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@ Made Easy

UNIT 1 : CALCULUS

1.1 — FUNCTIONS AND THEIR TYPES

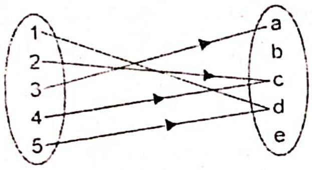
FUNCTIONS

Let us consider two sets A and B. Then a relation R from set A to set B is said to be a function from A to B if "For each element in set A, there exist only one image in set B",

i.e., $\forall x \in A$. There exist unique $y \in B$ such that $y = f(x)$.

Here A = Domain, B = Co-domain and Image set of A is called Range denoted by R so $R \subseteq$ Codomain (B).

Consider,



Then A = {1, 2, 3, 4, 5} is domain of f.

B = {a, b, c, d, e} is codomain of f

R = {a, c, d} is range of f.

TYPES OF FUNCTIONS

(i) **One-One function / Injective Mapping :** If different elements have different images then f is said to one-one. Mathematically if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is one-one.

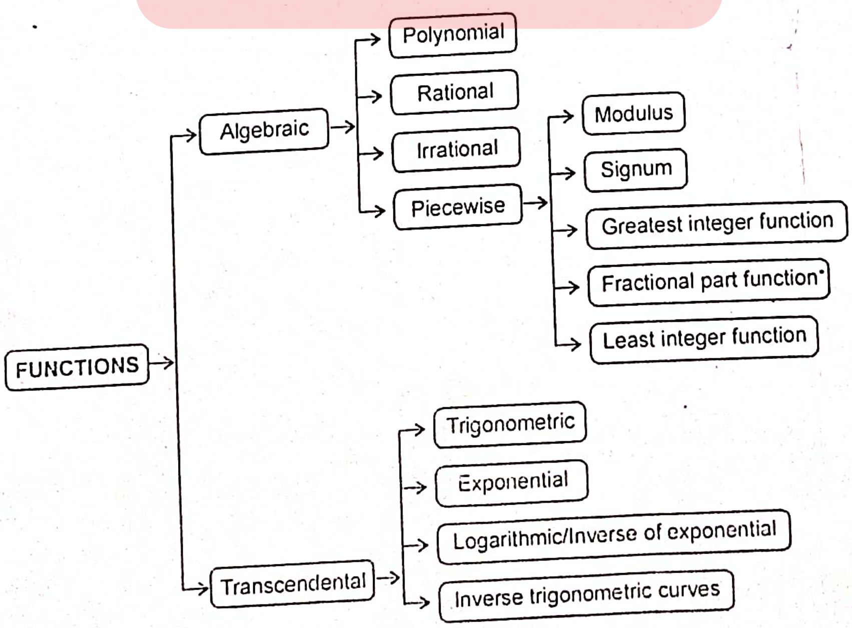
(ii) **Many One Function :** If different elements have same images then f is said to be many one. Mathematically, if $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$ then f is many one.

(iii) **ONTO Function :** If all the elements of codomain 'B' are used in mapping, i.e., Range f = Codomain then f is called ONTO.

(iv) **INTO Function :** If all the elements are not used in mapping i.e., Range $F \subset$ Codomain then f is INTO.

(v) **Bijective Function :** A function which is both one-one and onto is called Bijective, it is also called one-one correspondence between A and B.

TREE DIAGRAM OF FUNCTIONS



1.2 — INFINITE SERIES

Neighbourhood of Real Number 'a' : It means we are considering an open interval of very small length centered at point 'a', i.e.,

$$\text{Nod of 'a'} = (a - h,\ a + h) \text{ where } h \to 0 \text{ and } h > 0$$

TAYLOR SERIES

Consider a function f(x) which is defined in the domain D. If f(x) is differentiable any number of times in its domain, and a is any point in D then f(x) can be expanded in the neighbourhood of 'a' as follows :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots.. + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n}{n!}f^n(a) + \ldots..$$

Note :

1. If f(x) or any one of its derivative is discontinuous at x = a then we can not apply the Taylor series.
2. For n = 1, Taylor series reduces to Lagrange's Mean Value Theorem.
3. To find the approximate value in linear form, we should neglect 2nd and higher degree terms in Taylor series expansion, i.e., linear approximation of f(x) is defined as $f(x) = f(a) + (x-a)f'(a)$

MACLAURIN' SERIES

It is the Taylor Series expansion of f(x) in the neighbourhood of x = 0 and is given as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots.. + \frac{x^n}{x!}f^n(0) + \ldots..$$

Important Maclaurin's Expansion

1. $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \ldots..$
2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\infty$
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\infty$
4. $\sin h\, x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\infty$
5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\infty$
6. $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\infty$
7. $\tan x = x + \frac{x^2}{3} + \frac{2}{15}x^3 + \ldots\infty$
8. $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\infty$
9. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots..$

CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

1. If $\lim_{n\to\infty} \sum_{n=1}^{n} a_n$ = finite then $\sum_{n=1}^{\infty} a_n$ is called Convergent. (i.e., it is possible to find the sum of series).
2. If $\lim_{n\to\infty} \sum_{n=1}^{n} a_n = \pm\infty$ then $\sum_{n=1}^{\infty} a_n$ is called Divergent. (i.e., it is not possible to find the sum of series).
3. $\lim_{n\to\infty} \left(\sum_{n=1}^{n} a_n \right)$ = Neither finite nor infinite then $\sum_{n=1}^{\infty} a_n$ is called Oscillatory. (i.e., its value lies within some V range).

1.3 — LIMITS, CONTINUITY AND DIFFERENTIABILITY

LIMIT OF A FUNCTION

Consider the function $f(x) = \dfrac{x^2 - 9}{x - 3}$. Clearly, it is not possible to find f(3) as it comes in $\dfrac{0}{0}$ form. In such situation we try to find the value of f(x) in the neighbourhood x = 3, and this value is called limit of f(x), i.e.,

for the function $f(x) = \dfrac{x^2 - 9}{x - 3}$, f(3) = does not exist but $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{(x-3)(x+3)}{(x-3)} = \lim\limits_{x \to 3}(x+3) = 6$,

i.e., whenever x lies in the Nbd of 3, f(x) lies in the Nbd of 6

Definition : f(x) is said to have limit 'l' as x approaches to 'a' if

$\lim\limits_{x \to a} f(x) = l$ i.e., "Whenever x lies in the neighbourhood of a, f(x) lies in the neighbourhood of l".

Left Hand Limit : Value of f(x) in the left Nbd of 'a' is called LHL and it is denoted as : LHL = $\lim\limits_{x \to a^-} f(x)$

Right Hand Limit : Value of f(x) in the right nbd of 'a' is called RHL and it is denoted as : RHL = $\lim\limits_{x \to a^+} f(x)$

Note :

1. For the existence of limit, both LHL and RHL should be equal, i.e., if LHL $\neq$ RHL we say that limit does not exist.
2. Limit exist only when its value is unique and constant.

METHODS OF EVALUATING LIMITS

There are five methdos :

1. By Direct Substitution
2. By Factorisation
3. By Rationalisation
using Standard Results
5. By using Indeterminate Form Concept

INDETERMINATE FORMS

If value of the expression is neither unique nor infinite then expression is said to be in indeterminate form.

There are seven indeterminate forms : $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

The limiting value of indeterminate forms is known as true value

L-HOSPITAL RULE

This rule is applicable only for $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and rule is : "Differentiate numerator and denominator separately until we are free from $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form" i.e.,

when $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \ldots\ldots = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$

L-Hospital Rule for the Form $(\infty - \infty, 0 \times \infty)$

We will try to convert it into the form $\left(\frac{0}{0}\right)$ by simplification. Then use the L-Hospital Rule.

L-Hospital Rule for the Form $(0^0, 1^\infty, \infty^0)$

First take log and try to convert it into the form $\left(\frac{0}{0}\right)$. After that use L-Hospital Rule.

SOME STANDARD RESULTS ON LIMITS

1. $\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1$
2. $\lim_{x \to 0}\left(\frac{\sin^{-1} x}{x}\right) = 1$
3. $\lim_{x \to 0}\left(\frac{x}{\sin x}\right) = 1$
4. $\lim_{x \to 0}\left(\frac{\tan x}{x}\right) = 1$
5. $\lim_{x \to 0}\left(\frac{\sin(x-a)}{(x-a)}\right) = 1$
6. $\lim_{x \to 0}\left(\frac{\cos x}{x}\right) = \text{ONE}$
7. $\lim_{x \to 0}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$
8. $\lim_{x \to 0}\left(\frac{a^x - 1}{x}\right) = \log_e a$
9. $\lim_{x \to 0}\left(\frac{e^x - 1}{x}\right) = 1$
10. $\lim_{x \to 0}\left(\frac{1 - \cos(x)}{x^2}\right) = \frac{1}{2}$
11. $\lim_{x \to 0}\left(\frac{\log(1+x)}{x}\right) = 1$
12. $\lim_{x \to \infty}\left(\frac{\sin x}{x}\right) = 0$
13. $\lim_{x \to \infty}\left(\frac{\cos x}{x}\right) = 0$
14. $\lim_{x \to \infty}(1+ax)^{\frac{1}{x}} = e^a = \lim_{x \to \infty}\left(1 + \frac{a}{x}\right)^x$
15. $\lim_{x \to \infty}(1+x)^{\frac{1}{x}} = e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x$
16. $\lim_{x \to \infty}\left(\frac{a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots\ldots + a_m}{b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \ldots\ldots + b_n}\right) = \begin{cases} 0, & m < n \\ \frac{a_0}{b_0}, & m = n \\ \infty, & m > n \end{cases}$

CONTINUITY

A function $y = f(x)$ is said to be continuous if graph of function is continuous and if graph is broken at some point, we say that the function is discontinuous.

mathematically it is defined as "a function f(x) is said to be continuous at x = a, if

$$\lim_{x \to a} f(x) = f(a),$$

i.e., limit of function in the Nbd of 'a' = Functional Value at a."

Note :

1. A function f(x) is said to be discontinuous at x = a if we have any of the following cases:

   (i) $\lim_{x \to a} f(x)$ does not exist. (ii) $\lim_{x \to a} f(x) \neq f(a)$ (iii) f(a) is undefined

2. **Jump of a Function at a Point**

   Difference between LHL and RHL at x = a is called jump of function at that point.

   A function which has a finite number of jumps in a given interval is called **piecewise continuous.**

3. A function which is continuous in a closed interval is also bounded in that interval, i.e., we have finite values of y in that closed interval.

KINDS OF DISCONTINUITIES

Removable Discontinuity : If limit exist but not equal to functional value, i.e., $\lim_{x \to a} f(x) \neq f(a)$ then x = a is called Removable Discontinuity.

Discontinuity of First Kind/Jump Discontinuity : If both LHL and RHL exist but not equal, i.e., LHL ≠ RHL, then x = a is called Discontinuity of 1st kind.

Discontinuity of Second Kind : If neither LHL nor RHL exist then x = a is called Discontinuity of 2nd kind.

LIMIT OF A FUNCTION OF TWO VARIABLES

Consider a function of two variable z = f(x, y) then f(x, y) is said to have limit l as $(x, y) \to (a, b)$ if

$$\lim_{(x,y) \to (a,b)} f(x, y) = l$$

where (x, y) can approach to (a, b) along infinite number of paths.

If limit depends upon these paths we say that limit does not exist and if limit is same along infinite no. of paths then we say that limit exist.

Note : While solving questions based on double limits we can assume that all these path are contained in the radius vector y = mx if limit depends upon m we say that limit does not exist and if it is free from m (i.e., unique) then limit exists. Here m is representing path along which (x, y) is approaching to (a, b)

CONTINUITY OF FUNCTION OF TWO VARIABLES

A function f(x, y) is said to be continuous at a point (a, b) if $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$. i.e., limit must exist and it should be equal to functional value at (a, b).

DIFFERENTIABILITY

Rate of change of y with respect to x is called the derivative of y with respect to x and it is denoted by $\frac{dy}{dx}$ or $f'(x)$ and its mathematical definition is as follows :

Definition : f(x) is said to be differentiable at x = a if $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ exist and value of this limit is called derivative of f(x) at x = a and it is denoted as $f'(a)$, i.e.,

$$f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$$

Left Hand Derivative of f(x) is defined as $f'(a^-) = \lim_{x\to a^-}\left(\frac{f(x)-f(a)}{x-a}\right)$.

Right Hand Derivative of f(x) is defined as $f'(a^+) = \lim_{x\to a^+}\left(\frac{f(x)-f(a)}{x-a}\right)$

if LHD = RHD then also we say that f(x) is differentiable at x = a.

Note :

1. Geometrically $f'(x)$ is the representation of slope of tangent at any point 'x' in the graph of f(x), i.e.,

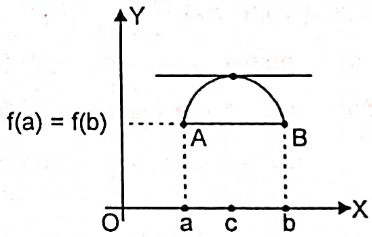
   $$\frac{dy}{dx} = f'(x) = \tan\theta = \text{Slope of tangent}$$

2. If x = a is the sharp point in the graph of f(x), then we say that f(x) is not differentiable there because in that situation we are not able to find unique tangent at x = a so it is not possible to find slope of tangent and consequently $f'(a)$ does not exist.
3. Continuity is a necessary conditions for the existence of derivative but it is not sufficient condition.

1.4 — MEAN VALUE THEOREMS

ROLLE'S THEOREM

Let f(x) be a function satisfying following properties :



(i) f(x) is continuous in the closed interval (a, b) i.e. $a \le x \le b$

(ii) $f'(x)$ exists in the open interval (a, b) i.e. a < x < b

(iii) f(a) = f(b)

then there exists at least one point c between a and b at which

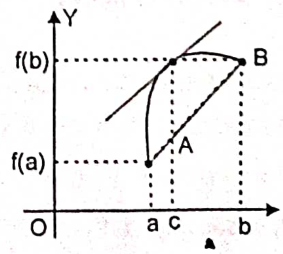
$$\boxed{f'(c) = 0}$$

i.e., tangent of f(x) at x = c becomes horizontal.

Note : Let f(x) is a polynomial having roots x = a and x = b, i.e., f(a) = f(b) = 0 then by Rolle's theorem there exist α in between a and b for which $f'(\alpha)$ = 0, i.e., α is the root of $f'(x)$.

Hence, between any two roots of an equation f(x) = 0 there is at least one root of $f'(x) = 0$.

LAGRANGE'S FIRST MEAN VALUE THEOREM



Let f(x) be a function satisfying following conditions

(i) f(x) is continuous in $a \le x \le b$.

(ii) f(x) is differentiable in the open interval a < x < b

then there exists at least one point c between a and b such that

$\boxed{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$, i.e., slope of tangent at 'C' = slope of chord AB

i.e., tangent at c, becomes parallel to the chord AB where A(a, f(a)), B(b, f(b)).

CAUCHY'S MEAN VALUE THEOREM

Let f(x) and g(x) are two functions satisfying following properties :

(i) Both f(x) and g(x) are continuous in [a, b].

(ii) Both f(x) and g(x) are differentiable in (a, b).

(iii) $g'(x) \neq 0 \forall x \in (a, b)$

Then there exist at least one point C in between a and b such that

$$\boxed{\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}$$

BOLZANO THEOREM

If f(x) is continuous and differentiable function and let x = a & x = b are any two points in the domain of f(x) such that f(a) and f(b) have opposite sign then there exist at least one 'α' between a & b such that $f(\alpha) = 0$.

i.e., "if f(a).f(b) < 0 then there exist at least one root α of f(x) in between a and b."

DARBOUX'S THEOREM

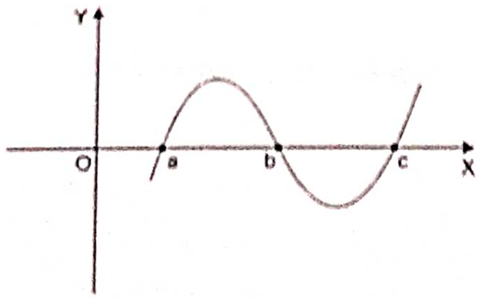
If f(x) is continuous and differentiable function such that $f'(a), f'(b)$ are of opposite signs, then there exists at least one $\alpha \in [a, b]$, such that $f'(\alpha) = 0$, i.e.,

if $f'(a).f'(b) < 0$ then there exist at least one α between a and b for which x = α is the root of $f'(x)$.

ROOTS OF f(x) = 0

Those points where graph of f(x) cuts X axis are called roots of f(x) = 0.

If x = a, b, c are the roots of f(x) then f(a) = f(b) = f(c) = 0.



Note : n^{th} degree polynomial has exactly n roots whether real or complex.

1.5 — MAXIMA AND MINIMA

INCREASING-DECREASING FUNCTION

Let f(x) be a function defined in D then f(x) is said to be

Increasing: If $f'(x) \geq 0$ and **Strictly increasing:** If $f'(x) > 0$

Decreasing: If $f'(x) \leq 0$ and **Strictly decreasing:** If $f'(x) < 0$

Note : $f'(x) - \tan\theta$ where θ is the angle makes by tangent with +ve X-axis. Hence,

1. For SI functions, $f'(x) > 0$, i.e. tangent, makes an acute angle θ. with +ve x-axis
2. For SD functions, $f'(x) < 0$ i.e. tangent, makes an obtuse angle θ. with +ve x-axis
3. **Monotonic function:** f(x) is said to be monotonic if it is either Strictly Increasing or Strictly Decreasing.

MAXIMA AND MINIMA OF FUNCTION OF SINGLE VARIABLE, i.e., of curve y = f(x)

Stationary Points

Those points where $f'(x) = 0$ are called stationary points, i.e., tangent must be horizontal at that points.

Critical points / Turning points

Those points for which $f'(x) = 0$ or undefined are called critical points, i.e., tangent is either horizontal or undefined.

Extreme Points/Optimal Points

Those critical points where we can obtain maxima or minima are called extreme points.

Extreme values / Extremas / Optimal Values :

Values of f(x) at extreme points (i.e. maximum or minimum values) are called extreme values.

Necessary condition for maxima or minima :

Tangent must be horizontal or undefined at that points, i.e., $f'(a) = 0$ or undefined where x = a is critical point.

Sufficient Condition for Maxima - Minima

Function should satisfy either first derivative test or second derivative test.

Ist Derivative Test : If x = a is critical point then check the symbol of $f'(x)$ according to the following diagram.

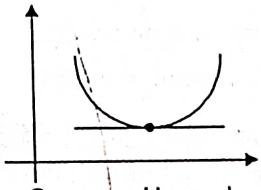
2nd derivative test : If x = a is critical point then we can obtain maxima-minima according to following chart

$f'(a) = 0$

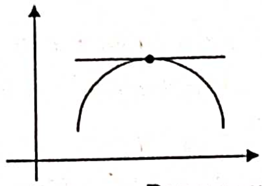
- $f''(a) < 0$ (Maxima)
- $f''(a) > 0$ (Minima)
- $f''(a) = 0$ $\downarrow$ $f'''(a) \neq 0$ (Saddle point)

Concave upward curve : If curve lies above the tangent always then it is called concave upward and its condition is $\boxed{f''(x) > 0}$

Concave Downward Curve : If curve lies below the tangent always then it is called Concave Downward and its condition is $\boxed{f''(x) < 0}$



Concave Upward

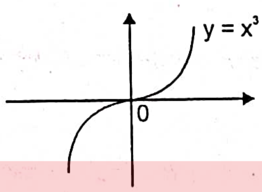


Concave Downward

Point of Inflexion / Saddle point : Points where curve is changing from concave upward to concave downward or vice-versa are called saddle points and these are the points where neither maxima nor minima occurs.

For example, $x = 0$ is the saddle point of $f(x) = x^3$

since in the LHS of $x = 0$, it is concave downward and in the RHS, $x = 0$ it is concave upward.



Note : 1. Concept of saddle point arises only when it must be a critical point.

2. If $f^n(c) = 0$ and $f^{n+1}(c) \neq 0$, n even, then $x = c$ is saddle point.

3. Maxima/Minima occurs alternatively.

4. An n^{th} degree polynomial bends at most $(n - 1)$ times, so can have maximum $(n - 1)$ extremas.

5. Corner points are always extreme points. So we should always try to check Maxima-Minima in case we have corner points in the domain of f(x).

MAXIMA – MINIMA OF FUNCTIONS OF TWO VARIABLES i.e. of surface $Z = f(x, y)$

Necessary Conditions for Maximum or Minimum

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$$

By solving these two equations we can obtain critical points.

Let after solving we get $x = a$ and $y = b$ then $P(a, b)$ will be critical point.

Note : In this chapter we have following notations :

Consider a function $z = f(x, y)$ and let $P(a, b)$ is a critical point then $\left(\frac{\partial^2 z}{\partial x^2}\right)_p = r; \left(\frac{\partial^2 z}{\partial x \partial y}\right)_p = s; \left(\frac{\partial^2 z}{\partial y^2}\right)_p = t$

SUFFICIENT CONDITION (LAGRANGE'S CONDITIONS) FOR MAXIMA-MINIMA :

Function must satisfy Lagrange's conditions as follows,

(i) If $rt - s^2 > 0$ and $r < 0$, then and $P(a, b)$ is a point of Maxima.

(ii) If $rt - s^2 > 0$ and $r > 0$, then and $P(a, b)$ is a point of Minima.

(iii) If $rt - s^2 < 0$, then $P(a, b)$ is saddle point.

