{A} = 0$



$\overline{\overline{A}} = A$

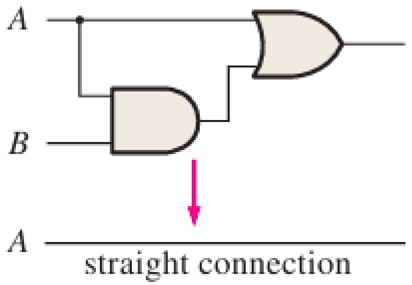
$A + AB = A$

$$
\begin{aligned}
A + AB &= A \cdot 1 + AB = A(1 + B) && \text{Factoring (distributive law)} \\
&= A \cdot 1 && \text{Rule 2: } (1 + B) = 1 \\
&= A && \text{Rule 4: } A \cdot 1 = A
\end{aligned}
$$

Rule 10: $A + AB = A$. Open file T04-02 to verify.

A	B	AB	$A + AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

equal



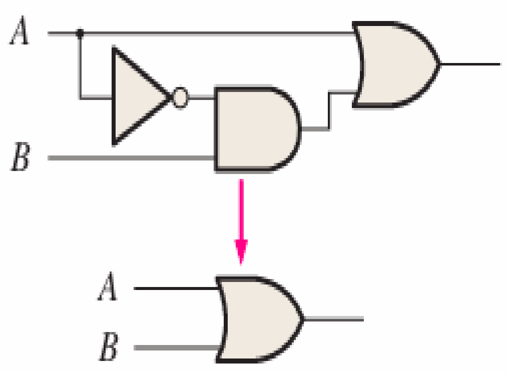
Rule 11: $A + \overline{A}B = A + B$ This rule can be proved as follows:

$$
\begin{aligned}
A + \overline{A}B &= (A + AB) + \overline{A}B && \text{Rule 10: } A = A + AB \\
&= (AA + AB) + \overline{A}B && \text{Rule 7: } A = AA \\
&= AA + AB + A\overline{A} + \overline{A}B && \text{Rule 8: adding } A\overline{A} = 0 \\
&= (A + \overline{A})(A + B) && \text{Factoring} \\
&= 1 \cdot (A + B) && \text{Rule 6: } A + \overline{A} = 1 \\
&= A + B && \text{Rule 4: drop the 1}
\end{aligned}
$$

Rule 11: $A + \overline{A}B = A + B$. Open file T04-03 to verify.

A	B	$\overline{A}B$	$A + \overline{A}B$	$A + B$
0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

equal



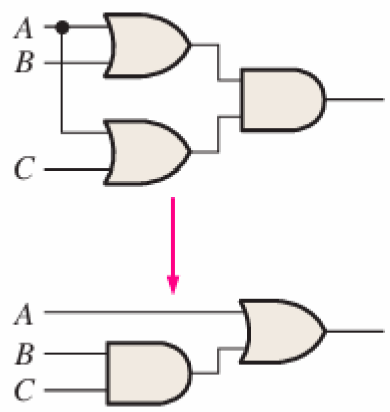
Rule 12: $(A + B)(A + C) = A + BC$ This rule can be proved as follows:

$$
\begin{aligned}
(A + B)(A + C) &= AA + AC + AB + BC && \text{Distributive law} \\
&= A + AC + AB + BC && \text{Rule 7: } AA = A \\
&= A(1 + C) + AB + BC && \text{Factoring (distributive law)} \\
&= A \cdot 1 + AB + BC && \text{Rule 2: } 1 + C = 1 \\
&= A(1 + B) + BC && \text{Factoring (distributive law)} \\
&= A \cdot 1 + BC && \text{Rule 2: } 1 + B = 1 \\
&= A + BC && \text{Rule 4: } A \cdot 1 = A
\end{aligned}
$$

Rule 12: $(A + B)(A + C) = A + BC$. Open file T04-04 to verify.

