Frequency Response

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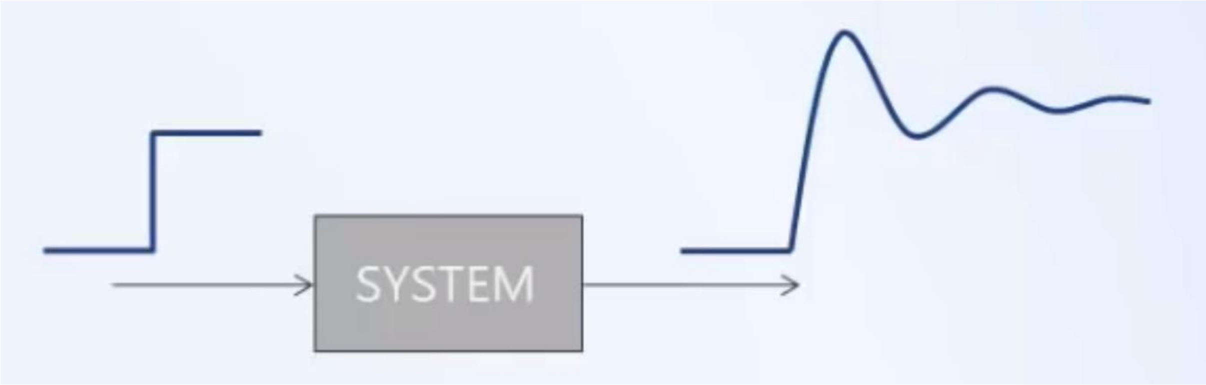
Most slides from Signals and Systems (MIT 6.003) by Prof. Denny Freeman

Time Response

- Previously, we have determined the time response of linear systems to arbitrary inputs and initial conditions

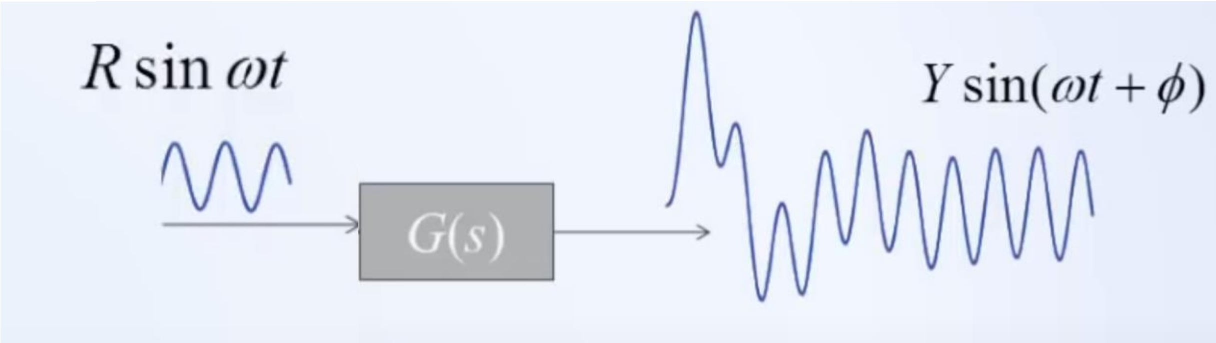
- We have also studied the character of certain standard systems to certain simple inputs



Frequency Response

- Only focus on steady-state solution
- Transient solution is not our interest any more

- Input sine waves of different frequencies and look at the output in steady state
- If $G(s)$ is linear and stable, a sinusoidal input will generate in steady state a scaled and shifted sinusoidal output of the same frequency



Response to a Sinusoidal Input

- When the input $x(t) = e^{j\omega t}$ to an LTI system

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ h(t)\end{array}} \longrightarrow y(t)$$

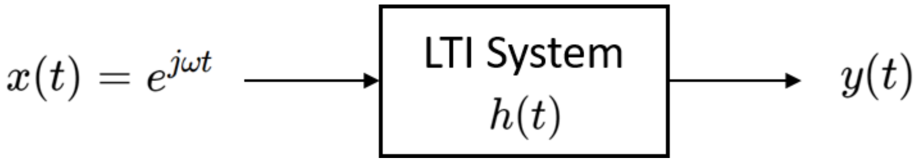
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}_{\text{complex function of } \omega}$$

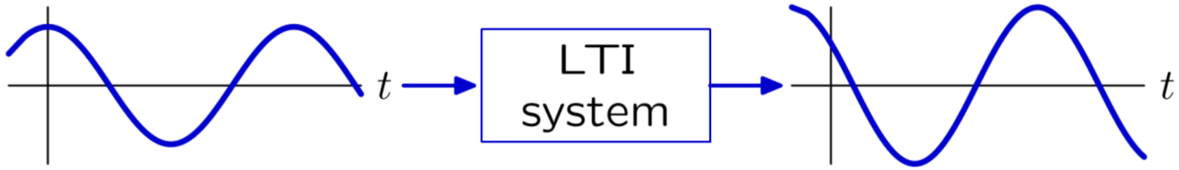
$$= e^{j\omega t} H(j\omega)$$

- Output is also a sinusoid
  - same frequency
  - possibly different amplitude, and
  - possibly different phase angle

Response to a Sinusoidal Input

- When the input $x(t) = e^{j\omega t}$ to an LTI system



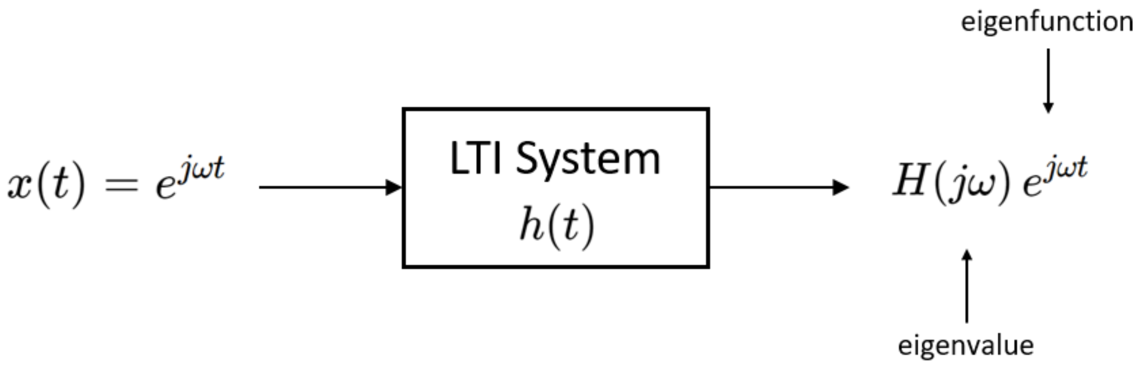


- Output is also a sinusoid
  – same frequency
  – possibly different amplitude, and
  – possibly different phase angle

Fourier Transform

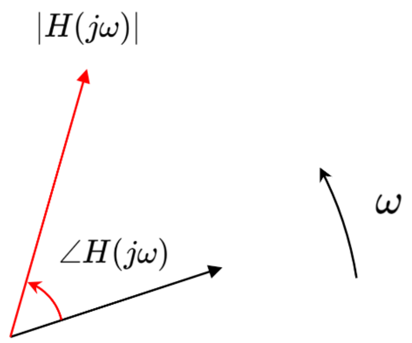
- Definition: Fourier transform

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$



- $H(j\omega)e^{j\omega t}$ rotates with the same angular velocity ω

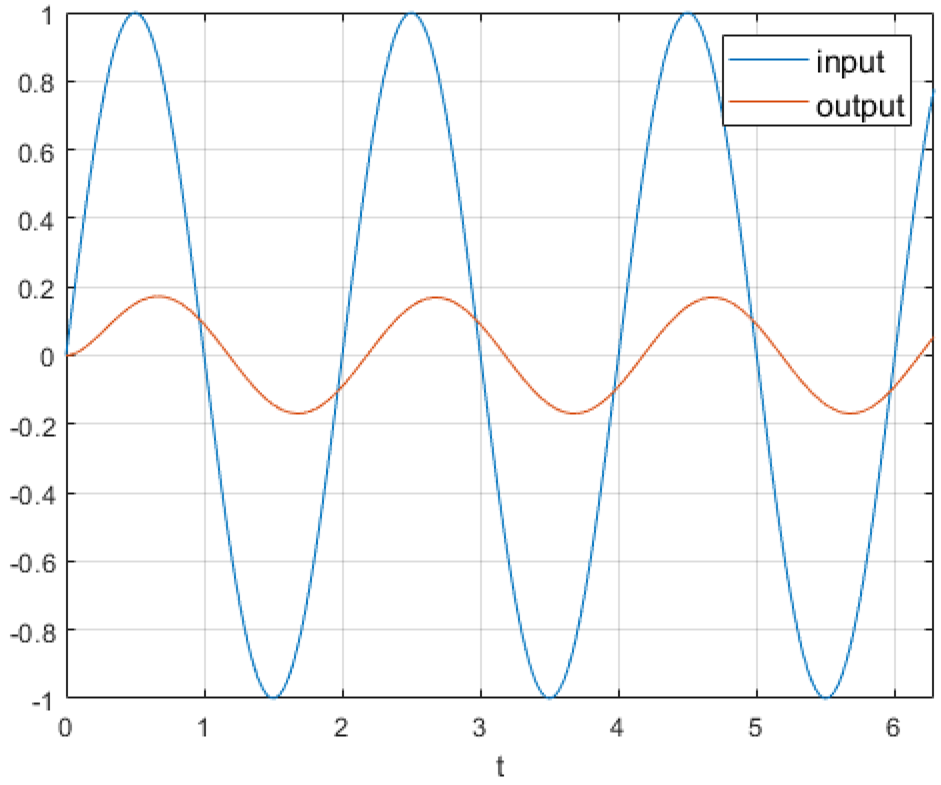
$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$



Response to a Sinusoidal Input: MATLAB

$$\dot{y} + 5y = x(t)$$

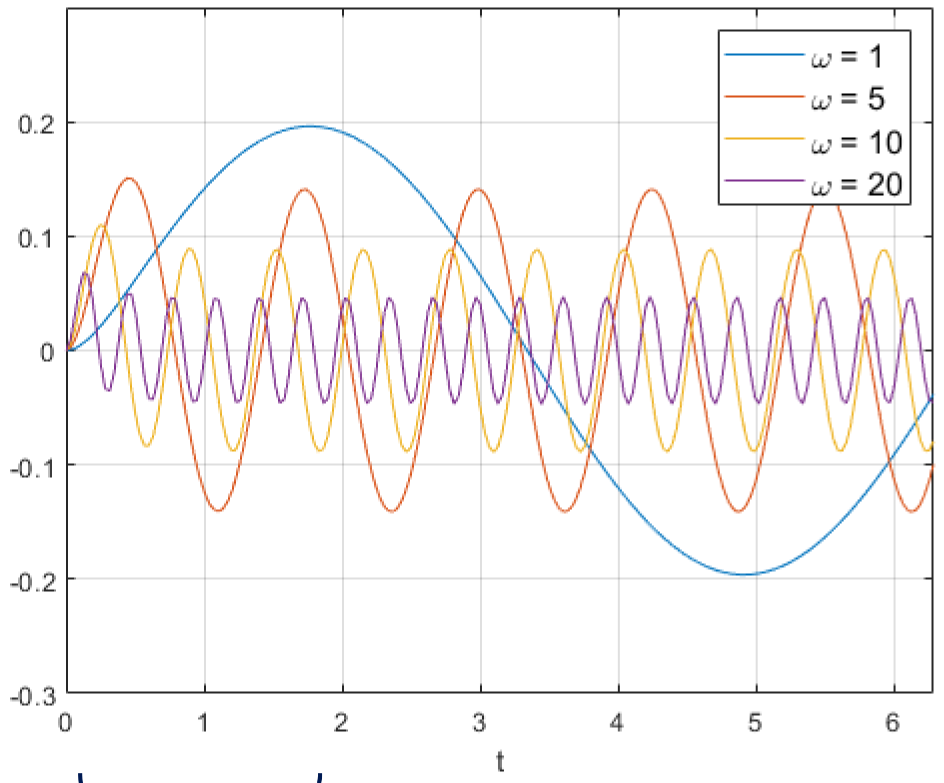
```
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

w = pi;
t = linspace(0,2*pi,200);

x0 = 0;

x = sin(w*t);

[y,tout] = lsim(G,x,t,x0);
```



Response to a Sinusoidal Input: MATLAB

$$\dot{y} + 5y = x(t)$$

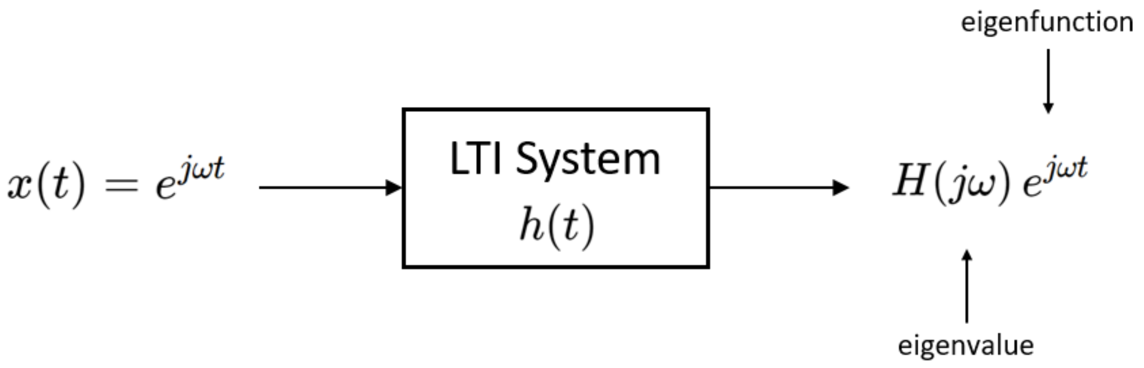
```
W = [1,5,10,20];
for w = W
    x = sin(w*t);
    [y,tout] = lsim(G,x,t,x0);
    plot(tout,y), hold on
end
```



transient

Frequency Response to a Sinusoidal Input

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$



- Two primary quantities of interest that have implications for system performance are:
  - The scaling = magnitude of $H(j\omega)$

$$\left| \frac{Y(s)}{X(s)} \right|_{s=j\omega} = |H(j\omega)|$$

  - The phase shift = angle of $H(j\omega)$

$$\phi = \angle H(j\omega)$$

Frequency Response to a Sinusoidal Input: MATLAB

- Given input $e^{j\omega t}$

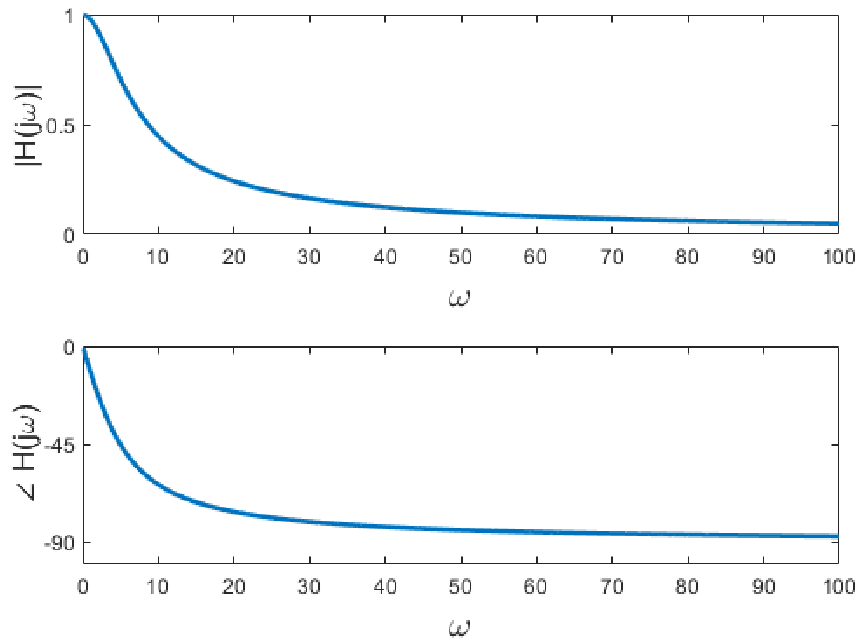
$$\dot{y} + 5y = 5e^{j\omega t}$$

- $y = Ae^{j(\omega t+\phi)}$

$$j\omega Ae^{j(\omega t+\phi)} + 5Ae^{j(\omega t+\phi)} = 5e^{j\omega t}$$
$$(j\omega + 5)\, Ae^{j\phi} = 5$$

$$|H(j\omega)| = A = \frac{5}{|j\omega + 5|}$$
$$\angle H(j\omega) = \phi = -\angle(j\omega + 5)$$

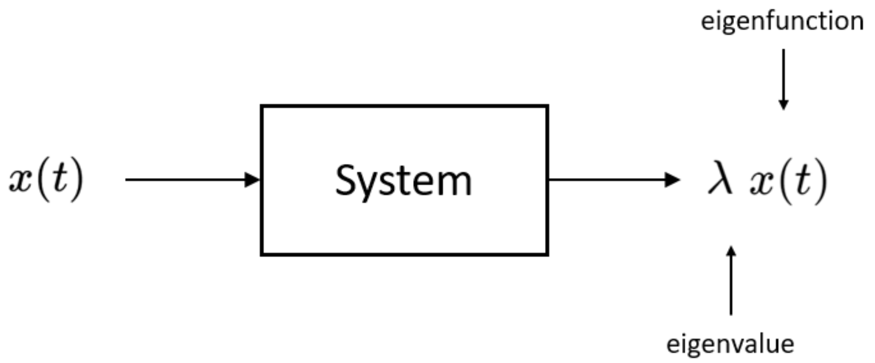
```
w = 0.1:0.1:100;

A = 5./abs(1j*w+5);
P = -angle(1j*w+5)*180/pi;
```



From Laplace Transform to Fourier Transform

Eigenfunctions and Eigenvalues

- Eigenfunctions
  - If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue

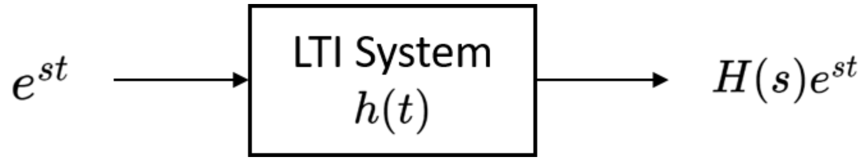


Eigenfunctions and Eigenvalues

- Fact: Complex exponentials are eigenfunctions of LTI systems.

- If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) e^{st}
\end{aligned}
$$



$$
H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau
$$

- The eigenvalue associated with eigenfunction e^{st} is $H(s)$

Rational Transfer Functions

- Eigenvalues are particularly easy to evaluate for systems represented by linear differential equations with constant coefficients.

- Then the transfer function is a ratio of polynomials in s

- Example

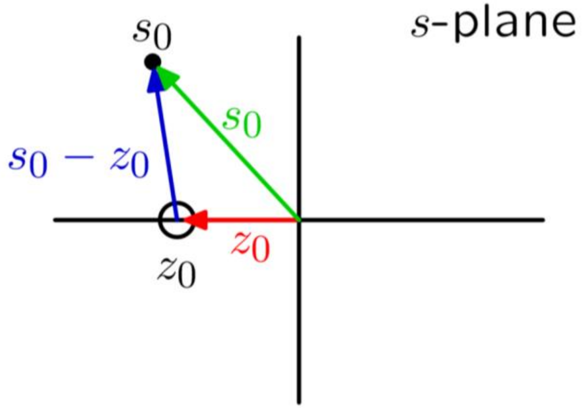
$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4}$$

Vector Diagrams

- The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.
- Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



- Each factor in the numerator/denominator corresponds to a vector from a zero/pole to s_0, the point of interest in the s-plane

Vector Diagrams

- The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributors of the vectors associated with each of the poles and zeros

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

  - The magnitude is determined by the product of the magnitudes

$$|H(s_0)| = |K|\frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

  - The angle is determined by the sum of the angles

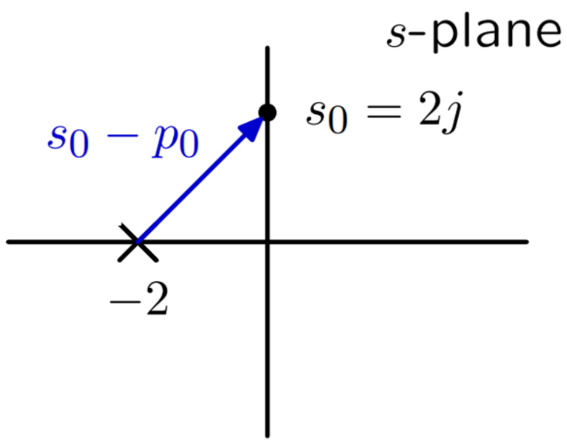
$$\begin{aligned}\angle H(s_0) = \angle K &+ \angle(s_0 - z_0) + \angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots \\ &- \angle(s_0 - p_0) - \angle(s_0 - p_1) - \angle(s_0 - p_2) - \cdots\end{aligned}$$

Frequency Response

- Given the system described by

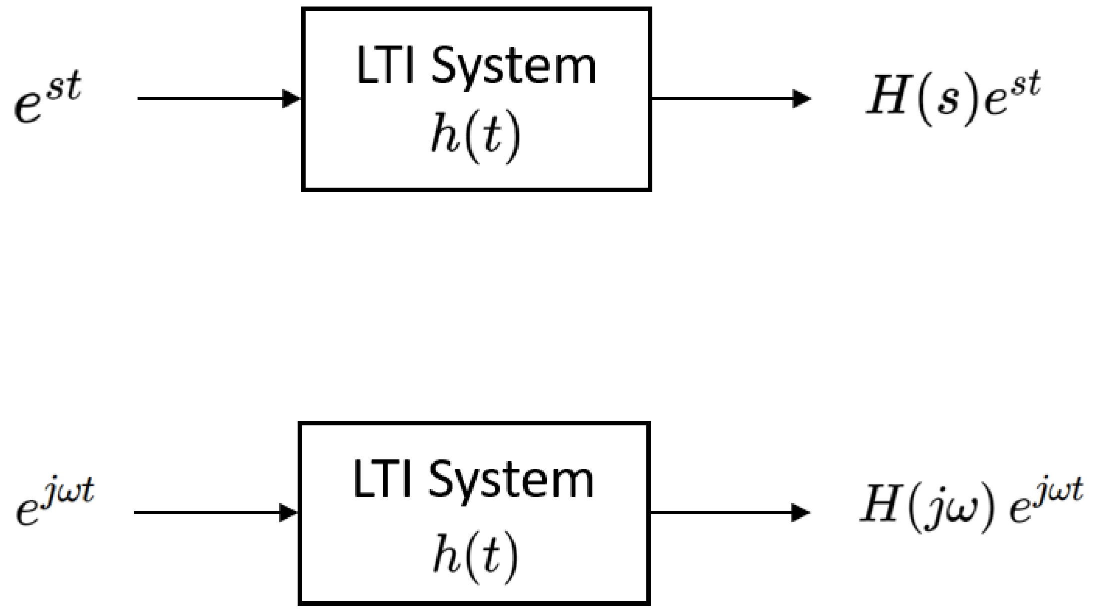
$$H(s) = \frac{1}{s+2}$$

- Find the response to the input $x(t) = e^{2jt}$



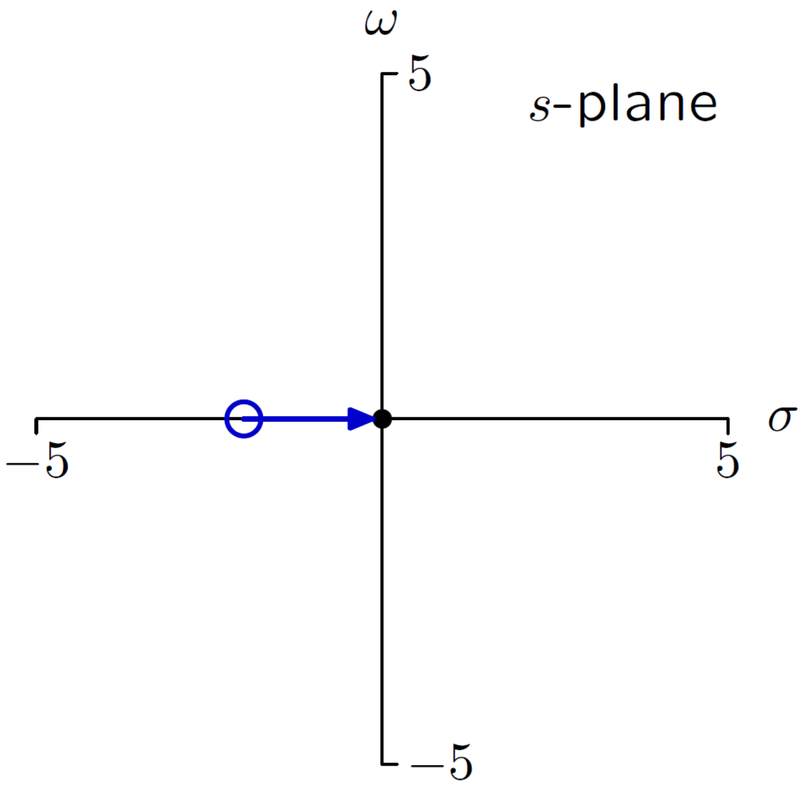
Vector Diagrams for Frequency Response

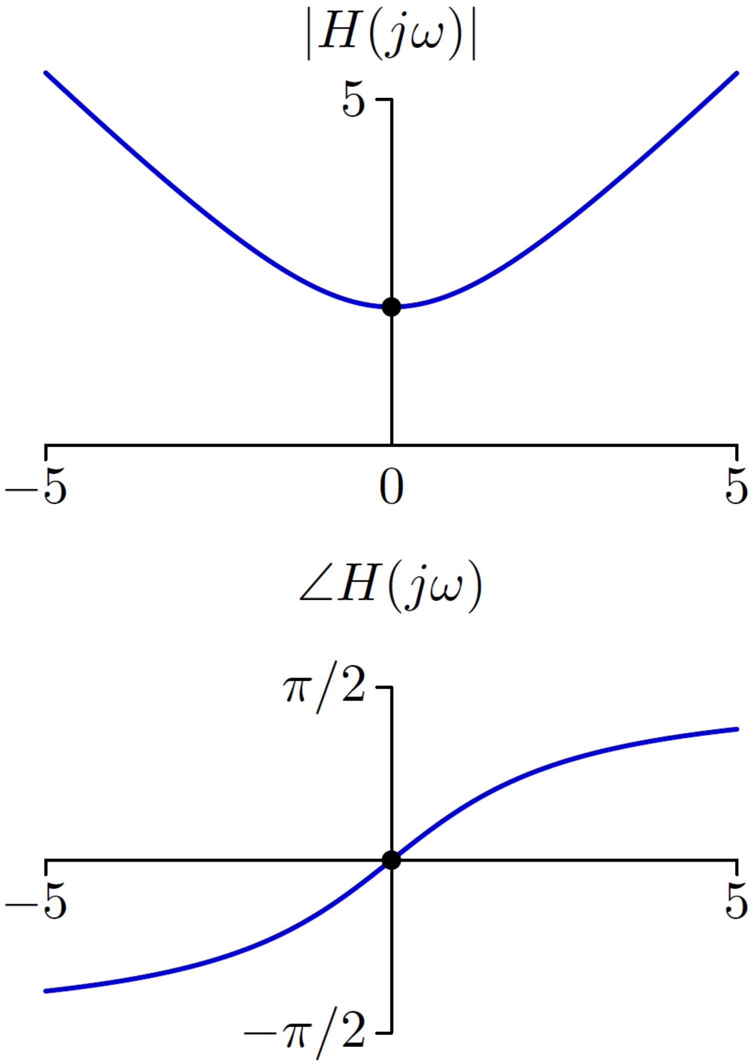
- The magnitude and phase of the response of an LTI system to $e^{j\omega t}$ is the magnitude and phase of $H(s)$ at $s = j\omega$



Vector Diagrams at $s = j\omega$

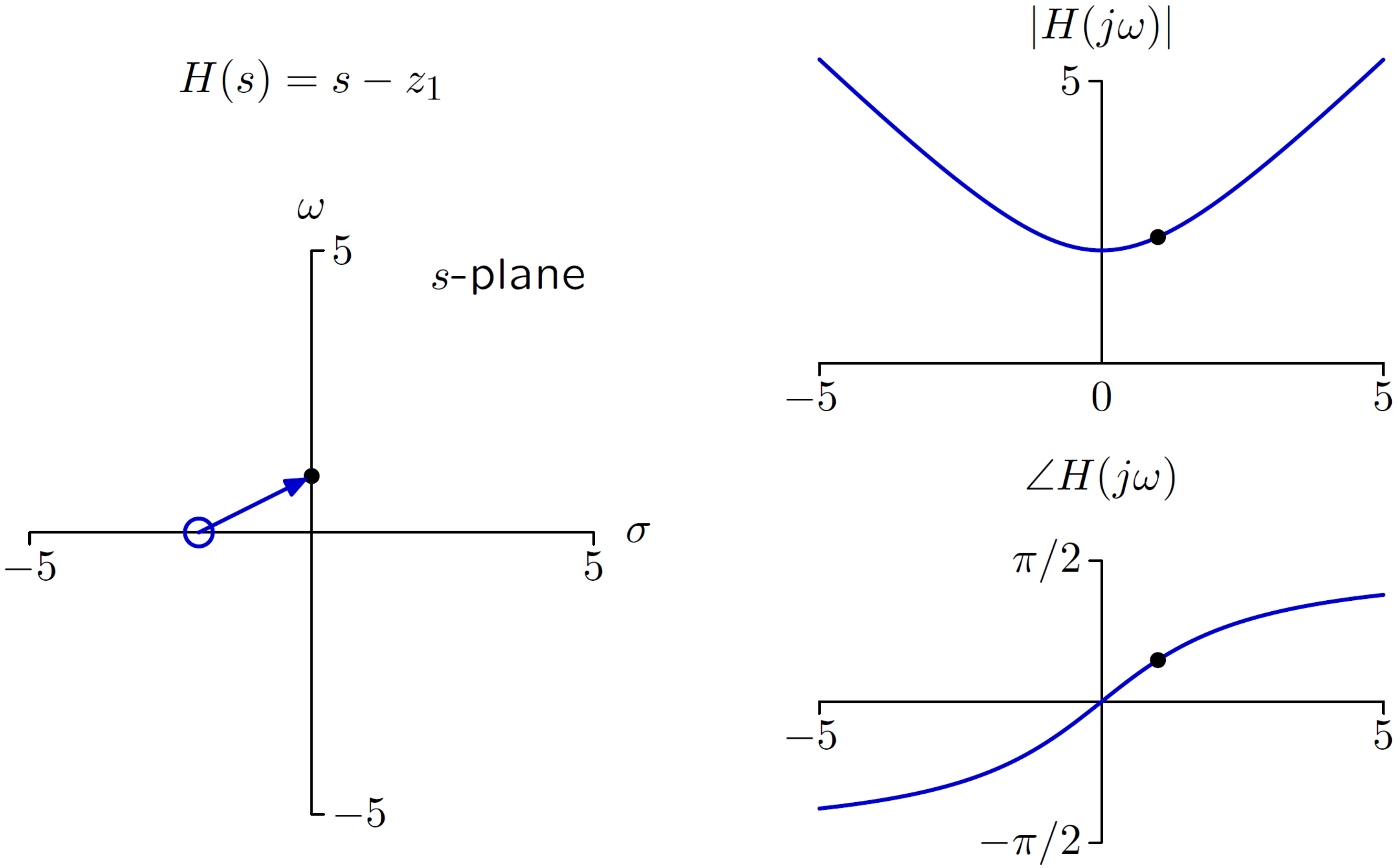
$$H(s) = s - z_1$$





From Signals and Systems (MIT 6.003) by Prof. Denny Freeman

Vector Diagrams at $s = j\omega$



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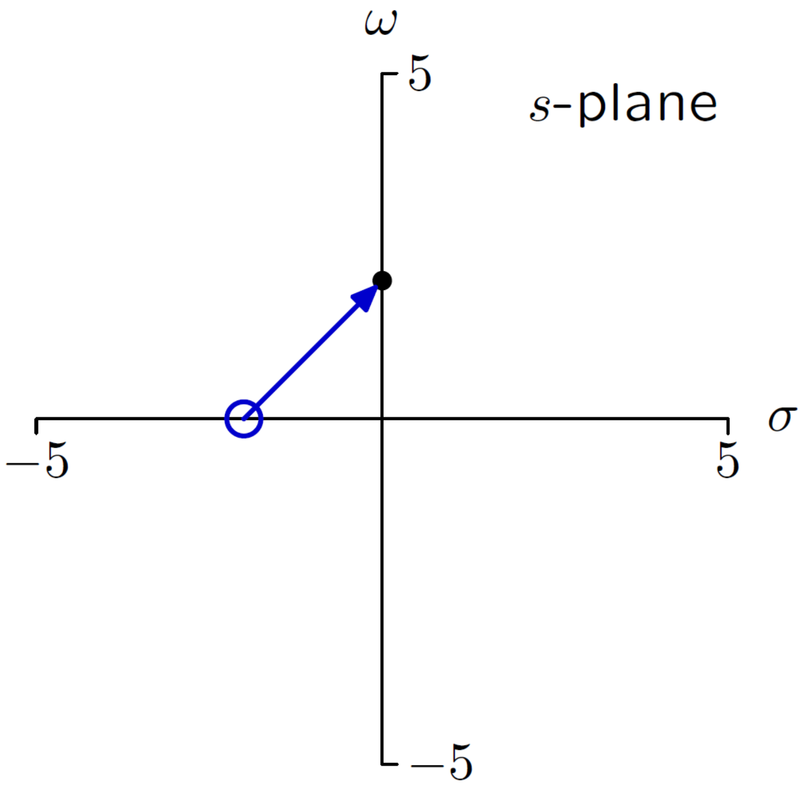