(iv) It $rt - s^2 = 0$, then case fail and we need further investingations to calculate maxima or minima.

1.6 — PARTIAL DIFFERENTIATION

PARTIAL DERIVATIVES

Let us consider a function of two variable z = f(x, y), then partial differential coefficients are defined as follows :

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = \lim_{h\to 0}\left[\frac{f(x+h, y)-f(x, y)}{h}\right] \text{ when } y = \text{constant}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = \lim_{k\to 0}\left[\frac{f(x+y+k)-f(x, y)}{k}\right] \text{ when } x \text{ constant}$$

Note :

1. In general, $f_{xy} = f_{yx}$, i.e., $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$
2. All the results that are used in case of ordinary derivatives are also valid in case of partial derivatives keeping other variable constant.

HOMOGENEOUS FUNCTIONS

A function f(x, y)is known as homogeneous function of x and y. If each term is of same degree.

For example, $f(x, y) = a_0y^n + a_1xy^{n-1} + a_2x^2y^{n-2} + \ldots + a_nx^n$ is Homogeneous of degree n.

Note : Mathematically, it is defined as If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then f(x, y) is called homogeneous function of degree 'n'.

where $n \in R$

EULER'S THEOREM

Let f(x, y) be a homogeneous function of degree 'n' then,

1. $\boxed{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf}$

2. $\boxed{x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$

Note :

1. If f = f(x, y, z) is homogeneous function in three variables x, y and z of degree n then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = nf$.
2. Consider the function z = u + v, where z is non-homogeneous but u & v are homogeneous function of degree n_1 and n_2 respectively then

   (i) $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = n_1u + n_2v$

   (ii) $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = n_1(n_1 - 1)u + n_2(n_2 - 1)v$

PARITAL DERIVATIVES BY USING CHANGE OF VARIABLE CONCEPT

If u = f(x, y, z) and x = x(r, s, t), y = y(r, s, t), z = z(r, s, t), i.e.,

$u \to (x, y, z) \to (r, s, t)$, then partial derivatives of u with respect to r, s & t are defined as follows :

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}\right)+\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\right)+\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)+\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

Above rule is also known as Chain Rule of Partial Derivatives.

TOTAL DIFFERENTIAL COEFFICIENT

Let u = f(x, y, z) then total derivative of u is defined as

$$du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz$$

Note :

1. If u = f(x, y, z) and x = x(t), y = y(t), z = z(t), i.e., (x, y, z) are the functions of 't' then total derivatives of u with respect to 't' is given as

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial u}{\partial z}\right)\frac{dz}{dt}$$

2. **Implicit Functions :** If we are not able to separate x and y then function is called implicit function and it is represented by the relation f(x, y) = c, e.g., $x^3 + y^3 + 3xy = 1$.
3. **Explicit Function :** When it is possible to separate x and y then it is called Explicit, it is of the type y = f(x).
4. Consider an implict function f(x, y) = C $\Rightarrow$ df = 0

   using total derivative concept in LHS.

$$\Rightarrow \quad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}.$$

   It is the slope of given curve when function is in implicit form.

1.7 — INTEGRATION

DEFINITION

In general the inverse process of differentiation is known as integration, i.e., if $\frac{d}{dx}f(x) = \phi(x)$ then $\int \phi(x) = f(x) + C$.

STANDARD RESULTS

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1}, n \neq -1$
2. $\int \frac{1}{ax+b}dx = \frac{1}{a}\log(ax+b)$
3. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}$
4. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$
5. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$
6. $\int \tan(ax+b)\,dx = \frac{1}{a}\log\sec(ax+b)$
7. $\int \cot(ax+b)\,dx = \frac{1}{a}\log\sin(ax+b)$
8. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$
9. $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$
10. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)$
11. $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)$
12. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$
13. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$
14. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} = \log\left(x+\sqrt{x^2+a^2}\right)$

15. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} = \log\left(x+\sqrt{x^2-a^2}\right)$

16. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

17. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

18. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}, x>a = \frac{1}{2a}\log\frac{a-x}{a+x}, x<a$

20. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

21. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

22. $\int f\cdot g\,dx = f\int g\,dx - \int\left[f'\cdot\int g\,dx\right]dx$, where f, g are functions of x.

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

3. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{when } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, & \text{when } f(x) \text{ is even} \end{cases}$

4. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

5. When graph of f(x) is symmetrical about y-axis then f(x) is said to be an **even function** and mathematically $f(-x) = f(x)$.

6. When graph of f(x) is symmetrical about origin, i.e., symmetry occurs in opposite quadrants then it is called an **odd function** and mathematically $f(-x) = -f(x)$.

LEIBNITZ RULE OF DIFFERENTIATION UNDER THE SIGN OF INTEGRATION

$$\frac{d}{dx}\left[\int_{\varphi(x)}^{\psi(x)} f(x,t)dt\right] = \int_{\varphi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)dt + \frac{d\psi}{dx}.f(x,\psi) - \frac{d\varphi}{dx}.f(x,\varphi)$$

Note:

If integrand is function of 't' alone then

$$\frac{d}{dx}\left[\int_{\varphi(x)}^{\psi(x)} f(t)dt\right] = \frac{d\psi}{dx}.f(\psi) - \frac{d\varphi}{dx}.f(\varphi)$$

BETA AND GAMMA FUNCTIONS

A. **Beta function (1st Eulerian Integral) :**

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \qquad m, n > 0 \text{ not necessarily an integer.}$$

B. **Gamma functions (2nd Eulerian Integral) :**

$$\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx, \qquad n > 0 \text{ and } n \text{ may not be an integral value.}$$

Properties of Beta & Gamma Functions

1. Beta function is symmetrical about m and n i.e. $\beta(m,n) = \beta(n,m)$

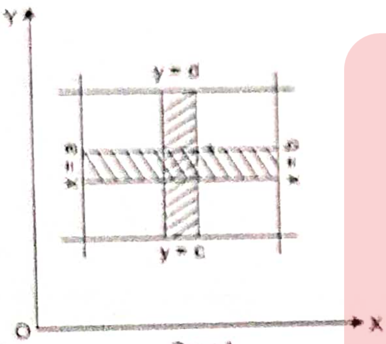
2. Relation between beta and gamma function $\beta(m, n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

3. $\displaystyle\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$ where $m > -1$ and $n > -1$

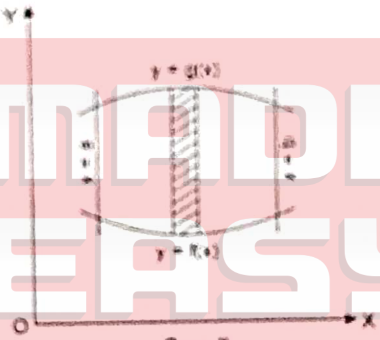
4. **Reflection or Complement Formulae** $\quad \Gamma n \, \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

5. $\Gamma(n+1) = \begin{cases} n!, & n \in I^{+} \\ n\Gamma n, & \text{otherwise} \end{cases}$

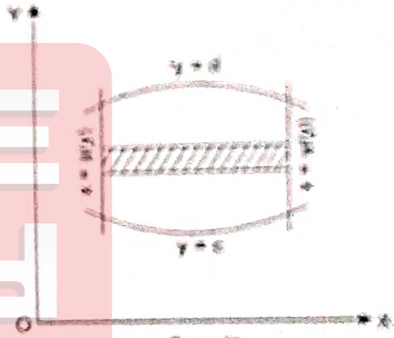
DOUBLE INTEGRAL WORKING RULE :



Case I



Case II



Case III

Case I : If both the variables x and y have constant limits, we can integrate in any order keeping other variable constant, i.e.,

$$I = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

Case II : If y has variable limits and x has constant limits then we should first integrate dy keeping x constant, i.e.,

$$I = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f(x)}^{g(x)} f(x,y)\,dy\right]dx$$

Case III : If x has variable limits and y has constant limits then we should first integrate dx keeping y constant, i.e.,

$$I = \int_{y=c}^{d}\int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\right]dy$$

Note :

1. If f(x, y) has discontinuities within the region of intergration then

$$\int_a^b\left\{\int_c^d f(x, y)\,dy\right\}dx \neq \int_c^d\left\{\int_a^b f(x, y)\,dx\right\}dy$$

2. If both the limits are constant and f(x, y) is an Explicit function of x and y then we can also integrate separately.

DOUBLE INTEGRAL IN POLAR COORDINATES

It is defined as $I = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta)\,dr\,d\theta = \int_{\theta_1}^{\theta_2}\left\{\int_{r_1}^{r_2} f(r, \theta)\,dr\right\}d\theta$

In general θ_1 and θ_2 are constant and r_1 and r_2 are variables, so we will first integrate dr keeping θ constant.

TRIPLE INTEGRALS WORKING RULE

Case I : If all the variables have constant limits then we can integrate in any order, i.e.,

$$I = \int_{z=e}^{f}\int_{y=c}^{d}\int_{x=a}^{b} f(x,y,z)dxdydz = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=e}^{f} f(x,y,z)dzdydx \text{ = and so on...}$$

Case II : If one or two variables have variable limits then process will be as follows :

$$I = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)}\int_{z=\phi(x,y)}^{\psi(x,y)} f(x,y,z)dxdydz = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)}\int_{z=\phi(x,y)}^{\psi(x,y)} f(x,y,z)dzdydx$$

i.e., which one is more variable should be integrated first.

CHANGE OF ORDER OF INTEGRATION

If limits are in variable form then by changing the order of integration we can easily solve the double integration.

Flow chart

Type-I

1. If variable limits are of y; it means it is given that strip is parallel to y-axis, i.e., vertical strip is given.
2. Now, make a strip parallel to x-axis, i.e., horizontal strip.
3. Now, put the limits according to the new strip.

Type-II

1. If variable limits are of x; it means it is given that strip is ||^r to x-axis, i.e., horizontal strip is given.
2. Now, make a new strip II^r to y-axis, i.e., vertical strip.
3. Now, put the limits according to new strip.

Symbolically,

$$I = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} f(x,y)dydx = \int_{y=c}^{d}\int_{x=\phi(x)}^{\psi(x)} f(x,y)dxdy$$

This process is known as change of order.

JOCOBIAN

1. If u = u(x, y) and v = v(x, y) then, Jocobian of (u, v) with respect to (x, y) in defined as, $J(u, v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

2. If u = u(x, y, z), v = v(x, y, z), w = w(x, y, z) then, Jocabion of (u, v, w) with respect to (x, y, z) is defined as,

$$J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note :

1. Two function u and v are called functionally **dependent** if exist a relationship between them, and its mathematical condition is "their Jacobian should be zero", i.e.,

   if u = u(x, y) and v = v(x, y) then u and v are dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$, i.e., J = 0.

2. Two functions u and v are **independent** if their jacobian is not zero i.e., $J \neq 0$.

Example : If u = x, v = x tan y, w = z then evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ = ?

Solution : $\because$ (u, v, w) are the functions of (x, y, z) so Jacobian of (u, v, w) with respect to (x, y, z) is

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan y & x\sec^2 y & 0 \\ 0 & 0 & 1 \end{vmatrix} = x \sec^2 y$$

MULTIPLE INTEGRAL USING JACOBIAN

Let x and y are the functions of u and v i.e., $x = \phi(u, v)$ and $y = \Psi(u, v)$ then we have following result :

$$\boxed{dx\,dy = |J|\,du\,dv}, \text{ where } J = \frac{\partial(x,y)}{\partial(u,v)}$$

Similarly, in case of three variables, we have dxdydz = |J| dudvdw, where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

1. Polar Coordinates

Let P(x, y) are the cartesian and $P(r, \theta)$ are Polar coordinates of any point that lies on circle then

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

where $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \text{ so } \boxed{dxdy = |J|drd\theta = rdrd\theta}$$

2. Spherical coordinates

Let P(x, y, z) and $P(r,\theta,\phi)$ are the cartesian and spherical coordinates of any point that lies on sphere then

$$\boxed{x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi} \text{ where } r^2 = x^2 + y^2 + z^2$$

and ϕ is the angle formed by any radius vector (of sphere) from z-axis and θ is the angle varies on XY-plane.

Now, $$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\theta\sin\phi & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} = r^2\sin\phi$$

Hence, $$\boxed{dxdydz = (r^2\sin\phi)drd\theta d\phi}$$

3. Cylindrical coordinates

Let P(x, y, z) are cartesian and $P(\rho, \phi, z)$ are cylindrical coordinates of any point that lies on cylindrical then

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

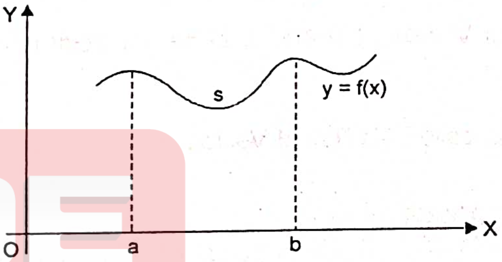
Now, $$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} x_\rho & x_\phi & x_z \\ y_\rho & y_\phi & y_z \\ z_\rho & z_\phi & z_z \end{vmatrix} = \rho \text{ and } \boxed{dx\,dy\,dz = |J|\,dr\,d\phi\,dz = \rho\,dr\,d\phi\,dz.}$$

NOTE :

1. In cartesian coordinate Area $= \iint_R (1)\, dx\, dy$; Volume $= \iiint_R (1)\, dx\, dy\, dz$
2. In polar coordinate Area $= \iint_R (r)\, dr\, d\theta$
3. In spherical coordinate Volume $= \iiint_R (r^2 \sin\phi) dr d\theta d\phi$
4. In cylindrical coordinate Volume $= \iiint_R (\rho) d\rho\, d\phi\, dz$

LENGTH OF CURVES

(i) When curve is given in cartesian form, i.e., in terms of y = f(x) then



$$s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad s = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

(ii) When curve is given in polar form, i.e., r = f(θ) then

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{or} \quad s = \int_{r_1}^{r_2} \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\, dr$$

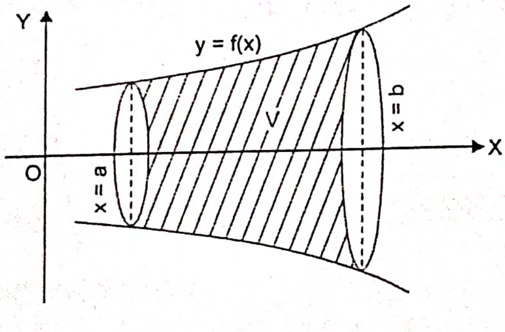
(iii) When the curve is given in parametric form, i.e., x = x(t), y = y(t), then

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt,$$

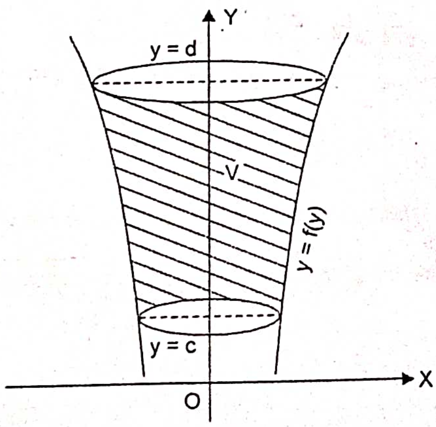
VOLUME OF SOLID FORMED BY REVOLUTION

Case I : The volume of the solid generated by the revolution of the arc of curve y = f(x), between x = a and x = b about the x-axis is $V = \int_a^b \pi y^2 dx$

Case II : The volume generated by the revolution of the arc of the curve x = ϕ(y), and the lines y = a, y = b about y axis is $V = \int_a^b \pi x^2 dy$



Case I



Case II

UNIT 2 : VECTOR CALCULUS

2.1 — VECTOR AND THEIR BASIC PROPERTIES

SCLAR AND VECTOR QUANTITY

A quantity which consist only magnitude but not the direction is called scalar quantity. e.g., Distance, Mass, Time, Speed, Temperture, Work, Energy, Potential etc.

A quantity consisting magnitude as well as direction is called Vector, e.g., Displacement, Velocity, Acceleration, Momentum, Impulse, Weight, etc.

Unit Vector

A vector whose magnitude is one is called unit vector, it is given as $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Position Vector : Let A(x, y, z) be any point in 3D then Position vector of point A is defined as

$$\overline{OA} = x\hat{i} + y\hat{j} + z\hat{k}$$

It is also called 3D Radial Vector.

DOT PRODUCT

It is defined as $\boxed{\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$ and it is used to find the angle between two vectors, i.e., $\theta = \cos^{-1}\left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$

CROSS PRODUCT

It is defined as $\boxed{\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta \times \hat{n}}$

where $\hat{n}$ is a unit vector perpendicular to the plane containing vectors $\vec{a}$ and $\vec{b}$.

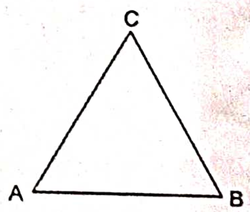
Note :

1. Unit normal on plane is $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

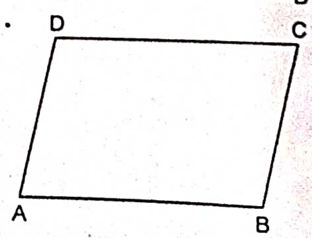
2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

   then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

3. **Area of Triangle :** ABC : Area $= \frac{1}{2}\left|\overline{AB}\times\overline{AC}\right| = \frac{1}{2}\left|\overline{BC}\times\overline{BC}\right| = \frac{1}{2}\left|\overline{CA}\times\overline{CB}\right|$



4. **Area of Parallelogram ABCD :** Area $= \left|\overline{AB}\times\overline{AD}\right|$ in terms of sides



$$= \frac{1}{2}\left|\overline{AC} \times \overline{BD}\right| \text{ in terms of diagonals}$$

SCALAR TRIPLE PRODUCT

It is defined as $\left[\bar{a},\bar{b},\bar{c}\right] = \bar{a}.\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ where $\begin{array}{l}\bar{a} \equiv a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ \bar{b} \equiv b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \\ \bar{c} \equiv c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\end{array}$

Note :

1. It represents the volume of rectangular parallelopiped.
2. $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar vectors then, $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

VECTOR TRIPLE PRODUCT

It is defined as $\boxed{\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})b - (\bar{a}\cdot\bar{b})\bar{c}}$

STRAIGHT LINES IN 3D

1. Equation of line passing through a point (x_1, y_1, z_1) and parallel to vector $a\hat{i} + b\hat{j} + c\hat{k}$ is $\boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}}$

2. Equation of line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$

3D PLANE

1. The general equation of plane is ax + by + cz = d where d is any scalar.
2. **Distance of Point from Plane :** Let ax + by + cz + d = 0 be any plane & let $P(x_1, y_1, z_1)$ be any point from where distance is to be measured. then required distance is $D = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}\right|$

2.2 — GRADIENT, DIVERGENCE AND CURL

DEL OPERATOR / HAMILTON OPERATOR / NABLA OPERATOR

It is defined as $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$. It is not a vector but combines both vectorial and differential properties.

THE LAPLACIAN OPERATOR

It is defined as $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ or $\nabla^2 = \bar{\nabla}\cdot\bar{\nabla}$

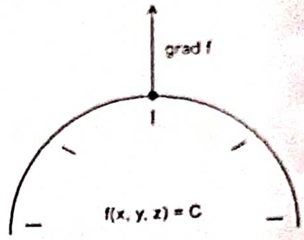
Laplace's Equation : It is defined as $\nabla^2\phi = 0$ where $\phi \rightarrow \phi(x,y,z)$ is called Harmonic function.

Harmonic Function: Function which satisfies Laplace equation is called Harmonic Function.

GRADIENT (NORMAL VECTOR)

If f(x,y,z) = c is a scalar point function then (grad f) is defined as

$$\text{grand } f = \nabla f = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)f = \left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right) = \Sigma\left(\hat{i}\frac{\partial f}{\partial x}\right)$$



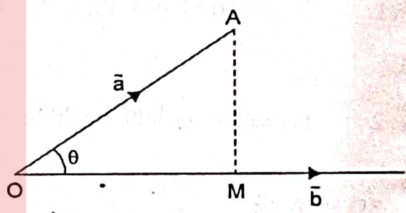
Geometrically gradient of f shows a vector normal to the surface f(x,y,z) = c.

Note :

1. Direction grad f = Direction of Normal Vector = Direction of Surface.
2. Angle between two surface ϕ_1 and ϕ_2 is nothing but angle between their normal vectors, i.e.,

   if $\bar{n}_1$ = grad ϕ_1 and $\bar{n}_2$ = grad ϕ_2 then their angle of intersection is given as $\theta = \cos^{-1}\left[\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}\right]$.

3. Unit vector normal to the surface f(x, y, z) = c is given as $\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}$
4. Component of $\bar{a}$ in the direction of $\bar{b}$ is given as the distance OM as shown in the diagram, i.e.,



$$\text{Required component} = OM = |\overline{OA}|\cos\theta = |\bar{a}|\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right) = \bar{a}\cdot\hat{b}$$

DIRECTIONAL DERIVATIVE (D.D.)

Component of grad f in the direction of vector $\bar{a}$ is known as directional derivative in that direction and it is defined as

$$\boxed{DD = (\text{grad } f).\hat{a}}$$

Note : Since component of any vector in its own direction is maximum. So, maximum DD will occur in the direction of grand f. And it is calculated as

$$\text{Max D.D} = (\text{grad} f).(\widehat{\text{grad} f}) = (\text{grad } f)\cdot\frac{\text{grad } f}{|\text{grad } f|} = \frac{|\text{grad } f|^2}{|\text{grad } f|} = |\text{grad } f|$$

DIVERGENCE

If $\bar{f} = f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$, be any vector point function then divergence of $\bar{f}$ is defined as

$$\text{div } \bar{f} = \nabla\cdot\bar{f} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(f_1\hat{i}+f_2\hat{j}+f_3\hat{k}) = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} \approx \sum\hat{i}\cdot\frac{\partial\bar{f}}{\partial x}$$

Solenoidal Vector : If div $\bar{f}$ is zero then $\bar{f}$ is called solenoidal vector.

