# SI120 Tutorial

Apr 8, 2022

# Contents

- HW5 Solutions
- Practice on examples

Q1

1. (20 points) Let $A$ and $B$ be any sets. Show that if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.
(**Remark:** $\mathcal{P}(A)$ is the power set of $A$, i.e., the set of all subsets of $A$)

Q1: Solution 1

To prove that if $P(A)=P(B)$, then $A=B$

We have $P(A)=P(B)$. Assume $A \neq B$

without loss of generality

$\therefore \exists x_0 \in A,\ x_0 \notin B$

$\because P(A)=\{x \mid x \subset A\},\ P(B)=\{x \mid x \subset B\}$

$\therefore \{x_0\} \in P(A),\ \{x_0\} \notin P(B)$

$\therefore P(A) \neq P(B)$

$\therefore$ contradiction

$\therefore A=B$

proved

$|A|=n$

Solution 2:

$\because A \subseteq P(A),\ B \subseteq P(B)$

$\therefore A \subseteq P(B),\ B \subseteq P(A)$

$\therefore A \subseteq B,\ B \subseteq A$

$\Rightarrow A = B$

$A \subseteq A,\ \therefore A \subseteq P(A)$

$\because P(A)=P(B)$

$\therefore A \subseteq P(B)$

$\therefore A \subseteq B$

W.L.O.G.

$B \subseteq A$

$\therefore A=B.$

Solution 3:

证逆否命题成立

即 “$A \neq B$, then $P(A) \neq P(B)$”

1. original set 是它 Power set 的 Subset

2. $x \in$ Powerset, $x \subseteq$ original set

1. $\Rightarrow$

2. $\Rightarrow$ $A \subseteq B,\ B \subseteq A$

$\in$

$\subseteq$

扣分点:

(1) most common: 默认 A.B (P(A), P(B)) 为可列有限集, 写作 $A=\{a_1, a_2, \cdots\}$ 或 $|A|=n$, 扣 5'

(2) 过程过于简洁, 扣 10'

(3) 解答十分 confusing, 扣 15'

(4) 部分符号错误, 扣 1-2' [e.g. 错误区分 '$\in$' 和 '$\subset$', 将 '$\in$' 写成 '$\ni$']

Q2 2. (20 points) Construct a bijection from $A = (0,1) \cup [2,3) \cup (4,5]$ to $B = (6,7) \cup [8,+\infty)$.

$f: A \to B$

$f^{-1}: B \to A$

$A \to B$

## 2 Problem 2

Solution:

$$f : A \mapsto B \quad x \mapsto \begin{cases} \frac{x+12}{2} & x \in (0,1) \\ \frac{x+11}{2} & x \in [2,3) \\ \frac{8}{x-4} & x \in (4,5] \end{cases}$$

Proof:

Surjective:
When $x \in (0,1)$, $f(x) = \frac{x+12}{2} \in (\frac{0+12}{2}, \frac{1+12}{2}) = (6, \frac{13}{2})$
When $x \in [2,3)$, $f(x) = \frac{x+11}{2} \in [\frac{2+11}{2}, \frac{3+11}{2}) = [\frac{13}{2}, 7)$
When $x \in (4,5]$, since $\lim_{x\to 4^+} \frac{8}{x-4} = +\infty$, $f(x) = \frac{8}{x-4} = [\frac{8}{5-4}, +\infty) = [8, +\infty)$
So when $x \in A = (0,1) \cup [2,3) \cup (4,5]$, $f(x) \in (6, \frac{13}{2}) \cup [\frac{13}{2}, 7) \cup [8, +\infty) = B$
So $f$ is surjective.

Injective:
If $\exists x_1, x_2 \in A, f(x_1) = f(x_2)$,
then we have that $x_1, x_2 \in (0,1)$ or $x_1, x_2 \in [2,3)$ or $x_1, x_2 \in (4,5]$, otherwise $x_1 \neq x_2$ obtained by proven.
And in every interval of A, $f$ is monotone,i.e.$\forall x_1, x_2, \frac{f(x_2)-f(x_1)}{x_2-x_1} \neq 0$ and is continous.
So $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
So $f$ is injective.

So $f$ is bijective.

Q3 3. (20 points) Prove or disprove $|\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}| = |\mathbb{R}|$.

