VECTOR CALCULUS

Q1) $\phi = x^2yz + 4xz^2$, $P(1,-2,-1)$

$grad\,\phi = \vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$

$(grad\,\phi)_P = 8\hat{i} - \hat{j} - 10\hat{k}$

Req. DD $= (grad\,\phi)_P \cdot \hat{a}$, $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

$= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}} = \frac{37}{3}$

Q2) $\phi = x^{2/3} + y^{2/3}$, $P(8,8)$,

$grad\,\phi = \left(\frac{2}{3}x^{-1/3}\right)\hat{i} + \left(\frac{2}{3}y^{-1/3}\right)\hat{j}$

$(grad\,\phi)_P = \frac{\hat{i}}{3} + \frac{\hat{j}}{3}$

Now vector along the line $y = x$ is

$\vec{OP} = 8\hat{i} + 8\hat{j}$ & $\hat{OP} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$

Req. DD $= (grad\,\phi)_P \cdot \hat{OP}$

$= \left(\frac{\hat{i}}{3} + \frac{\hat{j}}{3}\right) \cdot \left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{3}$

Q3): $\phi = x^3 - y^3 + x^2z$, $P(1,+1,-2)$

$grad\,\phi = \vec{\nabla}\phi = (3x^2 + 2xz)\hat{i} + (-3y^2)\hat{j} + (x^2)\hat{k}$

$\vec{n} = (grad\,\phi)_P = -\hat{i} - 3\hat{j} + \hat{k}$

$\hat{n} = \frac{grad\,\phi}{|grad\,\phi|} = \frac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$

Q4) $\phi = \ln(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$

W.K.T. $grad\,\phi = \phi'(r)\hat{r}$

$\Rightarrow grad(\ln r) = \left(\frac{1}{r}\right)\hat{r} = \frac{\vec{r}}{r^2} = \frac{\vec{r}}{\vec{r}\cdot\vec{r}}$

Q5): $\phi = xy^2 + yz^3$, $P(2,-1,1)$

$grad\,\phi = \vec{\nabla}\phi = y^2\hat{i} + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$

$(grad\,\phi)_P = \hat{i} - 3\hat{j} - 3\hat{k}$

Max DD $= |grad\,\phi|_P = \sqrt{1 + 9 + 9}$

$= \sqrt{19}$

Q6) Let $\phi = \frac{1}{r}$ then

$grad\,\phi = \phi'(r)\hat{r}$

$\Rightarrow grad\left(\frac{1}{r}\right) = \left(-\frac{1}{r^2}\right)\hat{r} = \frac{-\vec{r}}{r^3}$

$div\, grad\left(\frac{1}{r}\right) = div\left[\frac{-\vec{r}}{r^3}\right]$

$\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = -div\left[\frac{\vec{r}}{r^3}\right]$

$\nabla^2\left(\frac{1}{r}\right) = 0$

Note: WKT $div\,\vec{f} = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 |\vec{f}|\right)$

$\Rightarrow div\left[\frac{\vec{r}}{r^3}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \left|\frac{\vec{r}}{r^3}\right|\right]$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$

Q7) $\phi = kxy - x^3y - xy^3$

$grad\,\phi = \vec{\nabla}\phi =$

$= (ky - 3x^2y - y^3)\hat{i} + (kx - x^3 - 3xy^2)\hat{j}$

$(grad\,\phi)_P = (2k - 14)\hat{i} + (k - 13)\hat{j}$

Given, $\hat{a} = \frac{-\hat{i} - \hat{j}}{\sqrt{2}}$

so Req DD $= (grad\,\phi)_P \cdot \hat{a}$

$\frac{15}{\sqrt{2}} = \left[(2k - 14)\hat{i} + (k - 13)\hat{j}\right] \cdot -\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right)$

$15 = 14 - 2k + 13 - k \Rightarrow k = 4$

Q8) $\phi = e^{3x}\sin(yz^4)$

$grad\,\phi = \hat{i}(3e^{3x}\sin yz^4)$

$+ \hat{j}(e^{3x}\cdot\cos(yz^4)\cdot z^4)$

$+ \hat{k}(e^{3x}\cdot\cos(yz^4)\cdot 4yz^3)$

$(grad\,\phi)_{P(0,\frac{\pi}{2},1)} = 3\hat{i} + 0\hat{j} + 0\hat{k}$

Max DD $= |grad\,\phi|_P = 3$

Q9) $\phi_1 = x^2+y^2+z^2-9, \quad P(2,-1,2)$

$\text{grad}\,\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\vec{n}_1 = (\text{grad}\,\phi_1)_P = 4\hat{i} - 2\hat{j} + 4\hat{k}$

$\phi_2 = x^2+y^2-z-3, \quad P(2,-1,2)$

$\text{grad}\,\phi_2 = 2x\hat{i} + 2y\hat{j} - (1)\hat{k}$

$\vec{n}_2 = (\text{grad}\,\phi_2)_P = 4\hat{i} - 2\hat{j} - \hat{k}$

$\vec{n}_1 \cdot \vec{n}_2 = (4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k}) = 16$

$\theta = \cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) = 0.95$

Q10) $\vec{V} = 5xy\,\hat{i} + 2y^2\hat{j} + 3yz^2\hat{k}$

$\text{div}\,\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}(5xy) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(3yz^2)$

$= 5y + 4y + 6yz$

$(\text{div}\,\vec{V})_{P(1,1,1)} = 5(1)+4(1)+6(1) = 15$

Q11) $\vec{f} = 3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\hat{k}$

$\text{div}\,\vec{f} = \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-yz^2)$

$= 3z + 2x - 2yz$

$(\text{div}\,\vec{f})_{(1,1,1)} = 3(1)+2(1)-2(1) = 3$

Q12) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\text{div}\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

Q13) w.k.t $\text{div}\,\vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2|\vec{f}|)$

and for solenoidal vector

$\text{div}\,\vec{f} = 0$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\cdot|r^n\vec{r}|\right] = 0$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+3}) = 0$

$(n+3)\,r^{n+2}/r^2 = 0$

$\Rightarrow n = -3$

Q14) $\vec{f} = x^2z\,\hat{i} + xy\,\hat{j} - yz^2\hat{k}$

$\text{div}\,\vec{f} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(-yz^2)$

$= 2xz + x - 2yz$

$(\text{div}\,\vec{f})_{(1,-1,1)} = 2(1)(1)+1-2(-1)(1) = 5$

Q15) $|\vec{F}| = r^n$ & $\vec{\nabla}\cdot\vec{F} = 0$

ie $\text{div}\,\vec{F} = 0$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2|\vec{F}|) = 0$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^n) = 0$

$\frac{(n+2)}{r^2}\,r^{n+1} = 0 \Rightarrow n = -2$

Q16) $\vec{V} = x^2\hat{i} + 2y^3\hat{j} + z^4\hat{k}$

$\text{div}\,\vec{V} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y^3) + \frac{\partial}{\partial z}(z^4)$

$= 2 + 6y^2 + 4z^3$

$(\text{div}\,\vec{V})_{P(1,2,3)} = 2 + 6(2)^2 + 4(3)^3 = 134$

Q17) $\vec{V} = 2yz\,\hat{i} + 3xz\,\hat{j} + 4xy\,\hat{k}$

$$\text{curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2yz & 3xz & 4xy \end{vmatrix}$$

$= (4x-3x)\hat{i} - \hat{j}(4y-2y) + \hat{k}(3z-2z)$

ie $\vec{\nabla}\times\vec{V} = x\hat{i} - 2y\hat{j} + z\hat{k}$

$\text{div}(\text{curl}\,\vec{V}) = \vec{\nabla}\cdot\vec{\nabla}\times\vec{V}$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(z)$

$= 1 - 2 + 1 = 0$

Q18) By vector Identity, WKT

$\text{div}(\text{curl}\,\vec{F}) = 0$

$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$

Note: this identity can also be used in Q17.

Q19) $\vec{f} = 2x^2y\,\hat{i} + 5z^2\hat{j} - 4yz\,\hat{k}$

$$\text{Curl}\,\vec{f} = \vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2y & 5z^2 & -4yz \end{vmatrix}$$

$$= \hat{i}(-4z - 10z) - \hat{j}(0-0) + \hat{k}(0 - 2x^2)$$

$$= -14\hat{i} + 0\hat{j} - 2x^2\hat{k}$$

Q20) $\vec{V} = 2x^2y + 3y^2z + 4z^2x$

By Vector identity, WKT

$\text{Curl}(\text{grad}\,\phi) = 0$.

So in this question,

$\text{Curl}(\text{grad}\,v) = 0$

Note: Above question can also be solved by using conventional Method.

Q21) $\vec{V} = y\hat{i} + z\hat{j} + x\hat{k}$ & let f be any scalar function, then

$f\vec{V} = (yf)\hat{i} + (zf)\hat{j} + (xf)\hat{k}$

$\text{div}(f\vec{V}) = \frac{\partial}{\partial x}(yf) + \frac{\partial}{\partial y}(zf) + \frac{\partial}{\partial z}(xf)$

$\vec{\nabla}\cdot(f\vec{V}) = y f_x + z f_y + x f_z$

ATQ,

$yx^2 + zy^2 + xz^2 = y f_x + z f_y + x f_z$

$\Rightarrow f_x = x^2,\ f_y = y^2,\ f_z = z^2$

So we can take

$f = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3}$

$\text{grad}\,f = (x^2)\hat{i} + (y^2)\hat{j} + (z^2)\hat{k}$

$\vec{V}\cdot\text{grad}\,f = (y\hat{i} + z\hat{j} + x\hat{k})\cdot(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$

$= yx^2 + zy^2 + xz^2$

(M-II) W.K.T, $\text{div}(f\vec{V}) = f\,\text{div}\,\vec{V} + \vec{V}\cdot\text{grad}\,f$

$(x^2y + y^2z + z^2x) = 0 + \vec{V}\cdot(\nabla f)$

ie $\vec{V}\cdot(\vec{\nabla} f) = x^2y + y^2z + z^2x$

Q22) $\vec{F} = (3y + k_1 z)\hat{i} + (k_2 x - 2z)\hat{j} - (k_3 y + z)\hat{k}$

For Irrotational, Curl $\vec{F} = \vec{0}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y - k_1 z & k_2x - 2z & -k_3y - z \end{vmatrix} = \vec{0}$$

$\hat{i}[-k_3 + 2] - \hat{j}[0 + k_1] + \hat{k}[k_2 - 3] = \vec{0}$

$(k_3 - 2)\hat{i} - k_1\hat{j} + (k_2 - 3)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow k_1 = 0,\ k_2 = 3,\ k_3 = 2$

Q23) $\vec{F} = 5x^2\hat{i} + (xz - y)\hat{j} + 3z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\vec{F}\cdot d\vec{r} = 5x^2dx + (xz - y)dy + 3z\,dz$

Now eqn of st. line OP where $O(0,0,0)$ & $P(1,1,1)$ is

$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} \Rightarrow \boxed{x = y = z = t}$

$dx = dy = dz = dt, \quad 0 \le t \le 1$.