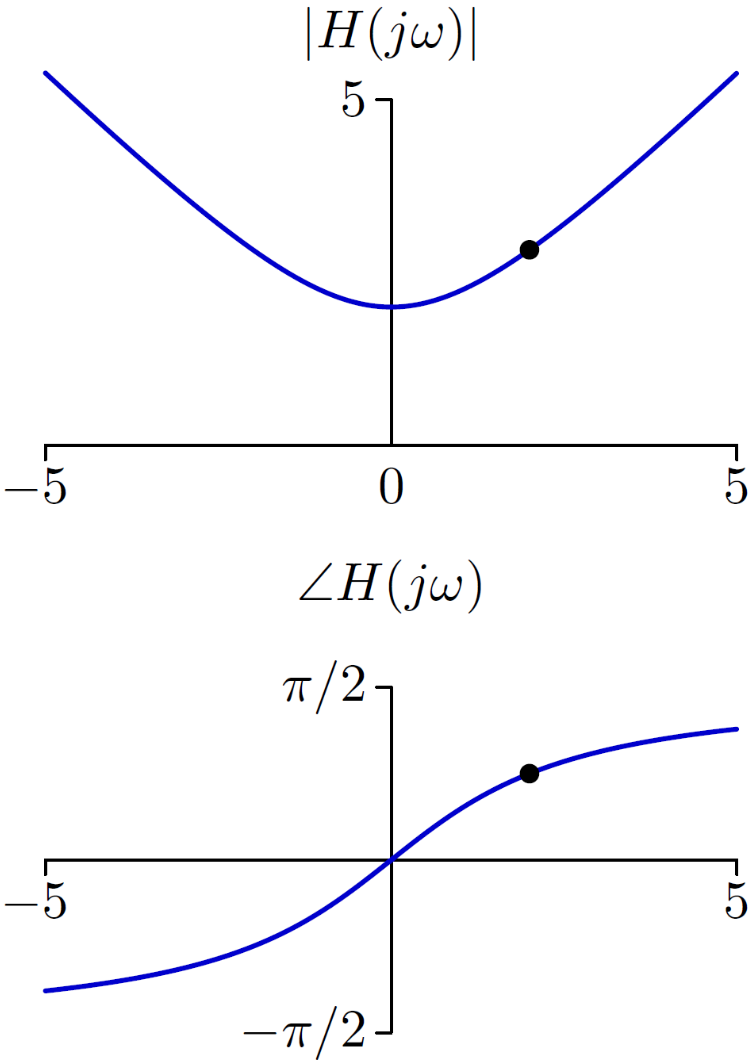

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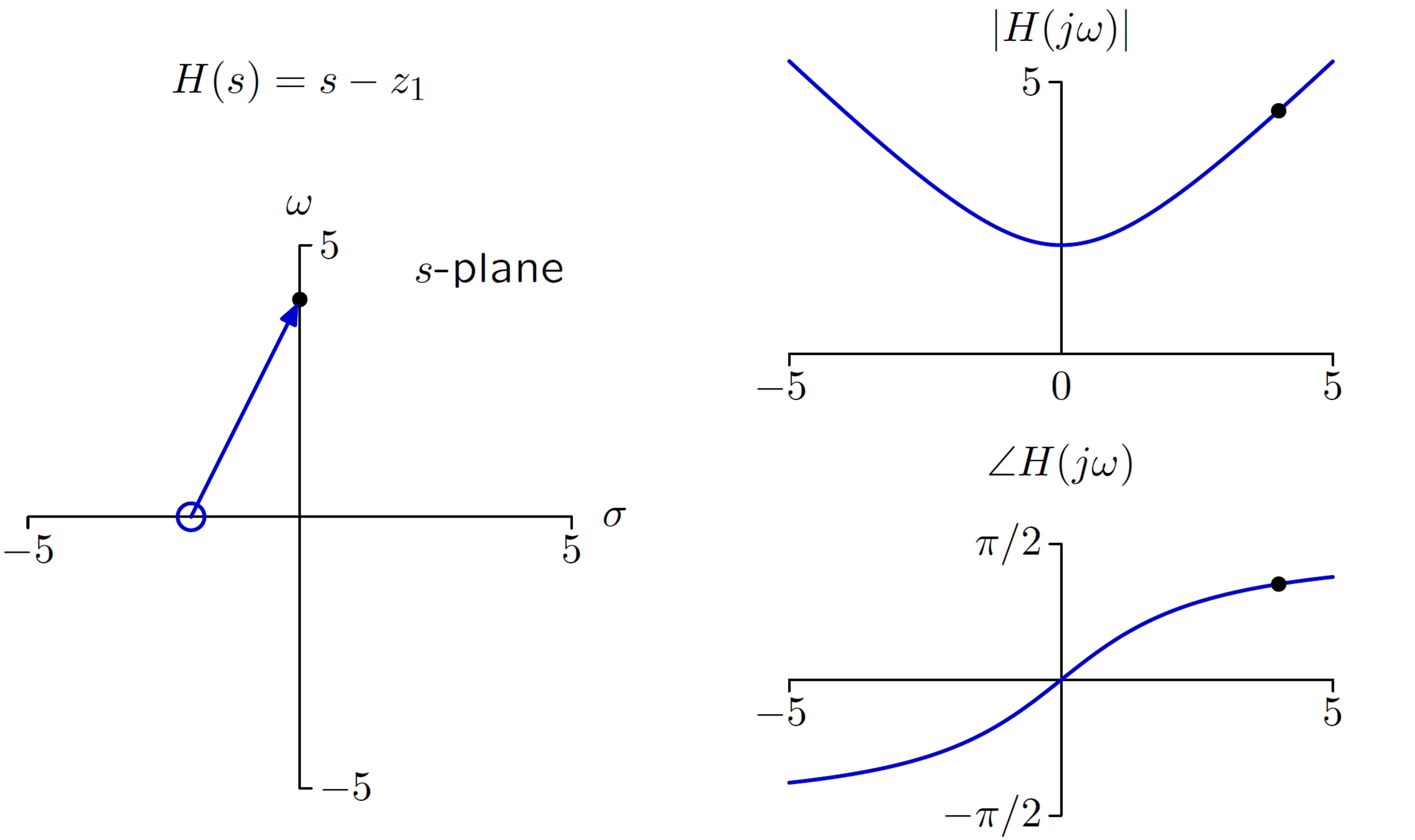

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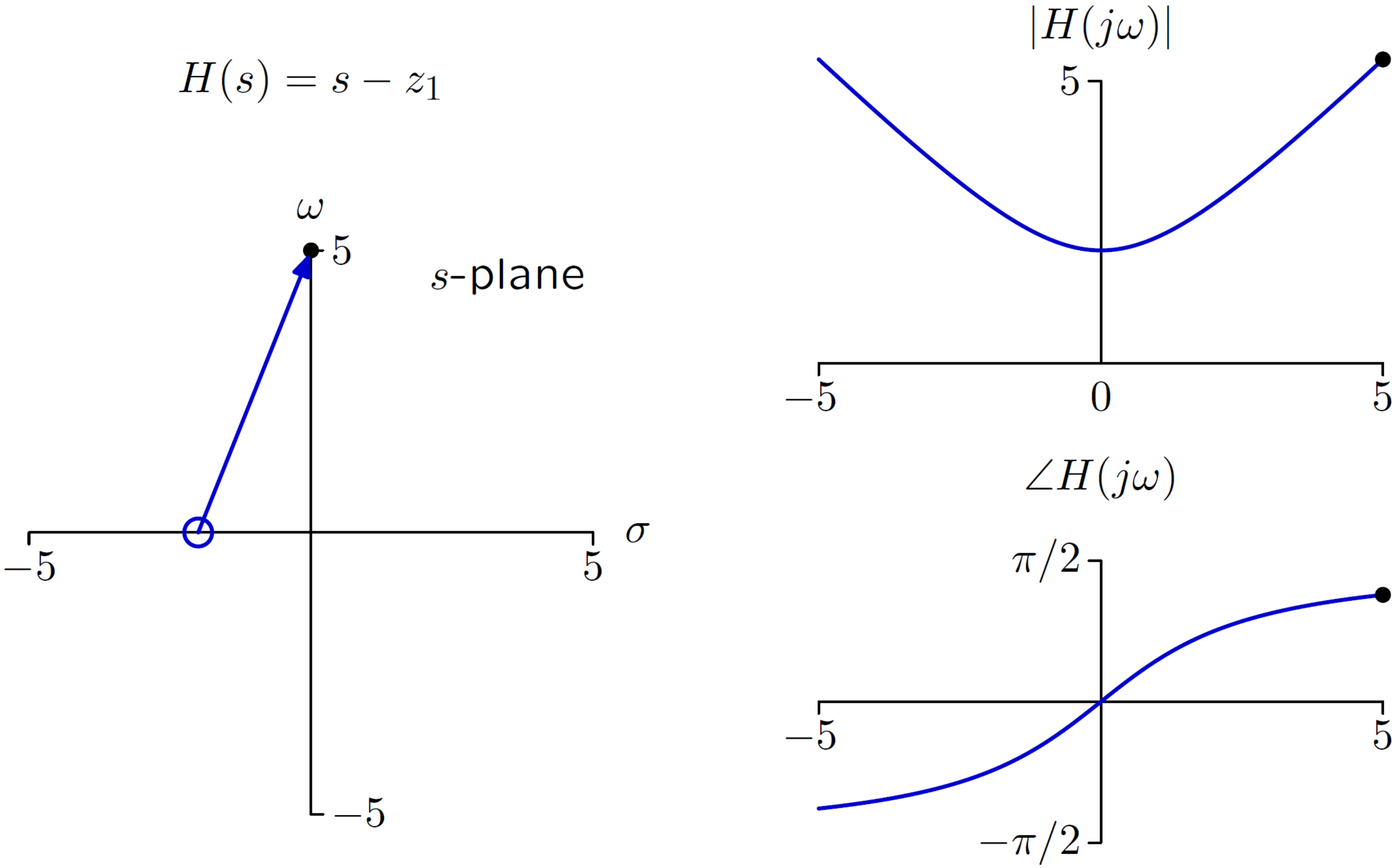

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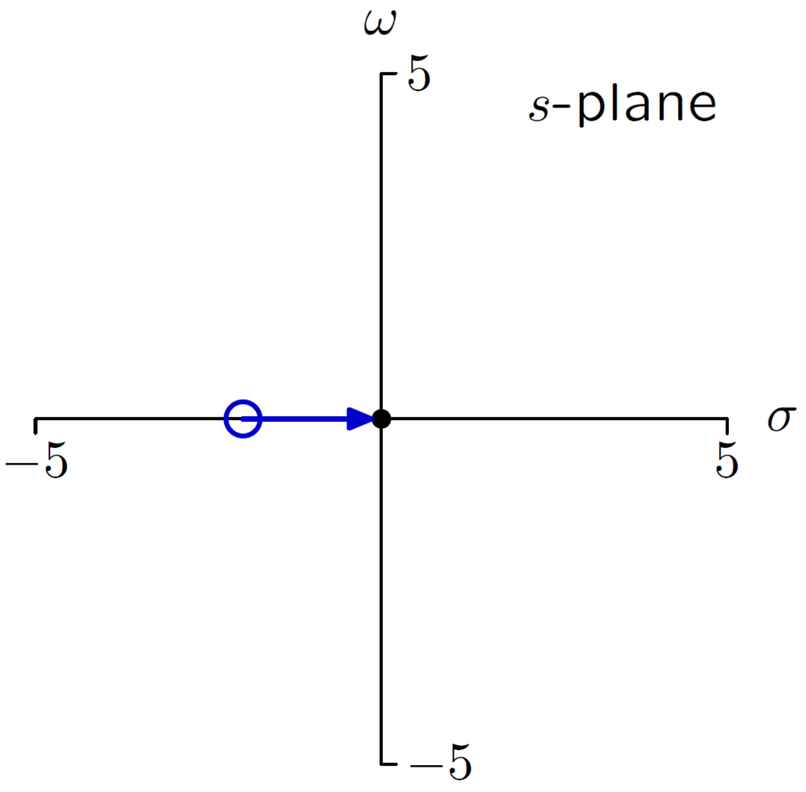

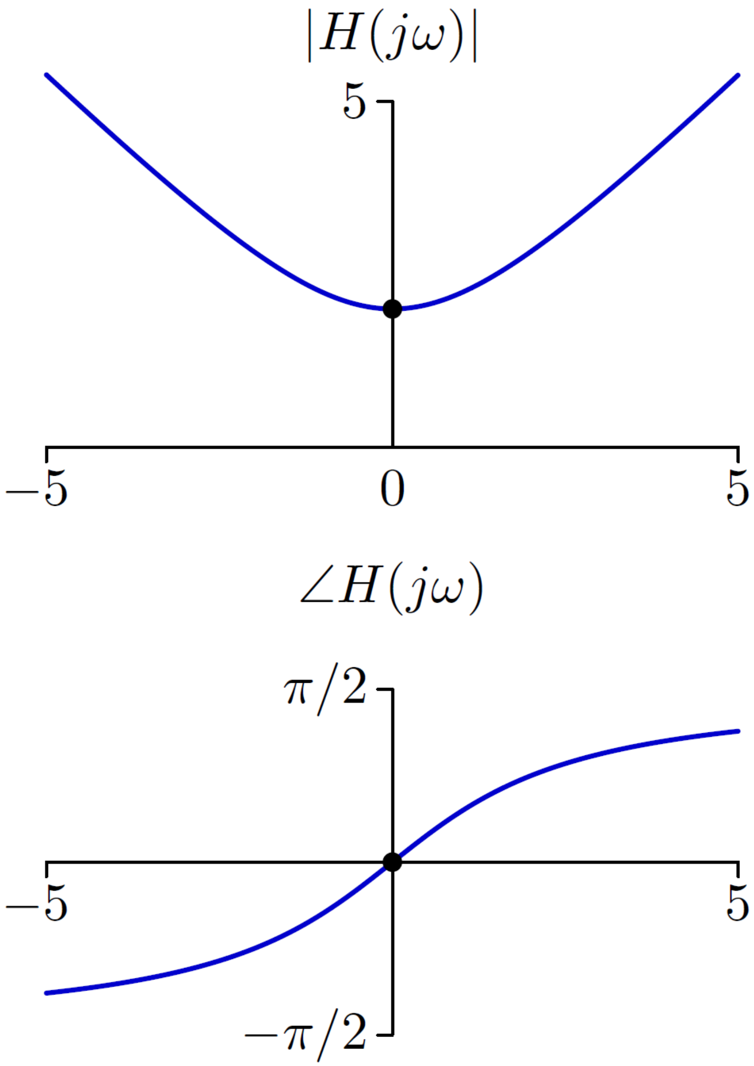

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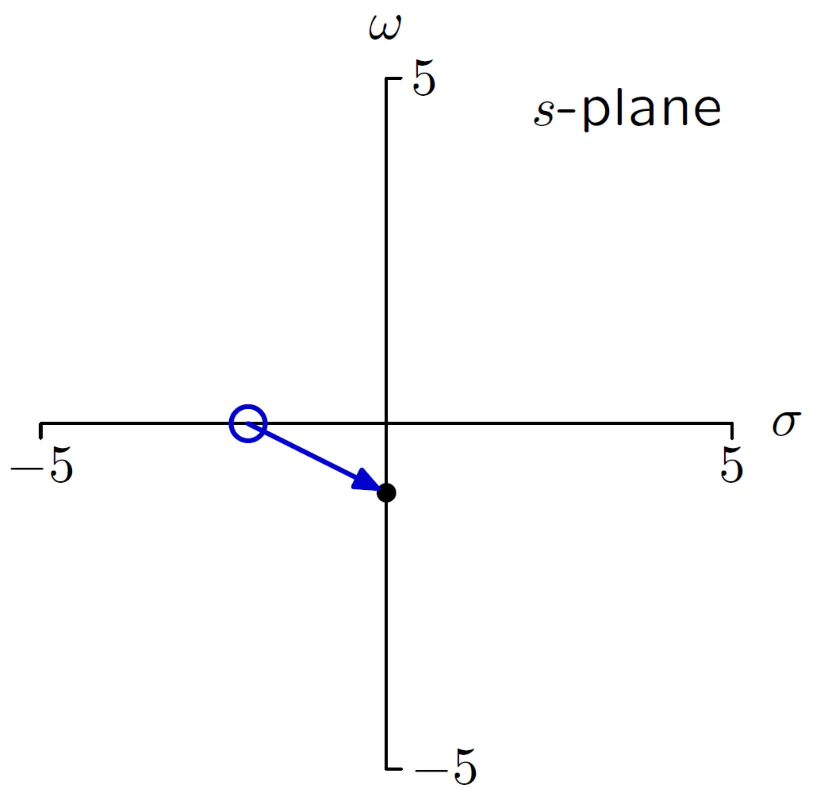

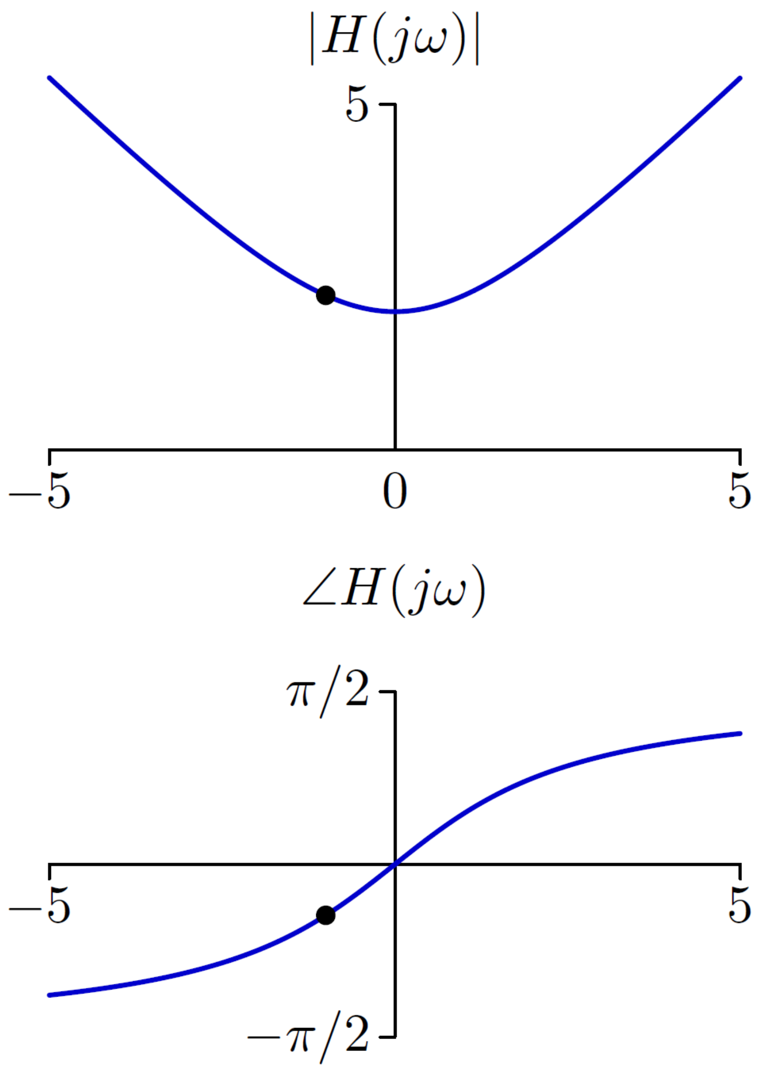

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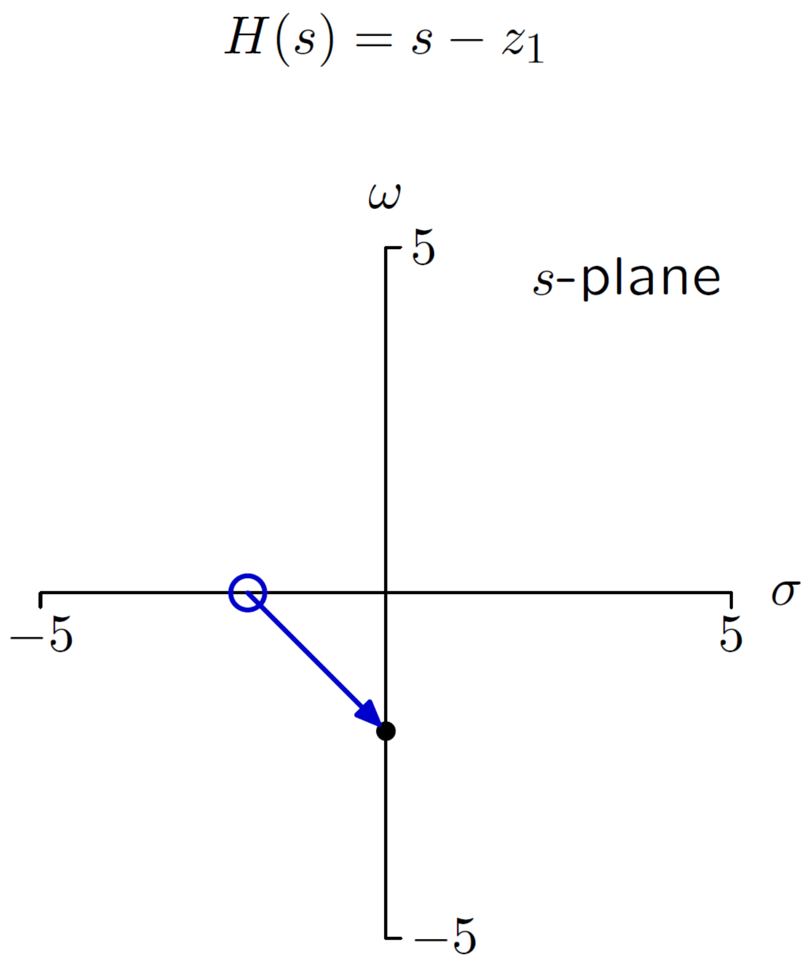

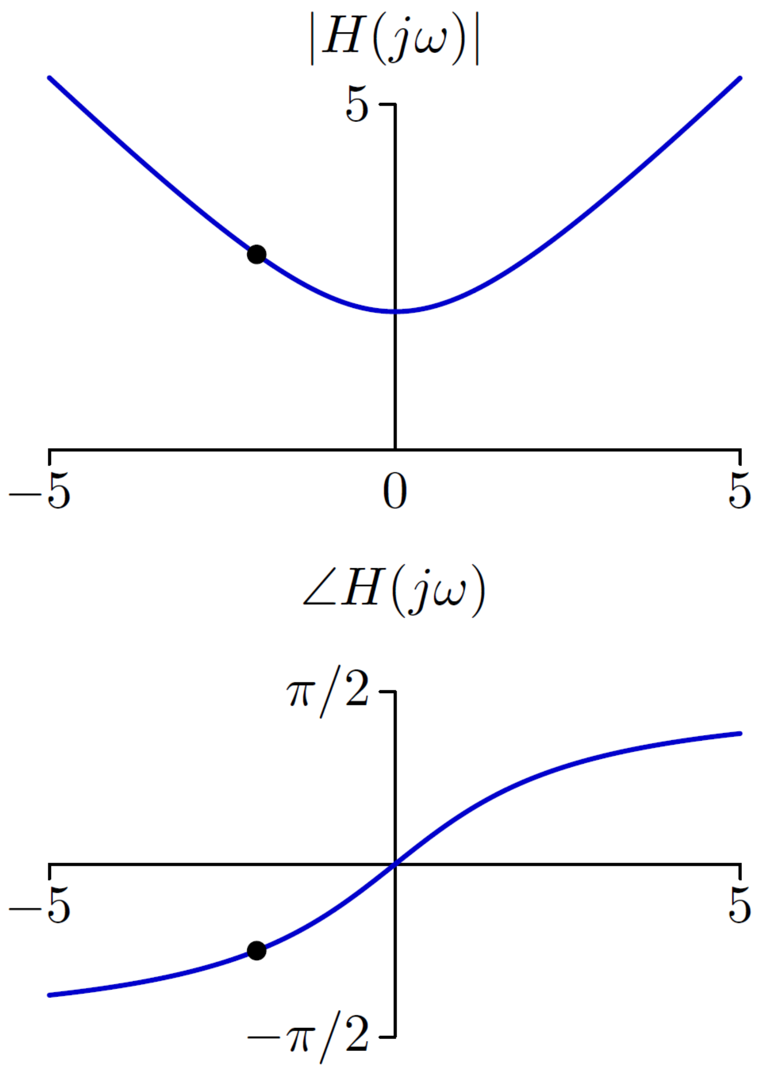

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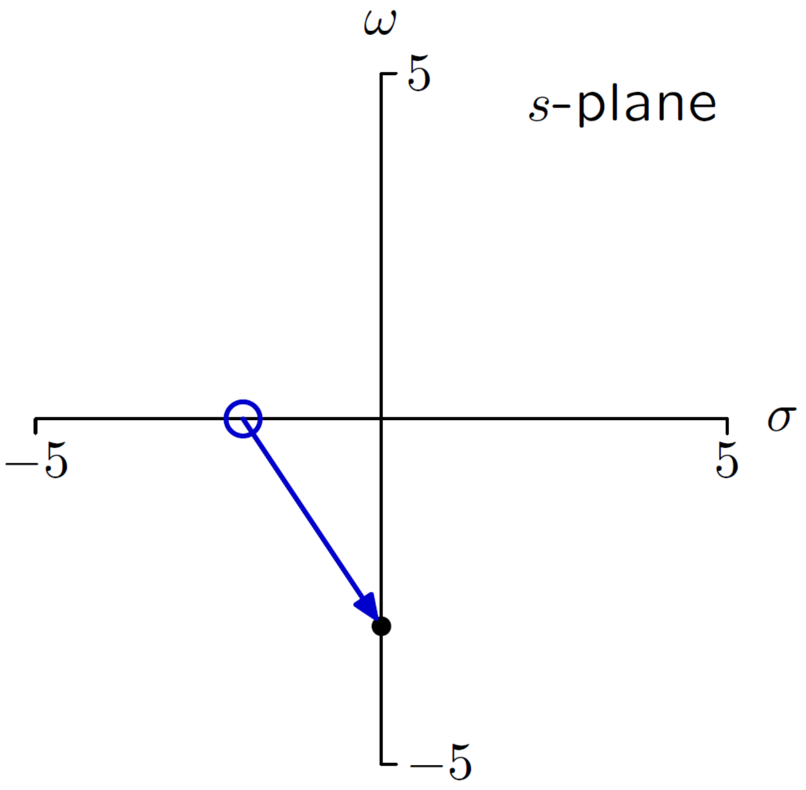

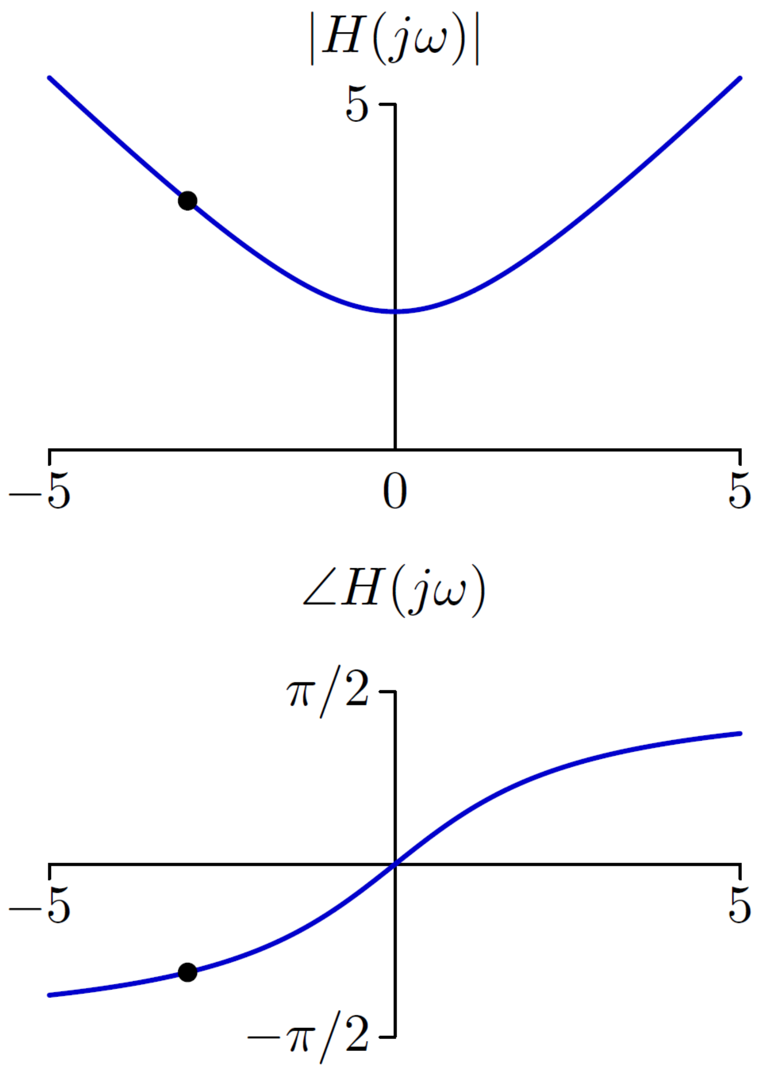

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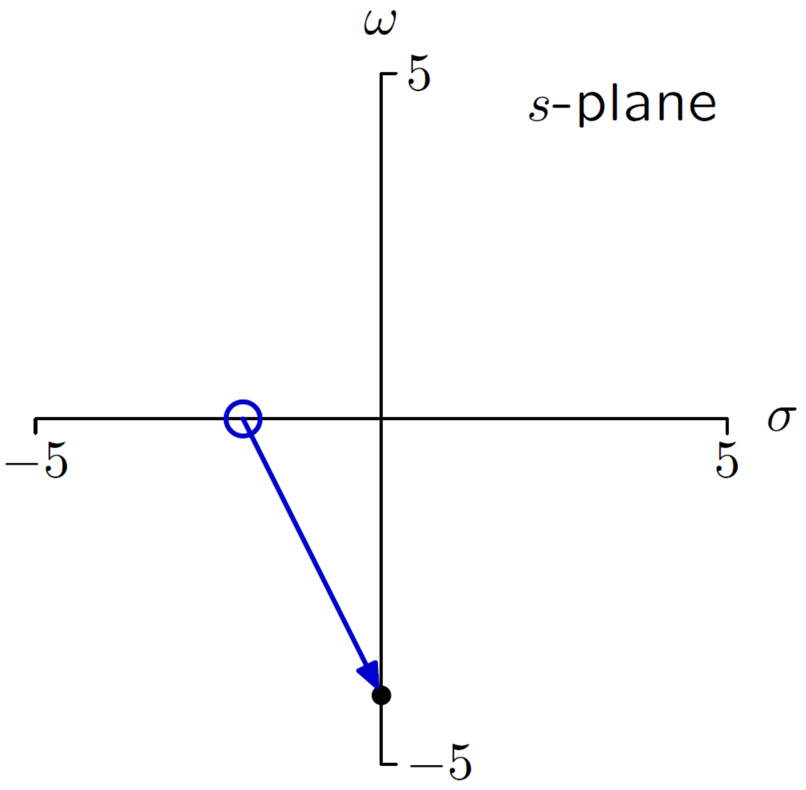

