Forced Response

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Outline

- LTI Systems

- Time Response to Constant Input
- Time Response to Singularity Function Inputs
- Response to General Inputs (in Time)

- Response to Sinusoidal Input (in Frequency)
- Response to Periodic Input (in Frequency)
- Response to General Input (in Frequency)

- Fourier Transform

Linear Time-Invariant (LTI) Systems

Systems

- H is a transformation (a rule or formula) that maps an input signal $x(t)$ into a time output signal $y(t)$

$$x \longrightarrow \boxed{H} \longrightarrow y$$

- System examples

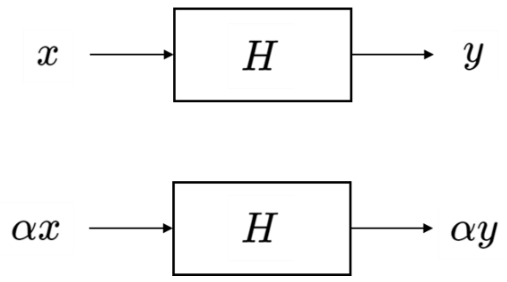
Identity	$y(t) = x(t)$	$\forall t$
Scaling	$y(t) = 2\,x(t)$	$\forall t$
Offset	$y(t) = x(t) + 2$	$\forall t$
Square signal	$y(t) = (x(t))^2$	$\forall t$
Shift	$y(t) = x(t+2)$	$\forall t$
Decimate	$y(t) = x(2t)$	$\forall t$
Square time	$y(t) = x(t^2)$	$\forall t$

Linear Systems

- A system H is linear if it satisfies the following two properties:

- Scaling

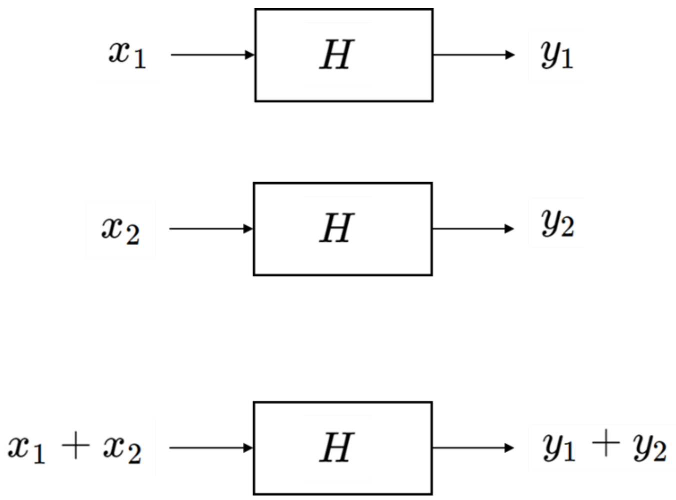
$$H\{\alpha x\} = \alpha H\{x\} \quad \forall\, \alpha \in \mathbb{C}$$



- Additivity

$$\text{If } y_1 = H\{x_1\} \text{ and } y_2 = H\{x_2\}$$

$$\text{then } H\{x_1 + x_2\} = y_1 + y_2$$



Time-Invariant Systems

- A system H processing infinite-length signals is time-invariant (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$x(t-s) \longrightarrow \boxed{H} \longrightarrow y(t-s)$$

Linear Time-Invariant (LTI) Systems

- We will only consider Linear Time-Invariant (LTI) systems

- Examples

Identity	$y(t) = x(t)$	$\forall t$	Linear	Time Invariant
Scaling	$y(t) = 2\,x(t)$	$\forall t$	Linear	Time Invariant
Offset	$y(t) = x(t) \; + \; 2$	$\forall t$	Non Linear	Time Invariant
Square signal	$y(t) = (x(t))^2$	$\forall t$	Non Linear	Time Invariant
Shift	$y(t) = x(t+2)$	$\forall t$	Linear	Time Invariant
Decimate	$y(t) = x(2t)$	$\forall t$	Linear	Time Variant
Square time	$y(t) = x(t^2)$	$\forall t$	Linear	Time Variant

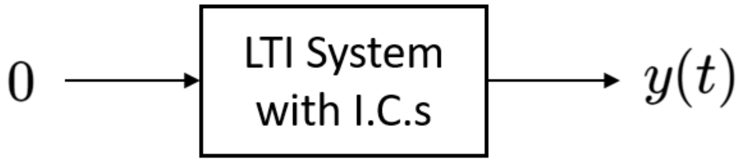
Time Response to Constant Input

Natural Response

- So far, natural response of zero input with non-zero initial conditions are examined

$$\dot{y} + \frac{1}{\tau} y = 0$$

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = 0$$



Response to Non-Zero Constant Input

$$\text{constant} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

- Assume all the systems are stable
- Inhomogeneous ODE

$$\dot{y} + \frac{1}{\tau}y = 0 \qquad \Longrightarrow \qquad \dot{y} + \frac{1}{\tau}y = q'$$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \qquad \qquad \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = p'$$

- Same dynamics, but it reaches different steady state
- Good enough to sketch

$$(\dot{y} - \dot{q}) + \frac{1}{\tau}(y - q) = 0$$

$$(\ddot{y} - \ddot{p}) + 2\zeta\omega_n(\dot{y} - \dot{p}) + \omega_n^2(y - p) = 0$$

Response to Non-Zero Constant Input

- Dynamic system response = transient + steady state

- Transient response is present in the short period of time immediately after the system is turned on
  - It will die out if the system is stable

- The system response in the long run is determined by its steady state component only

- In steady state, all the transient responses go to zero

$$\frac{dy}{dt} \to 0$$

Example

- Example

$$\dot{y} + \frac{1}{\tau} y = q, \qquad y(0) = 0$$

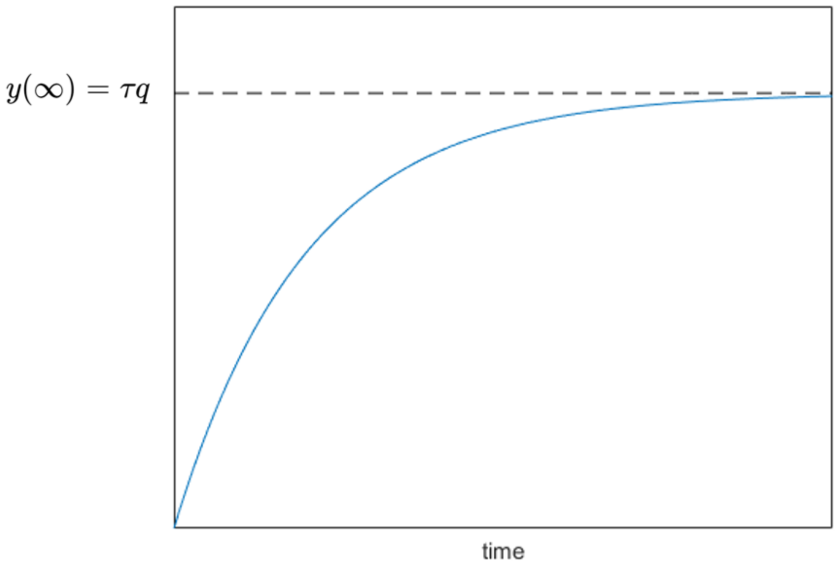
$$\dot{y}(\infty) = 0$$

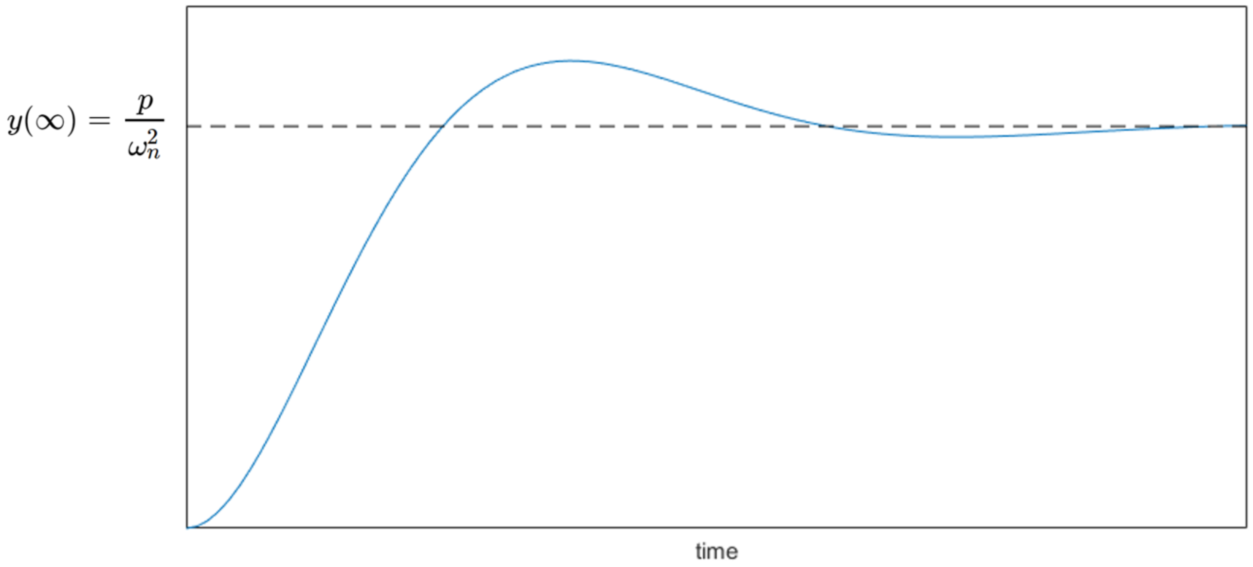
$$\implies \quad y(\infty) = \tau q$$

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = p, \qquad y(0) = 0,\ \dot{y}(0) = 0$$

$$\ddot{y}(\infty) = \dot{y}(\infty) = 0$$

$$\implies \quad y(\infty) = \frac{p}{\omega_n^2}$$



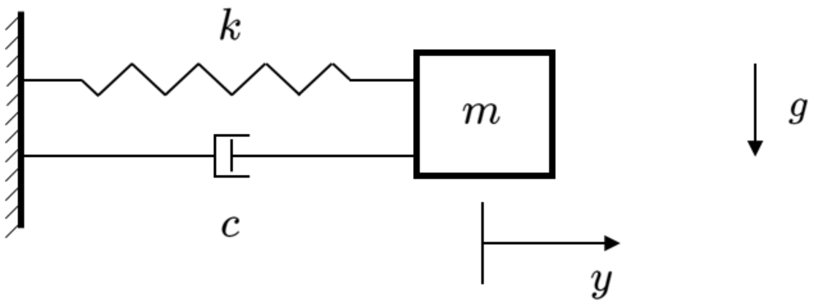


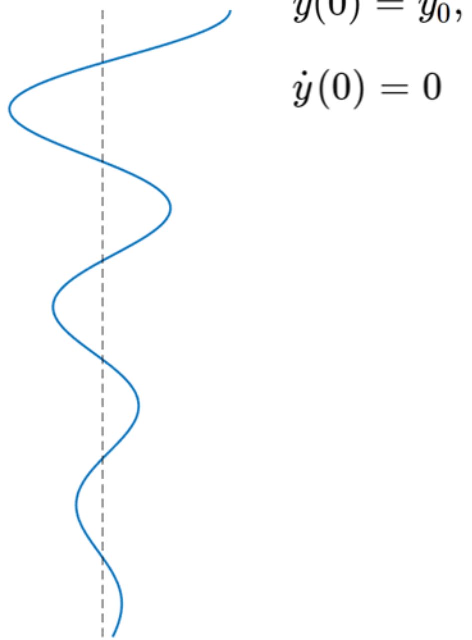
Response to Non-Zero Constant Input

- Think about mass-spring-damper system in horizontal setting

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0,$$

where $y(0) = y_0, \; \dot{y}(0) = 0$





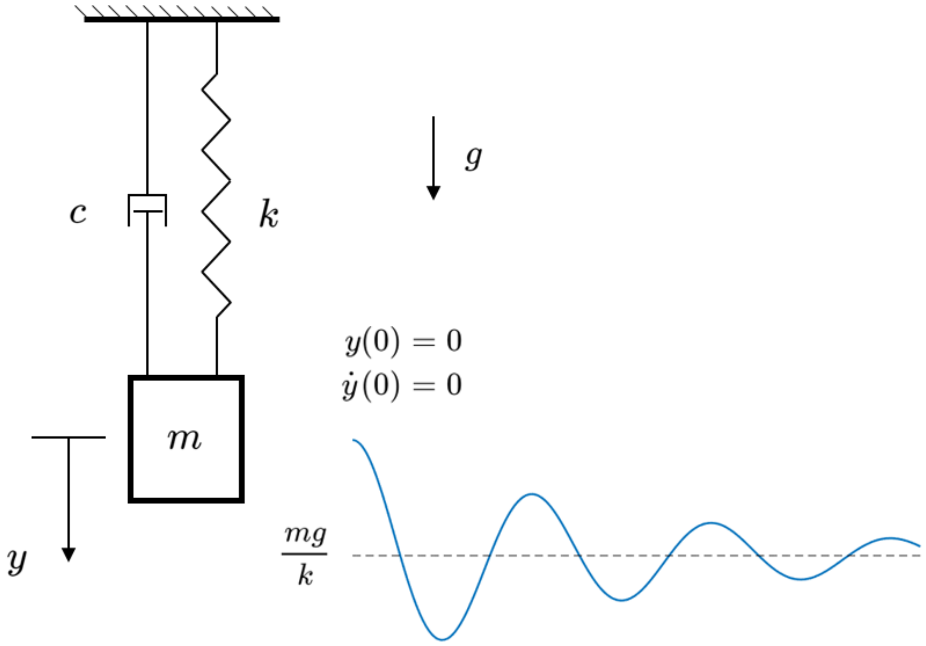
Response to Non-Zero Constant Input

- Mass-spring-damper system in vertical setting
- $y(0) = 0$ no initial displacement
- $\dot{y}(0) = 0$ initially at rest

$$m\ddot{y} + c\dot{y} + ky = mg$$

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = g,$$

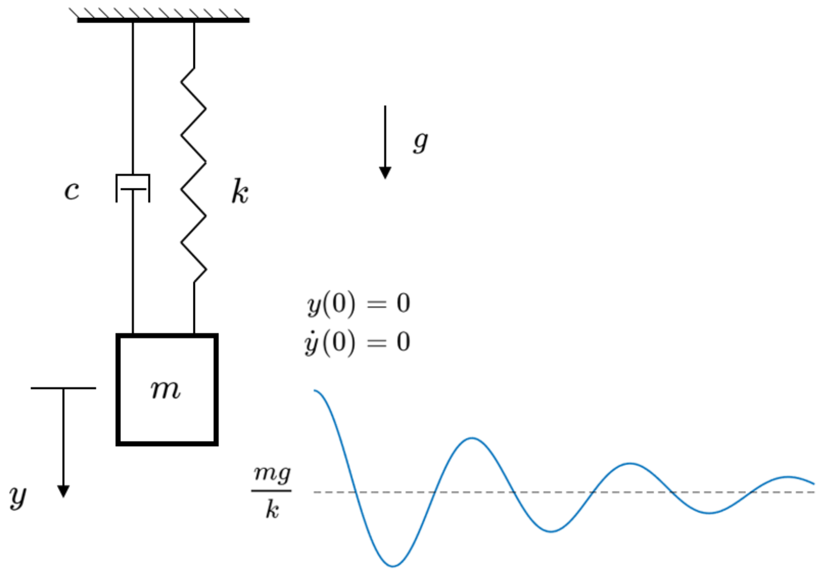
$$\text{where } y(0) = 0, \ \dot{y}(0) = 0$$

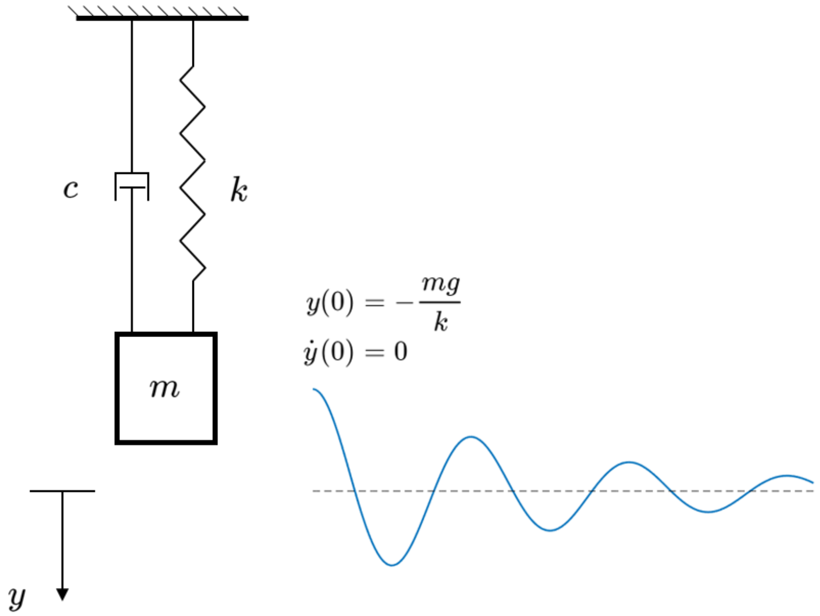


Response to Non-Zero Constant Input

- Shift the origin of y axis to the static equilibrium point, then act like a natural response with
- $y(0) = -\frac{mg}{k}$ and $\dot{y}(0) = 0$ as initial conditions

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 p \quad \longrightarrow \quad (\ddot{y} - \ddot{p}) + 2\zeta\omega_n(\dot{y} - \dot{p}) + \omega_n^2(y - p) = 0$$



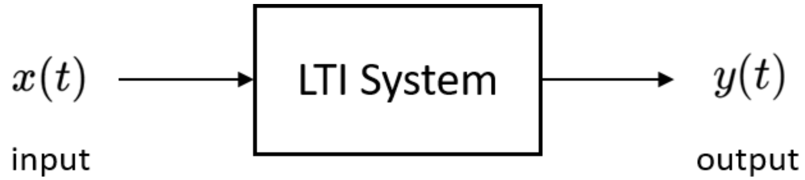


Time Response to Singularity Function Inputs

Time Response to General Inputs

- We studied output response $y(t)$ when input $x(t)$ is constant

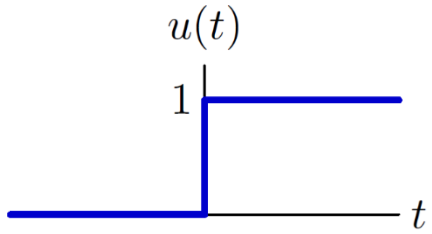
- Ultimate Goal: output response of $y(t)$ to general input $x(t)$



- Consider singularity function inputs first
  - Step function
  - Impulse function (Delta Dirac function)

Step Function

- Step function



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Step Response

- Start with a step response example

$$\dot{x} + 5x = 1 \quad \text{for} \quad t \geq 0, \qquad x(0) = 0$$

- Or

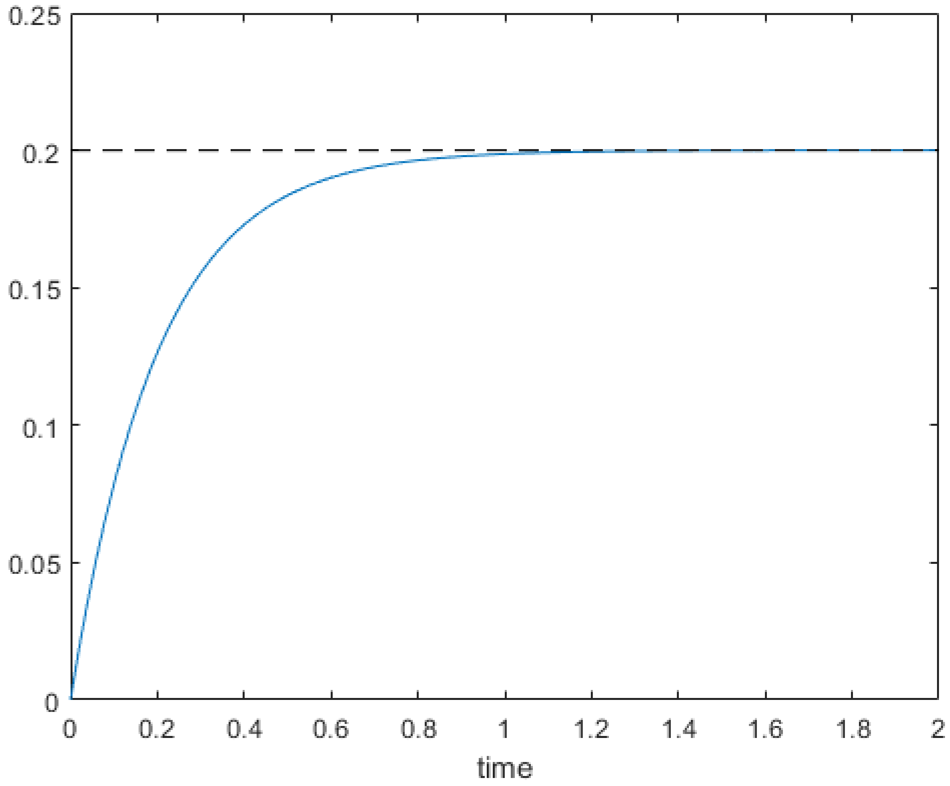
$$\dot{x} + 5x = u(t), \qquad x(0) = 0 \qquad \text{where} \;\; u(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- The solution is given:

$$x(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$$

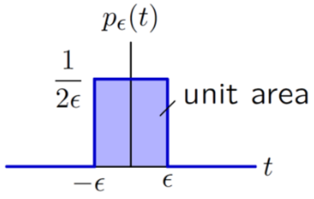
Step Response

```
t = linspace(0,2,100);
x = 1/5*(1-exp(-5*t));

plot(t,x,t,0.2*ones(size(t)),'k--')
ylim([0,0.25])
xlabel('time')
```

