Lecture 11: Model-Free Control

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Outline

1. Introduction
2. On-Policy Monte-Carlo Control
3. On-Policy Temporal-Difference Learning
4. Off-Policy Learning: Importance Sampling
5. Off-policy Learning: Q-learning
6. Summary
7. References

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Model-Free Reinforcement Learning

- Last lecture:
  - Model-free prediction
  - *Estimate* the value function of an *unknown* MDP
- This lecture:
  - Model-free control
  - *Optimize* the value function of an *unknown* MDP

Uses of Model-Free Control

Some example problems that can be modeled as MDPs

- Elevator
- Parallel Parking
- Ship Steering
- Bioreactor
- Helicopter
- Aeroplane Logistics
- Robocup Soccer
- Quake
- Portfolio management
- Protein Folding
- Robot walking
- Game of Go

For most of these problems, either:

- MDP model is unknown, but experience can be sampled
- MDP model is known, but is too big to use, except by samples

Model-free control can solve these problems

On & Off-Policy Learning

- Behavior-policy: determines which action to take, from which we determine the next state to visit (also called "sampling policy")
- Target-policy: determines the action that appears to be best (also called "learning policy")
- Goal in reinforcement learning:
  - improve the target policy
  - while using a behavior policy to ensure that we visit states often enough (exploration schemes such as ϵ-greedy)
- On-policy learning: when the learning policy and the sampling policy are the same
- Off-policy learning: when the learning policy and the sampling policy are different

On & Off-Policy Learning

- On-policy learning
  - "Learn on the job"
  - Learn the value of the target policy π from experience sampled from behavior policy π
  - May not ensure the enough exploration of state space
- Off-policy learning
  - "Look over someone's shoulder"
  - Learn the value of the target policy π from experience sampled from behavior policy μ
  - Learning is from experience(data) "off" the target policy
  - Compared to on-policy learning, off-policy learning is
    - ★ more powerful & general
    - ★ often of greater variance & slower to converge

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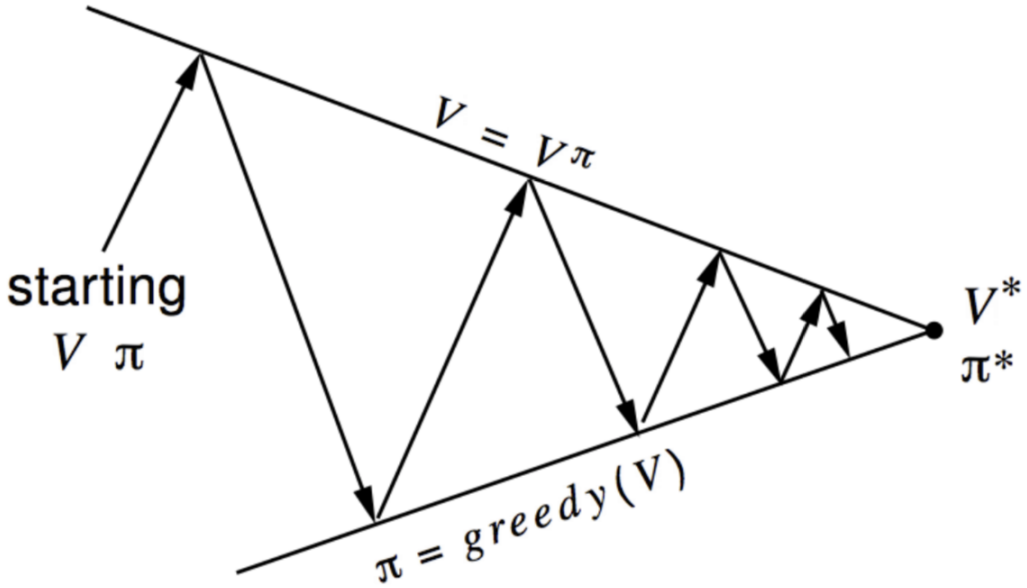
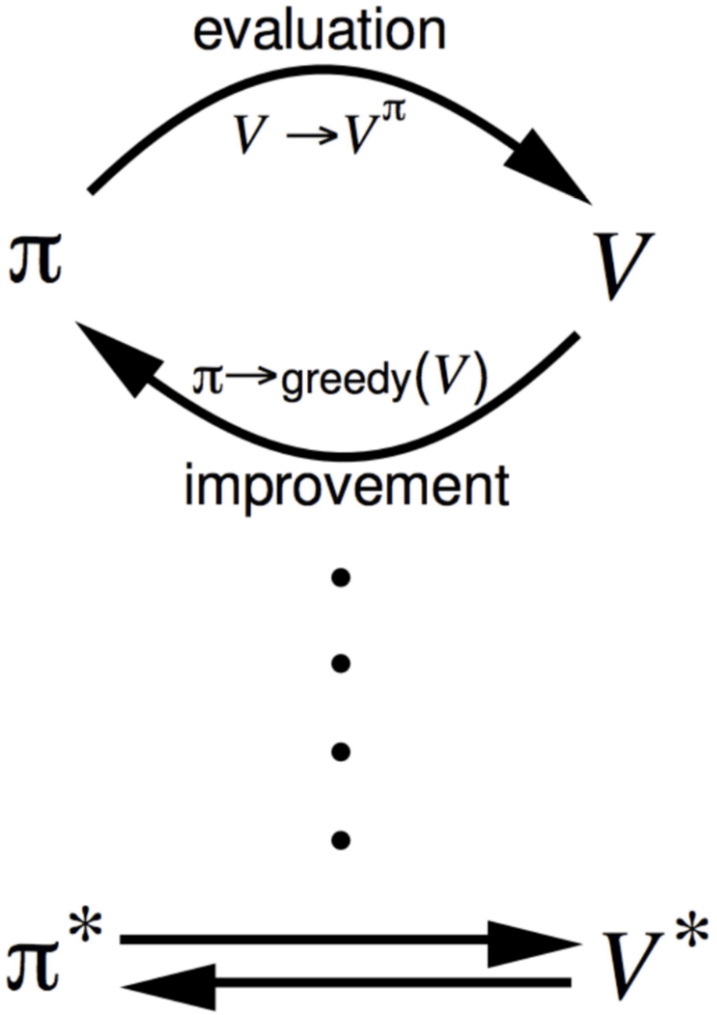
Generalized Policy Iteration (Refresher)





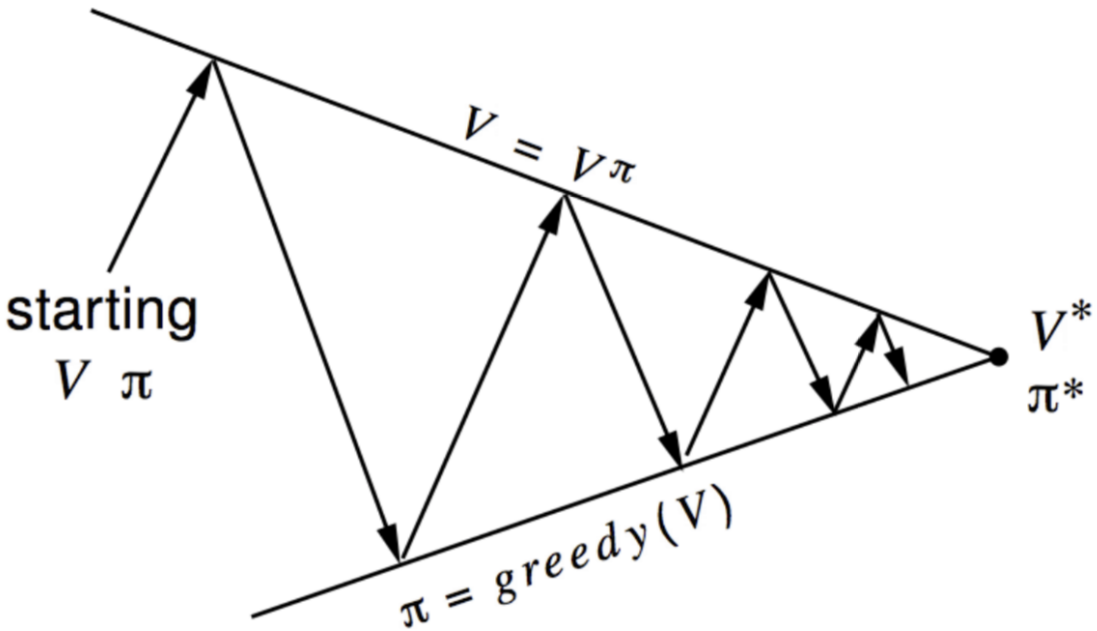
Policy evaluation Estimate v_π

e.g. Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

e.g. Greedy policy improvement

Generalized Policy Iteration With Monte-Carlo Evaluation



Policy evaluation Monte-Carlo policy evaluation, $V = v_\pi$?
Policy improvement Greedy policy improvement?