So work done $= \int_C \vec{F}\cdot d\vec{r}$

$= \int_{OP} (5x^2dx + (xz - y)dy + 3z\,dz)$

$= \int_{t=0}^{1} (5t^2dt + (t^2 - t)dt + 3t\,dt)$

$= \int_0^1 (6t^2 + 2t)\,dt = 3$

Q24) Path OP: $\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$

So $dx = dy = dz = dt$ & $0 \le t \le 1$

$\int_C (2xy^2dx + 2x^2y\,dy + dz)$

$= \int_{OP} (2t\,t^2dt + 2t^2t\,dt + dt)$

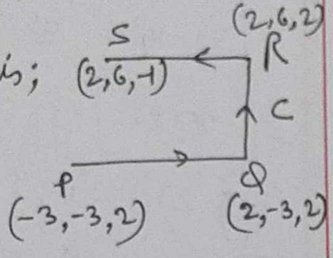
$= \int_{t=0}^{1} (4t^3 + 1)\,dt = 2$

Q25) $\int_C (yz\,dx + zx\,dy + xy\,dz)$

$= \int_C d(xyz) = (xyz)_{PQ}$

$= (xyz)_{P(1,1,0)}^{Q(2,3,2)} = 2\times3\times2 - 0 = 12$

Q26) Given path is;



$C: PQ + QR + RS$

Now $f = 2x^3 + 3y^2 + 4z$

grad $f = 6x^2\,\hat{i} + 6y\,\hat{j} + 4\,\hat{k}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\int_C (\text{grad} f \cdot d\vec{r}) = \int_{PQ} + \int_{QR} + \int_{RS}$

$= \int_P^Q (6x^2 dx + 6y\,dy + 4\,dz)$

$+ \int_Q^R (6x^2 dx + 6y\,dy + 4\,dz)$

$+ \int_R^S (6x^2 dx + 6y\,dy + 4\,dz)$

$= (2x^3 + 3y^2 + 4z)_{P(-3,-3,2)}^{Q(2,-3,2)}$

$+ (2x^3 + 3y^2 + 4z)_{Q(2,-3,2)}^{R(2,6,2)} + (2x^3 + 3y^2 + 4z)_{R(2,6,2)}^{S(2,6,-1)}$

$= 70 + 81 - 12 = 139$

Q27) $I = \int_C (2xy + y^2)dx + (2xy + x^2)dy$

$= \int_C d(x^2y + xy^2) = (x^2y + xy^2)_{P(3,0)}^{Q(0,3)} = 0$

Note: In above Question,
$\vec{f} = (2xy + y^2)\hat{i} + (2xy + x^2)\hat{j}$ & Curl $\vec{f} = \vec{0}$
so above integral will be path independent

Q28) $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

so $\vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$

& $\vec{r} = t\,\hat{i} + t^2\hat{j} + t^3\hat{k}$

$\Rightarrow x = t,\ y = t^2,\ z = t^3$

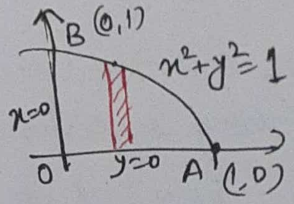
$dx = dt,\ dy = 2t\,dt,\ dz = 3t^2 dt$

$\int_C \vec{F}\cdot d\vec{r} = \int (xy\,dx + yz\,dy + zx\,dz)$

$= \int_C t^3 dt + t^5\cdot 2t\,dt + t^4\cdot 3t^2 dt$

$= \int_{t=0}^{1} (t^3 + 5t^6)\,dt = \frac{27}{28}$

Q29)



$I = \int_C (2x^2 + y^2)\,dx + (e^y)\,dy = ?$

By Green's theorem

$\int_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$

$\int_C [(2x^2+y^2)\,dx + (e^y)\,dy] = \iint_S (0 - 2y)\,dx\,dy$

$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} (-2y)\,dy\,dx$

$= -\int_{x=0}^{1} (1-x^2)\,dx = -2/3$

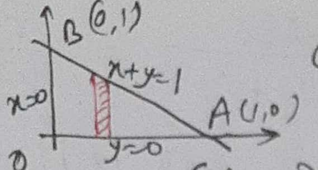
Q30) By G-theorem

$\int_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$

$\int_C (xy\,dx + x^2 dy) = \iint_S (2x - x)\,dx\,dy$

$= \int_{x=\frac{1}{\sqrt{3}}}^{2/\sqrt{3}} \int_{y=1}^{3} x\,dx\,dy = 1$

Note: Here $\vec{A} = xy\,\hat{i} + x^2\hat{j}$
& $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

Q31) 

C : OABO.

$\oint_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$

$\oint_C [(3x-8y^2)dx + (4y-6xy)dy] = \iint_S (-6y+16y)\,dx\,dy$

$= \int_{x=0}^{1}\int_{y=0}^{1-x} (+10y)\,dx\,dy = 5\int_{x=0}^{1} (y^2)_0^{1-x}\,dx$

$= 5\int_{x=0}^{1} (1-x)^2\,dx = 1.67.$

Q32) By Green's theorem,

$\oint_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$

$\oint_C (xy^2 dx + x^2 y\,dy) = \iint_S (2xy - 2xy)\,dx\,dy$

$= 0.$

Q33) By Stoke's theorem,

$\oint_C \vec{r}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{r}\cdot d\vec{S} = 0.$

$[\because \text{Curl}\,\vec{r} = \vec{\nabla}\times\vec{r} = \vec{0}]$

Q34) $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = -\hat{i} - \hat{j} - \hat{k}$ & $\hat{n} = \hat{k}$

$\because$ surface lies on XY plane so $\hat{n} = \hat{k}$

$\text{Curl}\,\vec{F}\cdot\hat{n} = -1$

By Stoke's theorem

$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{Curl}\,\vec{F}\cdot\hat{n})\,ds = \iint_S -1\cdot ds$

$= \iint_S (-1)\,dx\,dy = -$ Area of S

$= -[\pi(1)^2] = -\pi$

(Here S is disc of unit radius in XY Plane)

Q35) On comparision,

$\int_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}]\cdot d\vec{r} \simeq \int_C \vec{F}\cdot d\vec{r}$

$\Rightarrow \vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$

$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = 0\hat{i} + 0\hat{j} + \hat{k}$

Here surface 's' is in the form of Hemisphere (of radius 1) which lies on XY plane so we can take $\hat{n} = \hat{k}$ so $\text{Curl}\,\vec{F}\cdot\hat{n} = 1$

Now By Stoke's theorem,

$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{Curl}\,\vec{F}\cdot\hat{n})\,ds$

$= \iint_S (1)\,ds = \iint_S (1)\,dx\,dy$

$=$ Area of $S = \pi(1)^2 = \pi.$

Q36) By G-Div theorem,

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\text{div}\,\vec{F})\,dv$

$\iint_S (\vec{r}\cdot\hat{n})\,ds = \iiint_V (\text{div}\,\vec{r})\,dv$

$= 3\iiint_V (1)\,dv = 3V$

Q37) $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \text{div}\,\vec{F} = 3$

& W.K.T, Volume of Ellipsoid is

$V = \frac{4}{3}\pi abc$

So By G-D theorem

Flux $= \iint_S \vec{F}\cdot d\vec{S} = \iiint_V (\text{div}\,\vec{F})\,dv$

$= 3V = 4\pi abc$

Q38) $\vec{F} = x\sin y\,\hat{i} + \cos^2 x\,\hat{j} + 2z - 3\sin y\,\hat{k}$

$\text{div}\,\vec{F} = \sin y + 0 + 2 - \sin y = 2$

so $\iint_S (x\sin y, \cos^2 x, 2z - 3\sin y)\cdot(x,y,z)\,ds$

$= \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv = 2V$

$= 2\left(\frac{4}{3}\pi(1)^3\right) = \frac{8\pi}{3}$

Q39) on Comparision with $\iint_S (\vec F\cdot\hat n)\, ds$

$\simeq \iint_S [(2x^2+3x) - y^2 + 5z^2]\, ds$.

$\Rightarrow \vec F\cdot\hat n = 2x^2+3x-y^2+5z^2$

$(f_1\hat i+f_2\hat j+f_3\hat k)\cdot\vec r = (2x+3)x - yy + (5z)z$

{∵ in Case of unit sphere $\hat n = \vec r$}

$f_1x+f_2y+f_3z = (2x+3)x+(-y)y+(5z)z$

$\Rightarrow \vec f = f_1\hat i+f_2\hat j+f_3\hat k$

$= (2x+3)\hat i + (-y)\hat j + (5z)\hat k$

$\text{div}\,\vec F = 2+(-1)+(5) = 6$

Now $\oiint_S (\vec F\cdot\hat n)\, ds = \iiint_V (\text{div}\,\vec F)\, dv$

$= 6V = 6\left(\frac{4}{3}\pi(1)^3\right) = 8\pi$

Q40) on comparision with $\iint_S (\vec F\cdot\hat n)\, ds$

we get $\vec f = 4x\hat i - 2y^2\hat j + z^2\hat k$

$\text{div}\,\vec f = (4-4y+2z)$

S: $x^2+y^2=4$ & $z=0$ to 3

in Cylindrical Coordinates

$x^2+y^2=\rho^2,\ 0\le\rho\le 2,\ 0\le\phi\le 2\pi$

By G-D theorem,

$\oiint_S (\vec F\cdot\hat n)\, ds = \iiint_V \text{div}\,\vec F\, dv$

$= \iiint_V (4-4y+2z)\, dx\, dy\, dz$

$= \int_{z=0}^{3}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{2} (4-4\rho\sin\phi+2z)\,\rho\, d\rho\, d\phi\, dz$

$= \int_{z=0}^{3}\int_{\phi=0}^{2\pi} \left(2\rho^2 - \frac{4}{3}\sin\phi\,\rho^3 + z\rho^2\right)_0^2 d\phi\, dz$

$= \int_{z=0}^{3}\int_{\phi=0}^{2\pi} \left(8 - \frac{32}{3}\sin\phi + 4z\right) d\phi\, dz$

$= \int_{z=0}^{3} \left(8\phi + \frac{32}{3}\cos\phi + 4z\phi\right)_{\phi=0}^{2\pi} dz$

$= \int_0^3 \left(16\pi + \frac{32}{3} + 8\pi z - \frac{32}{3}\right) dz = 84\pi$

Q41) $\vec F = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} = \frac{\vec r}{r^3}$

& $\text{div}\,\vec F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2|\vec F|)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\left|\frac{\vec r}{r^3}\right|\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$

By G-Div theorem,

$\iint_{\partial W} \vec F\cdot\hat n\, dw = \iiint_V (\text{div}\,\vec F)\, dV = 0$

Q42) $\vec F = (x+y)\hat i + (x+z)\hat j + (y+z)\hat k$

$\text{div}\,\vec F = 1+0+1 = 2$

Sphere S: $x^2+y^2+z^2=9$ i.e $r=3$.

By G-D theorem,

$\oiint_S \vec F\cdot\hat n\, ds = \iiint_V (\text{div}\,\vec F)\, dv$

$= 2V = 2\left(\frac{4}{3}\pi(3)^3\right)$

$= 72\pi$

T-6) $I = \int_C (2z\,dx + 2y\,dy + 2x\,dz)$

$= 2\int_C (x\,dz + z\,dx) + 2\int_C y\,dy$

$= 2\int d(xz) + 2\int y\,dy$

$= \left[2xz + y^2\right]_{A(0,3,1)}^{B(4,1,-1)} = -11$

T-8) C: Max$(|x|,|y|=1) \Rightarrow x=\pm1,\ y=\pm1$

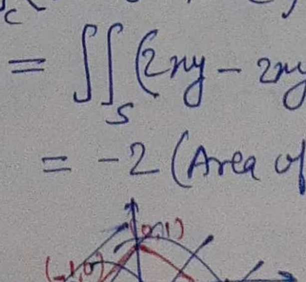
~~x+y=1, x-y=1, -x+y=1, -x-y=1~~

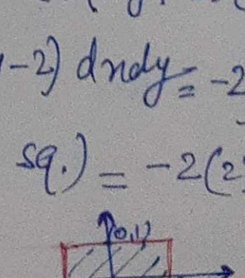
$\int_C (f_1\,dx + f_2\,dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$

$\int_C (xy^2 + 2y + \sin e^x)\,dx + (x^2y + \cos e^y)\,dy$

$= \iint_S (2xy - 2xy - 2)\,dx\,dy = -2\iint_S (1)\,dx\,dy$

$= -2\,(\text{Area of sq.}) = -2(2)^2 = -8$





COMPLEX VARIABLES

Q1) $\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{(x+iy)\to(0,0)} \left(\frac{x-iy}{x+iy}\right)$

$= \lim_{(x,y)\to(0,0)} \left(\frac{x-iy}{x+iy}\right) = \lim_{x\to 0}\left(\frac{x-imx}{x+imx}\right)$

$= \lim_{x\to 0}\left(\frac{1-im}{1+im}\right) = \frac{1-im}{1+im}$

i.e limit depends upon path ($y=mx$) choosen hence DNE.