Proof:

Denote $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Define $f : [0, 1) \mapsto A \quad t \mapsto (\cos(2\pi t), \sin(2\pi t))$

sol1 For the definition of the radian, $\forall P \in A$, denote $\theta = \angle POx \in [0, 2\pi)$, $\cos\theta = \frac{x}{r} = x, \sin\theta = \frac{y}{r} = y$

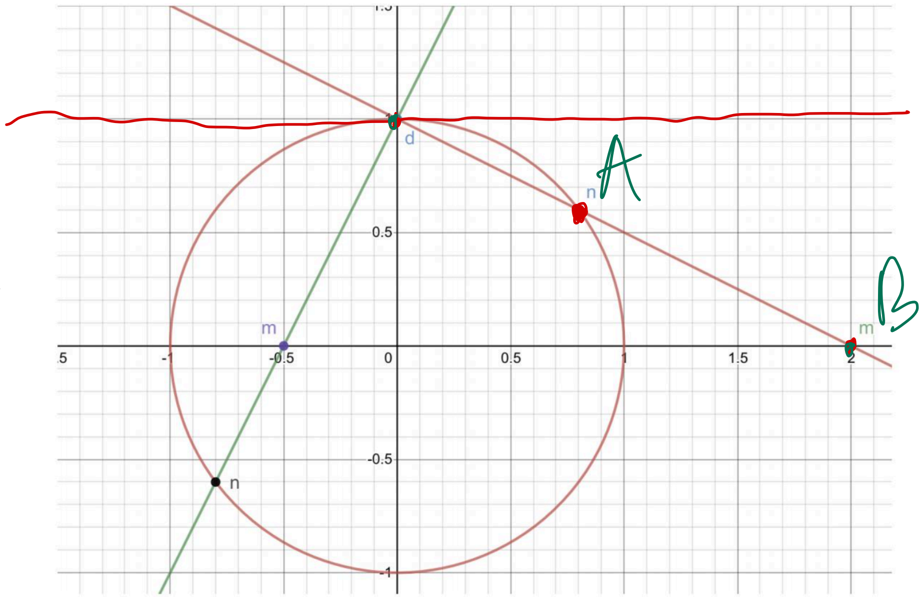
$(x_1, y_1) = (x_2, y_2) \Rightarrow \cos\theta_1 = \cos\theta_2$, $\sin\theta_1 = \sin\theta_2$, for $2\pi t \in [0, 2\pi)$, $\theta_1 = \theta_2 \Rightarrow \frac{\theta_1}{2\pi} = \frac{\theta_2}{2\pi} \Rightarrow t_1 = t_2$

And angle of a whole circle is $2\pi$, so according to the definition of radian, $f([0, 1)) = A$

So $f$ is both injective and surjective, so $f$ is bijective.

So $|A| = |[0, 1)|$,and $|[0, 1)| = |\mathbb{R}|$,so $|\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}| = |\mathbb{R}|$

sol2



Q4 4. (20 points) Prove or disprove $|\{(a_1, a_2, a_3, \ldots) : a_i \in \{1, 2, 3\} \text{ for all } i = 1, 2, 3, \ldots\}| = |\mathbb{Z}^+|$.

$A$

$0.\left(\sum_{i=1}^{\infty} (a_i - 1) 3^{i-1}\right)$

$[0, 1]$

十进制中, $a_3 a_2 a_1 a_0$

$\Rightarrow a_0 \times 10^0 + a_1 \times 10^1 + a_2 \times 10^2 + a_3 \times 10^3$

三进制, $2012$

$0.123 = 1 \times 10^{-1} + 2 \times 10^{-2} + 3 \times 10^{-3} \Rightarrow 2 \times 3^0 + 1 \times 3^1 + 0 \times 3^2 + 2 \times 3^3$

$\sum_{i=1}^{\infty} (a_i - 1) 3^{-i}$ $\quad A \to [0, 1]$ $\quad |A| = |[0,1]| \neq |\mathbb{Z}^+|$

$\forall a_1, a_2 \quad a_1 < a_2, \exists a_3$ s.t. $a_1 < a_3 < a_2$

$\mathbb{Q}$

Q5

5. (20 points) Find a countably infinite number of subsets of $\mathbb{Z}^+$, say $A_1, A_2, \ldots \subseteq \mathbb{Z}^+$ such that the following requirements are simultaneously satisfied:

- $|A_i| = |\mathbb{Z}^+|$ for all $i = 1, 2, \ldots$;
- $A_i \cap A_j = \emptyset$ for all $i \neq j$;
- $\cup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

$A_i$：每个元素都恰好出现在个 I.

1.$A_i = \{n : n \text{ has } i \text{ ones in its binary representation}\}$.