Model-Free Policy Iteration Using Action-Value Function

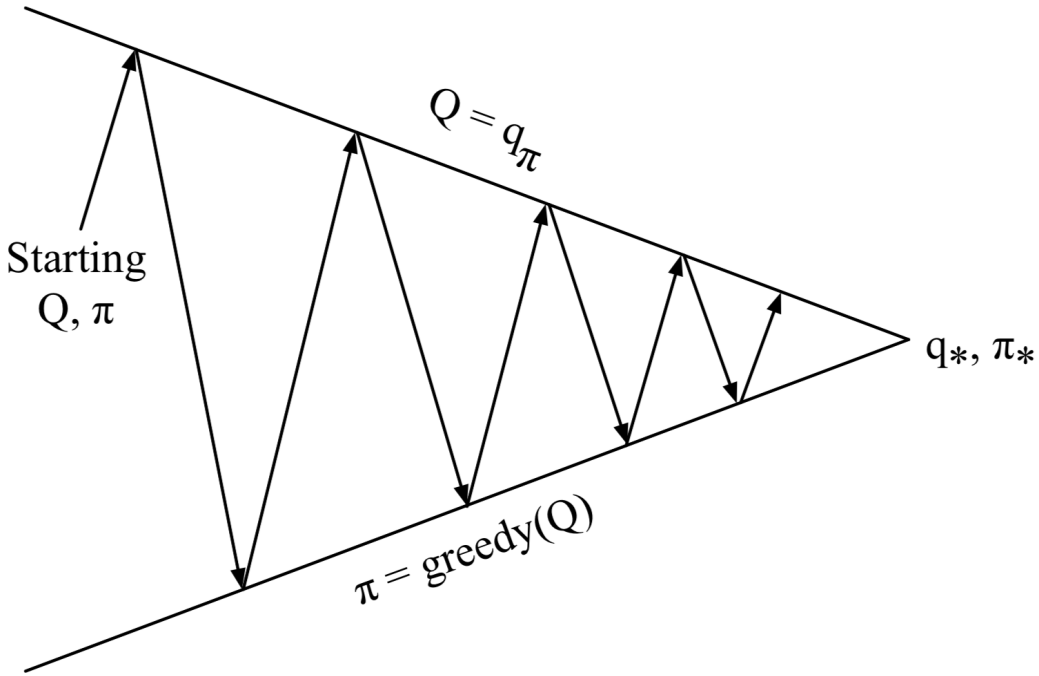
- Greedy policy improvement over $V(s)$ requires model of MDP

$$\pi'(s) = \arg\max_{a \in \mathcal{A}} \{\mathcal{R}_s^a + \mathcal{P}_{ss'}^a V(s')\}$$

- Greedy policy improvement over $Q(s, a)$ is model-free

$$\pi'(s) = \arg\max_{a \in \mathcal{A}} Q(s, a)$$

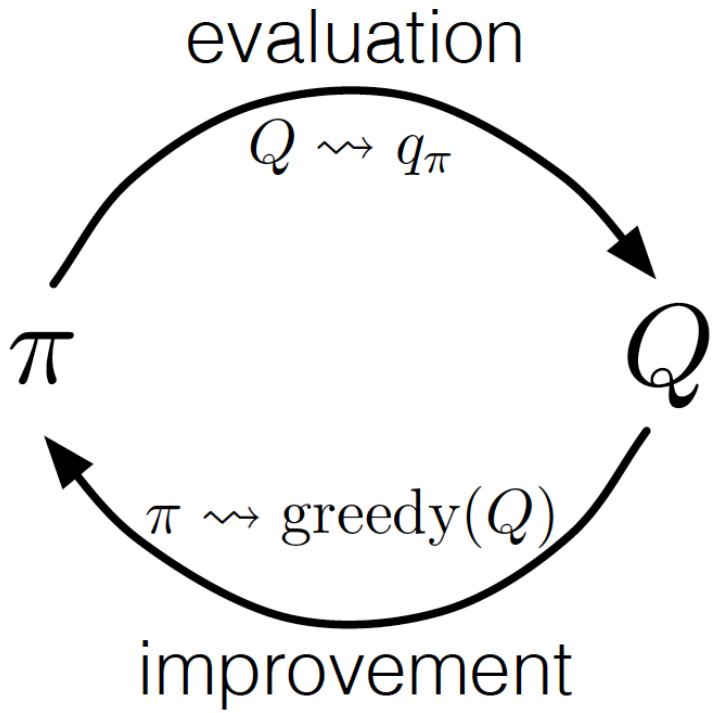
Generalized Policy Iteration with Action-Value Function



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement Greedy policy improvement?

Monte-Carlo Control



- MC policy iteration: policy evaluation using MC methods followed by policy improvement
- Policy improvement: greedify with respect to value (or action-value) function

Example of Greedy Action Selection


"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

- There are two doors in front of you.
- You open the left door and get reward 0
  $V(left) = 0$
- You open the right door and get reward $+1$
  $V(right) = +1$
- You open the right door and get reward $+3$
  $V(right) = +3$
- You open the right door and get reward $+2$
  $V(right) = +2$

$\vdots$

- Are you sure you've chosen the best door?

Monte Carlo ES (Exploring States)

Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$

Initialize:

$\pi(s) \in \mathcal{A}(s)$ (arbitrarily), for all $s \in \mathcal{S}$

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Loop forever (for each episode):

Choose $S_0 \in \mathcal{S}$, $A_0 \in \mathcal{A}(S_0)$ randomly such that all pairs have probability > 0

Generate an episode from S_0, A_0, following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

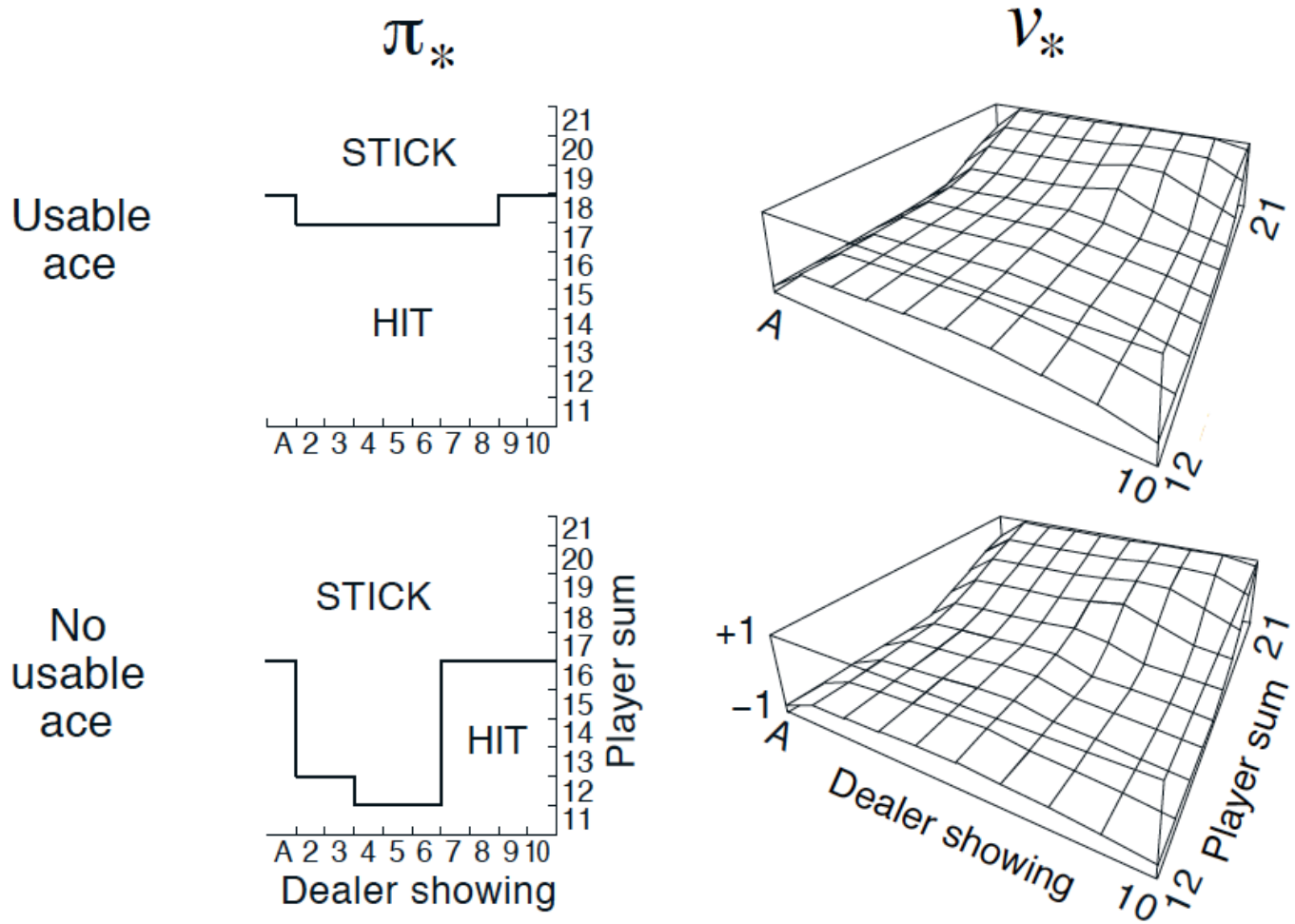
Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$

Back to the Blackjack Example

Monte-Carlo ES for Blackjack



ϵ-Greedy Exploration

- How to avoid the unlikely assumption of exploring starts?
- Simplest idea for ensuring continual exploration
- All m actions are tried with non-zero probability
- With probability $1 - \epsilon$ choose the greedy action
- With probability ϵ choose an action at random

$$\pi(a|s) = \begin{cases} \epsilon/m + 1 - \epsilon & \text{if } a^* = \arg\max_{a \in \mathcal{A}} Q(s, a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

ϵ-Greedy Policy Improvement

Theorem

For any ϵ-greedy policy π, the ϵ-greedy policy π' with respect to q_π is an improvement, $v_{\pi'}(s) \geq v_\pi(s)$

$$
\begin{aligned}
q_\pi(s, \pi'(s)) &= \sum_{a \in \mathcal{A}} \pi'(a|s) q_\pi(s, a) \\
&= \epsilon/m \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \epsilon) \max_{a \in \mathcal{A}} q_\pi(s, a) \\
&\geq \epsilon/m \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \epsilon) \sum_{a \in \mathcal{A}} \frac{\pi(a|s) - \epsilon/m}{1 - \epsilon} q_\pi(s, a) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a) = v_\pi(s)
\end{aligned}
$$