Q2) $f(z) = z^2 = (x+iy)^2$

$\Rightarrow u+iv = x^2-y^2+2ixy$

i.e $u = x^2-y^2$ & $v = 2xy$

Now in 1st Quadrant $x>0, y>0$

$\Rightarrow xy>0 \Rightarrow 2xy>0 \Rightarrow v>0$

which represents upper Half plane

Q3) W.K.T. $\log z = \log r + i\theta$

where $r = |z|$ & $\theta = \mathrm{Amp}(z)$

so $\log(i)^{1/2} = \frac{1}{2}\log i = \frac{1}{2}\log(0+i)$

$= \frac{1}{2}\left[\log\sqrt{0^2+1^2} + i\tan^{-1}\left(\frac{1}{0}\right)\right]$

$= \frac{1}{2}\left[\log 1 + i\left(\frac{\pi}{2}\right)\right] = 0 + i\frac{\pi}{4}$

Q4) Let $z = x+iy \approx (x,y)$

ATQ, $|z|<1 \Rightarrow \frac{1}{|z|}>1$

& $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \approx \mathrm{IV}$

$\because$ z lies in 1st Quad so $\frac{1}{z}$ will lie in IV Quad

& $|z|<1 \Rightarrow \left|\frac{1}{z}\right|>1$ so. (d)

Q5) $x = \sqrt{-1} = i$ then

$x^x = (i)^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$

Q6) $I = \int_5^{3i} \frac{dz}{z} = (\ln z)_5^{3i}$

$= \log(3i) - \log 5$

$= \left[\log 3 + i\frac{\pi}{2}\right] - \log 5$

$= \log\left(\frac{3}{5}\right) + i\frac{\pi}{2}$

$= -0.5111 + 1.57i$

Q7) $z_1 = 5 + (5\sqrt{3})i$ & $z_2 = \frac{2}{\sqrt{3}} + 2i$

$\mathrm{Arg}\, z_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = \frac{\pi}{3}$

$\mathrm{Arg}\, z_2 = \tan^{-1}\left(\frac{2}{2/\sqrt{3}}\right) = \frac{\pi}{3}$

$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2 = 0$

Q8) $\lim_{z\to i} \frac{z^2+1}{z^3+2z-i(z^2+2)} \quad \left(\frac{0}{0}\right)$

$= \lim_{z\to i} \frac{2z}{3z^2+2-i(2z)}$

$= \frac{2i}{3(i^2)+2-i(2i)} = \frac{2i}{1}$

Q9) $f(z) = u+iv$, $if(z) = iu-v$

Adding $(1+i)f(z) = (u-v) + i(u+v)$

$\Rightarrow \phi(z) = U + iV$

where $U = u-v = 2xy + x^2-y^2$

Now using Case I of MILNE THOMSON

$\frac{\partial U}{\partial x} = 2y+2x \approx \phi_1(x,y)$

$\Rightarrow \phi_1(z,0) = 2z$

Again $\frac{\partial U}{\partial y} = 2x-2y \approx \phi_2(x,y)$

$\Rightarrow \phi_2(z,0) = 2z$

So $\phi(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz + C$

$(1+i)f(z) = \int(2z-i2z)dz + C = (1-i)z^2+C$

$f(z) = \left(\frac{1-i}{1+i}\right)z^2 + \frac{C}{1+i}$

$= -iz^2 + C$

Q10) $f(z) = |z|^2 + i\bar{z} + 1$

$u + iv = x^2 + y^2 + i(x - iy) + 1$

$u + iv = (x^2 + y^2 + y + 1) + ix$

$\Rightarrow u = x^2 + y^2 + y + 1$ & $v = x$

By CR equ^n, $u_x = v_y$ & $u_y = -v_x$

$2x = 0$ & $2y + 1 = -1$

$\Rightarrow x = 0,\ y = -1$ i.e at $(0,-1)$.

i.e CR equ^n are satisfied only at $(0,-1)$ or at $-i$

i.e diff. at $-i$ but Not Analytic

Q11) $f(z) = (x^2 + ay^2) + ibxy$

$\Rightarrow u = x^2 + ay^2,\ v = bxy$

By CR equ^n. $u_x = v_y$ & $u_y = -v_x$

$2x = bx$ & $2ay = -by$

$b = 2$ & $a = -1$

Q12) $u = x^2 + axy + by^2$ & $v = cx^2 + dxy + y^2$

By CR equ^n, $u_x = v_y$ & $u_y = -v_x$

$2x + ay = dx + 2y$ & $ax + 2by = -2cx - dy$

$\Rightarrow d = 2,\ a = 2,\ c = -1,\ b = -1$

Q13) option (d) $\because$ CR equ^n are not satisfied by any one of them.

Q14) $u = 2x^2 - 2y^2 + 4xy$

By Case I of MILNE THOMSON,

$u_x = 4x + 4y \approx \phi_1(x,y)$

$u_y = -4y + 4x \approx \phi_2(x,y)$

so $\phi_1(z,0) = 4z$ & $\phi_2(z,0) = 4z$

$f(z) = \int(\phi_1 - i\phi_2)\,dz + C$

$= \int(4z - i4z)\,dz + C$

$= 2(1-i)z^2 + C$

$f(z) = 2(1-i)[x^2 - y^2 + 2ixy] + C$

$u + iv = (2x^2 - 2y^2 + 4xy)$

$\cancel{+ i(2x^2 - 2y^2 + xy) + C}$

$+ i(-2x^2 + 2y^2 + 4xy) + C$

so $v = -2x^2 + 2y^2 + 4xy + C$

i.e Ans (d)

Q15) $w = \phi + i\psi$. & $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

Here Img part is given so using Case II of Milne Thomson

$\psi_x = 2x + \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ & $\psi_x(z,0) = 2z - \dfrac{1}{z^2} = \psi_2$

$\psi_y = -2y - \dfrac{2xy}{(x^2+y^2)^2}$ & $\psi_y(z,0) = 0 = \psi_1$

so $f(z) = \int(\psi_1 + i\psi_2)\,dz + C$

$= \cancel{\int\left(2z - \frac{1}{z^2}\right)} \int 0 + i\left(2z - \frac{1}{z^2}\right)dz + C$

$= i\left(z^2 + \frac{1}{z}\right) + C$

$= i\left(x^2 - y^2 + 2ixy + \dfrac{1}{x+iy}\right) + C$

$\phi + i\psi = i\left[x^2 - y^2 + 2ixy + \dfrac{x - iy}{x^2+y^2}\right] + C$

$= \left(-2xy + \dfrac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \dfrac{x}{x^2+y^2}\right) + C$

$\Rightarrow \phi = -2xy + \dfrac{y}{x^2+y^2} + C$

Q16) CR equ^n, $u_r = \frac{1}{r} v_\theta$ & $u_\theta = -r v_r$

Q17) $u = 2kxy$ & $v = x^2 - y^2$

By CR equ^n, $u_x = v_y$ & $u_y = -v_x$

$2ky = -2y$ & $2kx = -2x$

$\Rightarrow k = -1$

Q18) $f(z) = \sin z$, $f(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$, $f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$, $f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z$, $f'''(\pi/4) = -1/\sqrt{2}$

& so on.....

Taylor series about $z = \pi/4$ is

$$f(z) = f(\pi/4) + \left(z - \frac{\pi}{4}\right) f'(\pi/4) + \left(z - \frac{\pi}{4}\right)^2 \frac{f''(\pi/4)}{2!} + \cdots\cdots$$

$$\sin z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} + \cdots\cdots$$

Q19) $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1$ & $\frac{|z|}{2} < 1$

Now $f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$

$$= \frac{-1}{z(1-\frac{1}{z})} - \frac{1}{2} \frac{1}{(1-\frac{z}{2})}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right] - \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \left(-\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots\right) + \left(-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \cdots\right)$$

so coeff of $\frac{1}{z^2}$ is -1

Q20) $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)}$

$$= \frac{-1}{z-1} + \frac{1}{z-2} = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$\because |z| > 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ & $\left|\frac{2}{z}\right| < 1$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \cdots\cdots\right]$$

$$= \left(0 + \frac{1}{z^2} + \frac{3}{z^3} + \cdots\cdots\right)$$

coeff of $\frac{1}{z^3} = 3$

Q21) $\log\left(\frac{z}{z-1}\right) = -\log\left(\frac{z-1}{z}\right)$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$= -\left[-\frac{1}{z} - \frac{(1/z)^2}{2} - \frac{(1/z)^3}{3} + \cdots\right]$$

so coeff of $\frac{1}{z}$ is 1

Q22) $\underset{(z=3)}{\text{Res}} f(z) = \lim_{z\to 3} (z-3) f(z)$.

$$= \lim_{z\to 3} \frac{z^3}{(z-1)^4(z-2)} = \frac{27}{16}$$

Q23) $f(z) = \frac{1}{\cos z - \sin z}$.

Putting $\cos z - \sin z = 0$ we get $\tan z = 1 \Rightarrow z = \pi/4$.

so it is simple pole bcoz it is obtained by putting D^r equal to zero.

Q24) $f(z) = \frac{\sin z}{z^8}$ at $z=0$

$$= \frac{1}{z^8}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= \frac{1}{z^7} - \frac{1}{3!\,z^5} + \frac{1}{5!\,z^3} - \frac{1}{7!\,z} + \frac{z}{9!} - \frac{z^3}{11!} + \cdots\cdots$$

$\underset{z=0}{\text{Res}} f(z) = \text{coeff of } \frac{1}{z} = \frac{-1}{7!}$

Q25) $\beta = e^{\frac{i\pi}{10}}$ & $f(z) = \frac{1}{1+z^{10}}$

Pole is $1 + z^{10} = 0 \Rightarrow z = (-1)^{\frac{1}{10}}$

$$= (e^{\pi i})^{\frac{1}{10}} = \beta$$

$\underset{z=\beta}{\text{Res}} f(z) = \lim_{z\to\beta} \left(\frac{z-\beta}{1+z^{10}}\right)$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{z\to\beta} \frac{1}{10 z^9} = \frac{1}{10\beta^9}$$

$$= \frac{\beta}{10 \cdot \beta^{10}} = \frac{\beta}{10(-1)} = \frac{-\beta}{10}$$

Q26) $f(z) = \cot z = \frac{\cos z}{\sin z}$

Poles are $\sin z = 0 \Rightarrow z = n\pi$

i.e $z = \cdots -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Let us find the Residue at $z=0$

i.e $\underset{(z=0)}{\text{Res}}\, f(z) = \lim_{z\to 0} (z-0) f(z)$

$= \lim_{z\to 0} \frac{z \cdot \cos z}{\sin z} \quad \left(\frac{0}{0}\right)$

$= \lim_{z\to 0} \left(\frac{-z\sin z + \cos z}{\cos z}\right) = 1$

Q27) $f(z) = \frac{z - \sin z}{z^3}$

$= \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right]$

$= \left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\right) + 0$

∵ Principal Part of Laurent Series Contains No terms so $z=0$ is the Removable singularity of $f(z)$.

