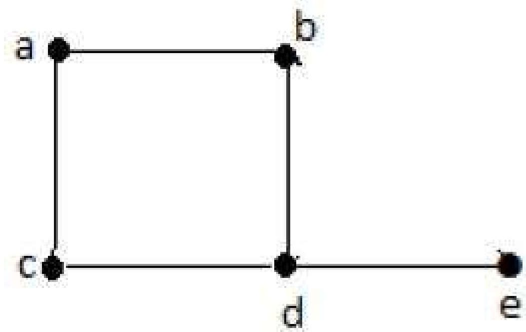
Lecture

NOTE: FOR FURTHER DETAILS AND MORE COMPREHENSIVE STUDY, PLEASE SEE RECOMMENDED BOOKS OR INTERNET.

Graph

A graph is a pictorial representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by points termed as ***vertices*** also known as ***Point, Node***, and the links that connect the vertices are called ***edges*** also known as ***Line, Link, Arc***.

Formally, a graph G is a pair of sets ***G(V, E),*** where ***V*** is the set of vertices and ***E*** is the set of edges, connecting the pairs of vertices. Take a look at the following graph:



In the above graph,

V = {a, b, c, d, e}

E = {ab, ac, bd, cd, de}

Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering:

- ***Electrical Engineering*** – The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.
- ***Computer Science*** – Graph theory is used for the study of algorithms. For example,
  - Kruskal's Algorithm
  - Prim's Algorithm
  - Dijkstra's Algorithm
- ***Computer Network*** – The relationships among interconnected computers in the network follows the principles of graph theory called topologies.
- ***Science*** – The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.
- ***Linguistics*** – The parsing tree of a language and grammar of a language uses graphs.
- ***General*** – Routes between the cities can be represented using graphs.

Graph Theory Fundamentals

A graph is a diagram of points and lines that connect the points. It has at least one line joining a set of two vertices with no vertex connecting itself. The concept of graphs in graph theory stands up on some basic terms such as point, line, vertex, edge, degree of vertices, properties of graphs, etc. Here we will cover these fundamentals of graph theory.

Point

A point is a particular position in a one-dimensional, two-dimensional, or three-dimensional space. For better understanding, a point can be denoted by an alphabet. It can be represented with a dot or circle.

Here, the dot is a point named ***'a'***.

Line

A Line is a connection between two points. It can be represented with a solid line.

Here, ***'a'*** and ***'b'*** are the points. The link between these two points is called a ***line***.

Vertex

A vertex is a point where multiple lines meet. It is also called a node. Similar to points, a vertex is also denoted by an alphabet.

Here, the vertex is named with an alphabet ***'a'***.

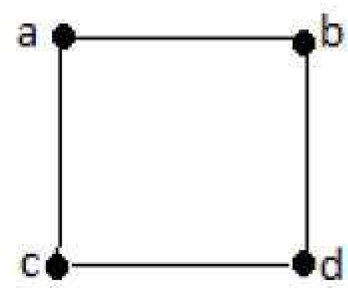
Edge

An edge is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.

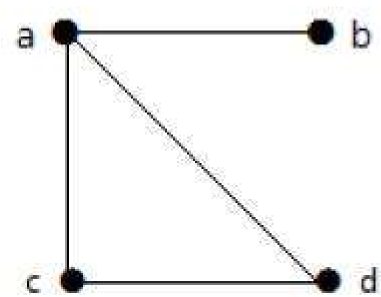
Here, ***'a'*** and ***'b'*** are the two vertices and the link between them is called an ***edge***.

Graph

A graph ***'G'*** is defined as ***G=(V, E)*** Where ***V*** is a set of all vertices and ***E*** is a set of all edges in the graph.



In the above example, ***ab***, ***ac***, ***cd***, and ***bd*** are the edges of the graph. Similarly, ***a***, ***b***, ***c***, and ***d*** are the vertices of the graph.