- $A_1 = \{1, 10, 100, \cdots\}$, $A_2 = \{11, 110, 101, 1100, \cdots\}$

2.$A_i = \{n : n \text{ has } (i+1) \text{ factors}\}$, $A_1 = A_1 \cup \{1\}$.

- $n = \prod_{k=1}^{r} p_k^{\alpha_k}$, $\prod_{k=1}^{r} (\alpha_k + 1) = i$. $\left(\sum_{k=1}^{r} \alpha_k = i \text{ also OK.}\right)$

3.$A_{i+1} = \{n : n \text{ is the power of the } i^{\text{th}} \text{ prime}\}$, $A_1 = \mathbb{Z}^+ \backslash \bigcup_{i=2}^{\infty} A_i$.

- $A_2 = \{2^1, 2^2, 2^3, \cdots\}$, $A_3 = \{3^1, 3^2, \cdots\}$, $A_4 = \{5^1, 5^2, \cdots\}$, $\cdots$ $A_1 =$ others

4.$A_i = \{n : n = \frac{i^2 - i + 2}{2} + \frac{(2i-3)k + k^2}{2}, k \in \mathbb{Z}^+\}$

- $A_1 = \{1, 2, 4, 7, 11 \cdots\}$
  $A_2 = \{3, 5, 8, 12, 17 \cdots\}$
  $A_3 = \{6, 9, 13, 18, \cdots\}$.
  $\vdots$

# Contents

- HW5 Solutions
- Practice on examples

EX1. Prove or disprove $|\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}| = |\mathbb{Z}^+|$.

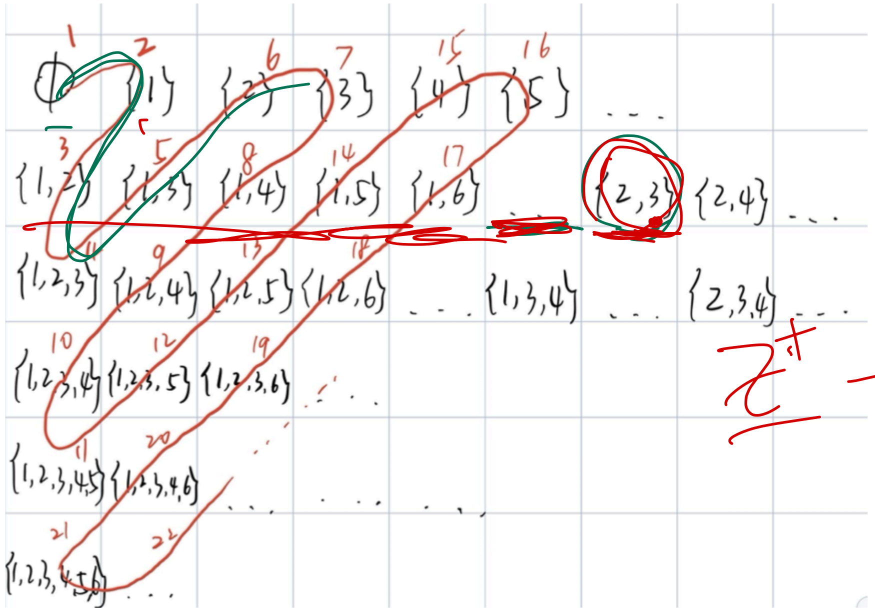
**Proof:**

Because $X \subseteq \mathbb{Z}^+$ and $X$ is finite $\Rightarrow |X| \leqslant |\mathbb{Z}^+|$ and $X$ can be shown as $X = \{a_1, a_2, a_3, a_4, a_5, \ldots, a_n\}$, $(n \in \mathbb{Z}^+)$. Therefore

Then order those '$X$'s in a series of specific rules, looks like this:

$$X_0 = \emptyset \quad X_1 = \{1\} \quad X_2 = \{2\} \quad X_3 = \{3\} \quad \ldots$$
$$X_i = \{1,2\} \quad X_{i+1} = \{1,3\} \quad \ldots \quad X_{i+j} = \{2,3\} \quad X_{i+j+1} = \{2,4\} \quad \ldots$$
$$X_{i+j+k+\ldots} = \{1,2,3\} \quad X_{i+j+k+\ldots+1} = \{1,2,4\} \quad \ldots \quad X_{i+j+k+\ldots+l} = \{1,3,4\} \quad X_{i+j+k+\ldots+l+1} = \{1,3,5\} \quad \ldots$$
$$\vdots$$

And put those '$X$'s into a table like this:




$\{2,3\}$.