A	B	C	$A + B$	$A + C$	$(A + B)(A + C)$	BC	$A + BC$
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

equal



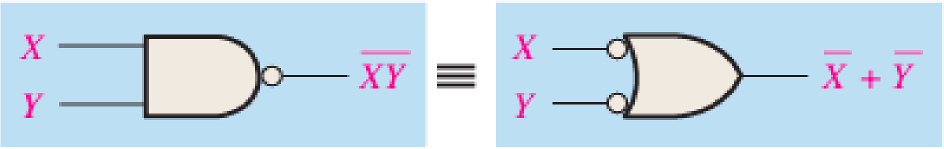
De Morgan's Theorems

The complement of a product of variables is equal to the sum of the complements of the variables.

The complement of two or more ANDed variables is equivalent to the OR of the complements of the individual variables.

$$\overline{XY} = \overline{X} + \overline{Y}$$

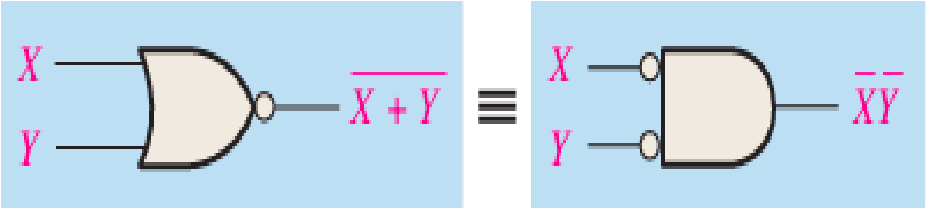
To apply De Morgan's theorem, break the bar over the product of variables and change the sign from AND to OR.



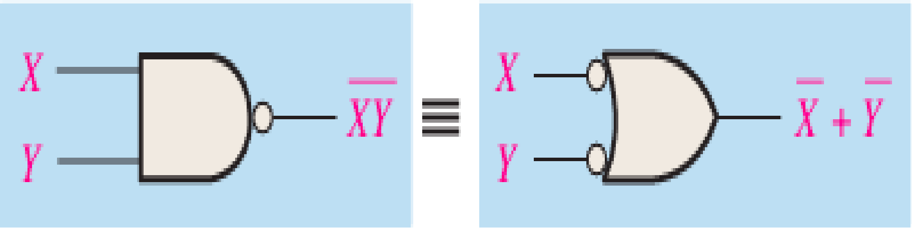
NAND Negative-OR

$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

The complement of a sum of variables is equal to the product of the complements of the variables.

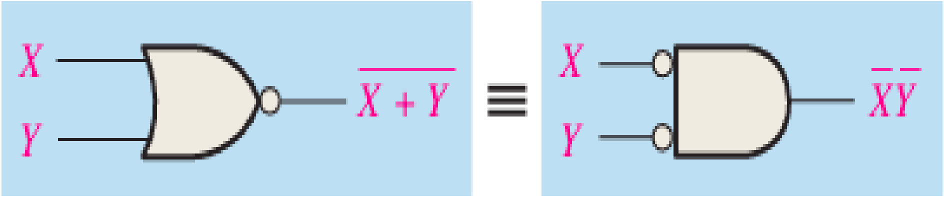


NOR Negative-AND



NAND　　Negative-OR

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



NOR　　Negative-AND

Inputs		Output	
X	Y	$\overline{X + Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Apply DeMorgan's theorems to the expressions $\overline{XYZ}$ and $\overline{X + Y + Z}$.

Solution

$$\overline{XYZ} = \overline{X} + \overline{Y} + \overline{Z}$$
$$\overline{X + Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$$

Related Problem

Apply DeMorgan's theorem to the expression $\overline{\overline{X} + \overline{Y} + \overline{Z}}$.

Apply DeMorgan's theorems to the expressions $\overline{WXYZ}$ and $\overline{W + X + Y + Z}$.