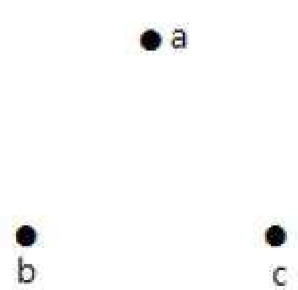
In this graph, there are four vertices ***a***, ***b***, ***c***, and ***d***, and four edges ***ab***, ***ac***, ***ad***, and ***cd***.

Types of Graph

There are various types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure. We will discuss only a certain few important types of graphs.

Null Graph

A graph having ***no edges*** is called a ***Null Graph***.



In the above graph, there are three vertices named ***'a'***, ***'b'***, and ***'c'***, but there are no edges among them. Hence it is a ***Null Graph***.

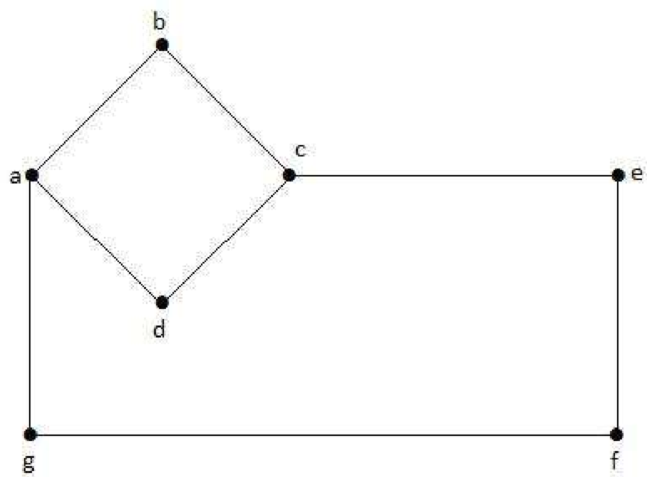
Trivial Graph

A graph with ***only one vertex*** is called a ***Trivial Graph***.

In the above shown graph, there is only one vertex ***'a'*** with no other edges. Hence it is a ***Trivial graph***.

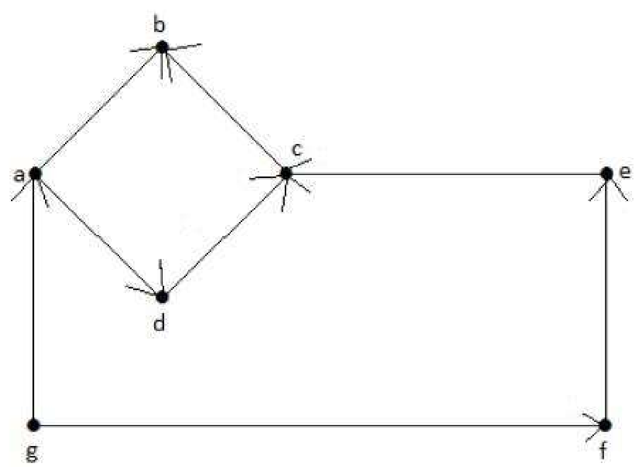
Undirected Graph

An undirected graph contains ***edges*** but the edges are ***not directed*** ones.



Directed Graph

In a directed graph, each ***edge*** has a ***direction***.

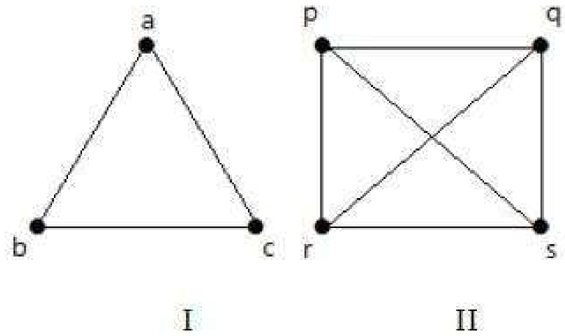


Complete Graph

A simple graph where each vertex should have ***edges with all*** other vertices, then it called a ***complete graph***.