CURL

If $\bar{f} = f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ be any vector point function then, curl of $\bar{f}$ is defined as

$$\text{Curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \text{Vector Quantity} = \sum \hat{i} \times \frac{\partial \vec{f}}{\partial x}$$

Note : If $\vec{v}$ and $\vec{w}$ are linear and angular velocity of rotating body then $\boxed{\text{curl } \vec{v} = 2\vec{w}}$

Proof : Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$

then $\vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = i(w_2 z - w_3 y) - \hat{j}(w_1 z - w_3 x) + \hat{k}(w_1 y - w_2 x)$

Now Curl $\vec{v}$ = curl$(\vec{w} \times \vec{r})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (w_2 z - w_3 y) & (w_3 x - w_1 z) & (w_1 y - w_2 x) \end{vmatrix} = 2(w_1\hat{i} + w_1\hat{j} + w_3\hat{k}) = 2\vec{w}.$$

Irrotational Vector : If curl $\vec{f} = \vec{0}$ then $\vec{f}$ is called irrotational vector.

Conservative Field : If the work done in moving the particle from one point to another is independent of the path then field $\vec{F}$ is called conservative field and the necessary and sufficient condition for a field to be conservative is curl $\vec{F}$ = $\vec{0}$.

VECTOR IDENTITIES

Following are the standard results known as vector identities :

1. div $(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl}\vec{a} - \vec{a} \cdot \text{curl}\vec{b}$
2. Curl $(\vec{a} \times \vec{b}) = \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a} + (\vec{b} \cdot \nabla)\vec{a} + (\vec{a} \cdot \nabla)\vec{b}$
3. div Curl $\vec{f} = \nabla \cdot (\nabla \times \vec{f}) = 0$
4. grad div $\vec{f}$ = Curl curl $\vec{f} + \nabla^2 \vec{f}$
5. grad (u v) = u grad v + v grad u
6. grad $(\vec{a} \cdot \vec{b}) = \vec{a} \times \text{curl}\vec{b} + \vec{b} \times \text{curl}\vec{a} + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a}$
7. div grad $f = \nabla^2 f$
8. Curl grad f = $\nabla \times \nabla f = \vec{0}$
9. $\text{div}(u\vec{a}) = u\,\text{div}\vec{a} + \vec{a}.\text{grad}u$
10. $\text{Curl}(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}u) \times \vec{a}$
11. Curl Curl $\vec{f}$ = grad div $\vec{f} - \nabla^2 \vec{f}$

Theorem 1 if f = f(r) be any scalar function of r then $\boxed{\text{grad } f = f'(r)\hat{r}}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Proof : $\because$ $r = \sqrt{x^2 + y^2 + z^2}$ $\Rightarrow$ $r^2 = x^2 + y^2 + z^2$ then $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now : grad (f) = $\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$= \hat{i}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial f}{\partial r}\frac{\partial r}{\partial z} = f'(r)\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right] = \frac{f'(r)}{r}(\vec{r}) \approx f'(r)\hat{r}$$

Theorem 2 : If $\vec{f} = f(\vec{r})$ be any vector function in terms of $\vec{r}$ then $\boxed{\text{div } \vec{f} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 |\vec{f}|\right)\right]}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

2.3 — VECTOR INTEGRATION

Consider the following Notations :

1. $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ 2. $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ 3. $d\bar{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}$

4. Consider the small surface $d\bar{s}$ in vector form and $\hat{n}$ is the outward drawn unit normal on $d\bar{s}$ then

$\boxed{d\bar{s} = (ds)\hat{n}}$ where $ds = |d\bar{s}|$, $\hat{n} = \dfrac{\text{grads}}{|\text{grads}|}$

Here, $d\bar{S} = (dS)\hat{n}$ is valid because in general we know that $\bar{a} = |\bar{a}|\hat{a}$.

LINE INTEGRAL

An integral which is to be evaluated over the curve is called line integral and it is defined as

$$\boxed{L.I. = \int_C \bar{f}\cdot d\bar{r}} \text{ or } \int_C (f_1dx + f_2dy + f_3dz)$$

SURFACE INTERGRAL

An integral which is to be evaluated over the surface is called surface integral and it is defined as

$$\boxed{SI = \iint_C \bar{f}\cdot d\bar{s}} \approx \iint_S (\bar{f}\cdot\hat{n})ds \approx \iint_S (f_1dydz + f_2dzdx + f_3dxdy)$$

VOLUME INTEGRAL

An integral which is to be evaluated over the volume is called volume integral and it is defined as

$$\boxed{VI = \iiint_V \bar{f}dv} \approx \iiint_V \bar{f}dxdydz$$

GAUSS' DIVERGENCE THEOREM (connects surface to volume integral)

Let S is any **closed** surface encloses volume V then

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_V \text{div}\,\bar{F}dV \text{ or } \boxed{\iint_s \bar{F}\cdot\hat{n}dS = \iiint_v \nabla\cdot\bar{F}dv}$$

Cartesian Form

$$\iint_s (F_1dydz + F_2dzdx + F_3dxdy) = \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz$$

STOKE'S THEOREM (connects line integral into surface integral)

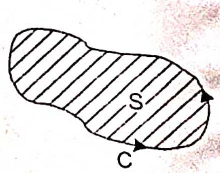
Let S is an **open** surface bounded by a simple closed curve C. Then,

$$\boxed{\oint_c \bar{F}\cdot d\bar{r} = \iint_s (\text{curl}\bar{F})\cdot d\bar{S}} \text{ or } \oint_c (F_1dx + F_2dy + F_3dz) = \iint_s (\nabla\times\bar{F})\cdot\hat{n}\,ds$$

where C is in anti-clockwise sense.

GREEN'S THEOREM (also connects line into surface integral but in 2D only)

Let S is any surface in 2D bounded by closed curve C then



$$\oint_c \bar{F}\cdot d\bar{r} = \iint_s \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy \text{ or } \boxed{\oint_c (F_1dx + F_2dy) = \iint_s \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy}$$

UNIT 3 : COMPLEX ANALYSIS

3.1 — COMPLEX NUMBER & ITS PROPERTIES

IOTA

It is denoted by i and is defined as $\boxed{i=\sqrt{-1}}$, i.e., square root of -1. E.g. $x^2+1=0 \Rightarrow x=\pm\sqrt{-1}=\pm i$

Note :

1. $\boxed{i^n = i^r}$ where r = Remainder when n is divided by 4

2. $\boxed{i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0}$, $n \in I$ i.e. sum of four consecutive powers of iota is always zero.

EULER NOTATION

It is defined as : $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ where $\left|e^{i\theta}\right| = 1$

e.g., $1 = e^{2\pi i}$ or $e^{2n\pi i}$, $-1 = e^{\pi i}$, $i = e^{\frac{\pi}{2}i}$, $-i = e^{-\frac{\pi}{2}i}$

Note :

1. Circular function: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

2. Hyperbolic function: $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

REPRESENTATION OF COMPLEX NUMBER :

We have three forms :

1. **Cartesian Representation :** $z = x + iy \approx (x, y)$

2. **Polar Representation :** $z = r(\cos\theta + i\sin\theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$

3. **Euler Representation :** $z = re^{i\theta}$ where $r = |z| = \sqrt{x^2+y^2}$ and θ = Amp (z)

COMPLEX CONJUGATE

Let $z = x + iy$ then $\bar{z} = x - iy$ and it has following properties :

1. $z + \bar{z} = 2\mathrm{Re}(z) = 2x$
2. $z - \bar{z} = 2i\,\mathrm{Im}(z) = 2iy$
3. $|z|^2 = z\bar{z} = x^2 + y^2$
4. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
5. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
6. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

MODULUS AND ARGUMENT

Let $z = x + iy$ then $|z| = \sqrt{x^2+y^2}$ is modulus of z and $\theta = \tan^{-1}(y/x)$ is amplitude of z

Properties of Modulus and Argument

1. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left||z_1|^2 + |z_2|^2\right|$

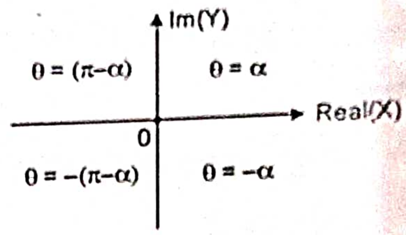
2. Product of Complex Numbers $|z_1 z_2 \ldots\ldots z_n| = |z_1||z_2|\ldots\ldots|z_n|$ and arg $(z_1 z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots\ldots + \arg z_n$

3. **Quotient of Two Complex Numbers :** $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2$

METHOD OF CALCULATING PRINCIPAL VALUES OF ARGUEMENT

Those values of θ that lies in $(-\pi, \pi]$ are called Principal Values.

Let $z = x + iy$ then to calculate principal value of θ first find α, using the result $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$ then θ is according to quadrant, as shown in diagram.



COMPLEX FORM OF EQUATION OF CIRCLE

$|Z - Z_o| = R$ is a circle with centre Z_o and radius R

e.g. $|z - (3 + 4i)| = 5$. It is circle with centre (3, 4) and radius = 5.

CUBE ROOTS OF UNITY

Consider the equation $z = (1)^{1/3}$ or $z^3 - 1 = 0$. Roots of this equation are called Cube Roots of Unity, i.e.,

$z^3 - 1 = 0 \Rightarrow (z - 1)(z^2 + z + 1) = 0 \Rightarrow z = 1$ and $z^2 + z + 1 = 0$

or $z = 1$ and $z = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$. So we have three roots as follows

$$z_1 = 1,\ z_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2},\ z_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

These are known as cube rotos of unity.

Note :

1. Cube roots of unity can also be defined as

$z_1 = 1$

$$z_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = e^{i\frac{2\pi}{3}} \approx w \text{ (let)}$$

$$z_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = e^{i\frac{4\pi}{3}} \approx w^2$$

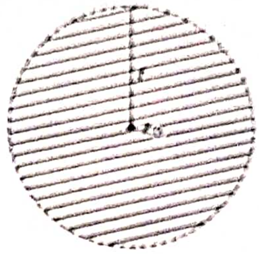
2. $1 + w^n + w^{2n} = \begin{cases} 3, & n = \text{Multiple of } 3 \\ 0, & \text{Otherwise} \end{cases}$, e.g. $1 + w + w^2 = 0$

3. $w^{3n} = 1$, e.g., $w^3 = w^6 = w^9 = \ldots\ldots = 1$

4. $\overline{w} = w^2$, $\overline{w^2} = w$, $|w| = |w^2| = 1$

3.2 — ANALYTIC FUNCTIONS

NEIGHBOURHOOD OF COMPLEX NUMBER z_0



We know that geometrically z_0 is the presentation of any point (x_0, y_0) in complex plane. So neighbourhood of z_0 means we are considering an open disc centered at z_0 having sufficiently small radius r, i.e., Nbd of z_0 is $|z - z_0| < r$.

GRAPH OF COMPLEX FUNCTIONS

Consider the complex function $w = f(z)$, then $u + iv = f(x + iy)$.

After comparison $u = u(x, y)$ and $v = v(x, y)$, i.e., there exist four variables in which (x, y) are independent and (u, v) are dependent variables. So, we need two planes to plot the graph of complex function on Argand plane, one is known as z-plane and another one is w-plane.

CONTINUITY OF COMPLEX FUNCTION

$w = f(z)$ is said to be continuous at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$,

i.e., whenever z lies in the neighbourhood of z_0, f(z) should lies in the neighbourhood of $f(z_0)$.

DIFFERENTIABLILITY OF COMPLEX FUNCTION

Function $w = f(z)$ is said to be differentiable at z_0 if $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ provided this limit exists.

ANALYTIC FUNCTIONS

f is said to be analytic at z_0 if it is not only differentiable at z_0 but also differentiable in the neighbourhood of z_0, i.e., for a function to be analytic at z_0, it should be differentiable at each and every point that liesin the neighbourhood of z_0.

CAUCHY RIEMANN EQUATIONS IN CARTESIAN FORM

Consider the function $w = f(z)$, i.e., $u + iv = f(x + iy)$ then

Cauchy-Riemann equations are given as : $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

Note : If f(z) is an analytic function then $\dfrac{d}{dz} f(z) = u_x + iv_x = v_y - iu_y = e^{-i\theta} \dfrac{\partial w}{\partial r}$

POLAR FORM OF CAUCHY-RIEMANN EQUATIONS

Consider $w = f(z)$ is a function of r and θ i.e., $u + iv = f(re^{i\theta})$ then $u = u(r,\theta)$ and $v = v(r,\theta)$

then C-R equation are defined as follows. $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \text{ and } \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}}$

Note :

1. Continuity is the necessary condition for differentiability and differentiability is the necessary condition for analyticity.
2. If C-R equation are not satisfied at z_0, then f(z) is non-differentiable at z_0 and hence non-analytic.
3. If C-R equation are satisfied at z_0 but not in its neighbourhood then f(z) is non-analytic at z_0.
4. If C-R equations are satisfied not only at z_0 but also in the nbd of z_0 and also $f'(z_0)$ is unique then f(z) is an analytic function at z_0.

ORTHOGONAL SYSTEM

If w = f(z) is an analytic function of x & y then real and imaginary parts of f(z) together constitutes an orthogonal system, i.e., they intersect at 90°.

Proof : w = f(z) $\Rightarrow$ u + iv = f(x + iy), i.e., after comparison u & v will be functions of x & y.

Let us assume them as $u(x, y) = C_1$ and $v(x, y) = C_2$.

$\Rightarrow$ $du = d(c_1)$ and $dv = d(c_2)$ or du = 0 and dv = 0

Using total derivative concept in LHS of both equations

$$\left(\frac{\partial u}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}\right)dy = 0 \text{ and } \left(\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial v}{\partial y}\right)dy = 0$$

or
$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y} \text{ and } m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}$$

Now,
$$m_1m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = \left(\frac{v_y}{-v_x}\right)\left(\frac{v_x}{v_y}\right) = -1. \text{ Hence Proved.}$$

HARMONIC FUNCTIONS

Those function that satisfies Laplace Equation are called Harmonic functions.

Note :

1. If $f(z)=u+iv$ is an analytic function then u and v both are Harmonic functions, i.e., they satisfies Laplace equations

   For, $\because$ f(z) is analytic so by C-R equation we have $u_x = v_y$ ...(1) and $u_y = -v_x$ ...(2)

   Differentiating (1) partially w.r.t. x and (2) w.r.t. y we have

   $$u_{xx} = v_{xy} \text{ and } u_{yy} = -v_{yx}$$

   Adding $u_{xx} + u_{yy} = 0$ which is Laplace equation so u is harmonic.

   Similarly, we can show that v is also harmonic.

2. If u & v are harmonic functions such that u + iv is an analytic function then u & v are called harmonic. conjugate of each other.

CONSTRUCTION OF AN ANALYTIC FUNCTION f(z) = u + iv using Milne Thomson Method :

Case I: (When real part 'u' is given)

Step 1 : Find $\frac{\partial u}{\partial x}$ and write it as $\phi_1(x,y)$

Step 2 : Find $\phi_1(z,0)$

Step 3 : Find $\frac{\partial u}{\partial y}$ and write it as $\phi_2(x,y)$

Step 4 : Find $\phi_2(z,0)$

Step 5 : $f(z) = \int[\phi_1(z,0)-i\phi_2(z,0)]dz+c$

Case II : (When imaginary part v is given)

Step 1 : Find $\frac{\partial v}{\partial x}$ and write it as $\phi_2(x,y)$

Step 2 : Find $\phi_2(z,0)$

Step 3 : Find $\frac{\partial v}{\partial y}$ and write it as $\phi_1(x,y)$

Step 4 : Find $\phi_1(z,0)$

Step 5 : $f(z) = \int[\phi_1(z,0)+i\phi_2(z,0)]dz+c$

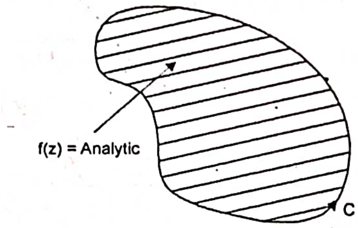
3.3 — COMPLEX INTEGRATION

To obtain the integrals of complex functions, we have following methods :

CAUCHY INTEGRAL THEOREM

If $f(z)$ is an analytic function within and on a closed contour C, then

$$\boxed{\oint_C f(z)dz = 0} = 0.$$



Note : if C is an open contour then $\int_C f(z)dz$ = path independent.

CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C and 'a' is any point within C, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz}.$$

Cauchy's Integral Formula for Higher Order Derivative of an Analytic Function

If a function f(z) is analytic within and on closed contour c and 'a' is any point within c, then it's n^{th} derivative at point a is given as

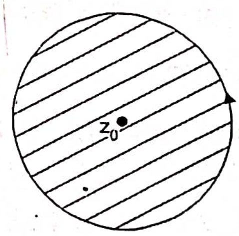
$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}}$$

e.g. $f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^2}$, $f''(a) = \frac{\cancel{2r}\;2!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^3}$ and so on....

TAYLOR SERIES

Let f(z) be analytic inside the circle C having centre at z_0 then f(z) can be expanded in the neighbourhood of z_0 as follows:

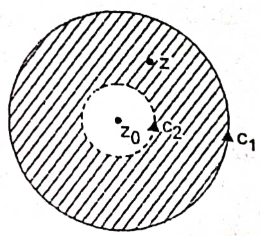
$$f(z) = f(z_0)+(z-z_0)f'(z_0)+\frac{(z-z_0)^2}{2!}f''(z_0)+\frac{(z-z_0)^3}{3!}f'''(z_0)+.....$$



LAURENT'S SERIES

Let f(z) be non-analytic at z_0 then it can be expanded in the deleted neighbourhood of z_0 (i.e., in the ring shaped region as shown in the diagram) as follows :

$$f(z) = (a_0+a_1(z-z_0)+a_2(z-z_0)^2+a_3(z-z_0)^3+.....)+\left(\frac{b_1}{(z-z_0)^2}+\frac{b_2}{(z-z_0)^2}+.....\right)$$



where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)dz}{(z-z_0)^{n+1}}$ and $b_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)dz}{(z-z_0)^{-n+1}}$

Note :

1. An analytic functionc ~~an~~ can be expanded with the help of Taylor series while non-analytic function can be expanded with the help of Laurent series.
2. Decreasing powers of $(z - z_0)$ in Laurent series is called principal part of Laurent series, i.e., series involving b_1, b_2, b_3 ..... is called Principal Part.
3. Expansion of function with the standard result of Laurent series is not an easy task. So, we should follow Binomial theorem concept to find its expansion.
4. $(1 + x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-\overset{2}{\cancel{3}})}{3!}x^3+.....$ e.g., $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$
5. In Taylor series, we get increasing powers of $(z - z_0)$ while in Laurent series we get both increasing and decreasing powers of $(z - z_0)$.

SINGULAR POINTS

z_0 is called singular point of f(z) if f(z) is not analytic at z_0 but should be analytic in every neighbourhood of z_0. for e.g.

(i) z = 0 is the singular point of f(z) = log z because it is analytic in every neighbourhood of z = 0 but not analytic at 0.

(ii) $f(z) = \bar{z}$ has no singular points since it is nowhere analytic.

TYPES OF SINGULARITIES

(i) Isolated singularity (ii) Non Isolated singularity

If z_0 is the only singularity in its neighbourhood then it is called Isolated and if there exist so many singularities in the neighbourhood of z_0 then it is called Non Isolated Singularity.

Example : $f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)}$ then find nature of singularities if exist.

Solution : It is not analytic where $\frac{\pi}{z} = n\pi$ or, $z = \frac{1}{n}$. Thus, $z = 1, \frac{1}{2}, \frac{1}{3}.....$ are isolated singularities. But z = 0 is the non-isolated singularity of f(z) because in the neighbourhood of z = 0 there exist so many singularities tending to 0.

TYPES OF ISOLATED SINGULAR POINTS

1. **Removable Singularity :** z_0 is called removable singularity of f(z) if Principal Part of Laurent series about z_0 contains no terms.
2. **Pole :** z_0 is called pole of f(z) if principal part of Laurent series expansion of f(z) about z_0 has finite number of terms and highest –ve power of $(z - z_0)$ in principal part is called order of Pole.
3. **Essential Singularity :** z_0 is called essential singularity of f(z) if principal part of Laurent series in the neighbourhood of z_0 has infinite number of terms.

Example : Check the nature of singularities of f(z) where (i) $f(z) = \frac{z-\sin z}{z^3}$ (ii) $f(z) = \frac{\sin z}{z^9}$ (iii) $f(z) = e^{\frac{1}{z^2}}$

Solution : (i) $f(z) = \dfrac{z-\sin z}{z^3} = \dfrac{1}{z^3}\left[z-\left(z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+.....\right)\right] = \left(\dfrac{1}{3!}-\dfrac{z^2}{5!}+\dfrac{z^4}{7!}-\dfrac{z^6}{9!}+.....\right) + (0)$

$\because$ Principal part of Laurent expansion of f(z) in the neighbourhood of z = 0 has no terms, so z = 0 is the removable singularity.

