



Linear Algebra 1

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Linear Equations

- Set of linear equations (two equations, two unknowns)

$$4x_1 - 5x_2 = -13$$

$$-2x_1 + 3x_2 = 9$$

Linear Equations

- Solving linear equations

- Two linear equations

$$4x_1 - 5x_2 = -13$$

$$-2x_1 + 3x_2 = 9$$

- In a vector form, $Ax = b$, with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

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- Solution using inverse

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

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- Solution using inverse

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

- Don't worry here about how to compute matrix inverse
- We will use a numpy to compute

Linear Equations in Python

$$\begin{aligned}4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9\end{aligned}$$

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

```
import numpy as np
```

```
A = np.array([[4, -5],  
              [-2, 3]])  
b = np.array([[-13],  
              [9]])  
  
x = np.linalg.inv(A).dot(b)  
print(x)
```

```
[[ 3.]  
 [ 5.]]
```

```
A = np.asmatrix(A)  
b = np.asmatrix(b)  
  
x = A.I*b  
print(x)
```

```
[[ 3.]  
 [ 5.]]
```

System of Linear Equations

- Consider a system of linear equations

$$\begin{aligned}y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\&\vdots \\y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{aligned}$$

- Can be written in a matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Elements of a Matrix

- Can write a matrix in terms of its columns

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

- Careful, a_i here corresponds to an entire vector $a_i \in \mathbb{R}^m$
- Similarly, can write a matrix in terms of rows

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}$$

- $b_i \in \mathbb{R}^n$

Vector-Vector Products

- Inner product: $x, y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$$

```
x = np.array([[1],  
              [1]])  
y = np.array([[2],  
              [3]])
```

```
print(x.T.dot(y))
```

```
[[5]]
```

```
x = np.asmatrix(x)  
y = np.asmatrix(y)
```

```
print(x.T*y)
```

```
[[5]]
```

Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$
- Writing A by rows, each entry of Ax is an inner product between x and a row of A

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_m^T x \end{bmatrix}$$

Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$
- Writing A by columns, Ax is a linear combination of the columns of A , with coefficients given by x

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \sum_{i=1}^n a_i x_i$$

Norms (Strength or Distance in Linear Space)

- A vector norm is any function $f: \mathbb{R}^n \Rightarrow \mathbb{R}$ with

1. $f(x) \geq 0$ and $f(x) = 0 \iff x = 0$
2. $f(ax) = |a|f(x)$ for $a \in \mathbb{R}$
3. $f(x + y) \leq f(x) + f(y)$

$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

- l_2 norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

- l_1 norm

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

- $\|x\|$ measures length of vector (from origin)

Norms in Python

```
x = np.array([[4],  
              [3]])  
np.linalg.norm(x, 2)
```

5.0

```
np.linalg.norm(x, 1)
```

7.0

Orthogonality

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if

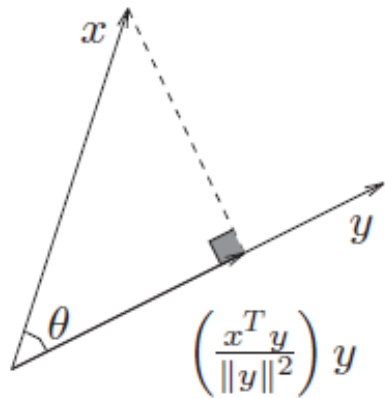
$$x^T y = 0 \quad \text{and} \quad \|x\|_2 = \|y\|_2 = 1$$