In other words, if a vertex is ***connected to all*** other vertices in a graph, then it is called a ***complete graph***.

In the following graphs, each vertex in the graph is connected with all the remaining vertices in the graph except by itself.



In graph I,

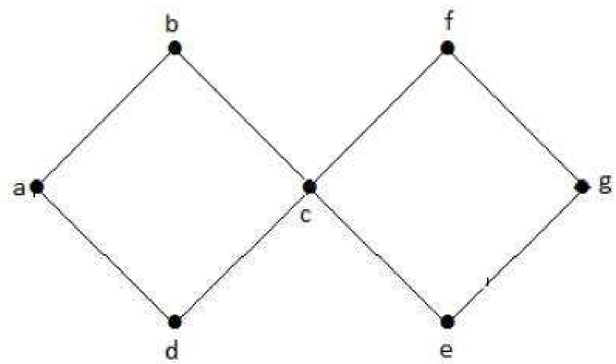
	a	**b**	**c**
a	Not Connected	Connected	Connected
b	Connected	Not Connected	Connected
c	Connected	Connected	Not Connected

In graph II,

	p	**q**	**r**	**s**
p	Not Connected	Connected	Connected	Connected
q	Connected	Not Connected	Connected	Connected
r	Connected	Connected	Not Connected	Connected
s	Connected	Connected	Connected	Not Connected

Cyclic Graph

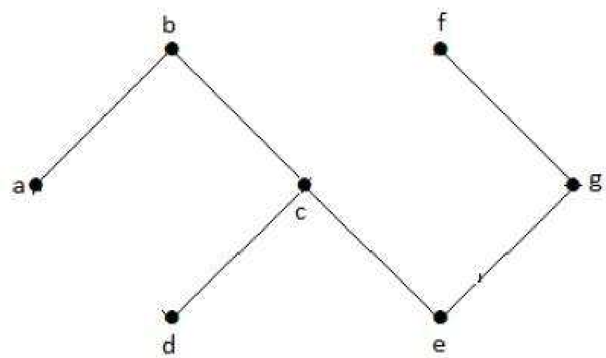
A graph with at least ***one cycle*** is called a ***cyclic graph***.



In the above example graph, we have two cycles ***a-b-c-d-a*** and ***c-f-g-e-c***. Hence it is called a ***cyclic graph***.

Acyclic Graph

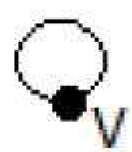
A graph with ***no cycles*** is called an ***acyclic graph***.



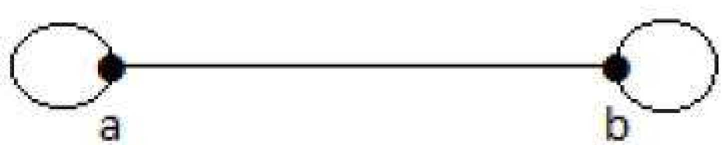
In the above example graph, we ***do not*** have any ***cycles***. Hence it is an ***acyclic graph***.

Loop

In a graph, if an edge is drawn from vertex to itself, it is called a ***loop***.



In the above graph, ***V*** is a vertex for which it has an ***edge (V, V)*** forming a ***loop***.



In this graph, there are two loops which are formed at vertex ***a***, and vertex ***b***.

Pendent Vertex

By using degree of a vertex, a vertex with ***degree one*** is called a ***pendent vertex***.

Here, in this example, vertex ***'a'*** and vertex ***'b'*** have a connected edge ***'ab'***. So with respect to the vertex ***'a'***, there is only one edge towards vertex ***'b'*** and similarly with respect to the vertex ***'b'***, there is only one edge towards vertex ***'a'***. Finally, ***vertex 'a'*** and vertex ***'b'*** has degree as one which are also called as the ***pendent vertex***.