Solution

$$\overline{WXYZ} = \overline{W} + \overline{X} + \overline{Y} + \overline{Z}$$
$$\overline{W + X + Y + Z} = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}$$

Related Problem

Apply DeMorgan's theorem to the expression $\overline{\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}}$.

Apply DeMorgan's theorems to each of the following expressions:

(a) $\overline{(A + B + C)D}$

(b) $\overline{ABC + DEF}$

(c) $\overline{A\overline{B} + \overline{C}D + EF}$

Solution

(a) Let $A + B + C = X$ and $D = Y$. The expression $\overline{(A + B + C)D}$ is of the form $\overline{XY} = \overline{X} + \overline{Y}$ and can be rewritten as

$$\overline{(A + B + C)D} = \overline{A + B + C} + \overline{D}$$

Next, apply DeMorgan's theorem to the term $\overline{A + B + C}$.

$$\overline{A + B + C} + \overline{D} = \overline{A}\,\overline{B}\,\overline{C} + \overline{D}$$

(b) Let $ABC = X$ and $DEF = Y$. The expression $\overline{ABC + DEF}$ is of the form $\overline{X + Y} = \overline{X}\,\overline{Y}$ and can be rewritten as

$$\overline{ABC + DEF} = (\overline{ABC})(\overline{DEF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{ABC}$ and $\overline{DEF}$.

$$(\overline{ABC})(\overline{DEF}) = (\overline{A} + \overline{B} + \overline{C})(\overline{D} + \overline{E} + \overline{F})$$

(c) Let $A\overline{B} = X$, $\overline{C}D = Y$, and $EF = Z$. The expression $\overline{A\overline{B} + \overline{C}D + EF}$ is of the form $\overline{X + Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$ and can be rewritten as

$$\overline{A\overline{B} + \overline{C}D + EF} = (\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{A\overline{B}}$, $\overline{\overline{C}D}$, and $\overline{EF}$.

$$(\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF}) = (\overline{A} + B)(C + \overline{D})(\overline{E} + \overline{F})$$

Apply DeMorgan's theorems to each expression:

(a) $\overline{(\overline{A + B}) + \overline{C}}$

(b) $\overline{(\overline{A} + B) + CD}$

(c) $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}}$

Solution

(a) $\overline{(\overline{A + B}) + \overline{C}} = (\overline{\overline{A + B}})\overline{\overline{C}} = (A + B)C$

(b) $\overline{(\overline{A} + B) + CD} = (\overline{\overline{A} + B})\overline{CD} = (\overline{\overline{A}}\,\overline{B})(\overline{C} + \overline{D}) = A\overline{B}(\overline{C} + \overline{D})$

(c) $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}} = ((\overline{(A + B)\overline{C}\,\overline{D}})(\overline{E + \overline{F}}) = (\overline{A}\,\overline{B} + C + D)\overline{E}F$

Using Boolean algebra techniques, simplify this expression:

$$AB + A(B + C) + B(B + C)$$

Solution

The following is not necessarily the only approach.

Step 1: Apply the distributive law to the second and third terms in the expression, as follows:

$$AB + AB + AC + BB -$$

Step 2: Apply rule 7 ($BB = B$) to the fourth term.