Angle between Vectors

- For any $x, y \in \mathbb{R}^n$,

$$|x^T y| \leq \|x\| \|y\|$$

- (unsigned) angle between vectors in \mathbb{R}^n defined as

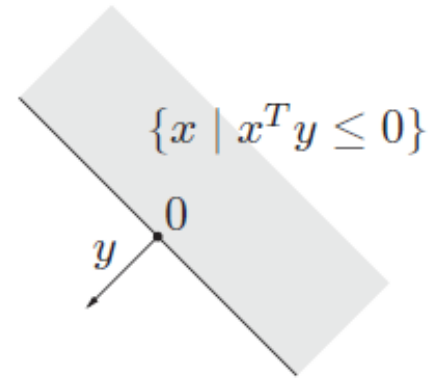
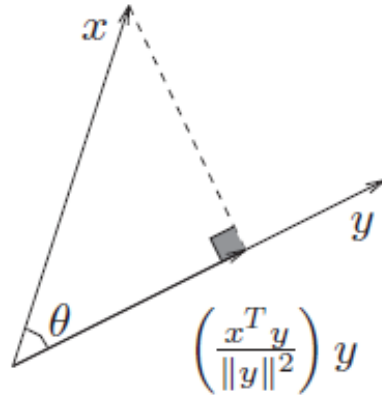


$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

$$\text{thus } x^T y = \|x\| \|y\| \cos \theta$$

Angle between Vectors

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$



- $\{x \mid x^T y \leq 0\}$ defines a half space with outward normal vector y , and boundary passing through 0



Linear Algebra 2

Industrial AI Lab.

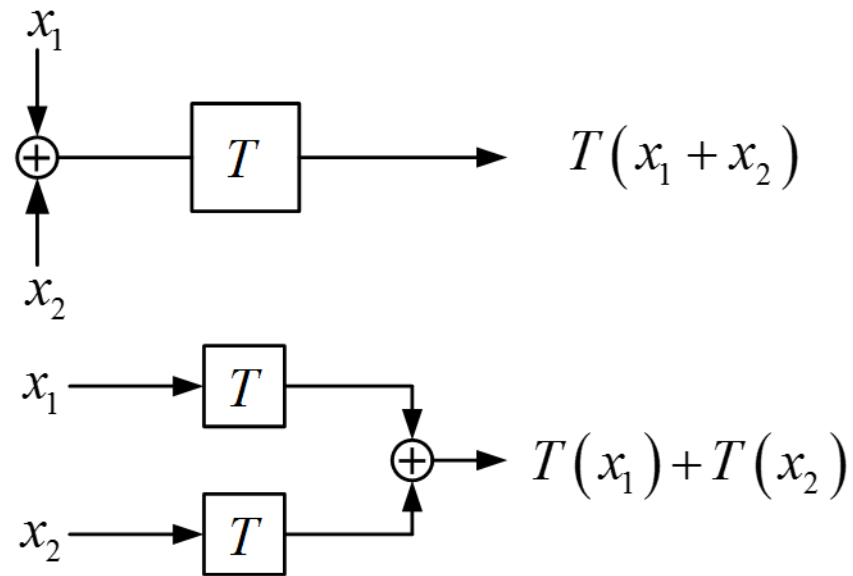
Changyun Choi, Juhyeong Jeon

Linear Transformation

- See if the given transformation is linear
 - A linear system makes our life much easier
- Superposition
- Homogeneity

Linear Transformation

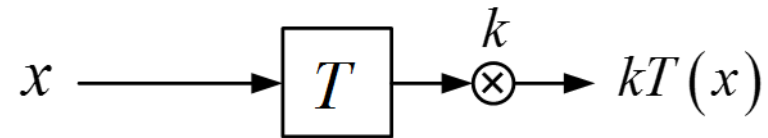
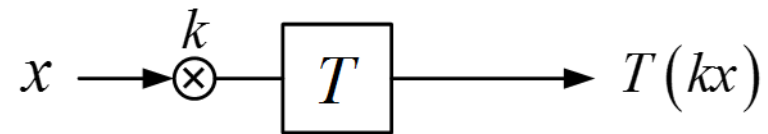
- Superposition



$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Linear Transformation

- Homogeneity



$$T(kx) = kT(x)$$

Matrix and (Linear) Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad \vec{y} = M\vec{x}$$

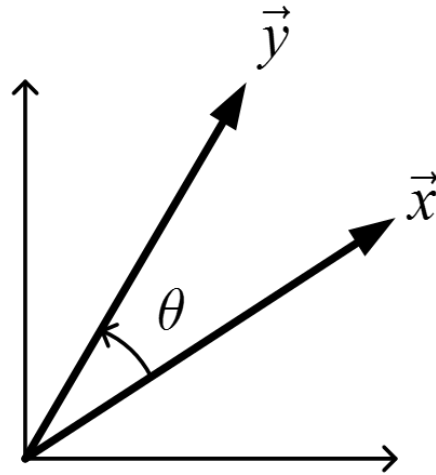
$$\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

Given	Interpret
linear transformation	→ matrix
matrix	→ linear transformation

$$\begin{array}{ccc} \vec{x} & \text{linear transformation} & \vec{y} \\ \text{input} & \Rightarrow & \text{output} \end{array}$$

Rotation

- Is a rotation operation linear?