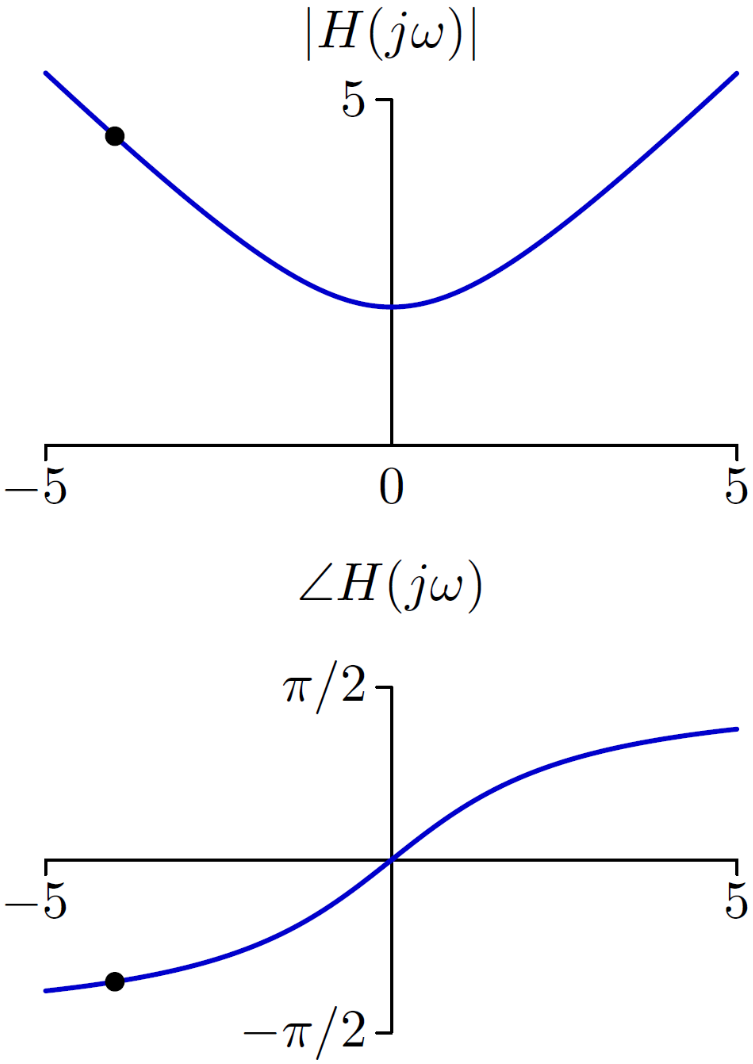

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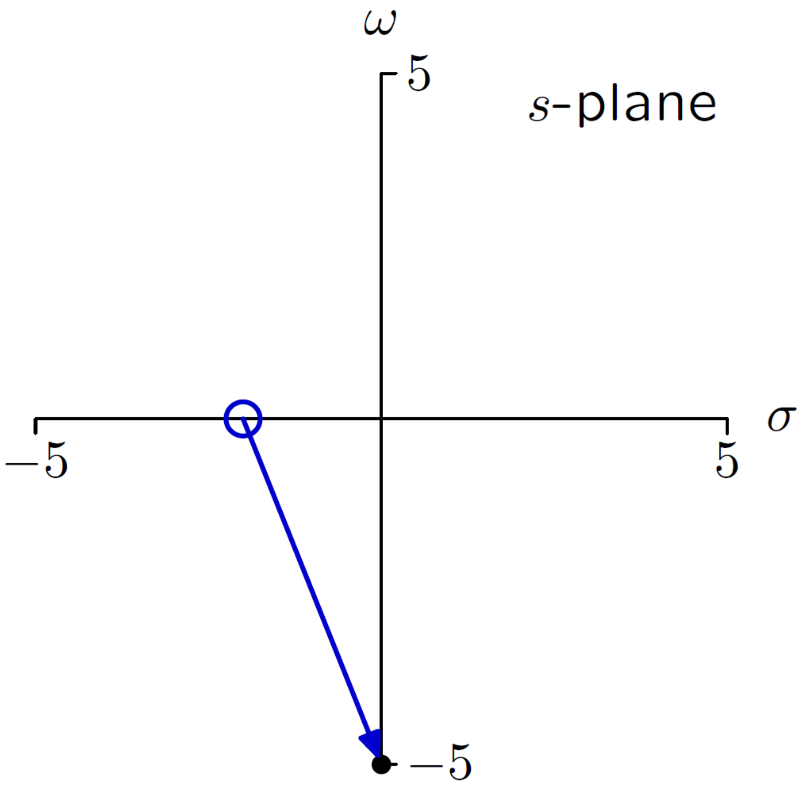

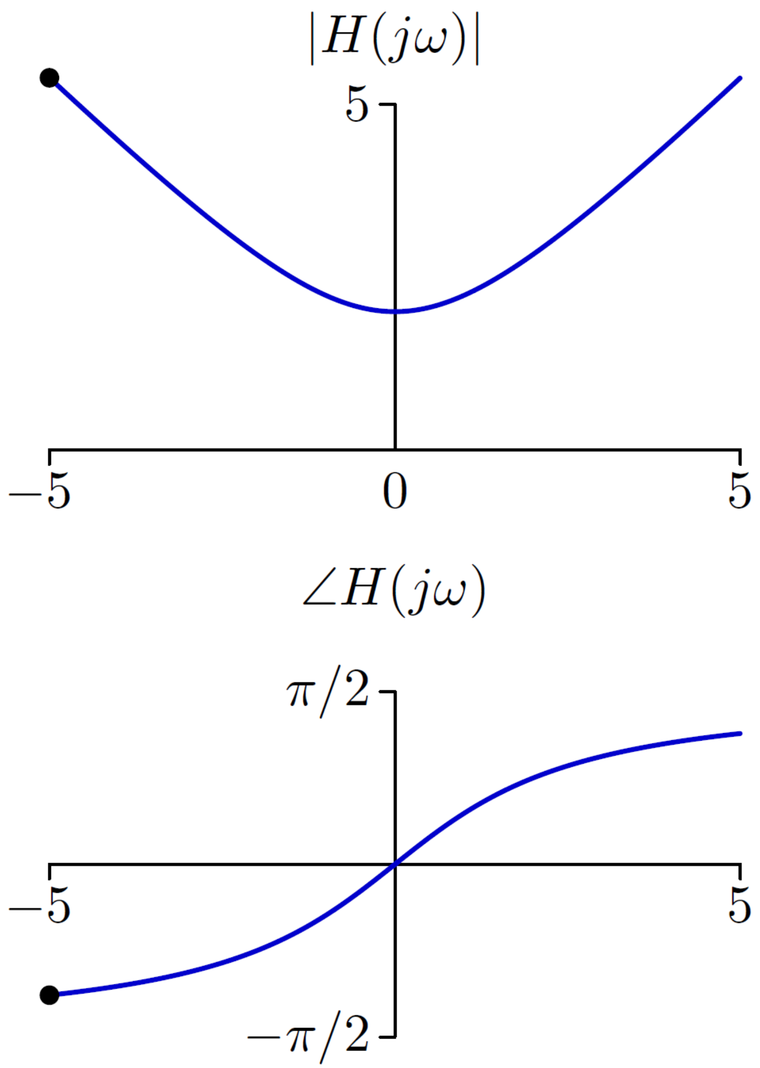

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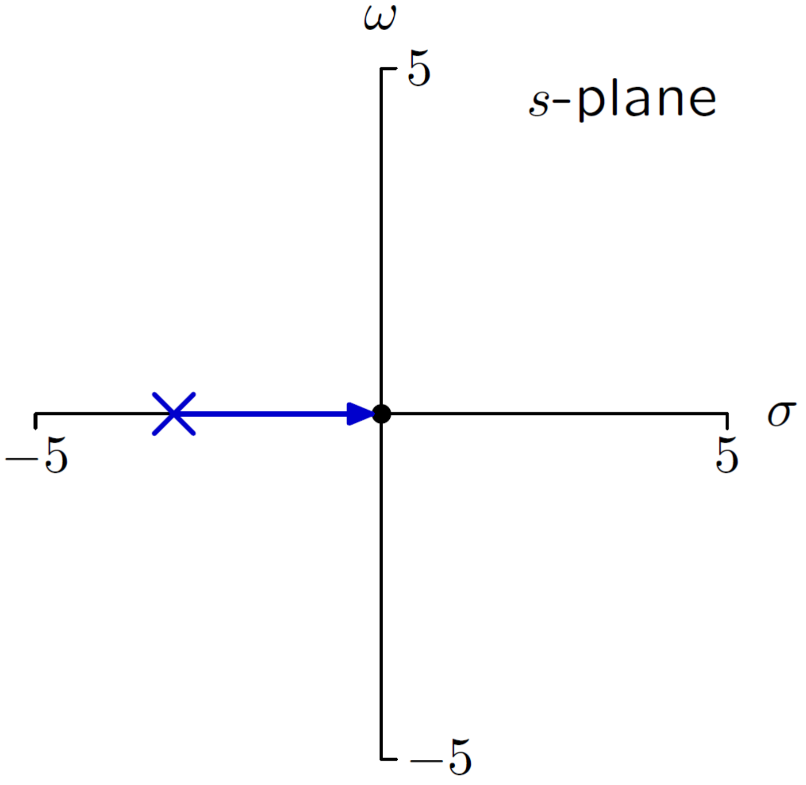
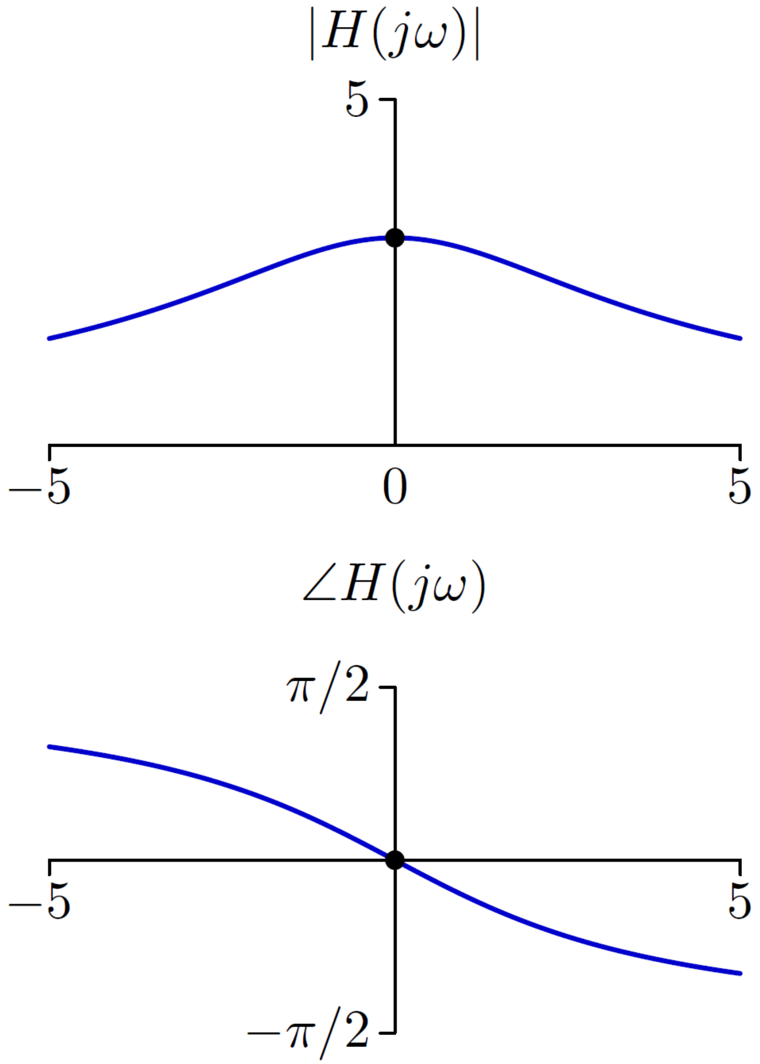
Vector Diagrams at $s = j\omega$

$$H(s) = \frac{9}{s - p_1}$$





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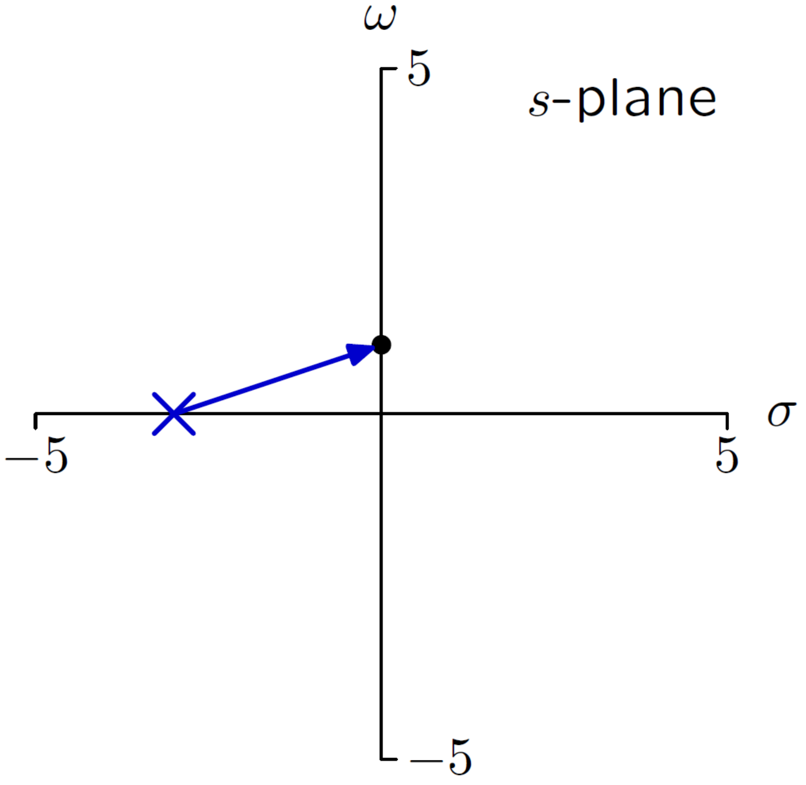

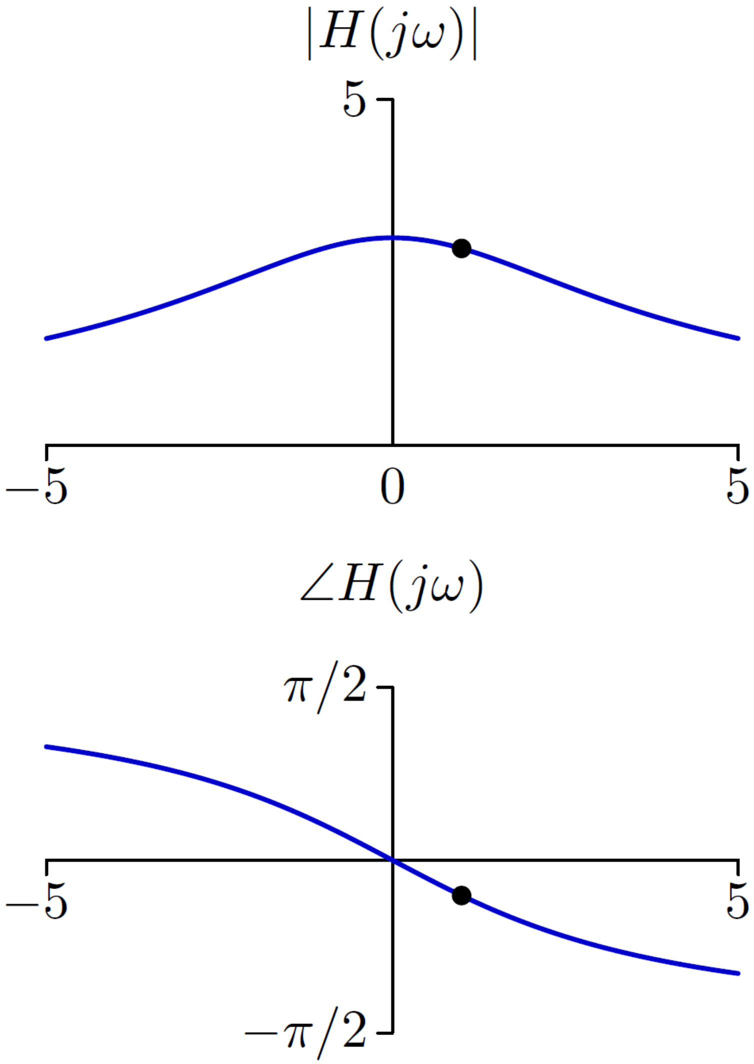

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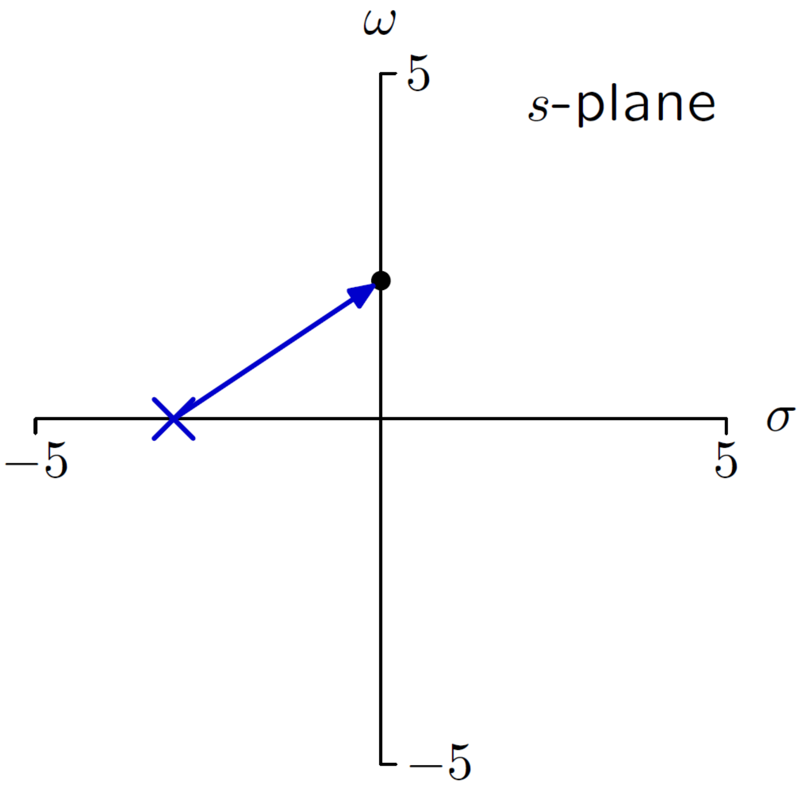

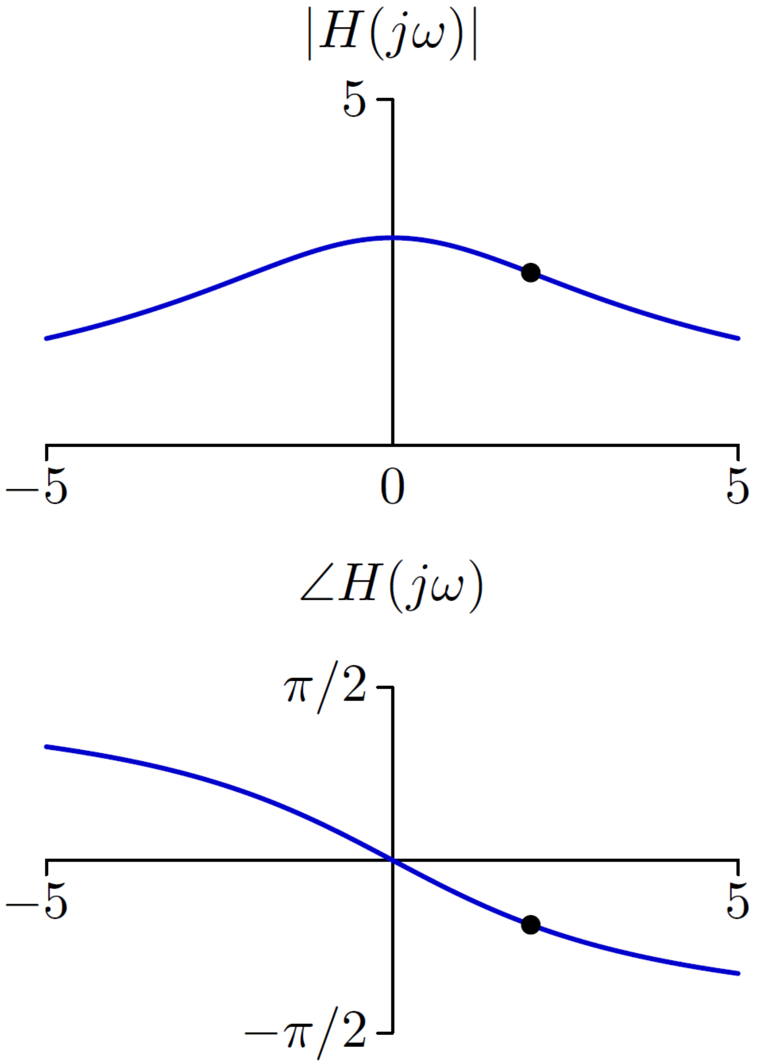

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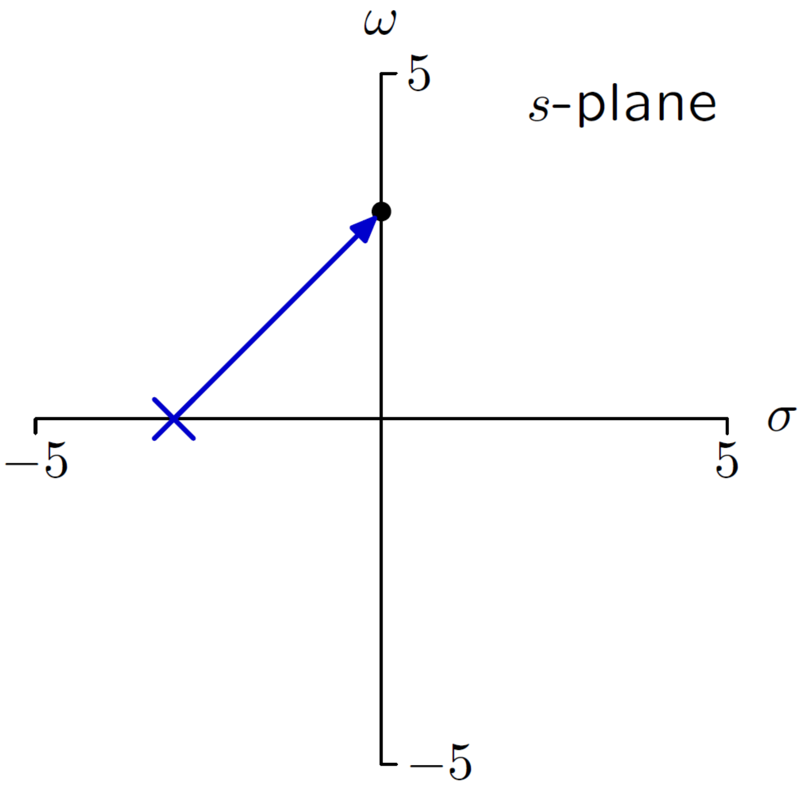

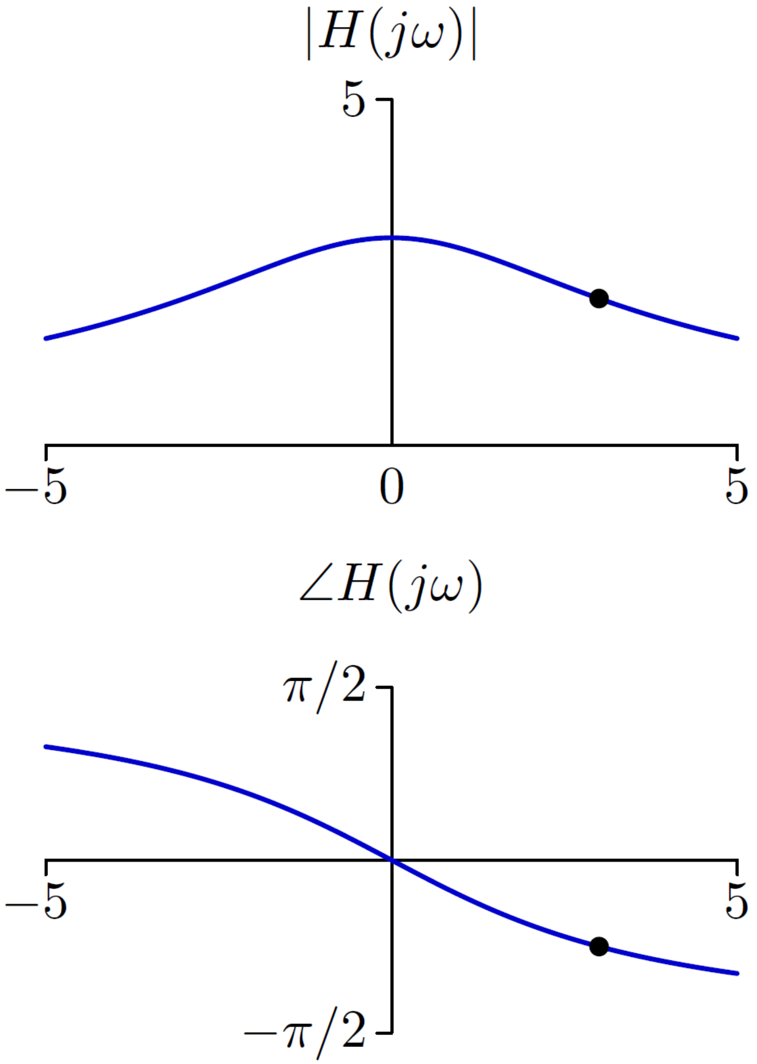

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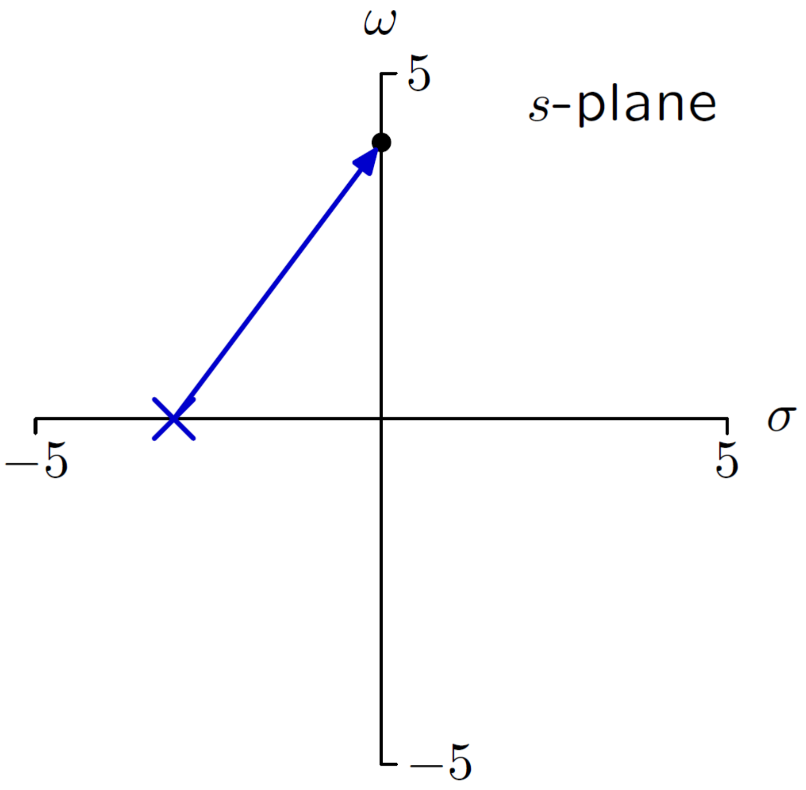