Isolated Vertex

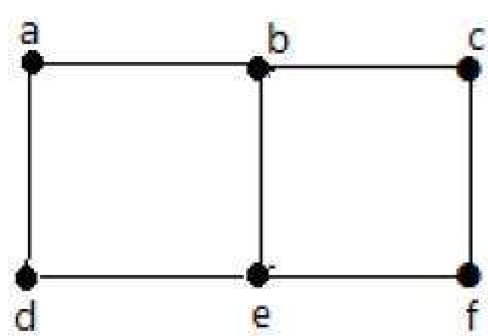
A vertex with degree zero is called an isolated vertex.

Here, the vertex ***'a'*** and vertex ***'b'*** has a ***no connectivity*** between each other and also to any other vertices. So the degree of both the vertices ***'a'*** and ***'b'*** are zero. These are also called as ***isolated vertices***.

Adjacency

Here are the norms of ***adjacency***,

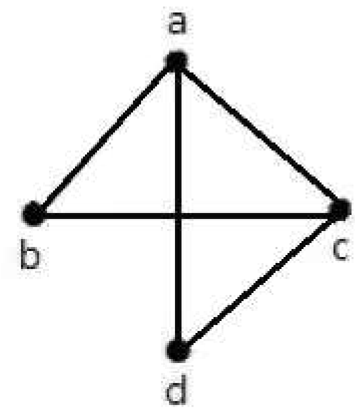
- In a graph, ***two vertices*** are said to be ***adjacen***t, if there is an ***edge*** between the two vertices. Here, the ***adjacency*** of vertices is maintained by the single edge that is connecting those two vertices.
- In a graph, ***two edges*** are said to be ***adjacent***, if there is a common ***vertex*** between the two edges. Here, the ***adjacency*** of edges is maintained by the single vertex that is connecting two edges.



In the above graph:

- ***'a'*** and ***'b'*** are the adjacent vertices, as there is a common edge ***'ab'*** between them.
- ***'a'*** and ***'d'*** are the adjacent vertices, as there is a common edge ***'ad'*** between them.

- ***'ab'*** and ***'be'*** are the adjacent edges, as there is a common vertex ***'b'*** between them.
- ***'be'*** and ***'de'*** are the adjacent edges, as there is a common vertex ***'e'*** between them.

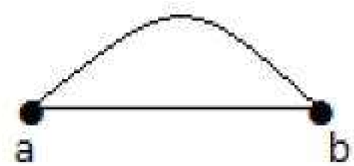


In the above graph:

- ***'a'*** and ***'d'*** are the adjacent vertices, as there is a common edge ***'ad'*** between them.
- ***'c'*** and ***'b'*** are the adjacent vertices, as there is a common edge ***'cb'*** between them.
- ***'ad'*** and ***'cd'*** are the adjacent edges, as there is a common vertex ***'d'*** between them.
- ***'ac'*** and ***'cd'*** are the adjacent edges, as there is a common vertex ***'c'*** between them.

Parallel Edges

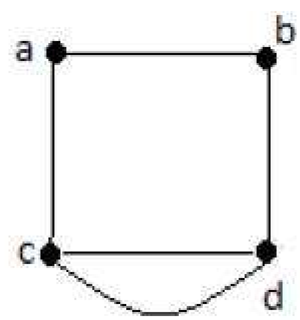
In a graph, if a pair of vertices is connected by ***more than one edge***, then those edges are called ***parallel edges***.



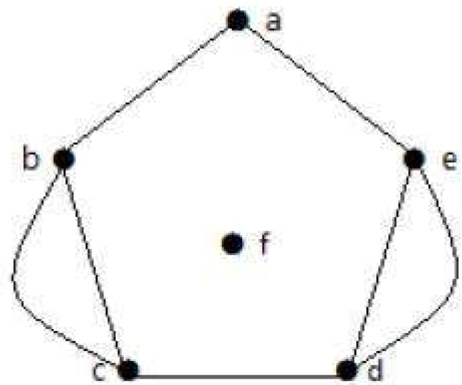
In the above graph, ***'a'*** and ***'b'*** are the two vertices which are connected by two edges ***'ab'*** and ***'ab'*** between them. So it is called as ***parallel edges***.