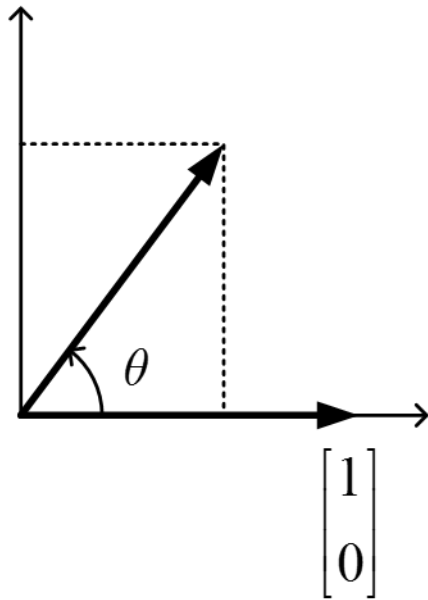


- Rotation matrix: $M = R(\theta)$
- Transformation: $\vec{y} = R(\theta)\vec{x}$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$

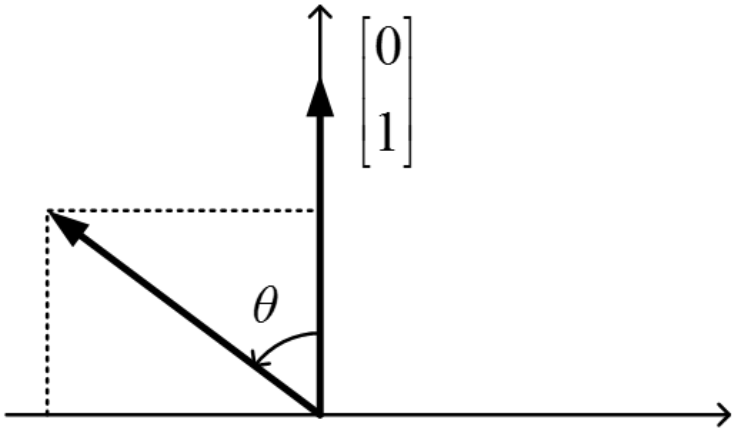


$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\begin{array}{l} M\vec{x}_1 = \vec{y}_1 \\ M\vec{x}_2 = \vec{y}_2 \end{array} \quad \Rightarrow \quad M \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R(\theta)$$

- Note on how to find a matrix from two vectors and their linearly-transformed ones

Stretch/Compress

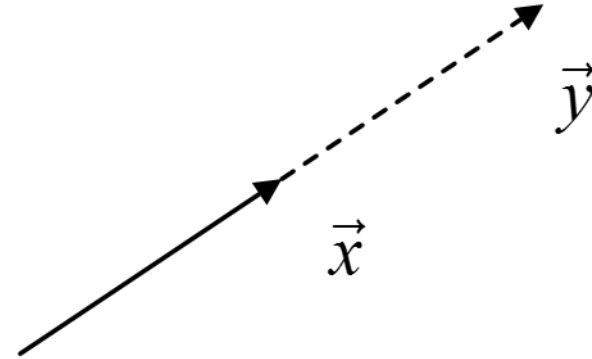
- Stretch/Compress
 - keep the direction

$$\vec{y} = k\vec{x}$$

↑
scalar (not matrix)

$$\vec{y} = kI\vec{x} \quad \text{where } I = \text{Identity matrix}$$

$$\vec{y} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$$



- Still represented by a matrix

Stretch/Compress: Example

- T : stretch by a along \hat{x} -direction & stretch by b along \hat{y} -direction
- Compute the corresponding matrix A