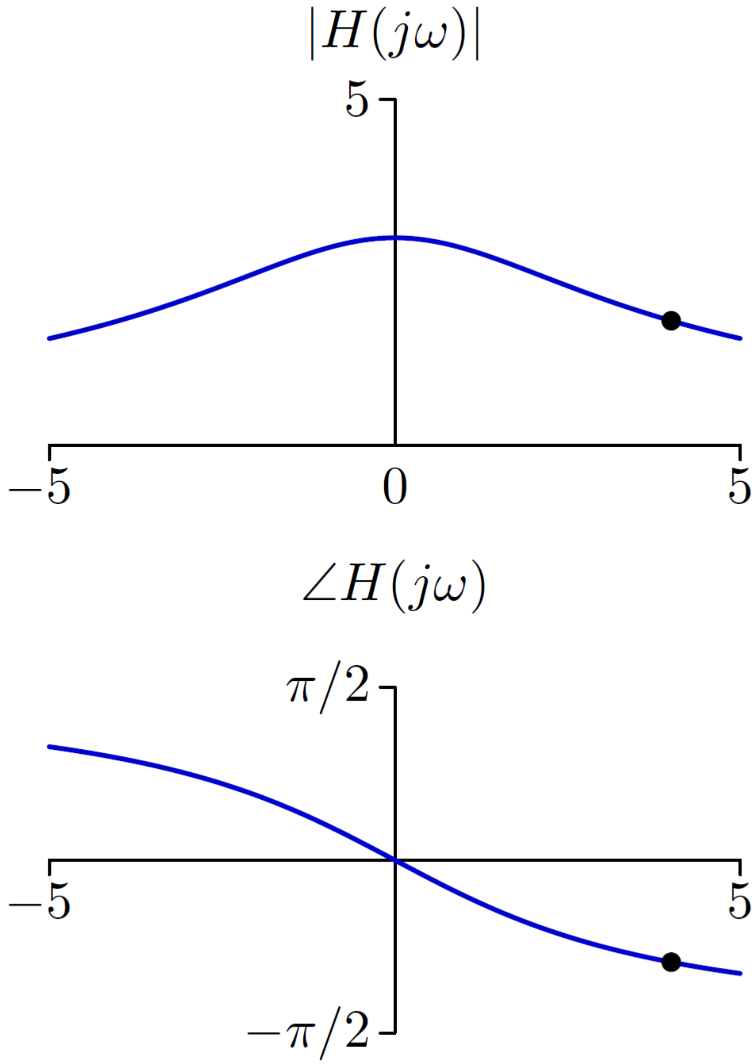

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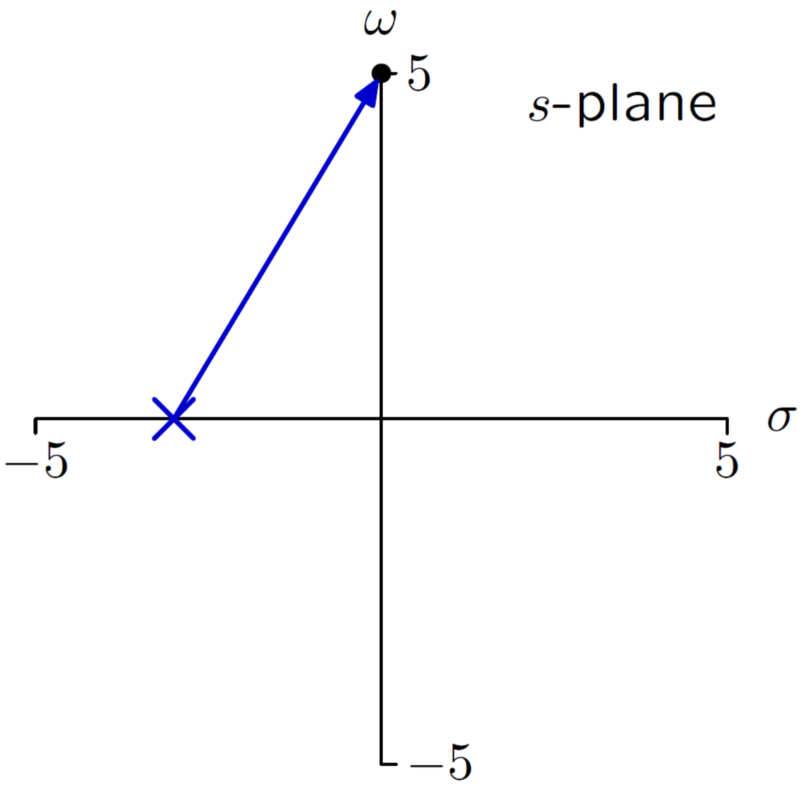
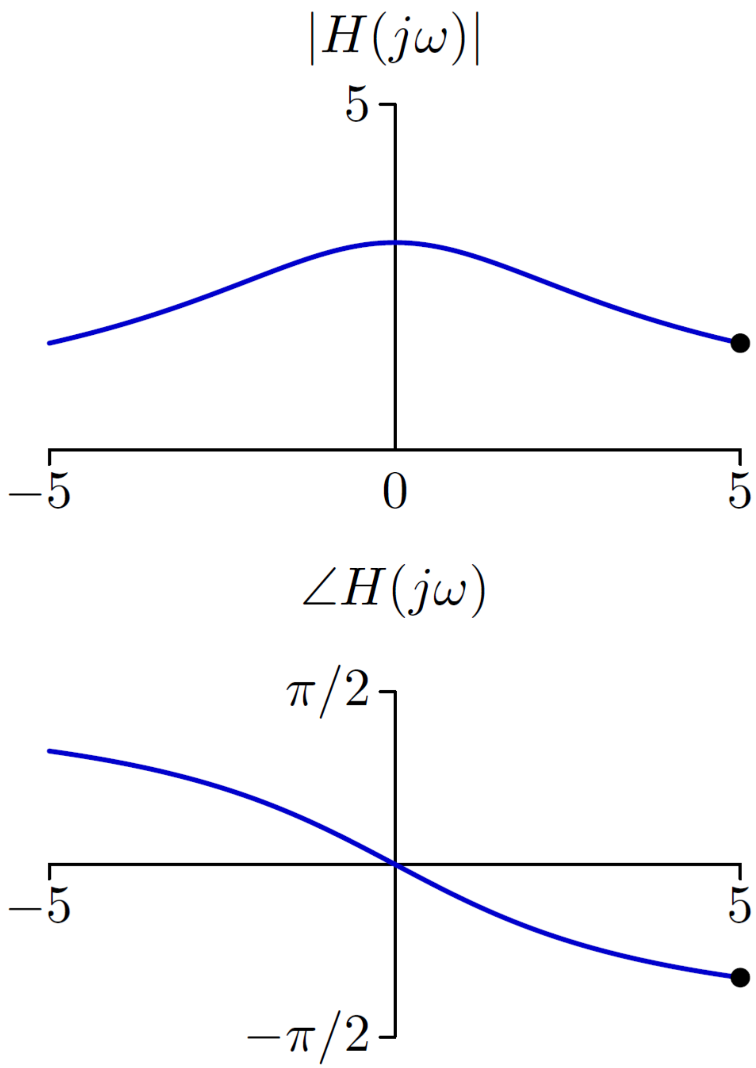
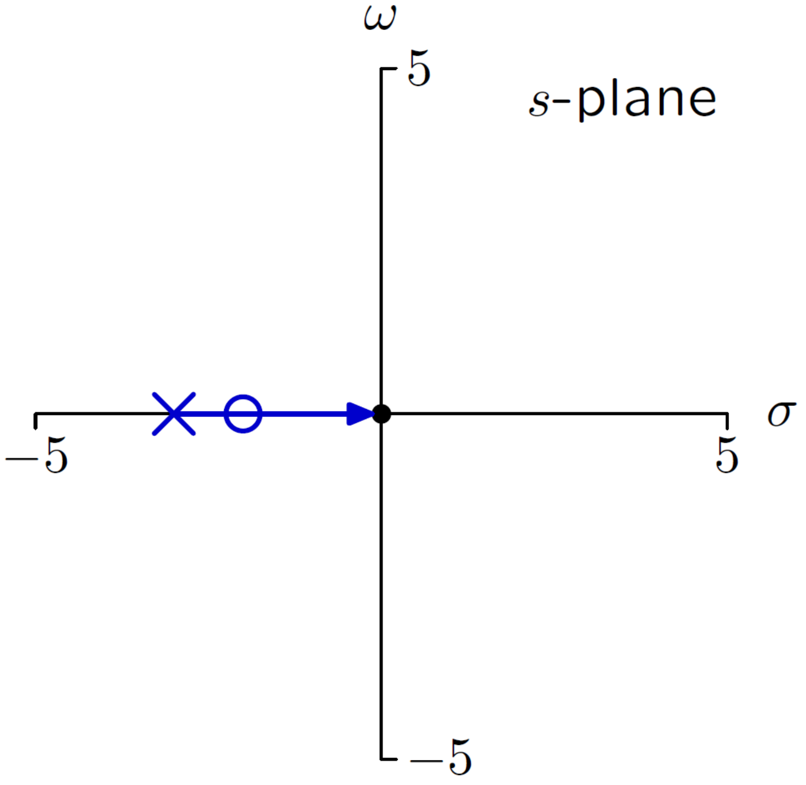
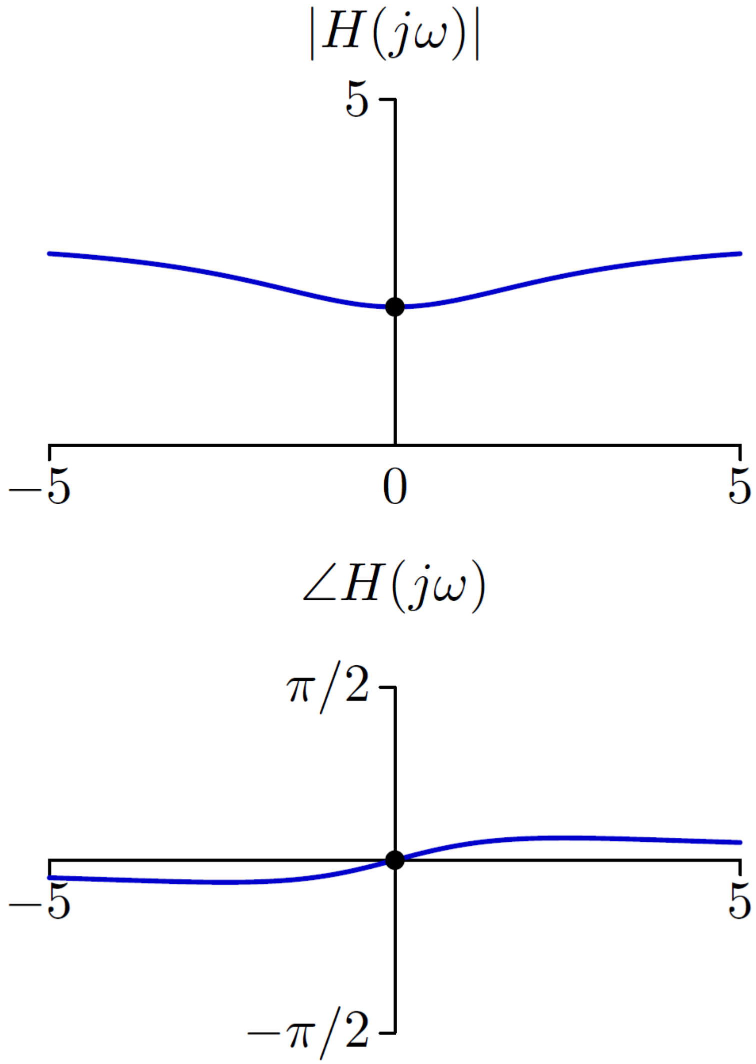




Vector Diagrams at $s = j\omega$

$$H(s) = 3\frac{s - z_1}{s - p1}$$





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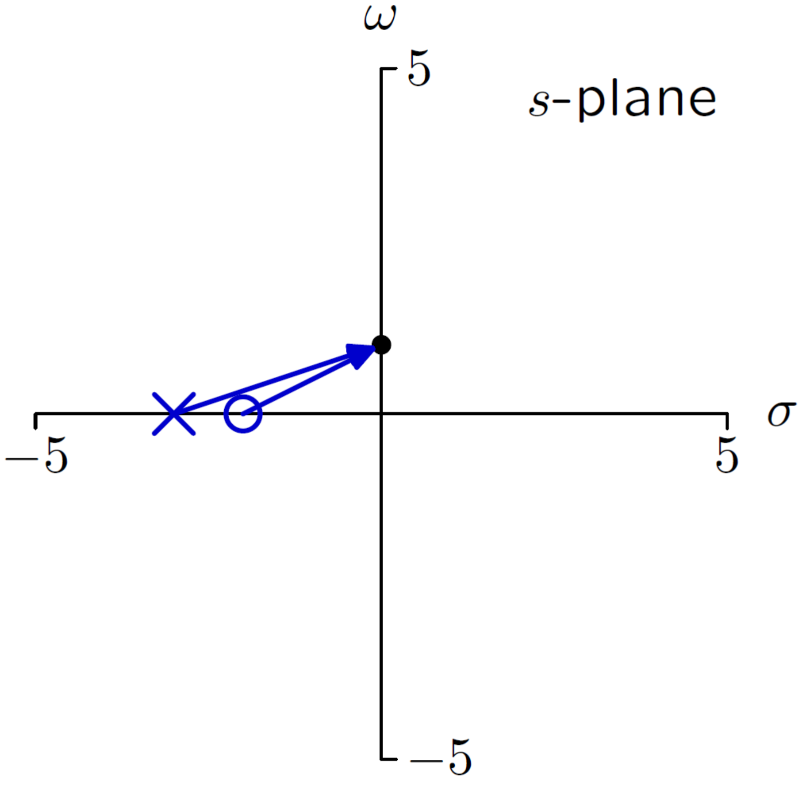

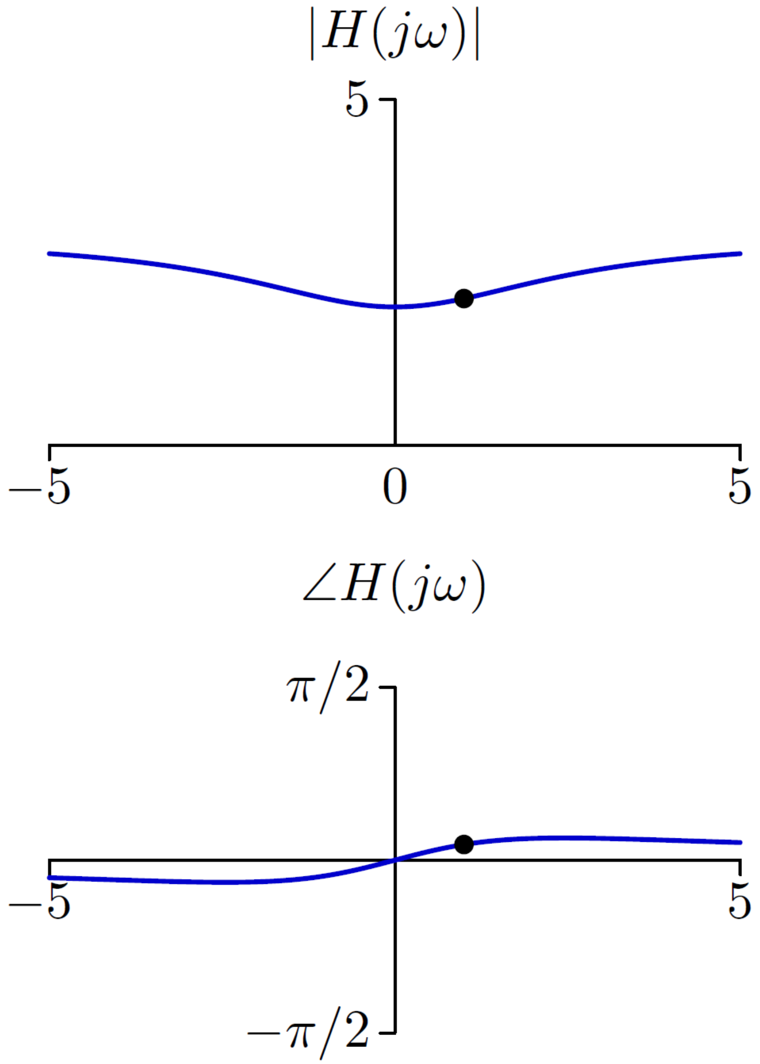

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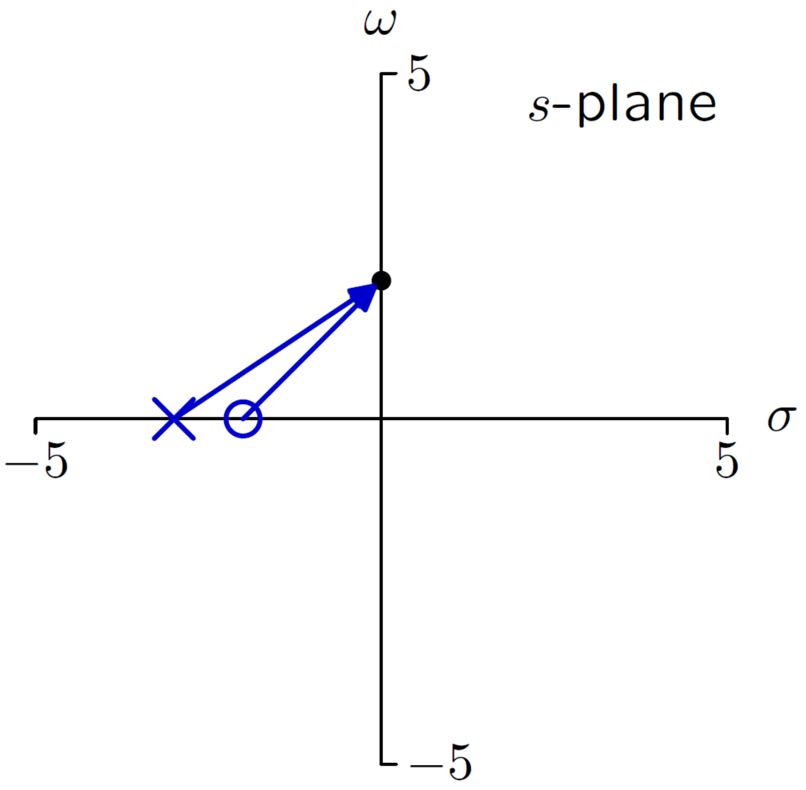
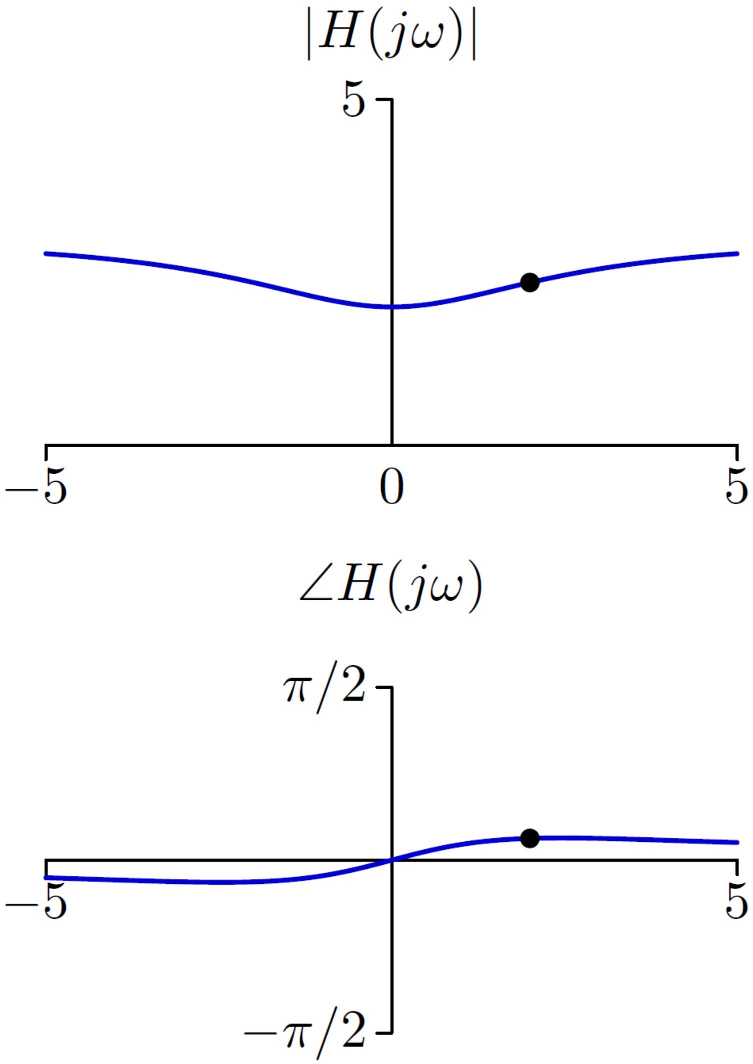
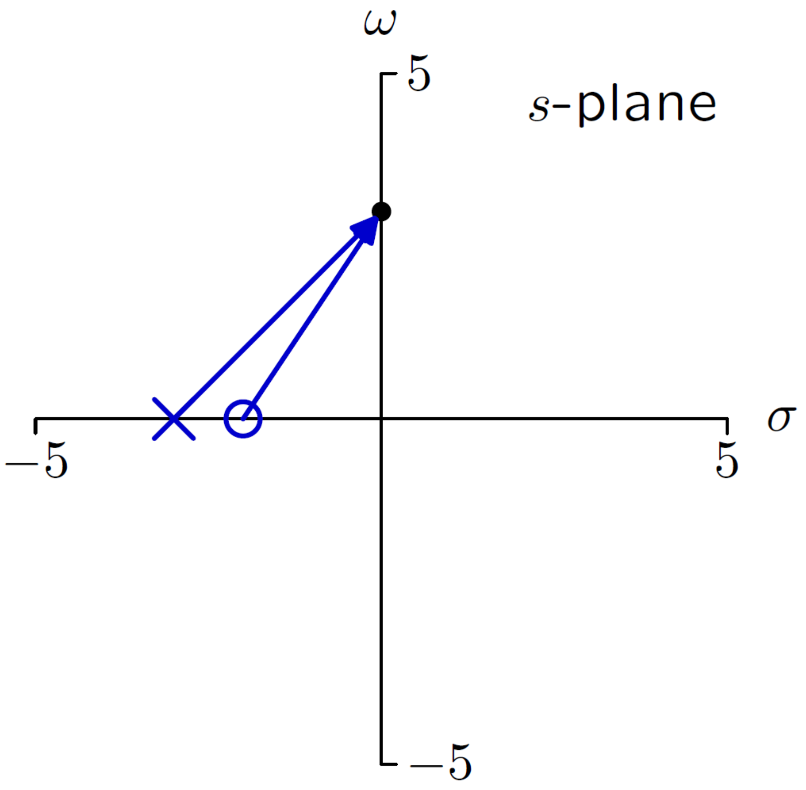
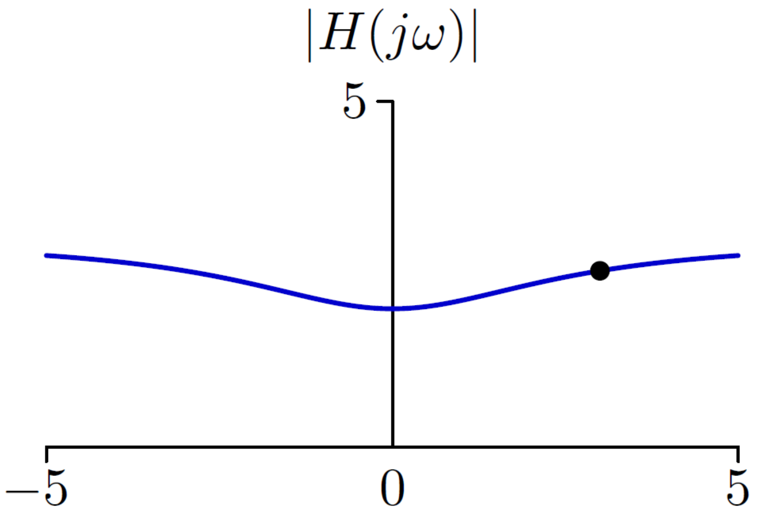


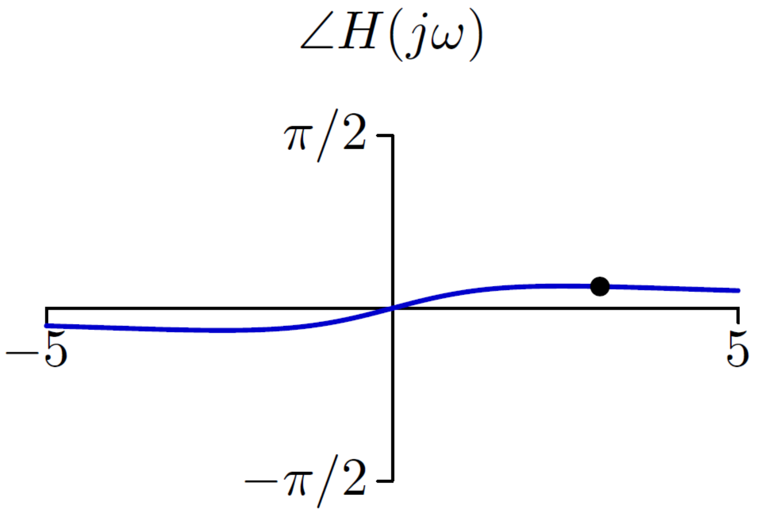


Vector Diagrams at $s = j\omega$

$$H(s) = 3\frac{s - z_1}{s - p1}$$

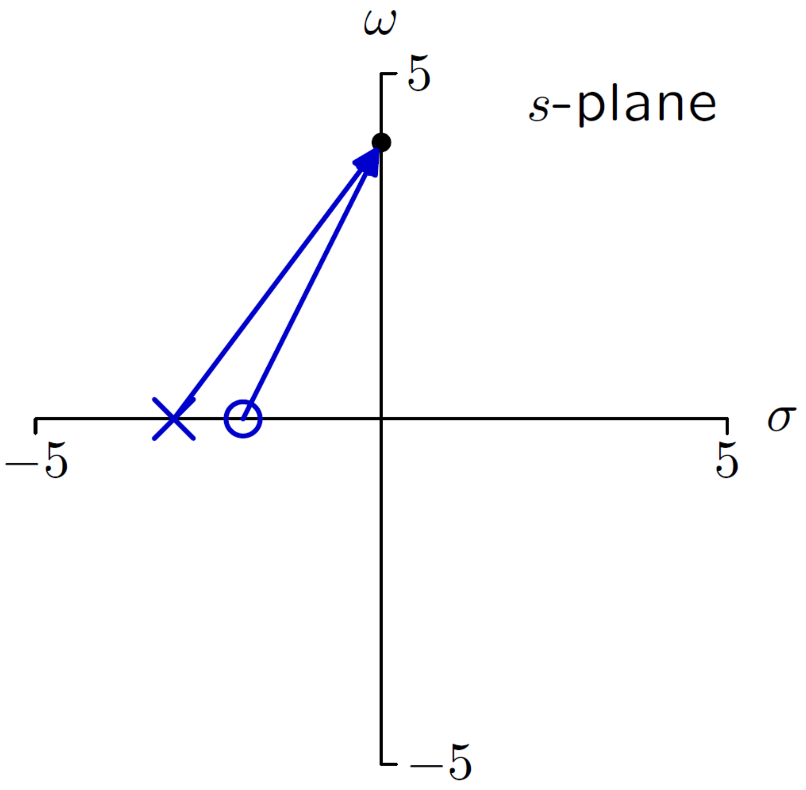


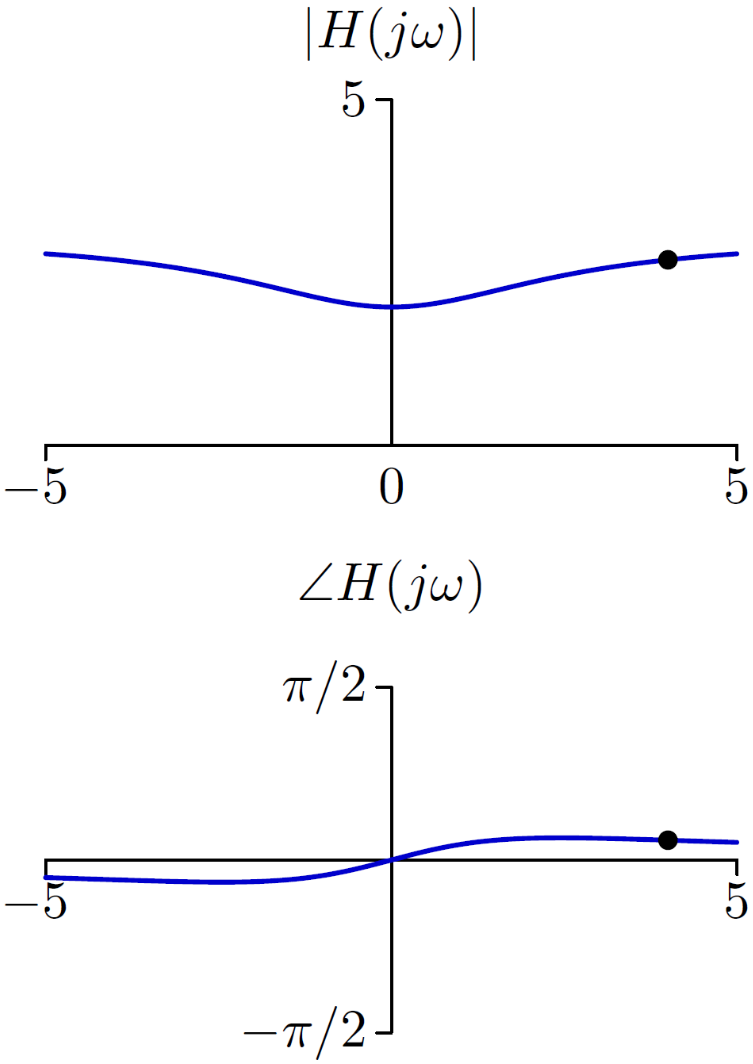




Vector Diagrams at $s = j\omega$

$$H(s) = 3\frac{s - z_1}{s - p1}$$





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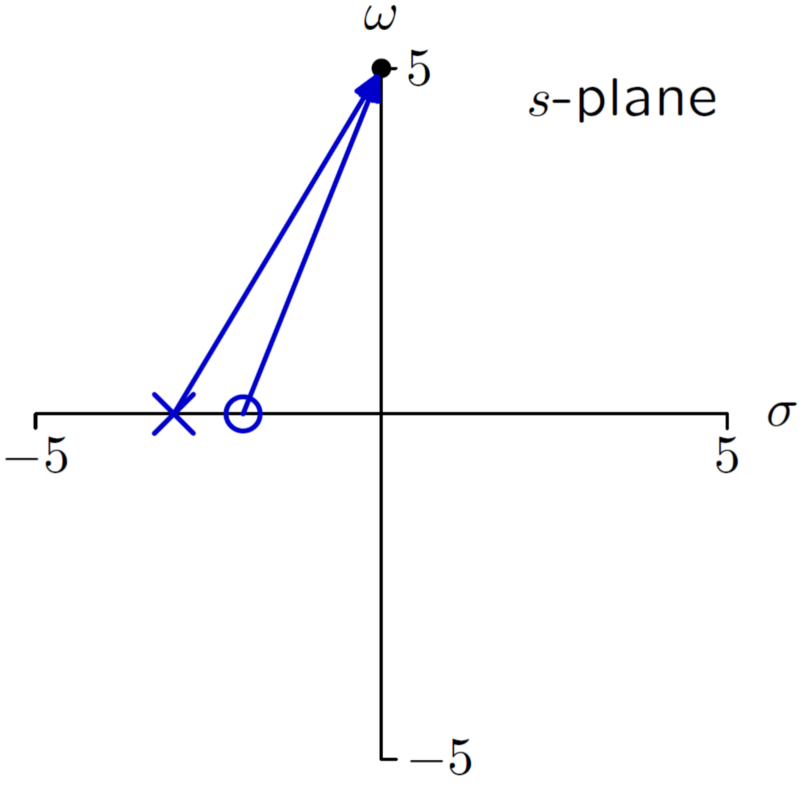
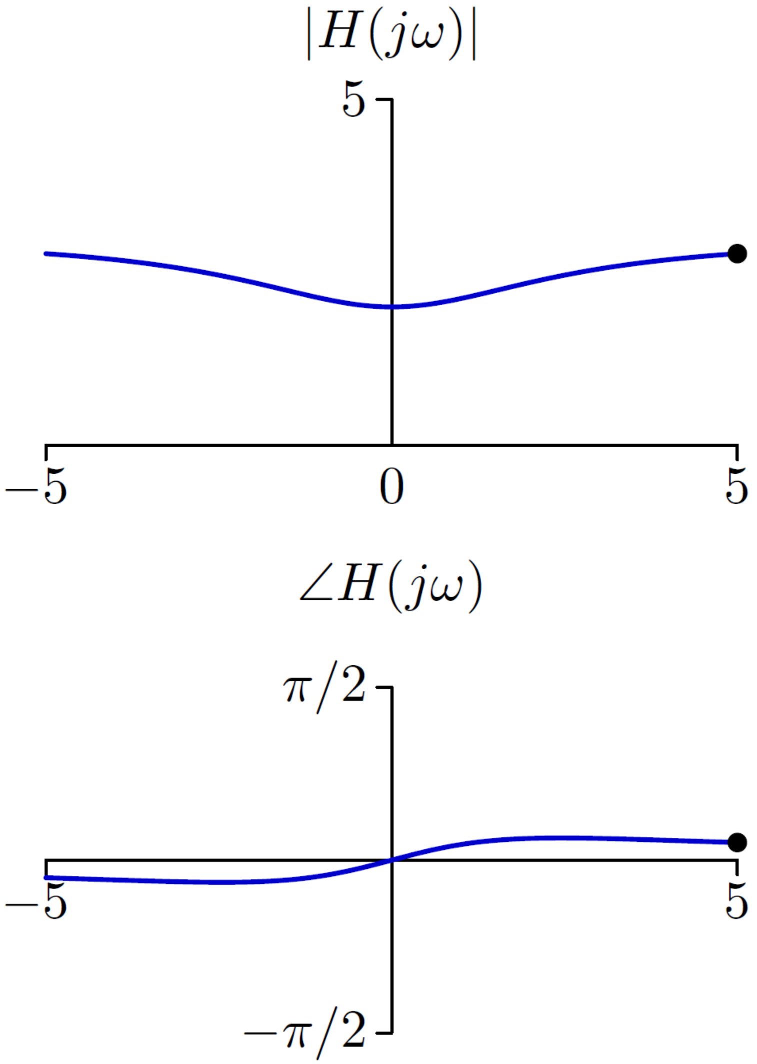
Vector Diagrams at $s = j\omega$