(ii) $f(z) = \dfrac{\sin z}{z^9}$. Obviously z = 0 is the singular point of f(z) as f(z) is not analytic there. Now we will try to find its Laurent series expansion in the neighbourhood of z = 0 as follows.

$$f(z) = \frac{\sin z}{z^9} = \frac{1}{z^6}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\frac{z^9}{9!}-.....\right] = \left(\frac{1}{9!}-\frac{z^2}{11!}+\frac{z^4}{13!}-\frac{z^6}{15!}+.....\right)+\left(-\frac{1}{7!z^2}+\frac{1}{5!z^4}-\frac{1}{3!z^6}+\frac{1}{z^8}\right)$$

$\because$ Principal part of Laurent series consist finite number of terms so z = 0 is the pole of f(z) and it is a pole of order eight.

(iii) $f(z) = e^{\frac{1}{z^2}}$. $\because$ at z = 0, $f(0) = e^{\frac{1}{0}} = e^{\infty} = \infty$. So, clearly z = 0 is the singularity of f(z).

$$\text{Now, } f(z) = e^{\frac{1}{z^2}} = 1+\frac{\left(\frac{1}{z^2}\right)}{1!}+\frac{\left(\frac{1}{z^2}\right)^2}{2!}+\frac{\left(\frac{1}{z^2}\right)^3}{3!}+..... = (1)+\underbrace{\left[\frac{1}{z^2}+\frac{1}{2z^4}+\frac{1}{6z^6}+.....\right]}_{\text{Principal Part}}$$

$\because$ Principal Part of Laurent series consist infinite number of terms. So, z = 0 is the essential singularity of f(z).

RESIDUE

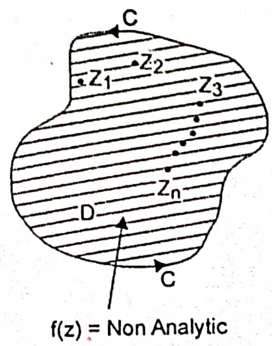
Let z_0 is the singularity of f(z) then Residue of f(z) at $(z = z_0)$ is denoted as $\underset{(z=z_0)}{\text{Res}} f(z)$ and is defined as :

$$\underset{(z=z_0)}{\text{Res}} f(z) = \text{Coefficient of } \left(\frac{1}{z-z_0}\right) \text{ i.e., } b_1 \text{ in the Laurent series expansion of f(z) about } z_0.$$

Using standard definition of Laurent series, $b_1 = \dfrac{1}{2\pi i}\oint_C f(z)dz$ is called Residue at z_0.

CAUCHY-RESIDUE THEOREM

Let f(z) is non-analytic at finite number of poles $z_1, z_2, z_3, ..., z_n$ that lies with in the region D bonded by closed contour 'C' (as shown in diagram) then



$$\boxed{\oint_C f(z)dz = 2\pi i \text{ (sum of residues at poles that lies inside C)}}$$

$$= 2\pi i[R_1+R_2+R_3+.....+R_n]$$

where $R_1, R_2, R_3, ..., R_n$ are the Residues at $z_1, z_2, z_3, ..., z_n$ respectively.

METHODS OF EVALUATING RESIDUES

(1) If f(z) has a simple pole at $z = z_0$ then $\boxed{\underset{(z=z_0)}{\text{Res}}\, f(z) = \lim_{z \to z_0} (z - z_0) f(z)}$

(2) If f(z) has a pole of order n at $z = z_0$ then, $\boxed{\underset{(z=z_0)}{\text{Res}}\, f(z) = \frac{1}{\lfloor n-1} \left[\frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \right]_{z=z_0}}$

(3) If it is not possible to find the value of residue using above two results, we can use the concept of Laurent series as follows :

Residue at z_0 = Coefficient of $\left(\frac{1}{z - z_0} \right)$ in Laurent series expansion of f(z) in the deleted neighbourhood of z_0.

UNIT 4 : DIFFERENTIAL EQUATIONS

4.1 — ORDINARY DIFFERENTIAL EQUATIONS

An equation which consist differential coefficient is known as differential equation.

Order : The highest order derivative occurs in a differential eqn is known as its order.

Degree : Degree is the exponent of highest order derivative when it is made free from fractions and radicals.

OR

It is the exponent of highest order derivative when it is written in polynomial form in terms of derivatives.

NON-LINEAR DIFFERENTIAL EQUATION

If a differential equation consist at least one of the following properties then it is called non-linear.

O.D.E.	P.D.E.
1. If degree is more than one.	1. Same as LHS.
2. If exponent of dependent variable is more than one.	2. Same as LHS.
3. If exponent of any derivative is more than one.	3. Same as LHS.
4. If product of dependent variable with its derivative exist.	4. Same as LHS.
	5. If product of any two partial derivatives occurs.

Here ODE $\simeq$ Ordinary Differential Equation, and PDE $\simeq$ Partial Differential Equation

Linear Differential Equation

If a differential equation is free from any of the above properties then it is linear.

SOLUTION OF DIFFERENTIAL EQUATIONS

Relation, between dependent and independent variable satisfying the given differential equation is known as its solution.

(i) **General solution :** It contains same no. of arbitrary constant as the order of differential equation.

(ii) **Particular solution :** If a solution has no arbitrary constant then it is known as particular solution.

FORMATION OF DIFFERENTIAL EQUATION

By eliminating the given number of arbitrary constants from general solution, we can form a differential equation and for this purpose we should differentiate general solution as many times as the number of Arbitrary Constants.

WRONSKIAN

If, y_1, y_2, y_3 are three solutions of third order differential equation then their Wronskian is defined as;

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

LINEAR COMBINATIONS OF SOLUTIONS :

Let y_1, y_2, y_3, y_n are the solutions of n^{th} order differential equation and let $c_1, c_2, c_3, \ldots, c_n$ are the arbitrary constants, then the relation

$$c_1y_1 + c_2y_2 + c_3y_3 + \ldots\ldots + c_ny_n = 0 \qquad \ldots(1)$$

is called linear combination of solutions.

(i) **Linearly Dependent Solutions :** Solutions are called LD if there exist a linear combination between them and its condition is :

All the scalars $c_1, c_2, c_3, \ldots, c_n$ should not all zero simultaneously.

(ii) **Linearly Independent Solutions :** Solutions are called L.I. if there does not exist any linear combination between them and its condition is :

All the scalars $c_1, c_2, c_3, \ldots, c_n$ should be zero simultaneously i.e., $c_1 = c_2 = c_3 = \ldots = c_n = 0$.

Note :

(i) If $W = 0$, solutions are called LD and if $W \neq 0$ solutions are called L.I.

(ii) If $y_1, y_2, y_3, \ldots, y_n$ are L.I. solutions of any n^{th} order differential equation then its general solution is

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \ldots + c_ny_n.$$

EXACT DIFFERENTIAL EQUATION

A differential equation $Mdx + Ndy = 0$ ...(1) is said to be exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

and its solution is given as $\boxed{\displaystyle\int_{y\,\text{constant}} Mdx + \int (\text{those terms of N which are free from x})\, dy = C}$

Integrating Factor (IF)

A special term such that by multiplying it in a given DE it can be solved easily.

EQUATION REDUCIBLE TO EXACT FORM/METHODS OF EVALUATING INTEGRATING FACTORS :

Rule 1 : If $Mx + Ny \neq 0$ and equation is homogeneous then IF = $\dfrac{1}{Mx+Ny}$

Rule 2 : If $Mx - Ny \neq 0$ and Equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$ then IF = $\dfrac{1}{Mx-Ny}$

Rule 3 : If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$ alone then $IF = e^{\int f(x)\,dx}$

Rule 4 : If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = f(y)$ alone then $If = e^{\int f(y)\,dy}$

Method of Solving First Order - First Degree Differential Equation

(i) Variable separable method; (ii) Homogeneous DE method; (iii) Linear differential eqn. method

VARIABLE SEPARABLE METHOD

In this method, we try to separate variables so that it can be integrated easily.

e.g., consider the differential equation $\frac{dy}{dx} = f(x).g(y)$ then process is as follows :

$$\frac{dy}{g(y)} = f(x)dx$$

Integrating both sides,
$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

Now try to solve above integrals by using basic concepts.

HOMOGENEOUS DIFFERENTIAL EQUATION OF 1ST ORDER

A differential equation of the type $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is called Homogeneous differential equation of 1st order, if both f(x, y) and g(x, y) are the homogeneous functions of same degree.

To solve above equation, put y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

By using these transformations, differential equation (1) can be solved by using variable-separable methods.

Note : f(x, y) is said to be homogeneous function of degree 'n' if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ where $n \in R$ i.e., after replacing x with λx and y with λy, if it is possible to take out common 'λ' from the expression then f(x, y) is called homogeneous function.

LINEAR DIFFERENTIAL EQUATION OF 1ST ORDER

A differential equation of the type $\frac{dy}{dx} + Py = Q$ is called Linear differential equation of 1st order in y & x, where P and Q are functions of x alone or may be constant. Its integrating factor is defined as $\boxed{\text{I.F.} = e^{\int Pdx}}$.

Multiplying equation (1) by I.F., we get

$$e^{\int Pdx}\cdot\frac{dy}{dx} + P\cdot e^{\int Pdx}\cdot y = Qe^{\int Pdx}$$

By observation,
$$\frac{d}{dx}\left[y\cdot e^{\int Pdx}\right] = Qe^{\int Pdx}$$

Integrating both sides
$$\int \frac{d}{dx} y(\text{I.F.})dx = \int Q(\text{I.F.})dx + C$$

or
$$\boxed{y(\text{I.F.}) = \int Q(\text{I.F.})dx + C}$$

This is the required solution given differential equation (1).

EQUATION REDUCIBLE TO LINEAR FORM

A differential equation which can be reduced to linear form by assuming suitable assumption is called Bernoullie differential equation. Its standrd form is

$$\frac{dy}{dx}+Py = Q.y^n,\ n>0 \qquad ...(1)$$

Right now it is not linear, but we can convert it into linear form as follows :

$$\frac{dy}{dx}+Py = Qy^n \Rightarrow \frac{1}{y^n}\frac{dy}{dx}+P\cdot y^{1-n}=Q$$

Put $\boxed{y^{1-n}=v}$ i.e., $\frac{d}{dx}(y^{1-n}) = \frac{dv}{dx} \Rightarrow \frac{1}{1-n}y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

Using these transformations in above differential equations, we get

$$(1-n)\frac{dv}{dx}+P(v) = Q \text{ or } \frac{dv}{dx}+(1-n)P.v = (1-n)Q$$

which is linear differential equation in V and x.

4.2 — LINEAR DIFFERENTIAL EUQATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

A differential equation of the form $a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}...+a_ny = Q$ ...(1)

where $a_0, a_1, a_2, ..., a_n$ are cosntants and Q is a function of x only or constant is known as LDE of n^{th} order with constant coefficients.

Let $\frac{d}{dx}$ = D then (1) becomes $a_0D^ny + a_1D^{n-1}y + ... + a_ny = Q$

or $$\left(a_0D^n+a_1D^{n-1}+a_2D^{n-2}...+a_n\right)y = Q$$

$\Rightarrow$ $$\boxed{f(D)\cdot y=Q} \qquad ...(2)$$

Its complete solution is given as y = CF + PI where CF = solution of LHS, PI = solution of RHS

Steps for Finding Auxiliary Equation

Just replace $D \to m$ and $y \to 1$ in LHS only. We will get auxiliary euqation in terms of m.

RULES FOR FINDING THE COMPLEMENTARY FUNCTION (i.e., CF)

Let $m_1, m_2, m_3, ..., m_n$ are the roots of Auxilliary equation.

Case 1: When all roots are real and distinct :

$$C.F. = c_1e^{m_1x}+c_2e^{m_2x}+...+c_ne^{m_nx} \qquad ...(iv)$$

Case 2 : When the roots are equal i.e., $m_1 = m_2$ and $m_3 = m_4 = m_5$ and rest are real and distinct.

$$CF = (c_1+c_2x)e^{m_1x}+(c_3+c_4x+c_5x^2)e^{m_3x}+c_6e^{m_6x}+.....$$

Case 3: When roots are imaginary.

Let $m_1 = \alpha+i\beta$, $m_2 = \alpha-i\beta$ and rest are real and distinct.

$$\boxed{C.F. = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + c_3e^{m_3x} + \ldots + c_ne^{m_nx}}$$

Case 4 : When imaginary roots are repeated.

Let $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$, and rest are real and distinct.

$$C.F. = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + c_5e^{m_5x} + \ldots + c_ne^{m_nx}$$

Case 5 : When roots are irrational.

$$m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta} \text{ and rest are distinct.}$$

$$C.F. = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots c_ne^{m_nx}$$

PARTICULAR INTEGRAL (P.I.)

Consider n^{th} order linear differential equation with constant coefficient as follows : $f(D).y = Q$

then PI is defined as $PI = y = \frac{1}{f(D)}Q$ where $\frac{1}{f(D)}$ represents an operator.

METHODS OF EVALUATING (P.I.)

Case 1 : (When $Q = e^{ax}$) Replace $D \to$ 'a' in Denominator only.

i.e.,

$$PI = \boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \text{ provided } f(a) \neq 0.}$$

Note : (a) if $f(a) = 0$ then $PI = \frac{1}{f(D)} \cdot e^{ax} = \frac{x}{f'(a)} \cdot e^{ax}$ provided $f'(a) \neq 0$

(b) if $f'(a) = 0$ then $PI = \frac{1}{f(D)} \cdot e^{ax} = \frac{x^2}{f''(a)} \cdot e^{ax}$ provided $f''(a) \neq 0$ and so on

Case 2 : (When Q is of the form sin ax or cos ax). Replace $D^2 \to -a^2$ in Denominator only.

i.e., $PI = \boxed{\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)}, \text{ if } f(-a^2) \neq 0}$

Note :

(a) if $f(-a^2) = 0$, then, $PI = \frac{1}{f(D^2)} \cdot \sin ax = \frac{x}{f'(D^2)} \cdot \sin ax = \frac{x}{f'(-a^2)} \cdot \sin ax$

(b) if $f'(-a^2) = 0$, then, $P.I. = \frac{1}{f(D^2)} \cdot \sin ax = \frac{x^2}{f''(-a^2)}\sin ax$ and so on.

Similar results are applicable in case of cos ax.

Case 3 : When $Q = x^m$, m being a positive integer.

Steps :

1. Take out common the lowest degree term from denominator to make the first term unity.
2. Take this factor in the numerator. It takes the form $[1+\varphi(D)]^{-1}$ or $[1-\varphi(D)]^{-1}$.
3. Expand it in ascending powers of D by using Binomial theorem as follows.

Note : $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$ (where n is –ve integrer or fraction)

Case 4 : When $Q = e^{ax}.V$ where V is a function of x.

Replace $D \to D + a$ after taking out common e^{ax} i.e., PI = $\boxed{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V}$

Case 5 : When Q = xV where V is any function of x : In this case use the following result

$$\boxed{PI = \frac{1}{f(D)}\cdot(x\cdot V) = x\cdot\frac{1}{f(D)}\cdot V - \frac{f'(D)}{[f(D)]^2}\cdot(V)}$$

Case 6 : General Method

Resolve f(D) into linear factors then apply Partial Fraction and finally use standard results, i.e.,

$$P.I. = \frac{1}{f(D)}Q = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots(D-m_n)}Q = \left(\frac{A_1}{D-m_1}+\frac{A_2}{D-m_2}+\ldots\ldots+\frac{A_n}{D-m_n}\right)Q$$

$$= A_1\frac{1}{D-m_1}Q + A_2\frac{1}{D-m_2}Q+\ldots\ldots+A_n\frac{1}{D-m_n}Q$$

Now use following result for each factor

$$\boxed{\begin{array}{l}\text{(i)}\ \frac{1}{D-\alpha}Q = e^{\alpha x}\int e^{-\alpha x}Q\,dx \\ \text{(ii)}\ \frac{1}{D+\alpha}Q = e^{-\alpha x}\int e^{\alpha x}Q\,dx\end{array}}$$

CAUCHY'S HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

Consider the differential equation

$$a_0x^n\frac{d^xy}{dx^n}+a_1x^{n-1}\frac{d^{x-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}}{dx^{n-2}}+\ldots\ldots+a_{n-1}x\cdot\frac{dy}{dx}+a_ny = Q$$

or $[a_0x^nD^n+a_1x^{n-1}D^{n-1}+a_2x^{n-2}D^{n-1}+\ldots..+a_{n-1}x)+a_n]y = Q$ ...(1)

where $a_0, a_1, a_2, \ldots.., a_n$ are constants and Q is a functions of x alone or may be constant, then it is called Cauchy's Homogeneous Linear Differential equation of n^{th} order. It is not with constant coefficients so it cannot be solved by using (CF + PI) methods. Hence to solve it we have to use following transformations :

$x = e^z$ or $z = \log x$,

$xD = D_1$, $x^2D^2 = D_1(D_1-1)$, $x^3D^3 = D_1(D_1-1)(D_1-2)$ and so on ..... where $D = \frac{d}{dx}$, $D_1 = \frac{d}{dz}$

By using above transformations, differential equation (1) leads to linear differential equation with constant coefficient in y and z and hence can be solved by using (CF + PI) methods.

Note : If $x = e^z$ or $z = \log x$ then prove that $\boxed{xD = D_1}$

We can write,

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\cdot\left(\frac{1}{x}\right) \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x\frac{d}{dx} = \frac{d}{dz} \Rightarrow xD = D_1.$$ Hence Proved.

4.3 — PARTIAL DIFFERENTIAL EQUATION

The differential equation which contains one or more partial derivatives is called P.D.E.

Note : In this chapter we have following notations : $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial y^2} = t$ and $\frac{\partial^2 z}{\partial x \partial y} = s$

CLASSIFICATION OF 2ND ORDER LINEAR P.D.E.

Consider the P.D.E $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = f(x, y, u, u_x, u_y)$.

This DE will be (a) Elliptic if $B^2 - 4AC < 0$; (b) Hyperbolic if $B^2 - 4AC > 0$; (c) Parabolic if $B^2 - 4AC = 0$

ONE DIMENSIOANL WAVE EQUATION

It measures the deflection in an oscillating string at any time t and at any point x and it is given as

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \qquad ...(1)$$

Boundary conditions are : $y(0, t) = 0$ and $y(\ell, t) = 0$

Initial conditions are : $y(x, 0) = f(x)$; $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$

D-Alembert's Solution of Wave Equation

Consider the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ such that $y(x, 0) = f(x)$ and $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = g(x)$

Then it's solution can be taken as

$$\boxed{y(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)dx}$$

ONE DIMENSIONAL HEAT EQUATION

It measures the temperature distribution in a uniform rod at any point 'x' and at any time 't' and it is defined as :

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad ...(1)$$

Boundary conditons are : $u(0, t) = 0 = u(\ell, t)$ (temperature at corner points)

& Initial temperature is: $u(x, 0) = f(x)$

TWO DIMENSIONAL HEAT EQUATION

It is defined as $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

It is a temperature distribution in a metal plate in transient state.

LAPLACE EQUATION

It is defined as $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

Laplace Equation represents temperature distribution in rectangular plate at steady state, i.e., when $\frac{\partial u}{\partial t} = 0$, 2-D heat equation converts into Laplace equation.

Boundary conditions are : $u(0, y) = u(\ell, y) = u(x, 0) = 0$ & $u(x, a) = f(x)$

SEPARATION OF VARIABLES METHOD

Suppose we have to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(1)

then in this method we assume $u = f(x).g(t)$ ...(2)

as the solution of (1) where variables are assumed as separated from each other. Now we try to find the values of f(x) and g(t) using (1) and (2) simultaneously.

This process is called separation of variables method.

UNIT 5 : NUMERICAL METHODS / NUMERICAL TECHNIQUES (N.T.)

- N.T.
  - Solution of Linear Equations
    - Gauss Jacobi Method
    - Gauss Seidel Method
  - Solution of Non-Linear & Transcendental Equations
    - Bisection Method
    - Regular Falsi Method
    - Secant Method
    - N-R (Newton-Rapson) Method
  - Numerical Differentiation
    - Using Newton Forward Formula
    - Using Newton Backward Formula
    - Lagrange's Interpolation Method
    - Using Newton Divided Difference Formula
  - Numerical Integration
    - Trapezoidal
    - Simpson's 1/3rd Rule
    - Simpson's 3/8th Rule
  - Numerical Solution of Ordinary D.E.
    - Picard Method
    - Euler Method
    - Runge-Kutta Method of 4th Order

5.1 — NUMERICAL SOLUTIONS OF LINEAR EQUATIONS

1. Direct Methods (or Exact Methods / Ellimination Methods)

Those methods in which the solution is obtained in a finite number of steps. e.g. Gauss elimination methods, Gauss-Jordan method, Crouts method, Dolittle's method are called direct methods.

2. Iterative Methods

These methods converges when the number of steps tends to infinity e.g. **Gauss-Jacobi** method, **Gauss-Seidel** method. In these methods, we start the process by assuming a solution nearer to the exact solution and try to obtain the solution by successive repetition of the process.

Diagonally Dominant System : Consider the system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ $\quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ $\quad a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

if we have $\quad |a_{11}| > |a_{12}| + |a_{13}|$, $\quad |a_{22}| > |a_{21}| + |a_{23}|$, $\quad |a_{33}| > |a_{31}| + |a_{32}|$

then above system is called Diagonally Dominant.

i.e., In diagonally dominant system, diagonal element has the largest numerical value in each row.