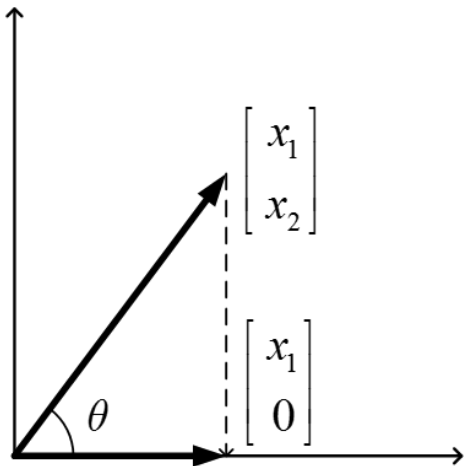
$$\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies A = ?$$
$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$
$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

- More importantly, can you think of the corresponding transformation T by looking at $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$?

Projection

- Is a projection operation linear?
- Suppose P: Projection onto \hat{x} - axis



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\vec{x}} \xRightarrow{P} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}_{\vec{y}}$$

$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ P \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Multiple Transformations

- T_1 : transformation 1 of matrix M_1
- T_2 : transformation 2 of matrix M_2
- T : Do transformation 1, followed by transformation 2

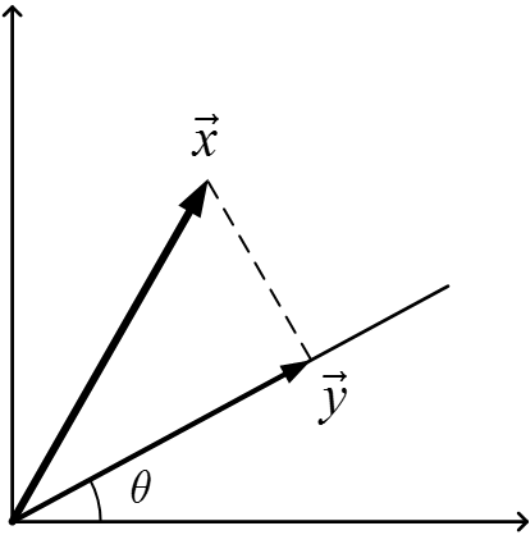
$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned}\vec{y} &= M_1 \vec{x} \\ \vec{z} &= M_2 \vec{y} = M_2 M_1 \vec{x} \\ &= M \vec{x}\end{aligned}$$

$$\therefore M = M_2 M_1$$

←

Example: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$



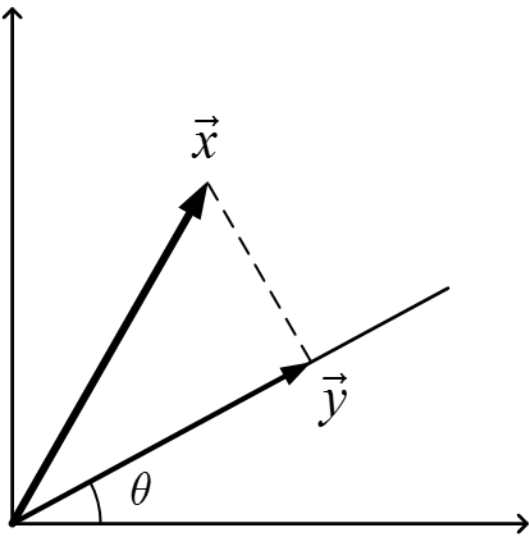
$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta \\ \cos \theta \sin \theta \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

Example: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

- Another way to find this projection matrix



$$\begin{aligned} \vec{x} &\xRightarrow{R(-\theta)} \vec{x}' \xRightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \vec{x}'' \xRightarrow{R(\theta)} \vec{y} \\ \vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta) \vec{x} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \end{aligned}$$

Eigenvalue and Eigenvector

$$A\vec{v} = \lambda\vec{v}$$

$A\vec{v}$ parallel to \vec{v}

$$\lambda = \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases}$$

$\lambda\vec{v}$: stretched vector
(same direction with \vec{v})

$A\vec{v}$: linearly transformed vector
(generally rotate + stretch)