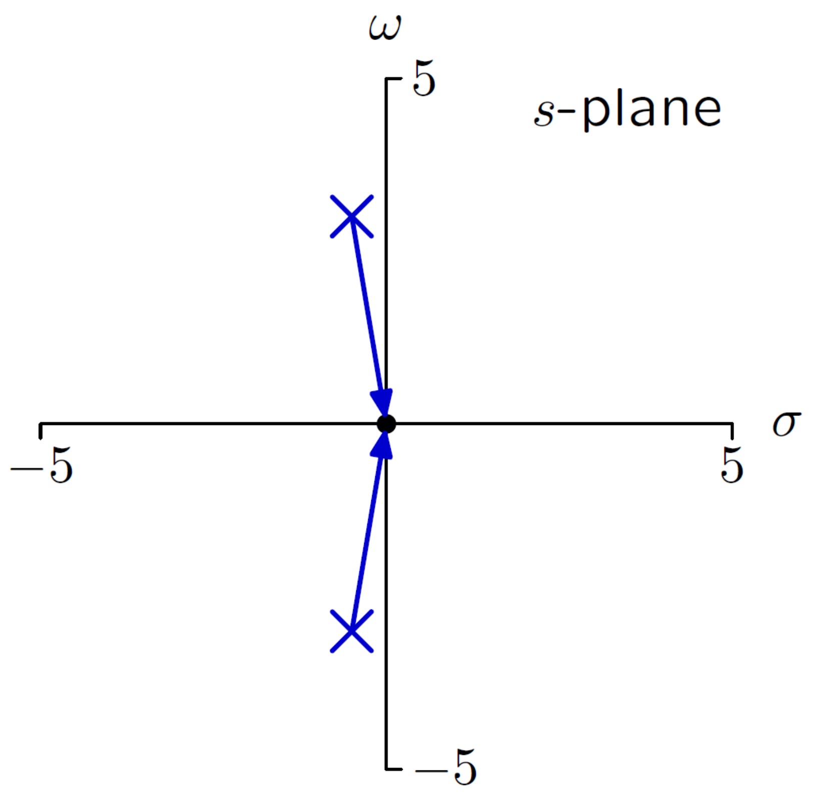
$$H(s) = 3\frac{s - z_1}{s - p1}$$

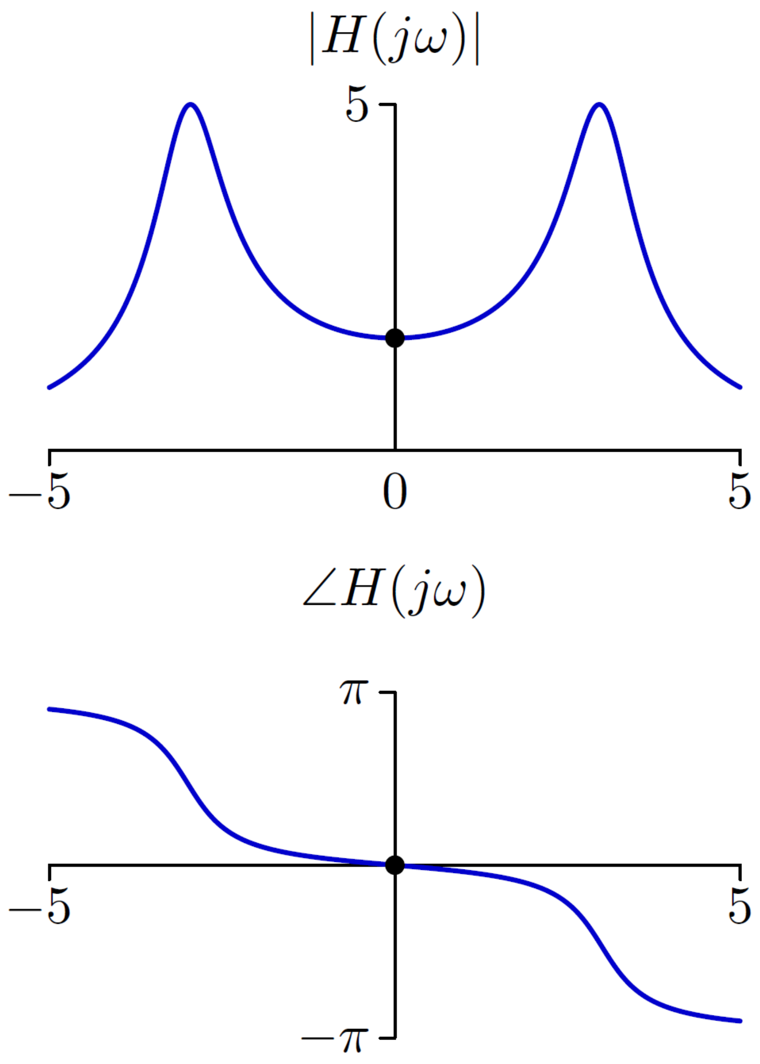




Vector Diagrams at $s = j\omega$

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$





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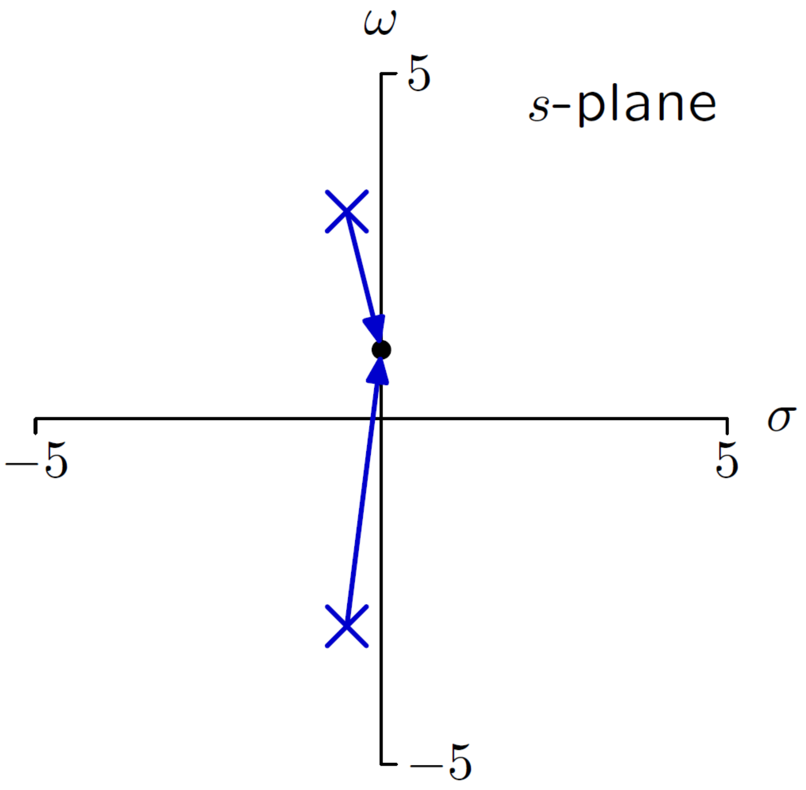

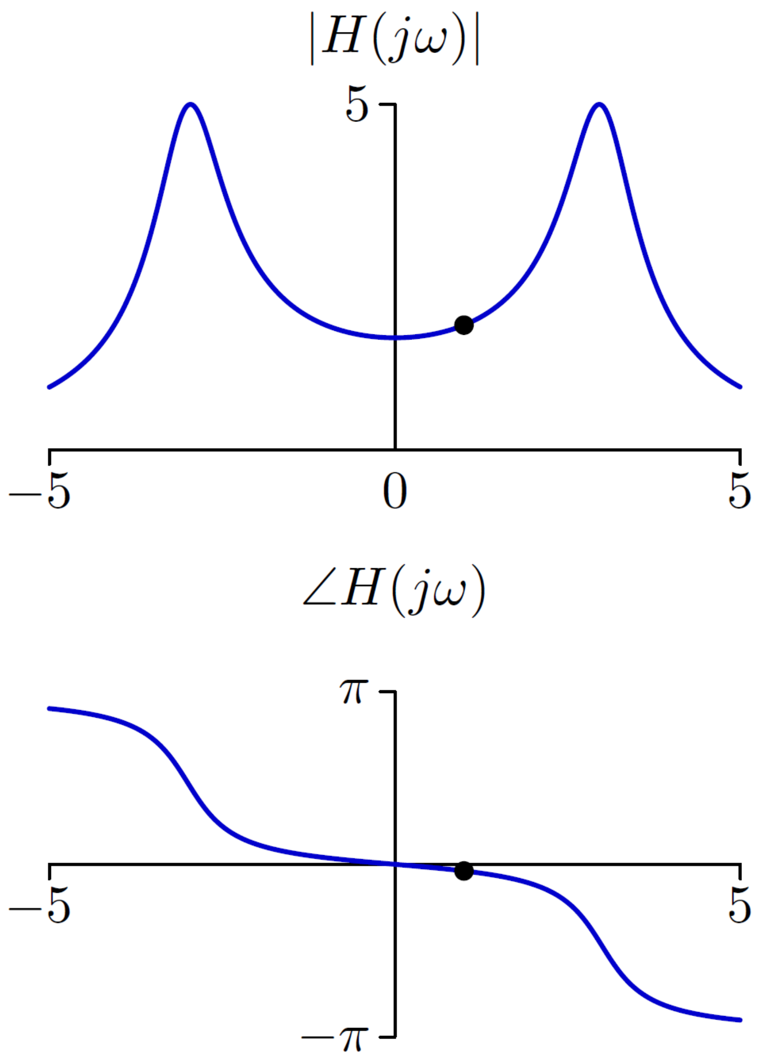

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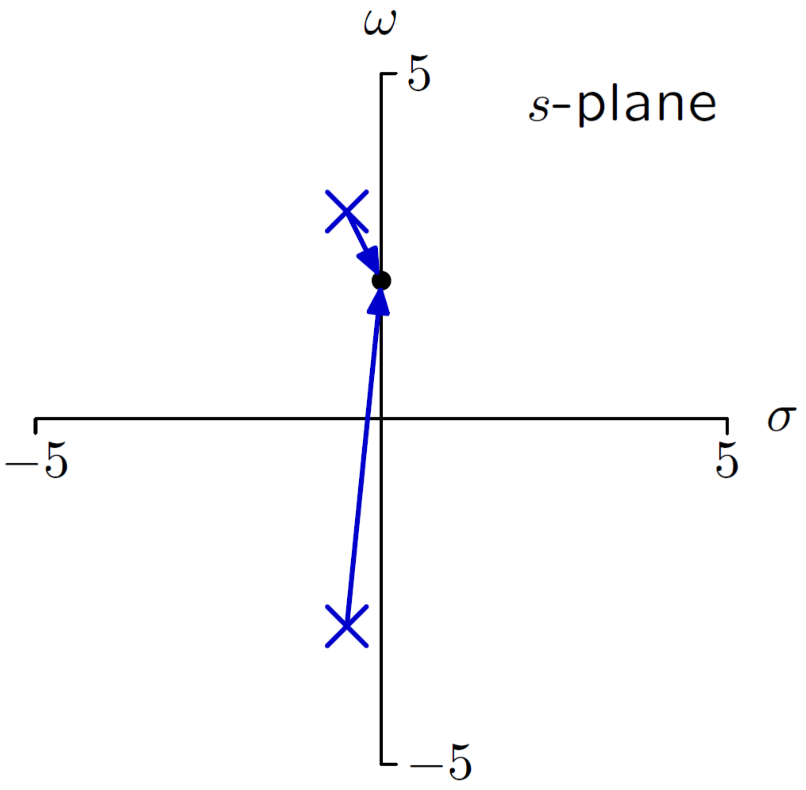

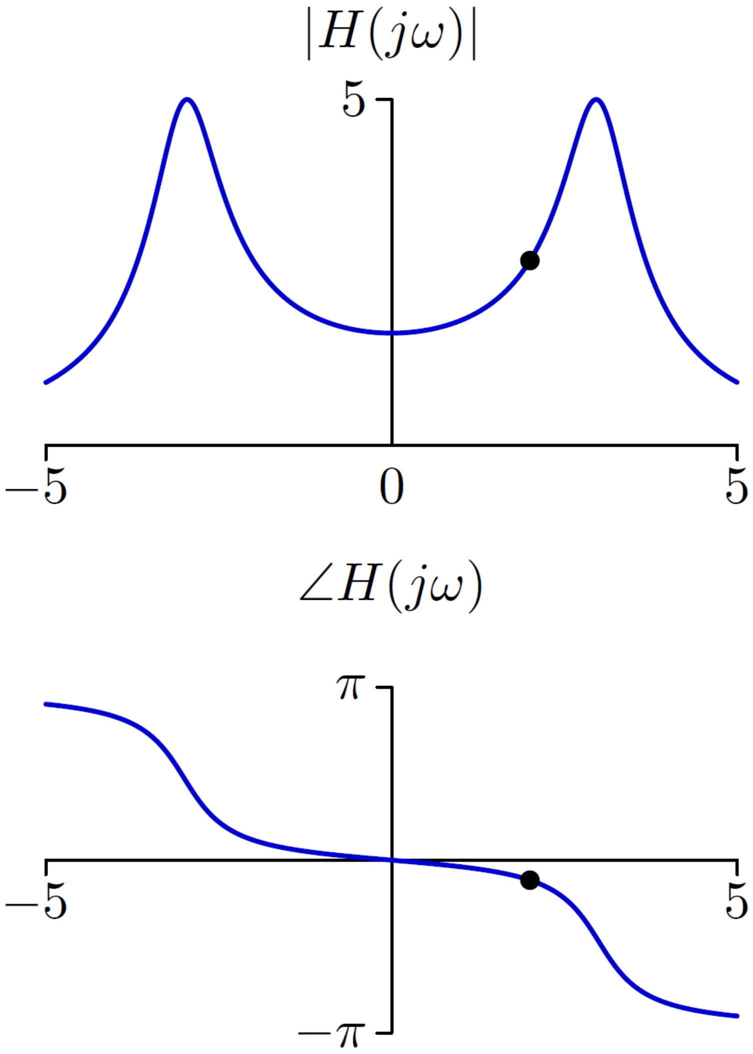

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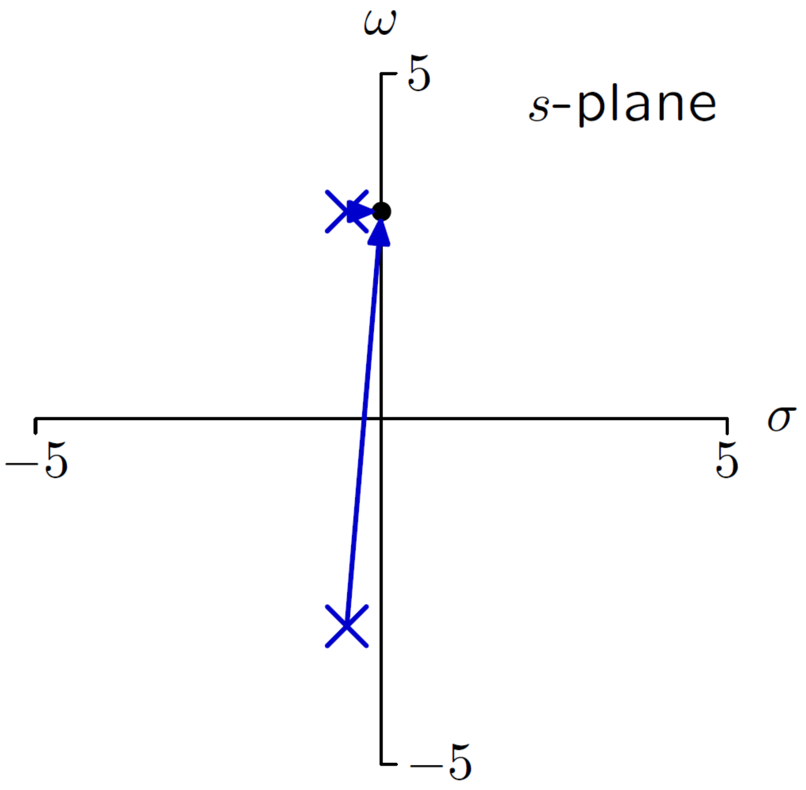

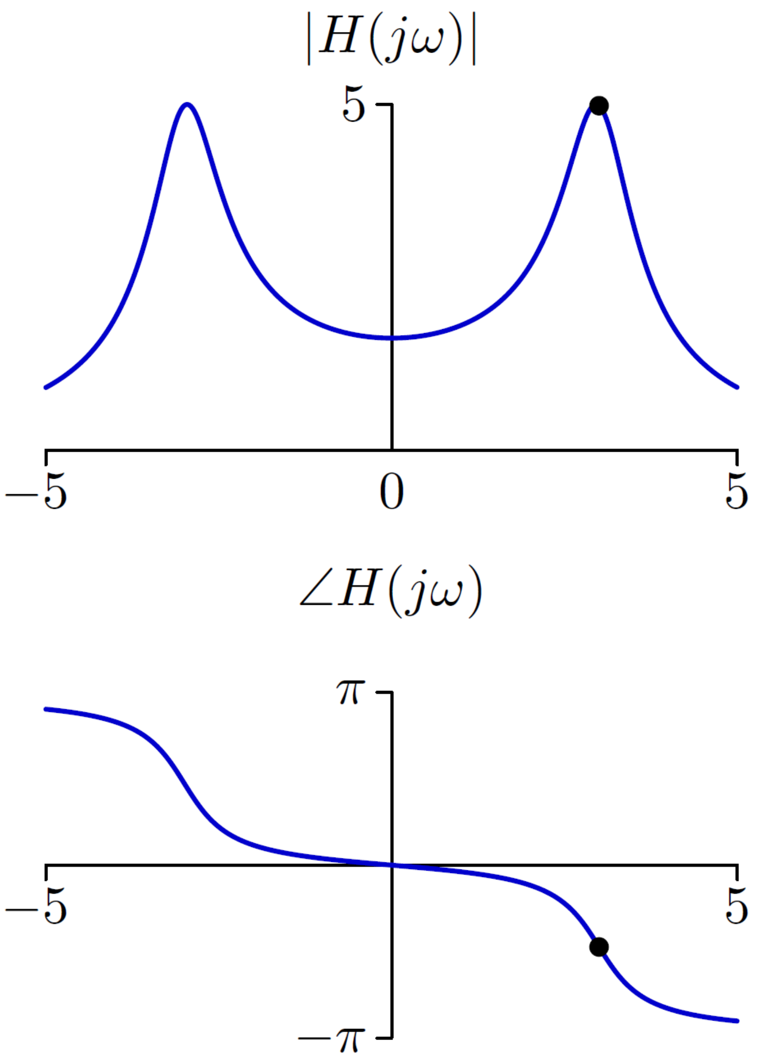

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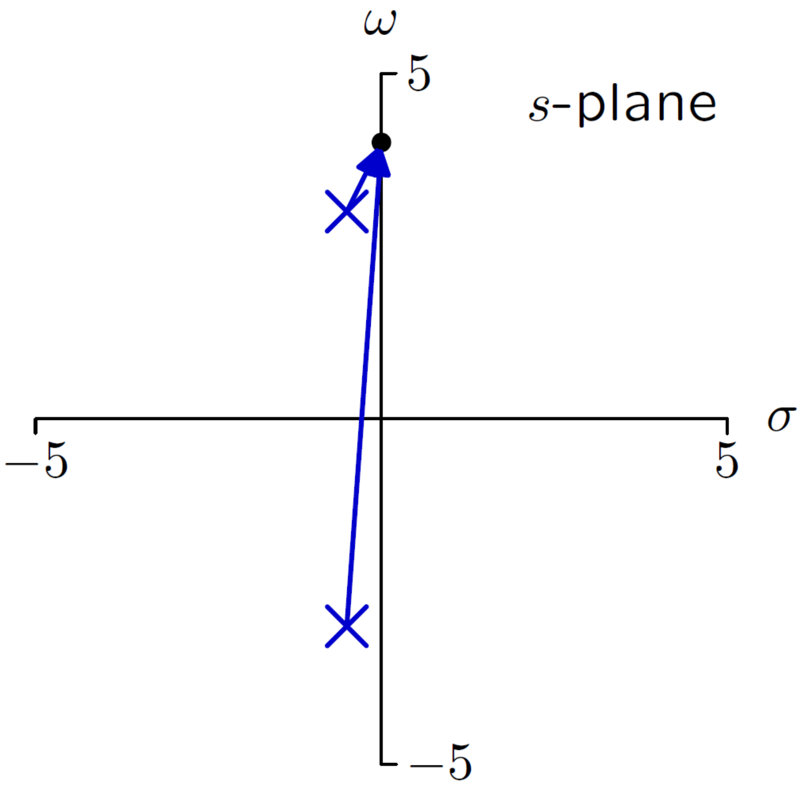

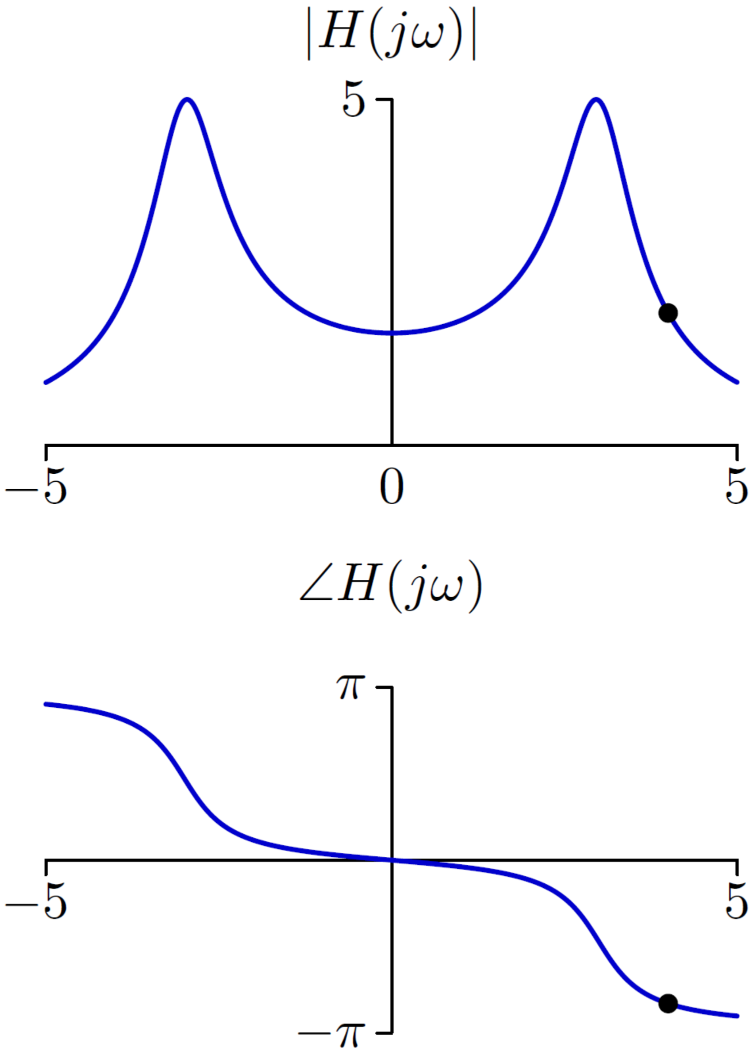

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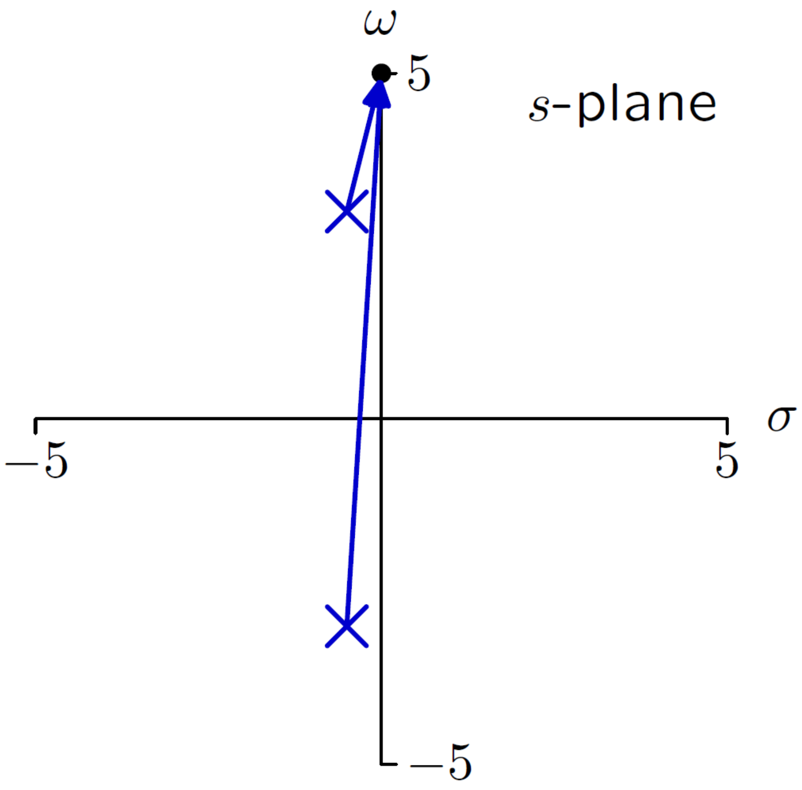
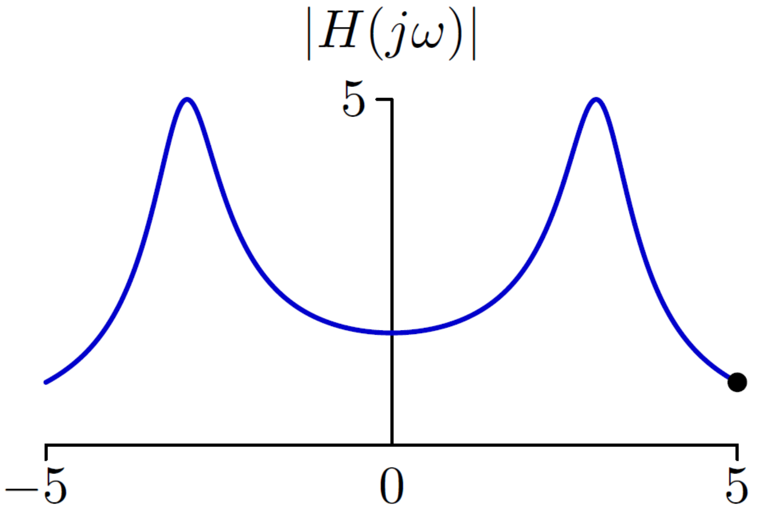
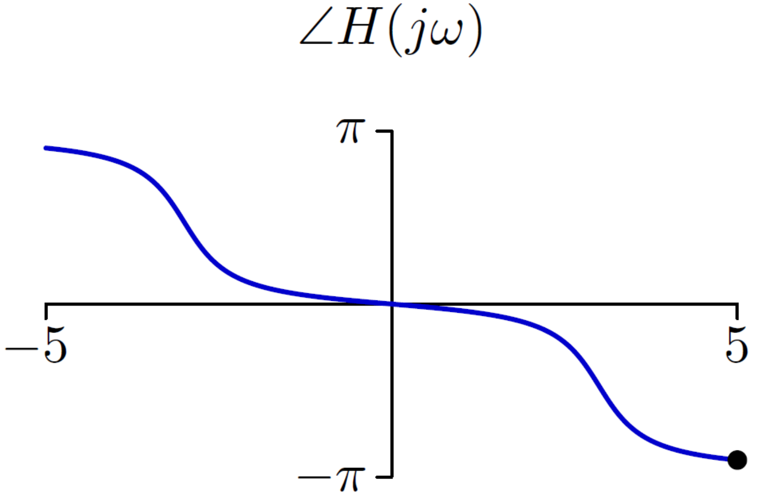
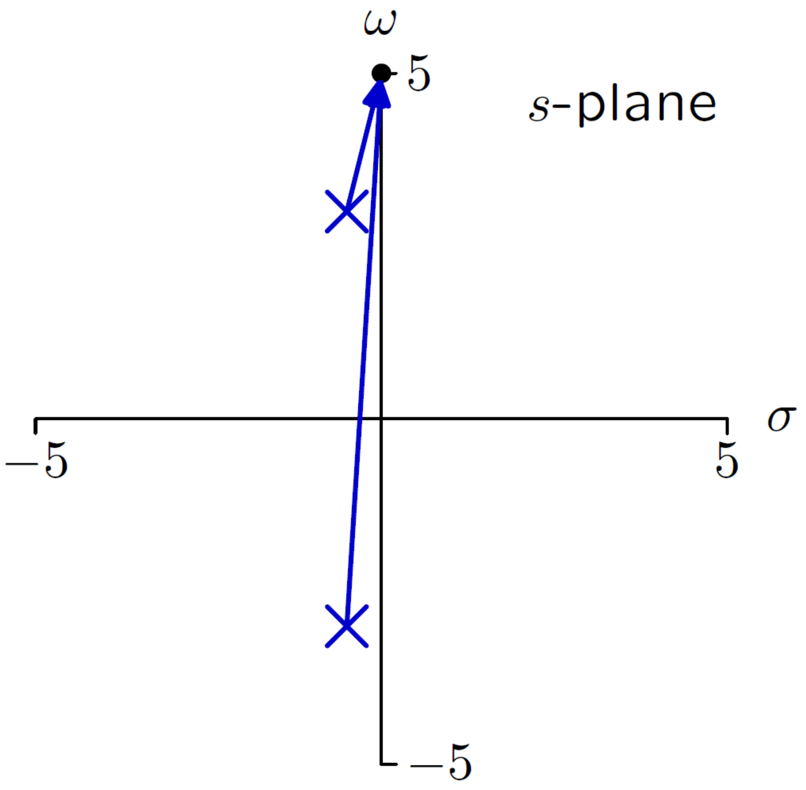


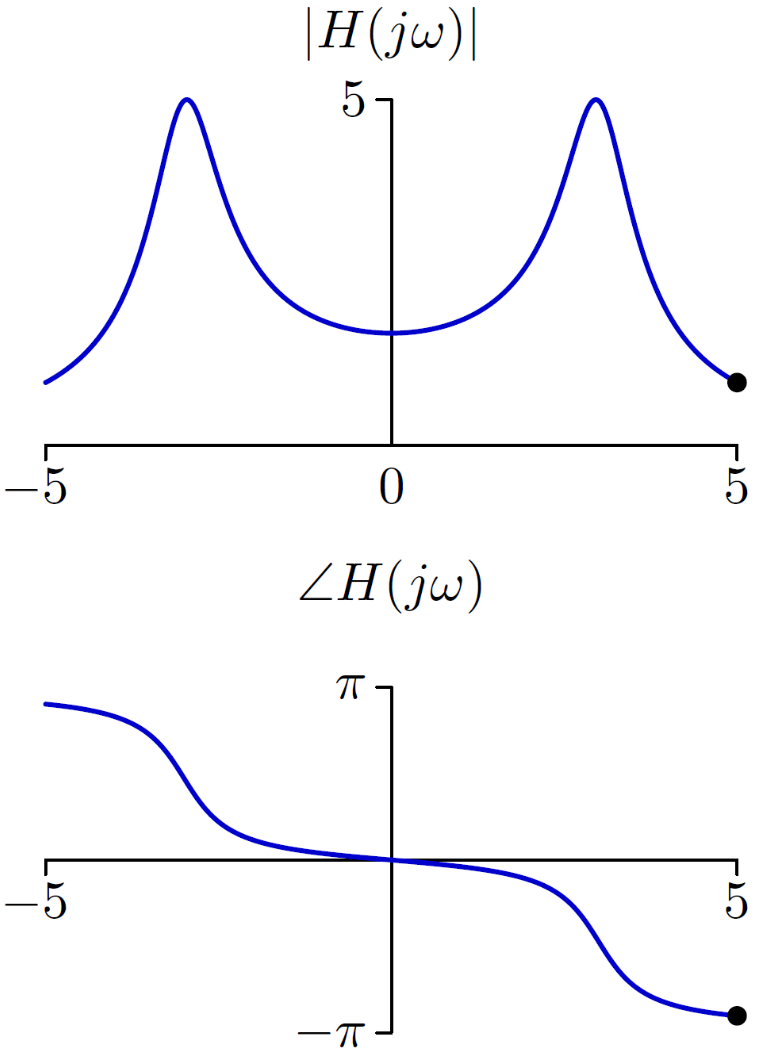




System Design in S-plane

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$





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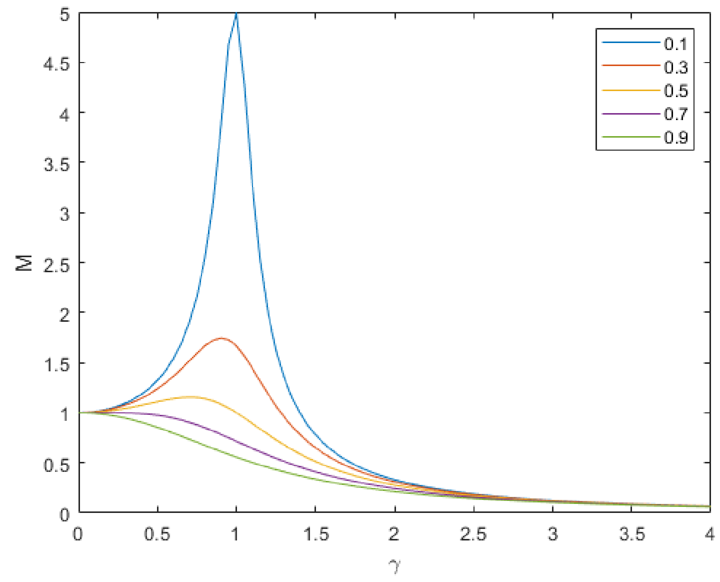
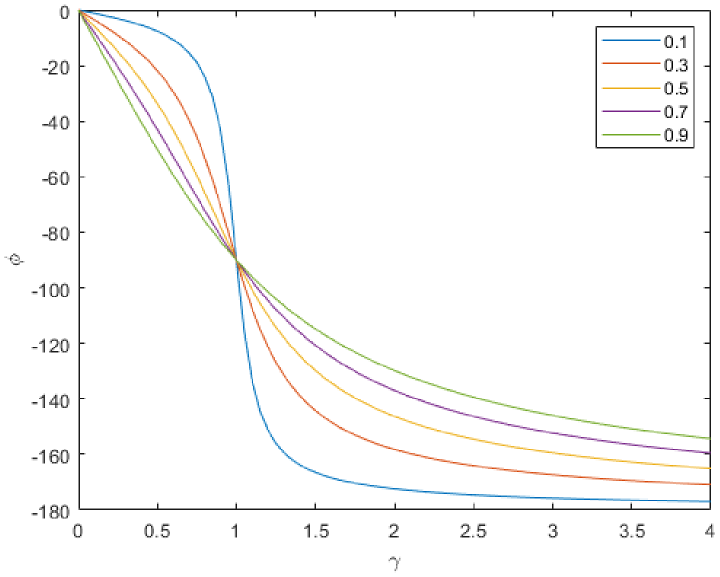
Frequency Response (Frequency Sweep): MATLAB

```
r = 0:0.05:4;

zeta = 0.1:0.2:1;
A = [];
for i = 1:length(zeta)
    A(i,:) = 1./sqrt((1-r.^2).^2 + (2*zeta(i)*r).^2);
end
```