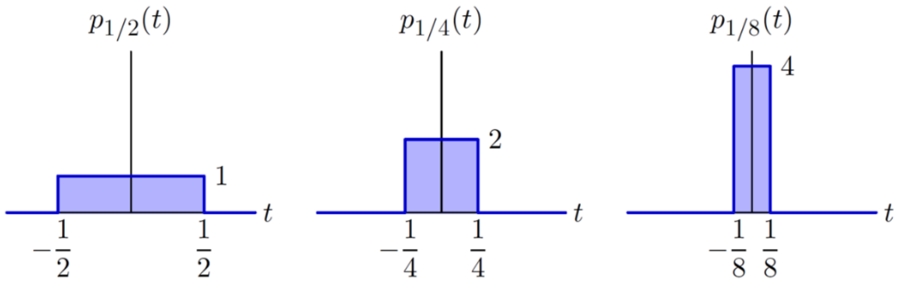


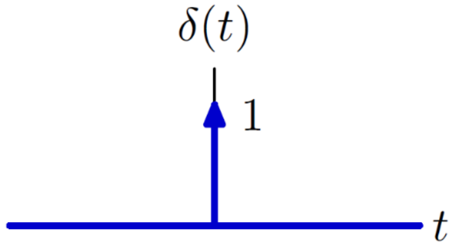
Impulse

- Impulse: difficult to image
- The unit-impulse signal acts as a pulse with unit area but zero width



$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$





- The unit-impulse function is represented by an arrow with the number 1, which represents its area
- It has two seemingly contradictory properties :
  - It is nonzero only at $t = 0$ and
  - Its definite integral $(-\infty, \infty)$ is 1

Properties of Dirac Delta Function

- The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(t) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

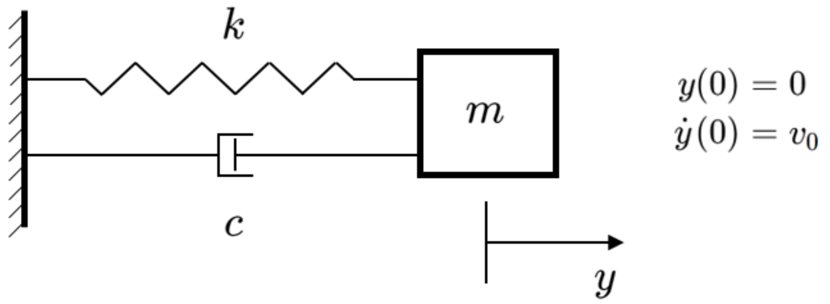
- Sifting property

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)\delta(t)dt &= \int_{-\infty}^{0^-} x(t)\delta(t)dt + \int_{0^-}^{0^+} x(t)\delta(t)dt + \int_{0^+}^{\infty} x(t)\delta(t)dt \\ &= 0 + x(0)\int_{0^-}^{0^+} \delta(t)dt + 0 \\ &= x(0) \end{aligned}$$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Impulse Response

- Impulse response: difficult to image
- Question: how to realize initial velocity of $v_0 \neq 0$



- Momentum and impulse in physics I
  - Consider an "impulse" which is a sudden increase in momentum $0 \rightarrow mv$ of an object applied at time 0
  - To model this,

$$mv - 0 = \int_{0_-}^{0^+} f(t)dt$$

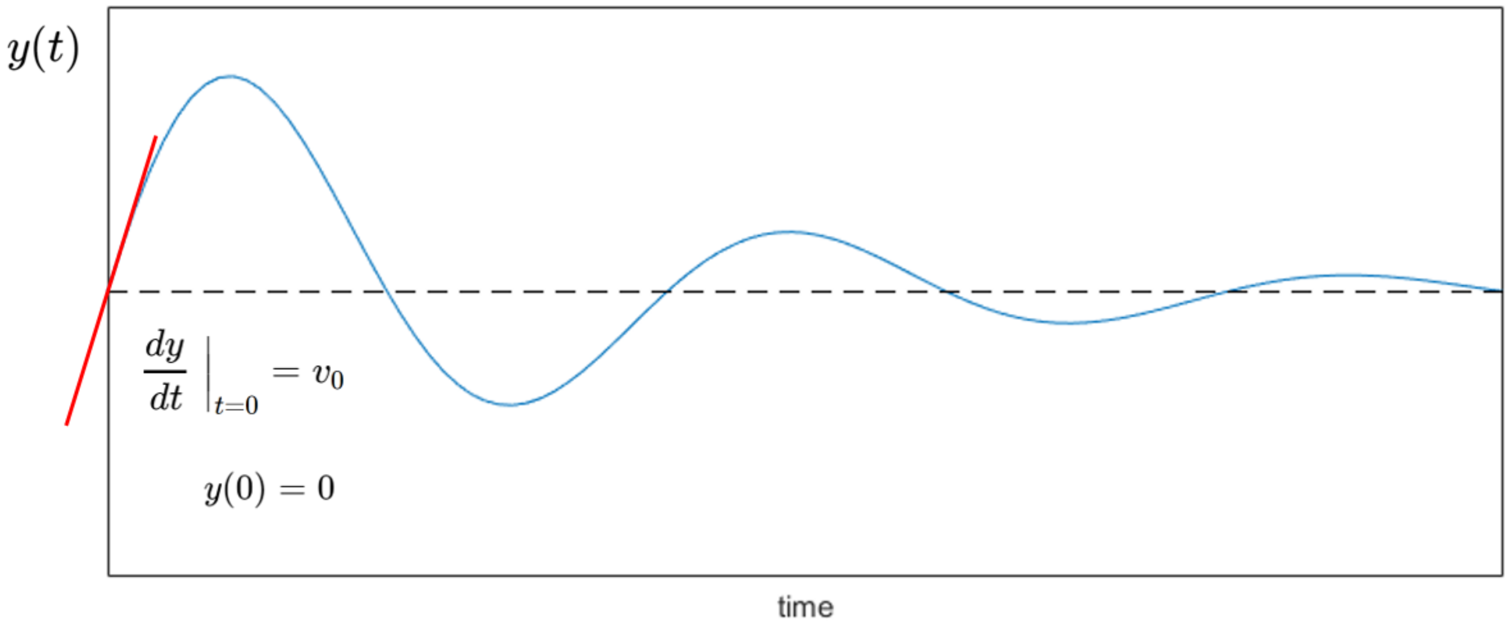
  - where force $f(t)$ is strongly peaked at time 0
  - Actually the details of the shape of the peak are not important, what is important is the area under the curve

$$f(t) = mv\delta(t)$$

  - This is the motivation that mathematician and physicist invented the delta Dirac function

$$m\ddot{y} + c\dot{y} + ky = mv_0\delta(t) \qquad \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = v_0\delta(t)$$

Impulse Response



$$v_0\delta(t) \quad \longrightarrow \quad \begin{matrix} y(0) = 0 \\ \dot{y}(0) = v_0 \end{matrix} \quad \longrightarrow \quad \text{natural response}$$

Impulse Response to LTI system

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t)$$

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \delta(t),$$

$$\begin{aligned} \delta(t) &: \text{input} \\ y &: \text{output} \end{aligned}$$

- Later, we will discuss why the impulse response is so important to understand an LTI system

Impulse Response to LTI system

- Example: now think about the impulse response

$$\dot{y} + 5y = \delta(t), \qquad y(0) = 0$$

- The solution is given: (why?)

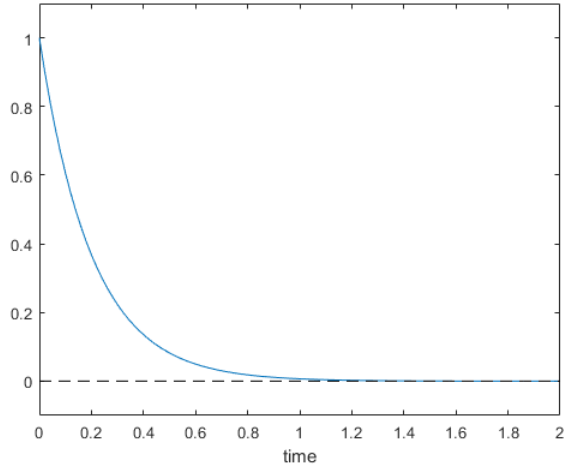
$$y(t) = h(t) = e^{-5t}, \quad t \geq 0$$

- Impulse input can be equivalently changed to zero input with non-zero initial condition

$$\int_{0^-}^{0^+} \delta(t)\, dt = u(0^+) - u(0^-) = 1$$

$$\dot{y} + 5y = 0, \qquad y(0) = 1$$

```
t = linspace(0,2,100);
h = exp(-5*t);

plot(t,h,t,zeros(size(t)), 'k--'),
ylim([-0.1,1.1])
xlabel('time')
```



Step Response Again

- Relationship between impulse response and unit-step response

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t) \qquad \text{or}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 1, \qquad t > 0$$

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y_1(t)$$

$$u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y_2(t)$$

$$\frac{du(t)}{dt} = \delta(t) \qquad\qquad \frac{dy_2(t)}{dt} = y_1(t)$$

- Impulse response is the derivative of the step response

Response to General Inputs (in Time)

Response to a General Input (in Time)

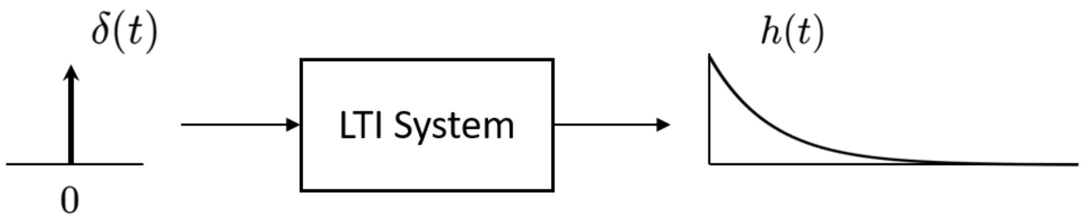
- Finally, think about response to a "general input" in time

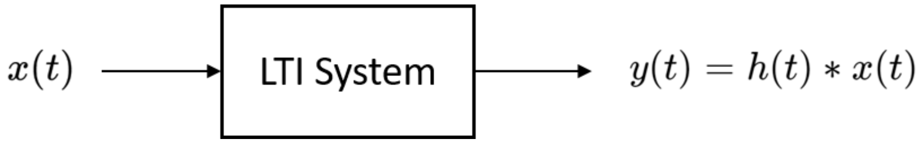
$$\dot{y} + 5y = x(t), \quad y(0) = 0$$

- The solution is given

$$\boxed{y(t) = h(t) * x(t) \qquad \text{where } h(t) \text{ is impulse response}}$$

- If this is true, we can compute output response to any general input if an impulse response is given
  - Impulse response = LTI system



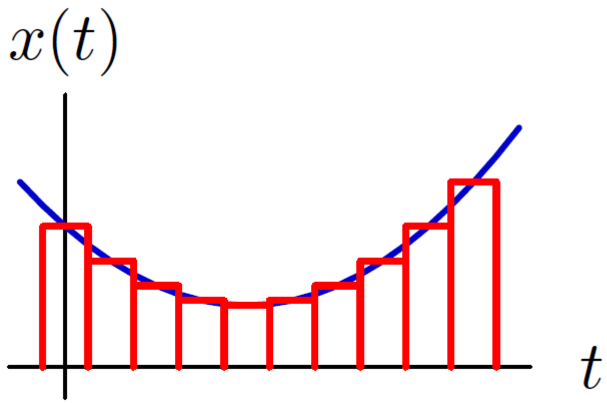


Convolution: Definition

- $y(t)$ is the integral of the product of two functions after one is reversed and shifted by t

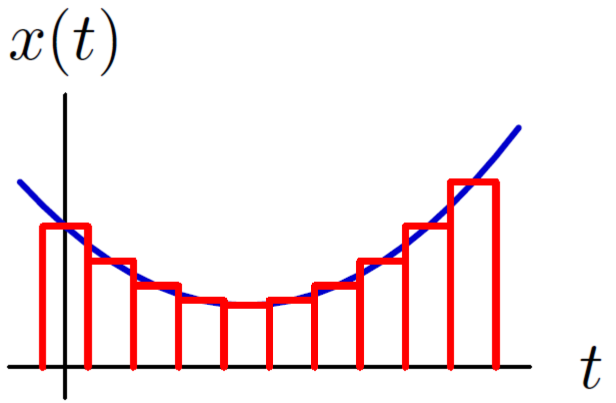
$$
\begin{aligned}
y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
\end{aligned}
$$

Easier Way to Understand Continuous Time Signal

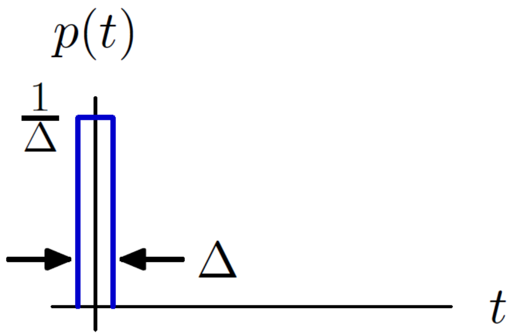


$$x(t) = \lim_{\Delta \to 0} \sum_k x(k\Delta)p(t - k\Delta)\Delta$$

Easier Way to Understand Continuous Time Signal



$$x(t) = \lim_{\Delta \to 0} \sum_k x(k\Delta)p(t - k\Delta)\Delta$$

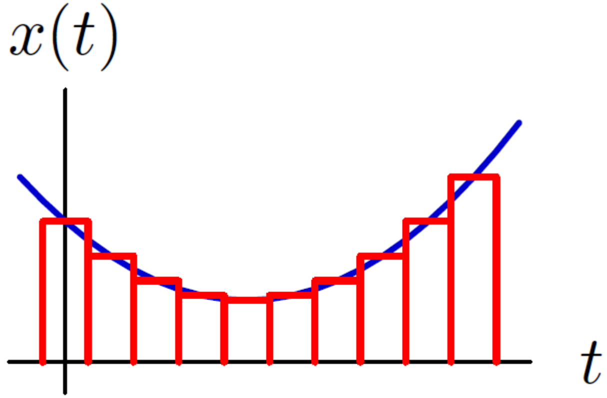


As $\Delta \to 0$, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:

$$x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

Structure of Superposition

- If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.



$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

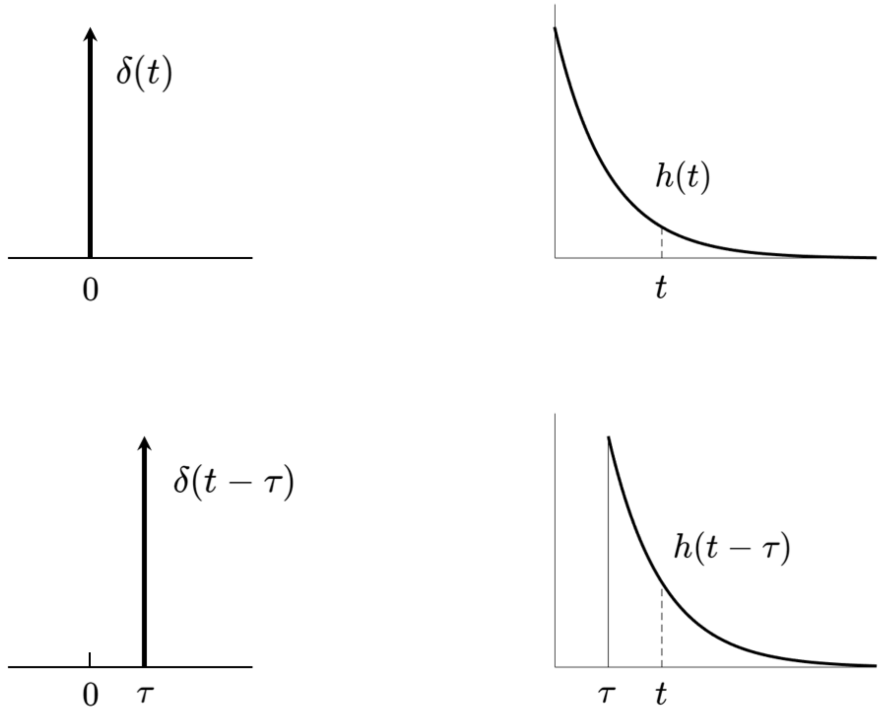
$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

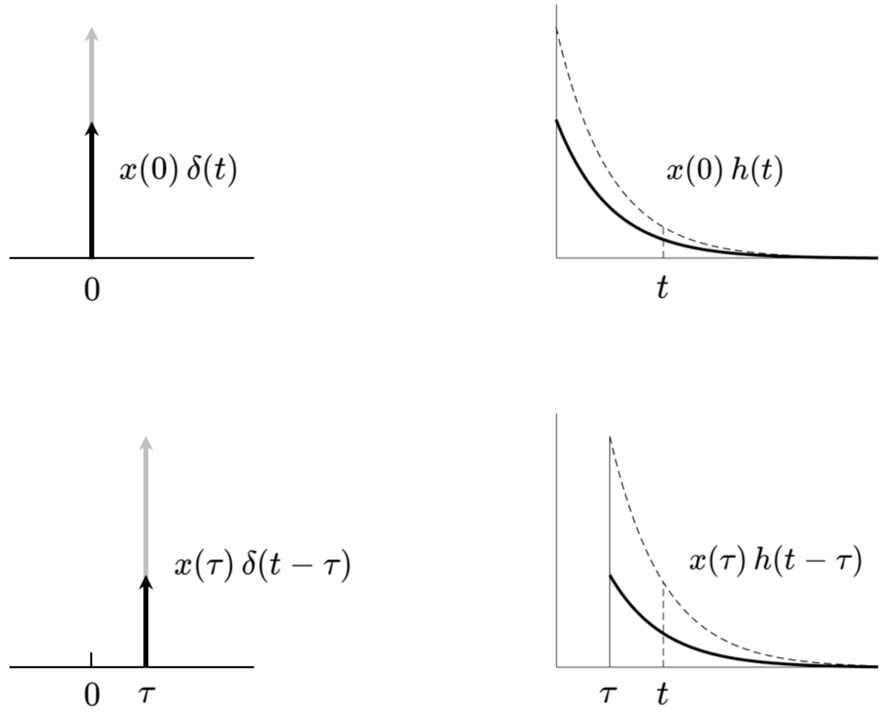
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Impulse Response to LTI System

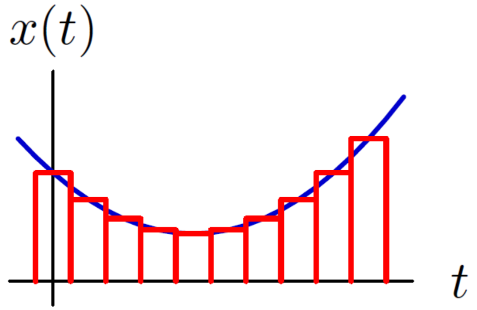
Time-invariant

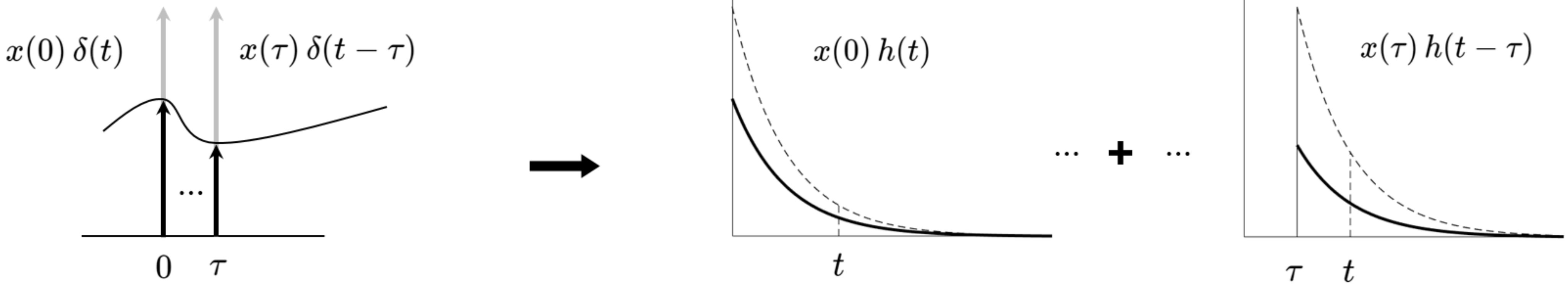


Linear (scaling)



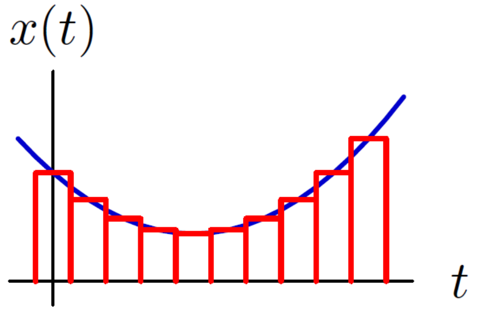
Response to Arbitrary Input $x(t)$ (1/2)

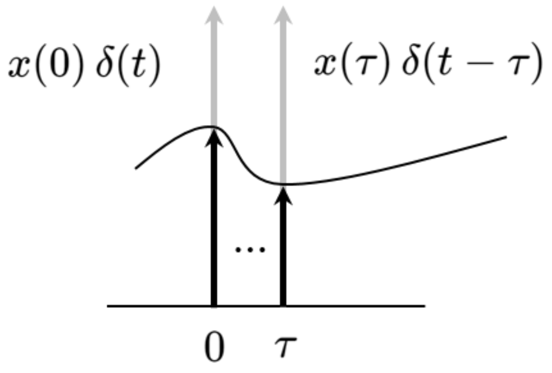


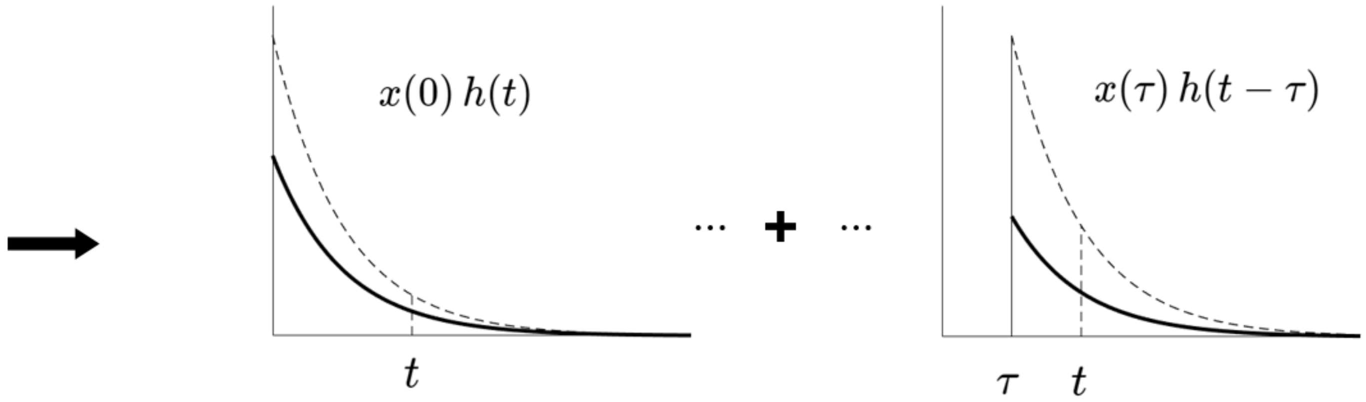


$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \quad \rightarrow \quad \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \;=\; x(t) * h(t)}$$