How to Compute Eigenvalue and Eigenvector

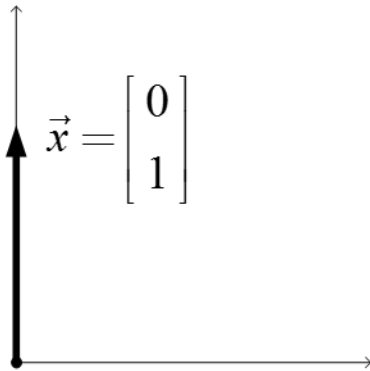
$$\begin{aligned} A\vec{v} &= \lambda\vec{v} = \lambda I\vec{v} \\ A\vec{v} - \lambda I\vec{v} &= (A - \lambda I)\vec{v} = 0 \end{aligned}$$

$$\begin{aligned} \implies A - \lambda I &= 0 \text{ or} \\ \vec{v} &= 0 \text{ or} \\ (A - \lambda I)^{-1} &\text{ does not exist} \end{aligned}$$

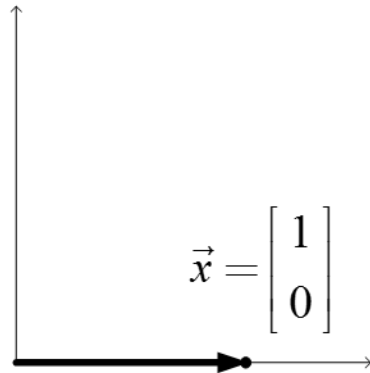
$$\implies \det(A - \lambda I) = 0$$

Example: Eigen Analysis of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

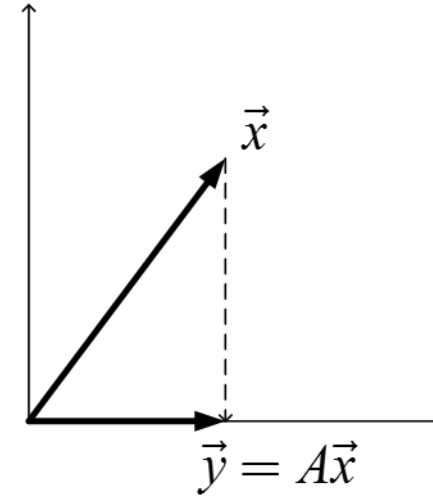
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection onto \hat{x} - axis
- Find eigenvalues and eigenvectors of A .



$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$
$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

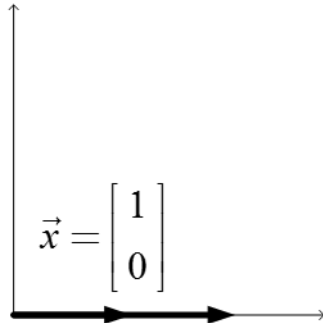


$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$
$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

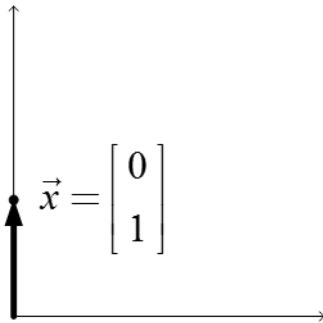


Example: Eigen Analysis of $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

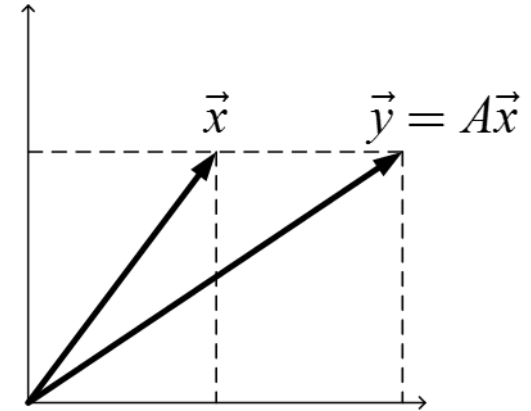
- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$: stretch by 2 along \vec{x} - axis
stretch by 1 along \vec{y} - axis
- Find eigenvalues and eigenvectors.