Therefore from policy improvement theorem, $v_{\pi'}(s) \geq v_\pi(s)$

Remark

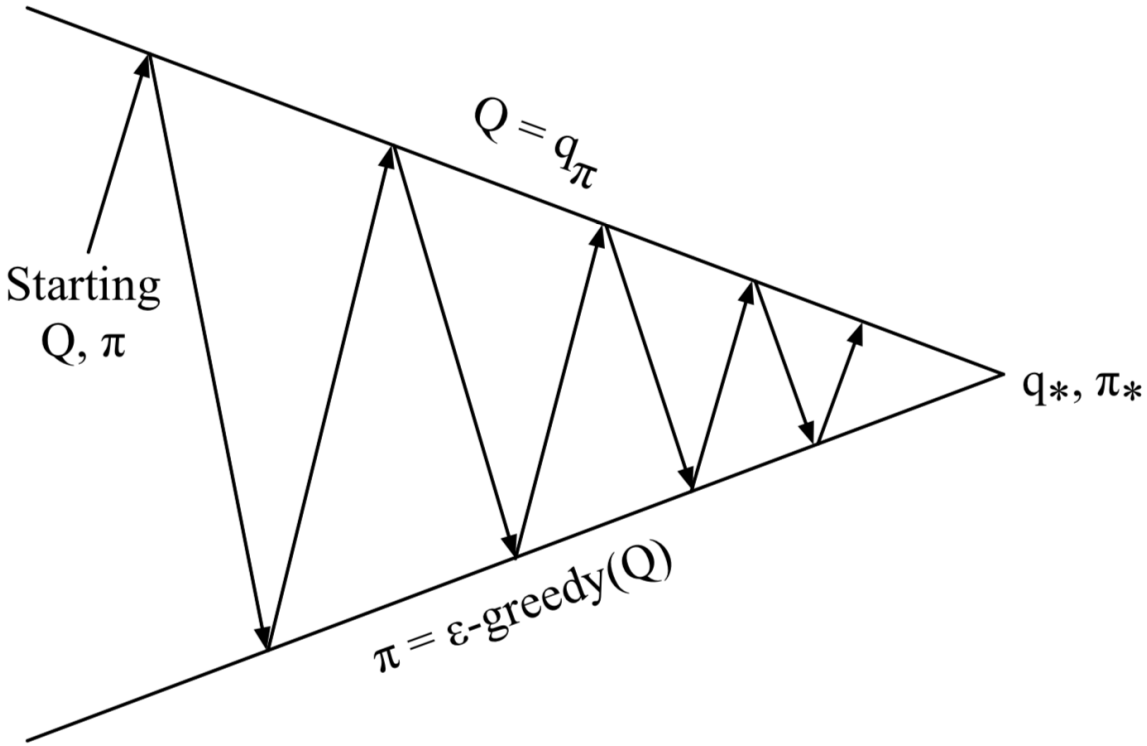
1°. $\sum_{a\in A} \frac{\pi(a|s)-\varepsilon/m}{1-\varepsilon} = \frac{\sum_{a\in A}\pi(a|s) - \varepsilon/m \cdot m}{1-\varepsilon} = \frac{1-\varepsilon}{1-\varepsilon} = 1$

2°. $\sum_{a\in A} \frac{\pi(a|s)-\varepsilon/m}{1-\varepsilon} \cdot q_\pi(s,a) \le \sum_{a\in A} \frac{\pi(a|s)-\varepsilon/m}{1-\varepsilon} \cdot \max_{a\in A} q_\pi(s,a)$

$= \left(\sum_{a\in A} \frac{\pi(a|s)-\varepsilon/m}{1-\varepsilon}\right) \cdot \max_{a\in A} q_\pi(s,a)$

$= \max_{a\in A} q_\pi(s,a)$

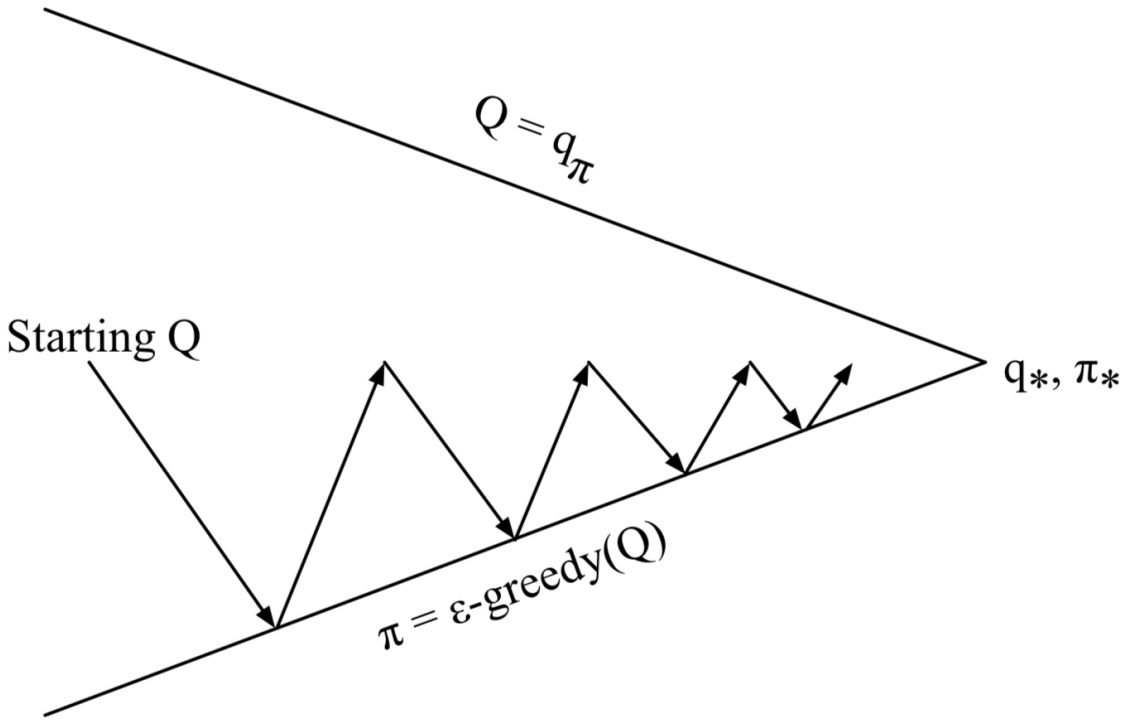
Monte-Carlo Policy Iteration



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement ϵ-greedy policy improvement

Monte-Carlo Control



Every episode:

Policy evaluation Monte-Carlo policy evaluation, $Q \approx q_\pi$

Policy improvement ϵ-greedy policy improvement

GLIE

Definition

Greedy in the Limit with Infinite Exploration (GLIE)

- All state-action pairs are explored infinitely many times,

$$\lim_{k \to \infty} N_k(s, a) = \infty$$

- The policy converges on a greedy policy,

$$\lim_{k \to \infty} \pi_k(a|s) = \mathbf{1}(a = \arg\max_{a' \in \mathcal{A}} Q_k(s, a'))$$

GLIE

- For example, ϵ-greedy is GLIE if ϵ reduces to zero at $\epsilon_k = \frac{1}{k}$
- Keep a declining exploration probability at a sufficiently slow rate that try all actions infinitely ofen
- In practice, $\epsilon_k = \frac{1}{k^\beta}$ with $\beta \in (0.5, 1]$

GLIE Monte-Carlo Control

- Sample kth episode using π: $\{S_1, A_1, R_2, \ldots, S_T\} \sim \pi$
- For each state S_t and action A_t in the episode,

$$N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)}(G_t - Q(S_t, A_t))$$

- Improve policy based on new action-value function

$$\epsilon \leftarrow 1/k$$

$$\pi \leftarrow \epsilon\text{-greedy}(Q)$$

Theorem

GLIE Monte-Carlo control converges to the optimal action-value function, $Q(s, a) \rightarrow q_*(s, a)$

On-Policy First-Visit MC Control

On-policy first-visit MC control (for ε-soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter: small $\varepsilon > 0$

Initialize:

$\pi \leftarrow$ an arbitrary ε-soft policy

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)

For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

Partial Summary of MC

- MC has several advantages over DP
  - can learn directly from interaction with environment
  - No need for full models
  - No need to learn about ALL states (no bootstrapping)
  - Less harmed by violating Markov property
- MC methods provide an alternate policy evaluation process
- One issue to watch for: maintaining sufficient exploration
  - exploring starts, soft policies

Boltzmann Exploration

- A limitation of ϵ-greedy: when we explore, we choose actions at random without regard to their estimated values
- For larger action spaces, this can mean that we are spending a lot of time evaluating actions that are quite poor
- Boltzmann Exploration (also known as "Gibbs sampling" & "soft-max"): choosing an action based on its estimated value.

$$\pi(a|s) = \frac{e^{\beta \hat{Q}(s,a)}}{\sum_{a'} e^{\beta \hat{Q}(s,a')}}$$

- $\hat{Q}(s, a)$ is an estimate of the value of being in state s and taking action a.
- β is a tunable parameter
  - $\beta = 0$ produces a pure exploration policy
  - $\beta \to \infty$ produces a greedy policy

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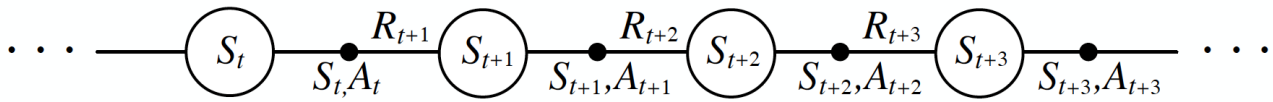
MC vs. TD Control

- Temporal-difference (TD) learning has several advantages over Monte-Carlo (MC)
  - Lower variance
  - Online
  - Incomplete sequences
- Natural idea: use TD instead of MC in our control loop
  - Apply TD to $Q(S, A)$
  - Use ϵ-greedy policy improvement
  - Update every time-step

Learning An Action-Value Function

Estimate q_π for the current policy π



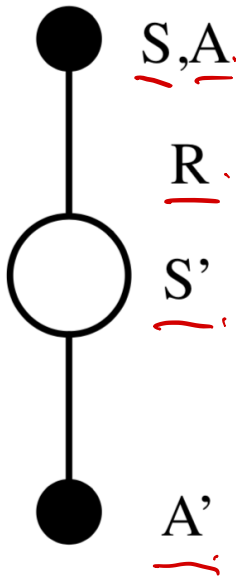
After every transition from a nonterminal state, S_t, do this:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\right]$$