Response to Arbitrary Input $x(t)$ (2/2)







$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \;=\; x(t) * h(t)}$$

Response to Arbitrary Input: MATLAB (1/2)

- Example

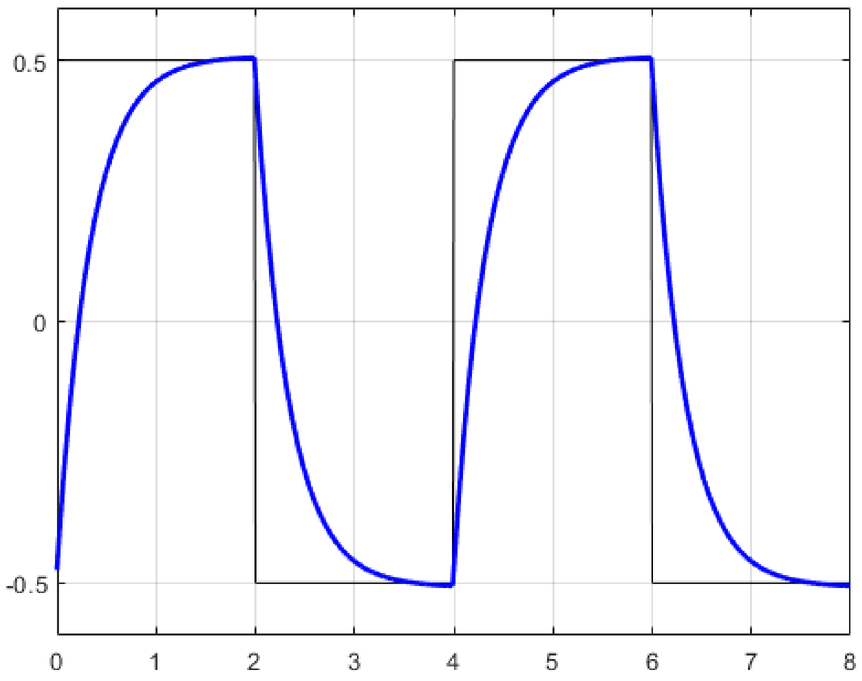
$$\dot{y} + 3y = 3x(t)$$

- The solution is given:

$$y(t) = h(t) * x(t), \qquad \text{where } h(t) = 3e^{-3t}$$

```
w0 = 2*pi/T;
x = 1/2*square(w0*t);
```

```
% convolution (circular, will be discussed later)

h = 3*exp(-3*t);
y = cconv(x,h,length(t))*Ts;

plot(t,x,'k'), hold on
plot(t,y,'b', 'linewidth',2), hold off
grid on
```



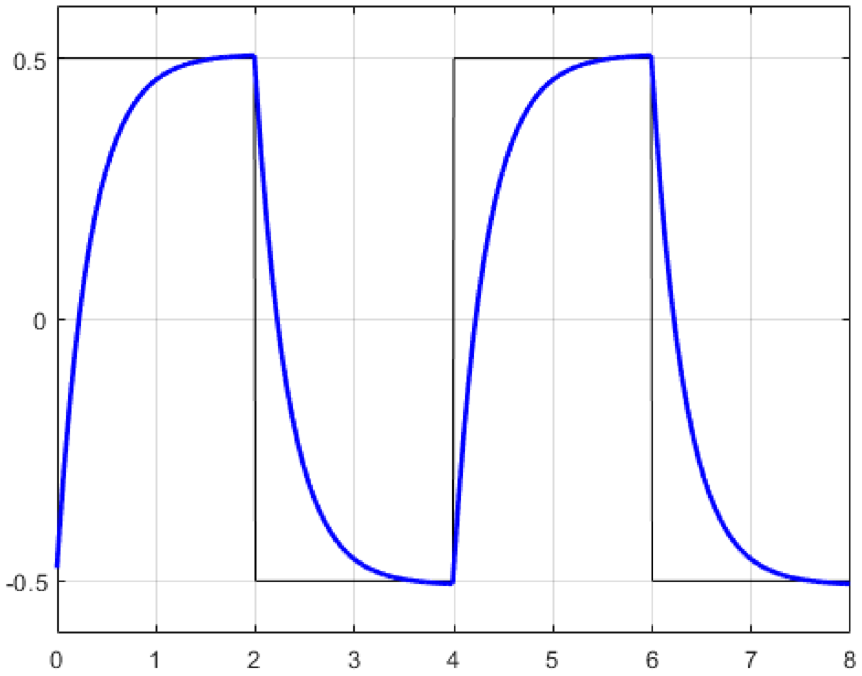
Response to Arbitrary Input: MATLAB (2/2)

- Example

$$\dot{y} + 3y = 3x(t)$$

- The solution is given:

$$y(t) = h(t) * x(t), \qquad \text{where } h(t) = 3e^{-3t}$$



Response to Sinusoidal Input (in Frequency)

Response to a Sinusoidal Input

- When the input $x(t) = e^{j\omega t}$ to an LTI system

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ h(t)\end{array}} \longrightarrow y(t)$$

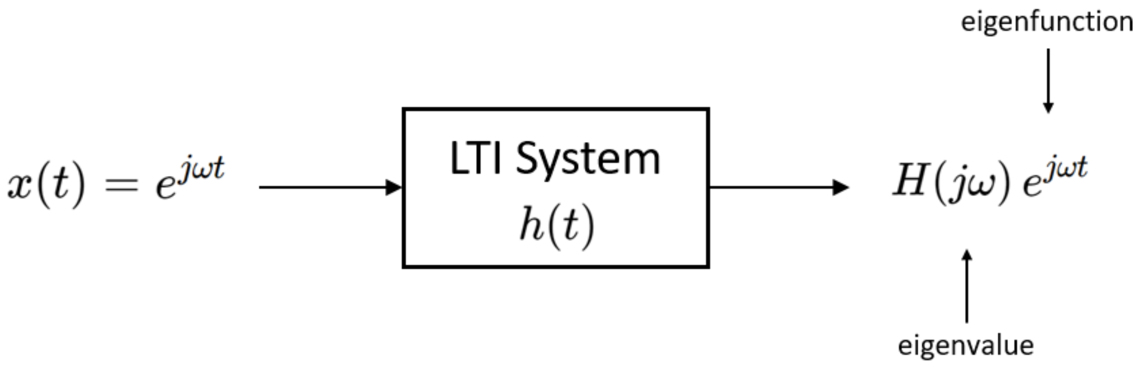
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau = e^{j\omega t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}_{\text{complex function of }\omega}$$

$$= e^{j\omega t}H(j\omega)$$

Fourier Transform

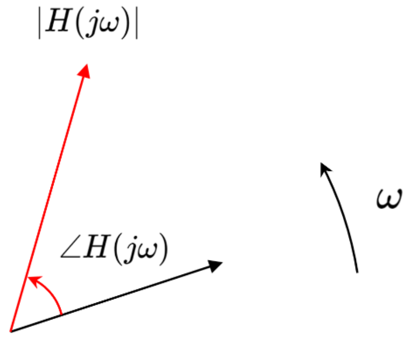
- Definition: Fourier transform

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$



- $H(j\omega)e^{j\omega t}$ rotates the same angular velocity ω

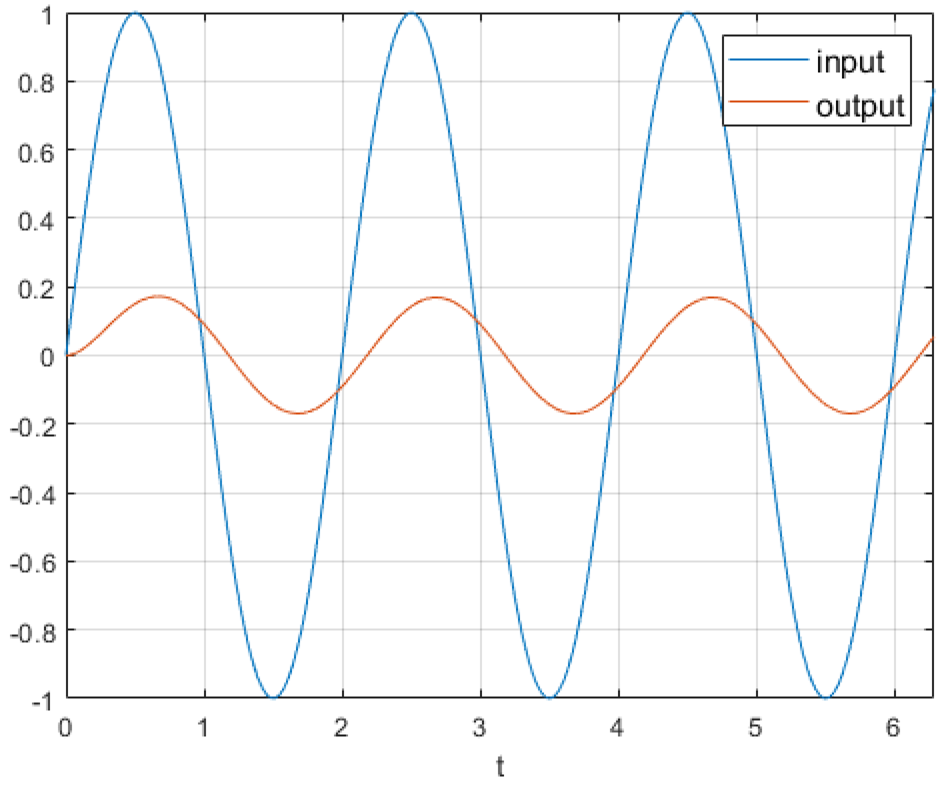
$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$



Response to a Sinusoidal Input: MATLAB

$$\dot{y} + 5y = x(t)$$

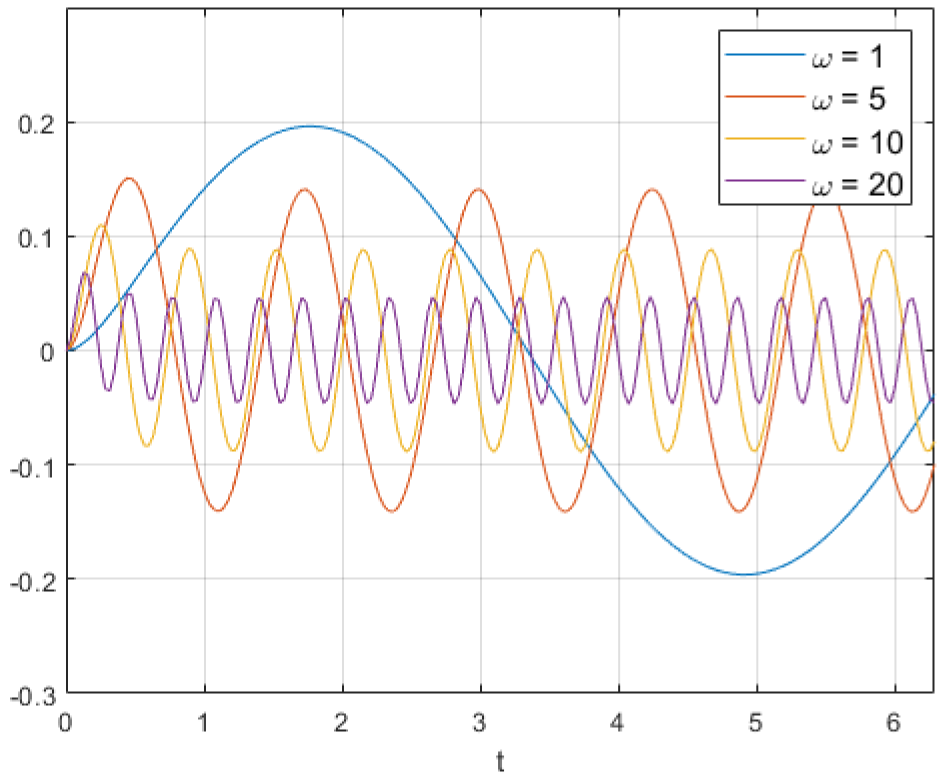
```
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

w = pi;
t = linspace(0,2*pi,200);

x0 = 0;

x = sin(w*t);

[y,tout] = lsim(G,x,t,x0);
```



Response to a Sinusoidal Input: MATLAB

$$\dot{y} + 5y = x(t)$$

```
W = [1,5,10,20];
for w = W
    x = sin(w*t);
    [y,tout] = lsim(G,x,t,x0);
    plot(tout,y), hold on
end
```



Response to Periodic Input (in Frequency)

Response to a Periodic Input (in Frequency Domain)

- Periodic signal: Definition

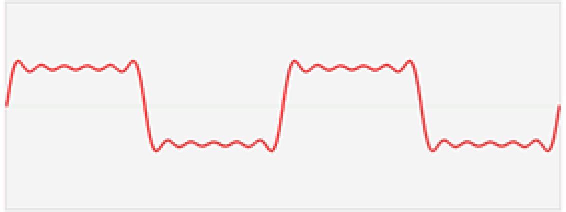
$$x(t) = x(t+T)$$

$$\text{Fundamental frequency} \quad \omega_0 = 2\pi f$$

$$\text{Fundamental period} \quad T = \frac{2\pi}{\omega_0}$$

- Fourier series represent periodic signals in terms of sinusoids (or complex exponential of $e^{j\omega t}$)
- Fourier series represent periodic signals by their harmonic components

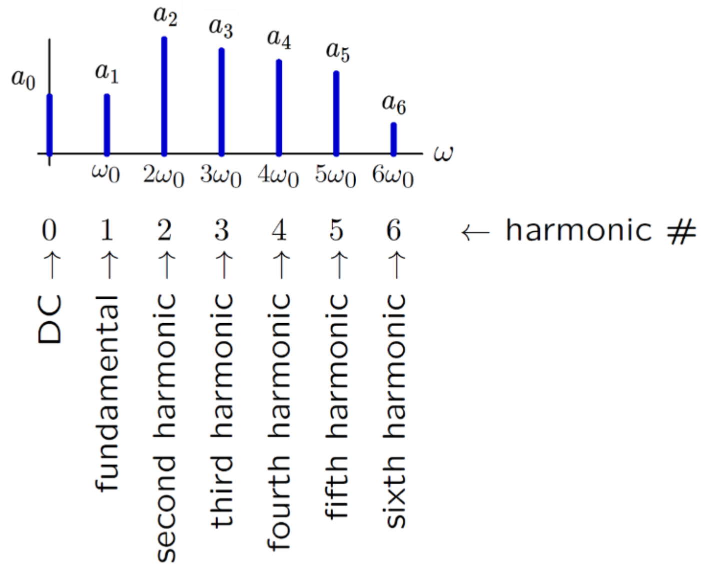
$$\begin{aligned} x(t) &= a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + \cdots \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \end{aligned}$$



Response to a Periodic Input (in Frequency)

- Fourier series represent periodic signals by their harmonic components

$$
\begin{aligned}
x(t) &= a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + \cdots \\
&= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}
\end{aligned}
$$

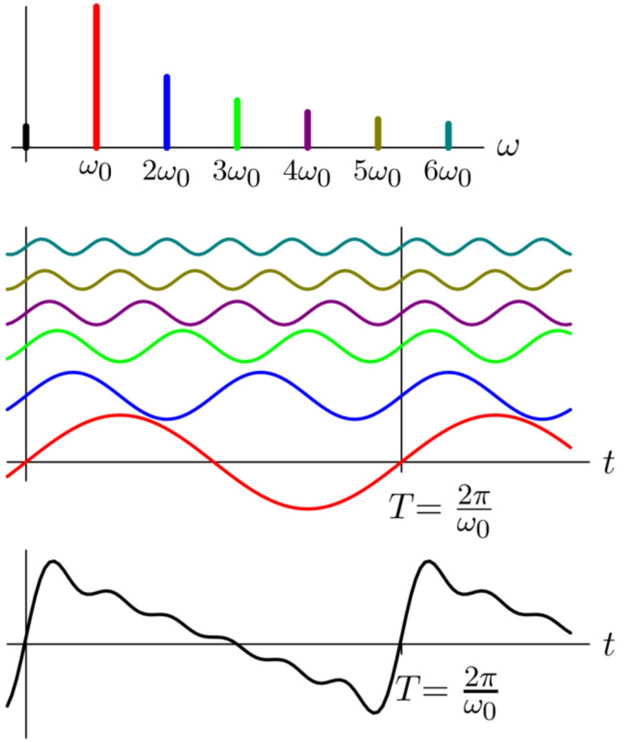


Response to a Periodic Input (in Frequency)

- What signals can be represented by sums of harmonic components?

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + \cdots$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$



Harmonic Representations

- It is possible to represent all periodic signals with harmonics
- Question: how to separate harmonic components given a periodic signal

- Underlying properties

$$e^{jk_1\omega_0 t} \cdot e^{jk_2\omega_0 t} = e^{j(k_1+k_2)\omega_0 t}$$

$$\int_t^{t+T} e^{jk\omega_0\tau}\, d\tau = \begin{cases} 0 & k \neq 0 \\ T & k = 0 \end{cases}$$

$$= T\delta[k] \quad \text{where } \delta[k] \text{ is Kronecker delta}$$

Harmonic Representations

- Assume that $x(t)$ is periodic in T and is composed of a weighted sum of harmonics of $\omega_0 = \frac{2\pi}{T}$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Then

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-n)\omega_0 t} dt$$

$$= a_n T$$

$$\therefore a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} = \frac{1}{T} \int_T x(t) e^{-jk\frac{2\pi}{T}t}$$

Fourier Series

- Fourier series: determine harmonic components of a periodic signal

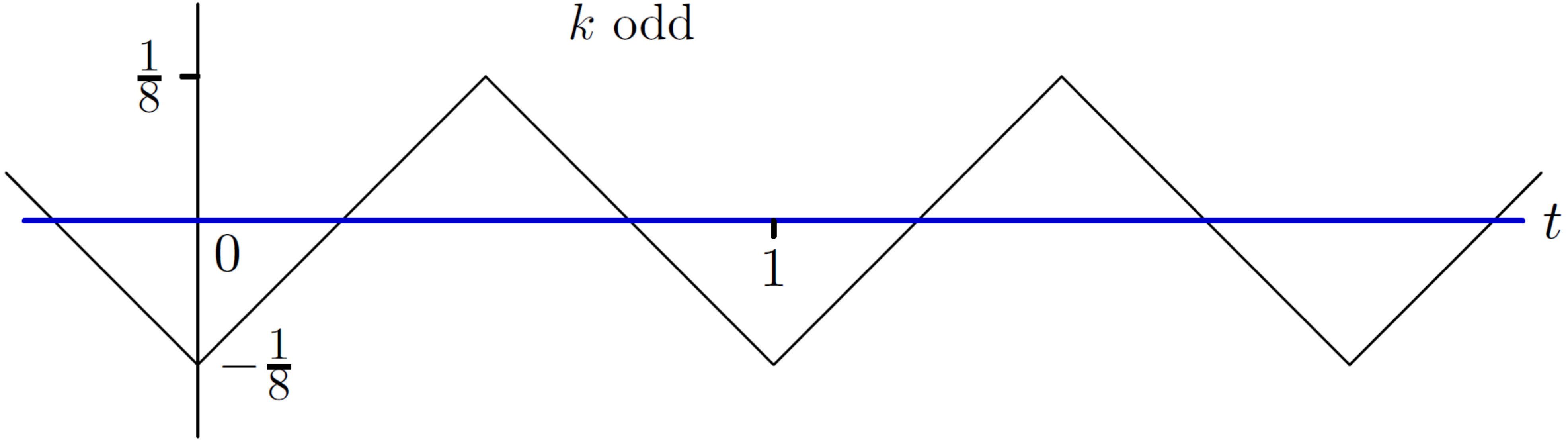
$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 kt}dt, \qquad \text{analysis}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}, \qquad \text{synthesis}$$

Example: Triangle Waveform

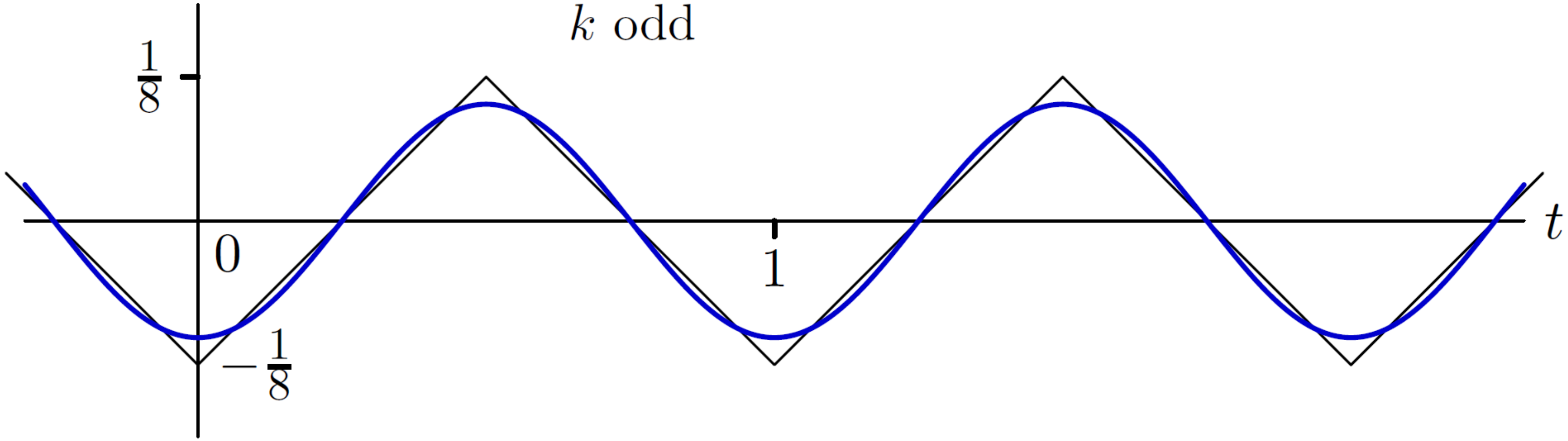
- One can visualize convergence of the Fourier Series by incrementally adding terms.