$$AB + AB + AC + B +$$

Step 3: Apply rule 5 ($AB + AB = AB$) to the first twc

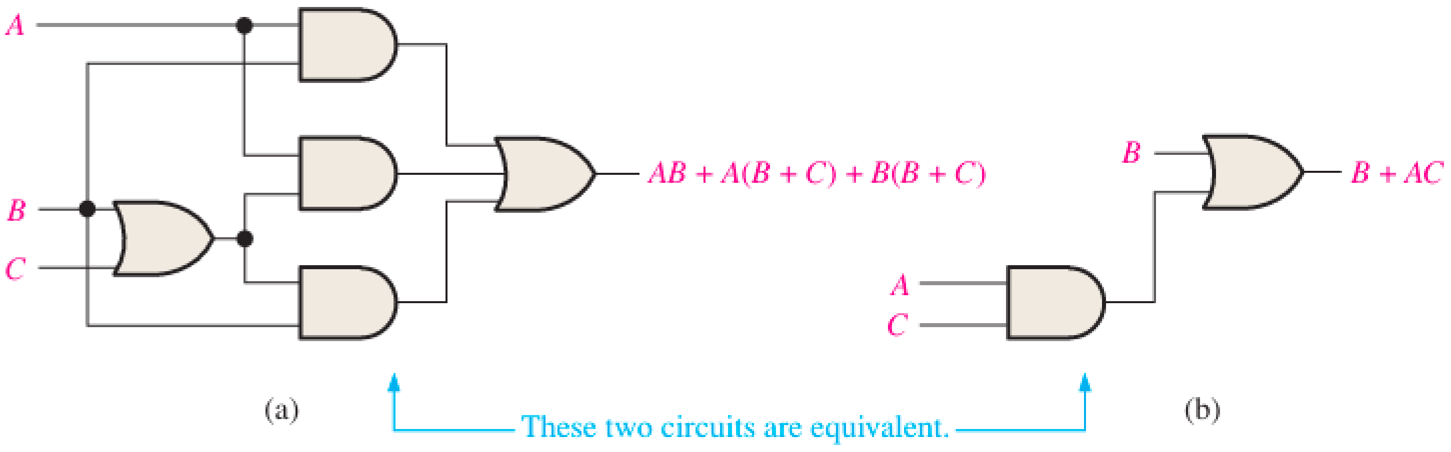
$$AB + AC + B + B($$

Step 4: Apply rule 10 ($B + BC = B$) to the last two t

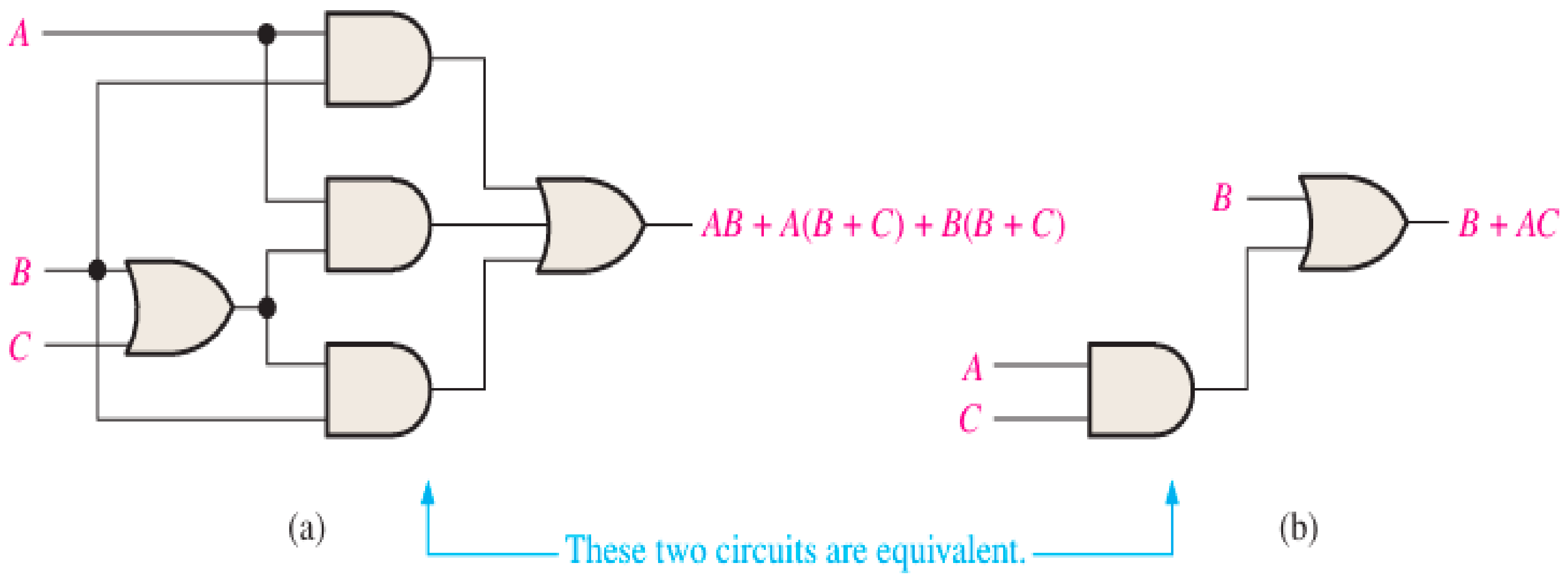
$$AB + AC + B$$

Step 5: Apply rule 10 ($AB + B = B$) to the first and third terms.

$$B + AC$$



At this point the expression is simplified as much as possible. Once you gain experience in applying Boolean algebra, you can often combine many individual steps.


A
B
C
AB + A(B + C) + B(B + C)
(a)
These two circuits are equivalent.
B
A
C
B + AC
(b)

EXAMPLE

$$[A\overline{B}(C + BD) + \overline{A}\,\overline{B}]C$$