If S_{t+1} is terminal, then define $Q(S_{t+1}, A_{t+1}) = 0$

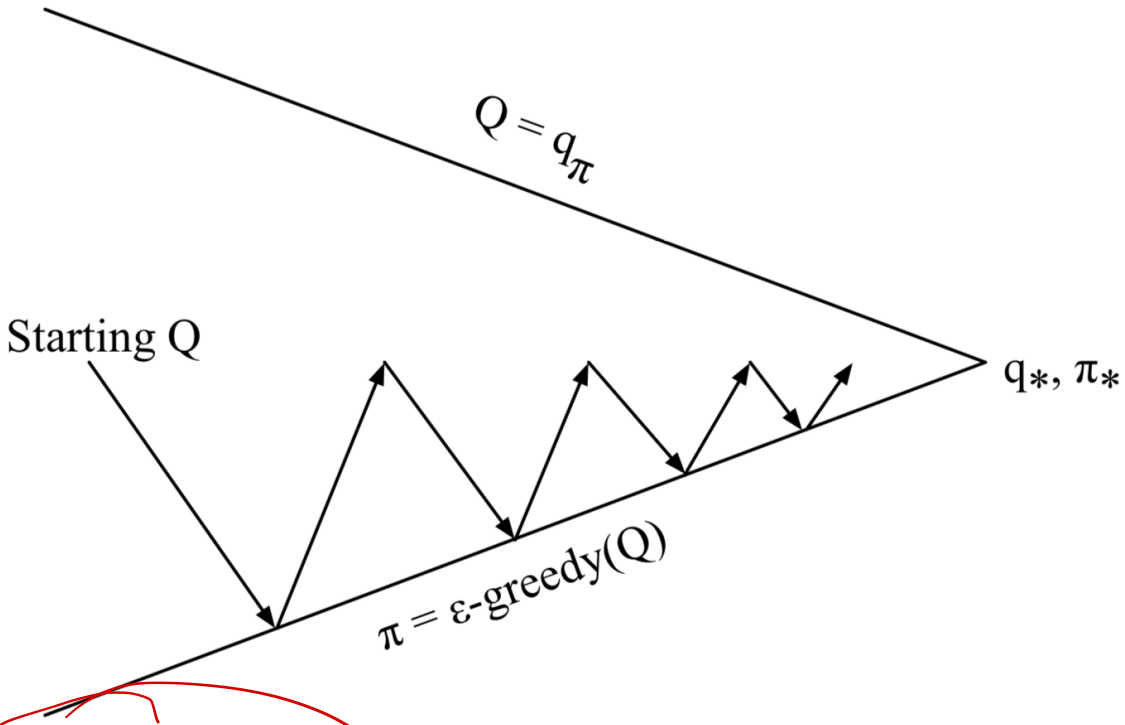
$$q_\pi(s, a) = \mathbb{E}_\pi\left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

Updating Action-Value Functions with Sarsa



$$Q(S,A) \leftarrow Q(S,A) + \alpha(R + \gamma Q(S',A') - Q(S,A))$$

On-Policy Control With Sarsa



Every time-step:

Policy evaluation Sarsa, $Q \approx q_\pi$

Policy improvement ϵ-greedy policy improvement

Sarsa Algorithm for On-Policy Control

Sarsa (on-policy TD control) for estimating $Q \approx q_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Loop for each step of episode:
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma Q(S', A') - Q(S, A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

Convergence of Sarsa

Theorem

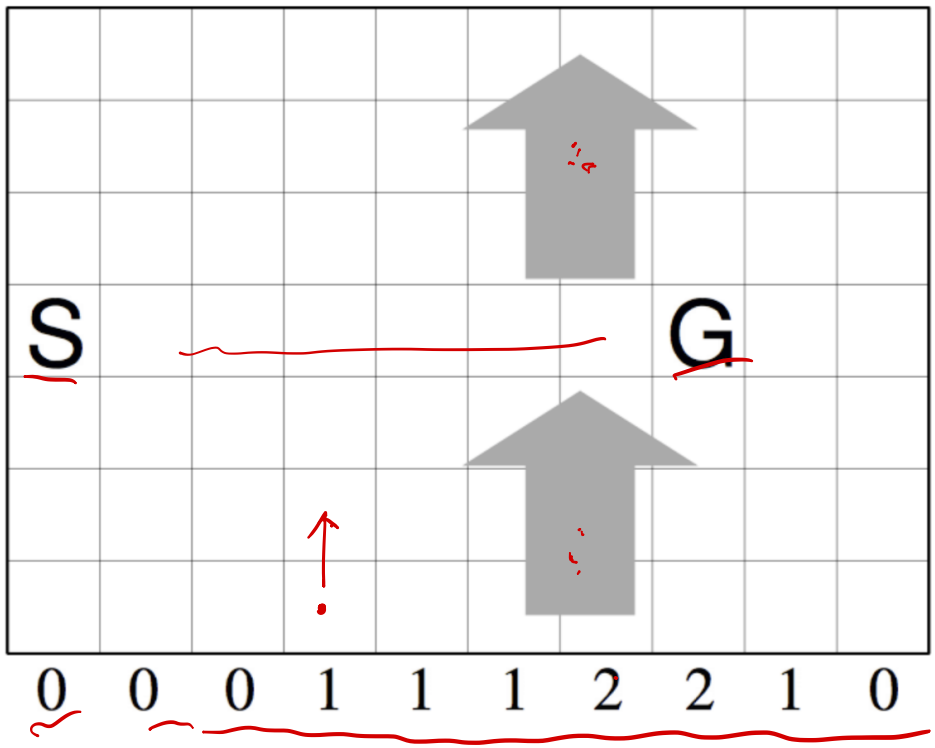
Sarsa converges to the optimal action-value function, $Q(s,a) \rightarrow q_*(s,a)$*, under the following conditions:*

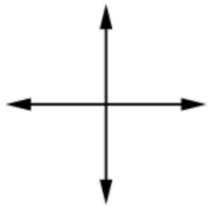
- *GLIE sequence of policies* $\pi_t(a|s)$
- *Robbins-Monro sequence of step-sizes* α_t

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

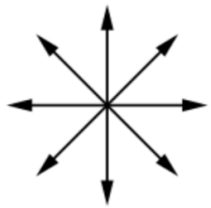
$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

Windy Gridworld Example





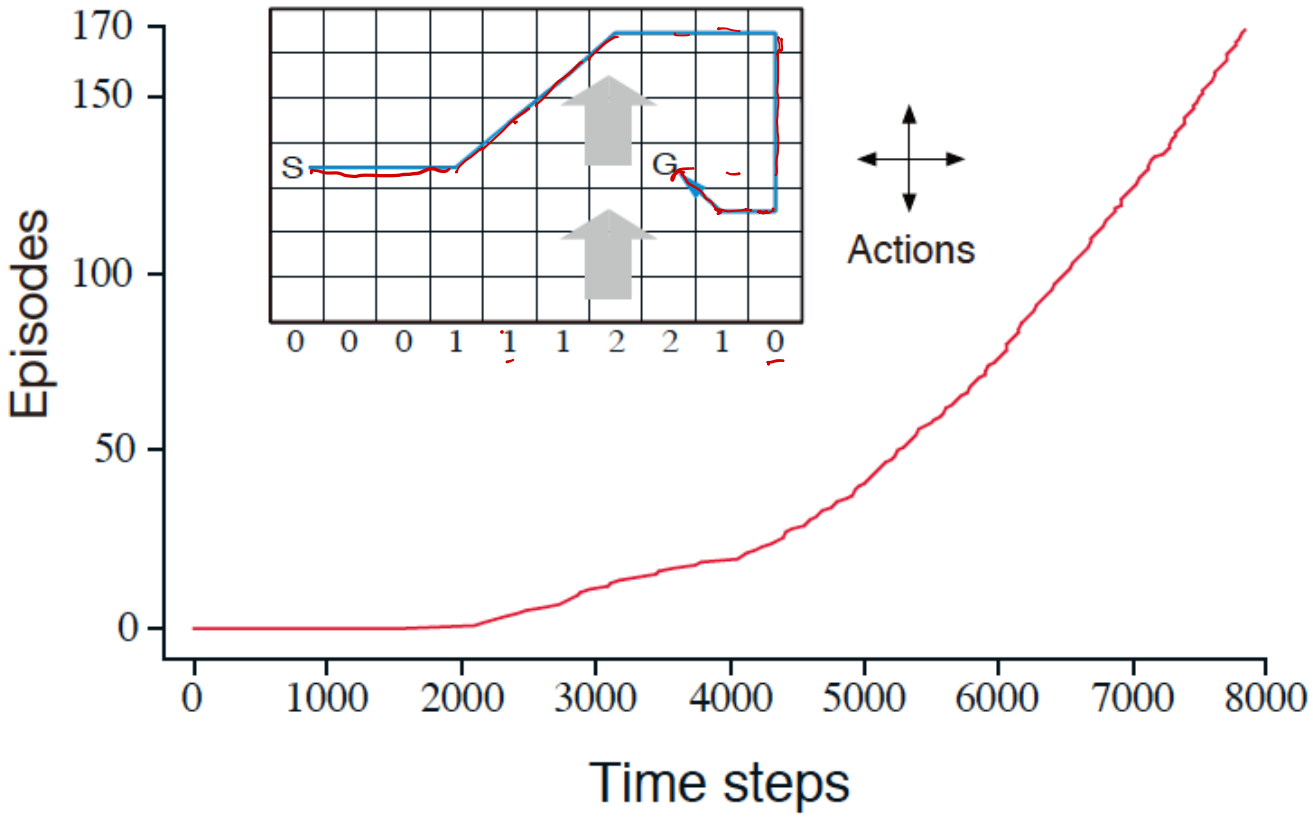
standard moves



king's moves

- Reward = -1 per time-step until reaching goal
- Undiscounted

Sarsa on the Windy Gridworld



$\epsilon = 0.1$

$\alpha = a_5$

initial $Q(s,a) = 0$

n-Step Sarsa

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{aligned}
n = 1 \quad (Sarsa) \quad & q_t^{(1)} = R_{t+1} + \gamma Q(S_{t+1}) \\
n = 2 \quad & q_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2}) \\
\vdots \quad & \vdots \\
n = \infty \quad (MC) \quad & q_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T
\end{aligned}
$$