Note :

1. If system is not diagonally dominant, we can make it diagonally dominant by interchanging rows.
2. Gauss Jacobi and Gauss Seidal methods converges only for diagonally dominant system.
3. If initial assumption is not given, we can take it as $(x_0, y_0, z_0) = (0, 0, 0)$

5.2 — NUMERICAL SOLUTIONS OF NON-LINEAR AND TRANSCENDENTAL EQUATIONS

ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS

(i) Polynomial function, rational function, irrational function, modulus function, signum function etc. are the examples of algebraic functions.

(ii) Exponential function, logarithmic functions, trignometric functions, inverse trignometric functions are the examples of transcendental functions.

Hence, those functions which are not transcendental are known as Algebraic Functions.

ROOT OF AN EQUATION

The real number α is called root of the equation $f(x) = 0$ if $f(\alpha) = 0$

or the real root is the value of x where graph of f(x) meets x-axis.

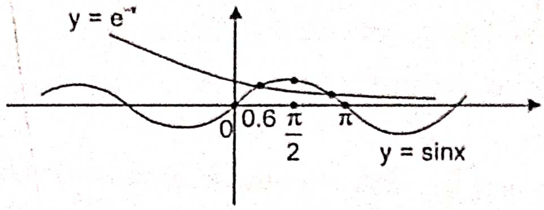
or when f(x) is divided by $(x - \alpha)$ and remainder is zero then $x = \alpha$ is the root of $f(x) = 0$.

There are two methods to find the roots of the equation $f(x) = 0$.

1. **Direct Methods:** These methods determine all the roots at the same time. If the equation $f(x) = 0$ is written as $f_1(x)$ and $f_2(x)$, then roots are the points of intersection of the curves $y = f_1(x)$ and $y = f_2(x)$.
2. **Iterative Methods:** These methods determine only one root at a time and these are based on successive approximation concept.

e.g. 1 : The number of real roots of the equation $e^{-x} = \sin x$ is/are ?

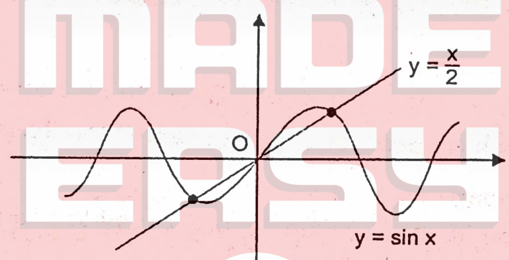
Solution: Let us draw $y = e^{-x}$ and $y = \sin x$.



$\because$ There are two intersecting points so we have two roots of above equations.

e.g. 2 : The number of real roots of $\sin x = \frac{x}{2}$ is/are?

Solution : Let us draw $y = \sin x$ and $y = \frac{x}{2}$. Since there are three intersecting points so we have three roots of given equations.



Note : It is not always possible to find the roots of an equation by using graphical method so in such situations we use numerical methods which are as follows :

Bolzano Theorem : If f(x) is real valued continuous function defined in [a, b] such that f(a) and f(b) have opposite sign then there exist at least one root α of f(x) between a and b.

Note : Bisection method is the repeated application of this theorem.

BISECTION METHOD OR HALF INTERVAL METHOD

Consider f(x) = 0 be any equation and let α is the root of f(x). By hit and trial method, try to find two values of x (say x = a and x = b) for which f(a) and f(b) have opposite sign then by Bolzano theorem $\alpha \in (a, b)$.

Now, we will bisect this interval as follows :

Iteration 1 : $x_1 = \frac{a+b}{2}$ (Here, we have assumed that f(a) = –ve and f(b) = +ve).

Now two possibilies arises :

(i) if $f(x_1)$ = +ve then by Bolzano theorem $\alpha \in (a, x_1)$

(ii) if $f(x_1)$ = –ve then by Bolzano theorem $\alpha \in (x_1, b)$

Let $\alpha \in (a, x_1)$ then we will bisect this interval as follows :

Iteration 2 : $x_2 = \frac{a+x_1}{2}$

Again we have two possibilities :

(i) if $f(x_2)$ = +ve then by Bolzano theorem $\alpha \in (a, x_2)$

(ii) if $f(x_2)$ = –ve then by Bolzano theorem $\alpha \in (x_2, x_1)$

Let $\alpha \in (x_2, x_1)$ then again we will bisect this interval as follows :

Iteration 3 : $x_3 = \dfrac{x_2 + x_1}{2}$ and so on .....

We continue this process of bisection until the root with desired accuracy is obtained.

Note : 1. Number of iterations (n) required to achieve an accuracy of $\in$ is given as $\boxed{\dfrac{|b-a|}{2^n} < \in}$ because at the end of the n^{th} step, the new interval will be $[a_n, b_n]$ of length $\dfrac{|b-a|}{2^n}$. Here $\in$ is very small +ve number

2. Order of Convergence of Bisection Method is one i.e., it is first order convergent or linearly convergent.

REGULA–FALSI METHOD OR THE METHOD OF FALSE POSITION / METHODS OF LINEAR INTERPOLATION / METHODS OF CHORDS

Consider the equation f(x) = 0 and let α is the root of f(x). By using hit and trial method, try to find two values of x (say x = a and x = b) where f(a) and f(b) have opposite sign then by Bolzano theorem $\alpha \in (a, b)$ {Here, we have assumed that f(a) = +ve and f(b) = –ve}.

In this method, instead of bisecting the interval containing root, we try to find the equation of chord joining A(a, f(a)) and B(b, f(b)). Hence equation of AB is y – f(a) = m[x – a]

i.e.,
$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

Let this chord passes through the point $(x_1, 0)$. So, this point will satisfy the equation of chord as follows :

$$0 - f(a) = \frac{f(b)-f(a)}{b-a}(x_1 - a)$$

So **Iteration 1 :**
$$\boxed{x_1 = a - \left(\frac{b-a}{f(b)-f(a)}\right)f(a)}$$

Again we have two possibilities :

(i) if $f(x_1)$ = –ve then $\alpha \in (a, x_1)$

(ii) if $f(x_2)$ = +ve then $\alpha \in (x_2, b)$

Let $\alpha \in (a, x_1)$ then equation of chord joining A(a, f(a) and $C(x_1, f(x_1))$ is given as

$$y - f(a) = \left(\frac{f(x_1)-f(a)}{x_1 - a}\right)(x-a)$$

This passes through $(x_2, 0)$ so this point will satisfy the equatio of chord as follows :

$$0 - f(a) = \frac{f(x_1)-f(a)}{x_1 - a}(x_2 - a)$$

So **Iteration 2 :**
$$x_2 = a - \left(\frac{x_1 - a}{f(x_1)-f(a)}\right)f(a)$$

Continue in the same manner until we get the root with desired accuracy.

Note : **Order of Convergence** of Regula-Falsi Method is one i.e., it is also linearly convergent.

THE SECANT METHOD

It is the particular case of Regula-Falsi method in which there is no need to verify Bolzano thoerem in each step and the iterative scheme for secant method is as similar as that of Regula-Falsi method, but the only difference is that : if $\alpha \in (a,b)$ and once x_1 is obtained, it is directly assumed that $\alpha \in (x_1,b)$ for 2nd iteration.

and when x_2 is obtained then we can assume $\alpha \in (x_2,b)$ for 3rd iteration and so on.

Vartious iterations are as follows :

Iteration 1 : Let initial guess is $\alpha \in (a,b)$ then $x_1 = a - \left(\frac{b-a}{f(b)-f(a)}\right)f(a)$

Iteration 2 : Let $\alpha \in (x_1,b)$ then $x_2 = x_1 - \left[\frac{b-x_1}{f(b)-f(x_1)}\right]f(a)$

Iteration 3 : Let $\alpha \in (x_2,b)$ then $x_3 = x_2 x_2 - \left[\frac{b-x_2}{f(b)-f(x_2)}\right]f(a)$ and so on.

Note : 1. Secant method is not necessarily convergent because in any step if f(a) = f(b) the whole process of iteration becomes meaningless.

2. **Order of convergence** of secant method is 1.62, i.e., it is 1.62 times faster than Regula Falsi method.

NEWTON RAPHSON METHOD (METHOD OF TANGENT)

Consider the equation f(x) = 0 and let α is the exact root of f(x). In this method we assume that root is x_0, i.e., initial assumption of root is 'x_0' or we can say that root lies near x_0.

To find the root 'α' we draw a tangent at $A(x_0, f(x_0))$, i.e., equation of tangent at A is

$$y - f(x_0) = m[x - x_0]$$

$$y - f(x_0) = f'(x_0)[x - x_0]$$

Let this tangent passes through $(x_1, 0)$ so satisfying the equation of tangent by this point, we get

$$0 - f(x_0) = f'(x_0)[x_1 - x_0]$$

or

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \text{...(Iteration 1)}$$

Now equation of tangent at $B(x_1, f(x_1))$ is

$$y - f(x_1) = f'(x_1)[x - x_1]$$

Let this tangent passes through $(x_2, 0)$ so this point will satisfy above equation as follows :

$$0 - f(x_1) = f'(x_1)[x_2 - x_1]$$

or

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{...(Iteration 2)}$$

Continuing in the same fashion, we can obtain the root upto desired accuracy, and iterative scheme of Newton Raphson method is

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

for x = 0, 1, 2, 3, ... we can obtain the various approximated roots, which are improving respectively.

Necessary Conditions for the convergence of N-R Method :

There are two necessary conditions :

1. N-R method converges when initial assumption x_0 is assumed sufficiently close to the exact root α.
2. N-R method converges for large values of $f'(x)$, i.e., where tangent is nearly vertical at initial assumption x_0.

Note : 1. The Newton-Raphson method is quadratic convergent, i.e., its order of convergence is 2.

2. The convergence of Newton's method for double root is linear.
3. This method is also used to obtain complex roots.
4. When $f'(x)$ is large, near the root, N-R method should be applied.
5. When $f'(x)$ is small, near the root, Regula-Falsi method should be applied.

5.3 — NUMERICAL DIFFERENTIATION

VARIOUS OPERATORS :

In this chapter, we have following operators :

Forward Difference Operator

First difference is denoted as $\Delta f(a)$ is defined as $\boxed{\Delta f(a) = f(a+h) - f(a)}$

Second difference is denoted by $\Delta^2 f(a)$ and is defined as $\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a)$. and so on.

Here Δ^2 does not represent square of the quantity but denotes repetition of the operation Δ.

Backward Difference Operator : It is denoted by ∇ and is defined as

$$\nabla f(a) = f(a) - f(a-h), \quad \nabla^2 f(a) = \nabla f(a) - \nabla f(a-h) \text{ and so on.}$$

NEWTON-FORWARD FORMULA (FOR EQUAL INTERVALS)

If y = f(x) takes the values f(a), f(a + h), f(a + 2h), ...f(a + nh) at x = a, a + h, a + 2h, ..., (a + xh) respectively then

$$f(x) = f(a) + \frac{(x-a)}{h}\Delta f(a) + \frac{(x-a)(x-a-h)}{2!h^2}\Delta^2 f(a) + \ldots + \frac{(x-a)(x-a-h)\ldots(x-a(\overline{n+1})h)}{n!h^n}\Delta^n f(a) \qquad \ldots(1)$$

Writing $u = \dfrac{x-a}{h}$, the forward formula reduces to

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \ldots + \ldots + \frac{u(u-1)\ldots(u-\overline{n-1})}{n!}\Delta^n f(a)$$

NEWTON-BACKWARD FORMULA (FOR EQUAL INTERVALS)

If y = f(x) takes the values f(a), f(a + h), f(a + 2h),...., f(a + nh) at x = a, a + h, a + 2h, ... a + nh then

$$f(x) = f(a+nh) + \frac{x-a-nh}{1!h}\nabla f(a+nh) + \frac{(x-a-nh)(x-a-\overline{n+h})}{2!h^2}\nabla^2 f(a+nh) + \ldots$$

$$\ldots + \frac{(x-a-nh)(x-a-\overline{n-1}h)\ldots(x-a-h)}{n!h^n}\nabla^n f(a+nh) \qquad \ldots(1)$$

Writing $u = \dfrac{x-x_n}{h}$, the backward interpolation formula reduces to

$$f(x+uh) = (x_0+hu) = f(x_n) + \frac{u}{1!}\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \dots + \frac{u(u+1)\dots(u+n-1)}{n!}\nabla^n f(x_n)$$

Note : Consider the polynomial of degree n as follows

$$y = f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

Hence, in case of n^{th} degree polynomial, we have (n + 1) unknowns so we need (n + 1) values of y to evaluate it.

Also, we know that, $\frac{d^ny}{dx^n} = f^n(x) =$ Constant and $f^{n+1}(x) = 0$, i.e., $(n+1)^{th}$ derivative is zero.

Similarly, in numerical techniques, if we have n^{th} degree polynomial then $(n+1)^{th}$ order differences will be zero, i.e.,

$$\Delta^{n+1}f(x) = 0$$

5.4 — NUMERICAL INTEGRATION

Suppose have to calculate $\int_a^b f(x)dx$. Sometimes it is not possible to integrate f(x) by using Analytical methods (as used in Calculus). So, in such types of situations we try to find approximate value of above integral by using some values of f(x), and these values are given in a tabular form.

Hence "Numerical integration is a process of evaluating the value of definite integral, when some set of numerical values of y = f(x) is given."

When this process is applied to function of single variable it is called **Quadrature** and when applied to the function of two variables, it is called **Cubature.**

GENERAL QUADRATURE FORMULA / NEWTON-COTE'S FORMULA

Consider the set of numerical values as follows :

x =	a	a + h	a + 2h	a + 3h	....	a + nh = b
y =	f(a)	f(a + h)	f(a + 2h)	f(a + 3h)	....	f(a + nh)

Here we have taken (n + 1) values of f(x) and size of each interval is h. Let us take any random point in this table as follows :

x = a + uh where $0 \le u \le n$ then dx = h du and (when x = a, u = 0), (when x = a + nh, u = n)

Now,

$$I = \int_a^b f(x)dx = \int_a^{a+nh} f(x)dx = \int_{u=0}^{n} f(a+uh)h\,du = h\int_0^n f(a+uh)du$$

$$= h\int_0^n \left[f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \dots\right]du$$

Here we have used Newton Forward Interpolation Formula.

Let $f(a) = y_0$ then above integral reduces to

$$I = h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \dots\right]_{u=0}^{n}$$

$$\boxed{I = h\left[ny_0 + \left(\frac{n^2}{2}\right)\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots\right]}$$

It is called General Quadrature formula for numerical integration and by substituting different values of n, we can derive different integration formula such as Trapezoidal Rule, Simpson's Rule etc.

TRAPEZOIDAL RULE

If we take n = 1 strip at a time and assuming the curve (x_0, y_0) to (x_1, y_1) as a **linear** polynomial then we can neglect the 2nd and higher order differences in General Quadrature Formula, i.e.,

$$\int_{x_0}^{x_1} f(x)dx = h\left[1 \cdot y_0 + \left(\frac{1}{2}\right)\Delta y_0 + \text{Neglect}\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

$$\int_{x_1}^{x_2} f(x)dx = h\left[1 \cdot y_1 + \left(\frac{1}{2}\right)\Delta y_1 + \text{Neglect}\right] = h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] = \frac{h}{2}(y_1 + y_2)$$

Similarly, $\int_{x_2}^{x_3} f(x)dx = \frac{h}{2}(y_2 + y_3)$, ..., $\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}(y_{n-1} + y_n)$

Now, $I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$\boxed{\int_a^b f(x)dx = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})]}$$

This is known as Trapezoidal rule.

SIMPSON'S ONE THIRD RULE

If we take n = 2 strips at a time and assuming the curve from (x_0, y_0) to (x_2, y_2) as a **Quadratic** polynomial then we can neglect the 3rd and higher order differences in General Quadrature Formula, i.e.,

$$\int_{x_0}^{x_2} f(x)dx = h\left[2 \cdot y_0 + \left(\frac{2^2}{2}\right)\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \text{Neglect}\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{\Delta^2 y_0}{6}\right] = 2h\left[y_0 + (y_1 - y_0) + \frac{\Delta(y_1 - y_0)}{6}\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right] = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\int_{x_2}^{x_4} f(x)dx = h\left[2 \cdot y_2 + \left(\frac{2^2}{2}\right)\Delta y_2 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_2}{2!} + \text{Neglect}\right] = \frac{h}{3}[y_1 + 4y_3 + y_4]$$

Similarly, $\int_{x_4}^{x_6} f(x)dx = \frac{h}{3}[y_4 + 4y_5 + y_6]$, ..., $\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Adding all these,

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\boxed{\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2})\right]}$$

This is known as Simpson's 1/3rd rule.

SIMPSON'S 3/8TH RULE

If we take n = 3 strips at a time and assuming the curve from (x_0, y_0) to (x_3, y_3) as a **Cubic** polynomial then we can neglect the 4th and higher order differences in General Quadrature Formula, i.e.,

$$\int_{x_0}^{x_3} f(x)dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!} + ...\right]$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 + \text{Neglect}\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly, we can obtain other integrals varies from x_3 to x_6, x_6 to x_9 and so on.

$$\text{Hence, } I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + ... + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$\boxed{\int_a^b f(x)dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + ... + y_{n-3})\right]}$$

This is known as Simpson's 3/8th Rule.

CLASSIFICTION OF ERROR IN NUMERICAL METHOD

1. Error = Exact Value – Approx. Value
2. Absolute Error = |Exact Value – Approx. Value|
3. Relative Error = $\left|\dfrac{\text{Exact Value} - \text{Approx. Value}}{\text{Exact Value}}\right|$
4. Percentage Error = (Relative Error) × 100
5. Round Off Error = It is obtained either by chopping off or by rounded off the approx. value after some significant digits.
6. Truncation Error = It is obtained by eliminating infinite number of terms in an infinite series.

   For example, apprxomation of f(x) by using Taylor series gives Truncation error.

TRUNCATION ERROR IN VARIOUS ITERATIVE METHODS

Consider a function f(x) defined in [a, b] and let this interval is subdivided into n equal length sub-intervals having size h units then

For T-Rule : $TE = \left|\frac{(b-a)}{12}h^2M\right|$ where M = Max. value of $f''(x)$ in [a, b]

For 1/3rd Rule : $TE = \left|\frac{(b-a)}{180}h^4M\right|$ where M = Max. value of $F^{iv}(x)$ in [a, b]

For 3/8th Rule : $TE = \left|\frac{(b-a)}{80}h^4M\right|$ where M = Max. value of $f^{iv}(x)$ in [a, b]

Note : 1. Here $f^{iv}(x) \approx \frac{d^4}{dx^4}f(x)$

2. For 1/3 Rule : T.E. $= \frac{nh^5}{90}M = \frac{2nh^5}{180}M = \frac{(b-a)}{180}h^4M$ $\left(\because \frac{b-a}{2n} = h\right)$

3. For 3/8 Rule : $TE = \frac{3nh^5}{80}M = \frac{(b-a)}{80}h^4M$ $\left(\because \frac{b-a}{3n} = h\right)$

4. For both, simpson $\frac{1}{3}$rule and Simpson $\frac{3}{8}$rule, error is of order h^4. But the coefficient in error of $\frac{1}{3}$rule is $\frac{1}{90}$, which is smaller than the coefficient in error of $\frac{3}{8}$rule which is - $\frac{3}{80}$.

5.5 — NUMERICAL SOLUTION OF DIFFERENTIAL EQUATION

FORWARD EULER METHOD / EXPLICIT EULER METHOD

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ and step size = h ...(1)

and let its solution is y = f(x).

The equation of tangent at (x_0, y_0) to the curve y = f(x) is given as

$$y - y_0 = y'_{(x_0,y_0)}(x - x_0)$$

$$y - y_0 = f(x_0,y_0)(x - x_0) \qquad (\because y' = f(x,y) \text{ is given})$$

$$\Rightarrow \quad y = y_0 + f(x_0,y_0)(x - x_0)$$

Now, the value of y corresponding to $(x = x_1)$ is

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

So, **Iteration (1) :** $y_1 = y_0 + hf(x_0, y_0)$

Similarly, **Iteration (2) :** $y_2 = y_1 + hf(x_1, y_1)$ and so on.

In general, $\boxed{y_{n+1} = y_n + hf(x_n, y_n) : n = 0,1,2,......(4)}$

BACKWARD EULER'S METHOD / IMPLICT EULER METHOD

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ with step size h.

then iterations are as follows :

Iteration 1. $y_1 = y_0 + hf(x_1, y_1)$

Iteration 2. $y_2 = y_0 + hf(x_2, y_2)$

Iteration 3. $y_3 = y_0 + hf(x_3, y_3)$ and so on....

i.e., **Iteration equation :** $\boxed{y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})}$

RUNGE-KUTTA METHOD OF ORDER 2 (MODIFIED EULER METHOD)

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ and step size = h then iterations are as follows :

Iteration 1 : $\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$ where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$

Iteration 2 : $\boxed{y_2 = y_1 + \frac{1}{2}(k_1 + k_2)}$ where $k_1 = hf(x_1, y_1)$ and $k_2 = hf(x_1 + h, y_1 + k_1)$.

Similarly, we can find other iterations.

RUNGE-KUTTA METHOD OF ORDER 3

Consider the differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and step size = h

first Iteration is $\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)}$

where $k_1 = hf(x_0, y_0)$,

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$,

$k_3 = hf(x_0 + h, y_0 + k')$ where $k' = hf(x_0 + h, y_0 + k_1)$.

Similarly, we can find other iterations.

RUNGE-KUTTA METHOD OF ORDER 4 (CLASSICAL RUNGE KUTTA METHOD)

Let us consider the differential equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ with h as step size.