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{-1}{2k^2\pi^2} e^{j2\pi k t}$$

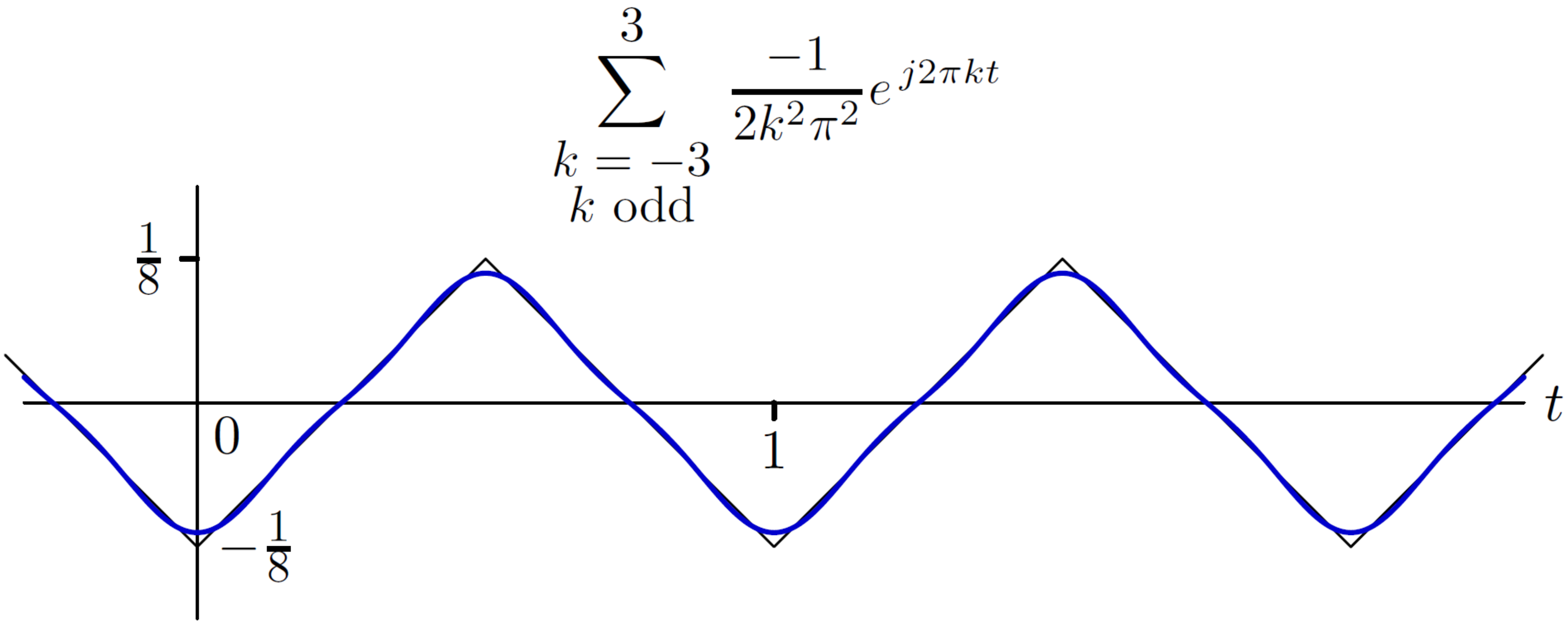


Example: Triangle Waveform

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

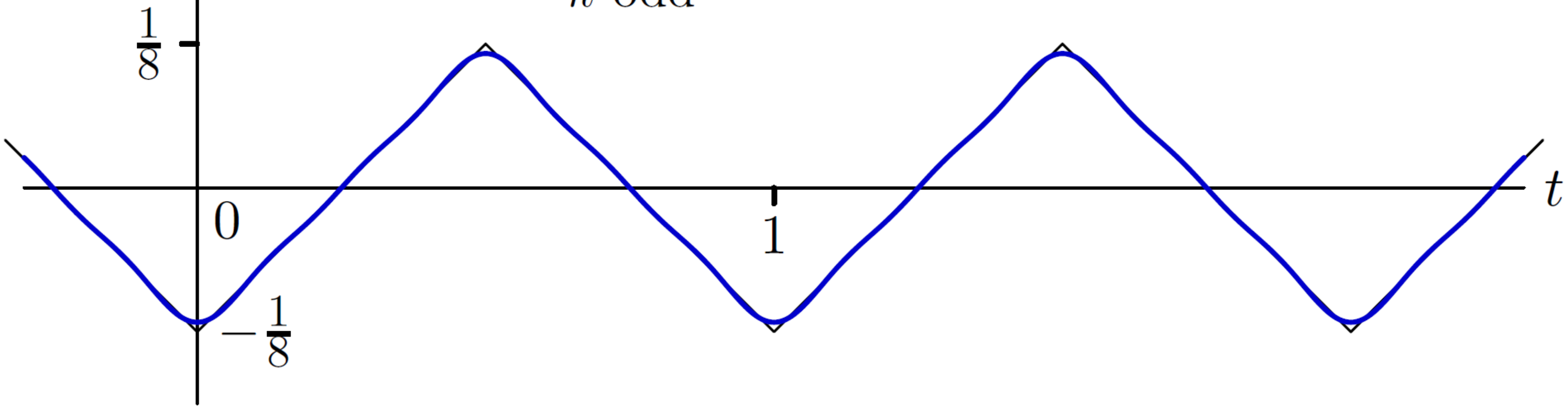


Example: Triangle Waveform



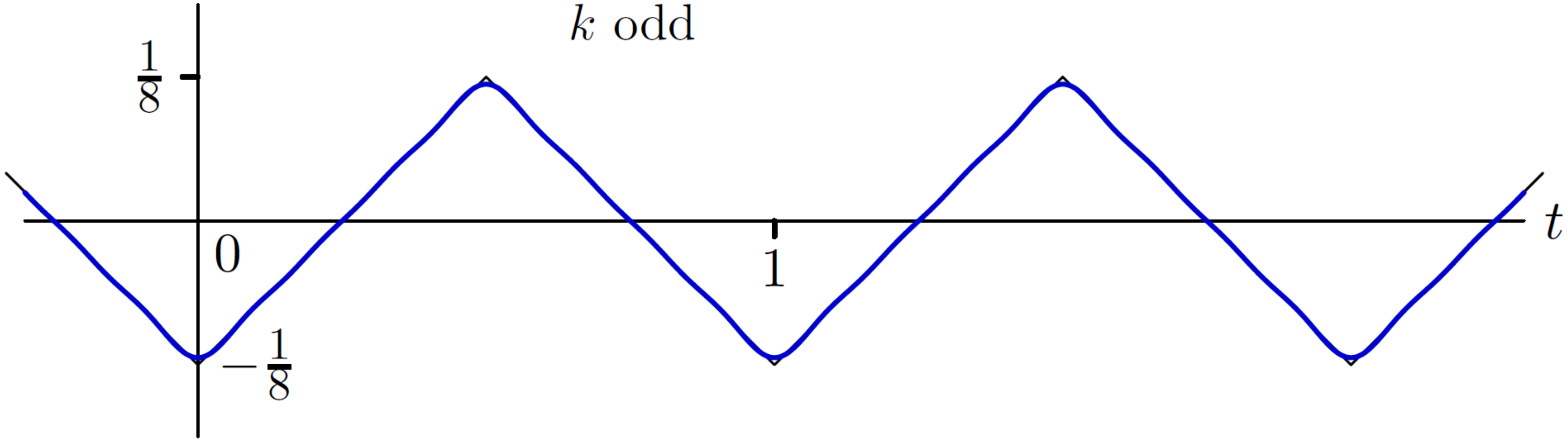
Example: Triangle Waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi k t}$$



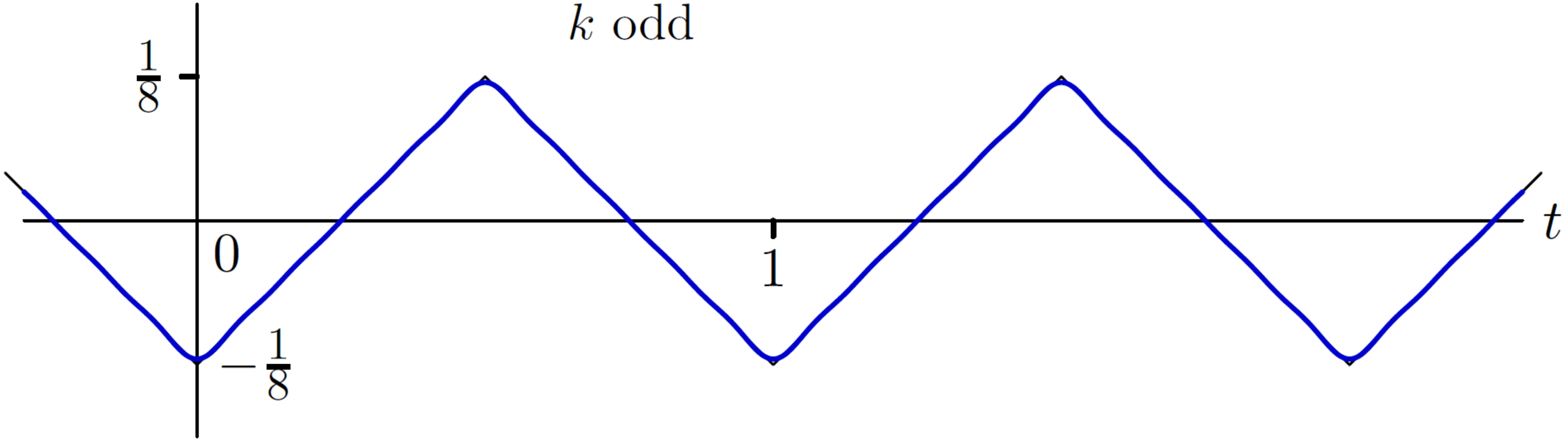
Example: Triangle Waveform

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

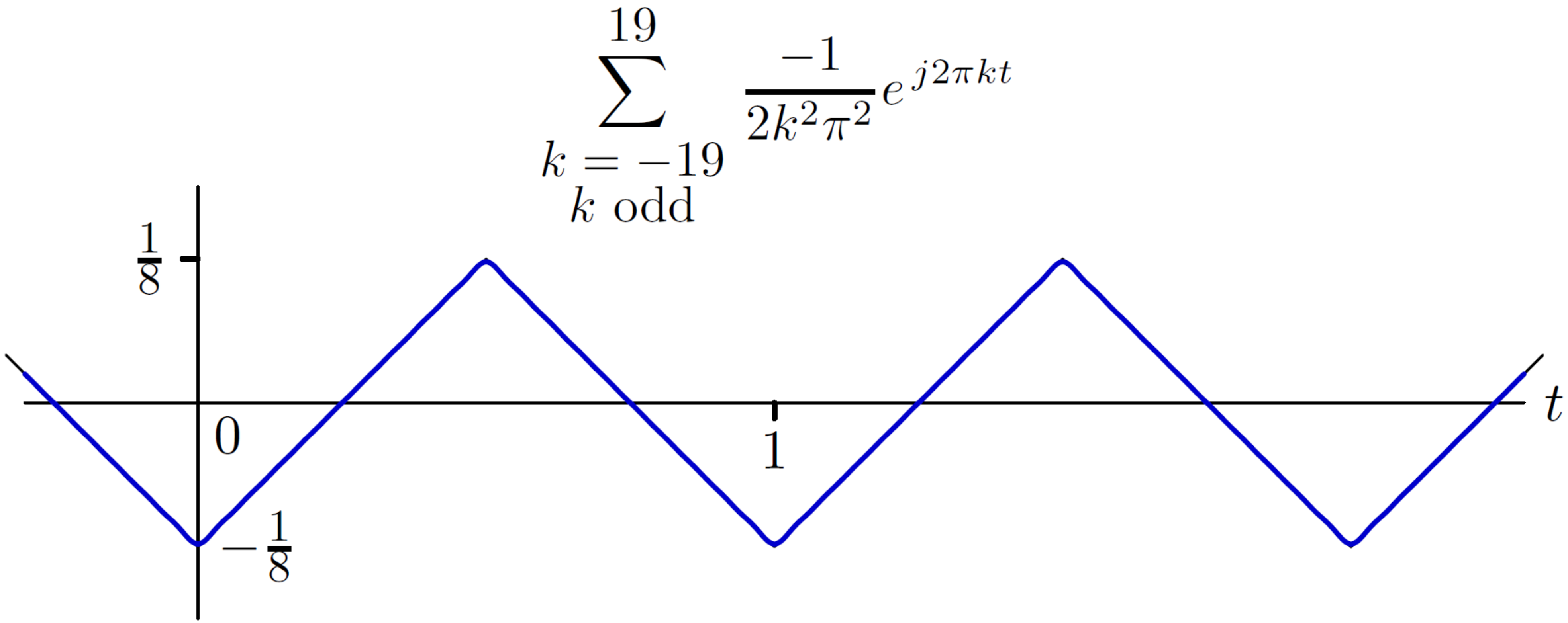


Example: Triangle Waveform

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{-1}{2k^2\pi^2} e^{j2\pi k t}$$

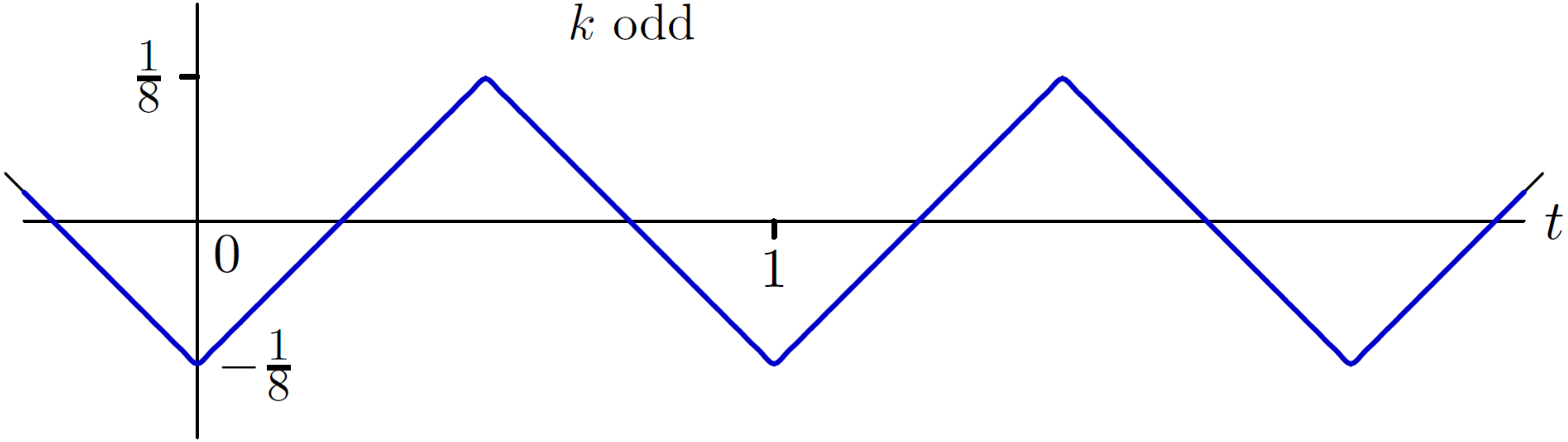


Example: Triangle Waveform



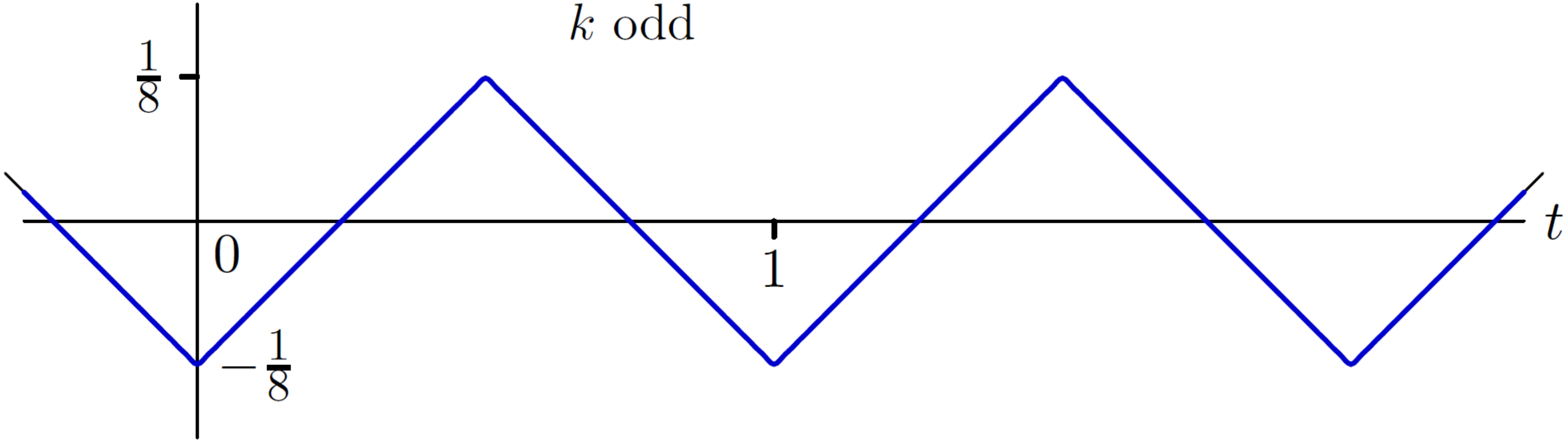
Example: Triangle Waveform

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi k t}$$



Example: Triangle Waveform

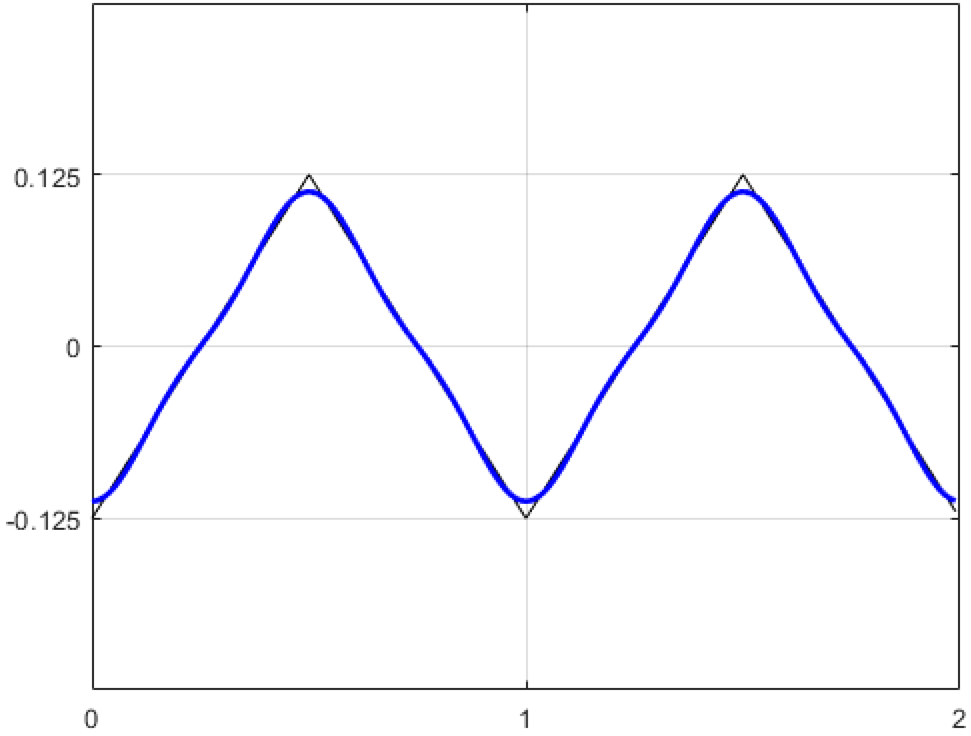
$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



Example: Triangle Waveform: MATLAB

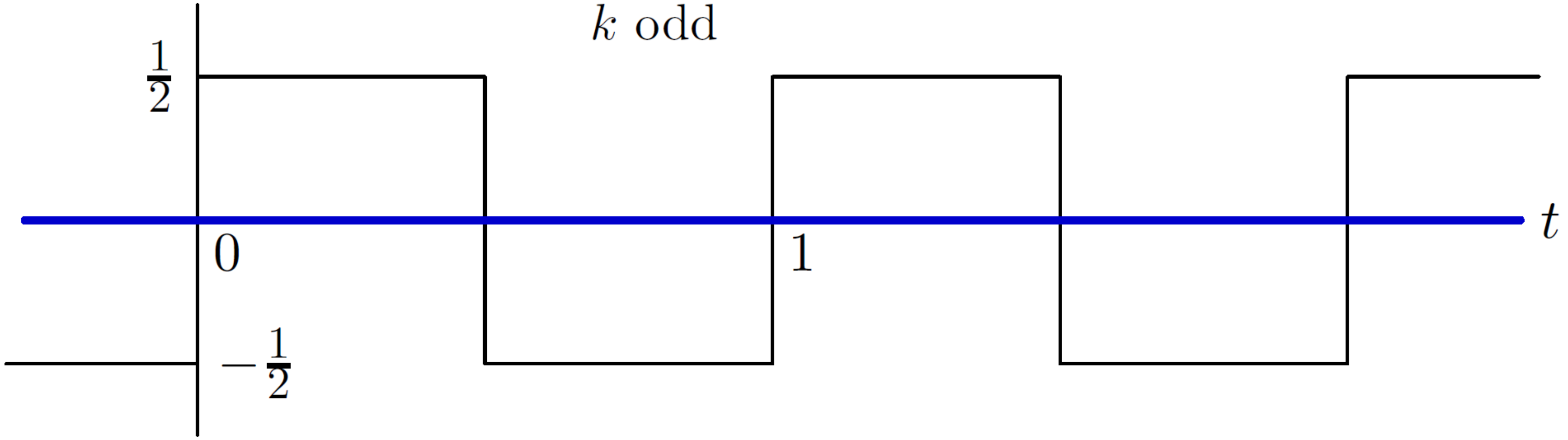
$$x(t) = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{-1}{2k^2\pi^2} e^{j2\pi kt} = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{-2}{\omega_0^2 k^2} e^{j\omega_0 kt}$$

```
T = 1;

fs = 100;
t = 0:1/fs:2*T-1/fs;

w0 = 2*pi/T;
x = 1/8*sawtooth(w0*t, 1/2);

% xhat = Fourier series of triangle ware
xhat = zeros(size(t));

for k = -3:2:3
    xhat = xhat - 2/(w0^2*k^2)*exp(1j*w0*k*t);
end

plot(t, x, 'k'), hold on
plot(t, xhat, 'b', 'linewidth', 2), hold off
grid on
ylim([-2/8,2/8])
xticks([0,1,2])
yticks([-1/8,0,1/8])
```