$$(A\overline{B}C + A\overline{B}BD + \overline{A}\,\overline{B})C$$

$$(\overline{B}B = 0)$$

$$(A\overline{B}C + A \cdot 0 \cdot D + \overline{A}\,\overline{B})C$$

$$(A \cdot 0 \cdot D = 0)$$

$$(A\overline{B}C + 0 + \overline{A}\,\overline{B})C$$

$$(A\overline{B}C + \overline{A}\,\overline{B})C$$

Apply the distributive law.

$$A\overline{B}CC + \overline{A}\,\overline{B}C$$

$$(CC = C)$$

$$A\overline{B}C + \overline{A}\,\overline{B}C$$

$$\overline{B}C(A + \overline{A})$$

$$(A + \overline{A} = 1).$$

$$\overline{B}C \cdot 1$$

$$\overline{B}C$$

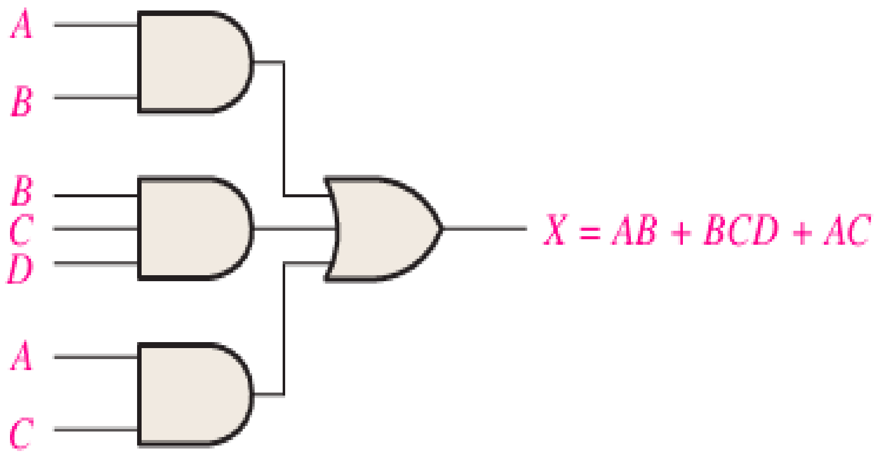
$$[A\overline{B}(C + BD) + \overline{A}\,\overline{B}]C = \overline{B}C$$