$$\lambda_1 = 2 \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Eigen Analysis in Python

```
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

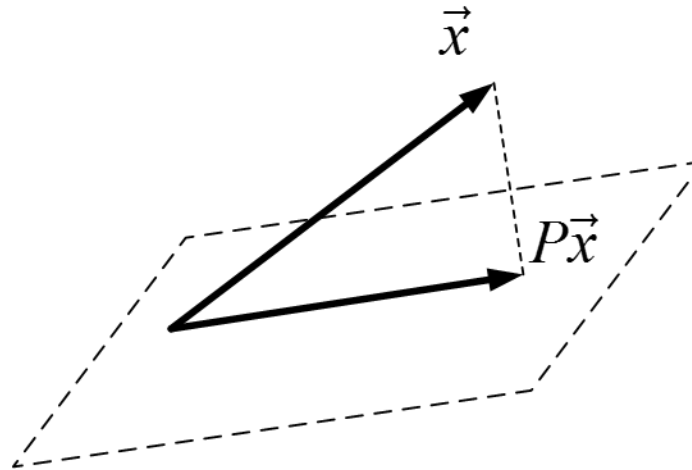
idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D :', D)
print('V :', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
     [ 0.  1.]
```

Example: Eigen Analysis of Projection

- Projection onto the plane
- Find eigenvalues and eigenvectors



- For any \vec{x} in the plane, $P\vec{x} = \vec{x} \rightarrow \lambda = 1$
- For any \vec{x} perpendicular to the plane, $P\vec{x} = \vec{0} \rightarrow \lambda = 0$



Linear Algebra 3

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System of Linear Equations

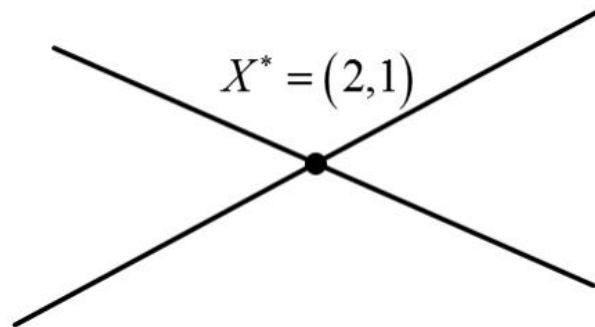
- Well-determined linear systems
- Under-determined linear systems
- Over-determined linear systems

Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{l} x_1^* = 2 \\ x_2^* = 1 \end{array}$$

- Geometric point of view



Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{l} x_1^* = 2 \\ x_2^* = 1 \end{array}$$

- Matrix form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \quad \begin{array}{l} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

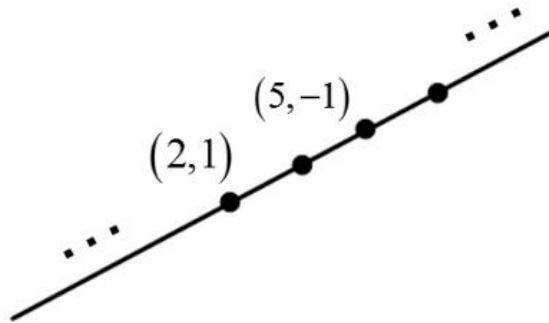
$$\therefore X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Geometric point of view



Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Matrix form

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \begin{array}{c} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

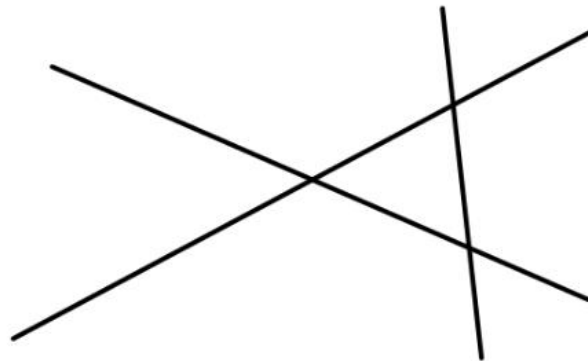
\therefore Many Solutions when A is fat

Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Geometric point of view



Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Matrix form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 = b_3 \end{array} \quad \begin{array}{l} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