Loop : $\{1, x\}$

$x : 2 \to \infty$

$\mathbb{Z}^+ \to \{2,3\}$

$\{2,3\}$

Naturally, we have an bijection there. Every element in $\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}$ matches an integer in $\mathbb{Z}^+$; And for every integer in $\mathbb{Z}^+$, it matches an element in $\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}$ as well.

So we have $|\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}| = |\mathbb{Z}^+|$.

EX1. Prove or disprove $|\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}| = |\mathbb{Z}^+|$.

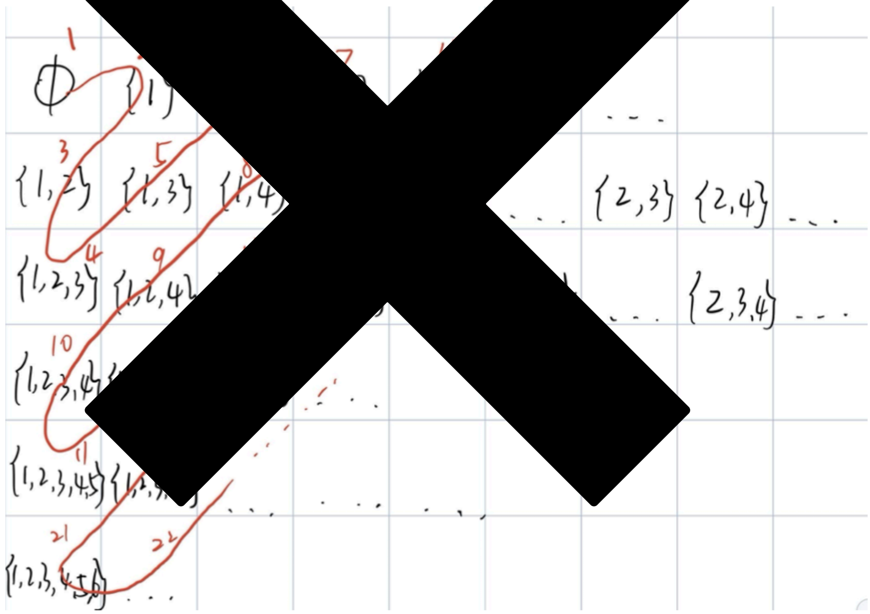
**Proof:**

Because $X \subseteq \mathbb{Z}^+$ and $X$ is finite $\Rightarrow |X| \leqslant |\mathbb{Z}^+|$ and $X$ can be shown as $X = \{a_1, a_2, a_3, a_4, a_5, \ldots, a_n\}$, $(n \in \mathbb{Z}^+)$. Therefore

Then order those '$X$'s in a series of specific rules, looks like this:

$X_0 = \emptyset \quad X_1 = \{1\} \quad X_2 = \{2\} \quad X_3 = \{3\} \quad \ldots$
$X_i = \{1,2\} \quad X_{i+1} = \{1,3\} \quad \ldots \quad X_{i+j} = \{2,3\} \quad X_{i+j+1} = \{2,4\} \quad \ldots$
$X_{i+j+k+\ldots} = \{1,2,3\} \quad X_{i+j+k+\ldots+1} = \{1,2,4\} \quad \ldots \quad X_{i+j+k+\ldots+l} = \{1,3,4\} \quad X_{i+j+k+\ldots+l+1} = \{1,3,5\} \quad \ldots$
$\vdots$

And put those [redacted]e like this:




Naturally, we have an bijection there. Every element in $\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}$ matches an integer in $\mathbb{Z}^+$; And for every integer in $\mathbb{Z}^+$, it matches an element in $\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}$ as well.

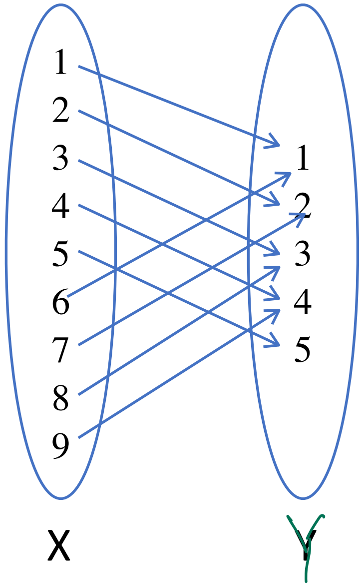
So we have $|\{X \subseteq \mathbb{Z}^+ : X \text{ is a finite set}\}| = |\mathbb{Z}^+|$.

陪域　　X domain　　Y codomain

Ex2. What is the number of **surjections** with a domain containing 9 elements and a codomain containing 5 elements?