Multi Graph

A graph having ***parallel edges*** is known as a ***Multigraph***.



In the above graph, there are five edges ***'ab'***, ***'ac'***, ***'cd'***, ***'cd'***, and ***'bd'***. Since ***'c'*** and ***'d'*** have two ***parallel edges*** between them, it a ***Multigraph***.



In the above graph, the vertices ***'b'*** and ***'c'*** have two edges. The vertices ***'e'*** and ***'d'*** also have two edges between them. Hence it is a ***Multigraph***.

Degree of Vertex

It is the number of vertices incident with the vertex ***v***.

Notation: deg(v).

In a simple graph with n number of vertices, the degree of any vertices is:

$$\mathbf{deg(v) \leq n - 1 \ \forall \ v \in G}$$

A vertex can form an edge with all other vertices except by itself. So the degree of a vertex will be up to the number of vertices in the graph minus 1. This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a Simple Graph.

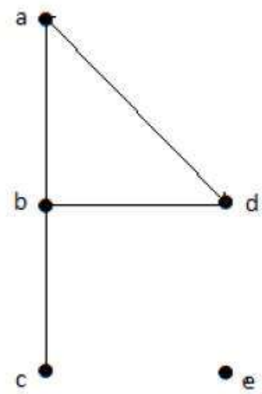
Degree of vertex can be considered under two cases of graphs:

- *Undirected Graph*
- *Directed Graph*

Degree of Vertex in an Undirected Graph

An undirected graph has no directed edges. Consider the following examples.

Take a look at the following graph:



In the above Undirected Graph,

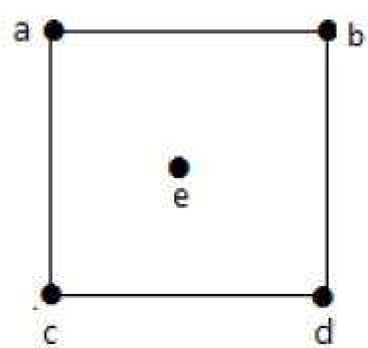
- ***deg(a) = 2***, as there are 2 edges meeting at vertex ***'a'***.
- ***deg(b) = 3***, as there are 3 edges meeting at vertex ***'b'***.
- ***deg(c) = 1***, as there is 1 edge formed at vertex ***'c'.***

So ***'c'*** is a ***pendent vertex***.

- ***deg(d) = 2***, as there are 2 edges meeting at vertex ***'d'***.
- ***deg(e) = 0***, as there are 0 edges formed at vertex ***'e'***.

So ***'e'*** is an ***isolated vertex***.

Take a look at the following graph



In the above graph,

*deg(**a**) = 2, deg(**b**) = 2, deg(**c**) = 2, deg(**d**) = 2, and deg(**e**) = 0*

The vertex ***'e'*** is an ***isolated vertex***. The graph does not have any ***pendent vertex***.

Degree of Vertex in a Directed Graph

In a directed graph, each vertex has an ***indegree*** and an ***outdegree***.

Indegree of a Graph

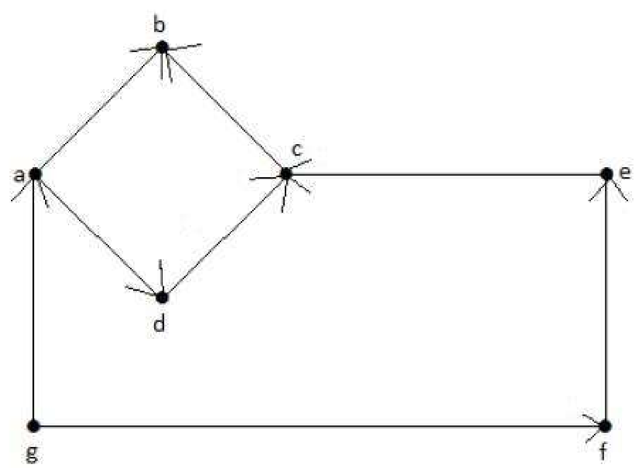
- ***Indegree*** of vertex ***v*** is the number of edges which are coming into the vertex ***v***.
- **Notation**: $deg^{+}(v)$.