\therefore No Solutions when A is skinny

Summary of Linear Systems

$$AX = B$$

- Square: Well-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Fat: Under-determined

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

- Skinny: Over-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

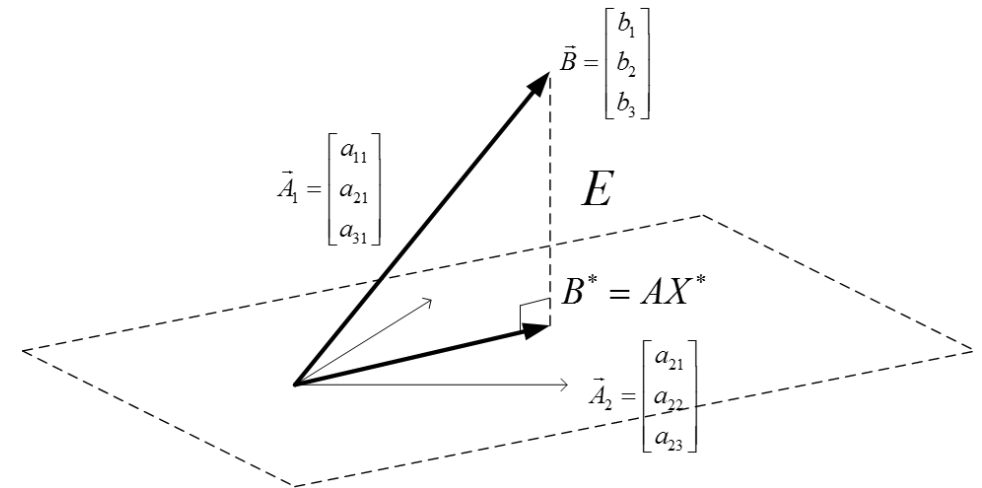
Least-Square Solution

- For over-determined linear system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{or} \quad AX \neq B$$
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Find X that minimizes $\|E\|$ or $\|E\|^2$ (error)
- i.e. optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$



Least-Square Solution

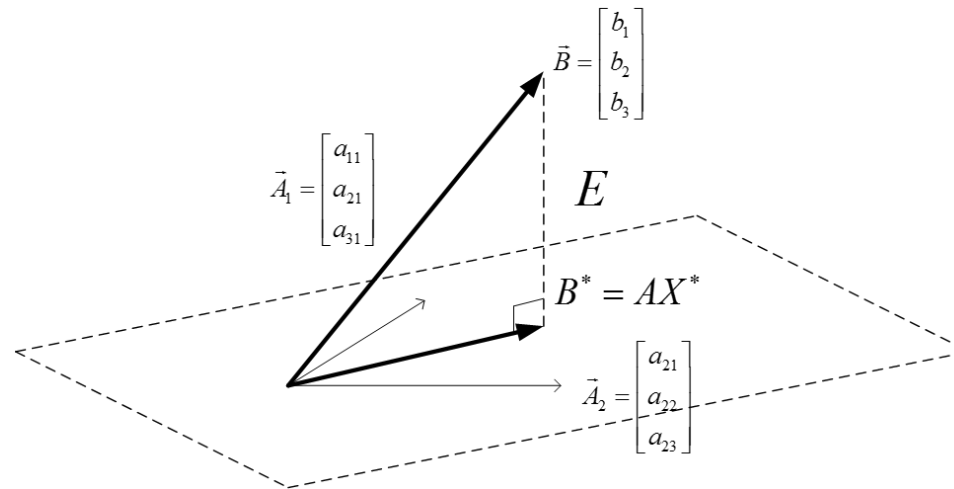
- *i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = (A^T A)^{-1} A^T B$$

$$B^* = AX^* = A(A^T A)^{-1} A^T B$$

- Geometric interpretation



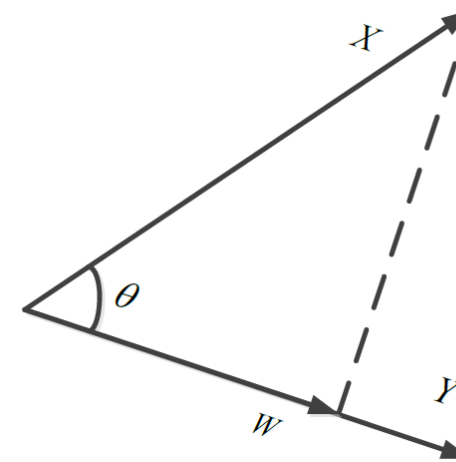
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Often estimation problem