First Iteration is $\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$ ...(1)

where, $k_1 = hf(x_0, y_0)$, $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$, $k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$, $k_4 = hf(x_0 + h, y_0 + k_3)$

Similarly other iterations can be calculated.

TRUNCATION ERROR IN VARIOUS ITERATIVE METHODS :

S.No.	Iterative Method	Error $\propto (h^n)$
1.	Forward Euler Method/R.K Method of 1st Order	h^2
2.	Backward Euler Method	h^2
3.	Modified Euler Method/R.K. Method of 2nd Order	h^3
4.	R.K. Method of 3rd Order	h^4
5.	R.K. Method of 4th Order	h^5

UNIT 6 : TRANSFORM THEORY

6.1 — LAPLACE TRANSFORM

INTEGRAL TRANSFORM

Let f(t) be a function of 't' then the integral, $\int_{-\infty}^{\infty} k(s,t)\, f(t)\, dt$ is called integral transform of f(t).

LAPLACE TRANSFORM (MAIN DEFINITION)

If we define $k(s,t) = \begin{cases} 0, & t<0 \\ e^{-st}, & t \ge 0 \end{cases}$ then above integral is known as Laplace Transform of f(t), i.e.,

$$\boxed{L\{f(t)\} = f(s) = \int_0^{\infty} e^{-st} f(t)\, dt}$$

Standard Results :

f(t)	L{f(t)}	$e^{at}f(t)$	$L\{e^{at}f(t)\}$
1	1/s	e^{at}	$\frac{1}{(s-a)}, s>a$
$t^n, n = 0,1,2\ldots$	$n!/(s)^{n+1}$	$e^{at}\cdot t^n, n = 0,1\ldots$	$n!/(s-a)^{n+1}, s>a$
$t^n, n>-1$	$\frac{\Gamma(n+1)}{(s)^{n+1}}$	$e^{at}\cdot t^n, n>-1,$	$\frac{\Gamma(n+1)}{(s-a)^{n+1}}, s>a$
$\sin Kt$	$\frac{K}{s^2+K^2}$	$e^{at}\sin Kt$	$\frac{K}{(s-a)^2+K^2}, s>a$
$\cos Kt$	$\frac{s}{s^2+K^2}$	$e^{at}\cos Kt$	$\frac{s-a}{(s-a)^2+K^2}, s>a$
$\sinh Kt$	$\frac{K}{s^2-K^2}$	$e^{at}\sinh Kt$	$\frac{K}{(s-a)^2-K^2}, s>a$
$\cosh Kt$	$\frac{s}{s^2-K^2}$	$e^{at}\cosh Kt$	$\frac{s-a}{(s-a)^2-K^2}, s>a$

PROPERTIES OF LAPLACE TRANSFORM

First Shifting Property : If L{f(t)} = f(s) then $L\{e^{at}f(t)\} = f(s-a)$

Second Shifting Property : Let L{f(t)} = f(s), and $G(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$ then $L\{G(t)\} = e^{-as}f(s)$.

Change of Scale Property : If L{f(t)} = f(s) then $L\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

LAPLACE TRANSFORM OF PERIODIC FUNCTION

If f(t) is a periodic function with period T, then $L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$

SOME STANDARD RESULTS

Laplace Transform of Derivatives : if $L\{f(t)\} = f(s)$ then $L\{f'(t)\} = sf(s) - f(0)$

Note : Similarly, $L\{f''(t)\} = s^2f(s) - sf(0) - f'(0)$, $L\{f'''(t)\} = s^3f(s) - s^2f(s) - sf'(0) - f''(0)$ and so on .....

Laplace Transform of Integrals: If $L\{f(t)\} = f(s)$ then, $L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}f(s)$

Laplace Transform of Functions Multiplied by Some Powers of 't':

If $L\{f(t)\} = f(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$, for $n = 1, 2, 3, \ldots$

Laplace Transform of Functions Divided by 't'

if $L\{f(t)\} = f(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds$, provided $\lim_{t \to 0} \frac{f(t)}{t}$ exist

INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = f(s)$ then $f(t)$ is defined as the inverse Laplace transform of $f(s)$ and is written as, $f(t) = L^{-1}\{f(s)\}$

Standard Results :

	f(s)	$L^{-1}\{f(s)\}$ i.e. f(t)
1.	$1/s, s > 0$	1
2.	$\frac{1}{s^n}, n = 1, 2, 3\ldots$	$\frac{t^{n-1}}{(n-1)!}$
3.	$\frac{1}{s^n}, s > 0, n > 0$	$\frac{t^{n-1}}{\Gamma(n)}$
4.	$\frac{1}{s-a}, s > a$	e^{at}
5.	$\frac{1}{s^2+a^2}, s > 0$	$\frac{1}{a}\sin at$
6.	$\frac{s}{s^2+a^2}, s > 0$	$\cos at$
7.	$\frac{1}{s^2-a^2}, s > 0$	$\frac{1}{a}\sinh at$
8.	$\frac{s}{s^2-a^2}, s > 0$	$\cosh at$

PROPERTIES OF INVERSE LAPLACE TRANSFORM

First Shifting Property : If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{f(s-a)\} = e^{at} \cdot L^{-1}\{f(s)\} = e^{at}f(t)$

2nd Shifting Property : if $L^{-1}\{f(s)\} = f(t)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then, $L^{-1}\{e^{-as}f(s)\} = G(t) = f(t-a)U(t-a)$ where $u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$

Change of Scale Property : if $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{f(as)\} = \frac{1}{a}f(t/a)$

Inverse Laplace Transform of Derivatives : $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\} = (-1)^n t^n f(t)$

Inverse Laplace Transform of Integrals : if $L^{-1}\{f(s)\} = f(t)$ then, $L^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{f(t)}{t}$

Inverse Laplace Transforms of Functions Multiplied by Some Powers of s

$$L^{-1}\{f(s)\} = f(t) \text{ then } L^{-1}\{s^n f(s)\} = \frac{d^n}{dt^n}f(t)$$

Inverse Laplace Transform of Functions Divided by s

if $$L^{-1}\{f(s)\} = f(t) \text{ then } L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(t)\,dt$$

Convolution Theorem (for inverse Laplace Transform)

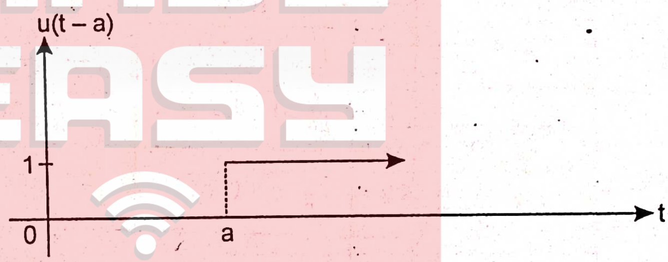
If $$L^{-1}\{f(s)\} = f(t) \text{ and } L^{-1}\{g(s)\} = g(t) \text{ then}$$

$$L^{-1}\{f(s).g(s)\} = f * g = \int_0^t f(u)\cdot g(t-u)\,du$$

or $$\boxed{L\{f * g\} = F(s)\cdot G(s)}$$

UNIT STEP FUNCTION

It is defined as $U(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$



Laplace Transform of Unit Step Function :

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\cdot U(t-a)\,dt = \int_0^a e^{-st}\cdot(0)\,dt + \int_a^\infty e^{-st}\cdot(1)\,dt = \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{1}{s}e^{-as}$$

UNIT IMPULSE FUNCTION (DIRAC DELTA FUNCTION)

Unit Impulse function $\delta(t-a)$ is defined as, $\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$

Laplace Transform of Unit Impulse Function : $L\{\delta(t-a)\} = \int_0^\infty e^{-st}\cdot\delta(t-a)\,dt = e^{-sa}$

Here, we have used the result $\int_0^\infty f(t)\delta(t-a)dt = f(a)$

Functions of Exponential Order

A function f(t) is said to be of exponential order 'α' if $\lim_{t\to\infty}[e^{-\alpha t}]\cdot f(t)$ exists finitely.

EXISTENCE THEOREM FOR LAPLACE TRANSFORM

If a function f(t) is piecewise continuous function and is of exponential order α as $t \to \infty$ then $L\{f(t)\}$ exist for $s > \alpha$,

INITIAL VALUE THEOREM

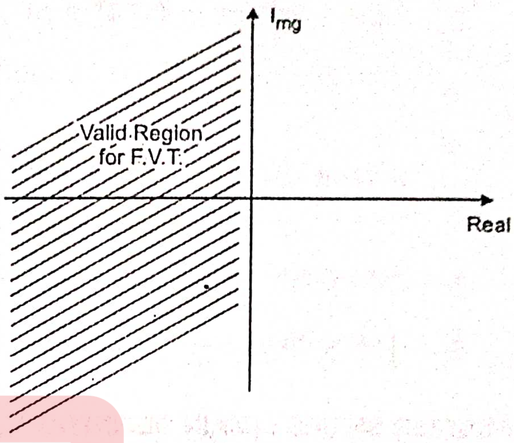
if $L\{f(t)\} = F(s)$ then $\lim_{t \to 0} f(t) = \lim_{s \to \infty} [sf(s)]$

FINAL VALUE THEOREM

If $L\{f(t)\} = F(s)$ then, $\lim_{t \to \infty} [f(t)] = \lim_{s \to 0} [s\,f(s)]$

Note :

1. Final Value Theorem does not exist in following cases :
   (i) s.F(s) has poles on RHP of s-Plane.
   (ii) s.F(s) has poles on $j\omega$ axis, i.e., on imaginary axis.
   (iii) s.F.(s) has poles on origin.
2. Initial Value Theorem is applicable only when F(s) is a proper fraction, i.e., degree of numerator < degree of denominator.



APPLICATION OF LAPLACE TRANSFORM

Case I : (In Differential Equation) : With the help of Laplace Transform technique we can directly calculate the particular solution of differential equation without finding its general solution. Let y = y(t) is the solution of differential equation and $\bar{y}$ = y(s) be its Laplace Transform, i.e., $\boxed{L\{y\} = \bar{y}}$ then we can use following results :

(i) $L\{y'\} = s\bar{y} - y(0)$ (ii) $L\{y'\} = s^2\bar{y} - sy(0) - y'(0)$

(iii) $L\{y'''\} = s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)$ and so on ....

By using above results, first we find the Laplace transform of solution y(t) in terms of $\bar{y}$ and then by taking Inverse Laplace Transform we can evaluate y(t), which will be the solution of given differential equation.

Case II : (In the Evaluation of Real Integrals) : Suppose we have to evaluate $\int_0^\infty f(t)dt$ then process will be as follows:

Step 1 : First find the Laplace transform of f(t) and write it as L{f(t)} = f(s).

Step 2 : Now use the main definition of Laplace Transform in LHS only, i.e., $\int_0^\infty e^{-st} f(t)dt = f(s)$.

Step 3 : Taking limits both sides when $s \to 0$, i.e., $\int_0^\infty f(t)dt = f(0)$. $\because$ LHS is the question, so RHS will be the answer.

6.2 — FOURIER SERIES

PERIODIC FUNCTION

f(x) is said to be periodic if $f(x+T) = f(x)$, and T is called its period.

e.g. : sinx, cosx are periodic functions with period 2π.

(Some useful results) :

1. If Dashes denotes differentiation and suffixes denotes integration with respect to x, then

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots \text{ where u and v are functions of x.}$$

2. If f(x) is defined in $(-l, l)$ then $\int_{-l}^{l} f(x)dx = \begin{cases} 2\int_0^l f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$

3. If f(x) is defined in $(0, 2l)$ then $\int_0^{2l} f(x)dx = \begin{cases} 2\int_0^l f(x)dx, & \text{if } f(2l-x) = f(x) \\ 0, & \text{if } f(2l-x) = -f(x) \end{cases}$

4. For even function $f(-x) = f(x)$ and for an odd function $f(-x) = -f(x)$.

5. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

6. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin x]$

FOURIER SERIES : (MAIN DEFINITION)

Let f(x) be periodic function with period $2l$, then in the interval $(k, k+2l)$, Fourier series expansion of f(x) is given as,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

where, $a_0 = \frac{1}{l}\int_k^{k+2l} f(x)dx$, $a_n = \frac{1}{l}\int_k^{k+2l} f(x)\cos\frac{n\pi x}{l}dx$, and $b_n = \frac{1}{l}\int_k^{k+2l} f(x)\sin\frac{n\pi x}{l}dx$

where a_0, a_n, b_n are called Fourier coefficients.

Case-1: Put K = 0 : Then fourier series is expansion of f(x) in (0, 2l) is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$, $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$

Case-2: Put k = $-l$. Then Fourier series expansion of $f(x)$ in $(-l, l)$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$, $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

Case 3 : Euler's Formula

(Put $l = \pi$ in case II), then the Fourier series of f(x) in the interval $(-\pi, \pi)$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

DIRICHLET'S CONDITIONS (NECESSARY CONDITIONS OF F-SERIES)

1. f(x) is periodic, single valued and finite
2. f(x) has finite number of discontunities in any one period
3. f(x) has atmost finite number of maxima and minima
4. f(x) must be absolutely integrable over one period.

Note :

1. At point of discontinuity x_0, the Fourier series of f(x) converges to the Arithmetic mean of LHL and RHL of f(x) at x_0 i.e. $f(x_0) = \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$
2. If Fourier Series is definied in $[-\pi, \pi]$ then at corner point $\pm\pi$, Fourier Series converges at f(x)

   where $f(x) = \dfrac{f(-\pi^+) + f(\pi^-)}{2}$
3. If function is Even in the interval $(-l, l)$ then $b_n = 0$
4. If function is Odd in the interval $(-l, l)$ then $a_n = 0$
5. For the questions related to Half Range Fourier Sine and Cosine series, always compare given interval by $(0, l)$.

Half Range Fourier Sine Series

If we are required to obtain Fourier sine series in $(0,l)$ then take mirror image of f(x) about origin so that f(x) becomes odd in (–l, l) and in that case $a_0 = a_n = 0$ and half range Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Half Range Fourier Cosine Series

If we are required to obtained Fourier cosine series in $(0, l)$, then take the mirror image about y axis so that f(x) becomes even in (–l, l) and in that case $b_n = 0$ so half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{where } a_0 = \frac{2}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x)dx$$

UNIT 7 : LINEAR ALGEBRA

7.1 — ALGEBRA OF MATRICES

DEFINITION OF MATRIX

A set of m n numbers arranged in a rectangular array is called a matrix of order $m \times n$ matrix.

For example,
$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m_1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n};\ 1 \le i \le m, 1 \le j \le n$$

The vertical lines are column vectors, and the horizontal lines are row vectors.

INVERSE OF A MATRIX

Let A be a square matrix of order n such that A.B = B.A. = I, then

B is called the inverse of A and denoted by A^{-1} and is defined as $A^{-1} = \frac{\text{adj}\,A}{|A|}$. Therefore A^{-1} exists only if $|A| \neq 0$.

CONJUGATE OF A MATRIX

The matrix which is obtained from A by replacing each element by its conjugate is called the conjugate of A

CONJUGATE TRANSPOSE

The conjugate transpose of A is denoted by A^*, and is defined as $A^* = (\bar{A})'$

TYPES OF MATRICES

1. **Row Matrix:** A matrix having only one row is called a row matrix, e.g., $[a_{11}, a_{12}, \ldots a_{1n}]_{1\times n}$ is a row matrix.
2. **Column Matrix:** A matrix having only one column is called a column matrix e.g. $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$ is a column matrix.
3. **Null Matrix :** Matrix in which all the elements are zero is Null Matrix.
4. **Square Matrix:** Matrix in which the number of rows equal to the number of columns is square matrix.
5. **Diagonal Matrix:** $A = (a_{ij})_{n\times n}$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$; and at least one diagonal element should be non-zero.
6. **Scalar Matrix:** If all the elements of a diagonal matrix are equal, then the matrix is scalar.
7. **Identity Matrix:** If all the diagonal elements are unity then matrix is called Unit Matrix.
8. **Upper Triangular Matrix:** All the elements below the principal diagonal are zero.
9. **Lower Triangular Matrix :** All the elements above the principal diagonal are zero.
10. **Sub matrix :** A matrix obtained by deleting some rows or columns or both is called a sub matrix.
11. **Idempotent Matrix :** A is said to be idempotent if $A^2 = A$.
12. **Involutory Matrix :** A is said to be involutory if $A^2 = I$.

13. Nilpotent Matrix : Matrix A is called the nilpotent matrix of index k, if $A^k = 0$, k is least +ve integer.

14. Orthogonal Matrix : A is called an orthogonal matrix if $AA' = I$

Note : The value of determinant of an orthogonal matrix is either 1 or –1.

15. Unitary Matrix : Matrix A is said to be unitary matrix if $A^*.A = I$

16. Symmetric Matrix : Matrix is said to be symmetric if $A = A'$.

17. Skew-Symmetric Matrix : Matrix is said to be skew-symmetric if $A = -A'$

Note : 1. If $A_{n \times n}$ such that A is skew symmetric and n is odd then $|A| = 0$ (always).

2. Symmetric matrix is symmtrical about leading diagonal.

18. Hermitian Matrix : Matrix A is called Hermitian if $A = A^*$.

19. Skew-Hermitian Matrix : Matrix A is said to be **skew-Hermitian** if $A^* = -A$.

Note : The diagonal elements of a Hermitian matrix are necessarily real.

20. Trace of Matrix: Sum of the diagonal elements is called **trace**.

21. Real Matrix : If all the elements are real numbers then matrix is called Real Matrix and its condition is $\overline{A} = A$.

7.2 — RANK OF MATRICES

The rank of a matrix is the order of highest order non-singular minor of the matrix and it is denoted by $\rho(A)$

OR

A number r is said to be a rank of matrix $A_{m \times n}$ if,

(i) there exist at least one non-singular submatrix of order $r \times r$ in the given matrix.

(ii) All the square submatrices of higher orders should be singular.

ELEMENTARY OPERATION

Following three operations are called elementary transformation.

(a) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(b) $R_i \to kR_i$ or $C_i \to kC_i$

(c) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

EQUIVALENT MATRICES

A and B are said to be equivalent, if B can be obtained from A by applying elementary transformation.

OR

Matrix obtained by applying finite chain of E-operations are equivalent to each other.

ELEMENTARY MATRICES

A matrix obtained from unit matrix by applying single elementary transformations is called an elementary matrix.

ECHELON FORM / TRIANGULAR FORM

A matrix $A_{m \times n}$ is said to be in Echelon form if

(i) Number of zeros before the 1st non-zero element in a row should be in an increasing order in the subsequent rows.

(ii) All zero rows (if exist) should occur at the bottom of matrix.

Flow Chart to Convert a Matrix in an Echelon Form

1. Make a_{11} unity
2. Make all the elements of C_1 zero using E-row transformation only
3. Make a_{22} unity
4. Make all the elements of C_2 zero that lies below a_{22} again by using E-Row operations only.
5. Repeat the whole process until we get a matrix in an echelon form.

PROPERTIES OF RANK

1. If A_{mxn} then $\rho(A) \le \min(m,n)$ e.g. if $\rho(A_{5\times7}) = 4$ and $\rho(B_{7\times6}) = 3$ then $\rho(AB)_{5\times6} \le 3$.
2. If A is non singular matrix of order n then $\rho(A) = n$
3. $\rho(A) = \rho(A^{\theta}) = \rho(A^{T}) = \rho(A^{-1}) = \rho(AA^{T}) = \rho(AA^{\theta})$
4. Rank of a null matrix is not defined, i.e., if $\rho(\text{Null Matrix})$ = Not defined.
5. Elementary transformation do not alter the rank of a matrix i.e. equivalent matrices have same rank.
6. The rank of a product cannot exceed the rank of either matrix, i.e., $\rho(AB) \le \{\rho(A), \rho(B)\}$
7. $\rho(A+B) \ne \rho(A) + \rho(B)$
8. If $\rho(A) = r$ and B is any non-singular matrix, i.e., $|B| \ne 0$ then $\rho(AB) = \rho(BA) = r$ i.e., rank of matrix does not alter by pre-multiplication or post-multiplication by any non-singular matrix.
9. Rank of a matrix in echelon form is equal to no. of non zero rows.
10. Nullity of matrix in Echelon form = Number of zero rows = Order-Rank.

7.3 — SYSTEM OF EQUATIONS

Ordered n-tuple: A set of n numbers $(x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n)$ in which position of each number is fixed is called ordered n-tuple. It is also called n-vector. Generally it is represented in the form of Column Matrix.

LINEAR COMBINATION OF VECTORS

Let $X_1, X_2, X_3, \ldots\ldots\ldots, X_r$ are the n-dimensional vectors and let $K_1, K_2, K_3, \ldots\ldots, K_r$ are the scalars then the relationship of the type

$$K_1X_1 + K_2X_2 + K_3X_3 + \ldots\ldots\ldots + K_rX_r = O \qquad \ldots(1)$$

is called linear combination of vectors.

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

1. Given set of vectors $X_1, X_2, X_3, \ldots\ldots, X_r$ are called linearly dependent if one of them can be expressed as linear combination of others OR

   there exist scalars $K_1, K_2, K_3 \ldots\ldots, K_r$ (not all zero simultaneously) such that linear combination given by (1) exist between the vectors.

2. Given set of vectors are called linearly independent if it is not possible to express them as a linear combination of each other OR

Linear combination exist only when all the scalars K_1, K_2, K_3, ..., K_r should be zero simultaneously.