Q28) $f(z) = \cancel{\frac{1}{\sin(1-z)}}\ \sin\left(\frac{1}{1-z}\right)$

$= \frac{\left(\frac{1}{1-z}\right)}{1!} - \frac{\left(\frac{1}{1-z}\right)^3}{3!} + \frac{\left(\frac{1}{1-z}\right)^5}{5!} - \cdots$

$= \frac{1}{1-z} - \frac{1}{6(1-z)^3} + \frac{1}{120(1-z)^5} - \cdots$

∵ P. Part of Laurent series about $z=1$ Contains infinite No. of terms so $z=1$ is the Essential singularity

Q29) $z = t^2 + it \rightarrow \bar{z} = t^2 - it$, $dz = (2t + i)\,dt$, $x = t^2$ & $y = t$

∵ $z = 0 = 0 + 0i \Rightarrow x = 0, y = 0$

& $z = 4 + 2i \Rightarrow x = 4, y = 2$ so $0 \le t \le 2$

$I = \int_C \bar{z}\,dz = \int_0^2 (t^2 - it)(2t + i)\,dt = 10 - \frac{8i}{3}$

Q30) $f(z) = \frac{\cos 2\pi z}{(2z-1)(z-3)}, |z| = 1$

Poles are $z = \frac{1}{2}$ & 3

only $z = \frac{1}{2}$ lies inside $C: |z| = 1$

$\underset{(z=\frac{1}{2})}{\text{Res}}\, f(z) = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$

$= \lim_{z\to\frac{1}{2}} \frac{\cos 2\pi z}{2(z-3)} = \frac{-1}{-5}$

By Cauchy Residue theorem,

$\int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}\,dz = 2\pi i\,[R]$

$= 2\pi i\left(\frac{1}{5}\right)$

Q31) $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)}$

Poles are $z = 2, 4$ but only $z = 2$ lies with in $C: |z| = 3$

$\underset{(z=2)}{\text{Res}}\, f(z) = \lim_{z\to 2} (z-2) f(z)$

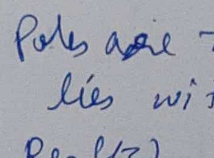
$= \lim_{z\to 2} \left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-4}\right)$

$= \frac{\sin 4\pi + \cos 4\pi}{2-4} = \frac{0+1}{-2} = 0$

By C-R-Theorem,

$\int_C f(z)\,dz = 2\pi i\,(R) = -\pi i$

Q32) $f(z) = \frac{1}{z^2+4}$ Poles are $z = \pm 2i$



only $z = 2i$ lies inside C

$R = \underset{(z=2i)}{\text{Res}}\, f(z) = \lim_{z\to 2i} (z - 2i)\left(\frac{1}{z^2+4}\right) = \frac{1}{4i}$

By C-R-Th, $\int_C f(z)\,dz = 2\pi i\,(R)$

$= 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$

Q33) $f(z)=\frac{z^2-1}{z^2+1}\cdot e^z$, $C: |z|=3$

Poles are $z^2+1=0 \Rightarrow z=\pm i$

Both lies inside 'C' so.

$R_1 = \underset{(z=i)}{\text{Res}}\, f(z) = \lim_{z\to i}(z-i)f(z) = ie^{i}$

$R_2 = \underset{(z=-i)}{\text{Res}}\, f(z) = \lim_{z\to -i}(z+i)f(z) = -ie^{-i}$

By C-R-Theorem,

$\int_c f(z)dz = 2\pi i\,[R_1+R_2]$

$= 2\pi i\,[ie^{i} - ie^{-i}]$

$= -2\pi\,[(\cos 1 + i\sin 1) - (\cos(-1)+i\sin(-1))]$

$= -2\pi\,[i\sin 1 + i\sin 1] = \cancel{2\pi i}$

$= -4\pi i \sin 1$

Q34) for $f(z)=e^{\frac{1}{z}}$, singularity $z=0$

By Laurent series expansion,

$f(z)=1+\frac{1/z}{\lfloor 1}+\frac{(1/z)^2}{\lfloor 2}+\frac{(1/z)^3}{\lfloor 3}+\dots\dots$

$\underset{(z=0)}{\text{Res}}\, f(z) = \text{coeff of}\left(\frac{1}{z}\right) = 1.$

By C-R-theorem,

$\int_c e^{\frac{1}{z}}dz = 2\pi i\,[1] = 2\pi\sqrt{-1}$

Q35) $C: |z|=1 \Rightarrow z=1.e^{i\theta}=\cos\theta+i\sin\theta$

& $dz = ie^{i\theta}d\theta$, $0\le\theta\le2\pi$

& $Re(z)=\cos\theta$

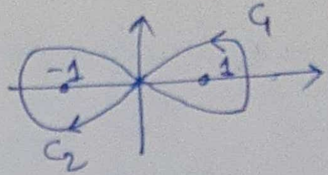
$\frac{1}{2\pi i}\int_c Re(z)\,dz = \frac{1}{2\pi i}\int_c \cos\theta\,.\,ie^{i\theta}d\theta$

$=\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\,(\cos\theta+i\sin\theta)\,d\theta$

$=\frac{1}{2\pi}\int_0^{2\pi}(\cos^2\theta + i\sin\theta\cos\theta)\,d\theta$

$=\frac{1}{2\pi}(\pi-0)=\frac{1}{2}$

Q36)



Poles of $f(z)=\frac{1}{z^2-1}$ are $z=\pm1$

Both lies inside 'C'

$R_1 = \underset{(z=1)}{\text{Res}}\, f(z) = \lim_{z\to1}(z-1)\frac{1}{z^2-1}=\frac{1}{2}$

$R_2 = \underset{(z=-1)}{\text{Res}}\, f(z) = \lim_{z\to-1}(z+1)\frac{1}{z^2-1}=-\frac{1}{2}$

~~But~~ By C-R-theorem,

$\frac{1}{\pi i}\oint_c \frac{dz}{z^2-1} = \frac{1}{\pi i}\left[2\pi i(R_1+R_2)\right]$

$=2\left[\frac{1}{2}+(-R_2)\right]$

$=2\left[\frac{1}{2}+\left(\frac{1}{2}\right)\right]=2$

Note: In above question C_2 is in clockwise direction that's why we have taken $(-R_2)$ in place of R_2.

Q37) $I=\int_0^{2\pi}\frac{d\theta}{2+\sin\theta} = \int_0^{2\pi}\frac{d\theta}{2+\frac{e^{i\theta}-e^{-i\theta}}{2i}}$

Let $z=e^{i\theta}\Rightarrow dz = ie^{i\theta}d\theta$

so $d\theta = \frac{dz}{iz}$

$I=\int_c\frac{dz/iz}{2+\left(\frac{z-\frac{1}{z}}{2i}\right)}$

$=\int_c\frac{2\,dz}{z^2+4iz-1}$ —— (1)

Poles are $z^2+4iz-1=0$

$z=\underbrace{(-2+\sqrt{3})i}_{=\alpha}$ & $\underbrace{(-2-\sqrt{3})i}_{=\beta}$

Only $z=\alpha$ lies inside $C: |z|=1$

$$\underset{(z=\alpha)}{\text{Res}}\, f(z) = \lim_{z\to\alpha} (z-\alpha) f(z).$$

$$= \lim_{z\to\alpha} \frac{(z-\alpha).2}{z^2+4iz-1}$$

$$= \lim_{z\to\alpha} \frac{2(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{2}{\alpha-\beta}$$

$$= \frac{2}{(-2+\sqrt{3})i - (-2-\sqrt{3})i} = \frac{1}{i\sqrt{3}}$$

By Cauchy Residue theorem

$$I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_c \frac{2dz}{z^2+4iz-1}$$

$$= 2\pi i[R] = 2\pi i\left(\frac{1}{i\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

T4) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$

$$= 2^n\left[\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^n + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^n\right]$$

$$= 2^n\left[\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n + \left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^n\right]$$

$$= 2^n\left(e^{\frac{n\pi}{6}i} + e^{-\frac{n\pi i}{6}}\right)$$

$$= 2^n.\left[2\cos\frac{n\pi}{6}\right] = 2^{n+1}.\cos\frac{n\pi}{6}.$$

T-15) $C: y=x$ & $z=x+iy$

$\Rightarrow dz = dx+idy = (1+i)dx$

($\because dy=dx$ & $0\le x\le 1$)

Now $I = \int_c (x^2+iy^2)\,dz$

$$= \int_{x=0}^{1} (x^2+ix^2)(1+i)\,dx$$

$$= (1+i)^2 \int_0^1 x^2 dx = \frac{(1+i)^2}{3}$$

$$= \frac{2i}{3}$$

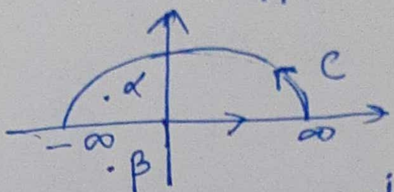
T-17) $I = \int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+2x+2}$

$$= \text{Img part of} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+2x+2}.\,dx$$

By definite integral property

$$= \text{Img Part of} \int_{-\infty}^{\infty} \frac{e^{iz}\,dz}{z^2+2z+2}$$

Let us assume a circle of ∞ radius in upper half plane



Now $f(z) = \frac{e^{iz}}{z^2+2z+2}$

Poles are $z^2+2z+2=0$

$$z = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$$

Let $\alpha=-1+i$ & $\beta=-1-i$ then only $z=\alpha$ lies inside C

$$\underset{(z=\alpha)}{\text{Res}}\, f(z) = \lim_{z\to\alpha} (z-\alpha) f(z).$$

$$= \lim_{z\to\alpha} \frac{(z-\alpha).e^{iz}}{(z-\alpha)(z-\beta)} = \frac{e^{i\alpha}}{\alpha-\beta}$$

$$= \frac{e^{i(-1+i)}}{2i} = \frac{e^{-i}e^{-1}}{2i}$$

By C-R theorem,

$$I = \text{IP of} \oint_c f(z)dz$$

$$= \text{IP of} \left[2\pi i\left(\frac{e^{-i}e^{-1}}{2i}\right)\right]$$

$$= \text{IP of} \left[\frac{\pi}{e}\left(\cos(-1) + i\sin(-1)\right)\right]$$

$$= \text{IP of} \left(\frac{\pi}{e}\left(\cos 1 - i\sin 1\right)\right)$$

$$= -\frac{\pi\sin 1}{e}$$

PROBABILITY & STATISTICS

Q1: 7 accident, 7 days

Total ways $= 7\times7\times7\times7\times7\times7\times7 = 7^7$

fav ways $= {}^7C_1 = 7$ So Ans $= \frac{{}^7C_1}{7^7} = \frac{1}{7^6}$

Q2: Total ways of picking a child

$= \left(\frac{50N}{100}\right)\times3 + \left(\frac{30N}{100}\right)\times2 + \left(\frac{20N}{100}\right)\times1$

So Prob $= \frac{\text{fav}}{\text{Total}} = \frac{\left(\frac{30N}{100}\right)\times2}{\left(\frac{N}{2}\times3 + \frac{3N}{5} + \frac{N}{5}\right)} = \frac{6}{23}$

Q3: $S = \{(1,1)(1,2) \ldots (6,6)\} \Rightarrow n(S) = 36$

$E = \{\text{sum} > 8\} = \{(3,6),(6,3),(4,5),(5,4),(5,5),(4,6),(6,4),(5,6),(6,5),(6,6)\} = 10$

Prob $= \frac{10}{36}$

Q4: ATQ, $H+T = n$ & $|H-T| = n-3$

ie $|H-(n-H)| = n-3 \Rightarrow |2H-n| = n-3$

Case I: $2H-n = +(n-3) \Rightarrow H = \frac{2n-3}{2}$

Not possible $\because$ H should be a Natural no.

Case II: $2H-n = -(n-3) \Rightarrow H = \frac{3}{2}$

Again Not Possible So Prob $= 0$.

