Propositional logic

Proposition:- A declarative sentence to which we can assign one and only one of the truth values TRUE (or) FALSE is called proposition.

Ex: Delhi is a city (T) // here Truth value TRUE (T) //

Ex: $2 \times 3 = 5$ (F) // here Truth value FALSE (F) //

Proposition can be classified into two categories

1. Atomic Proposition
2. Compound proposition

Atomic Proposition:- A proposition which can not be divided further.

Ex: $p: 2+8=11$

$q: 2 \times 10 = 20$

Compound Proposition:- Two (or) more atomic proposition can be combined.

Ex: $\underbrace{\text{Sun Rises in the east}}_{p}$ and $\underbrace{3+9=12}_{q}$

Ex: Sun Rises in the east (or) $3+9=12$

Connectivities:- There are five connectivities in the logic

not	$\sim$
or	$\vee$
and	$\wedge$
implies	$\longrightarrow$
if and only if	$\longleftrightarrow$

1. **Conjunction**:- $P \wedge q$ is True only when both P and q have Truth values are TRUE

2. **Disjunction**:- $P \vee q$ is False only when both P and q have truth values are FALSE

3) **Implication**:
- if p then q Can be written as $p \to q$
- $P \to q$ Truth value is False when P is True and q is False

Note:- $P \to q$ Can also be called as

p only if q
p is Sufficient Condition for q
q is necessary Condition for P
q if P
q follows from P
q provided P
q is Consequent of P

Note:- In $P \to q$,
- We Can call 'P' as antecedent and 'q' is Called Consequent
- We Can also call 'P' as premise (or) hypothesis and 'q' is Called Consequent (or) Conclusion.

Note:
1) The Converse of $(P \to q)$ is $(q \to P)$
2) The Inverse of $(P \to q)$ is $\sim P \to \sim q$
3) The Contrapositive of $(P \to q)$ is $\sim q \to \sim P$

Note $\boxed{P \to q \cong \sim q \to \sim P}$

Biconditional : 'p iff q' can be written as $p \leftrightarrow q$ is a proposition whose Truth value is True if p and q have Same True values

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note :-

$$(p \leftrightarrow q) \equiv ((p \to q) \wedge (q \to p))$$

$$\sim (p \leftrightarrow q) \equiv ((p \wedge \sim q) \vee (\sim p \wedge q))$$

$$\cong p \oplus q$$

$\therefore \oplus$ is XOR operation.

Tautology : A Compound proposition which is always True is called.

Ex: $(p \vee \sim p)$

Ex: $(p \vee (p \to q))$

Ex: $p \to (p \vee q)$

Ex: $(p \wedge q) \to (p \leftrightarrow q)$

Contradiction :- A compound proposition which is always False

Ex: $p \wedge \sim p$

Ex: $\sim \{(p \to q) \to (\sim p \vee q)\}$

Contingency :- A Compound proposition which is neither a tautology nor a Contradiction

Ex: $p \to q$

Ex: $p \vee q$

Ex: $p \leftrightarrow q$

Satisfiability:- A Compound proposition which is not a Contradiction is said to be Satisfiable

Ex:- $P \to q$

Ex $P \vee q$

Ex $P \leftrightarrow q$

Note: ① Every Contingency is Satisfiable

② Every tautology is Satisfiable

Tautological Implication

1) If P and Q are two Compound propositions and if $P \to Q$ is a tautology then we say that "P tautologically implies Q" and written as $P \Rightarrow Q$.

Method 1

*) Whenever 'P' is True, if 'Q' is also True, then $P \to Q$ is True, hence $P \Rightarrow Q$

Method 2

*) Whenever Q is False, if P is also False then $P \Rightarrow Q$.

Equivalence:- Let P and Q are two Compound propositions then "P is equivalent to Q" written as

$$P \cong Q \quad \text{(or)} \quad P \Leftrightarrow Q$$

if 'P' and 'Q' have same Truth Tables.