- Define the n-step Q-return

$$
q_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n Q(S_{t+n})
$$

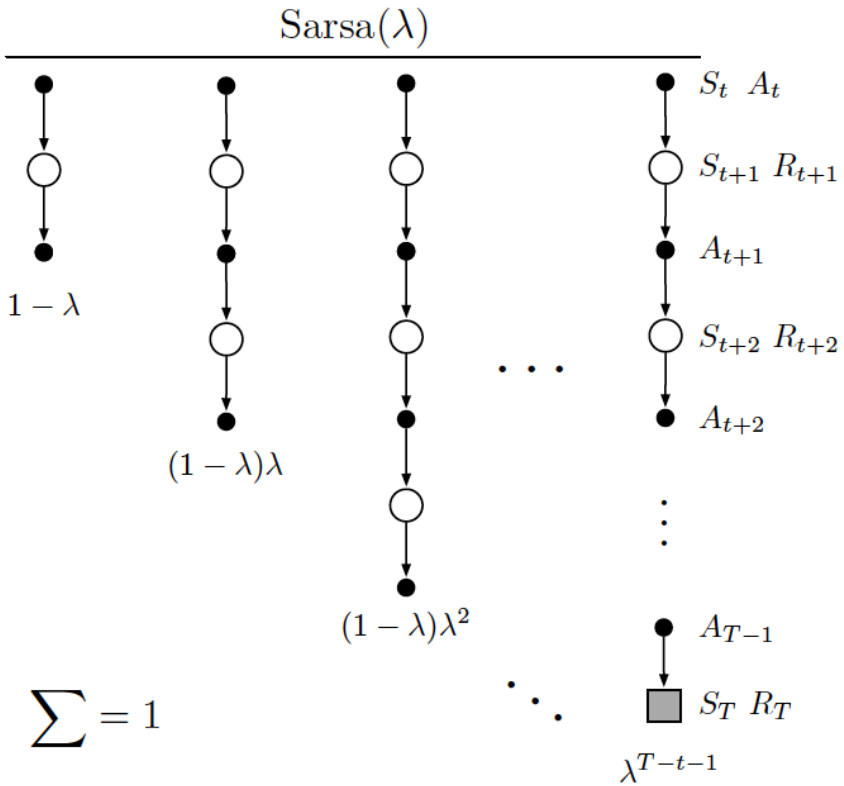
- n-step Sarsa updates $Q(s, a)$ towards the n-step Q-return

$$
Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^{(n)} - Q(S_t, A_t) \right)
$$

Forward View Sarsa(λ)

- Forward-view Sarsa(λ)

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^\lambda - Q(S_t, A_t) \right)$$



- The q^λ *return* combines all n-step Q-returns $q_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$q_t^\lambda = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} q_t^{(n)}$$

Backward View Sarsa(λ)

- Just like TD(λ), we use eligibility traces in an online algorithm
- But Sarsa(λ) has one eligibility trace for each state-action pair

$$E_0(s,a) = 0$$
$$E_t(s,a) = \gamma\lambda E_{t-1}(s,a) + \mathbf{1}(S_t = s, A_t = a)$$

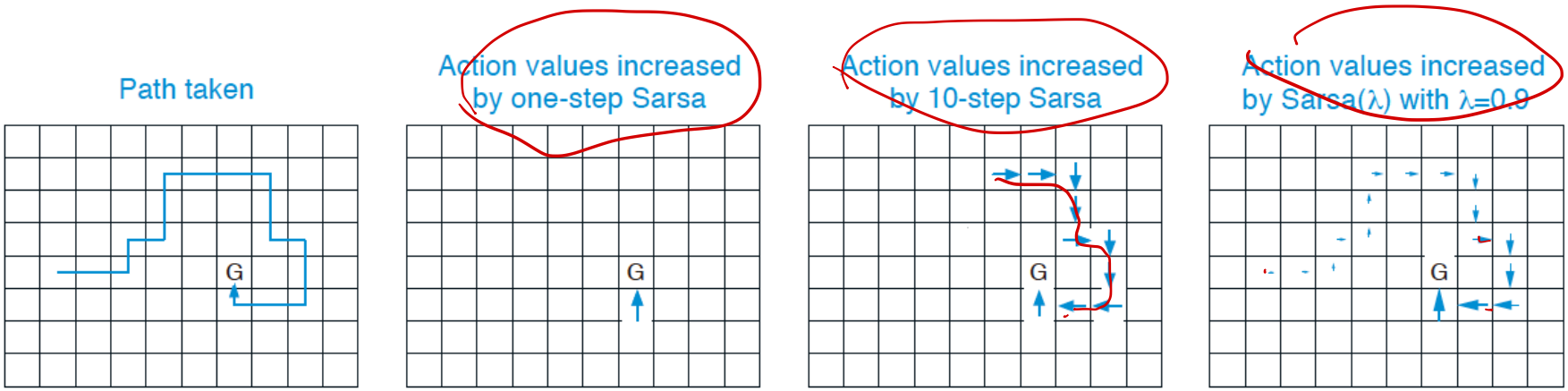
- $Q(s,a)$ is updated for every state s and action a
- In proportion to TD-error δ_t and eligibility trace $E_t(s,a)$

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$
$$Q(s,a) \leftarrow Q(s,a) + \alpha\delta_t E_t(s,a)$$

Sarsa(λ) Algorithm

Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
Repeat (for each episode):
    $E(s,a) = 0$, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
    Initialize S, A
    Repeat (for each step of episode):
        Take action A, observe R, S'
        Choose A' from S' using policy derived from Q (e.g., ε-greedy)
        $\delta \leftarrow R + \gamma Q(S', A') - Q(S, A)$
        $E(S, A) \leftarrow E(S, A) + 1$
        For all $s \in \mathcal{S}, a \in \mathcal{A}(s)$:
            $Q(s,a) \leftarrow Q(s,a) + \alpha \delta E(s,a)$
            $E(s,a) \leftarrow \gamma \lambda E(s,a)$
        $S \leftarrow S'$; $A \leftarrow A'$
    until S is terminal

Sarsa(λ) Gridworld Example



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Off-Policy Learning

- Learn the value of the target policy π from experience due to behavior policy μ
- In this sense, learning is from experience(data) "off" the target policy, and the overall process is termed off-policy learning.
- For example, π is the greedy policy (and ultimately the optimal policy) while μ is exploratory (e.g.,ϵ-soft)
- In general, we only require coverage, i.e., that μ generates behavior that covers, or includes, π

$$\mu(a|s) > 0 \text{ for every } s, a \text{ at which } \pi(a|s) > 0$$

Off-Policy Learning

- Evaluate target policy $\pi(a|s)$ to compute $v_\pi(s)$ or $q_\pi(s, a)$
- While following behaviour policy $\mu(a|s)$

$$\{S_1, A_1, R_2, \ldots, S_T\} \sim \mu$$

- Why is this important?
  - Learn from observing humans or other agents
  - Re-use experience generated from old policies $\pi_1, \pi_2, \ldots, \pi_{t-1}$
  - Learn about *optimal* policy while following *exploratory* policy
  - Learn about *multiple* policies while following *one* policy

Importance Sampling

- Estimate the expectation of a function

$$\mathbb{E}_{X\sim\pi}[g(X)] = \sum \pi(x)f(x) \approx \frac{1}{n}\sum_{k=1}^{n} g(x_k), x_k \sim \pi$$

- But sometimes it is difficult to sample x from π, then we can sample x from another distribution μ, then correct the weight

$$\mathbb{E}_{X\sim\pi}[g(X)] = \sum \pi(x)g(x) = \sum \mu(x)\frac{\pi(x)}{\mu(x)}g(x)$$

$$= \mathbb{E}_{X\sim\mu}\left[\frac{\pi(X)}{\mu(X)}g(X)\right] \approx \frac{1}{n}\sum_{k=1}^{n}\frac{\pi(x_k)}{\mu(x_k)}g(x_k), x_k \sim \mu$$

- For off-policy learning: weight each return by the ratio of the probabilities of the trajectory under the two policies

Importance Sampling for Off-Policy Monte-Carlo

- Use returns generated from μ to evaluate π
- Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along whole episode

$$G_t^{\pi/\mu} = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{\mu(A_{t+1}|S_{t+1})} \cdots \frac{\pi(A_T|S_T)}{\mu(A_T|S_T)} G_t$$

- Update value towards *corrected* return

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{\pi/\mu} - V(S_t) \right)$$

- Cannot use if μ is zero when π is non-zero
- Importance sampling can dramatically increase variance

Importance Sampling Ratio

- Probability of the rest of the trajectory, after S_t, under π

$$\begin{aligned}
&\Pr\{A_t, S_{t+1}, A_{t+1}, \ldots, S_T | S_t, A_{t:T-1} \sim \pi\} \\
&= \pi(A_t|S_t)\, p(S_{t+1}|S_t, A_t)\, \pi(A_{t+1}|S_{t+1}) \cdots p(S_T|S_{T-1}, A_{T-1}) \\
&= \prod_{k=t}^{T-1} \pi(A_k|S_k)\, p(S_{k+1}|S_k, A_k),
\end{aligned}$$

- In importance sampling, each arm is weighted by the relative probability of the trajectory under the two policies

$$\rho_t^T = \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)\, p(S_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} \mu(A_k|S_k)\, p(S_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}$$

- This is called the *importance sampling ratio*
- All importance sampling ratios have expected value 1

$$\mathbb{E}_{A_k \sim \mu}\left[\frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}\right] = \sum_a \mu(a|S_k) \frac{\pi(a|S_k)}{\mu(a|S_k)} = \sum_a \pi(a|S_k) = 1.$$

Off-Policy MC Policy Evaluation

Incremental off-policy every-visit MC policy evaluation (returns $Q \approx q_\pi$

Input: an arbitrary target policy π

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
$Q(s,a) \leftarrow$ arbitrary
$C(s,a) \leftarrow 0$

Repeat forever:
$\mu \leftarrow$ any policy with coverage of π
Generate an episode using μ:
$S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T$
$G \leftarrow 0$
$W \leftarrow 1$
For $t = T-1, T-2, \ldots$ downto 0:
$G \leftarrow \gamma G + R_{t+1}$
$C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$
$W \leftarrow W \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}$
If $W = 0$ then ExitForLoop

Off-Policy MC Control

Off-policy every-visit MC control (returns $\pi \approx \pi_*$)

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
$Q(s,a) \leftarrow$ arbitrary
$C(s,a) \leftarrow 0$
$\pi(s) \leftarrow \arg\max_a Q(S_t,a)$ (with ties broken consistently)