Q5 Req prob $= \underbrace{\left(\frac{2}{9}\times\frac{1}{8}\right)}_{\text{Washers}} \times \underbrace{\left(\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\right)}_{\text{Nuts}} \times \underbrace{1}_{\text{Bolts}} = \frac{1}{1260}$

Q6: Box 1 $\rightarrow$ 3, 6, 9, 12, 15

Box 2 $\rightarrow$ 6, 11, 16, 21, 26

Total ways of Selecting 2 Numbers $= {}^5C_1\times{}^5C_1$

fav ways $= \{\text{Product even}\}$

$=$ Even×Even + Even×odd + Odd×Even

$= {}^2C_1\times{}^3C_1 + {}^2C_1\times{}^2C_1 + {}^3C_1\times{}^3C_1 = 19$

Prob $= \frac{19}{25}$

Q7: $\underset{\text{I}}{\begin{bmatrix}4R\\3G\end{bmatrix}}, \underset{\text{II}}{\begin{bmatrix}3B\\4G\end{bmatrix}}$

Req. prob $= \underset{(\text{I})}{\text{Red}} \times \underset{(\text{II})}{\text{Blue}}$ ~~Blue × Red (I) (II)~~

$= \frac{4}{7}\times\frac{3}{7} = \frac{12}{49}$

Q8: Req. Prob $=$ (RBB) or (BRB) or (BBR)

$= \frac{4}{10}\times\frac{6}{9}\times\frac{5}{8} + \frac{6}{10}\times\frac{4}{9}\times\frac{5}{8} + \frac{6}{10}\times\frac{5}{9}\times\frac{4}{8} = \frac{1}{2}$

Q9: (4G & 5B)

Total ways of Selecting 4 members $= {}^9C_4$

fav ways $=$ (4G & 0B) or (3G & 1B)

$= {}^4C_4\times{}^5C_0 + {}^4C_3\times{}^5C_1 = 21$

Prob $= \frac{21}{{}^9C_4} = \frac{1}{6}$

Q10: Total ways $= \cancel{{}^nC_2}$ $(n-1)!$

fav ways when two particular persons always sit together are as follows:

$(1,2)\,3, 4, 5, \ldots, (n-1), n \longrightarrow \cancel{{}^{n-1}C_2}$ $(n-2)! \times 2!$

Here we have assumed (1,2) as a single person. So

P(sitting always together) $= \cancel{\frac{{}^{n-1}C_2}{{}^nC_2}}$ $\frac{(n-2)!\times2}{(n-1)!}$

So P(Do not sit together)

$= 1 - P(\text{always together}) = 1 - \cancel{\frac{{}^{n-1}C_2}{{}^nC_2}}$

$= \cancel{\frac{n-3}{n-1}} = 1 - \frac{(n-2)!\times2}{(n-1)!} = \frac{n-3}{n-1}$

Q11: B, B, I, I, P, R, O, L, T, Y, A $\rightarrow$ 11

Total arrangements $= \frac{11!}{2!\,2!}$

Now (BB), (II), P, R, O, L, T, Y, A $\longrightarrow$ 9

Fav arrangements $= 9!$

Prob $= \frac{\text{fav}}{\text{Total}} = \frac{9!}{11!/2!\,2!} = \frac{2}{55}$

Q12: 6 +ve & 8 −ve Numbers

Total ways of 4 Numbers $= {}^{14}C_4$

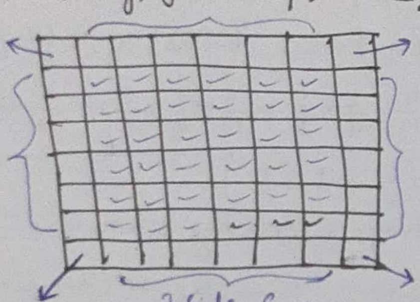
fav ways $= {}^6C_2\times{}^8C_2 + {}^6C_4 + {}^8C_4$

(2+ve & 2−ve) or (All 4+ve) or (All 4−ve)

Prob $= \frac{\text{fav}}{\text{Total}} = \frac{{}^6C_2\times{}^8C_2 + {}^6C_4 + {}^8C_4}{{}^{14}C_4}$

$= \frac{505}{1001}$

Q13: Total ways for 2 blocks = ${}^{64}C_2$



2 sides Common $({}^{4}C_1 \times {}^{2}C_1)$

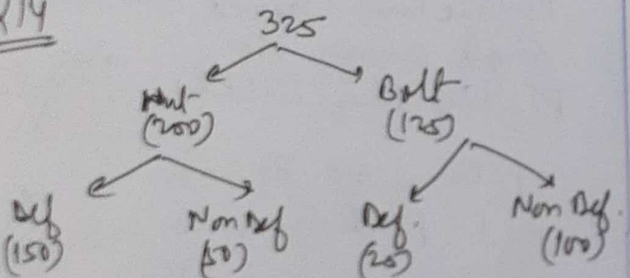
3 sides Common $({}^{24}C_1 \times {}^{3}C_1)$

Blocks with Tick (✓) mark having 4 sides common ie ${}^{36}C_1 \times {}^{4}C_1$

So Fav ways = $\dfrac{\left({}^{4}C_1 \times {}^{2}C_1 + {}^{24}C_1 \times {}^{3}C_1 + {}^{36}C_1 \times {}^{4}C_1\right)}{2}$

Prob = $\dfrac{\text{fav}}{\text{Total}} = \dfrac{1}{18}$

Q14



$P(D \cup N) = P(D) + P(N) - P(D \cap N)$

$= \dfrac{175}{325} + \dfrac{200}{325} - \dfrac{150}{325} = \dfrac{9}{13}$

Q15 $P(W) = \frac{1}{2}$, $P(L) = \frac{1}{2}$,

P(2nd win in 3rd test)

= P(exactly one W in 1st two) × P(W in 3rd)

$= \left(\dfrac{{}^{2}C_1}{2^2}\right) \times \dfrac{1}{2} = 0.25$

[for 1st two matches sample space is $S = \{(WW), (WL), (LW), (LL)\}$]

Q16 Total ways of Carrying 3 Pens

= (All different) or (All Same) or (1 diff & 2 Same)

$= {}^{4}C_3 + {}^{4}C_1 + {}^{4}C_1 \times {}^{3}C_1 = 20$ ways

fav ways (of Same Color) $= {}^{4}C_1 = 4$

Prob = $\dfrac{\text{fav}}{\text{Total}} = \dfrac{4}{20} = \dfrac{1}{5} = 0.2$

Q17: $n(S) = 6 \times 6 = 36$

fav Cases when Sum is divisible by 4 or 6 = {(1,3), (3,1), (2,2), (2,6), (6,2), (3,5), (5,3), (4,4), (6,6), (1,5), (5,1), (2,4), (4,2), (3,3), } = 14

Req prob = $\dfrac{\text{fav}}{\text{Total}} = \dfrac{14}{36} = \dfrac{7}{18}$

Q18: A = {Sum is 8} = {(2,6) (6,2) (3,5) (5,3) (4,4)} = 5

B = {Sum is 9} = {(3,6) (6,3) (4,5) (5,4)} = 4

$A \cap B = \phi$.

So $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{5}{36} + \dfrac{4}{36} - 0 = \dfrac{1}{4}$

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{1}{4} = \frac{3}{4}$

Q19: $P(X \cup \bar{Y}) = 0.7$ & $P(X) = 0.4$

$P(X) + P(\bar{Y}) - P(X \cap \bar{Y}) = 0.7$

$0.4 + P(\bar{Y}) - P(X) \cdot P(\bar{Y}) = 0.7$

$0.4 + (1 - 0.4)\, P(\bar{Y}) = 0.7$

$P(\bar{Y}) = \dfrac{0.3}{0.6} = \dfrac{1}{2} \Rightarrow P(Y) = 1 - \frac{1}{2} = \frac{1}{2}$

So $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= 0.4 + 0.5 - 0.4 \times 0.5$

$= 0.7$

Q20 $P(E_1 \cup E_2) = 1$

$P(E_1) + P(E_2) - P(E_1 \cap E_2) = 1$

$P(E_1) + P(E_2) - P(E_1) \cdot P(E_2) = 1$

$x + x - (x)^2 = 1$

$\cancel{x^2+2x}$ $x^2 - 2x + 1 = 0 \Rightarrow x = 1$

So $P(E_1) = P(E_2) = 1$.

Q21 $S = \{(1,1), (1,2), \ldots (6,6)\} = 36$

Reduced S space = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)}

= 15 Cases

fav Cases = {(4,3), (5,2), (6,1)} = 3

Req. Prob = $\dfrac{3}{15} = \dfrac{1}{5}$

Q22: Req. prob $= P[(H)HT \text{ or } (H)TH]$

$= 1\times\frac{1}{2}\times\frac{1}{2} + 1\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

Q23: $P(I) = 0.3$, $P(II) = 0.2$, & $P(I/II) = 0.6$

$\frac{P(I\cap II)}{P(II)} = 0.6 \Rightarrow P(I\cap II) = 0.6\times 0.2 = 0.12$

~~**Q24:**~~ Req. Prob $= \cancel{P[H, TH, TTH, TTTH, \ldots]}$

$\cancel{= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots}$

$\cancel{= \frac{1/2}{1-1/2}}$

Q24: $S = \{H, TH, TTH, TTTH, \ldots\}$

fav Cases = {odd tosses}

$= \{H, TTH, TTTTH, \ldots\}$

So $P = P[H, TTH, TTTTH, \ldots]$

$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots$

$= \frac{1/2}{1-\left(\frac{1}{2}\right)^2} = \frac{1/2}{1-\frac{1}{4}} = \frac{2}{3}$.

Q25 $P(p) = \frac{1}{4}$ ①, $P(p/q) = \frac{1}{2}$ ②, $P(q/p) = \frac{1}{3}$ ③

using ③, $\frac{P(q\cap p)}{P(p)} = \frac{1}{3} \Rightarrow P(q\cap p) = \frac{1}{12}$

using ②, $\frac{P(p\cap q)}{P(q)} = \frac{1}{2} \Rightarrow P(q) = \frac{1}{6}$

Now $P(\bar{p}/\bar{q}) = \frac{P(\bar{p}\cap\bar{q})}{P(\bar{q})} = \frac{1-P(p\cup q)}{1-P(q)}$

$= \frac{1-\{P(p)+P(q)-P(p\cap q)\}}{1-P(q)}$

$= \frac{1-\left(\frac{1}{4}+\frac{1}{6}-\frac{1}{12}\right)}{1-\frac{1}{6}} = \frac{4}{5}$

Q26: P(only 1st two tosses are head)

$= P(HHTTTTTTTT) = \left(\frac{1}{2}\right)^{10}$.