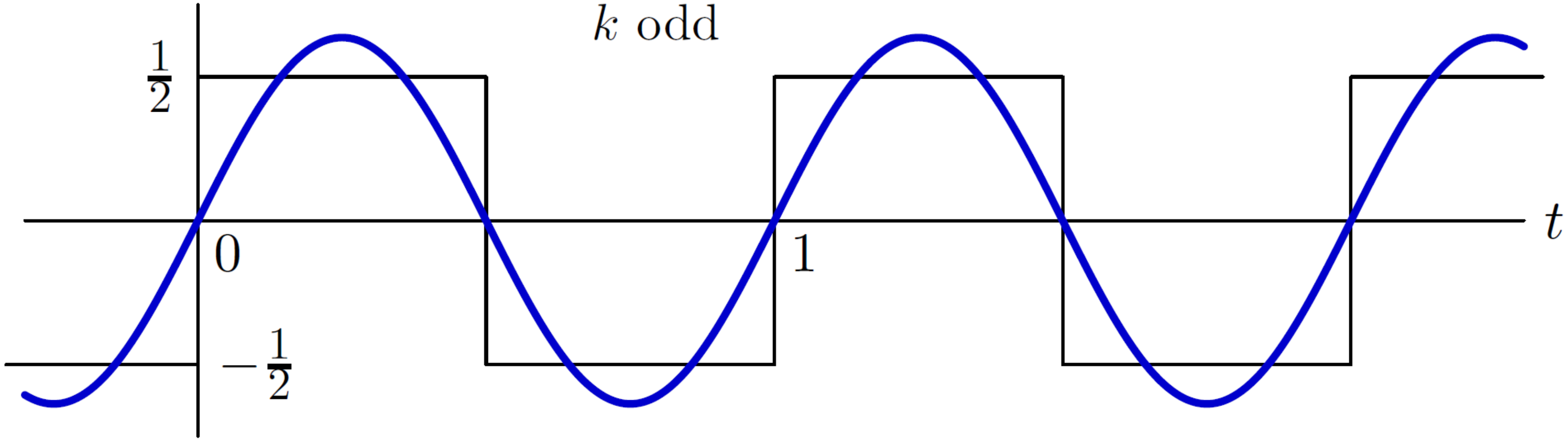
Example: Square Waveform

$$\sum_{\substack{k=-0\\ k\ \text{odd}}}^{0} \frac{1}{jk\pi} e^{j2\pi kt}$$



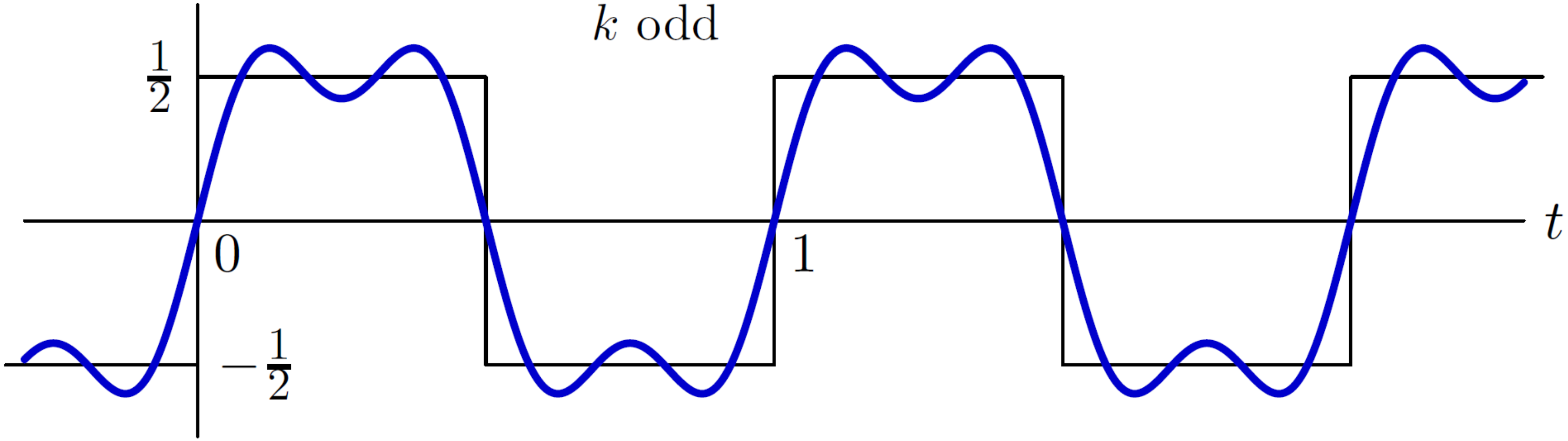
Example: Square Waveform

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{1}{jk\pi} e^{j2\pi kt}$$



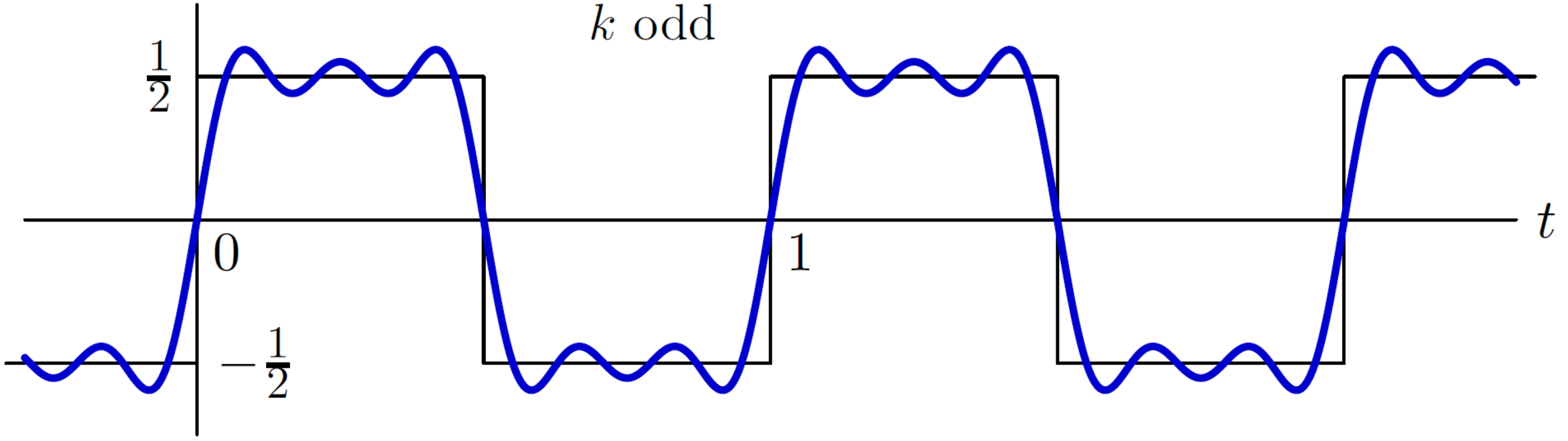
Example: Square Waveform

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$

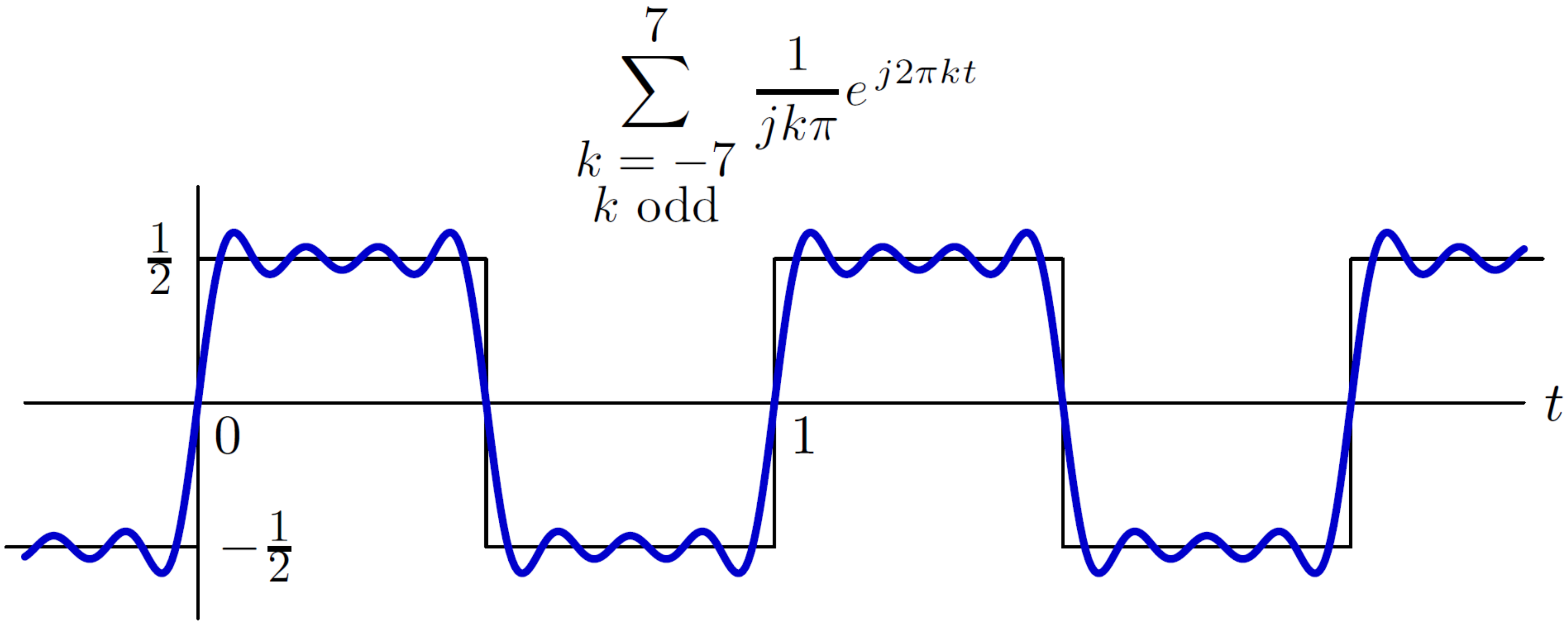


Example: Square Waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{1}{jk\pi} e^{j2\pi kt}$$

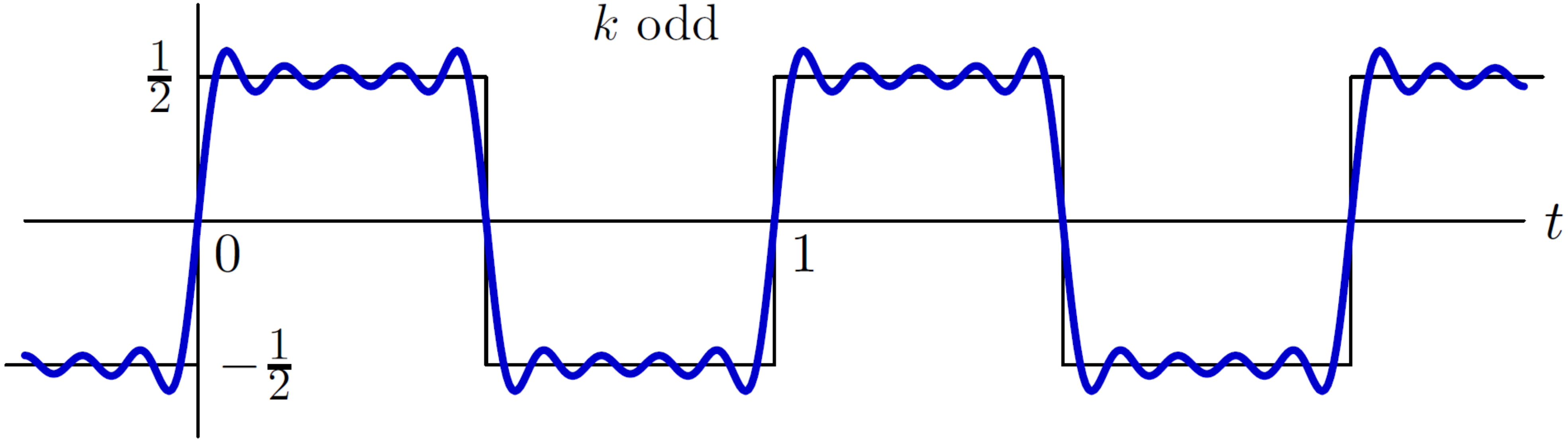


Example: Square Waveform

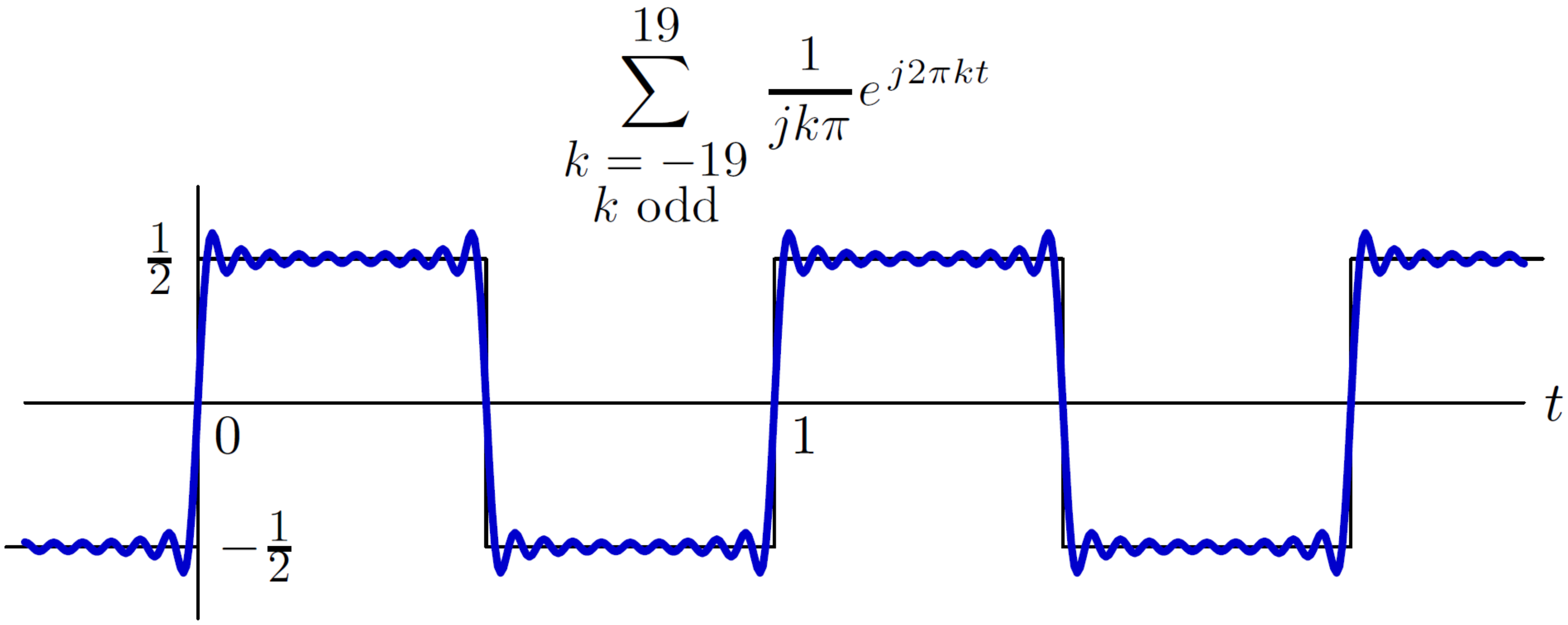


Example: Square Waveform

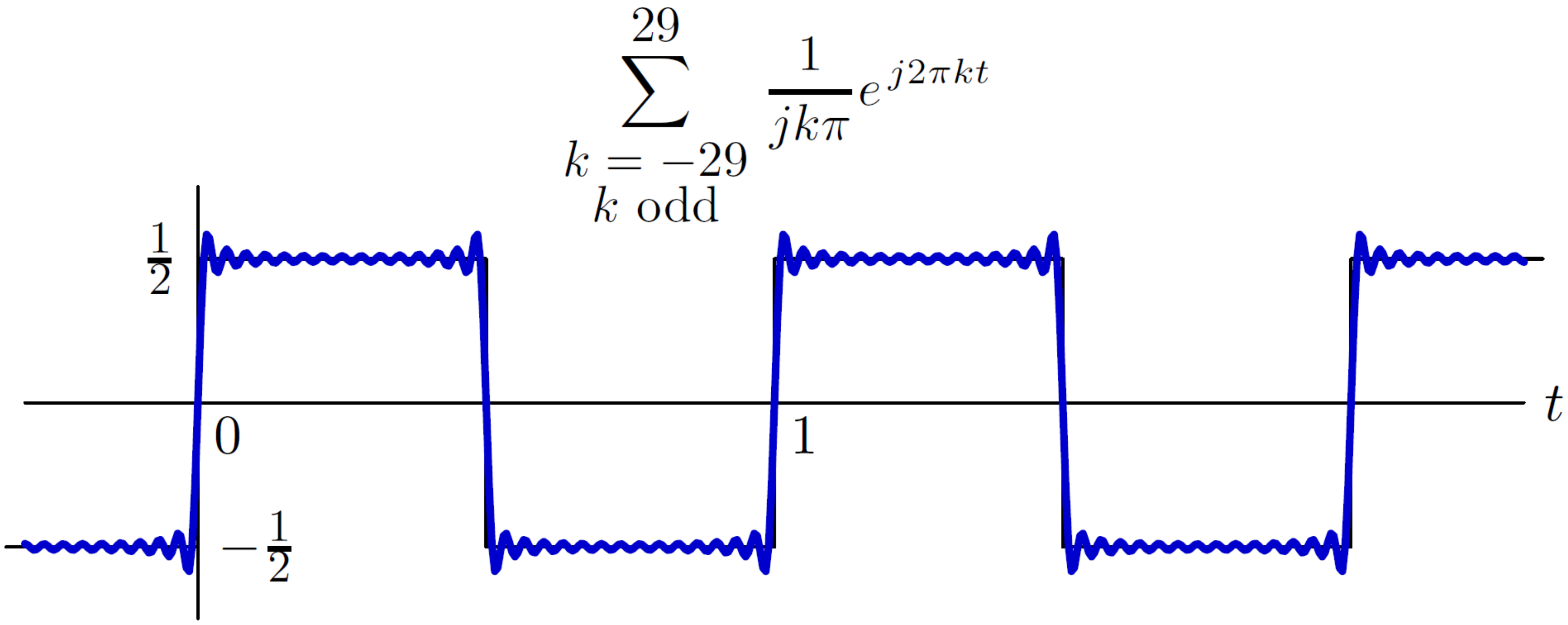
$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{1}{jk\pi} e^{j2\pi kt}$$



Example: Square Waveform

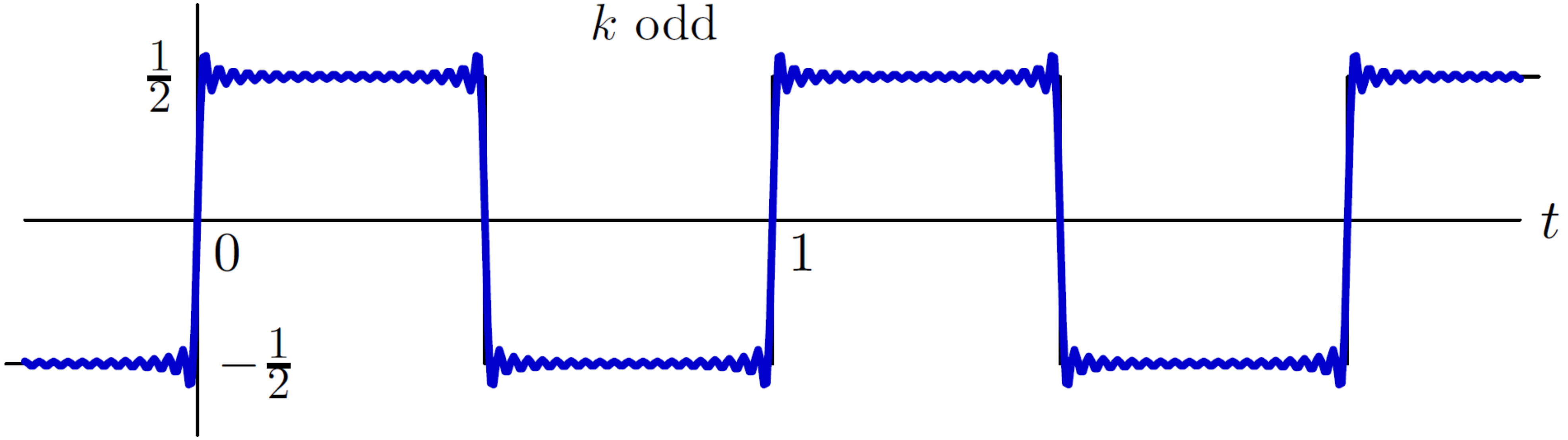


Example: Square Waveform



Example: Square Waveform

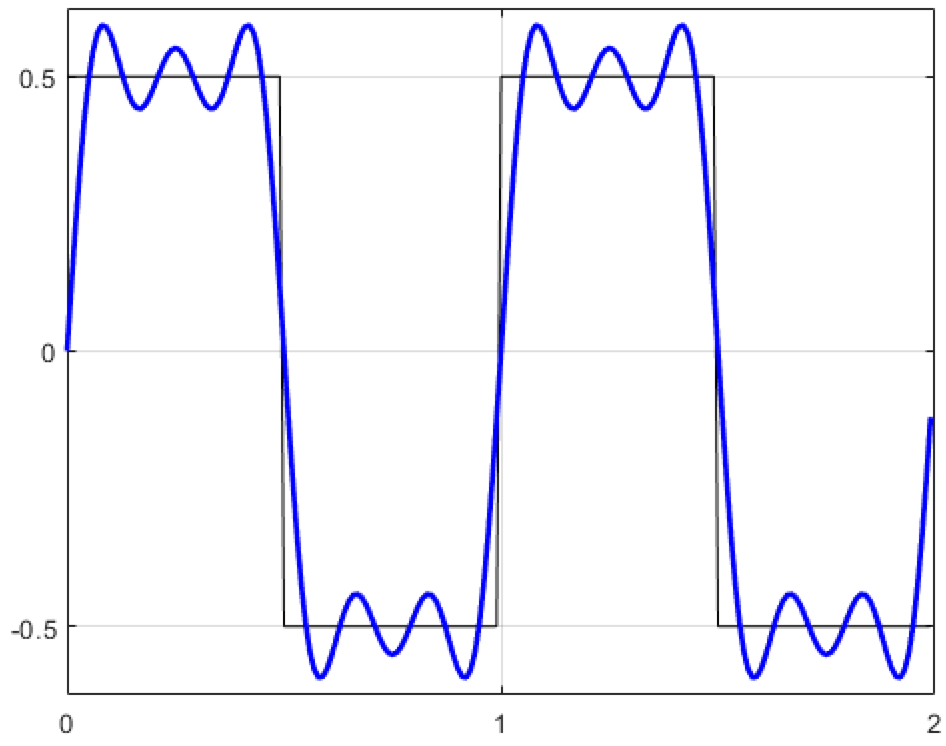
$$\sum_{\substack{k=-39\\ k\ \mathrm{odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$