Repeat forever:
$\mu \leftarrow$ any soft policy
Generate an episode using μ:
$S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T$
$G \leftarrow 0$
$W \leftarrow 1$
For $t = T-1, T-2, \ldots$ downto 0:
$G \leftarrow \gamma G + R_{t+1}$
$C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$
$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)
If $A_t \neq \pi(S_t)$ then ExitForLoop
$W \leftarrow W \frac{1}{\mu(A_t|S_t)}$

Target policy is greedy and deterministic

Behavior policy is soft, typically ε-greedy

Importance Sampling for Off-Policy TD

- Use TD targets generated from μ to evaluate π
- Weight TD target $R + \gamma V(S')$ by importance sampling
- Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha \left(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} (R_{t+1} + \gamma V(S_{t+1})) - V(S_t) \right)$$

- Much lower variance than Monte-Carlo importance sampling
- Policies only need to be similar over a single step

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Q-Learning

- We now consider off-policy learning of action-values $Q(s, a)$
- No importance sampling is required
- Next action to evaluate is chosen using behavior policy $A_{t+1} \sim \mu(\cdot|S_t)$
- But we consider alternative successor action $A' \sim \pi(\cdot|S_t)$
- Which means a separate policy is used to choose the alternative action in the future
- And update $Q(S_t, A_t)$ towards value of alternative action

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t))$$

Off-Policy Control with Q-Learning

- We now allow both behaviour and target policies to improve
- The target policy π is greedy w.r.t. $Q(s, a)$

$$\pi(S_{t+1}) = \arg\max_{a'} Q(S_{t+1}, a')$$

- The behaviour policy μ is e.g. ϵ-greedy w.r.t. $Q(s, a)$
- The Q-learning target then simplifies:

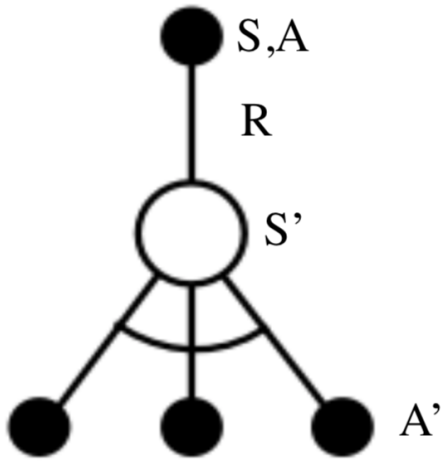
$$\begin{aligned} & R_{t+1} + \gamma Q(S_{t+1}, A') \\ = & R_{t+1} + \gamma Q(S_{t+1}, \arg\max_{a'} Q(S_{t+1}, a')) \\ = & R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a') \end{aligned}$$

Q-Learning Control Algorithm

Bellman optimality Equation

$q_*(S, a)$

$= E[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \mid S_t = S, A_t = a]$



$$Q(S, A) \leftarrow Q(S, A) + \alpha \left(R + \gamma \max_{a'} Q(S', a') - Q(S, A) \right)$$

Theorem

Q-learning control converges to the optimal action-value function,
$Q(s, a) \rightarrow q_*(s, a)$

Q-Learning Algorithm for Off-Policy Control

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

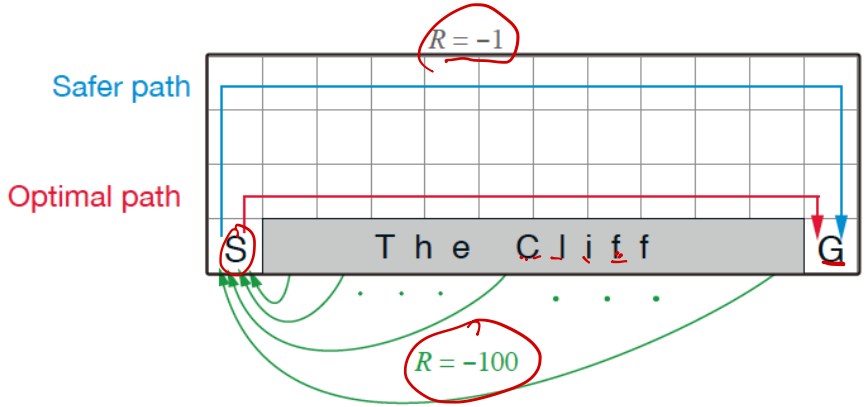
Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

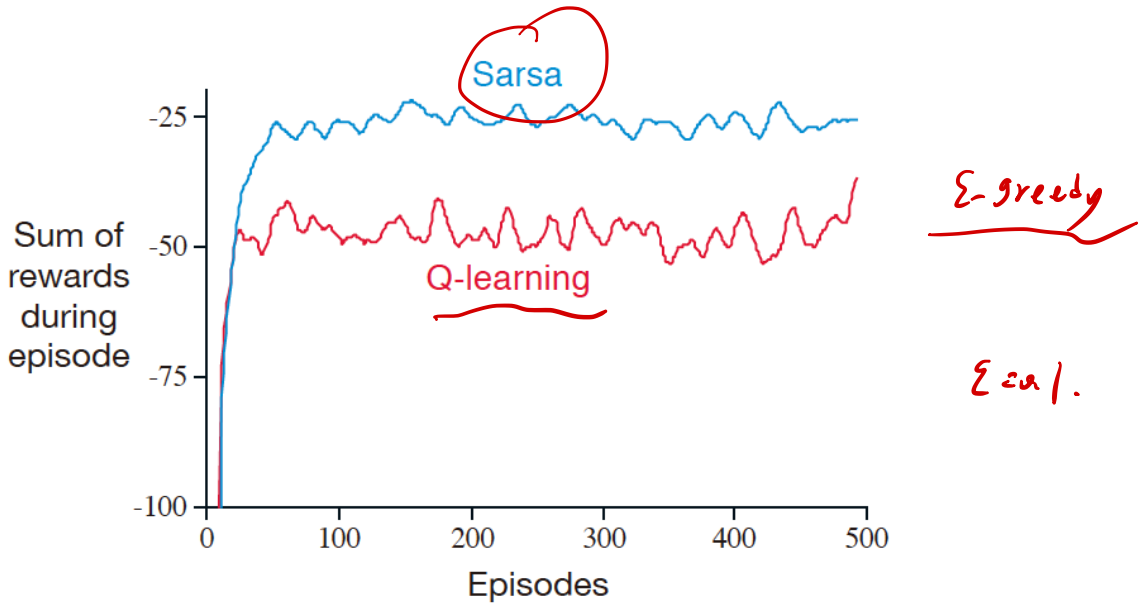
Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
    $S \leftarrow S'$
  until S is terminal

Q-Learning Demo

https://www.cs.ubc.ca/~poole/demos/rl/q.html

Cliff Walking Example

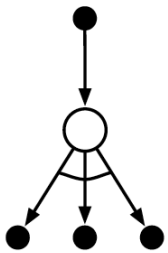




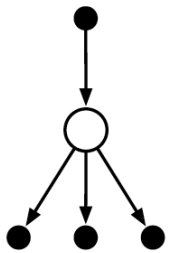
Expected Sarsa

- Instead of the sample value-of-next-state, use the expectation!
- Expected Sarsa's performs better than Sarsa (but costs more)

$$
\begin{aligned}
Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t)\Big] \\
&\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\Big]
\end{aligned}
$$

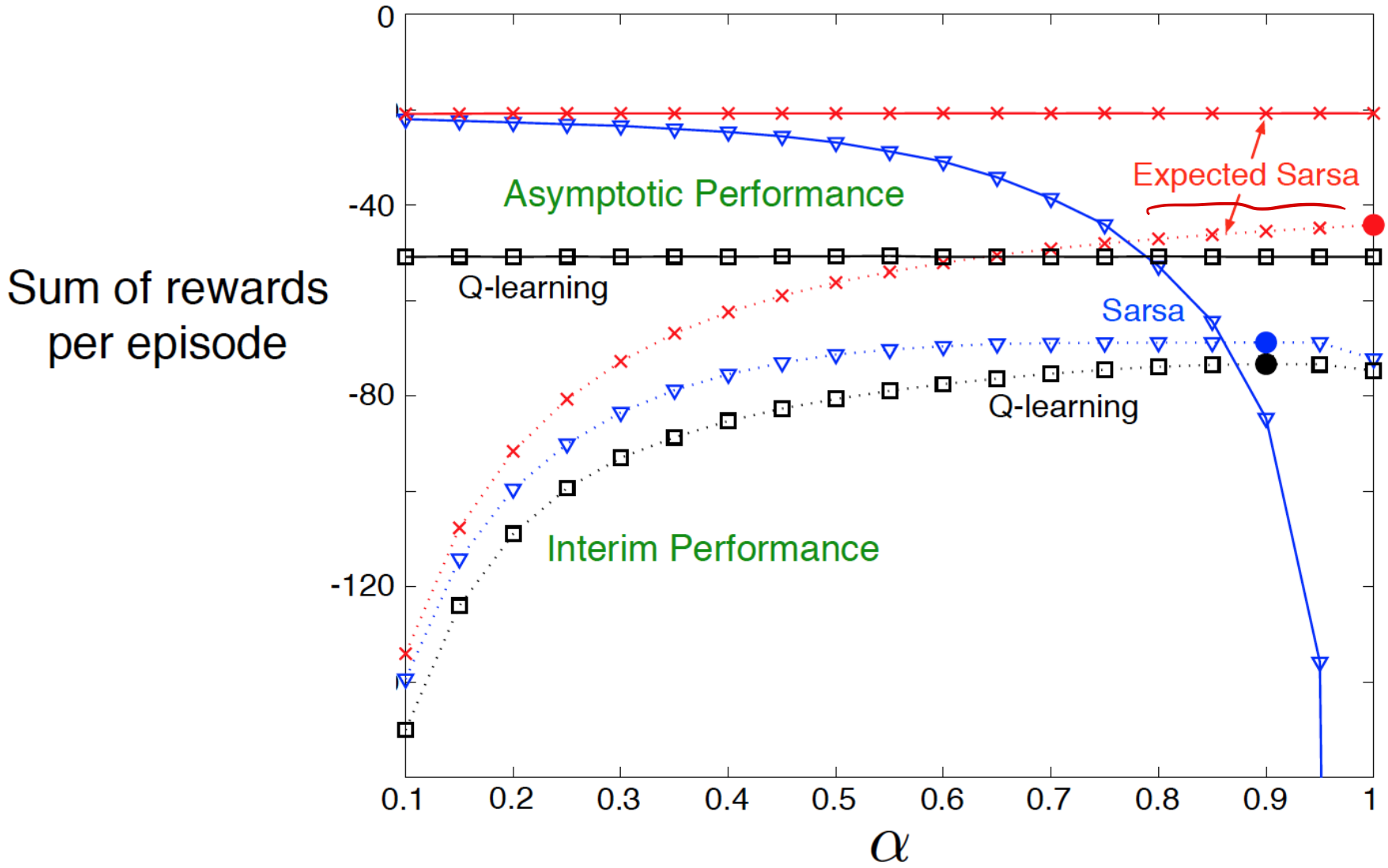


Q-learning



Expected Sarsa

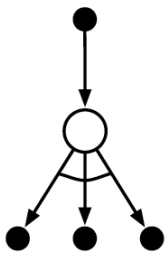
Performance on Cliff Walking Example



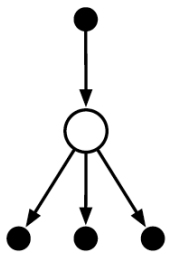
Off-Policy Expected Sarsa

- Expected Sarsa generalizes to arbitrary behavior policies μ
  - in which case it includes Q-learning as the special case in which π is the greedy policy
- This idea seems to be new