定义域

{1}
range
image




# Principle of IE ($n$ Sets)

Recall:

**THEOREM:** Let $S$ be a finite set. Let $A_1, A_2, \dots, A_n$ be subsets of $S$.

Then $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$

$5^9$ − #非满射 = #满射

$A_i = \{f : f(x) = y,\ x \in X,\ y \in Y / \{i\}\}$

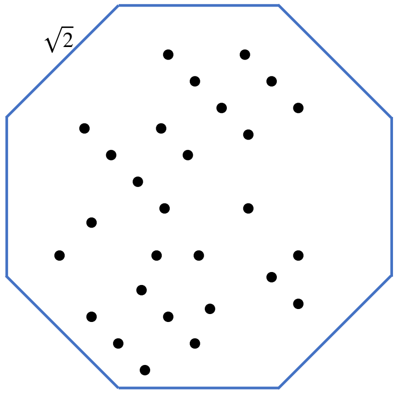
$A_1$ : 2~5　　$A_2$ 1, 3~5　　— — —　　$A_5$ 1~4

#非满射 $= |\bigcup_{i=1}^{5} A_i|$ = "至少有一个元素没有被映射到"的映射 ⇒ 非满射

$= \binom{5}{1}4^9 - \binom{5}{2}3^9 + \binom{5}{3}2^9 - \binom{5}{4}1^9$

$5^9 -$

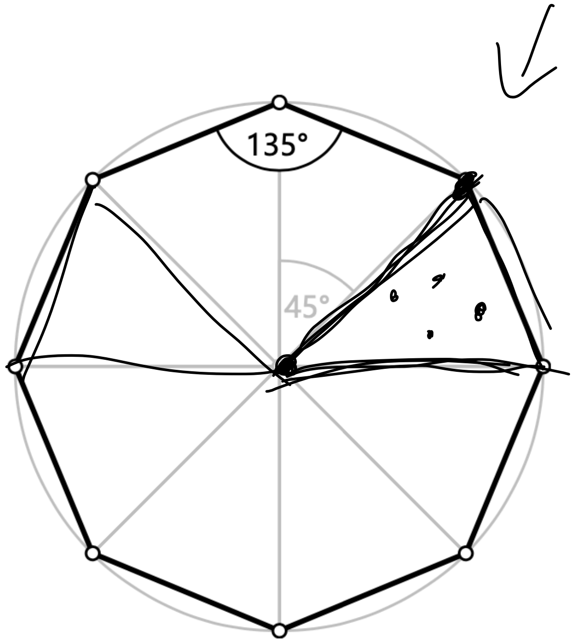
Ex3. Suppose we have a regular octagonal with side lengths equal to $\sqrt{2}$. Given 28 points in the octagonal, prove that there is a collection of 4 points such that the distance between any two in the collection is at most 2.




Recall: **THEOREM:** (general form) Let $A$ be a set with $\geq N$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of $A$. Then $\exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

Ex3. Suppose we have a regular octagonal with side lengths equal to $\sqrt{2}$. Given 28 points in the octagonal, prove that there is a collection of 4 points such that the distance between any two in the collection is at most 2.

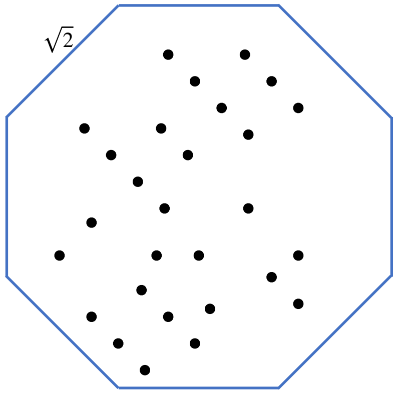
**Proof:**




8 4

28

$|A| \geq \left\lceil \frac{28}{8} \right\rceil = 4$




Recall: **THEOREM:** (general form) Let $A$ be a set with $\geq N$ elements. Let $\{A_1, A_2, \dots, A_n\}$ be a cover of $A$. Then $\exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

Divide the octagon into eight identical triangles. Inside each triangle, we can easily find that the maximum distance between two points is the length of a leg of the identical triangles which is $\frac{1}{2}\sqrt{6+4\sqrt{2}}$ and it is smaller than 2.

$< 2$

So we know that the distance between any two points inside such a triangle is smaller than 2.

From **The Pogeonhole Principle**, there exists such a triangle that contains at least $\lceil \frac{28}{8} \rceil = 4$ points if we place 28 points in this octagon. So we have at least 4 points in this octagon where the distance between any two points of them is not more than 2.

□