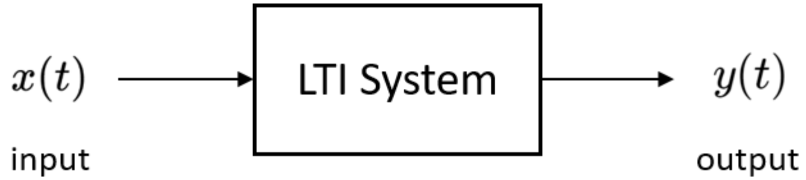
Example: Square Waveform: MATLAB

$$x(t) = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{1}{jk\pi} e^{j2\pi kt} = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{2}{j\omega_0 k} e^{j\omega_0 kt}$$

```
T = 1;

fs = 100;
t = 0:1/fs:2*T-1/fs;

w0 = 2*pi/T;
x = 1/2*square(w0*t);

% xhat = Fourier series of square ware
xhat = zeros(size(t));

for k = -5:2:5
    xhat = xhat + 2/(1j*w0*k)*exp(1j*w0*k*t);
end

plot(t, x, 'k'), hold on
plot(t, xhat, 'b', 'linewidth', 2), hold off
grid on
ylim([-5/8,5/8])
xticks([0,1,2])
yticks([-1/2,0,1/2])
```



Response to a Periodic Input (Filtering)



- Periodic input: Fourier series → sum of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

- Complex exponentials: eigenfunctions of LTI system

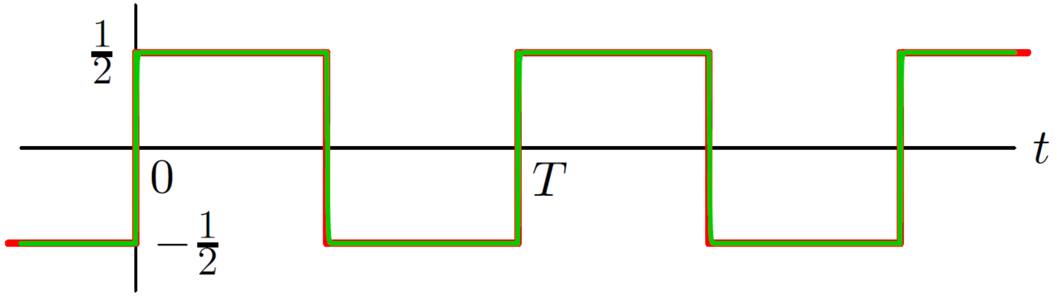
$$e^{j\omega_0 kt} \quad \longrightarrow \quad H(j\omega_0 k)e^{j\omega_0 kt}$$

- Output: same eigenfunctions, but amplitudes and phase are adjusted by the LTI system

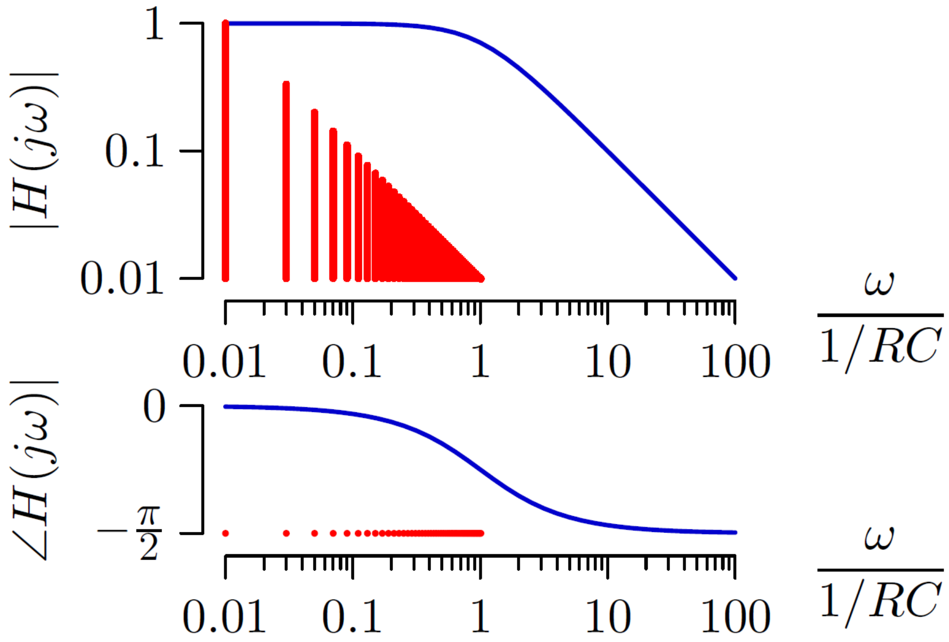
$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt}$$

- The output of an LTI system is a “filtered” version of the input

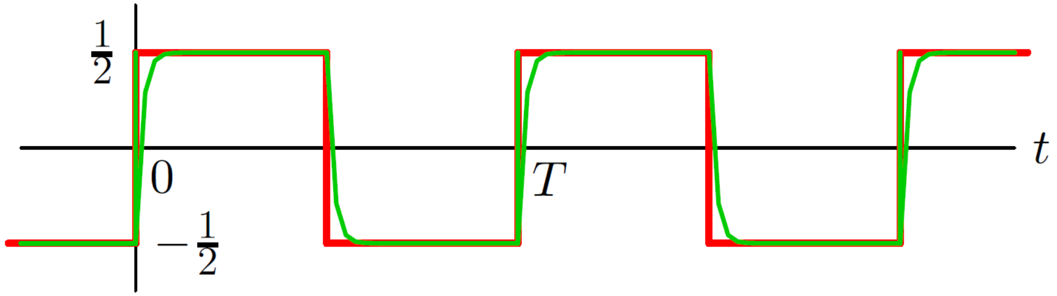
Output is a "Filtered" Version of Input



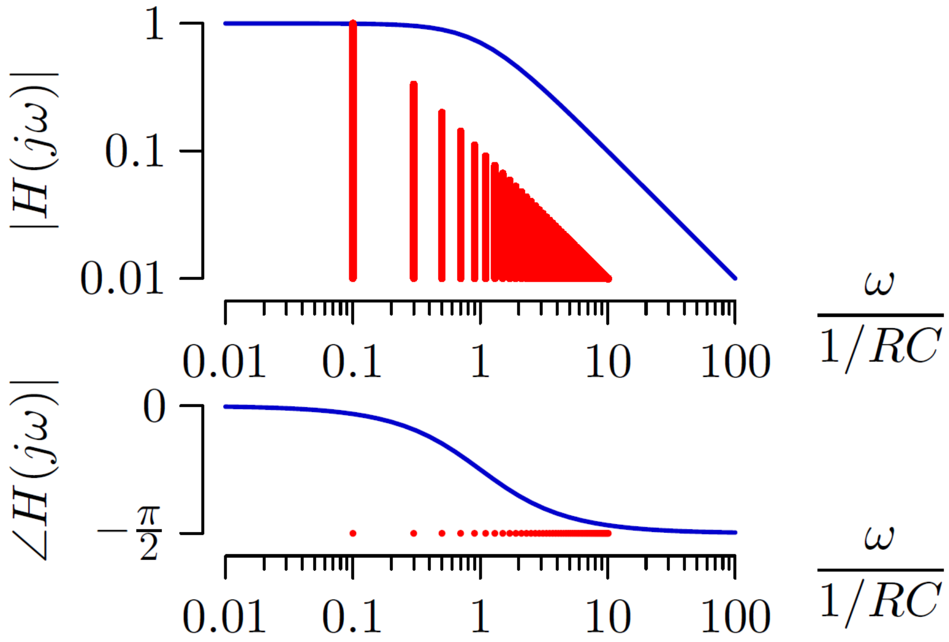
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$



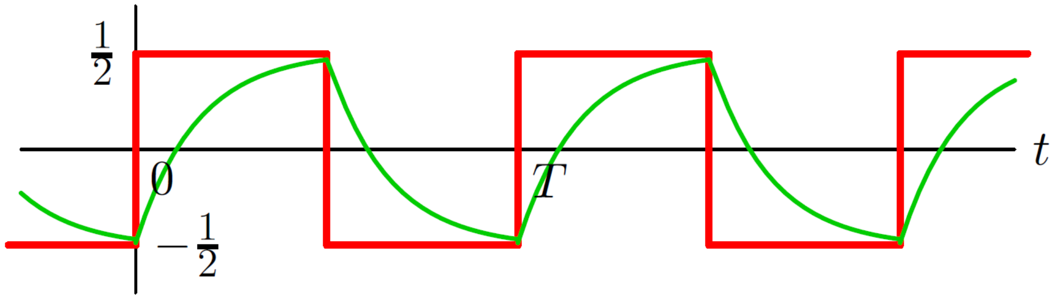
Output is a "Filtered" Version of Input



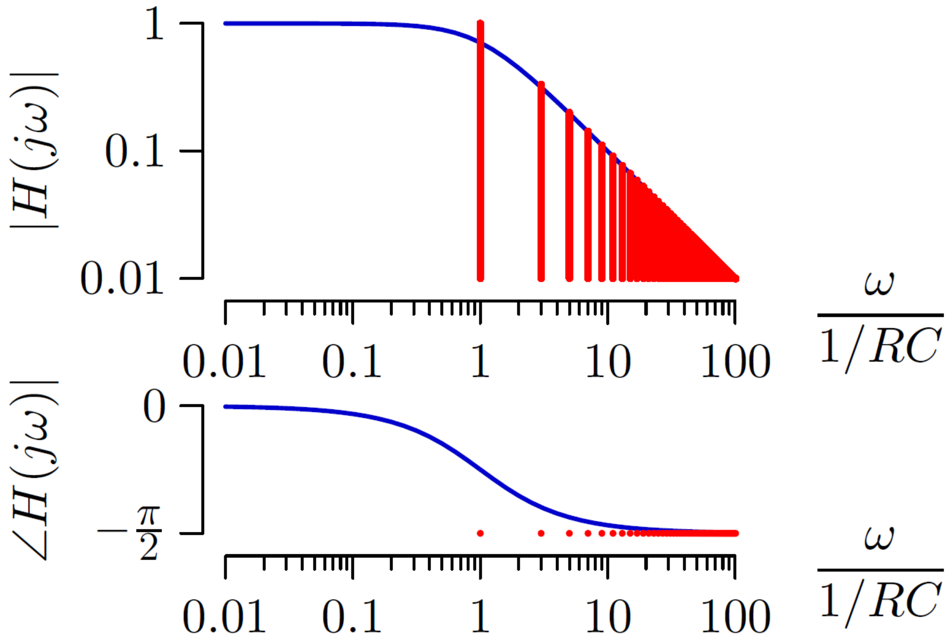
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



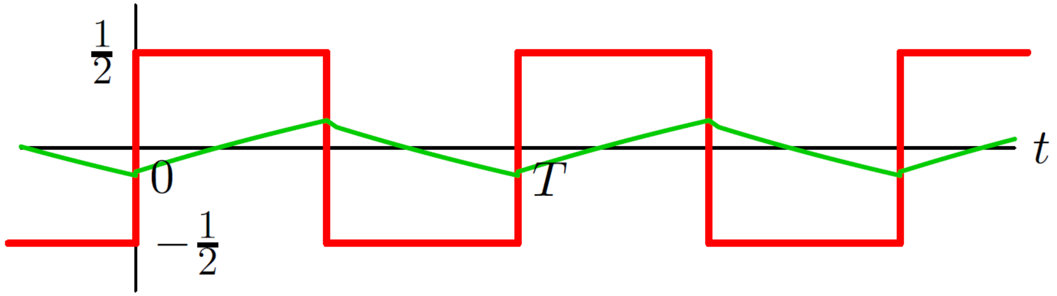
Output is a “Filtered” Version of Input



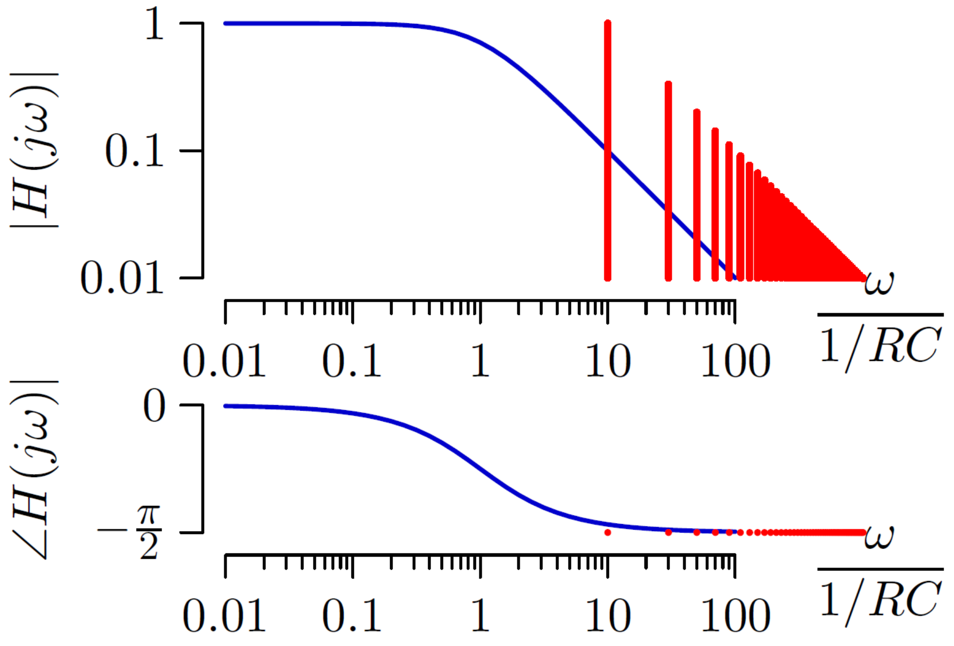
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \;; \qquad \omega_0 = \frac{2\pi}{T}$$



Output is a “Filtered” Version of Input



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$



Response to a Square Wave Input: MATLAB

- Decompose a square wave to a linear combination of sinusoidal signals

$$x(t) = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{1}{jk\pi} e^{j2\pi kt} = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{2}{j\omega_0 k} e^{j\omega_0 kt}$$

- The output response of LTI

$$\dot{y} + \frac{1}{\tau} y = \frac{1}{\tau} x(t)$$

Response to a Square Wave Input: MATLAB

- Decompose a square wave to a linear combination of sinusoidal signals

$$x(t) = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{1}{jk\pi} e^{j2\pi kt} = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{2}{j\omega_0 k} e^{j\omega_0 kt}$$

- The output response of LTI

$$\dot{y} + \frac{1}{\tau} y = \frac{1}{\tau} x(t)$$

- Given input $e^{j\omega t}$

$$\dot{y} + \frac{1}{\tau} y = \frac{1}{\tau} e^{j\omega t}$$

- $y = Ae^{j(\omega t+\phi)}$

$$j\omega A e^{j(\omega t+\phi)} + \frac{1}{\tau} A e^{j(\omega t+\phi)} = \frac{1}{\tau} e^{j\omega t}$$

$$\left(j\omega + \frac{1}{\tau}\right) A e^{j\phi} = \frac{1}{\tau}$$

$$\boxed{\begin{aligned} |H(j\omega)| &= A = \frac{1}{|j\tau\omega + 1|} \\ \angle H(j\omega) &= \phi = -\angle(j\tau\omega + 1) \end{aligned}}$$

Response to a Square Wave Input: MATLAB

- Linearity: input $\sum a_k x_k(t)$ produces $\sum a_k y_k(t)$

```
% xhat = Fourier series of square ware
xhat = zeros(size(t));

for k = -19:2:19
    xhat = xhat + 2/(1j*w0*k)*exp(1j*w0*k*t);
end
```

$$x(t) = \sum_{k=\cdots,-3,-1,1,3,\cdots} \frac{2}{j\omega_0 k} e^{j\omega_0 kt}$$

```
tau = 1/5;

yhat = zeros(size(t));

for k = -19:2:19
    w = w0*k;
    A = 1./abs(1j*tau*w + 1);
    Ph = -phase(1j*tau*w + 1);
    yhat = yhat + A*2/(1j*w0*k)*exp(1j*(w*t + Ph));
end
```

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt}$$

$$|H(j\omega)| = A = \frac{1}{|j\tau\omega + 1|}$$
$$\angle H(j\omega) = \phi = -\angle(j\tau\omega + 1)$$

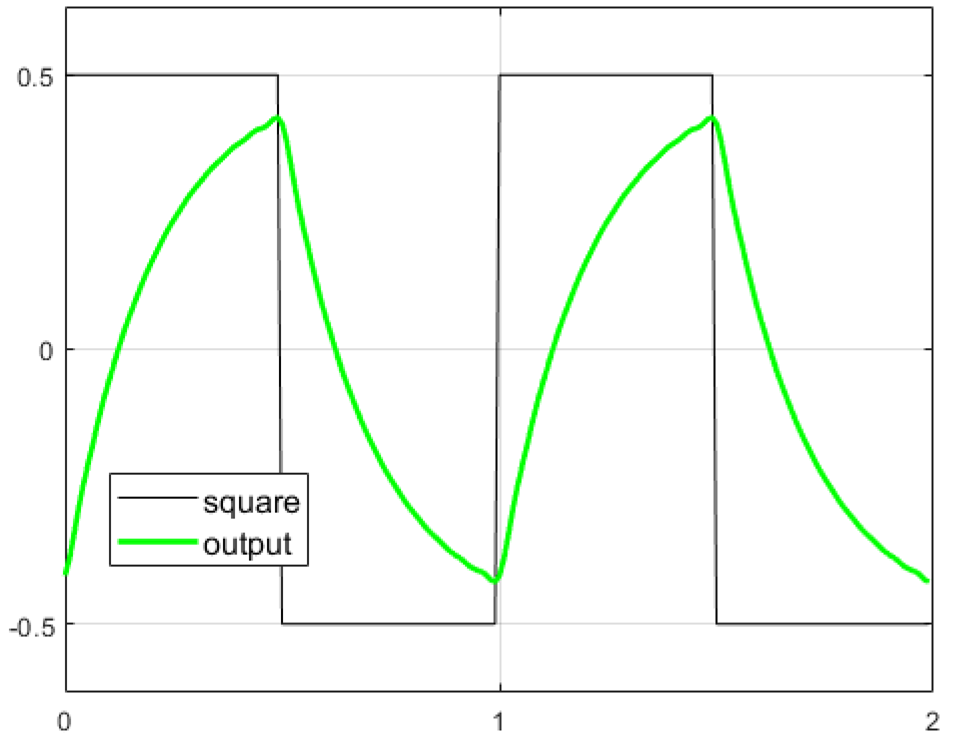
Response to a Square Wave Input: MATLAB

- Linearity: input $\sum a_k x_k(t)$ produces $\sum a_k y_k(t)$

$$\dot{y} + \frac{1}{\tau} y = \frac{1}{\tau} x(t)$$

```
% xhat = Fourier series of square ware
xhat = zeros(size(t));

for k = -19:2:19
    xhat = xhat + 2/(1j*w0*k)*exp(1j*w0*k*t);
end
```

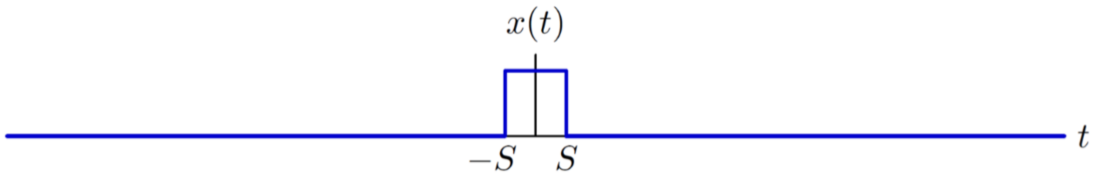
```
tau = 1/5;

yhat = zeros(size(t));

for k = -19:2:19
    w = w0*k;
    A = 1./abs(1j*tau*w + 1);
    Ph = -phase(1j*tau*w + 1);
    yhat = yhat + A*2/(1j*w0*k)*exp(1j*(w*t + Ph));
end
```



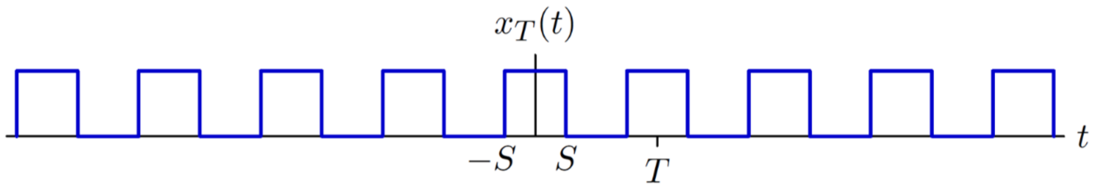
Response to General Input (in Frequency)

Response to a General Input (Aperiodic Signal) in Frequency Domain

- An aperiodic signal can be thought of as periodic with infinite period
- Let $x(t)$ represent an aperiodic signal