$$
\begin{aligned}
Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t)\Big] \\
&\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\Big]
\end{aligned}
$$

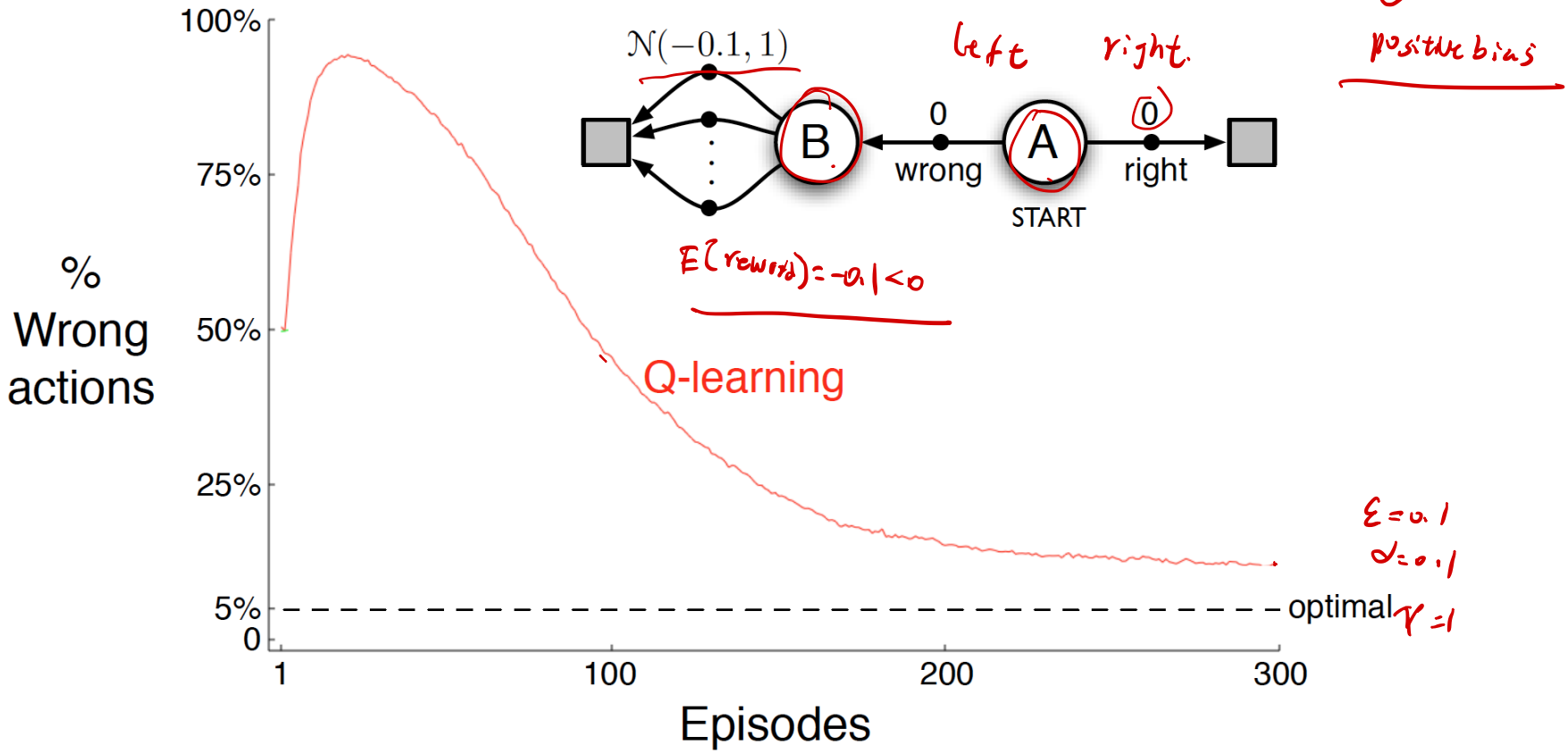


Q-learning



Expected Sarsa

Maximization Bias Example



Tabular Q-learning: $$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\Big]$$

Solution: Double Q-Learning

- Train 2 action-value functions Q_1 & Q_2
- Do Q-learning on both, but
  - never on the same time steps (Q_1 & Q_2 are independent)
  - pick Q_1 or Q_2 at random to be updated on each step
- If updating Q_1, use Q_2 for the value of the next state:

$$Q_1(S_t, A_t) \leftarrow Q_1(S_t, A_t) + \alpha\Big(R_{t+1} + Q_2\big(S_{t+1}, \arg\max_a Q_1(S_{t+1}, a)\big) - Q_1(S_t, A_t)\Big)$$

- Action selections are (say) ϵ-greedy with respect to the sum of Q_1 and Q_2

Applied to DQN Variants

Solution: Double Q-Learning

Double Q-learning, for estimating $Q_1 \approx Q_2 \approx q_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q_1(s, a)$ and $Q_2(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, such that $Q(\mathit{terminal}, \cdot) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose A from S using the policy ε-greedy in $Q_1 + Q_2$
    Take action A, observe R, S'
    With 0.5 probabilility:

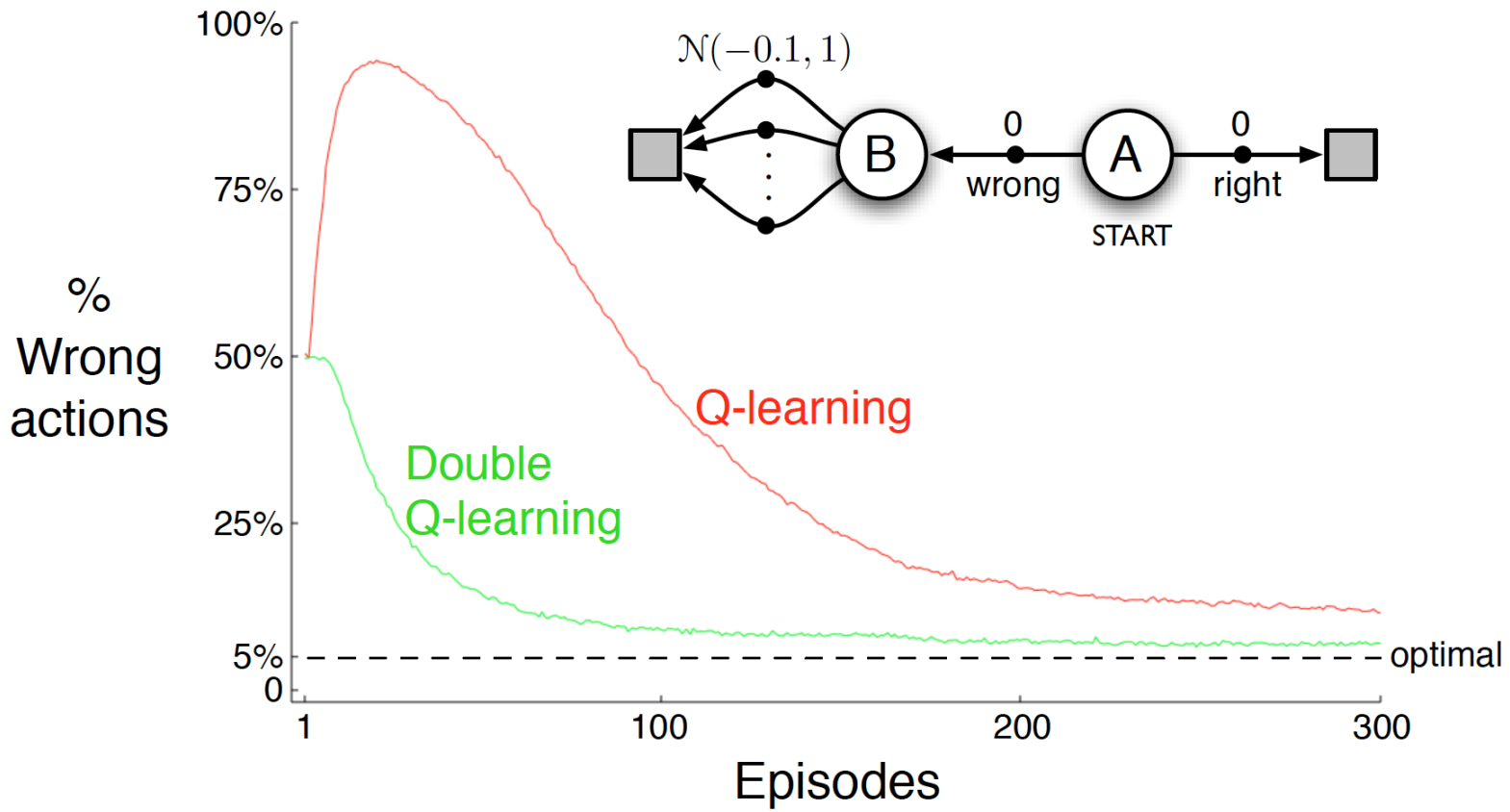
$$Q_1(S, A) \leftarrow Q_1(S, A) + \alpha\Big(R + \gamma Q_2\big(S', \operatorname{argmax}_a Q_1(S', a)\big) - Q_1(S, A)\Big)$$

    else:

$$Q_2(S, A) \leftarrow Q_2(S, A) + \alpha\Big(R + \gamma Q_1\big(S', \operatorname{argmax}_a Q_2(S', a)\big) - Q_2(S, A)\Big)$$