Q27: $S = \{W, LW, LLW, LLLW, LLLLW, \ldots\}$

Fav. Cases for $A = \{W, LLW, LLLLW, \ldots\}$

So $P(A \text{ win}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^4\frac{1}{6} + \ldots$

$= \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right] = \frac{6}{11}$

Q28: $A = \{\text{Total is } 5\} = \{(14)(41)(23)(32)\}$

$B = \{\text{Total is } 7\} = \{(16)(61)(25)(52)(34)(43)\}$

$N = \{\text{Total is Neither 5 Nor 7}\}$

So $P(A) = \frac{4}{36} = \frac{1}{9}$, $P(B) = \frac{6}{36} = \frac{1}{6}$ & $P(N) = \frac{26}{36}$

fav Cases $= \{A, NA, NNA, NNNA, \ldots\}$

So Prob $= P(A) + P(NA) + P(NNA) + \ldots$

$= \frac{1}{9} + \frac{26}{36}\times\frac{1}{9} + \left(\frac{26}{36}\right)^2\times\frac{1}{9} + \ldots$

$= \frac{1}{9}\left[\frac{1}{1-\frac{26}{36}}\right] = \frac{2}{5}$

Q29: fav. Cases =

$\{\underset{I\ II\ I}{WWW} \text{ or } \underset{I\ II\ I}{WBW} \text{ or } \underset{I\ II\ I}{BWW} \text{ or } \underset{I\ II\ I}{BBW}\}$

Req. Prob $= \frac{3}{7}\times\frac{5}{8}\times\frac{3}{7} + \frac{3}{7}\times\frac{3}{8}\times\frac{2}{7}$

$+ \frac{4}{7}\times\frac{1}{2}\times\frac{4}{7} + \frac{4}{7}\times\frac{1}{2}\times\frac{3}{7} = \frac{25}{56}$

Q30:

$$\begin{tikzcd}
 & \text{Total keys (2 Identical \& 4 Diff)} \arrow[dl, "(2/6)"'] \arrow[dr, "(4/6)"] & \\
\text{Identical lost (1 \& 4 Diff)} \arrow[d, "(1/5)"] & & \text{Different lost (2 Id \& 3 Diff)} \arrow[d, "(2/5)"] \\
\text{Lock open} & & \text{lock open}
\end{tikzcd}$$

Req. Prob = P(lock will open)

$= \frac{2}{6}\times\frac{1}{5} + \frac{4}{6}\times\frac{2}{5} = \frac{1}{3}$

Q31:

Question

0.7 → KNOWS, 0.3 → GUESS

Correct Ans: (A) $\quad 1 \qquad 1/5$

$P(A) = 0.7\times 1 + 0.3\times\frac{1}{5} = \frac{19}{25}$

Q32:

Person

0.01 → +ve Report, 0.99 → −ve Report

Incorrect Report (A): $\quad 0.12 \qquad 0.15$

$P(A) = 0.01\times 0.12 + 0.99\times 0.15 = 0.1497$

Q33:

Student

$2/3$ → KNOWS (E_1), $1/3$ → Guess (E_2)

Answered Correctly: (A) $\quad 1 \qquad 1/4$

$P(A) = \frac{2}{3}\times 1 + \frac{1}{3}\times\frac{1}{4} = \frac{3}{4}$

$P(E_1/A) = \frac{\frac{2}{3}\times 1}{P(A)} = \frac{\frac{2}{3}}{3/4} = \frac{8}{9}$

Q34:

Rainy Season

0.7 → Rain (E_1), 0.3 → Not Rain (E_2)

FMR has Not made loss A: $\quad 0.2 \qquad 0.9$

$P(A) = 0.7\times 0.2 + 0.3\times 0.9 = 0.41$

By Baye's theorem.

$P(E_2/A) = \frac{0.3\times 0.9}{P(A)} = \frac{27}{41}$

Q35) Mode = 17

∵ it ~~tems~~ has the maximum frequency in the given distribution i.e it is repeated maximum No. of times.

Q36) Given Data in increasing order:

32, 45, 49, 57, 53, 56, 60, 62, 66, 79

∵ $N=10$ so Median $= \frac{\left(\frac{N}{2}\right)^{th} + \left(\frac{N}{2}+1\right)^{th}}{2}$

$= \frac{53+56}{2} = \frac{109}{2} = 54.5$

Q37: Mean $= \frac{\Sigma X}{n} \Rightarrow x = \frac{3+x+2+4}{4}$

$x = 3$ ie Data is 3, 3, 2, 4

so Mode = 3

Q38: $P(x)\propto x$ & WKT $\Sigma p_i = 1$

ie $P(x) = kx$

so $k(1)+k(2)+k(3)+k(4)+k(5)+k(6) = 1$

$\Rightarrow k = \frac{1}{21}$ so $P(\text{odd No}) = k+3k+5k = \frac{3}{7}$

Q39: $P(x>1) = \int_1^\infty f(x)dx = \int_1^\infty e^{-x}dx = 0.368$

Q40: $E(x) = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x(2x)dx = 2/3$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2(2x)dx = \frac{1}{2}$

$Var(x) = E(x^2) - E^2(x) = \frac{1}{18}$

Q41: $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} k\frac{1}{1+x^2}dx = 1$

$\left[\tan^{-1}(x)\right]_{-\infty}^{\infty} = \frac{1}{k} \Rightarrow k = \frac{1}{\pi}$

Q42: ∵ $f(x)$ is p.d.f so $\int_{-\infty}^{\infty} f(x)dx = 1$

$\Rightarrow \int_0^\infty k x^n e^{-x}dx = 1 \Rightarrow \int_0^\infty e^{-x} x^{(n+1)-1}dx = \frac{1}{k}$

ie $\Gamma(n+1) = \frac{1}{k}$ or $k = \frac{1}{n!}$ —— (1)

Again $E(x) = 3 \Rightarrow \int_{-\infty}^{\infty} x f(x)dx = 3$

$\Rightarrow \int_0^\infty kx.x^n e^{-x}dx = 3 \Rightarrow \int_0^\infty e^{-x}.x^{(n+2)-1}dx = \frac{3}{k}$

ie $\Gamma(n+2) = \frac{3}{k}$ or $k = \frac{3}{(n+1)!}$ —— (2)

By (1) & (2), $k = \frac{1}{2}$ & $n = 2$

Q43: $\because f(x)$ is p.d.f so $\int_{-\infty}^{\infty} f(x)dx = 1$

ie Total Area under $f(x) = 1$

$\Rightarrow \frac{1}{2}\times 1\times h + \frac{1}{2}\times 1\times 2h + \frac{1}{2}\times 1\times 3h = 1$

or $h = 1/3$

Q44: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (-kx+6k)\,dx = 1$

$\Rightarrow k\left(\frac{x^2}{2}\right)_0^2 + 2k(x)_2^4 - \left(k\frac{x^2}{2}\right)_4^6 + 6k(x)_4^6 = 1$

$2k + 4k - 10k + 12k = 1 \Rightarrow k = 1/8$

Q45: ~~$E(x) = \sum p_i x_i$~~ Prob Dist is:

$$\boxed{\begin{array}{ccc} x & : & 0 & 1 \\ P(x) & : & 0.4 & 0.6 \end{array}}$$

Here $x_1 = 0$ & $x_2 = 1$

so $E(x) = \sum p_i x_i = p_1 x_1 + p_2 x_2 = 0.6$

$E(x^2) = \sum p_i x_i^2 = p_1 x_1^2 + p_2 x_2^2 = 0.6$

So $Var(x) = E(x^2) - E^2(x) = 0.6 - 0.36$

$= 0.24$

Q46: $N = 1000$ st., for single student

let $x = \{$Marks obtained$\} = \{1, -\frac{1}{4}\}$

$P(C) = \frac{1}{4}$, $P(W) = 3/4$

P. Dist as
$$\boxed{\begin{array}{cccc} x & : & 1 & -1/4 \\ P(x) & : & 1/4 & 3/4 \end{array}}$$

$E(x) = \sum p_i x_i = \frac{1}{4}\times 1 + \frac{3}{4}\times\left(\frac{-1}{4}\right) = \frac{1}{16}$

Av Marks obtained by single student in single Question $= \frac{1}{16}$

so Av Marks obtained by 1000 st. in 150 questions $= \frac{1}{16}\times 1000\times 150 = 9375$

Q47: WKT. $Var(x) = E(x^2) - E^2(x) = R$

and $Var \geq 0$ so $R \geq 0$

Q48: $x = \{$Length of word drawn$\} = \{3, 3, 4\}$

The Quick brown fox jumps over the lazy dog
3 5 5 3 5 4 3 4 3

i.e $x = \{3,3,3,3,4,4,5,5,5\}$

so $\bar{x} = \frac{\sum x}{n} = \frac{35}{9} = 3.8$

Q49: $x = \{$out come on a dice$\} = \{1,2,3,4,5,6\}$

$\bar{x} = \frac{\sum x}{n} = \frac{1+2+3+4+5+6}{6} = 3.5$

Q50: $p = 0.1$, $q = 1-p = 0.9$ & $n = 900$

Mean of Binomial distribution $= np = 90$

Variance " " $= npq = 81$

so SD $= \sqrt{81} = 9$

Q51: $x = \{$No. of Heads$\}$,

$n = 4$, $p = P(H) = \frac{1}{2}$ & $q = P(T) = \frac{1}{2}$

$P(x \geq 3) = P(x=3) + P(x=4)$

$= {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + {}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = \frac{5}{16}$

Q52: $P(H) = P(T) = \frac{1}{2}$

$P(4^{th}$ Head in 10^{th} throw$)$

$= P($exactly 3H in 1^{st} 9 throws$)$

$\times P($H in 10^{th} throw$)$

$= \left[{}^9C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^6\right]\times\frac{1}{2} = 0.082$

Q53: $P(\text{occuring } 3) = \frac{2}{6} = \frac{1}{3} = p$

$P(\text{Not occuring } 3) = \frac{4}{6} = \frac{2}{3} = q$

Total of 12 can be obtained only when there are exactly 4 'three' in 5 throw

ie ~~P(x=4)~~ $n = 5$ & $x = 4$.

$P(x=4) = {}^5C_4\, p^4 q^1$

$P(\text{Total } 12) = {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1$

Q54: $X = \{$No. of forward steps$\}$

$n = 11$, $p = P(FS) = 0.4$, $q = P(BS) = 0.6$

fav. Possibilities $= \left\{\begin{array}{l}\text{either 5 FS \& 6 BS} \\ \text{or 6 FS \& 5 BS}\end{array}\right\}$

so $P(x = 5 \text{ or } 6) = P(x=5) + P(x=6)$

$= {}^{11}C_5(0.4)^5(0.6)^6 + {}^{11}C_6(0.4)^6(0.6)^5$

$= 462\left(\frac{6}{25}\right)^5$

Q55: $X = \{$No. of Events in 2 yrs$\}$

Av No. of events $= 3/yr \approx 6/$two yrs

so $\lambda = 6$.

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-\lambda}\left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!}\right] = 0.062$$

Q56: $X = \{$No. of Accidents / month$\}$

Av $(\lambda) = 5.2$/month.

$$P(X<2) = P(X=0) + P(X=1)$$

$$= \frac{e^{-\lambda}\cdot\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} = 0.034$$

Q57: $X = \{$No. of individuals suffering from bad reaction$\}$

$n = 2000$, $p = 0.001$ so.

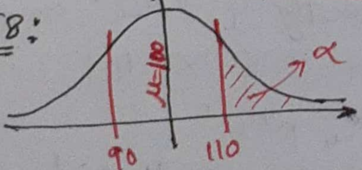
Av $(\lambda) = np = 2000 \times 0.001 = 2$

$$P(X>2) = 1 - P(X \le 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - e^{-\lambda}\left[\frac{\lambda^0}{\lfloor 0} + \frac{\lambda^1}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2}\right] = 1 - \frac{5}{e^2}$$

Q58:



$P(x > 110) = \alpha$ (given)

$P(100 < x < \infty) - P(100 < x < 110) = \alpha$

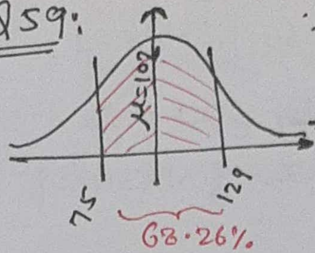
$\frac{1}{2} - P(100 < x < 110) = \alpha$

ie $P(100 < x < 110) = 0.5 - \alpha$

By symmetry $P(90 < x < 110) = 2P(100 < x < 110)$

$= 2(0.5 - \alpha) = 1 - 2\alpha$

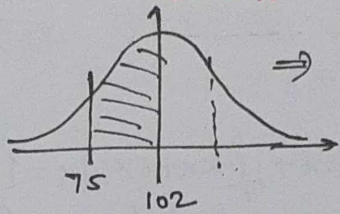
Q59:



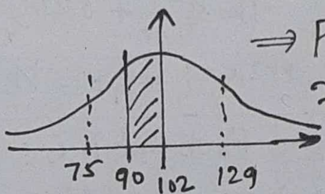
$\because \mu - \sigma = 102 - 27 = 75$

$\mu + \sigma = 102 + 27 = 129$

$\Rightarrow P(75 < x < 129) = 0.6826$



$\Rightarrow P(75 < x < 102) = \frac{0.6826}{2} = 0.3413$



$\Rightarrow P(90 < x < 102) \approx \frac{0.3413}{2} \approx 0.1706$

$\approx 17\%$.