METHODS TO FIND L I AND L D VECTORS

Let x_1, x_2, x_3, ... x_r are the given vectors and all are of n dimensions then consider

$$A = [X_1, X_2, X_3, \ldots\ldots\ldots X_r]$$

General Method:

(i) If $\rho(A)$ = No. of vectors then vectors are LI. (ii) If $\rho(A)$ < No. of vectors then vectors are LD.

Tricky Method:

Let A be a square matrix

(a) If $|A| \neq 0$ then vectors are LI.

(b) If $|A| = 0$ then vectors are LD.

PROPERTIES OF VECTORS

1. **Dot Product :** If $X_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$ and $X_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$ then dot product is defined as

$$X_1.X_2 = a_1b_1 + a_2b_2 + a_3b_3 + \ldots.. + a_nb_n.$$

2. **Norm of Vector :** It is represented as $||X_1||$ and is defined as $||X_1|| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots.. + a_n^2}$.

   In case of 2D or 3D vectors norm is also termed as length.

3. **Normalised Vector :** A vector whose norm is 1 is called Normalised Vector, i.e., if $||X_1|| = 1$ then X_1 is called Normalised Vector.

ORTHOGONAL AND ORTHONORMAL VECTORS

1. **Orthogonal Vectors :** Vectors X_1 and X_2 are called orthogonal if $X_1.X_2 = 0$.

2. **Orthonormal Vectors :** Vectors X_1 and X_2 are called orthonormal if (i) they should be orthogonal, (ii) each vector should be of unit norm.

NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

If in a system of equations AX = B, B is not a null matrix then system is called Non-Homogeneous System and in matrix form it is represented as follows.

Consider a system of linear equations as :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n &= b_m \end{aligned}$$

In matrix form this system can be represented as

$$\boxed{AX = B}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = X = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and $[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

where $[A : B]$ = Augmented Matrix

METHODS OF SOLVING NON-HOMOGENEOUS SYSTEM: AX = B WHERE B $\neq$ O

1. **Rank Method :** (always applicable)
   (i) If $\rho(A) \neq \rho[A:B]$ then system is incositent. (No solution)
   (ii) If $\rho(A) = \rho[A:B]$ = No. of unknows then; system has unique solution.
   (iii) If $\rho(A) = \rho[A:B]$ < No. of unknows; then system has infinite solution.
2. **Matrix Method :** (applicable only when A is a square matrix)
   (i) If $|A| \neq 0$, system has unique solution.
   (ii) If |A| = 0, and (adj. A)B = 0, system has infinite solutions.
   (iii) If |A| = 0 and (adj. A)B $\neq$ 0, system has no solution.

Note : 1. In the system of equations AX = B, X is called solution of system and it is in the form of column matrix.

2. If the system has one or more solutions then system is said to be **consistent**, and if it has no solution then it is said to be **inconsistent.**

3. Necessary condition for a system to be consistent is $\rho(A) = \rho(A:B)$.

HOMOGENEOUS SYSTEM :

The system AX = B is called Homogeneous if B is a null matrix, i.e., B = 0. Hence, standard notation of Homogeneous system is $\boxed{AX = 0}$.

In this system $\rho(A) = \rho(A:0)$ always. Hence, this system is always consistent and there is no need to write Augmented Matrix further.

METHODS OF SOLVING HOMOGENEOUS SYSTEM AX = O

1. **Rank Method (always applicable)**
   (i) If $\rho(A)$ = No. of unknowns; then system has unique sol. or Trivial solution.
   (ii) If $\rho(A)$ < No. of unknowns; then system has infinite solution i.e. Non trivial sol. exist.
2. **Matrix Method (applicable only when A is square matrix)**
   (i) If |A| $\neq$ 0 then system is consistent with unique sol.
   (ii) If |A| = 0 then system is consistent with infinite sol.

Note : 1. If we have homogeneous system AX = 0 in three variables x, y and z then (x, y, z) = (0, 0, 0) is always the solution of given system. This type of solution is called **Trivial Solution** or zero solution.

2. If there is a possibility of the existence of non-zero solution also then we say that **non-trivial solutions** (or infinite solutions) exist.

3. Homogeneous system is never inconsistent because (0, 0, 0) is always the solution the system.

LU DECOMPOSITION (FACTORIZATION) METHOD

Let A be any square matrix and we want to decompose it into the product of lower and upper triangular matrices then we have following two methods :

(i) **Dolittle Method :** In this method, we write

$$A = (\text{Unit LTM})(\text{U.T.M.}) \text{ or } A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

After comparing both sides, we can easily evaluate various values of l_{ij} and u_{ij} and hence L & U.

(ii) **CROUT Method :** In this method, we write

$$A = (\text{LTM})\ (\text{Unit UTM}) \text{ or } A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Again after comparison, we can evaluate L & U.

7.4 — EIGEN VALUES AND EIGEN VECTORS

DEFINITION

Consider the square matrix $A_{n \times n}$ then non-zero vector X is called Eigen vector of A corresponding to Eigen Value λ (real or complex) if there exist a relationship of the type

$$AX = \lambda X$$

i.e., Product of two matrices = Scalar multiplication in matrix.

Here, we are considering homogeneous system as follows :

$$AX - \lambda X = O \text{ or } \boxed{(A - \lambda I)X = O} \quad \text{...(1)}$$

where $A - \lambda I$ is known as characteristic matrix of A,

$|A - \lambda I|$ is known as characteristic polynomial of A,

and $|A - \lambda I| = 0$ is known as characteristic equation of A.

Method of Evaluating Eigen Values : Consider the equation

$$|A - \lambda I| = 0 \quad \text{...(2)}$$

This is known as **characteristic equation** of A and roots of this equation, i.e., values of 'λ' are called Eigen values. Hence, in order to find Eigen values of A we should try to solve characteristic equation in terms of λ.

PROPERTIES OF EIGEN VALUES

1. Eigen Values ≃ Eigen Roots ≃ Characteristic Values ≃ Characteritic Roots ≃ Latent Roots, i.e., all are same.
2. (i) Eigen values of Nilpotent Matrix is only zero.
   (ii) Eigen values of Idempotent matrix are 0 and 1.
   (iii) Eigen values of Involutary Matrix are 1 and –1.

(iv) Eigen values of Real symmetric matrix are all real numbers.

(v) Eigen values of Hermitian matrix are all real numbers.

(vi) Eigen values of Real skew symmetric are purely imaginary numbers.

(vii) Eigen values of skew hermitian matrix are purely imaginary numbers.

(viii) Eigen values of orthogonal matrix are of unit modulus.

(ix) Eigen values of unitary matrix are of unit modulus.

3. We can not apply E-operations in the given matrix, while calculating Eigen values, because in such situation Eigen values changes.

4. **Minimal Polynomial of A :** It is the subset of characteristic polynomial with the condition that each different factor should be taken at least once. Hence,

Minimum degree of Minimal Polynomial = Number of distinct Eigen values

and Maximum degree of minimal polynomial = Order of $A_{n \times n}$

5. The set of Eigen values of A is called spectrum of A and the largest Eigen value is called the **Spectral Radius** of A.

6. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n-eigen values of square matrix A, then

(i) Eigen values of A^T are same, i.e., $\lambda_1\ \lambda_2\ \lambda_3\ \ldots\ldots\ \lambda_n$

(ii) Eigen value of A^{-1} are $\frac{1}{\lambda}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ (Provided $|A| \neq 0$)

(iii) Eigen values of A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots \lambda_n^m$

(iv) Eigen values of adj. A are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \ldots\ldots, \frac{|A|}{\lambda_m}$

(v) $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n = |A|$, i.e., **product of Eigen Values = Determinant (A)**

(vi) $\lambda_1 + \lambda_2 + \lambda_3 \ldots \lambda_n$ = Trace (A), i.e., **Sum of Eigen Values = Trace (A)**

7. Zero is an Eigen Value of A iff Matrix is singular, i.e., if $\lambda = 0 \Leftrightarrow |A| = 0$

8. Eigen Values of U.T.M., L.T.M., Diagonal Matrix, Scalar Matrix are just the diagonal elements.

PROPERTIES OF EIGEN VECTORS

1. Consider $(A - \lambda I)X = 0$ then we can also write $(A - \lambda I)(KX) = 0$. Hence, if X is non-zero Eigen Vector, so will (KX) also and vice versa, i.e., we are free to multiply or divide with any constant in case of Eigen Vectors.
2. A matrix may have infinite number of Eigen Vectors but number of linearly independent Eigen Vectors are finite and rest are Linearly Dependent (LD) on them.
3. If all the Eigen Values are different, corresponding Eigen Vectors are also L.I.
4. If Eigen Values repeats then corresponding Eigen vectors may or may not be LI.
5. Eigen Vectors corresponding to different Eigen Values of real symmetric matrix are mutually orthgonal, i.e., if $A_{n \times n}$ is any symmetric matrix and λ_1, λ_2 are its Eigen values and corresponding Eigen vectors are X_1 and X_2 then for $\lambda_1 \neq \lambda_2$, $X_1.X_2 = 0$.

6. If λ and X are the eigen value and E-vector of any matrix A then to find Eigen vaues of any polynomial of A, we can replace A by λ in that polynomial and in such situation Eigen vector remains unaltered.
7. **Algebraic Multiplicity :** Number of times a particular Eigen value repeats is called its AM.
8. **Geometric Multiplicity :** Number of distinct LI eigen vectors for a particular eigen value is known as its GM.

SIMILAR MATRICES

Two matrices A and B are said to be similar if there exist an invertible matrix P such that $B = P^{-1}AP$

Note : Similar matrices have same trace, same determinant, same characteristic equations and same eigen values.

DIAGONALISATION (SIMILARITY TRANSFORMATION)

A matrix A is said to be diagonaligable if it is similar to a diagonal matrix, i.e.,
There exist an invertible matrix P such that we have a relationship of the type.

$$\boxed{P^{-1}AP = \text{Diagonal Matrix}}$$

where, Diagonal Matrix = D = $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and P is **Modal matrix** defined as $P = [X_1 X_2 X_3]$

Here, $\lambda_1, \lambda_2, \lambda_3$ are the Eigen Values of A and x_1, x_2, x_3 are Eigen Vectors.

Necessary conditions for Diagonalisation : $|A| \neq 0$, i.e., matrix should be non-singular.

Sufficient condition for Diagonalisation : No. of LI Eigen Vectors = Order of A.

CAYLEY-HAMILTON THEOREM

"Every square matrix satisfies its own characteristics equation".

Consider $A_{n\times n}$, then its char. Equn. is; $|A - \lambda I| = 0$. It will be n^{th} degree polynomial in λ as follows :

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots\dots + a_{n-1}\lambda + a_n = 0$$

Replace $\lambda \to A$, we get

$$\boxed{a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots\dots + a_{n-1}A + a_nI = 0} \quad \dots(1)$$

i.e. we can replace λ by A in Characteristic Equation.

Application of Cayley Hamilton Theorem :

1. Trace (A) = $-\dfrac{a_1}{a_0} = -\left[\dfrac{\text{Coefficient of } A^{n-1}}{\text{Coefficient of } A^n}\right]$

2. $|A| = (-1)^n \dfrac{a_n}{a_0} = (-1)^n \left[\dfrac{\text{Constant term}}{\text{Coefficient of } A^n}\right]$

3. Also, we can find A^{-1} by premultiplication in equation (1).

Note : If $A_{2\times 2}$ then its characteristic equation is $|A - \lambda I| = 0$ or $a_0\lambda^2 + a_1\lambda + a_2 = 0$

By Cayley Hamilton theorem we can replace $\lambda \to A$

i.e., $$a_0A^2 + a_1A + a_2I = 0 \Rightarrow A^2 + \left(\frac{a_1}{a_0}\right)A + \left(\frac{a_2}{a_0}\right)I = O$$

or $$\boxed{A^2 - (\text{Trace } A)A + (\text{Det } A)I = O}$$

UNIT 8 : PROBABILITY & STATISTICS

8.1 — PROBABILITY

STANDARD DEFINITIONS

Random Experiment : Those experiment for which outcome cannot be predicted is said to be random experiment.

Example : Throwing a dice is a random experiment because we can not predict which no will come.

Again, when a coin is tossed we are not sure which one of from H & T will actually be obtained.

Sample Space : The set of all possible outcomes of an experiment is called the sample space.

e.g. If a dice is thrown twice then

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix} \text{ and } n(S) = 36$$

Event : Any subset of sample space is called event

and total number of events associated with sample space S having n elements = 2^n.

E.g. if a coin is tossed twice then S = {(H,T), (H,H), (T,H), (T,T)}

and total number of events in this = 2^4 = 16 in which ϕ and S are two special events known as impossible and sure event.

Impossible Event :

Since null set is the subset of every set, so it can also be considered as an event. This event is called impossible event, denoted by ϕ and $P(\phi) = 0$.

e.g. : The appearance of number 8 in throwing a dice is an impossible event.

Sure Event :

Every set is the subset of itself, so it is also an event. This is called sure event denoted by S and P(S) = 1.

e.g. The appearance of any number in throwing a dice is a sure event.

Equally Likely Events :

Those events, for which probability of happening is same are called equally likely events.

e.g., if a dice is thrown once then S = {1, 2, 3, 4, 5, 6}.

Let $E_1 = \{1, 3, 5\}$ and $E_2 = \{2, 4, 6\}$. Then $P(E_1) = P(E_2) = \frac{1}{2}$. Hence, E_1 and E_2 are equally likely events.

Mutually Exclusive Events :

Two events are said to be mutually exclusive if they cannot occur together, i.e., occurrence of one event prevents the occurrence of other events then events are known as mutually exclusive events.

if A and B are mutually exclusive events then,

$$\boxed{A \cap B = \phi \text{ and } P(A \cap B) = 0 \text{ and } P(A \cup B) = P(A) + P(B)}$$

In general, for mutually exclusive events $A_1, A_2, A_3, \ldots\ldots A_n$

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n).$$

e.g. : If a dice is thrown once and let E_1 = {odd number} = {1, 3, 5} and E_2 = {even number} = {2, 4, 6}.

Since, $E_1 \cap E_2 = \phi$. So, E_1 and E_2 are mutually exclusive.

Exhaustive Events

If union of some events is equal to sample space then events are termed as exhaustive events.

If $E_1, E_2, \ldots, E_n$ are some events associated with sample space S such that

$$\boxed{E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S}$$

then $E_1, E_2, \ldots, E_n$ are called ***exhaustive events.***

Mutually Exclusive and Exhaustive Events

Let $E_1, E_2, E_3, \ldots, E_n$ are events associated with sample space S such that

$$\text{(i) } E_i \cap E_j = \phi \forall i \,\&\, j \text{ and (ii) } E_1 \cup E_2 \cup E_3 \ldots \cup E_n = S$$

then events are called mutually exclusive and exhaustive.

And in such situation $P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n) = 1$.

For example, If E_1, E_2 and E_3 are mutually exclusive and exhaustive events then $\boxed{P(E_1)+P(E_2)+P(E_3)=1}$.

Proof : $\because$ E_1, E_2, E_3 are exhaustive events so $E_1 \cup E_2 \cup E_3 = S$

$\Rightarrow$ $P(E_1 \cup E_2 \cup E_3) = P(S)$

i.e., $P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3) = 1$

Again, E_1, E_2, E_3 are mutually exclusive so $E_i \cap E_j = Q \Rightarrow P(E_i \cap E_j) = 0$

So $P(E_1) + P(E_2) + P(E_3) = 1$. Hence Proved.

PROBABILITY (DEFINITION)

If S be the sample space and E be any event associated with sample space S then

$$\boxed{P(E) = \frac{n(E)}{n(S)}} = \frac{\text{Number of elements in E}}{\text{Number of elements in S}} = \frac{\text{Favourable Cases}}{\text{Total Cases}}$$

ADDITION THEOREM OF PROBABILITY

(a) For two events : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) For three events : $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

INDEPENDENT EVENTS

Two events A and B are said to be independent, if occurrence or non occurrence of one event do not alter the occurrence or non occurrence of other event.

In case of independent events we can multiply their respective probabilities in order to find their combined probability, i.e., if A , B and C are three independent events then $P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)$

CONDITIONAL PROBABILITY

If A and B are two events such that B occurs only if A has already occured then P(B/A) is called the conditional probability of B under the condition that A has already occured and it is defined as

$$\boxed{P\left(B/A\right) = \frac{P(A \cap B)}{P(A)}}$$

Note : If A and B are independent events then

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) \cdot P(B)}{P(A)} = P(B) \text{ i.e., condition has no meaning.}$$

TOTAL PROBABILITY THEOREM

Let $E_1, E_2, E_3, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events associated with sample space S, and let A is an event which can occur with all $E_1, E_2, \ldots\ldots, E_n$ then

$$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A) \cup \ldots\ldots \cup (E_n \cap A)$$

$\because$ In case of mutually exclusive events, union splits into addition so we have

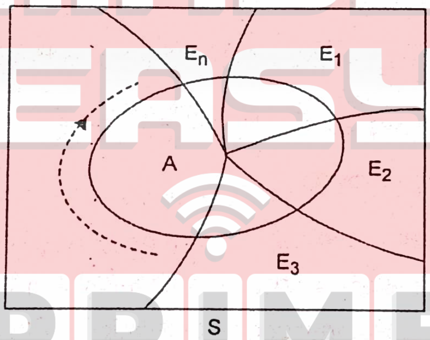
$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A) + \ldots\ldots + P(E_n \cap A)$$

Now using the basic definition of conditional probability, we can write

$$\boxed{P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \ldots\ldots + P(E_n)P(A/E_n)} \quad \ldots(1)$$

BAYE'S THEOREM (INVERSE PROBABILITY THEOREM)

Let $E_1, E_2, E_3, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events, and A is an event with can occur with all $E_1, E_2, E_3, \ldots\ldots, E_n$.



Now let us assume that A has been already occured then the probability that it is occuring due to the occurrence of E_i is given as

$$\boxed{P(E/A) = \frac{P(E_i)P(A/E_i)}{P(A)}}$$

where P(A) is given by equation (1) defined above.

Note : In Baye's theorem, assume A that event which is given as condition.

8.2 — Statistics-I (Probability Distributions)

RANDOM VARIABLE

Variable associated with Random Experiment is called Random Variable. It is categorised as follows :

Random Variable

Discrete Random Variable (X)	Continuous Random Variable (x)
(e.g. Number of students, number of vehicles)	(Height, Weight, Time)
↓	↓
Discrete Probability Distribution	Continuous Probability Distribution
(e.g. Binomial, Poisson)	(e.g. Normal, Uniform, Exponential)
↓	↓
Probability Mass Function (P.M.F.)	Probability Density Function (P.D.F.)
$p_i \geq 0$ and $\Sigma p_i = 1$	$f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$
↓	↓
1. Mean (X) = $E(X) = \Sigma P_i X_i$	1. $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)dx$
2. Var (X) = $E(X^2) - E^2(X)$	2. $Var(x) = E(x^2) - E^2(x)$
3. SD (σ) = $+\sqrt{Var(X)}$	3. SD (σ) = $+\sqrt{Var(x)}$

MEASURES OF CENTRAL TENDENCY

(i) **Mean/Central Value/Expected Value/Average Value :** It is the average of the probability distribution and is defined as

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \text{ or } E(X) = \frac{\sum_{i=1}^{n} p_i X_i}{\Sigma p_i} \approx \boxed{\sum p_i X_i}$$

(ii) **Median/Positional Average :** It is the middle most value of the distribution when values are arranged either in ascending order or in descending order. It divides the series into two equal parts.

(iii) **Mode :** In a set of observations, that value which is repeated maximum number of times (or that value of the distribution which has maximum frequency) is called Mode.

MEASURES OF DISPERSION (VARIANCE, SD AND COVARIANCE)

(i) **Variance :** It measures the spread of distribution about its central value, i.e., smaller the value of variance implies that individual values lies closer to average value.

"The average of squares of all the deviations from its central value is defined as variance", i.e.,

$$Var(X) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n} = E(X_i - \bar{X})^2 = E(X_i^2 + \bar{X}^2 - 2X_i\bar{X})$$

$$= E(X_i^2) + E(\bar{X}))^2 - 2E(X_i)\bar{X}$$

$$= E(X_i^2) + E(\bar{X})^2 - 2(\bar{X})(\bar{X}) = E(X^2) - (\bar{X})^2$$

$$\boxed{Var(X) = E(X^2) - E^2(X)}$$

(ii) **Standard Deviation :** It has same physical significance as that of variance and it is defined as

"Positive square root of variance is called S.D.", i.e., $\boxed{SD = +\sqrt{Var(X)}}$.

(iii) **Covariance :** It measures the simultaneous variation of two random variables x and y. For e.g., Variation in weight (x) with respect to height (y).

Mathematically it is defined as $\boxed{Cov(x,y) = E(xy) - E(x)E(y)}$.

If x and y are independent random variable then Cov(x, y) = 0.

Some Standard Results :

If a, b, c are constants and x, y are random variables then

1. $E(ax + by + c) = aE(x) \pm bE(y) \pm E(c) = aE(x) \pm bE(y) \pm C$
2. $Var(ax + b) = a^2 Var(x) + Var(b) = a^2 Var(x) + 0$
3. $Var(ax \pm by) = a^2 Var(x) + b^2 Var(y) \pm 2ab\, Cov(x, y)$
4. Var or SD $\propto \frac{1}{\text{Consistency}}$
5. Var or SD ≥ 0
6. $Cov(x, x) = Var(x)$
7. Coefficient of Variation $= \frac{\sigma}{\mu} = \frac{\text{S.D.}}{\text{Mean}}$
8. Standard Error of Mean $= \frac{\sigma}{\sqrt{N}}$ where N = sample size.