    $S \leftarrow S'$
  until S is terminal

Maximization Bias Example

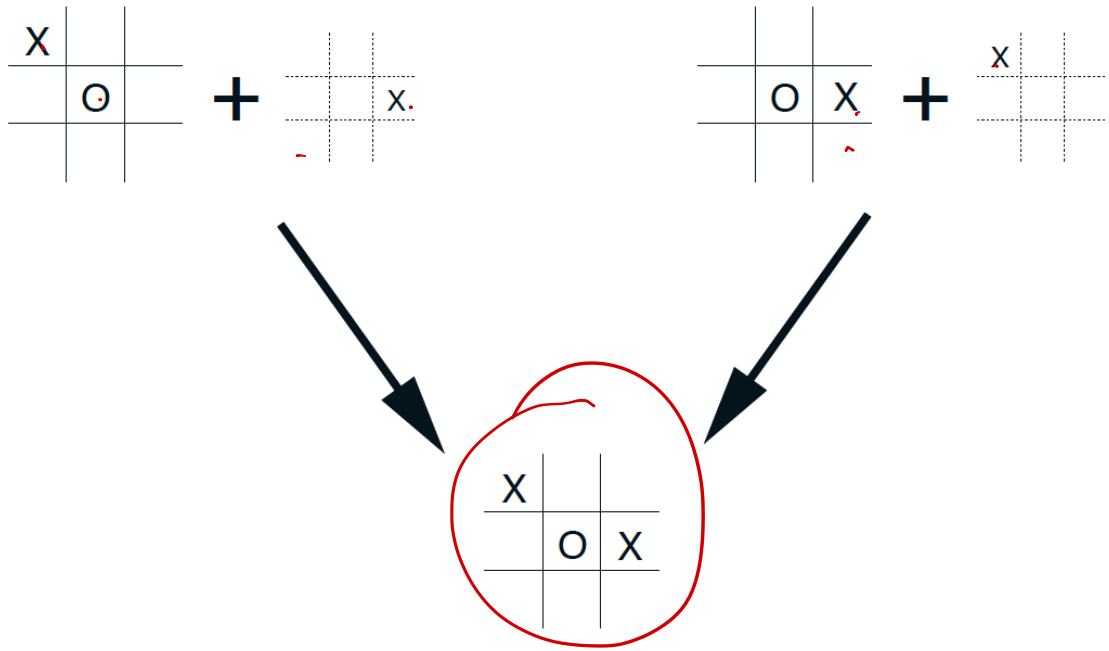


Double Q-learning:

$$Q_1(S_t, A_t) \leftarrow Q_1(S_t, A_t) + \alpha\Big(R_{t+1} + Q_2\big(S_{t+1}, \arg\max_a Q_1(S_{t+1}, a)\big) - Q_1(S_t, A_t)\Big)$$

Afterstates

- Usually, a state-value function evaluates states in which the agent can take an action
- But sometimes it is useful to evaluate states after agent has acted, as in tic-tac-toe
- More efficient to design algorithms



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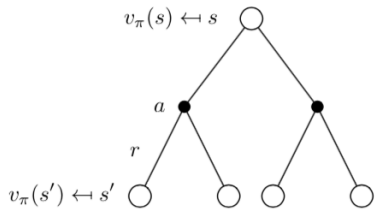
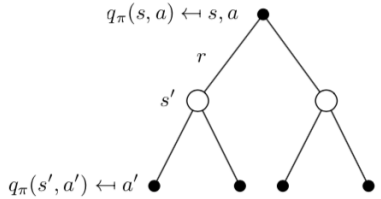
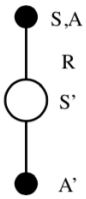
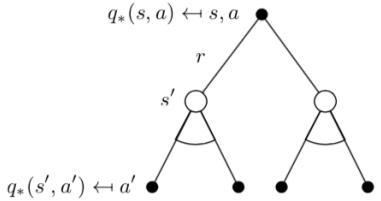
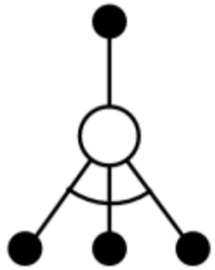
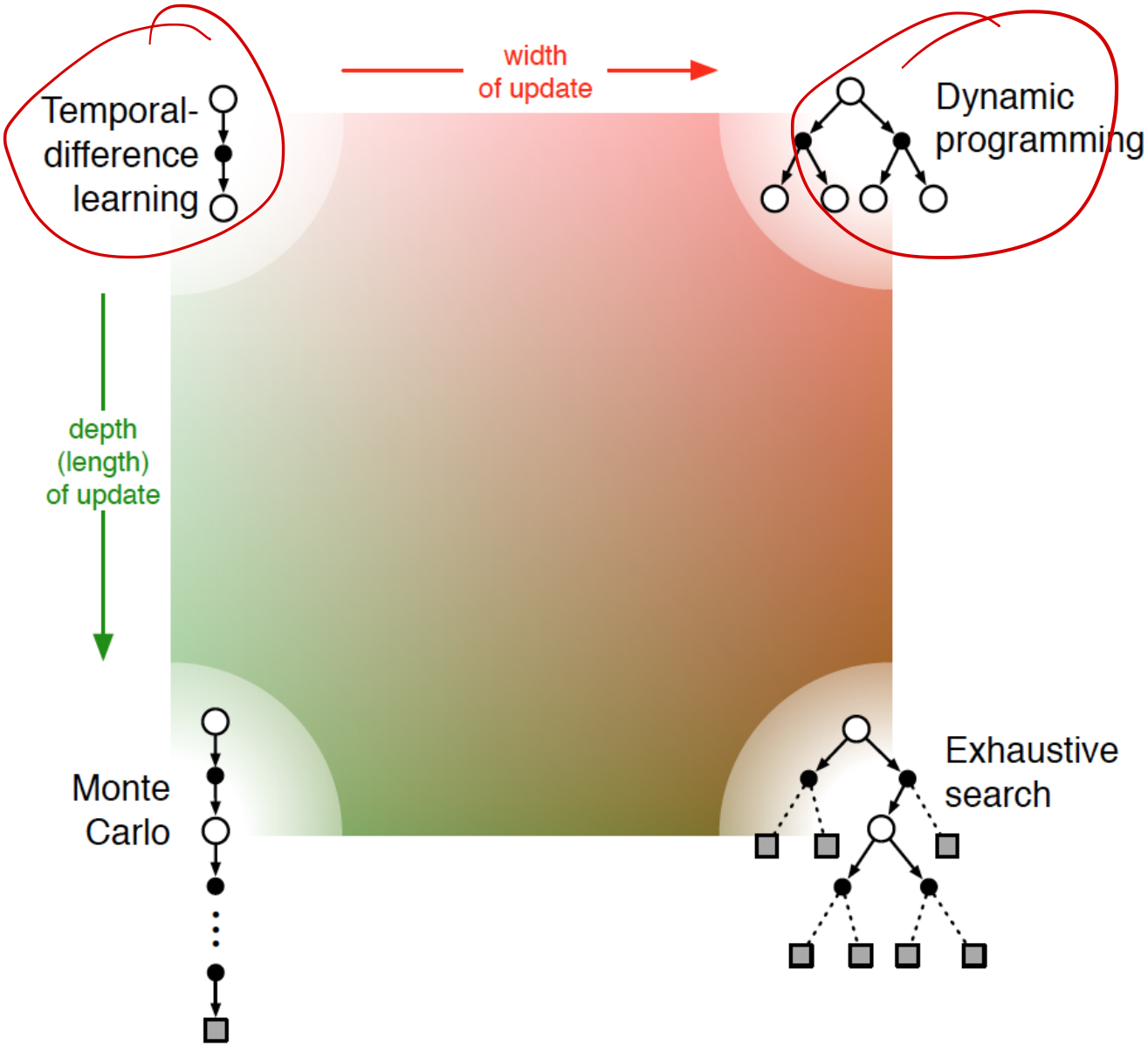
Relationship Between DP and TD

	Full Backup (DP)	*Sample Backup (TD)*
Bellman Expectation Equation for $v_\pi(s)$	 Iterative Policy Evaluation	 TD Learning
Bellman Expectation Equation for $q_\pi(s,a)$	 Q-Policy Iteration	 Sarsa
Bellman Optimality Equation for $q_*(s,a)$	 Q-Value Iteration	 Q-Learning

Relationship Between DP and TD(2)

Full Backup (DP)	*Sample Backup (TD)*
Iterative Policy Evaluation	TD Learning
$V(s) \leftarrow \mathbb{E}\left[R + \gamma V(S') \mid s\right]$	$V(S) \overset{\alpha}{\leftarrow} R + \gamma V(S')$
Q-Policy Iteration	Sarsa
$Q(s,a) \leftarrow \mathbb{E}\left[R + \gamma Q(S', A') \mid s, a\right]$	$Q(S,A) \overset{\alpha}{\leftarrow} R + \gamma Q(S', A')$
Q-Value Iteration	Q-Learning
$Q(s,a) \leftarrow \mathbb{E}\left[R + \gamma \max_{a' \in \mathcal{A}} Q(S', a') \mid s, a\right]$	$Q(S,A) \overset{\alpha}{\leftarrow} R + \gamma \max_{a' \in \mathcal{A}} Q(S', a')$

Unified View of Reinforcement Learning



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Main References

- Reinforcement Learning: An Introduction (second edition), R. Sutton & A. Barto, 2018.
- RL course slides from Richard Sutton, University of Alberta.
- RL course slides from David Silver, University College London.