So Ans $= 16.7\%$ (d)

Q60: on Comparision with the Standard formula of Normal distribution we have $\mu = a$, $\sigma = b$. So.

$$\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi} * b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx = \int_{-\infty}^{\mu} f(x)dx = \frac{1}{2}$$

Q61: $\mu = 30$, $\sigma = 5$,

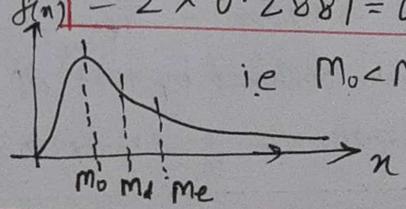
$$Z = \frac{x - \mu_x}{\sigma_x} \quad z_1 = \frac{26-30}{5} = -0.8, \quad z_2 = \frac{34-30}{5} = 0.8$$

$$P(26 \le x \le 34) = P(-0.8 \le z \le 0.8)$$

$$= 2P(0 \le z \le 0.8)$$

$$= 2 \times 0.2881 = 0.5762$$

Q62:



ie $M_o < M_d < M_e$

Q63: $\mu = 14$, $\sigma = 2.5$, $N = 100$ students.

For single students. $x = \{$Marks obtained$\}$

W.K.T. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$.

So $P(x=16) = f(16) = \frac{1}{2.5\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{16-14}{2.5}\right)^2\right\}$

$= 0.1157$

So % of Students getting Marks ~~the~~ '16'

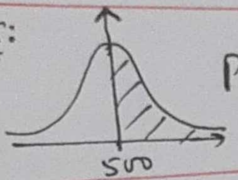
$= 100 \times 0.1157 = 12$

Q64: WKT. $P(\mu-3\sigma \le x \le \mu+3\sigma) = 0.997$

Put $\mu=0, \sigma=1$ then $x \simeq Z$

ie $P(-3 \le Z \le 3) = 0.997 = 99.7\%$

Q65:



$P(x>500) = \frac{1}{2}$

Q66: $t = \{\text{life of Bulb}\}$, $f(t) = \alpha e^{-\alpha t}$

$$P(100<t<200) = \int_{100}^{200} f(t)dt = \int_{100}^{200} \alpha e^{-\alpha t} dt$$

$$= \alpha\left(\frac{e^{-\alpha t}}{-\alpha}\right)_{100}^{200} = e^{-100\alpha} - e^{-200\alpha}$$

Q67: ATQ, $\frac{1}{\mu} = 1 \Rightarrow \mu = 1$. so p.d.f is

$$f(x) = \begin{cases} \mu e^{-\mu x}, & x \ge 0 \\ 0, & x<0 \end{cases} = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & x<0 \end{cases}$$

Hence c.d.f is.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} e^{-x}dx$$

$$= \left(\frac{e^{-x}}{-1}\right)_0^x = 1 - e^{-x}$$

Now $P(Z>2/Z>1) = \dfrac{P(Z>2 \cap Z>1)}{P(Z>1)}$

$$= \frac{P(Z>2)}{P(Z>1)} = \frac{1-P(Z\le 2)}{1-P(Z\le 1)}$$

$$= \frac{1-F(2)}{1-F(1)} = \frac{1-(1-e^{-2})}{1-(1-e^{-1})} = \frac{1}{e}$$

Q68: Av No. of vehicles arriving $= 360$ Veh/hr

ie $\lambda = 6$ Veh/min $= \frac{1}{10}$ Veh/sec.

so Av Gap {Headway} b/n two successive vehicles $= 10$ sec

$\because$ Inter arrival time follows exponential distribution so $\frac{1}{\mu} = 10$

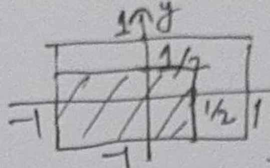
& p.d.f is $f(x) = \begin{cases} \mu e^{-\mu x}, & x \ge 0 \\ 0, & x<0 \end{cases}$

$$P(6<x<10) = \int_6^{10} f(x)dx = \int_6^{10} \mu e^{-\mu x}dx$$

$$= -\left(e^{-\mu x}\right)_6^{10} = e^{-6\mu} - e^{-10\mu} \quad (\because \mu = \tfrac{1}{10})$$

$$= e^{-0.6} - e^{-1} = 0.18.$$

Q69: $-1 \le x \le 1$ & $-1 \le y \le 1$



Total area $= (2)(2) = 4$.

Shaded area $= \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$

$$P[\max(x,y) < \tfrac{1}{2}] = P[-1 \le (x,y) < \tfrac{1}{2}]$$

$$= \frac{\text{Fav. Area}}{\text{Total area}} = \frac{9/4}{4} = \frac{9}{16}$$

Q70: $x = \{\text{Length of Shorter stick}\}$.

so $x \in (0, \frac{1}{2})$ i.e $0 \le x \le \frac{1}{2}$

$\because$ x is Uniform in b/n of $\frac{1}{2}$

so Av 'x' $= \bar{x} = \dfrac{0+\frac{1}{2}}{2} = \dfrac{1}{4}$

Q71: Let $x = \{\text{Arrival time}\}$, so $x \in (0,5)$ and it is URV so p.d.f is

$$f(x) = \begin{cases} \frac{1}{5-0}, & 0<x<5 \\ 0, & \text{otherwise.} \end{cases}$$

Now ATQ, waiting time will be a function of arrival time 'x' ie

waiting time $= g(x) = \begin{cases} 0, & 0 \le x \le 2 \\ 5-x, & 2 \le x \le 5 \end{cases}$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$= \int_0^5 g(x) \cdot \tfrac{1}{5} dx = \int_2^5 (5-x) \cdot \tfrac{1}{5} dx$$

$$= 0.9$$

T-14: ATQ $\frac{1}{\mu_1} = 0.5$ & $\frac{1}{\mu_2} = 0.25$

For Exponential Random Variables x_1 & x_2 we have

$$P(x_1 \le y) = 1 - P(x_1 > y) = 1 - \int_y^{\infty} f(x_1)dx_1 = 1 - e^{-\mu_1 y} \quad (A)$$

$$P(x_2 \le y) = 1 - P(x_2 > y) = 1 - \int_y^{\infty} f(x_2)dx_2 = 1 - e^{-\mu_2 y} \quad (B)$$

Now $P[\min(x_1,x_2) \le y] = 1 - P[\min(x_1,x_2) > y]$

$= 1 - P[x_1 > y] \cdot P[x_2 > y]$ ($\because x_1$ & x_2 are Ind)

$$= 1 - e^{-\mu_1 y} \cdot e^{-\mu_2 y} = 1 - e^{-(\mu_1+\mu_2)y}$$

Comparing above Result by (A) & (B) we say that $(\mu_1+\mu_2)$ is the parameter of y so

$$E(y) = \frac{1}{\mu_1+\mu_2} = \frac{1}{2+4} = \frac{1}{6}$$

LAPLACE TRANSFORM

Q1) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^k e^{-st}\left(\frac{1}{k}\right) dt + \int_k^\infty e^{-st}(1)\, dt$

$= \left(\frac{e^{-st}}{-ks}\right)_0^k + \left(\frac{e^{-st}}{-s}\right)_k^\infty$

$= -\frac{1}{ks}\left[e^{-ks} - 1\right] - \frac{1}{s}\left[e^{-\infty} - e^{-sk}\right]$

$= \frac{1 + (k-1)e^{-ks}}{ks}$

Q2) $f(t) = \begin{cases} 2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(2)\, dt$

$= 2\left(\frac{e^{-st}}{-s}\right)_0^1 = \frac{2 - 2e^{-s}}{s}$

Q3) $L\{t^{-1/2}\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

Q4) $L\{t^{1/2}\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

Q5) $L\{t \sinh at\} = ?$

wkt $L\{\sinh at\} = \frac{a}{s^2 - a^2}$

so $L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$

$= -\left[\frac{-2as}{(s^2 - a^2)^2}\right]$

Q6) wkt $L\{\sin 2t\} = \frac{2}{s^2 + 4} = F(s)$

so $L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty F(s)\, ds$

$= \int_s^\infty \frac{2}{s^2 + 4}\, ds$

$= 2\left(\frac{1}{2}\tan^{-1}\frac{s}{2}\right)_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$

$= \cot^{-1}\frac{s}{2}$

Q7) $L\{1 - e^{t}\} = \frac{1}{s} - \frac{1}{s-1} = F(s)$

so $L\left\{\frac{1 - e^{t}}{t}\right\} = \int_s^\infty F(s)\, ds$

$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$

$= \left(\log s - \log(s-1)\right)_s^\infty = \left(\log\frac{s}{s-1}\right)_s^\infty$

$= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right)$

Q8) w.k.t $L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{s}$

so $\int_0^\infty e^{-st} \cdot \frac{\sin at}{t}\, dt = \tan^{-1}\left(\frac{a}{s}\right)$

Put $s \to 0$ both sides

$\int_0^\infty \frac{\sin t}{t}\, dt = \tan^{-1}\left(\frac{a}{0}\right) = \frac{\pi}{2}$

Q9) $I = 2\int_{-\infty}^\infty \left(\frac{\sin 2\pi t}{\pi t}\right) dt$ even

$= 4\int_0^\infty \frac{\sin(2\pi t)}{\pi t}\, dt$

$= \frac{4}{\pi}\int_0^\infty \left(\frac{\sin(2\pi t)}{t}\right) dt = \frac{4}{\pi}\left(\frac{\pi}{2}\right)$

$= 2$

Q10) $\because L\{e^{t}\} = \frac{1}{s-1} = F(s)$

so $L\{t e^{t}\} = (-1)^1 \frac{d}{ds} F(s)$

$= -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$

Q11) W.K.T $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$

$\Rightarrow \int_0^\infty e^{-st}\cdot\left(\frac{\sin t}{t}\right)dt = \tan^{-1}\left(\frac{1}{s}\right)$

Put $s \to 1$ both sides

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \tan^{-1}(1) = \frac{\pi}{4}$$

Q12) $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{t\sin t\} = (-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$\int_0^\infty e^{-st}\cdot(t\sin t)dt = +\frac{2s}{(s^2+1)^2}$

Put $s \to 3$ both sides

$$\int_0^\infty e^{-3t}\cdot(t\sin t)dt = \frac{2\times3}{(9+1)^2} = \frac{3}{50}$$

Q13) $\frac{d^2f}{dt^2} + f = 0,\ f(0)=0,\ f'(0)=4$

or $f'' + f = 0$ —— (1)

Taking L.T both sides

$L\{f''\} + L\{f\} = 0$

$[s^2F(s) - sf(0) - f'(0)] + F(s) = 0$

$(s^2+1)F(s) - 0 - 4 = 0$

$F(s) = \frac{4}{s^2+1}$

Q14) $y'' + 4y = 12t,\ y(0)=0,\ y'(0)=9$

$L\{y''\} + 4L\{y\} = 12L\{t\}$

$[s^2F(s) - sf(0) - f'(0)] + 4F(s) = \frac{12}{s^2}$

$(s^2+4)F(s) - 0 - 9 = \frac{12}{s^2}$

$F(s) = \frac{12+9s^2}{s^2(s^2+4)}$

Taking Inverse L.T both sides

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{12+9s^2}{s^2(s^2+4)}\right\}$

$f(t) = L^{-1}\left\{\frac{3}{s^2} + \frac{6}{s^2+4}\right\}$ (PARTIAL FRACTION)

$= 3t + 3\sin 2t$

Q15) $\frac{d^2y}{dt^2} = 6\cos 2t,\ y(0)=3,\ y'(0)=1$

i.e $y'' - 6\cos 2t = 0$

$L\{y''\} - L\{6\cos 2t\} = 0$

$[s^2Y(s) - sy(0) - y'(0)] - \frac{6s}{s^2+4} = 0$

$s^2Y(s) - 3s - 1 - \frac{6s}{s^2+4} = 0$

$Y(s) = \left[\frac{6s}{s^2+4} + 1 + 3s\right]/s^2$

$Y(1) = \frac{\frac{6\times1}{1^2+4} + 1 + 3(1)}{1^2} = \frac{26}{5}$

Q16) $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \cancel{L^{-1}\left\{\frac{-1}{s+1} + \frac{1}{s+2}\right\}}$