Outdegree of a Graph

- ***Outdegree*** of vertex ***v*** is the number of edges which are going out from the vertex ***v***.
- **Notation**: $deg^{-}(V)$.

Consider the following examples.

Take a look at the following ***directed graph***. Vertex ***'a'*** has two edges, ***'ad'*** and ***'ab'***, which are going outwards. Hence its ***outdegree*** is 2. Similarly, there is an edge ***'ga'***, coming towards vertex ***'a'***. Hence the ***indegree*** of ***'a'*** is 1.

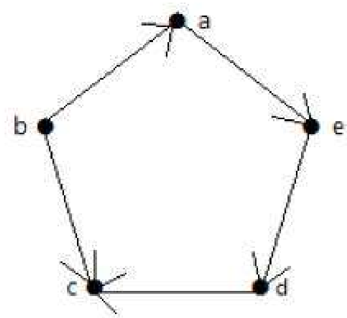


The ***indegree*** and ***outdegree*** of other vertices are shown in the following table:

Vertex	*Indegree*	*Outdegree*
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1

g	0	2

Take a look at the following directed graph. Vertex ***'a'*** has an edge ***'ae'*** going outwards from vertex ***'a'***. Hence its ***outdegree*** is 1. Similarly, the graph has an edge ***'ba'*** coming towards vertex ***'a'***. Hence the ***indegree*** of ***'a'*** is 1.

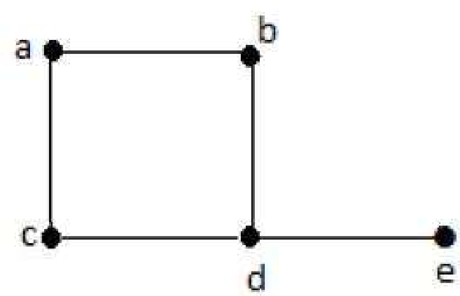


The ***indegree*** and ***outdegree*** of other vertices are shown in the following table:

Vertex	*Indegree*	*Outdegree*
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

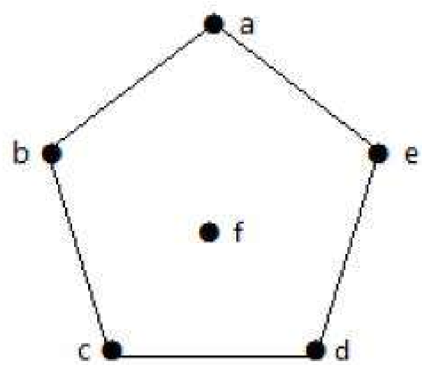
Degree Sequence of a Graph

If the degrees of all vertices in a graph are ***arranged*** in ***descending*** or ***ascending*** order, then the sequence obtained is known as the ***degree sequence*** of the graph.



Vertex	a	b	c	d	e
Connecting to	b,c	a,d	a,d	c,b,e	d
Degree	2	2	2	3	1

In the above graph, for the vertices ***{d, a, b, c, e},*** the degree sequence is ***{3, 2, 2, 2, 1}.***



Vertex	a	b	c	d	e	f
Connecting to	b,e	a,c	b,d	c,e	a,d	-
Degree	2	2	2	2	2	0

In the above graph, for the vertices ***{a, b, c, d, e, f},*** the degree sequence is ***{2, 2, 2, 2, 2, 0}.***