- Periodic extension



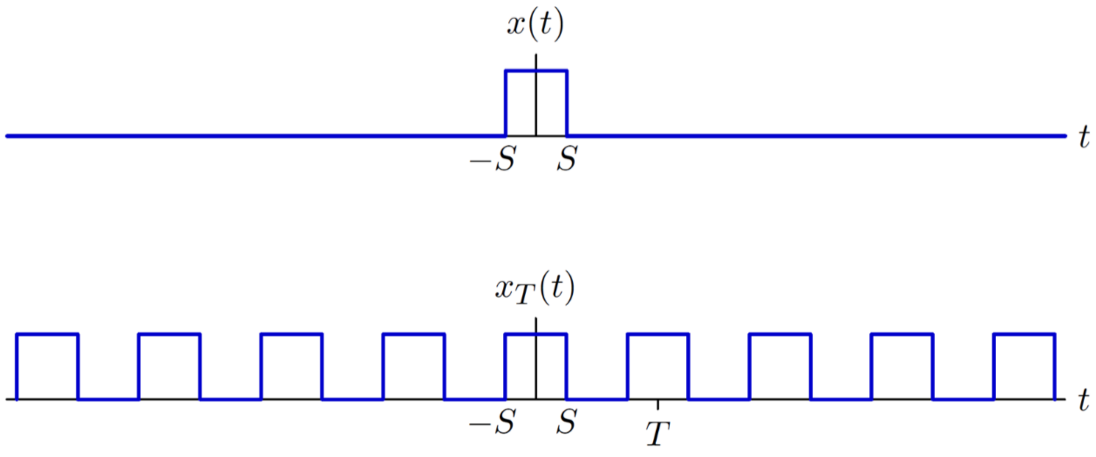
$$x_T(t) = \sum_{k=-\infty}^{\infty} x(t + kT)$$

- Then

$$x(t) = \lim_{T \to \infty} x_T(t)$$

Example: Periodic Square Wave

$$x(t) = \begin{cases} 1, & |t| < s \\ 0, & s < |t| < \frac{T}{2} \end{cases}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x_T(t)e^{-j\omega_0 kt}dt \\ &= \frac{1}{T}\left[\int_{-s}^{s} 1\cdot e^{-j\omega_0 kt}dt\right] = \frac{1}{T}\left[-\frac{e^{-j\omega_0 kt}}{j\omega_0 k}\Big|_{-s}^{s}\right] \\ &= -\frac{1}{T}\left[\frac{e^{-j\omega_0 ks}}{j\omega_0 k} - \frac{e^{j\omega_0 ks}}{j\omega_0 k}\right] = \frac{1}{T}\left[\frac{-2}{j\omega_0 k}\cdot\frac{e^{-j\omega_0 ks} - e^{j\omega_0 ks}}{2}\right] \\ &= \frac{1}{T}\frac{-2}{j\omega_0 k}(-j\sin\omega_0 ks) = \frac{1}{T}\frac{2\sin\omega_0 ks}{\omega_0 k} \\ &= \frac{2}{T}\frac{\sin\omega_0 ks}{\omega_0 k} \end{aligned}$$

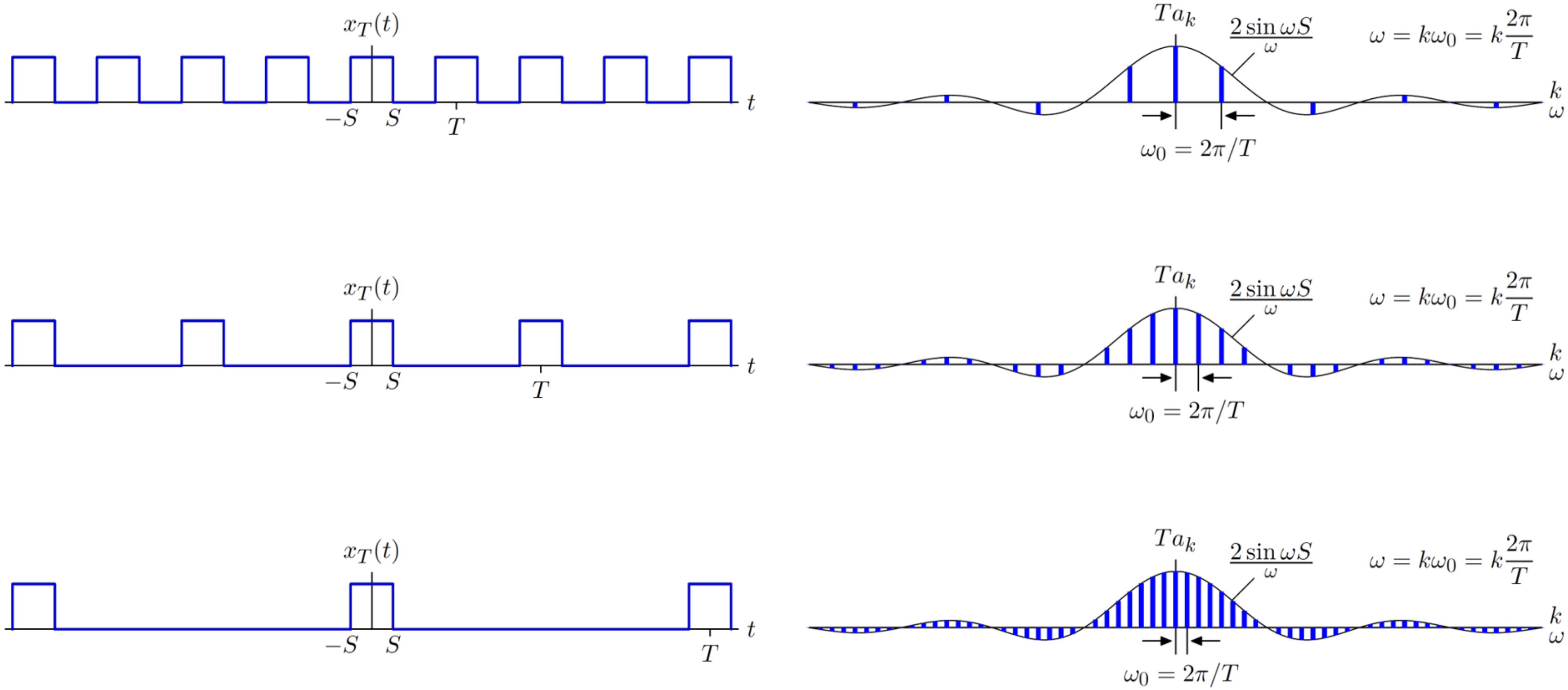
$$Ta_k = \frac{2\sin\omega s}{\omega}, \quad \omega = k\omega_0, \quad \omega_0 = \frac{2\pi}{T}$$

POSTECH

Example: Periodic Square Wave

$$Ta_k = \frac{2\sin\omega s}{\omega}, \quad \omega = k\omega_0, \quad \omega_0 = \frac{2\pi}{T}$$

- Doubling period doubles # of harmonics in given frequency interval



Example: Periodic Square Wave

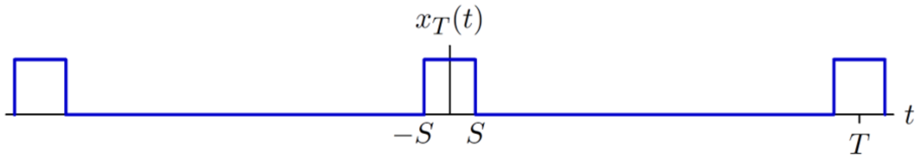
- As $T \to \infty$, discrete harmonic amplitudes $\to$ a continuum $X(j\omega)$

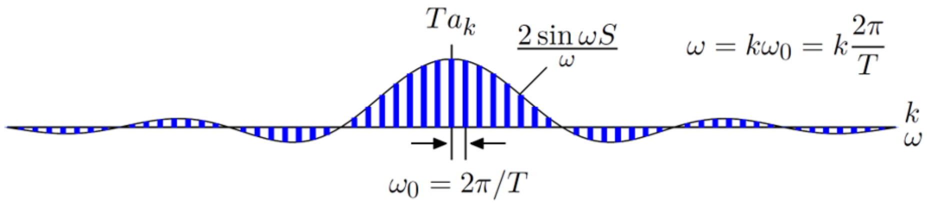
$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)e^{-j\omega t}dt = \frac{2\sin\omega s}{\omega} = X(j\omega)$$

- As alternative way of interpreting is as samples of an envelope function, specifically

$$Ta_k = \frac{2\sin\omega s}{\omega}\bigg|_{\omega=k\omega_0}$$

- That is, with ω thought of as a continuous variable, the set of Fourier series coefficients approaches the envelop function as $T \to \infty$





Fourier Transform

Fourier Transform

- As $T \to \infty$, synthesis sum $\to$ integral

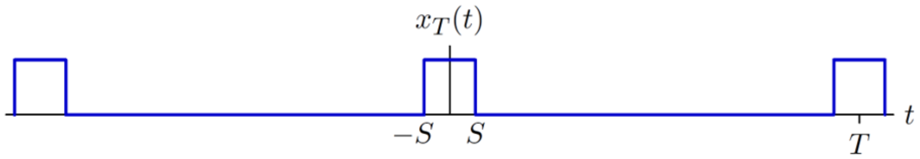
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} \frac{X(j\omega_0 k)}{T} e^{j\omega_0 kt}$$

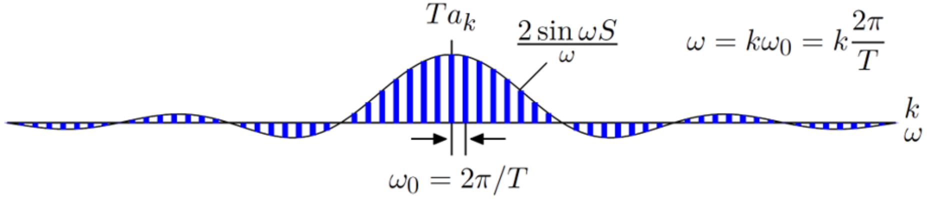
$$= \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(j\omega_0 k) e^{j\omega_0 kt},$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j\omega_0 k) e^{j\omega_0 kt} \omega_0, \quad \omega = k\omega_0, \quad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{2\pi} \int X(j\omega) e^{j\omega t} d\omega$$

- Aperiodic signal has all the frequency components instead of discrete harmonic components





Fourier Transform

- Definition: Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{analysis}$$

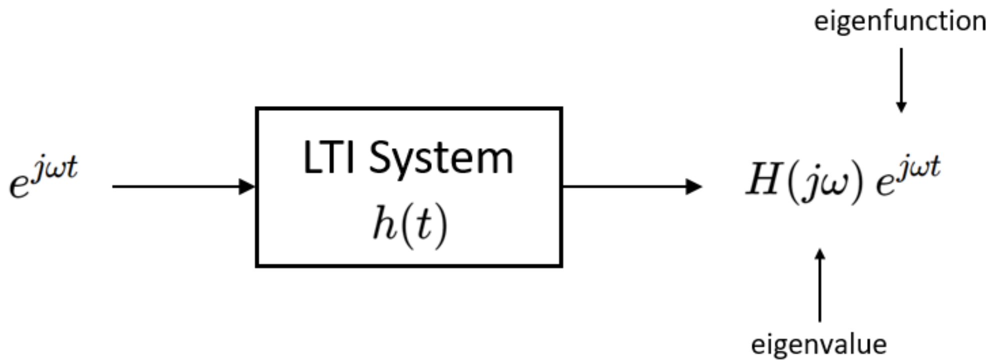
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{synthesis}$$

Response to General Input

- Response to LTI system with impulse response $h(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\underline{e^{j\omega t}}\,d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\underline{H(j\omega)e^{j\omega t}}\,d\omega$$



$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X(j\omega)H(j\omega)}e^{j\omega t}\,d\omega$$

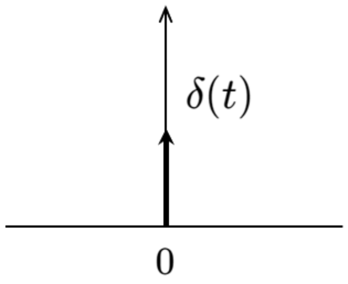
$$x(t) * h(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad X(j\omega)H(j\omega)$$

Magic of Impulse Response

- Fourier transform of Dirac delta function

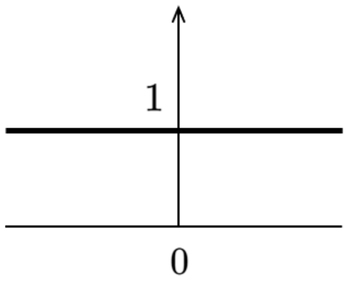
$$\delta(t) \xrightarrow{\text{LTI}} h(t)$$

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$



Time domain

=



Frequency domain

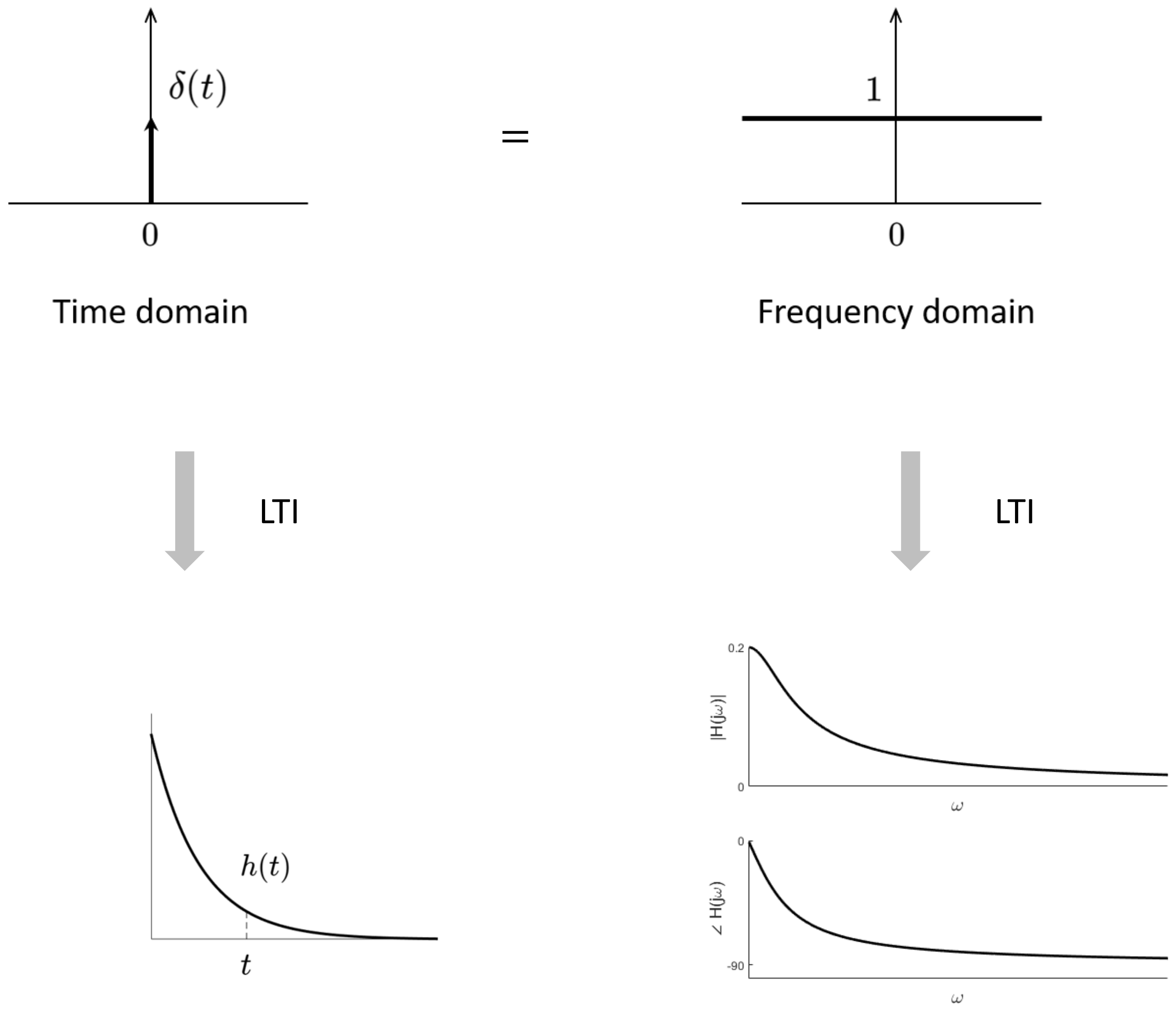
- Dirac delta function contains all the frequency components with 1
  - Convolution in time
  - Filtering in frequency

Magic of Impulse Response

- Impulse basically excites a system with all the frequency of $e^{j\omega t}$

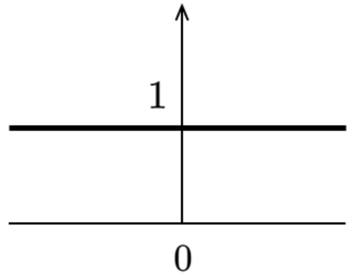
- Impulse response contains the information on how much magnitude and phase are filtered via the LTI system at all the frequency

$$x(t) * h(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad X(j\omega)H(j\omega)$$



Frequency Response (Frequency Sweep)

- Frequency sweeping is another way to collect LTI system characteristics
  - (same as the impulse response)



Frequency domain

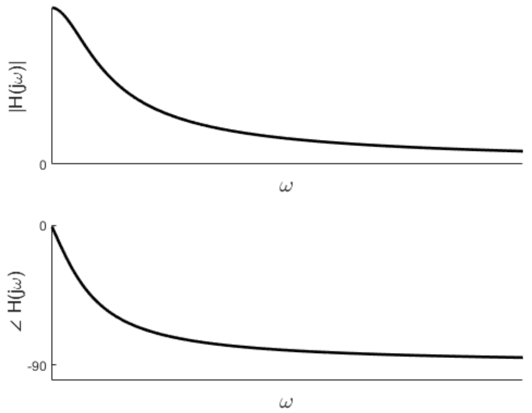
- Given input $e^{j\omega t}$

$$\dot{y} + 5y = 5e^{j\omega t}$$

- $y = Ae^{j(\omega t+\phi)}$

LTI

$$j\omega A e^{j(\omega t+\phi)} + 5Ae^{j(\omega t+\phi)} = 5e^{j\omega t}$$
$$(j\omega + 5)\, Ae^{j\phi} = 5$$



$$|H(j\omega)| = A = \frac{5}{|j\omega + 5|}$$
$$\angle H(j\omega) = \phi = -\angle(j\omega + 5)$$

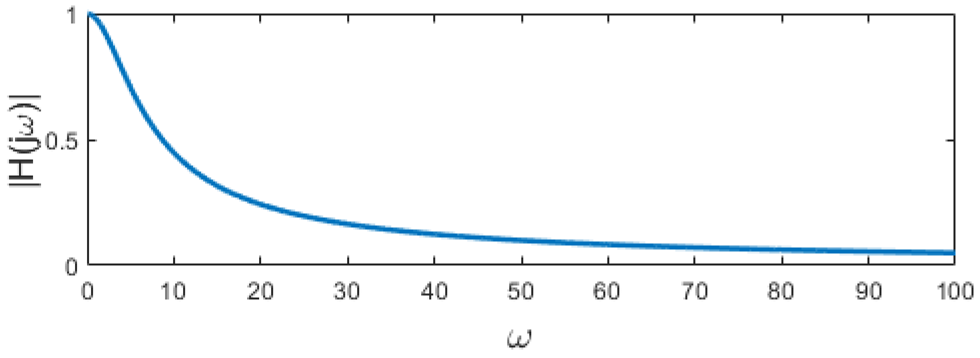
The First Order ODE: MATLAB

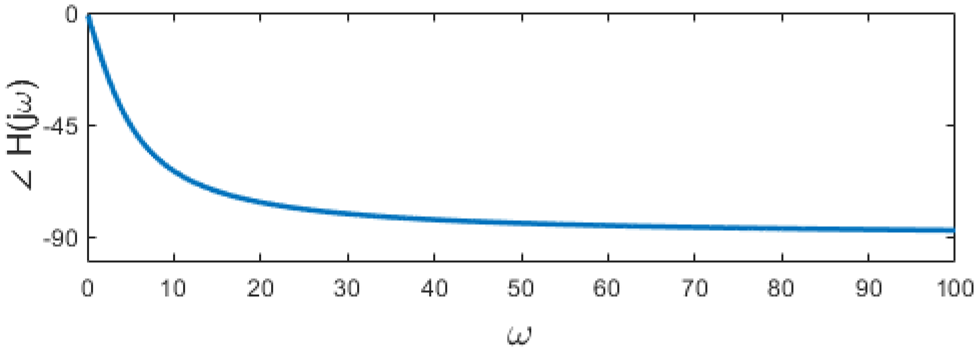
$$\dot{y} + 5y = 5e^{j\omega t}$$

$$|H(j\omega)| = A = \frac{5}{|j\omega + 5|}$$

$$\angle H(j\omega) = \phi = -\angle(j\omega + 5)$$

```
w = 0.1:0.1:100;

A = 5./abs(1j*w+5);
P = -angle(1j*w+5)*180/pi;
```





Example: The Second Order ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2\ x(t)$$
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2\ e^{j\Omega t}$$

$$y = Ae^{j(\Omega t+\phi)}$$

$$\left(-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2\right) Ae^{j\phi}e^{j\Omega t} = \omega_n^2 e^{j\Omega t}$$

$$Ae^{j\phi} = \frac{\omega_n^2}{-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2} = \frac{1}{1-\left(\frac{\Omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\Omega}{\omega_n}\right)}$$

$$A = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{\left(1-\gamma^2\right)^2 + 4\zeta^2\gamma^2}}, \quad \left(\gamma = \frac{\Omega}{\omega_n}\right)$$