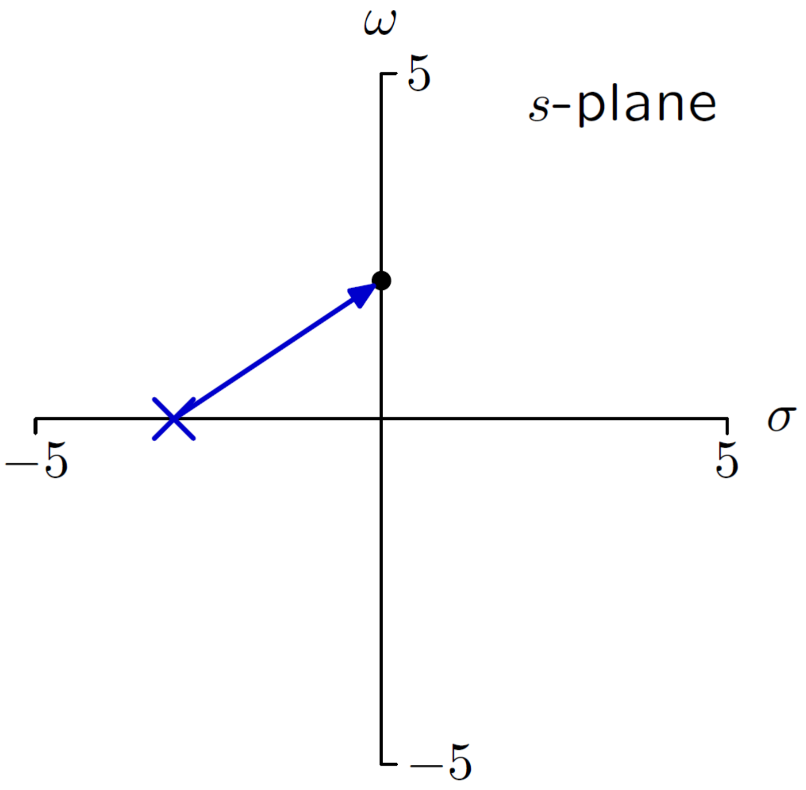

```
phi = [];
for i = 1:length(zeta)
    phi(i,:) = -atan2((2*zeta(i).*r),(1-r.^2));
end

plot(r,phi*180/pi)
xlabel('\gamma')
ylabel('\phi')
legend('0.1','0.3','0.5','0.7','0.9')
```

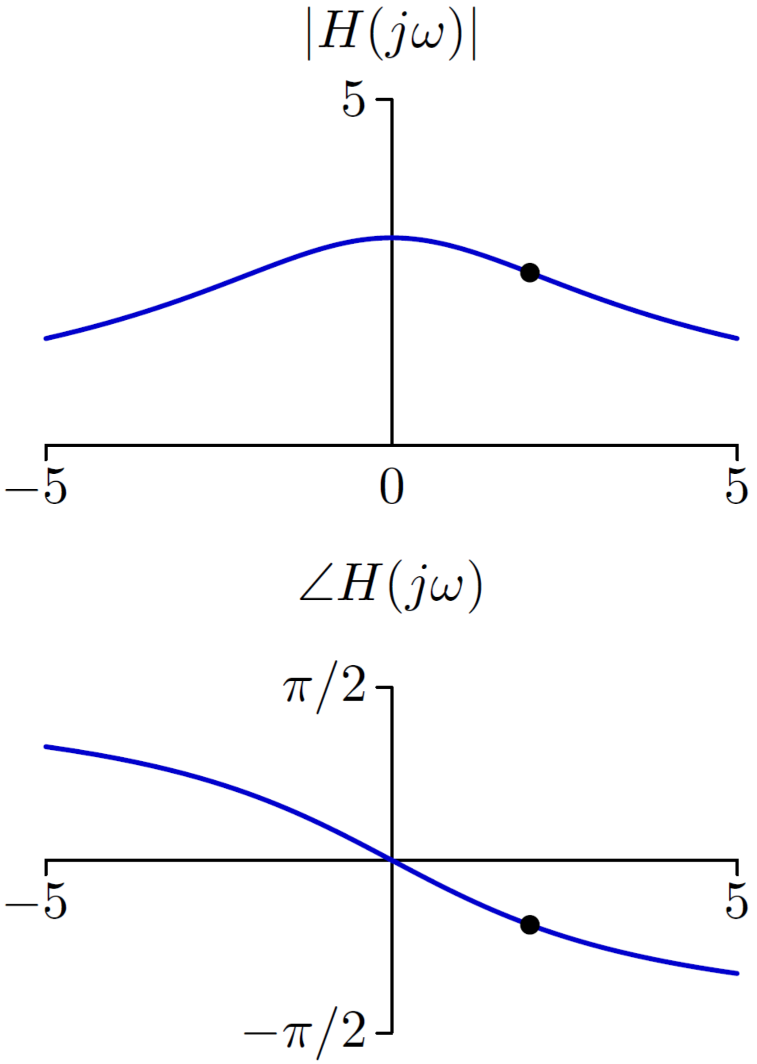


Frequency Response and Bode Plots

Frequency Response: $H(s)|_{s \leftarrow j\omega}$

$$H(s) = \frac{9}{s - p_1}$$





Poles and Zeros

- Frequency response

$$H(j\omega) = H(s)|_{s \leftarrow j\omega}$$

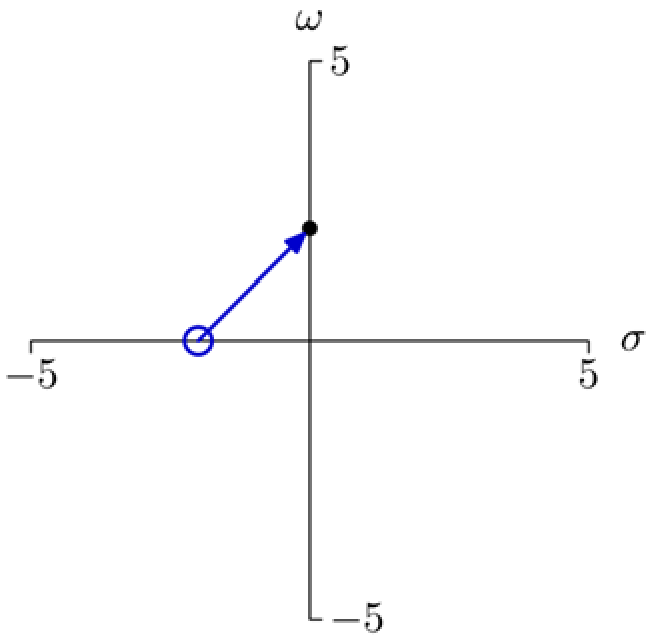
- Thinking about systems as collections of poles and zeros is an important design concept.
  - Simple: just a few numbers characterize entire system
  - Powerful: complete information about frequency response

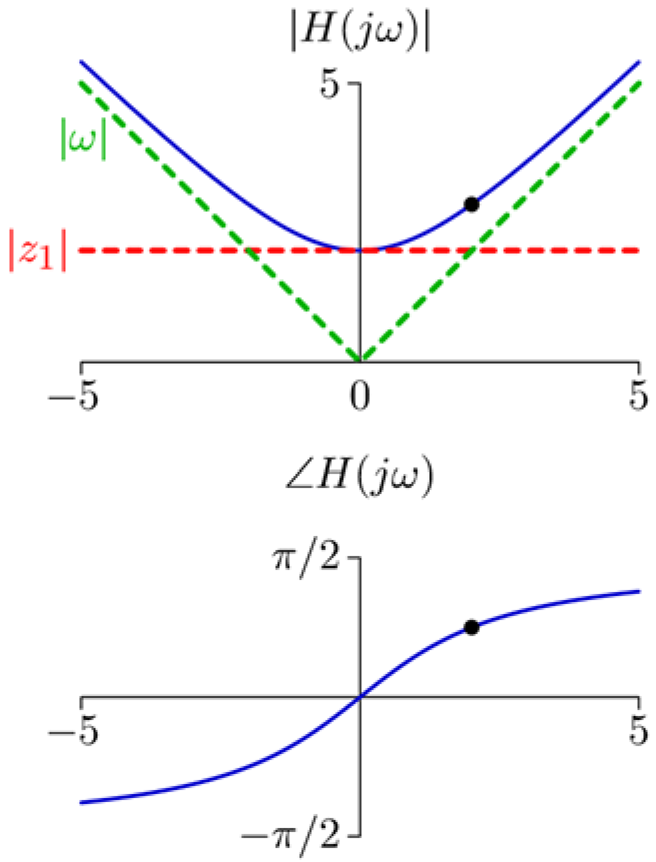
Bode Plots: Magnitude

Asymptotic Behavior: Isolated Zero

- The magnitude response is simple at low and high frequencies

$$H(s) = s - z_1$$

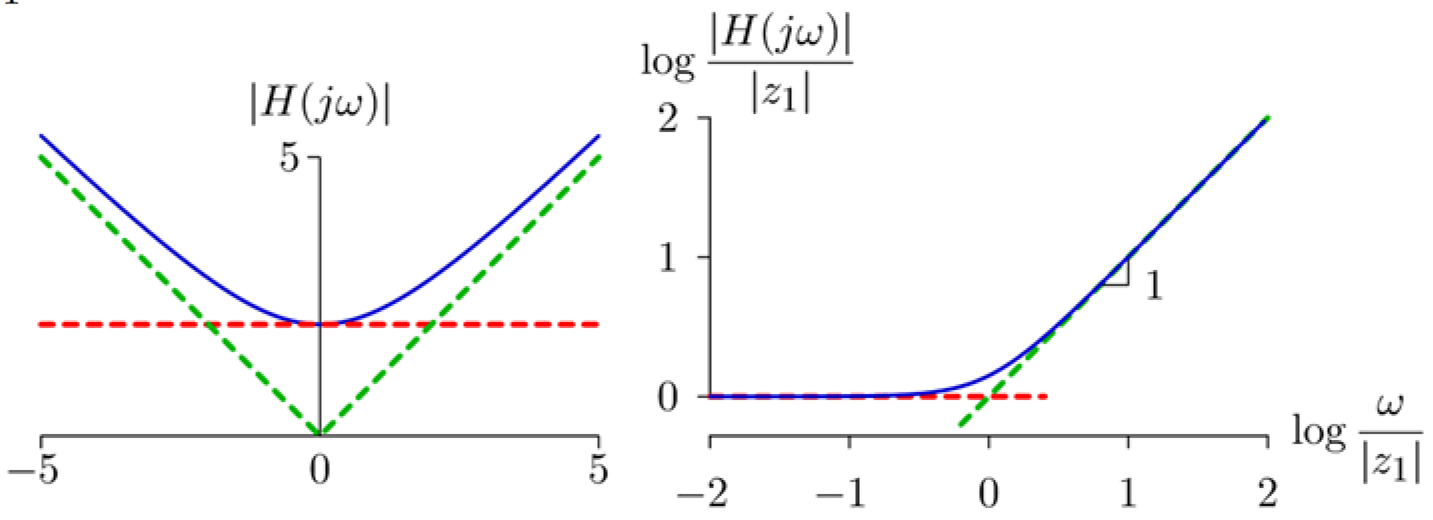




Asymptotic Behavior: Isolated Zero

- Two asymptotes provide a good approximation on log-log axes

$$H(s) = s - z_1$$



$$\lim_{\omega \to 0} |H(j\omega)| = |z_1|$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

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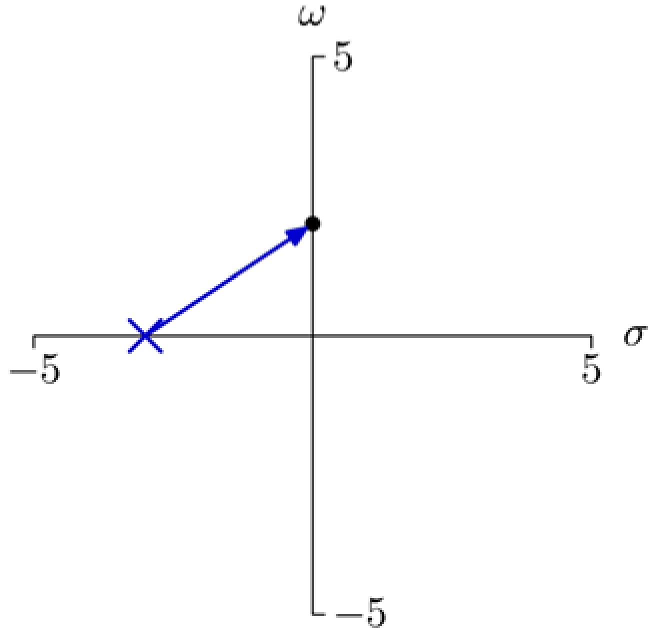
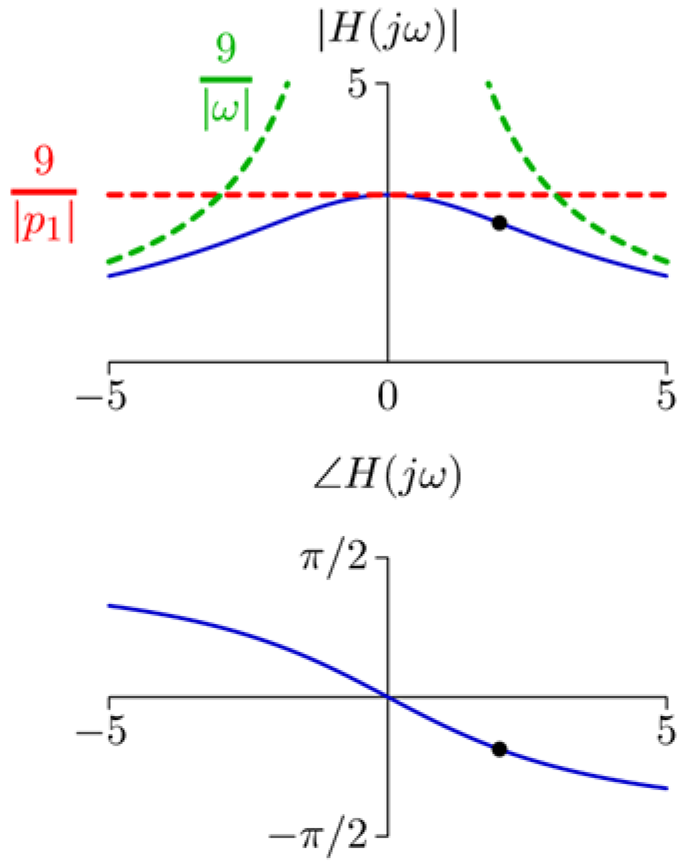
Asymptotic Behavior: Isolated Pole

- The magnitude response is simple at low and high frequencies

$$H(s) = \frac{9}{s - p_1}$$





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Asymptotic Behavior: Isolated Pole

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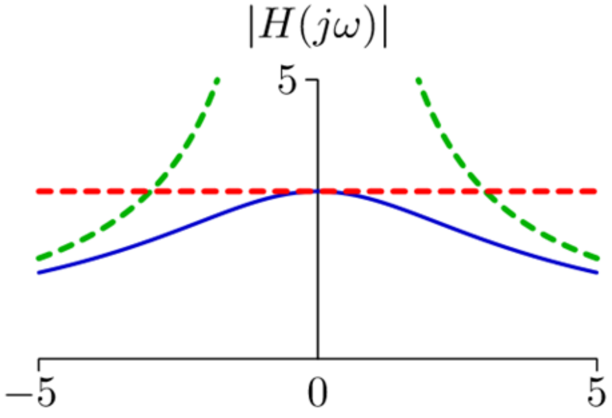

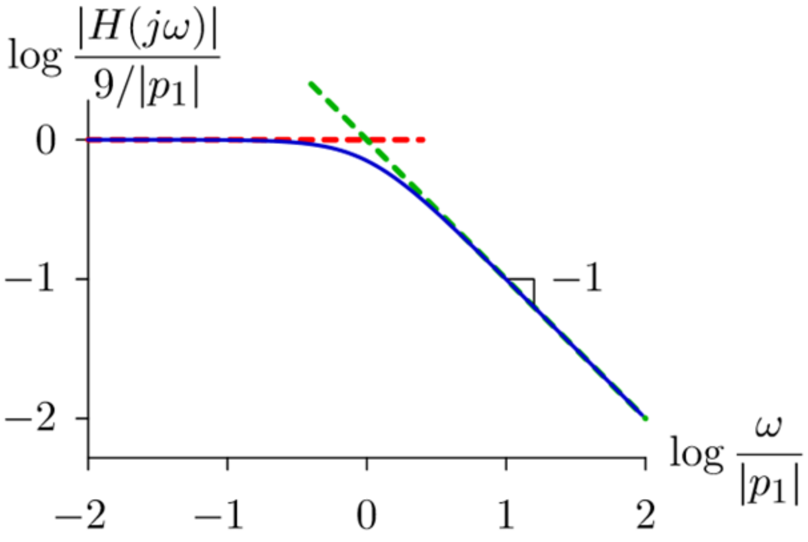

$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{|p_1|}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{9}{\omega}$$

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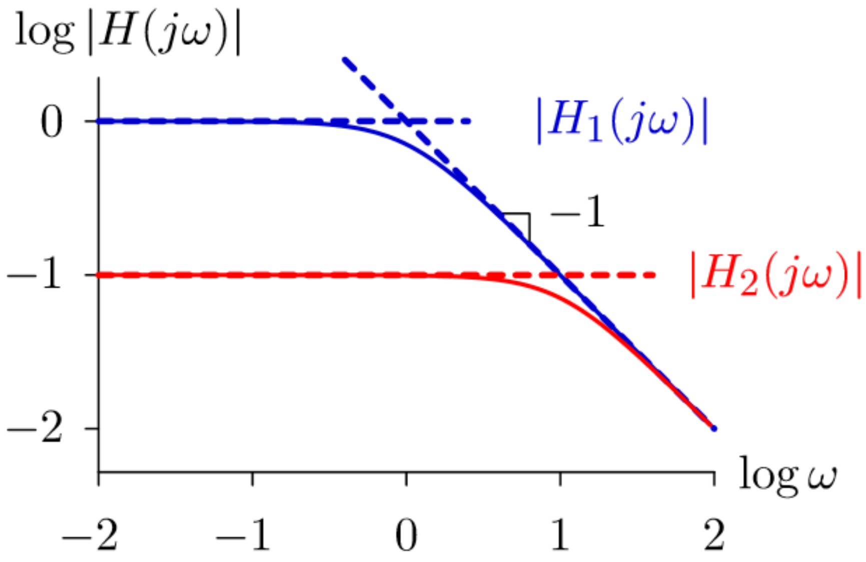
Check Yourself

- Compare log-log plots of the frequency-response magnitudes of the following system functions

$$H_1(s) = \frac{1}{s+1}$$

$$H_2(s) = \frac{1}{s+10}$$



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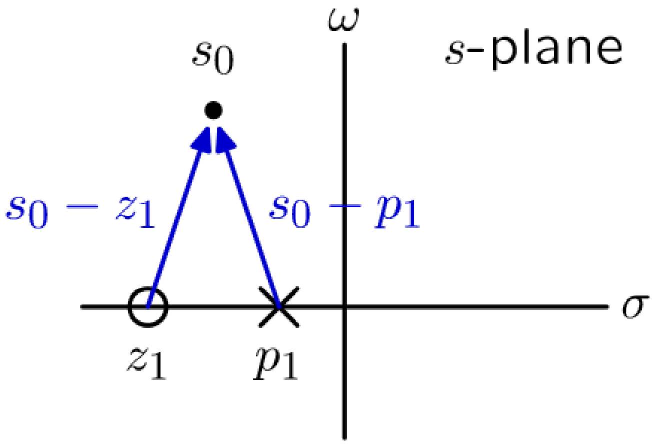
Asymptotic Behavior of More Complicated Systems

- Constructing $H(s_0)$

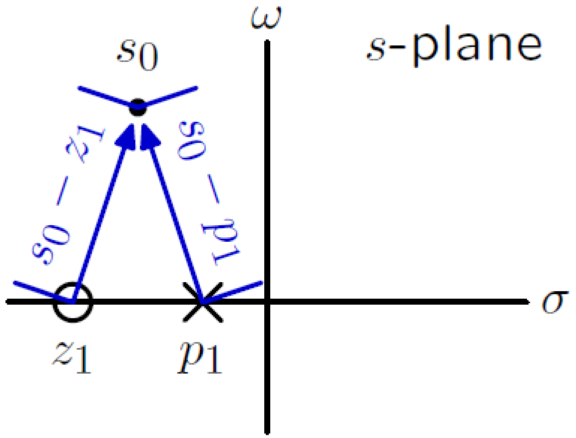
$$H(s_0) = K\frac{\prod_{m=1}^{M}(s_0 - z_m)}{\prod_{n=1}^{N}(s_0 - p_n)}$$



Asymptotic Behavior of More Complicated Systems

- The magnitude of a product is the product of the magnitudes

$$|H(s_0)| = |K| \frac{\prod_{m=1}^{M} |s_0 - z_m|}{\prod_{n=1}^{N} |s_0 - p_n|}$$



- The log of the magnitude is a sum of logs

$$\log|H(j\omega)| = \log|K| + \sum_{m=1}^{M} \log|j\omega - z_m| - \sum_{n=1}^{N} \log|j\omega - p_n|$$

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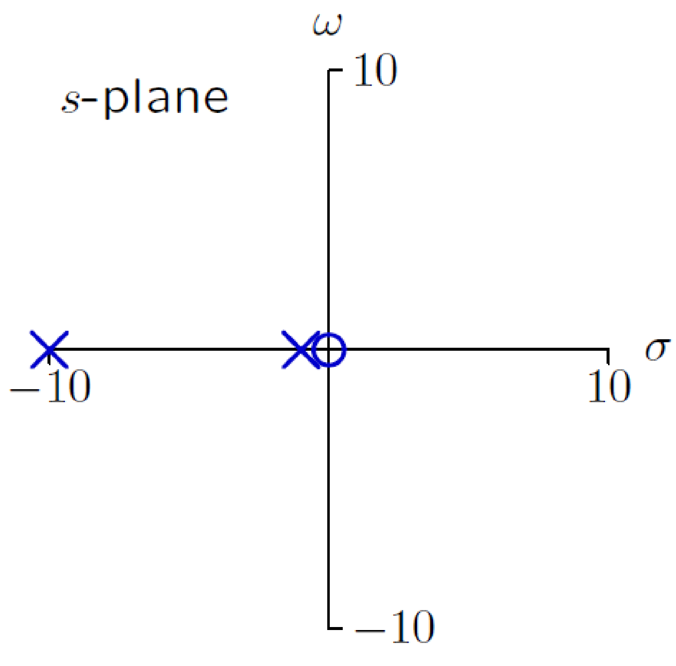
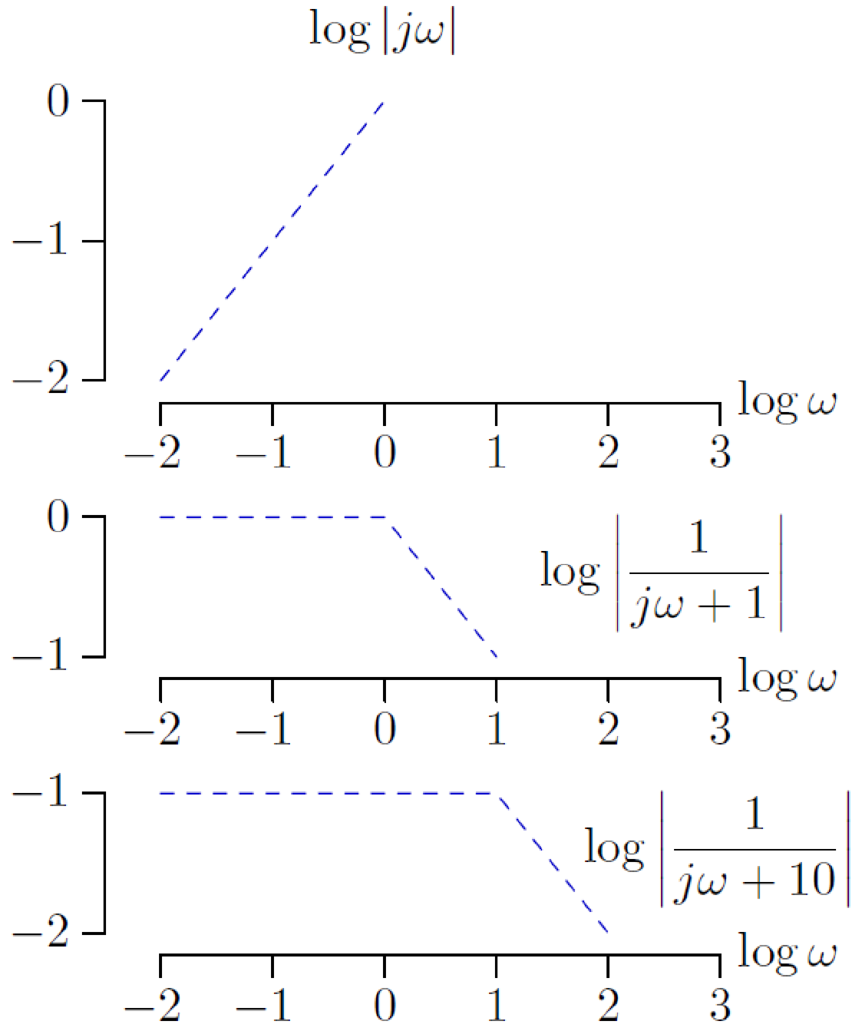
Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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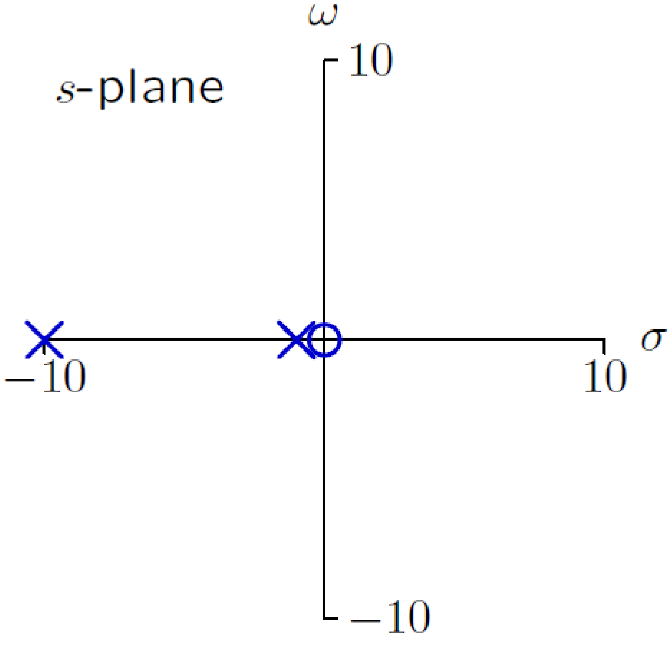

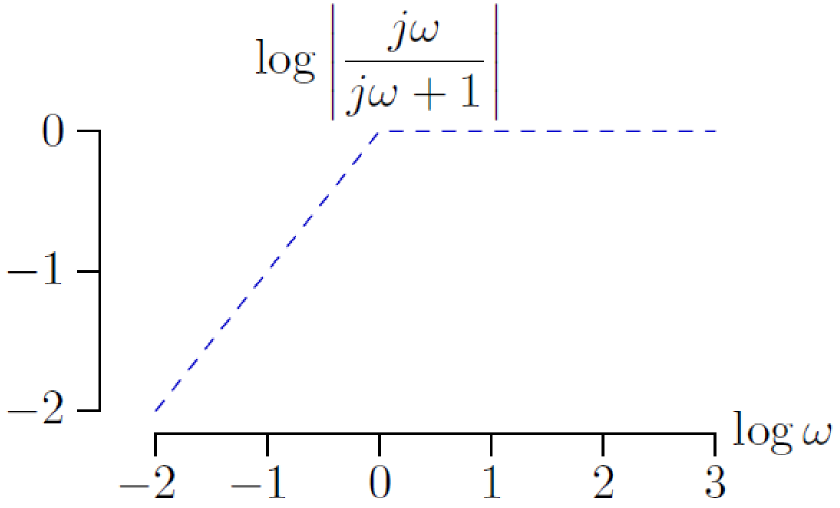

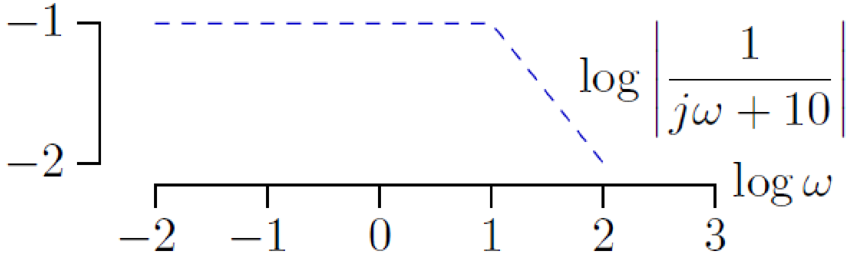

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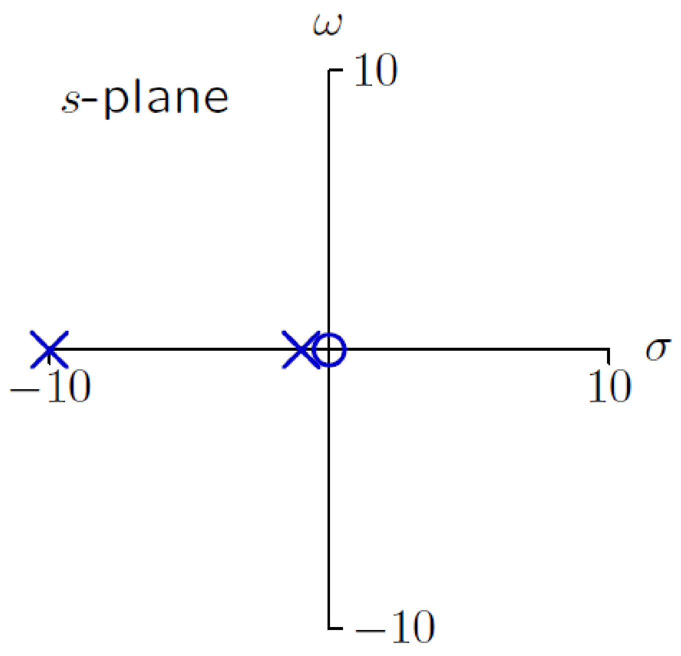

