Linear Algebra

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Linear Equations

- Two linear equations

$$\begin{aligned} 4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9 \end{aligned}$$

- In a vector form

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

- Solution using inverse

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

System of Linear Equations

- Consider a system of linear equations

$$
\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}
$$

- Can be written in a matrix form as $y = Ax$, where

$$
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

System of Linear Equations

1) Well-determined linear systems

2) Under-determined linear systems

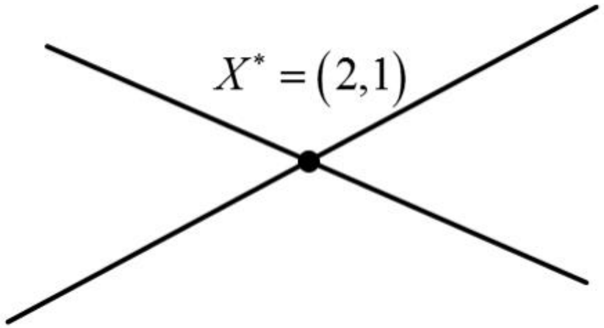
3) Over-determined linear systems

1) Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Geometric point of view



1) Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \implies \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Matrix form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned} \quad \overset{\text{Matrix form}}{\Longrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

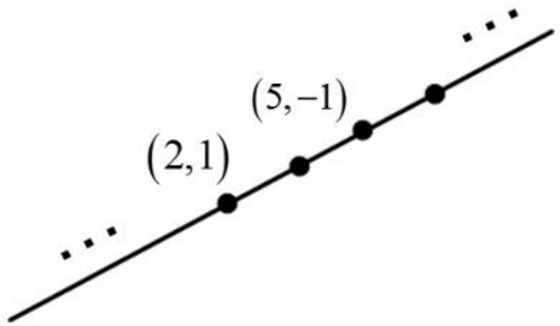
$$\therefore \quad X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

2) Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Geometric point of view



2) Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Matrix form

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \xRightarrow{\text{Matrix form}} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

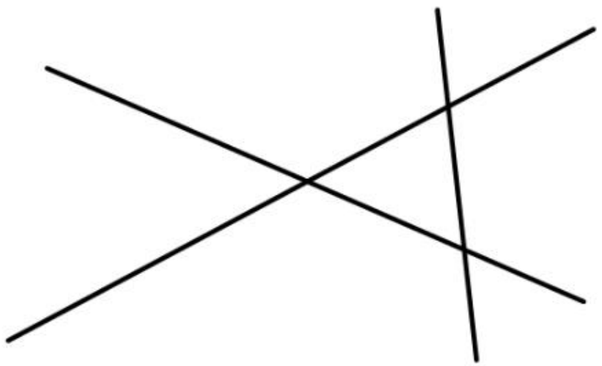
$$\therefore \ \text{Many Solutions when } A \text{ is fat}$$

3) Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \\ x_1 + x_2 &= 4 \end{aligned} \implies \text{No solutions}$$

- Geometric point of view



3) Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \\ x_1 + x_2 &= 4 \end{aligned} \implies \text{No solutions}$$

- Matrix form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 = b_3 \end{aligned} \quad \overset{\text{Matrix form}}{\implies} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

$$\therefore \text{ No Solutions when } A \text{ is skinny}$$

Vector-Vector Products

- Inner product: $x, y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^{n} x_i \, y_i \qquad \in \mathbb{R}$$

Norm

- l_2 norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- $\|x\|$ Measures length of vector (from the origin)

Orthogonality

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if

$$x^T y = 0 \quad \text{and} \quad ||x||_2 = ||y||_2 = 1$$

Angle between Vectors

- (unsigned) angle between vectors in $\mathbb{R}^n$ defined as

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

$$\text{thus } x^T y = \|x\| \|y\| \cos\theta$$

Half Space

- $\{x|x^T y \leq 0\}$ defines a half space with outward normal vector y, and boundary passing through 0