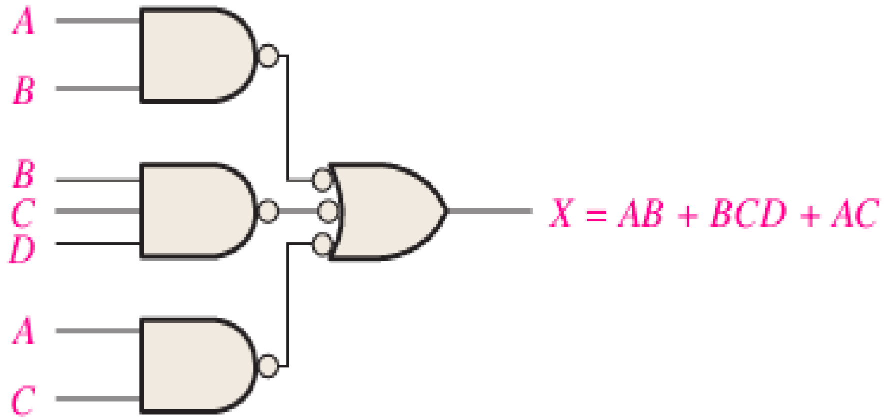
The Sum-of-Products (SOP) Form

$$AB + ABC$$

$$ABC + CDE + \overline{B}C\overline{D}$$

$$\overline{A}B + \overline{A}B\overline{C} + AC$$





Convert each of the following Boolean expressions to SOP form:

(a) $AB + B(CD + EF)$ (b) $(A + B)(B + C + D)$ (c) $\overline{\overline{(A + B)} + C}$

Solution

(a) $AB + B(CD + EF) = AB + BCD + BEF$

(b) $(A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$

(c) $\overline{\overline{(A + B)} + C} = \overline{\overline{(A + B)}}\,\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$

Binary Representation of a Standard Product Term

A standard product term is equal to 1 for only one combination of variable values. For example, the product term $A\overline{B}C\overline{D}$ is equal to 1 when $A = 1$, $B = 0$, $C = 1$, $D = 0$, as shown below, and is 0 for all other combinations of values for the variables.

$$A\overline{B}C\overline{D} = 1 \cdot \overline{0} \cdot 1 \cdot \overline{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

In this case, the product term has a binary value of 1010 (decimal ten).

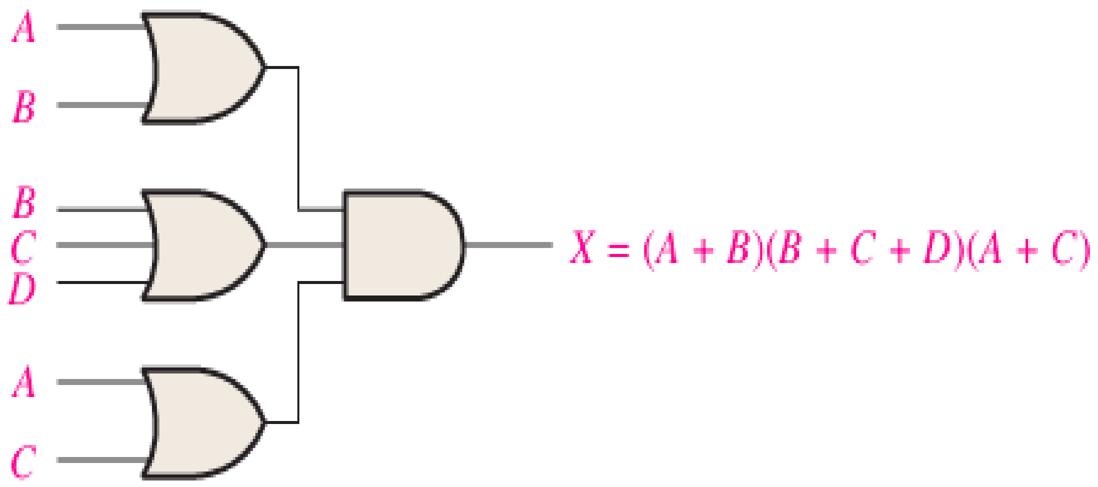
Remember, a product term is implemented with an AND gate whose output is 1 only if each of its inputs is 1. Inverters are used to produce the complements of the variables as required.