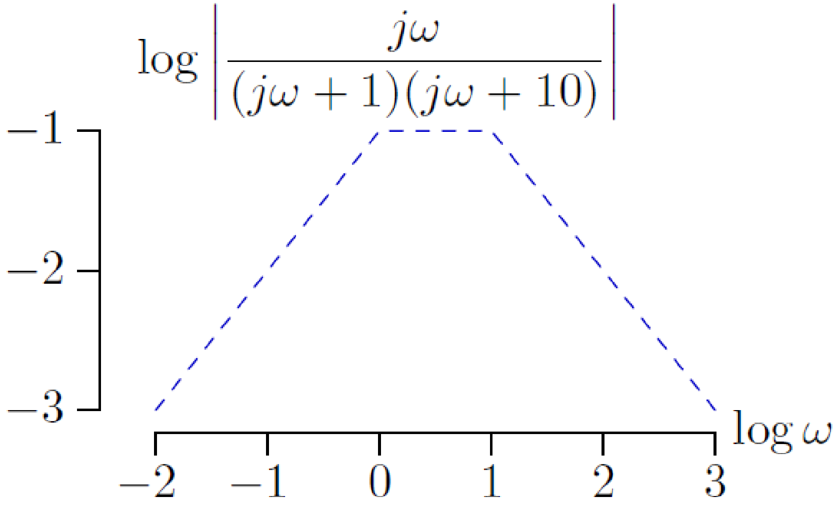

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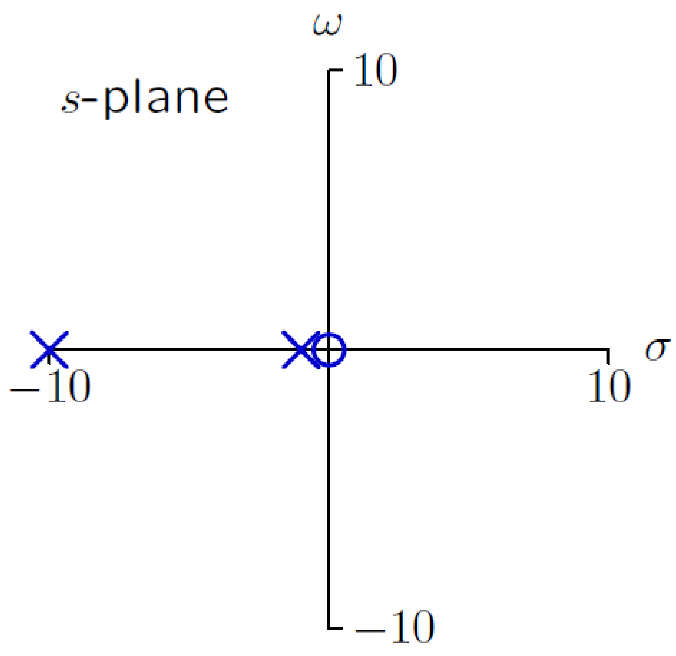
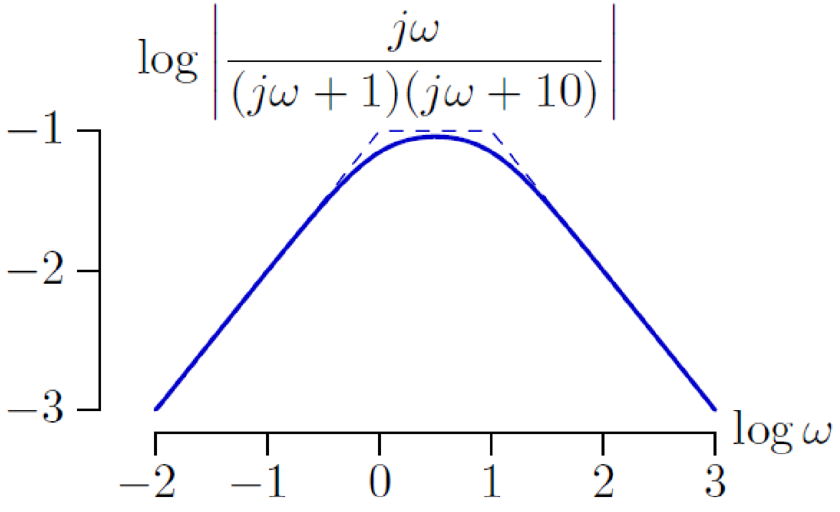


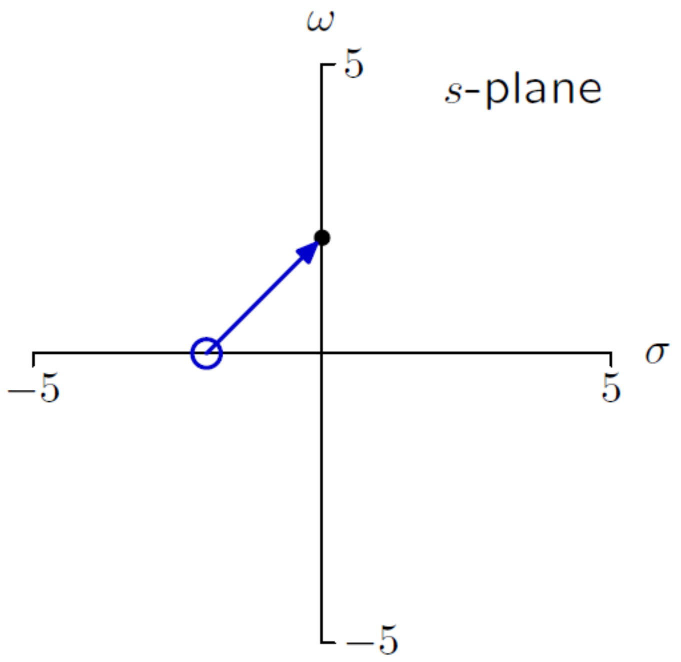


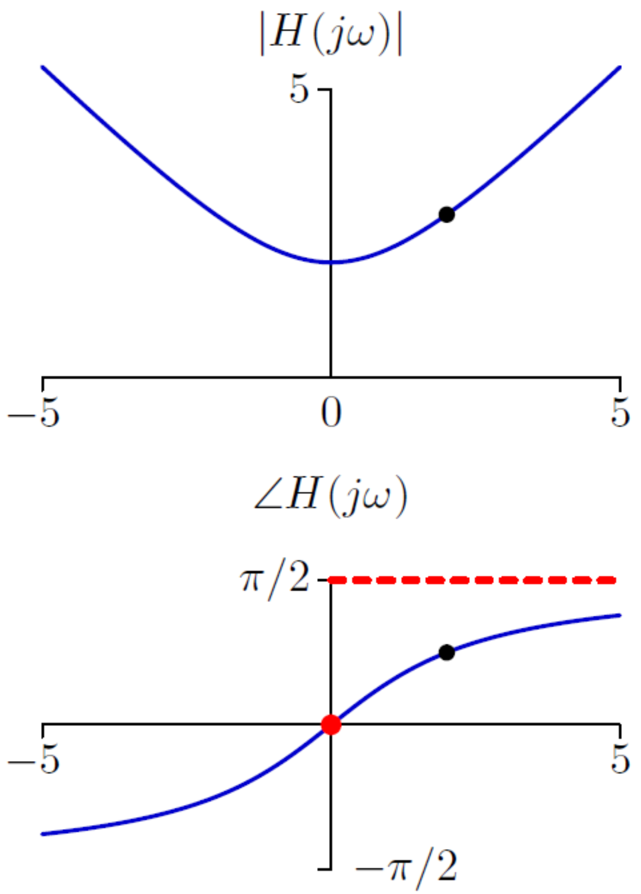
Bode Plots: Angle

Asymptotic Behavior: Isolated Zero

- The angle response is simple at low and high frequencies

$$H(s) = s - z_1$$

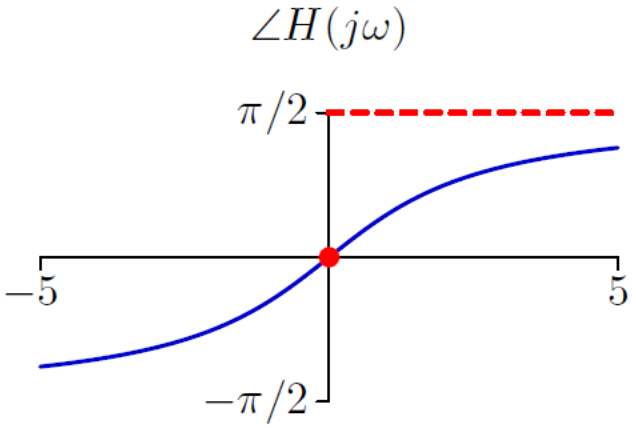


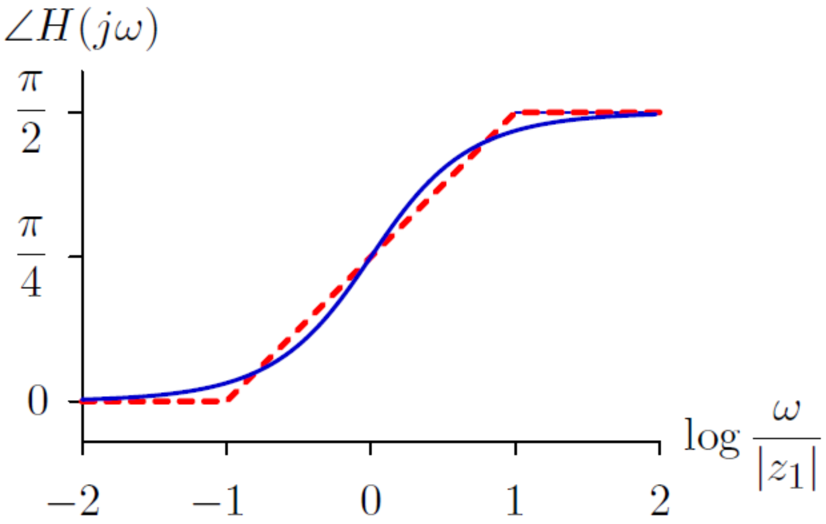


Asymptotic Behavior: Isolated Zero

- Three straight lines provide a good approximation versus log ω

$$H(s) = s - z_1$$





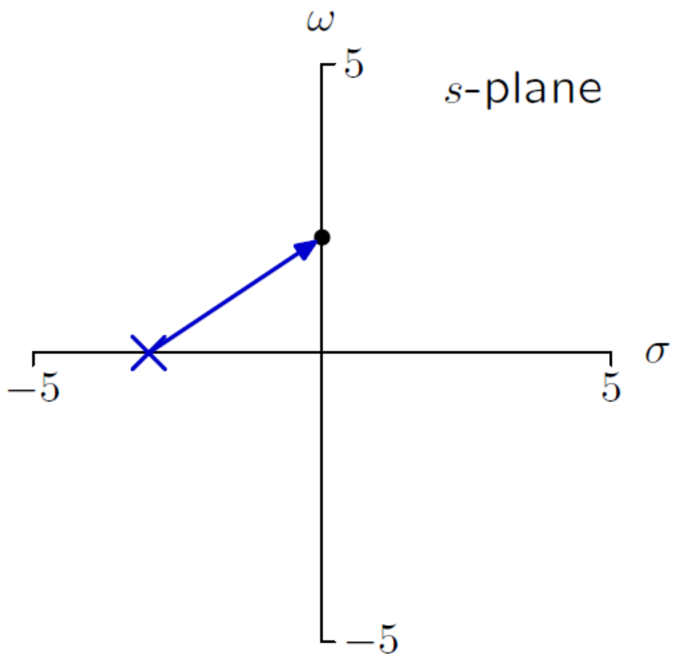
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

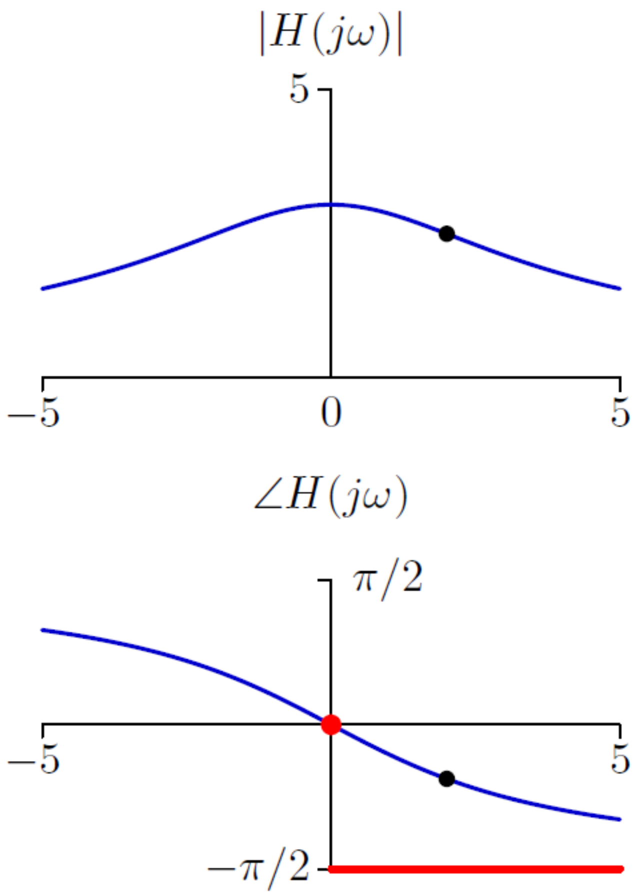
$$\lim_{\omega \to \infty} \angle H(j\omega) = \frac{\pi}{2}$$

Asymptotic Behavior: Isolated Pole

- The angle response is simple at low and high frequencies

$$H(s) = \frac{9}{s - p_1}$$





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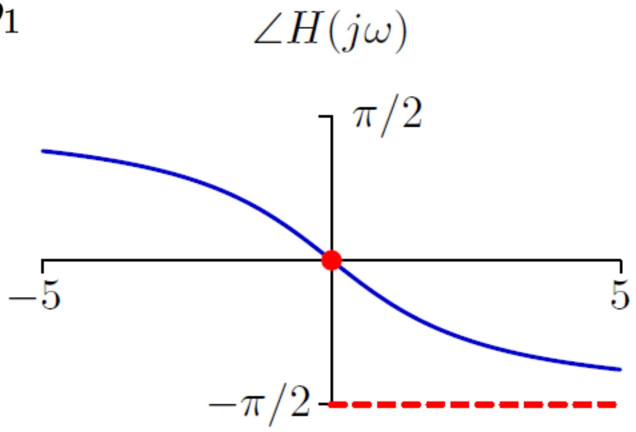
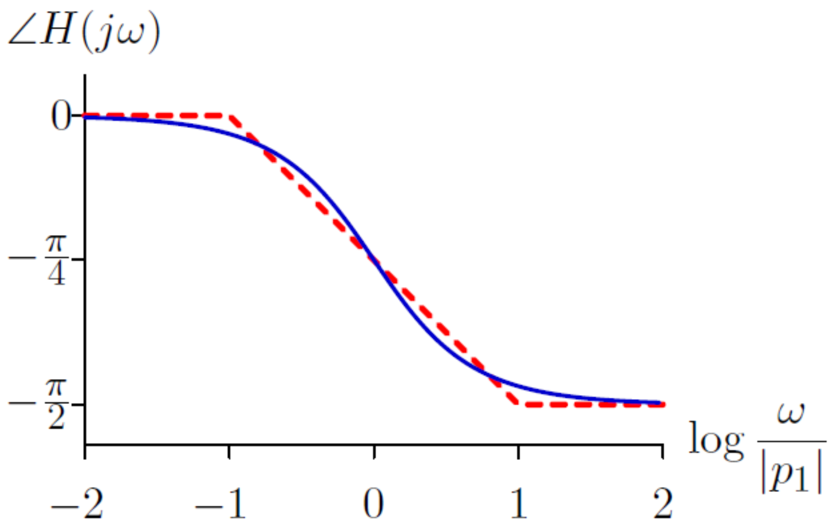




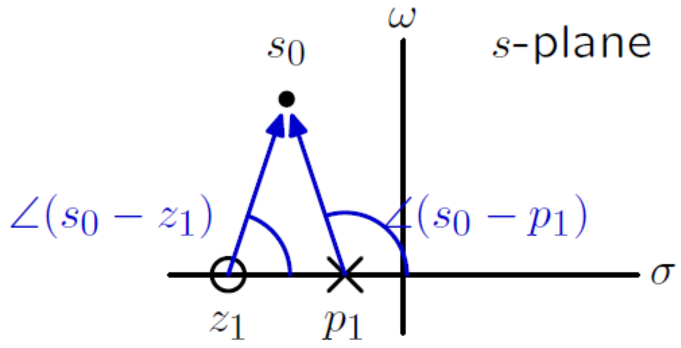
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\frac{\pi}{2}$$

Bode Plot: Adding

- The angle of a product is the sum of the angles

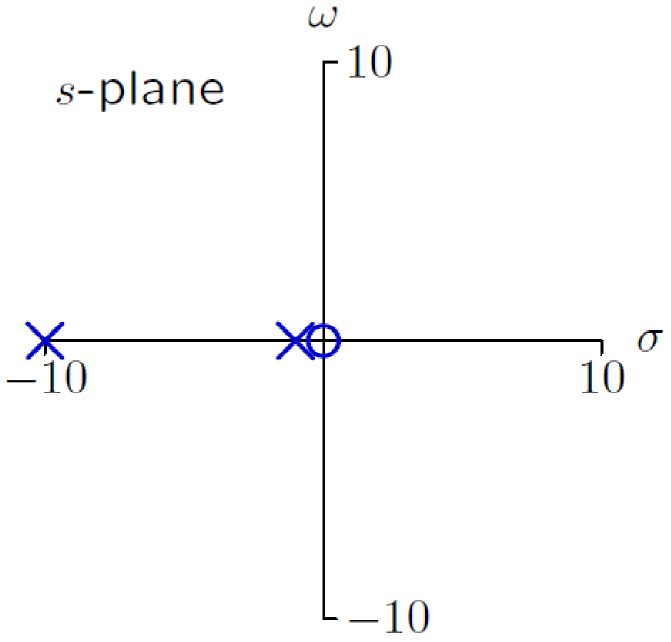
$$\angle H(s_0) = \angle K + \sum_{m=1}^{M} \angle(s_0 - z_m) - \sum_{n=1}^{N} \angle(s_0 - p_n)$$

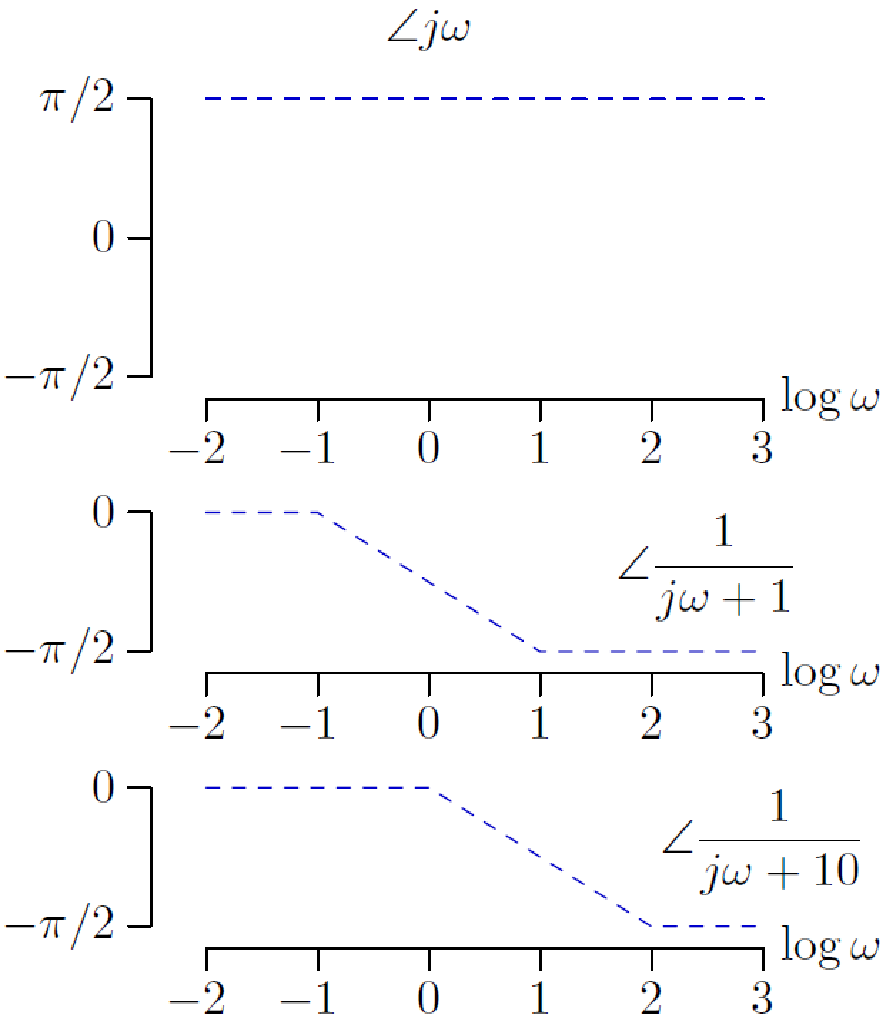


- The angle of K can be 0 or π for systems described by linear differential equations with constant, real-value coefficients

Bode Plot: Adding

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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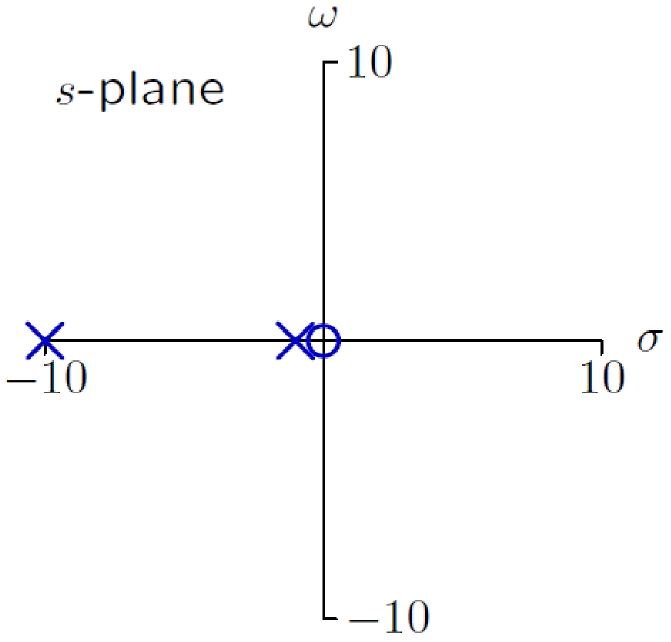
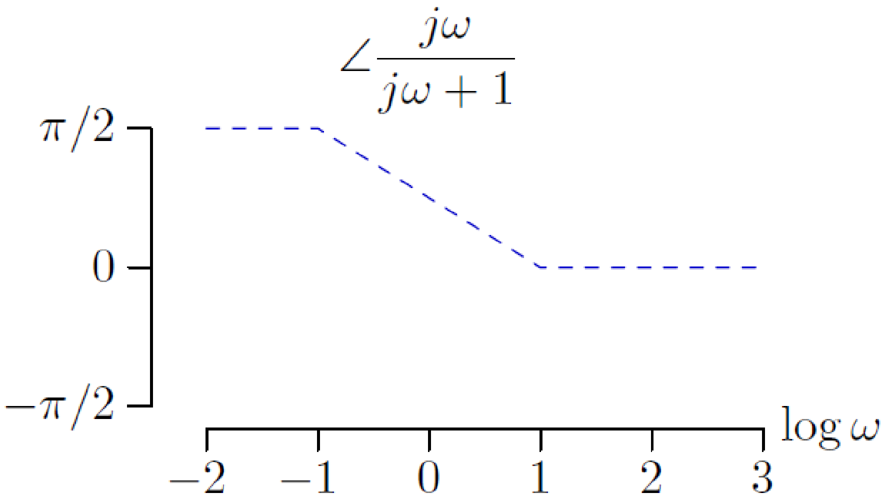
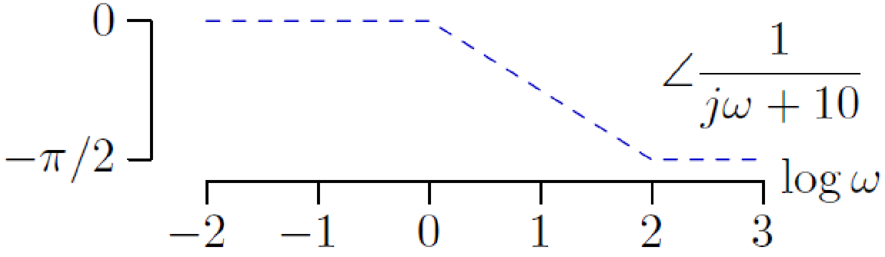
Bode Plot: Adding

$$H(s) = \frac{s}{(s+1)(s+10)}$$

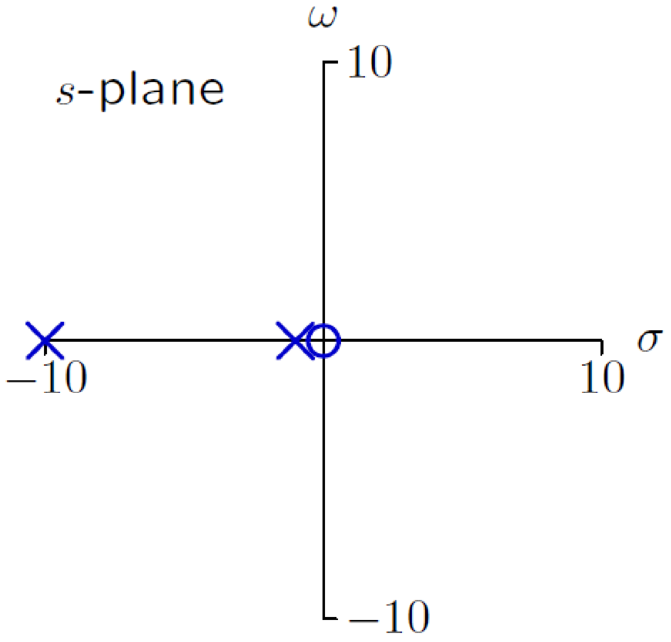


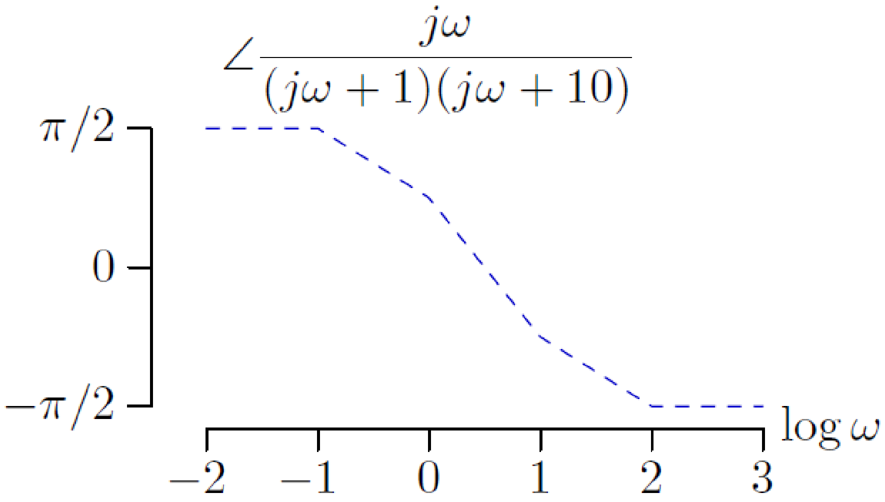




Bode Plot: Adding

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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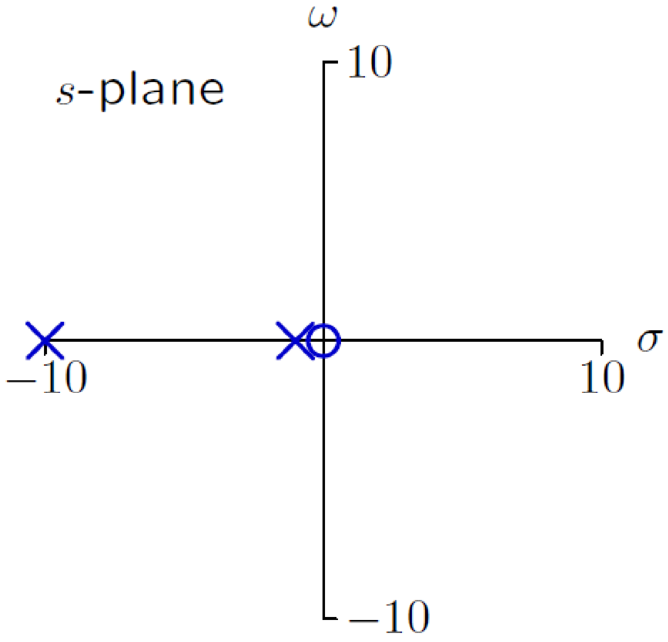
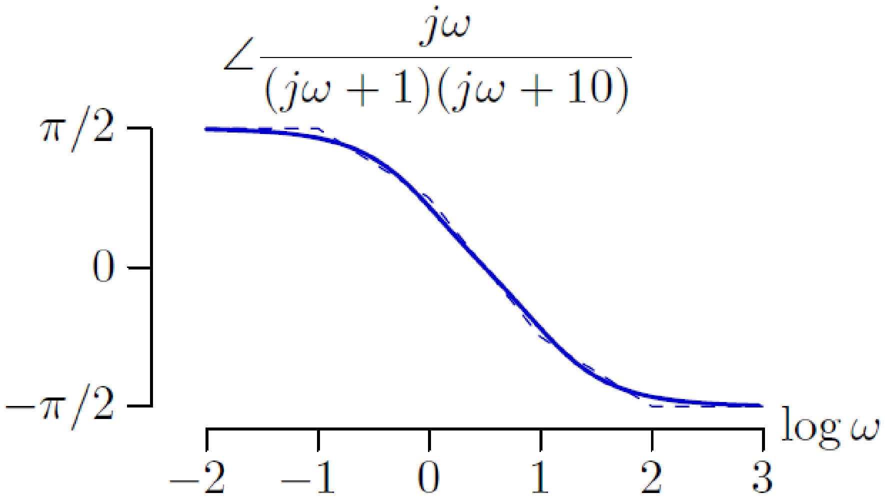




Summary: From Frequency Response to Bode Plot

- The log of the magnitude is a sum of logs

$$\log|H(j\omega)| = \log|K| + \sum_{m=1}^{M} \log|j\omega - z_m| - \sum_{n=1}^{N} \log|j\omega - p_n|$$

- The angle of $H(j\omega)$ is a sum of angles

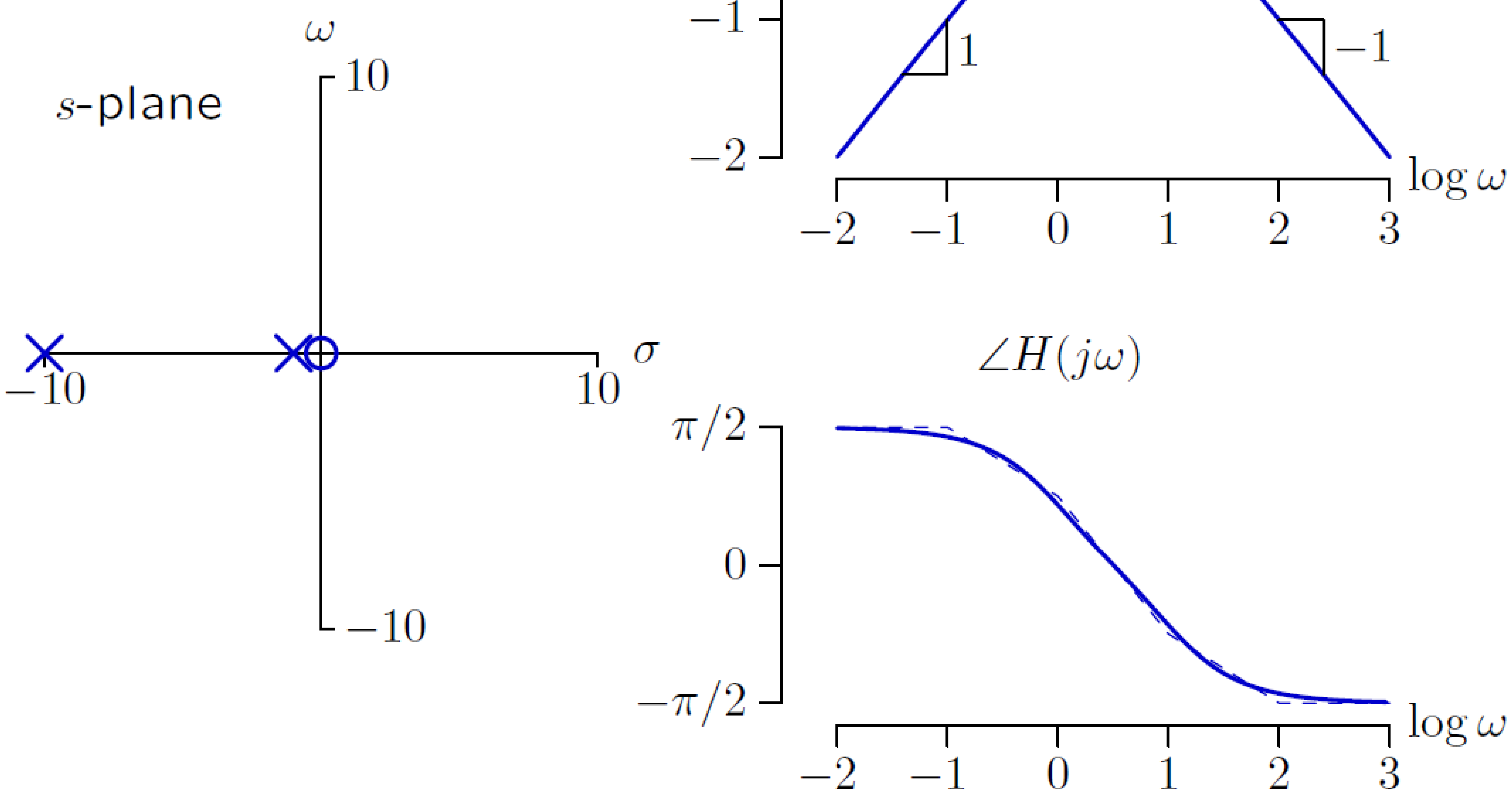
$$\angle H(j\omega) = \angle K + \sum_{m=1}^{M} \angle(j\omega - z_m) - \sum_{n=1}^{N} \angle(j\omega - p_n)$$