Note: If $\{(P \Rightarrow Q)$ and $(Q \Rightarrow P)\}$ then $P \Leftrightarrow Q$

Ex 1 $P \to Q \Leftrightarrow \sim P \vee Q$

Ex 2 $P \to Q \Leftrightarrow \sim Q \to \sim P$

Argument :-

If a set of premises $\{P_1, P_2, \dots P_n\}$ yield another proposition Q (conclusion) then the whole process is called an argument (inference) and denoted by

$$\{P_1, P_2, \dots P_n\} \rightarrow Q \text{ is a Tautology}$$

(or)

$$(P_1 \wedge P_2 \wedge P_3 \dots \wedge P_n) \rightarrow Q \text{ is a Tautology}$$

(or)

$$(P_1 \wedge P_2 \wedge P_3 \dots \wedge P_n) \Rightarrow Q$$

This symbol is Tautologically implication

(or)

$$\frac{P_1 \wedge P_2 \wedge P_3 \dots \wedge P_n}{\therefore Q \text{ [Conclusion]}}$$

Prove the following

Ex: $\{(P \vee Q), \sim P\} \Rightarrow Q$

We want to show that $[(P \vee Q) \wedge \sim P] \rightarrow Q$ is a Tautology

Now $\{(P \vee Q) \wedge \sim P\} \rightarrow Q$

Assum $(P \vee Q)$ is True, $(\sim P)$ is True then $(P \vee Q) \wedge (\sim P)$ is True, at this point of time 'Q' must be True

So according our discussion when we assume L.H.S is True then for those truth values, we have to get R.H.S is True then L.H.S → R.H.S is Tautology and hence L.H.S ⇒ R.H.S

So here whenever $(P \vee Q) \wedge (\sim P)$ is True then R.H.S value also True

$\therefore$ $\{(P \vee Q) \wedge \sim P\} \Rightarrow Q$.

Rules of Inference

1) Simplification Rule

a) $(P \wedge Q) \Rightarrow P$

b) $(P \wedge Q) \Rightarrow Q$

2) Addition Rule

a) $P \Rightarrow (P \vee Q)$

b) $Q \Rightarrow (P \vee Q)$

3) $\sim P \Rightarrow (P \rightarrow Q)$

4) $Q \Rightarrow (P \rightarrow Q)$

5) $\sim (P \rightarrow Q) \Rightarrow P$

6) $\sim (P \rightarrow Q) \Rightarrow \sim Q$

7) Disjunctive Syllogism

$$\{(P \vee Q) \wedge \sim P\} \Rightarrow Q$$

It is saying that whenever $\{(P \vee Q) \wedge \sim P\}$ is True then 'Q' is also True

(or)

whenever 'Q' is False then $\{(P \vee Q) \wedge \sim P\}$ also False.

8) Conjunctive Syllogism :-

$$\{\sim(P \wedge Q) \wedge P\} \Rightarrow \sim Q$$

i.e We have to Show that
$\{\sim(P \wedge Q) \wedge P\} \rightarrow Q$ is a tautology

Assume L.H.S is True, if we Can prove R.H.S also True then given statement is Valid

$$\therefore \{\sim(P \wedge Q) \wedge P\} \Rightarrow Q$$

9) Modus Pones

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array} \quad \text{(or)} \quad \{(P \rightarrow Q) \wedge P\} \Rightarrow Q$$

10) Modus Tollens

$$\{(P \rightarrow Q) \wedge \sim Q\} \Rightarrow \sim P$$

(OR)

$$\begin{array}{l} P \rightarrow Q \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

11) <u>Transitivity</u> (<u>Hypothetical Syllogism</u>)

$$\{(P \rightarrow Q) \wedge (Q \rightarrow R)\} \Rightarrow (P \rightarrow R)$$

12) <u>Dilemma</u>

$$\{(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)\} \Rightarrow R$$

13) <u>Constructive Dilemma</u>

$$\{(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)\} \Rightarrow (R \vee S)$$

14) Distructive Dilemma

$$\{(P \rightarrow R) \wedge (Q \rightarrow S) \wedge (\sim R \vee \sim S)\} \Rightarrow \sim P \vee \sim Q$$

15) <u>Demorgan law</u> :- i) $\sim (P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$

ii) $\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

16) <u>Idempotent law</u> :- i) $P \vee P \Leftrightarrow P$

ii) $P \wedge P \Leftrightarrow P$

17) Absortion law i) $P \vee (P \wedge Q) \Leftrightarrow P$

ii) $P \wedge (P \vee Q) \Leftrightarrow P$

* Conditional proof (CP Rule)

If $\{P_1, P_2, \cdots P_n\} \Rightarrow (Q \rightarrow R)$ then first convert it into $\{\{P_1, P_2, \cdots P_n\} \wedge Q\} \Rightarrow R$ and prove this argument is valid, hence by C.P Rule given argument also valid.

* Indirect proof (Proof by Contradiction)

Step1: To apply this rule, first we assume that the argument is not valid.

Step2: So now we take negation of the conclusion as a new premise.

Step3:- Now combined new premise with other premises if get any contradiction then given argument is valid.