



Linear Algebra

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Linear Equations

- Two linear equations

$$\begin{aligned}4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9\end{aligned}$$

- In a vector form

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

- Solution using inverse

$$\begin{aligned}Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b\end{aligned}$$

System of Linear Equations

- Consider a system of linear equations

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

- Can be written in a matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

System of Linear Equations

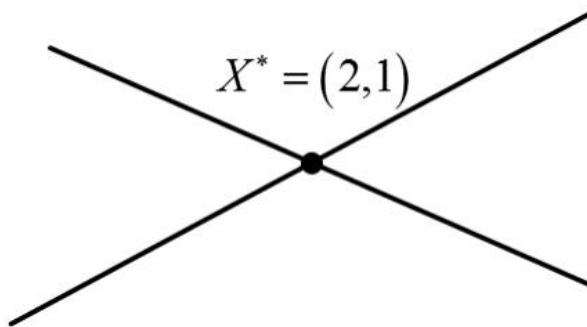
- 1) Well-determined linear systems
- 2) Under-determined linear systems
- 3) Over-determined linear systems

1) Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{rcl} x_1^* & = & 2 \\ x_2^* & = & 1 \end{array}$$

- Geometric point of view



1) Well-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \end{array} \implies \begin{array}{l} x_1^* = 2 \\ x_2^* = 1 \end{array}$$

- Matrix form

$$\begin{array}{lcl} a_{11}x_1 + a_{12}x_2 = b_1 & \text{Matrix form} & \\ a_{21}x_1 + a_{22}x_2 = b_2 & \implies & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{array}$$

$$AX = B$$

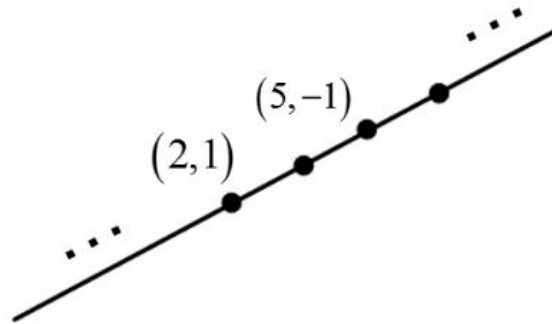
$$\therefore X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

2) Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Geometric point of view



2) Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \implies \text{Many solutions}$$

- Matrix form

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \begin{array}{c} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

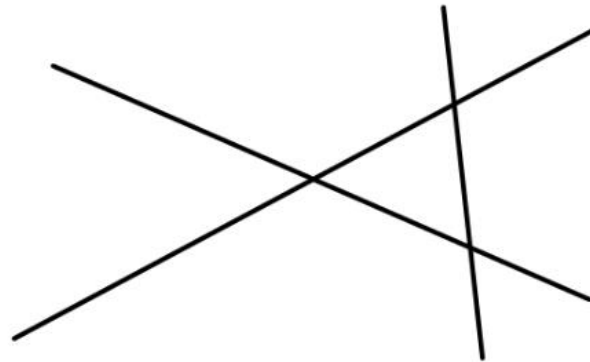
\therefore Many Solutions when A is fat

3) Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Geometric point of view



3) Over-Determined Linear Systems

- System of linear equations

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 7 \\ x_1 + 4x_2 & = & 6 \\ x_1 + x_2 & = & 4 \end{array} \implies \text{No solutions}$$

- Matrix form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 = b_3 \end{array} \quad \begin{array}{l} \text{Matrix form} \\ \implies \end{array} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

\therefore No Solutions when A is skinny

Vector-Vector Products

- Inner product: $x, y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$$

Norm

- l_2 norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

- $\|x\|$ Measures length of vector (from the origin)

Orthogonality

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if

$$x^T y = 0 \quad \text{and} \quad \|x\|_2 = \|y\|_2 = 1$$

Angle between Vectors

- (unsigned) angle between vectors in \mathbb{R}^n defined as

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

$$\text{thus } x^T y = \|x\| \|y\| \cos \theta$$

Half Space

- $\{x | x^T y \leq 0\}$ defines a half space with outward normal vector y , and boundary passing through 0