- Bode plot (logarithmic plot) : separate plots for magnitude and phase

$$20\log_{10}|H(j\omega)|$$

Bode Plot: dB

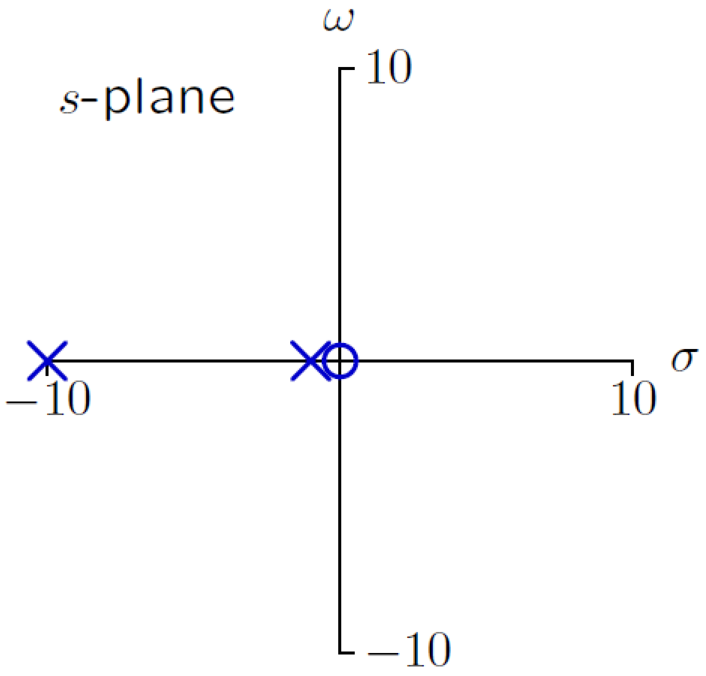
$$H(s) = \frac{10s}{(s+1)(s+10)}$$

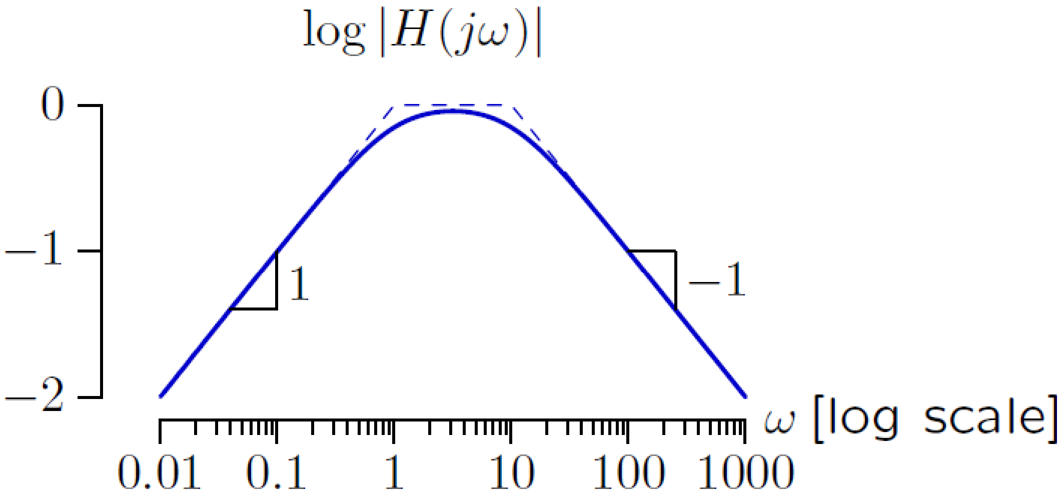


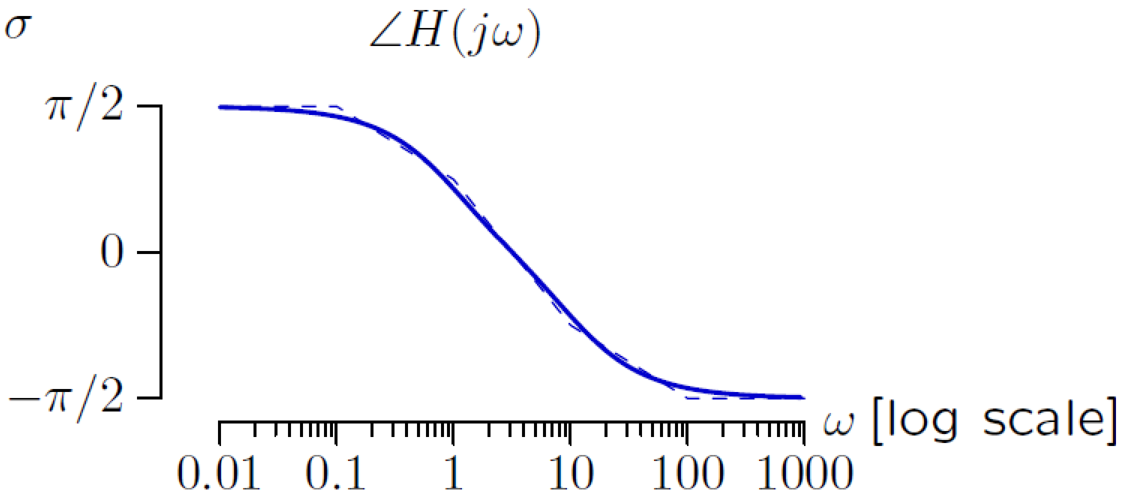
From Signals and Systems (MIT 6.003) by Prof. Denny Freeman

Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$







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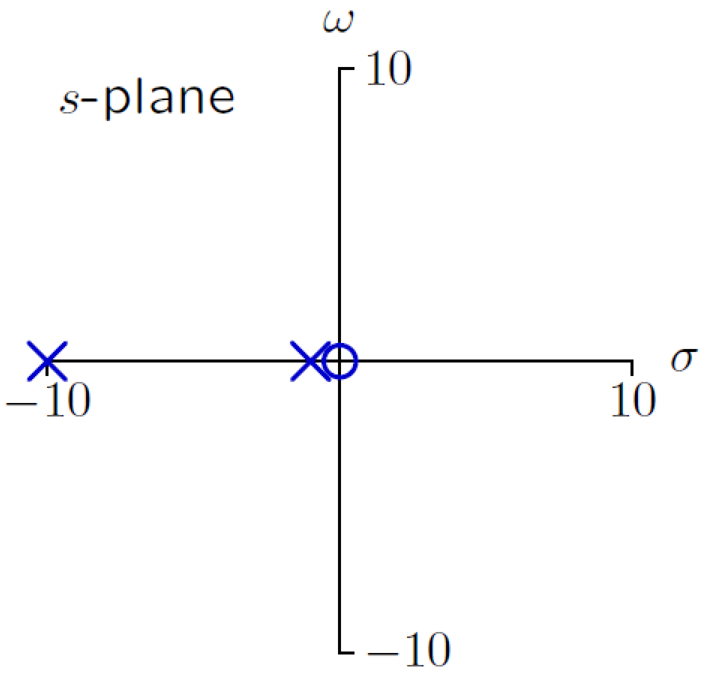
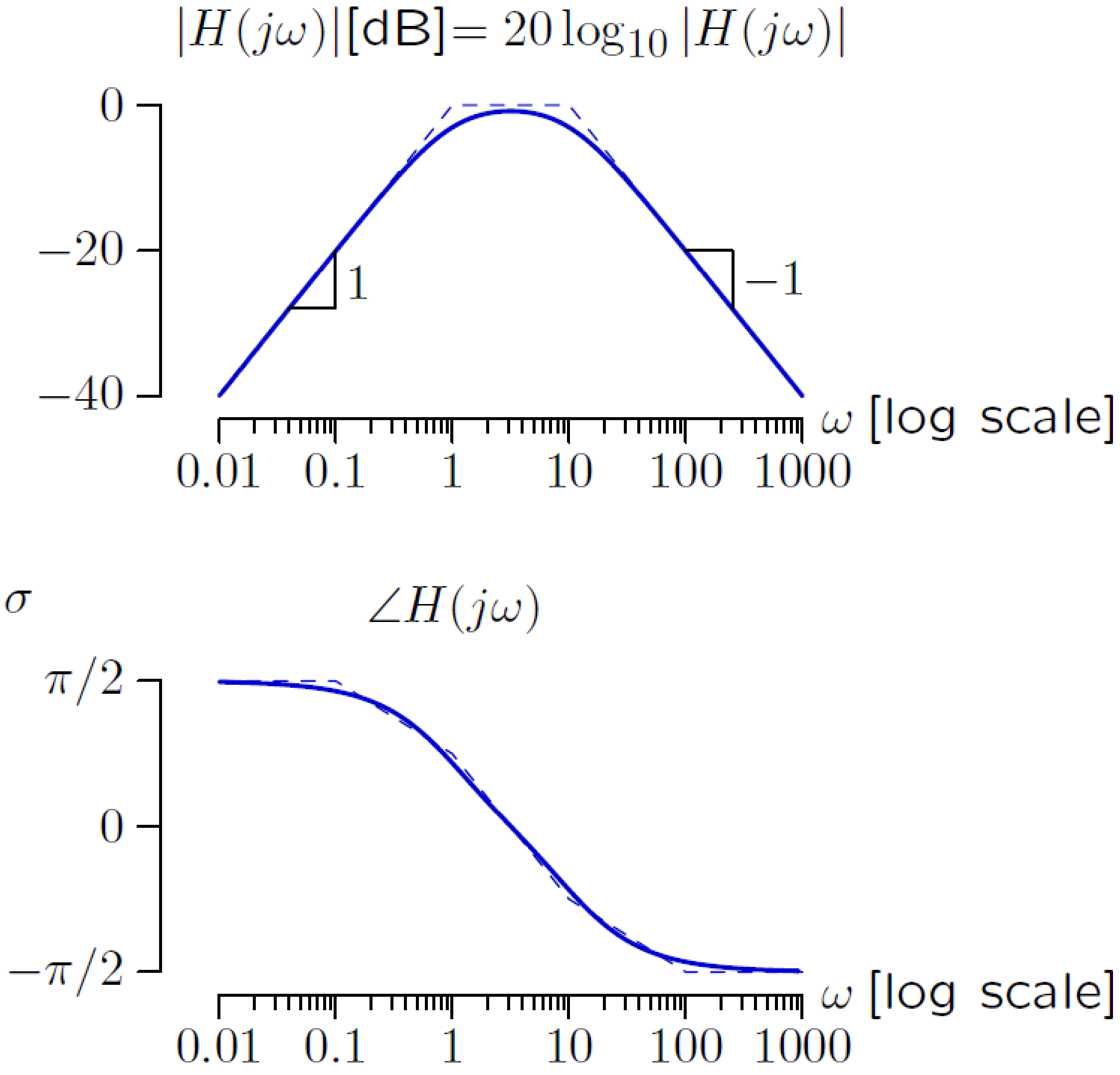
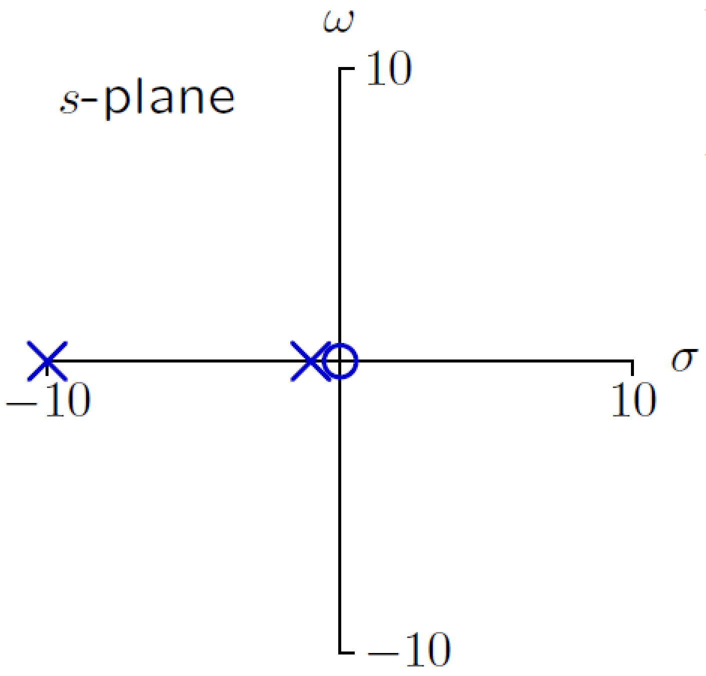
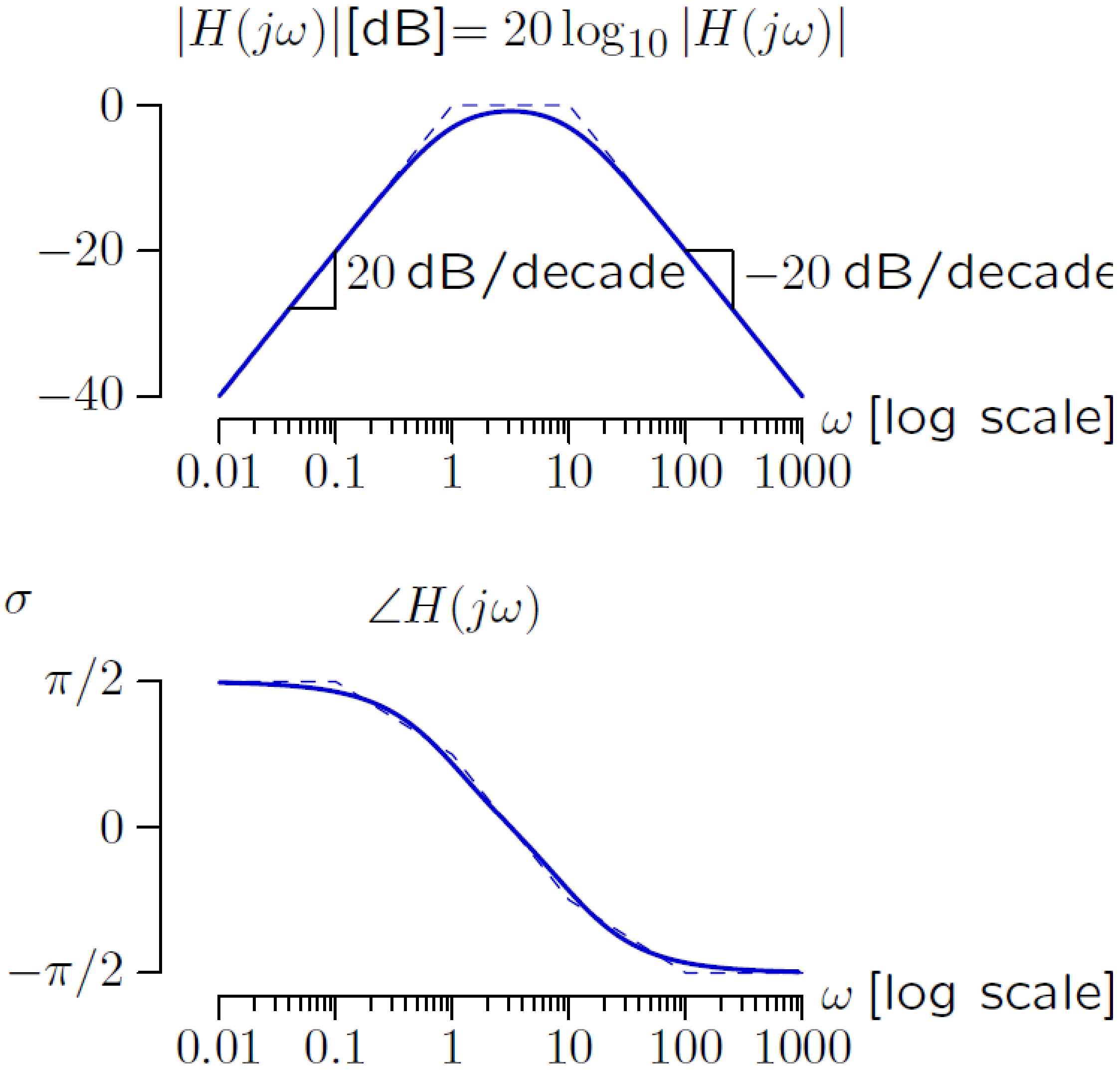




Bode Plot: dB

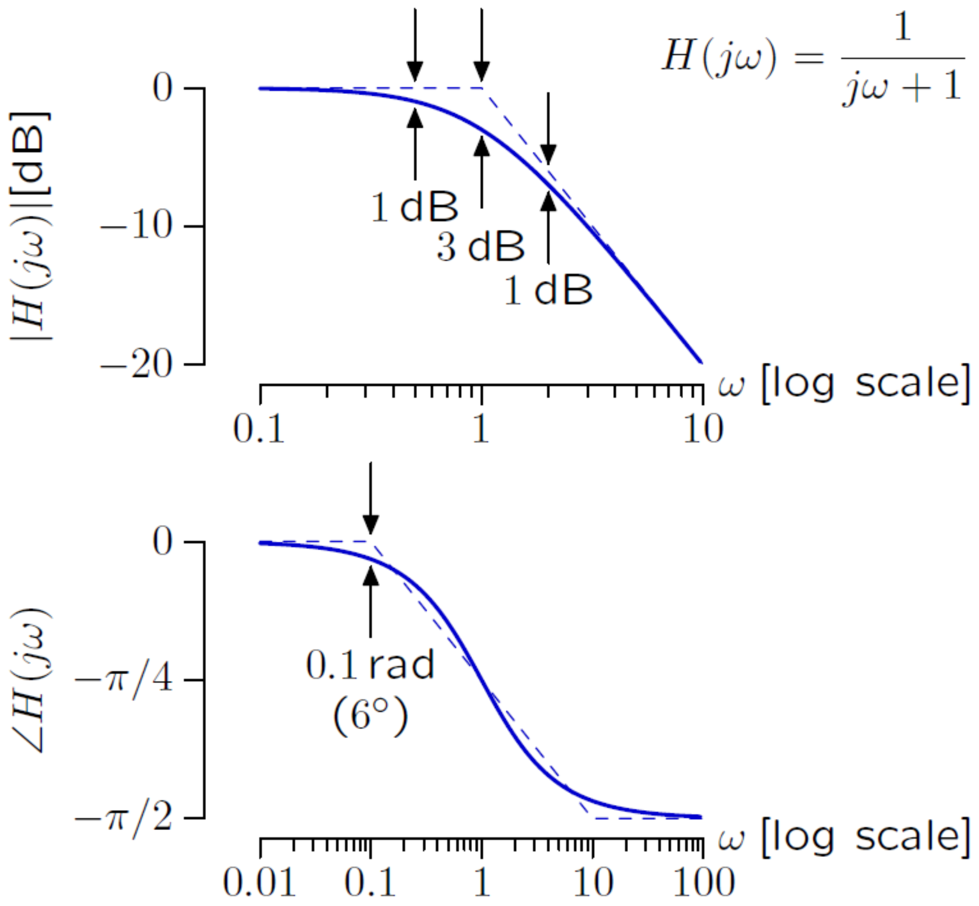
$$H(s) = \frac{10s}{(s+1)(s+10)}$$





Bode Plot: Accuracy

- The straight-line approximations are surprisingly accurate



X	$20 \log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB

How to Draw Bode Plots by Hands

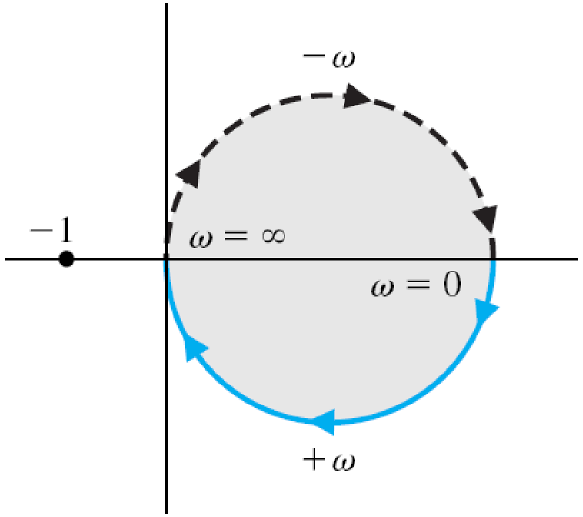
- You should watch the following video clips
  - https://www.youtube.com/watch?v=_eh1conN6YM&index=9&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk
  - https://www.youtube.com/watch?v=CSAp9ooQRT0&index=10&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk
  - https://www.youtube.com/watch?v=E6R2XUEyRy0&index=11&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk
  - https://www.youtube.com/watch?v=O2Cw_4zd-aU&index=12&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk
  - https://www.youtube.com/watch?v=4d4WJdU61Js&index=13&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk
  - https://www.youtube.com/watch?v=GIlx9Yu__y8&index=14&list=PLUMWjy5jgHK1NC52DXXrriwihVrYZKqjk

Frequency Response and Nyquist Plots

Nyquist Plot

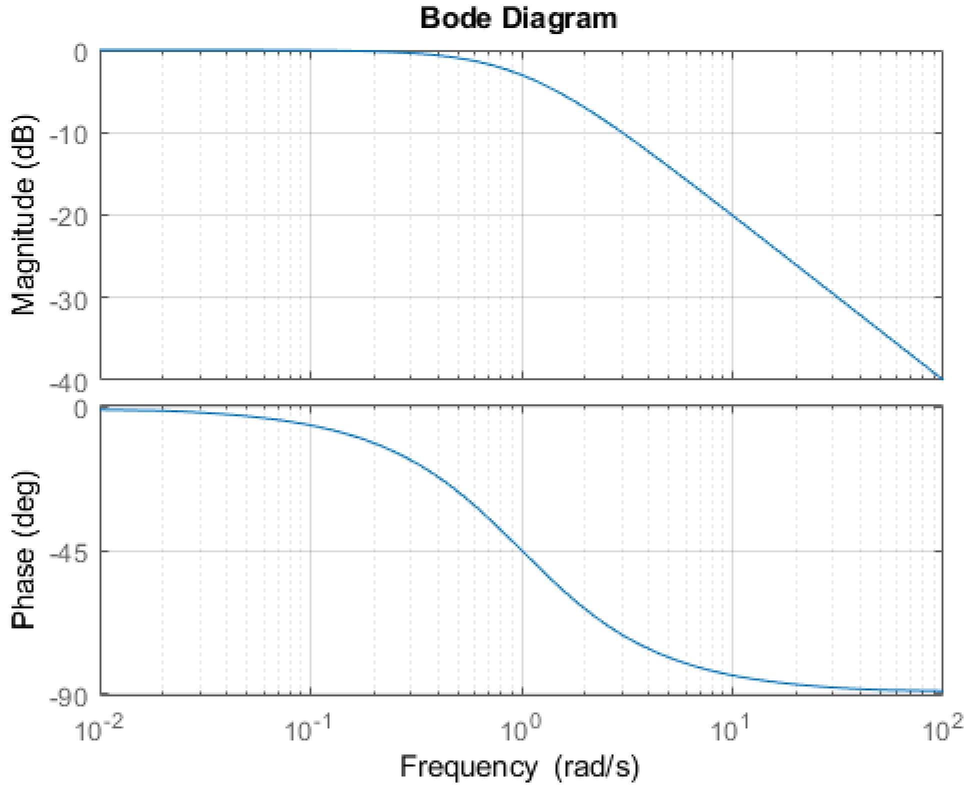
- If we only want a single plot we can use ω as a parameter

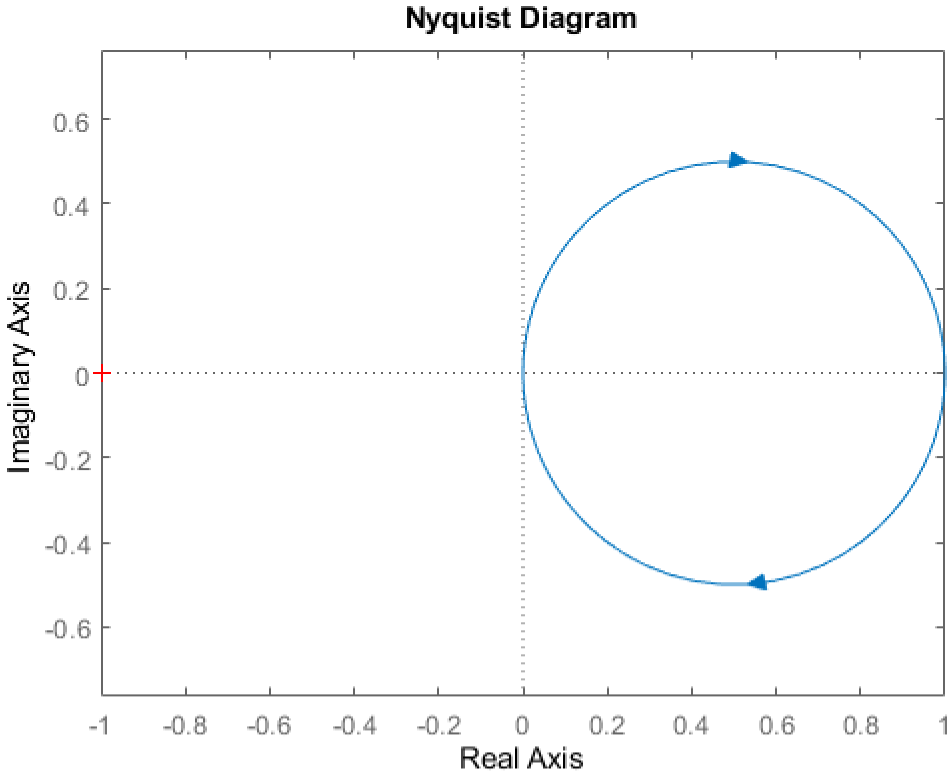
- A plot of $Re\{G(\omega)\}$ vs. $Im\{G(\omega)\}$ as a function of ω
  - Advantage: all information in a single plot



Example: Nyquist Plot

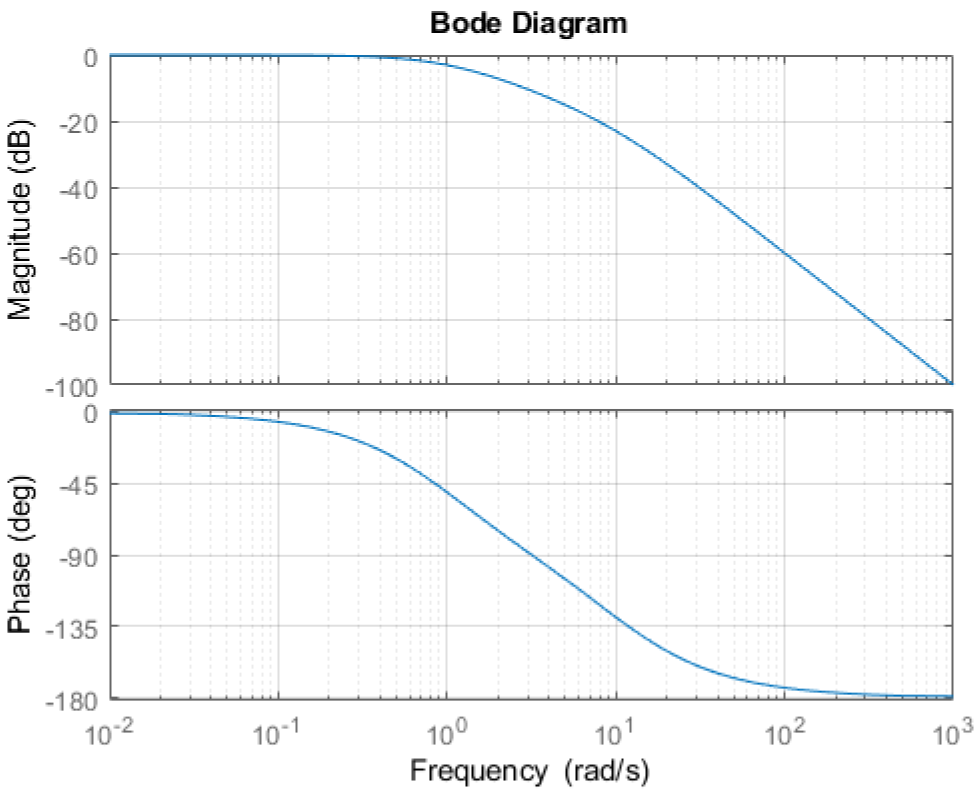
$$G(s) = \frac{1}{s+1}$$

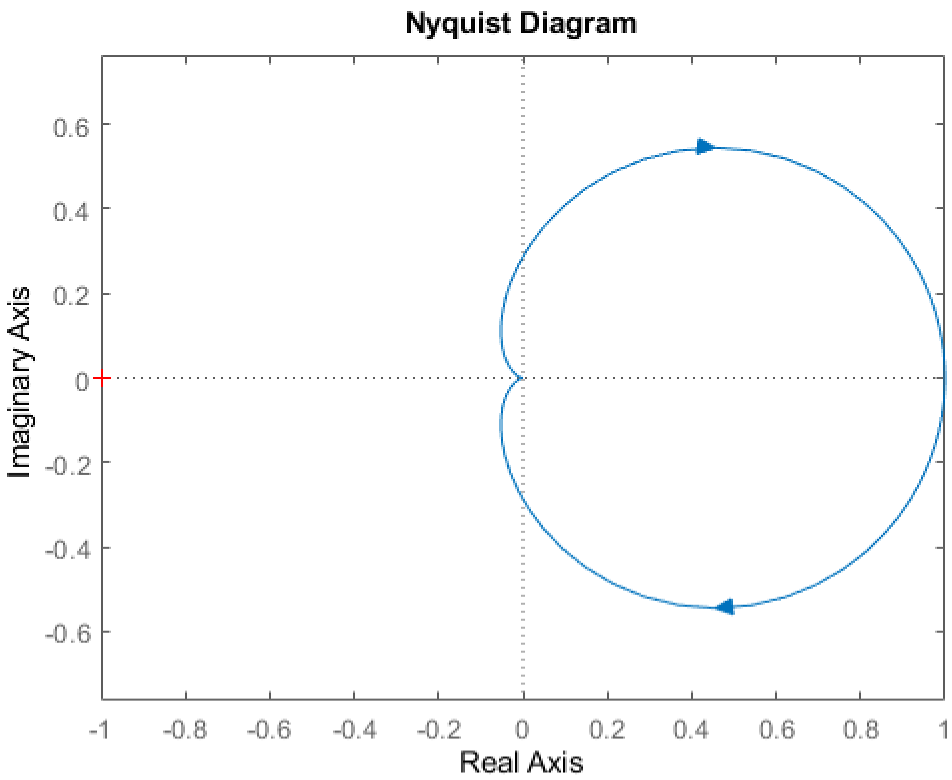




Example: Nyquist Plot

$$G(s) = \frac{1}{(s+1)(0.1s+1)}$$





Example: Nyquist Plot

```
s = tf('s');
G = 1/(s+1)^3;
```

$$G(s) = \frac{1}{(s+1)^3}$$

```
bode(G)
grid on
```

```
nyquist(G)
axis equal
```