An SOP expression is equal to 1 only if one or more of the product terms in the expression is equal to 1.

The Product-of-Sums (POS) Form

$$(\overline{A} + B)(A + \overline{B} + C)$$
$$(\overline{A} + \overline{B} + \overline{C})(C + \overline{D} + E)(\overline{B} + C + D)$$
$$(A + B)(A + \overline{B} + C)(\overline{A} + C)$$

Implementation of a POS Expression



A POS expression is equal to 0 only if one or more of the sum terms in the expression is equal to 0.

Binary Representation of a Standard Sum Term

$$A + \overline{B} + C + \overline{D} \text{ is } 0$$

when $A = 0$, $B = 1$, $C = 0$, and $D = 1$

$$A + \overline{B} + C + \overline{D} = 0 + \overline{1} + 0 + \overline{1} = 0 + 0 + 0 + 0 = 0$$

Determine the binary values of the variables for which the following standard POS expression is equal to 0:

$$(A + B + C + D)(A + \overline{B} + \overline{C} + D)(\overline{A} + \overline{B} + \overline{C} + \overline{D})$$

Solution

The term $A + B + C + D$ is equal to 0 when $A = 0$, $B = 0$, $C = 0$, and $D = 0$.

$$A + B + C + D = 0 + 0 + 0 + 0 = 0$$

The term $A + \overline{B} + \overline{C} + D$ is equal to 0 when $A = 0$, $B = 1$, $C = 1$, and $D = 0$.

$$A + \overline{B} + \overline{C} + D = 0 + \overline{1} + \overline{1} + 0 = 0 + 0 + 0 + 0 = 0$$

The term $\overline{A} + \overline{B} + \overline{C} + \overline{D}$ is equal to 0 when $A = 1$, $B = 1$, $C = 1$, and $D = 1$.

$$\overline{A} + \overline{B} + \overline{C} + \overline{D} = \overline{1} + \overline{1} + \overline{1} + \overline{1} = 0 + 0 + 0 + 0 = 0$$

The POS expression equals 0 when any of the three sum terms equals 0.

Boolean Expressions and Truth Tables

Develop a truth table for the standard SOP expression $\overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC$.

Inputs			Output	
A	***B***	***C***	***X***	**Product Term**
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

A	B	C	A’	B’	C’	A’B’C	AB’C’	ABC	A’B’C+AB’C’+ABC
0	0	0	1	1	1	0	0	0	0
0	0	1	1	1	0	1	0	0	1
0	1	0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	0	1	0	1
1	0	1	0	1	0	0	0	0	0
1	1	0	0	0	1	0	0	0	0
1	1	1	0	0	0	0	0	1	1

Determine the truth table for the following standard POS expression:

$$(A + B + C)(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + C)$$

Inputs			Output	
A	***B***	***C***	***X***	**Sum Term**
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \overline{B} + C)$
0	1	1	0	$(A + \overline{B} + \overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A} + B + \overline{C})$
1	1	0	0	$(\overline{A} + \overline{B} + C)$
1	1	1	1	

Determining Standard Expressions from a Truth Table

From the truth table determine the standard SOP expression and the equivalent standard POS expression.

Inputs			Output
A	*B*	*C*	*X*
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

There are four 1s in the output.

$011 \longrightarrow \overline{A}BC$
$100 \longrightarrow A\overline{B}\,\overline{C}$
$110 \longrightarrow AB\overline{C}$
$111 \longrightarrow ABC$

The resulting standard SOP expression for the output X is

$$X = \overline{A}BC + A\overline{B}\,\overline{C} + AB\overline{C} + ABC$$

For the POS expression, the output is 0.

$000 \longrightarrow A + B + C$
$001 \longrightarrow A + B + \overline{C}$
$010 \longrightarrow A + \overline{B} + C$
$101 \longrightarrow \overline{A} + B + \overline{C}$

$$X = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(\overline{A} + B + \overline{C})$$