Vector Projection onto Y

- The vector projection of a vector X on (or onto) a nonzero vector Y is the orthogonal projection of X onto a straight line parallel to Y

$$\begin{aligned} Y &\perp (X - W) \\ \implies Y^T (X - W) &= Y^T \left(X - \omega \frac{Y}{\|Y\|} \right) = 0 \\ \implies \omega &= \frac{Y^T X}{Y^T Y} \|Y\| \\ W &= \omega \frac{Y}{\|Y\|} = \frac{Y^T X}{Y^T Y} Y = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y \end{aligned}$$



Orthogonal Projection onto a Subspace

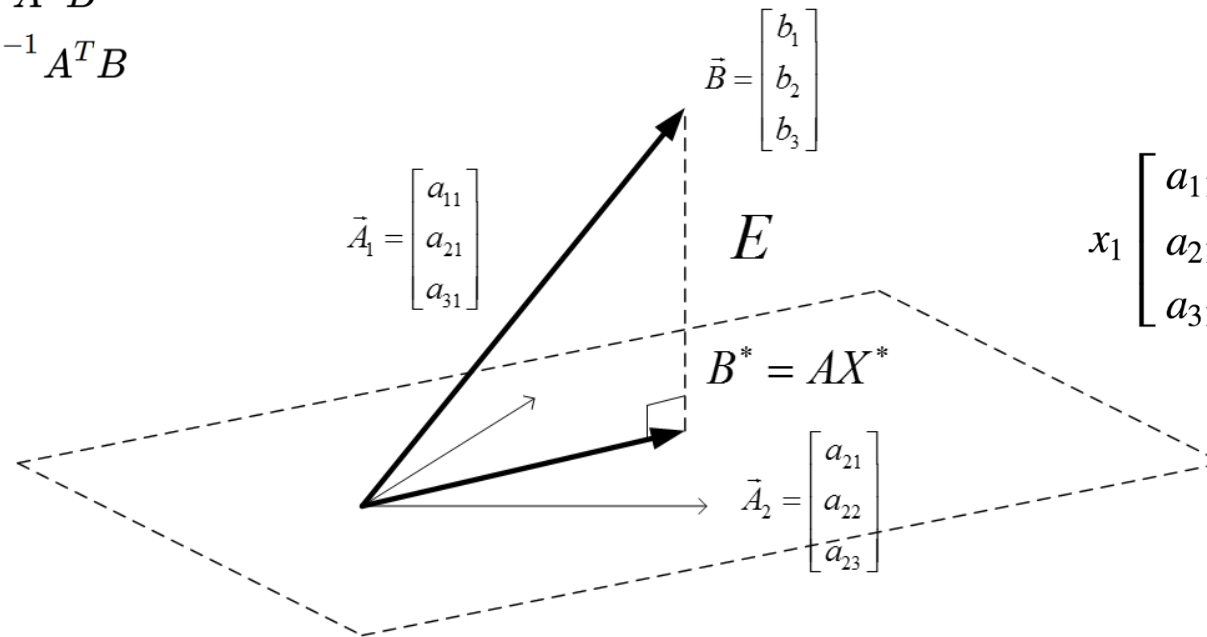
- Projection of B onto a subspace U of span of A_1 and A_2
- Orthogonality

$$\begin{aligned}
 A &\perp (AX^* - B) \\
 A^T (AX^* - B) &= 0 \\
 A^T AX^* &= A^T B \\
 X^* &= (A^T A)^{-1} A^T B \\
 B^* = AX^* &= A(A^T A)^{-1} A^T B
 \end{aligned}$$

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = (A^T A)^{-1} A^T B$$

$$B^* = AX^* = A(A^T A)^{-1} A^T B$$



$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$