DISCRETE PROBABILITY DISTRIBUTION

The table representing the distribution of probabilities at various values of x is called probability distribution.

For example, A couple has 3 children and we want to find the probability distribution of number of boys then random variable X can be taken as

$$X = \{\text{Number of Boys}\} = \{0, 1, 2, 3\}$$

and sample space is S = {(BBB), (BBG), (BGB), (BGG), (GBB), (GBG), (GGB), (GGG)}

Hence probability distribution is

X :	0	1	2	3
P(X) :	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Now, by using standard results, we can find mean of X, variance of X, S.D. of X and so on.

BINOMIAL DISTRIBUTION

Necessary Conditions for Binomial Distribution :

1. No. of trials 'n' should be **finite**.
2. Trials should be **independent.**
3. Each trial results in only 2 possible outcomes known as **Success** and **Failure**.
4. The probability of success for each trial should be **constant**.

Definition : Let X is discrete random variable such that its probability mass function is defined as

$\boxed{P(X = r \text{ success}) = {}^nC_r p^r q^{n-r}}$ where q + p = 1 and p = P (success) and q = P (failure).

Then X is called Binomial Random Variable with parameters n and p and it is denoted as $X \sim B\{n,p\}$

Note :

1. Mean (X) = E(X) = $\Sigma p_i X_i$ = ... = np
2. Var (X) = $E(X^2) - E^2(X)$ = ...= npq
3. SD(σ) = $+\sqrt{npq}$
4. Representation of complete binomial distribution is $\boxed{\sum_{n=0}^{n} {}^nC_r p^r q^{n-r} = (q+p)^n}$ = 1^n = 1 as n is finite.

POISSON DISTRIBUTION

It is a particular case of Binomial distribution under following restricton :

1. $n \to \infty$ (Very large)   2. $p \to 0$ (very small)   3. $np \to$ Constant (λ)

Definition : Let X is discrete random variable such that its probability mass function is defined as

$$\boxed{P(X = r \text{ success}) = \frac{e^{-\lambda}\lambda^r}{r!}}$$

then X is called Poissons Random Variable with parameter λ and is denoted as $X \sim P(\lambda)$

Note :

1. Mean (X) = E(X) = $\Sigma p_i X_i$ = ... = λ
2. Var (X) = $E(X^2) - E^2(X)$ = ... = λ
3. SD (σ) = $\sqrt{\lambda}$
4. Any question based on binomial can also be solved by poisson but converse is not necessarily true.
5. Here λ is average per unit time.

CONTINUOUS PROBABILITY DISTRIBUTION

Let x is continuous random variable and f(x) is its probability density function then following results are valid :

1. f(x) = probability at 'x'.
2. $P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$ = Total area under f(x)
3. $P(a < x < b) = \int_a^b f(x)dx$ = Total area under f(x) between x = a to x = b.
4. $E\{g(x)\} = \int_{-\infty}^{\infty} g(x)f(x)dx =$
$$\begin{cases} E(x) = \int_{-\infty}^{\infty} xf(x)dx = \text{Mean }(x) \\ E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \text{2nd Moment} \\ E(|x|) = \int_{-\infty}^{\infty} |x| f(x)dx \end{cases}$$
5. $Var(x) = E(x^2) - E^2(x)$
6. SD(σ) = $+\sqrt{Var(X)}$

CUMULATIVE DENSITY FUNCTION (C.D.F.)

Let x is continuous random variable, f(x) is its probability density function then its c.d.f. is denoted as F(x) and is defined as

$$F(x) = \int_{-\infty}^{\infty} f(x)dx \quad \begin{cases} F(-\infty) = \int_{-\infty}^{-\infty} f(x)dx = 0 \\ F(\infty) = \int_{-\infty}^{\infty} f(x)dx = 1 \end{cases}$$

In general, $F(a) = \int_{-\infty}^{a} f(x)dx$ = Sum of all probabilities from starting point till that point 'a' is called c.d.f. at 'a'.

EXPONENTIAL DISTRIBUTION

Definition : Let t be any continuous random variable such that its probability density function is defined as

$$\boxed{f(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$

then t is called Exponential Random Variable with parameter 'μ' and it is denoted as $t \sim E(\mu)$.

Note :

1. **Cross Check :** $\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{\infty} f(t)dt = 0 + \int_{0}^{\infty} \mu e^{-\mu t}dt = 0 + \mu\left(\frac{e^{-\mu t}}{-\mu}\right)_0^{\infty} dt = 0 + -\left[e^{-\infty} - 1\right] = 1$

   Hence, f(t) is p.d.f. for 't'.

2. Mean (t) = $E(t) = \int_{-\infty}^{\infty} tf(t)dt$ = ... = $\frac{1}{\mu}$ ≃ Average waiting time

3. Var (t) = $E(t^2) - E^2(t)$ = ... = $\frac{1}{\mu^2}$

4. SD = $\frac{1}{\mu}$

5. $\mu \rightarrow$ Service Rate per unit time, i.e., Service Rate $\propto \frac{1}{\text{Average waiting time}}$

6. $P(t_1 < t < t_2) = \int_{t_1}^{t_2} f(t)dt$ = Probability of waiting time lies between t_1 and t_2.

7. Traffic intensity in queue $(\rho) = \frac{\text{Arrival Rate } (\lambda)}{\text{Service Rate } (\mu)}$

UNIFORM DISTRIBUTION

Let x be continuous random variable in (a, b) such that its p.d.f. is defined as

$$\boxed{f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}}$$

then x is called uniform random variable between a and b.

Note :

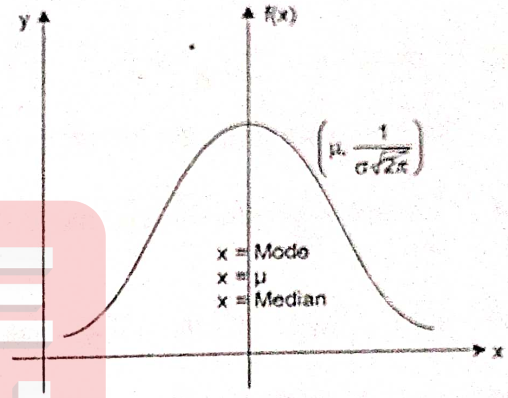
1. **Cross check :** $\because \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx = 0 + \int_{a}^{b}\left(\frac{1}{b-a}\right)dx + 0 = \frac{b-a}{b-a} = 1$

2. Mean (x) = $E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_{a}^{b} x\left(\frac{1}{b-a}\right)dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{b^2-a^2}{(b-a)\cdot 2} = \frac{a+b}{2}$

3. Var (x) = $E(x^2) - E^2(x) = \int_{-\infty}^{\infty} x^2 f(x)dx + \left(\frac{a+b}{2}\right)^2 = \int_{a}^{b} x^2\left(\frac{1}{b-a}\right)dx + \frac{(a+b)^2}{2} = \frac{b^3-a^3}{3(b-a)} + \frac{(a+b)^2}{2} = \frac{(b-a)^2}{12}$

4. S.D.(σ) = $+\sqrt{\text{Var}(x)} = \frac{b-a}{\sqrt{12}}$

5. $P(\alpha < x < \beta) = \int_{\alpha}^{\beta} f(x)dx$ = Area under f(x) from x = α to β

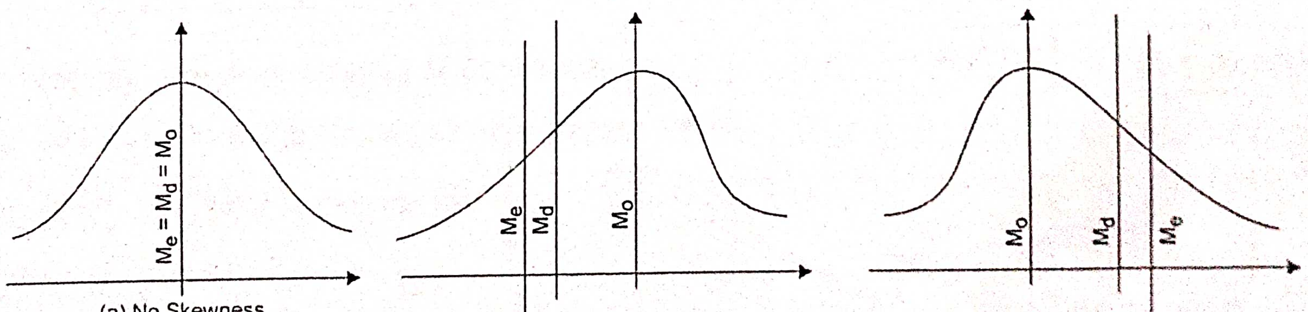


NORMAL / GAUSSIAN DISTRIBUTION

Whenever data has a tendency to accumulate about its central value then such types of distributions are called normal distribution and variable involved in, it is called as Normal Random Variable/Gaussian Variable. e.g. height, weight, marks, distribution, etc. are normal variables.

Note :

1. Normal curve is a symmetrical curve with symmetry about its mean (average value).
2. Normal curve is a unimodal curve with highest point occurs at average value μ.
3. Unimodal curve is that curve in which mean = median = mode.
4. Highest value of normal curve = $f(\mu) = \frac{1}{\sigma\sqrt{2\pi}}$ and it occurs at x = μ.
5. Normal curve is a Bell shaped curve where variance is represented by width of Bell.
6. Skewness in normal curve :



(a) No Skewness (b) Left Skew (c) Right Skew

(a) No skewness means graph is symmetrical about mean (i.e., $M_e = M_d = M_o$)

(b) Left skew (negatively skewed) when tail occuring in LHS ($M_e < M_d < M_o$)

(c) Right skew (positively skewed) when tail occuring in RHS ($M_e > M_d > M_o$)

Definition of Normal Random Variable :

Let x is continuous random variable such that its probability density function is defined as

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}$$

then x is called as Normal random variable with parameter μ and σ^2 and it is denoted as $\boxed{x \sim N\{\mu, \sigma^2\}}$

Note :

1. Mean (x) = E(x) = $\int_{-\infty}^{\infty} x f(x) dx = \ldots\ldots\ldots\ldots\ldots \mu$

2. Var (x) = $E(x^2) - E^2(x) = \ldots\ldots\ldots\ldots \sigma^2$

3. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx = \ldots\ldots\ldots\ldots\ldots\ldots = 1$. Hence verified.

   So, total area under the normal curve = 1 unit.

4. $\int_{-\infty}^{\mu} f(x)dx = \int_{\mu}^{\infty} f(x)dx = \frac{1}{2}$, i.e., half area under the normal curve is of 0.5 units.

5. $P(a \le x \le b) = \int_a^b f(x)dx$ = Area under f(x) from a to b.

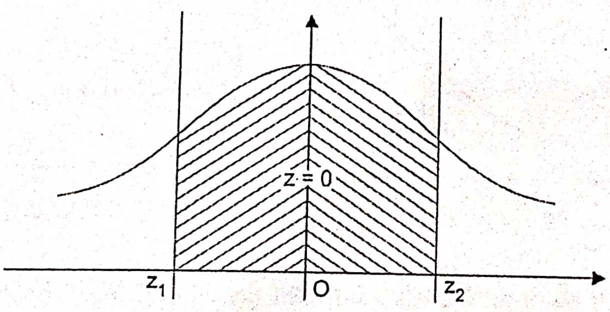
Standard Normal Variable

If we take $\mu = 0$, $\sigma = 1$ and $\boxed{z = \frac{x-\mu}{\sigma}}$ then z is called standard normal variable and its probability density function (p.d.f.) is defined as $\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}$ i.e., $Z \sim N\{0, 1\}$.

Note :

1. Standard Normal Variable is **free** from any parameter.

2. Mean (z) = E(z) = $\int_{-\infty}^{\infty} z \cdot f(z)dz = \ldots.. = 0$

3. 

   $P(a \le x \le b) = P(z_1 \le z \le z_2) = \int_{z_1}^{z_2} f(z)dz$ = area between z_1 to z_2.

4. Normal table starts from z = 0 and is defined only for positive values of z.

Flow Chart of Solving Questions :

Step 1 : 1st find X. (It is assumed as, which is required).

Step 2 : Convert X into Z using the transformation $\boxed{Z_x = \frac{z-\mu_x}{\sigma_x}}$

Step 3 : Use concept of symmetry.

Step 4 : Use normal table.

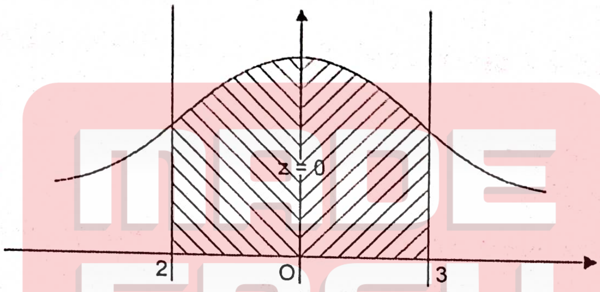
Concept of Symmetry in Normal Distribution :

1. $P(-\infty < z < \infty) = 1$ (Total Area)

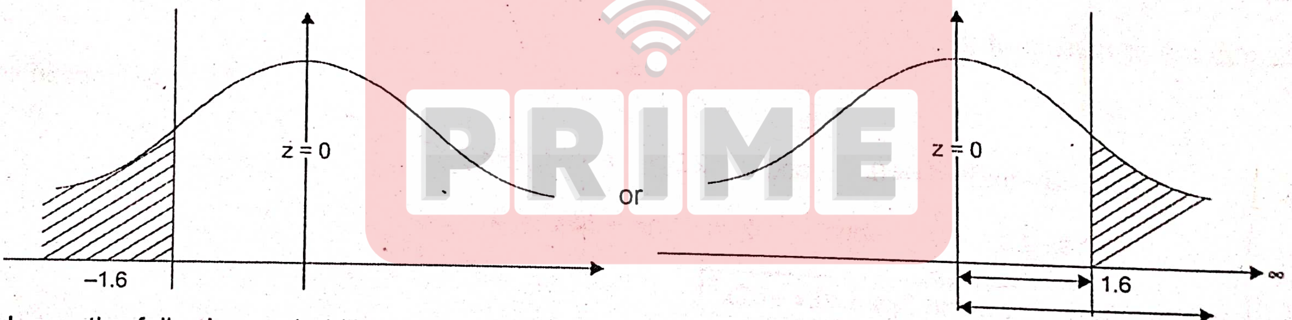
2. $P(-\infty < z \le 0) = P(0 \le z < \infty) = \frac{1}{2}$ i.e., (half area)

3. $P(-1 \le z \le 1) = 2P(0 \le z \le 1) = 2$ (Use N. Table) $= 2 \times (0.3413)$

4. $P(-2 \le z \le 3) = P(-2 \le z \le 0) + P(0 \le z \le 3) = P(0 \le z \le 2) + P(0 \le z \le 3) = \frac{0.95}{2} + \frac{0.997}{2}$ (Using N. Table)



5. $P(-\infty < z \le -1.6) = P(+1.6 \le z \le \infty) = \frac{1}{2} - P(0 \le z \le 1.6) = 0.5 - (0.4452)$ (Using N Table)



Note : Learn the following probabilities as Standard Results :

1. $P(\mu - \sigma \le x \le \mu + \sigma) = P(\mu - \sigma \le x \le \mu + \sigma) = 0.6826 \approx 68.26\%$

2. $P(\mu - 2\sigma \le x \le \mu + 2\sigma) = 0.955 \approx 95.5\%$

3. $P(\mu - 3\sigma \le x \le \mu + 3\sigma) = 0.997 \approx 99.7\%$

e.g. Evaluate $P(0 \le z \le 1) = ?$ where notations have their usual meaning

We know that, $P(\mu - \sigma \le x \le \mu + \sigma) = 0.6826$

At $\mu = 0$ and $\sigma = 1$, $x \approx z$, so we can write

$$P(-1 \le z \le 1) = 0.6826$$

$$2P(0 \le z \le 1) = 0.6826$$

i.e., $$P(0 \le z \le 1) = 0.3413$$

8.3 — STATISTICS : CORRELATION & REGRESSION

CURVE FITTING

Consider a set of numerical values as follows :

x =	a_1	a_2	a_3	a_4	....	a_n
y =	$f(a_1)$	$f(a_2)$	$f(a_3)$	$f(a_4)$	....	$f(a_n)$

If we want to find a curve in the form of y = f(x) then this curve is known as curve of best fit and the process of finding this curve is known as curve fitting.

Hence, "To find a relationship between two variables in the form of Algebraic function is called curve-fitting."

METHODS OF LEAST SQUARES

With the help of this method we can obtain constants, which occurs in the process of estimating curve in terms of y = f(x). Method will be cleared in the following examples :

Fitting of a straight line:

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the n set of observations and we have to find the line of best fit for this data. Let it be

$$y = a + bx \qquad \ldots(1)$$

then error at random point x_i is;

$$\text{error} = \text{exact value} - \text{approx value} \Rightarrow E_i = y_i - f(x_i) = y_i - (a + bx_i)$$

Now we will try to find the least value of square of this error.

Let
$$u = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

Now for maxima or minima of u

$$\frac{\partial u}{\partial a} = 0 \text{ and } \frac{\partial u}{\partial b} = 0 \qquad \{\because \text{ u is function of a \& b}\}$$

$$\Rightarrow \quad 2\Sigma(y_i - a - bx_i)(-1) = 0 \quad \text{and} \quad 2\Sigma(y_i - a - bx_i)(-x_i) = 0$$

$$\text{or} \quad \Sigma y_i - \Sigma a - \Sigma bx_i = 0 \quad \text{and} \quad \Sigma y_i x_i - \Sigma a x_i - \Sigma b x_i^2 = 0$$

$$\text{or} \quad \boxed{\Sigma y = na + b\Sigma x \text{ and } \Sigma xy = a\Sigma x + b\Sigma x^2}$$

These equations are called the normal equations and by solving these two equations we can find the best values of a and b and hence the line of best fit is y = a + bx

Fitting of a parabola :

Let $y = a + bx + cx^2$ be the curve of best it.

Then error at $x = x_i$ is defined as

$$\text{Error} = \text{Exact value} - \text{Approx. Value}$$

$$\text{or} \quad E_i = y_i - f(x_i) = y_i - (a + bx_i + cx_i^2)$$

Now let,
$$u = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)^2$$

Now using maxima-minima
$$\frac{\partial u}{\partial a} = 0, \ \frac{\partial u}{\partial b} = 0, \ \frac{\partial u}{\partial c} = 0$$

$$2\Sigma(y_i - a - bx_i - cx_i^2)(-1) = 0, \ 2\Sigma(y_i - a - bx_i - cx_i^2)(-x_i) = 0 \ \text{ and } \ 2\Sigma(y_i - a - bx_i - cx_i^2)(-x_i^2) = 0$$

or

$$\begin{cases}\Sigma y_i - \Sigma a - \Sigma bx_i - \Sigma cx_i^2 = 0 \\ \Sigma y_i x_i - \Sigma ax_i - \Sigma bx_i^2 - \Sigma cx_i^3 = 0 \\ \Sigma y_i x_i^2 - \Sigma ax_i^2 - \Sigma bx_i^3 - \Sigma cx_i^4 = 0\end{cases} \Rightarrow \boxed{\begin{aligned}\Sigma y &= na + b\Sigma x + c\Sigma x^2 \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \\ \Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + c\Sigma\end{aligned}}$$

These are called normal equations and solving these three, we can find the best values of a, b, c and hence the curve of best fit is $y = a + bx + cx^2$

CORRELATION

If two variables x and y are related in such a way that increase in one results in an increase or decrease in the other then x and y are said to be correlated.

Note :

1. It measures quantitative/quatitalive relationship between different variables like price and demand, beauty and bravery etc.
2. Correlation coefficients always lie between –1 and 1 ie $-1 \le r \le 1$
3. **Karl Pearson's Coeff. of correlation,**

$$r = \frac{\sum xy}{\sqrt{\sum(x^2)\sum(y^2)}} = \frac{\sum(xy)}{n\sigma_x\sigma_y} \text{ where } x = x_i - \bar{x},\ y = y_i - \bar{y},\ n = \text{no. of items}$$

4. If $r = 0$, there is no correlation i.e., variables x and y varies independently.

REGRESSION ANALYSIS

Consider the given set of observations as follows : $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$. If above set of observations are concentrated around the line $y = a + bx$ then this line is called as Regression line of y on x.

Similarly if above set of data is concentrated around the line $x = a + by$ then this line is called as Regression line of x on y.

LINE OF REGRESSION OF y ON x

It is given as $\boxed{y - \bar{y} = b_{yx}(x - \bar{x})}$ where $b_{yx} = \dfrac{n\sum xy - \sum x \cdot \sum y}{n\sum x^2 - (\sum x)^2} = r \cdot \dfrac{\sigma_y}{\sigma_x}$

LINE OF REGRESSION OF x ON y

It is given as $\boxed{x - \bar{x} = b_{xy}(y - \bar{y})}$ where $b_{xy} = \dfrac{n\sum xy - \sum x \cdot \sum y}{n\sum y^2 - (\sum y)^2} \approx \dfrac{\sigma_x}{\sigma_y}$

Note : b_{yx} and b_{xy} are called regression coefficient and regression lines are also known as **Regression Curve.**

PROPERTIES OF REGRESSION LINES

(i) Correlation coefficient is the GM between regression coefficient $\sqrt{b_{yx} \cdot b_{xy}} = r$

(ii) If one of the regression coefficient greater than unity, the other must be less than unity.

(iv) Correlation coefficient and two regression coefficient have same sign.

ANGLE BETWEEN TWO LINES OF REGRESSION

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$$