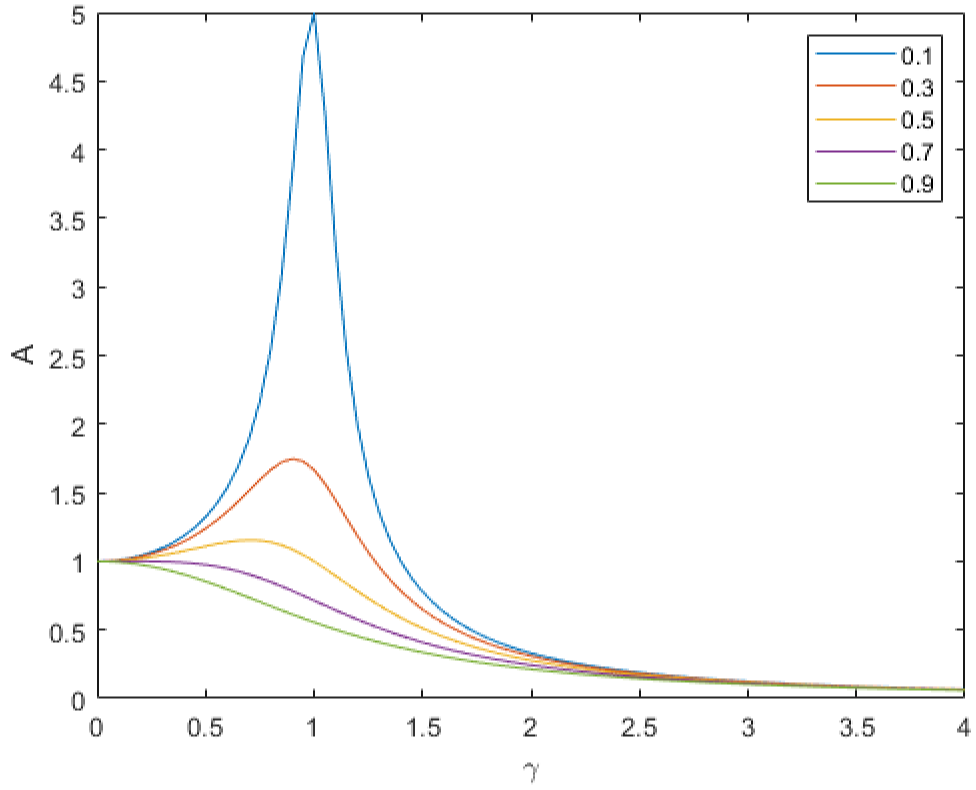
$$\phi = -\tan^{-1}\left(\frac{2\zeta\frac{\Omega}{\omega_n}}{1-\left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta\gamma}{1-\gamma^2}\right)$$

The Second Order ODE: MATLAB

$$A = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\zeta^2\gamma^2}}, \quad \left(\gamma = \frac{\Omega}{\omega_n}\right)$$

$$\phi = -\tan^{-1}\left(\frac{2\zeta\frac{\Omega}{\omega_n}}{1-\left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta\gamma}{1-\gamma^2}\right)$$

```
r = 0:0.05:4;

zeta = 0.1:0.2:1;
A = [];
for i = 1:length(zeta)
    A(i,:) = 1./sqrt((1-r.^2).^2 + (2*zeta(i)*r).^2);
end
```

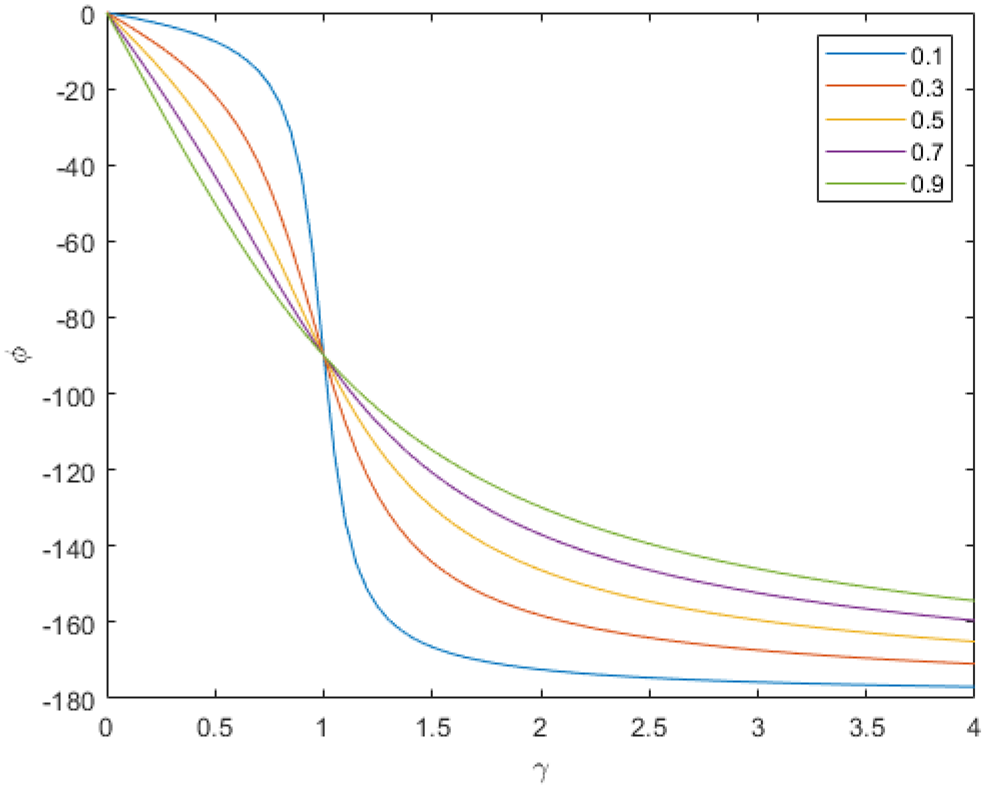


The Second Order ODE: MATLAB

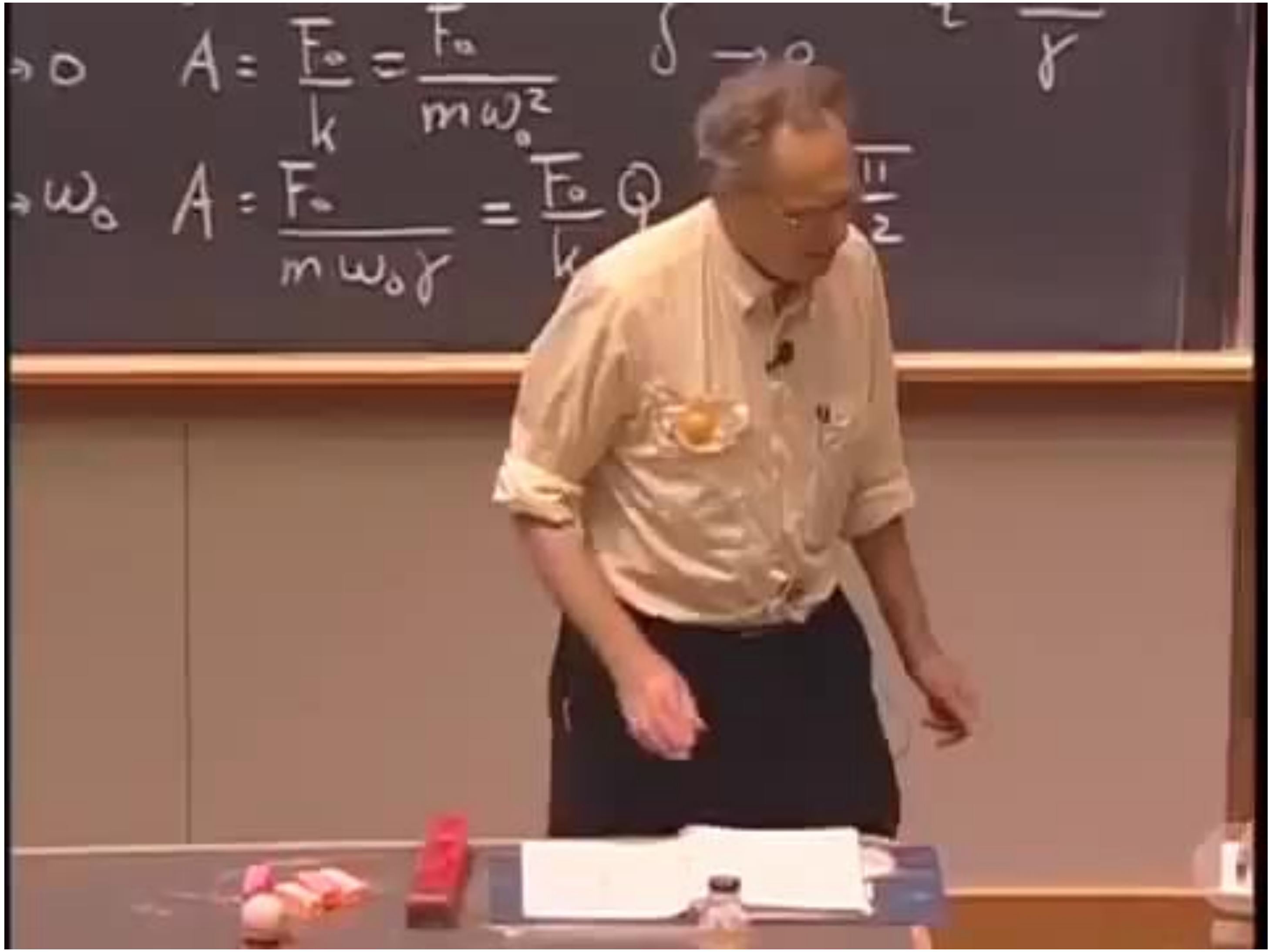
$$A = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\zeta^2\gamma^2}}, \quad \left(\gamma = \frac{\Omega}{\omega_n}\right)$$

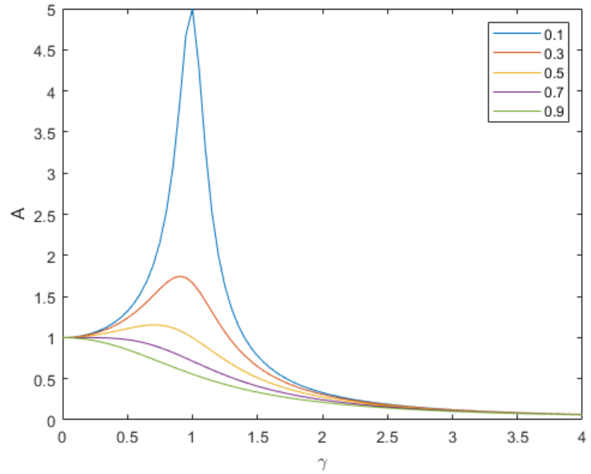
$$\phi = -\tan^{-1}\left(\frac{2\zeta\frac{\Omega}{\omega_n}}{1-\left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta\gamma}{1-\gamma^2}\right)$$

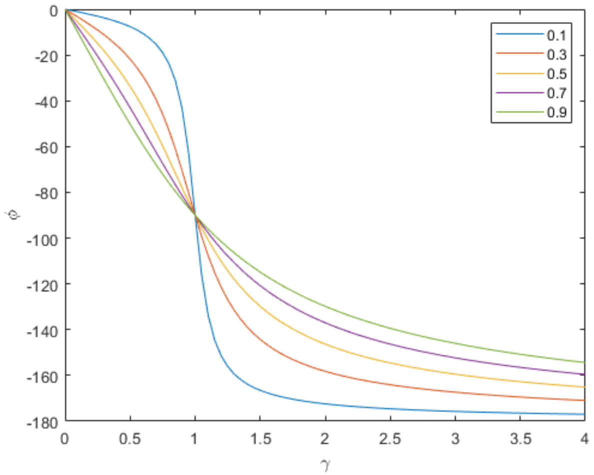
```
phi = [];
for i = 1:length(zeta)
    phi(i,:) = -atan2((2*zeta(i).*r),(1-r.^2));
end

plot(r,phi*180/pi)
xlabel('\gamma')
ylabel('\phi')
legend('0.1','0.3','0.5','0.7','0.9')
```



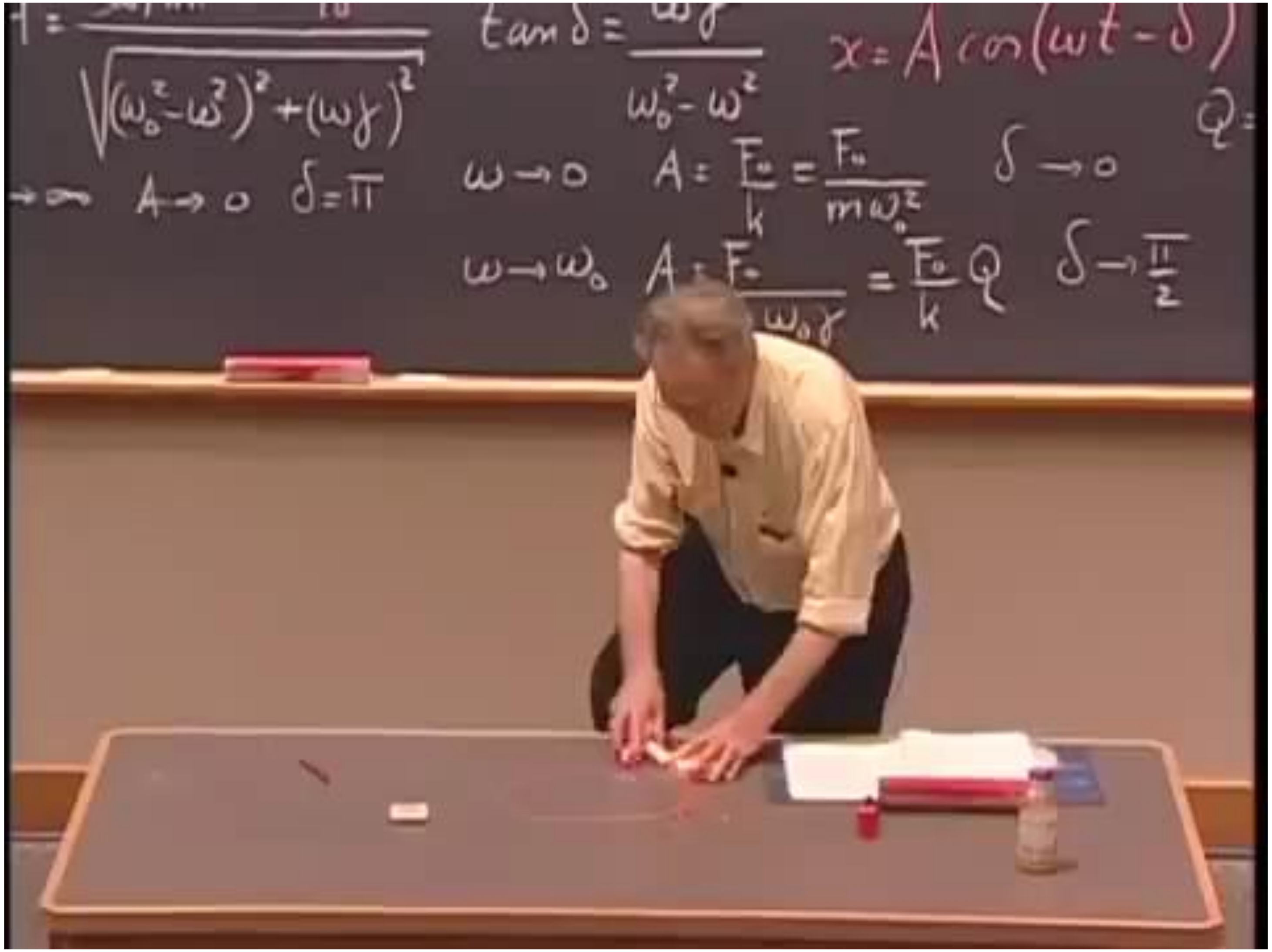
Experiment: The Second Order ODE

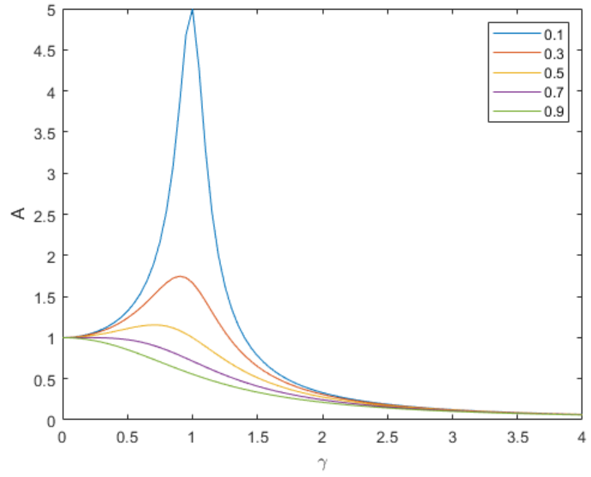


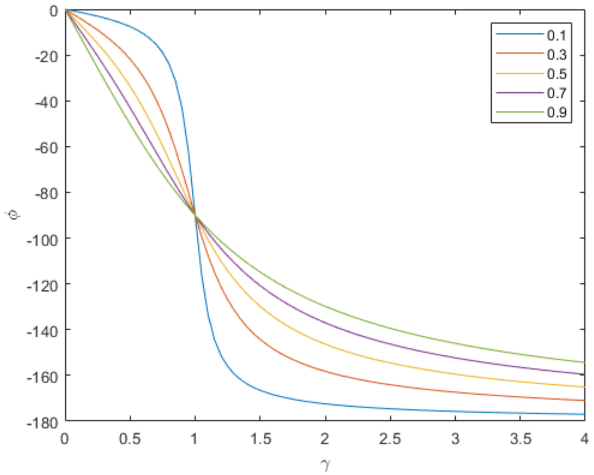




Experiment: The Second Order ODE

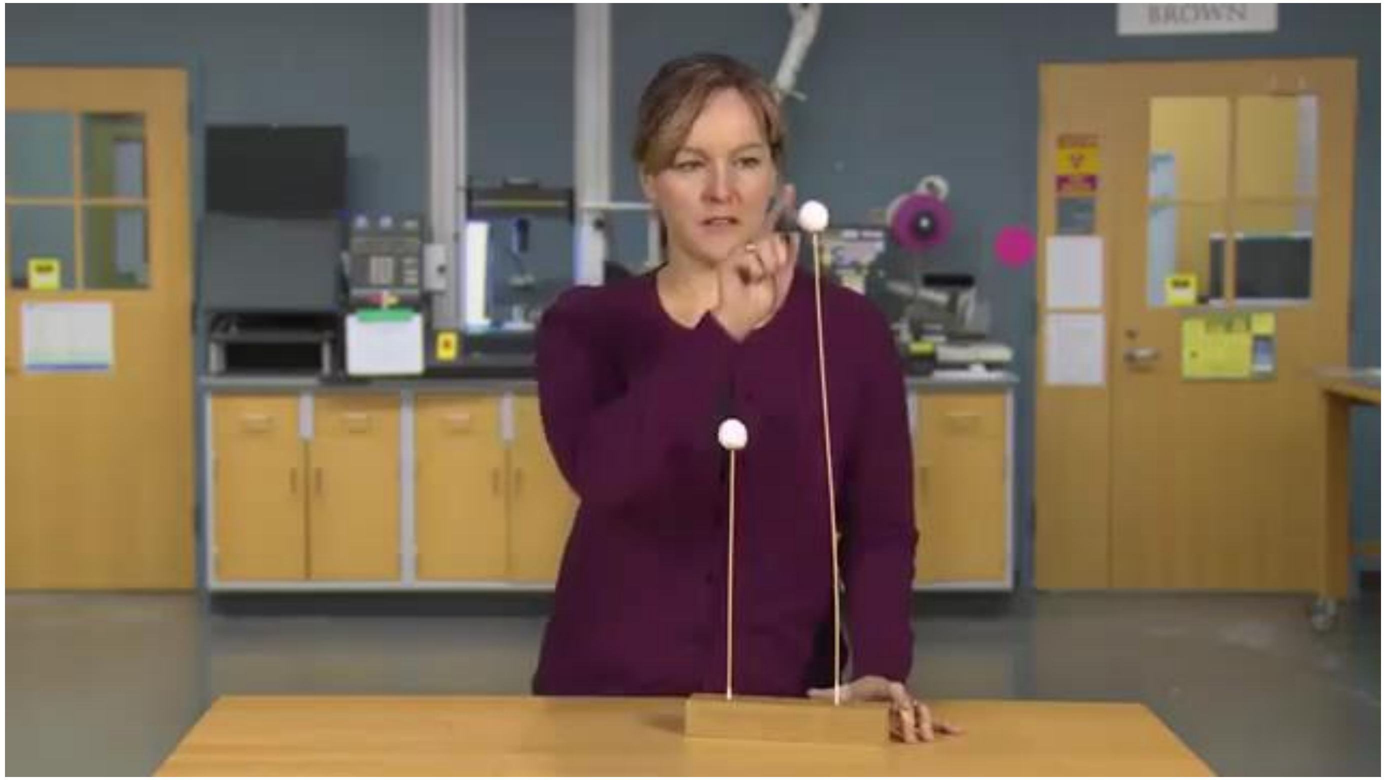




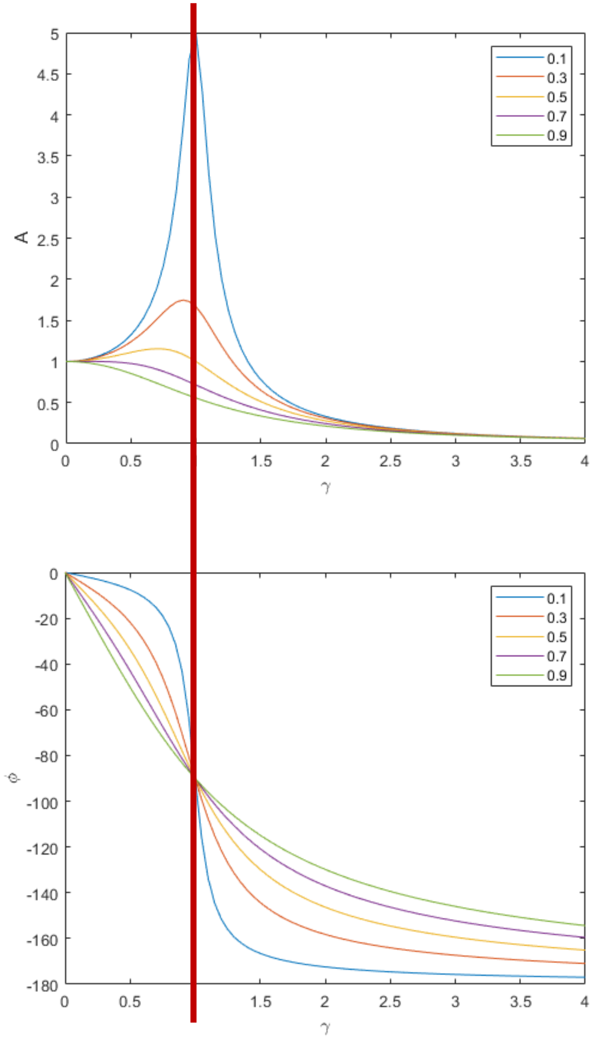


Resonance

- Input frequency near resonance frequency
- Resonance frequency is generally different from natural frequency, but they often are close enough



Resonance frequency



Summary

- To understand LTI system
- Impulse response
- Frequency sweep