$= L^{-1}\left\{\frac{-1/3}{s+1} + \frac{1/3}{s-2}\right\}$

$= -\frac{1}{3}(e^{-t}) + \frac{1}{3}(e^{2t}) = \frac{e^{2t} - e^{-t}}{3}$

Q17) $L^{-1}\{H(s)\} = L^{-1}\left\{\frac{s+3}{s^2+2s+1}\right\}$

$= L^{-1}\left\{\frac{(s+1)+2}{(s+1)^2}\right\}$

$= e^{-t}L^{-1}\left\{\frac{s+2}{s^2}\right\}$

$= e^{-t}\left[L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s^2}\right\}\right]$

$= e^{-t}(1+2t)$

Q18) Let $F(s) = \log(s-2) - \log(s-4)$

$$\frac{d}{ds}F(s) = \frac{1}{s-2} - \frac{1}{s-4}$$

$$(-1)^1 \frac{d}{ds} F(s) = \frac{1}{s-4} - \frac{1}{s-2}$$

$$L\{t \cdot f(t)\} = \frac{1}{s-4} - \frac{1}{s-2}$$

$$t\, f(t) = L^{-1}\left[\frac{1}{s-4} - \frac{1}{s-2}\right]$$

$$t\, f(t) = e^{4t} - e^{2t}$$

$$f(t) = (e^{4t} - e^{2t})/t$$

Q19) By F.V. Theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$f(\infty) = \lim_{s\to 0}\left(\frac{5s^2+23s+6}{s^3+2s+2}\right)$$

$$= 3$$

Q20) By F.V.Th.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

$$f(\infty) = \lim_{s\to 0}\left(\frac{2}{1+s}\right) = 2$$

I-1) $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2t\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4}\right) = \frac{2}{s(s^2+4)}$$

[Fourier Series]

Q1) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\cancel{x^2\frac{\sin nx}{n}} - \int 2x\frac{\sin nx}{n}dx\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\times\frac{2}{n}\left[\left(-x\frac{\cos nx}{n}\right) - \int 1\left(-\frac{\cos nx}{n}\right)dx\right]_0^{\pi}$$

$$= \frac{-4}{\pi n^2}\left[-x\cos nx + \cancel{\frac{\sin nx}{n}}\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2}\left[\pi\cos n\pi - 0\right] = \frac{4}{n^2}(-1)^n$$

Q2) ATQ, F.S of $f(x)$ is

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{4\cos(2n-1)\pi x}{(2n-1)^2\pi^2}$$

$$|x| = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right]$$

Put $x=0$ both sides

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

i.e. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

Q3) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$

$$a_0 = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{8\pi^2}{3}$$

But ATQ, $A_0 = \frac{a_0}{2} = \frac{4\pi^2}{3}$

Q4) For Half Range Fourier Cosine series, a_n is defined as

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \text{ in } (0,l)$$

~~where~~ In this question $l=1$.

so $a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$

$$= 2\int_0^1 (x-1)^2\cos(n\pi x)\,dx$$

$$= 2\left[(x-1)^2\frac{\cancel{\sin(n\pi x)}^{0}}{n\pi} - \int 2(x-1)\frac{\sin n\pi x}{n\pi}dx\right]_0^1$$

$$= \frac{-4}{n\pi}\left[(x-1)\left(-\frac{\cos n\pi x}{n\pi}\right) - \int\frac{\cos n\pi x}{n\pi}dx\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[-(x-1)\cos n\pi x + \frac{\cancel{\sin n\pi x}^{0}}{n^2\pi^2}\right]_0^1$$

$$= \frac{+4}{n^2\pi^2}\left[0-(0-1)\cos(0)\right] = \frac{4}{\pi^2 n^2}$$

Q5) $f(x) = \operatorname{cosec} x = \frac{1}{\sin x}, (-\pi,\pi)$

$x=0$ is point of Discontinuity &

$LHL = \lim_{x\to 0^-}\left(\frac{1}{\sin x}\right) = -\infty$ (DNE)

$RHL = \lim_{x\to 0^+}\left(\frac{1}{\sin x}\right) = +\infty$ (DNE)

For the existence of Fourier series, at point of Discontinuity x_0, both LHL & RHL should exist

But in this question these limits DNE so $f(x) = \operatorname{cosec} x$ can not be expanded in Fourier Series in the given interval.

Q6) $f(x) = x - x^3$ (odd function)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)\cos(nx)}_{\text{odd}}\,dx$$

$$= 0$$

Q7) Given F. Series is

$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + a_4\cos 4x + \dots$ —— (1)

Now $f(x) = \cos^2 x$

$$= \frac{1+\cos 2x}{2}$$

$= \frac{1}{2} + 0.\cos x + \frac{1}{2}\cos 2x + 0.\cos 3x + \dots$ —— (2)

On Comparision (1) & (2)

$$a_2 = \frac{1}{2} = 0.5$$

DIFFERENTIAL EQUATIONS

Q1) $[1+(y''')^2]^{4/3} = y''$

$[1+(y''')^2]^4 = (y'')^3$

$[1+(y''')^4+2(y''')^2]^2 = (y'')^3$

$(y''')^8 + 1 + 4(y''')^4 + 2(y''')^4$
$+ 4(y''')^6 + 4(y''')^2 = (y'')^3$

So order $= 3$ & Degree $= 8$

Q2) $[y + x(y'')^2]^{1/4} = y'''$

$[y + x(y'')^2] = (y''')^4$

order $= 3$, Degree $= 4$

Q3) Let centre be $(h, 0)$ & rad $= r$

then $(x-h)^2 + y^2 = r^2$ —— ①

Diff., $2(x-h) + 2yy' = 0$

or $x-h = -yy'$

Putting this value in ①

$(-yy')^2 + y^2 = r^2$

$y^2[1+(y')^2] = r^2$

Q4) $y = Ax + Bx^2$ —— ①

$y' = A + 2Bx$

$y'' = 2B$ $\Rightarrow$ $B = y''/2$

$y' = A + y''.x$ $\Rightarrow$ $A = y' - xy''$

Putting these values in ①

$y = (y' - xy'')x + \frac{y''}{2}x^2$

$y = xy' - \frac{x^2}{2}y''$

or $x^2y'' - 2xy' + 2y = 0$

Q5) A). $\frac{dy}{dx} = \frac{y}{x} \rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$

$\log y = \log x + \log c \Rightarrow \boxed{y = cx}$

B). $\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \int\frac{dy}{y} = -\int\frac{dx}{x}$

$\log y = -\log x + \log c \Rightarrow \boxed{xy = c}$

C). $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c$ or $\boxed{x^2 - y^2 = c}$

D) $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\,dy = -\int x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c \Rightarrow \boxed{x^2+y^2 = c}$

St. line $\downarrow$ A Hyperbola $\downarrow$ B & C circle $\downarrow$ D

Q6) $\frac{dy}{dx} = (1+y^2)x$

$\int\frac{dy}{1+y^2} = \int x\,dx + c$

$\tan^{-1} y = \frac{x^2}{2} + c$

$y = \tan\left(\frac{x^2}{2} + c\right)$

Q7) Put $x+y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

So $\frac{dy}{dx} = \cos(x+y)$

$\frac{dt}{dx} - 1 = \cos t$ or $\frac{dt}{dx} = 1 + \cos t$

$\frac{dt}{2\cos^2\frac{t}{2}} = dx \Rightarrow \frac{1}{2}\int \sec^2\frac{t}{2}\,dt = \int dx$

$\frac{1}{2}\frac{\tan\left(\frac{t}{2}\right)}{\left(\frac{1}{2}\right)} = x + c \rightarrow \tan\left(\frac{x+y}{2}\right) = x + c$

Q8) $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x} = \frac{\cos^2 y}{\sin^2 x}$

$\int \sec^2 y\,dy = \int \operatorname{cosec}^2 x\,dx + c$

$\tan y = -\cot x + c$

$\tan y + \cot x = c$

Q9 → $\frac{dy}{dx} = \frac{x^2+y^2}{2y} + \frac{y}{x}$ —— (1)

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

By (1), $v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2.vx} + \frac{vx}{x}$

$x\frac{dv}{dx} = \frac{x}{2}\left(\frac{1+v^2}{v}\right)$

$\int \frac{2v}{1+v^2}\,dv = \int dx + c$

$\log(1+v^2) = x + c$

$\log\left(1+\frac{y^2}{x^2}\right) = x + c$

using $y(1)=0 \Rightarrow c = -1$

Hence $\log\left(1+\frac{y^2}{x^2}\right) = x - 1$

Q10 → $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$

$\frac{y}{\sqrt{1-y^2}}\,dy = -\frac{x}{\sqrt{1-x^2}}\,dx$

Put $1-y^2 = t$ & $1-x^2 = \alpha$

$y\,dy = -\frac{dt}{2}$ & $x\,dx = -\frac{d\alpha}{2}$

So $-\frac{1}{2}\frac{dt}{\sqrt{t}} = -\left[-\frac{1}{2}\frac{d\alpha}{\sqrt{\alpha}}\right]$

$\int t^{-1/2}\,dt = -\int \alpha^{-\frac{1}{2}}\,d\alpha + c$

$t^{1/2} = -\alpha^{1/2} + c$

$\sqrt{1-x^2} + \sqrt{1-y^2} = c$

Q11 → Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So. $\frac{dy}{dx} = \left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)$

$v + x\frac{dv}{dx} = v + \tan v$

$\frac{dv}{\tan v} = \frac{dx}{x} \Rightarrow \int \cot v\,dv = \int \frac{dx}{x}$

$\log \sin v = \log x + \log c$

$\sin v = xc \Rightarrow \sin\left(\frac{y}{x}\right) = xc$

Q12 →

$(\alpha xy^3 + y\cos x)dx + (x^2y^2 + \beta\sin x)dy = 0$

on Comparision with $Mdx + Ndy = 0$

$M = \alpha xy^3 + y\cos x$, $N = x^2y^2 + \beta\sin x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$3\alpha xy^2 + \cos x = 2xy^2 + \beta\cos x$

$\Rightarrow \alpha = \frac{2}{3}$, $\beta = 1$

Q13 → $(\cos y\sin 2x)dx + (\cos^2 y - \cos^2 x)dy = 0$

$M = \cos y\sin 2x$, $N = \cos^2 y - \cos^2 x$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not exact

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{1}{\cos y\sin 2x}\left[\sin 2x + \sin y\sin 2x\right]$

$= \frac{1+\sin y}{\cos y} = \sec y + \tan y = f(y)$

So IF $= e^{\int f(y)dy} = e^{\int(\sec y + \tan y)dy}$

$= e^{\log(\sec y + \tan y) + \log\sec y}$

$= e^{(\log \sec^2 y + \sec y\tan y)}$

$= \sec^2 y + \sec y\tan y.$

Q14 → $M = x^2+y^2+2x$, $N = 2y$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so given DE is not exact

Now $\frac{1}{N}\left\{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right\} = \frac{1}{2y}(2y-0) = 1$

So IF $= e^{\int f(x)dx} = e^{\int 1dx} = e^x$

Multiplying the given DE by IF

$(x^2+y^2+2x)e^x dx + 2y e^x dy = 0$

$\underbrace{y^2 e^x dx + 2y e^x dy} + (x^2+2x)e^x dx = 0$

$\int d(y^2e^x) + \int (x^2+2x)e^x dx = \int 0$

$y^2e^x + x^2e^x - \int 2xe^x dx + \int 2xe^x dx = c$

$y^2e^x + x^2e^x = C$

$e^x(x^2+y^2) = C$

Q15 → $(1-xy)y\,dx + (-1-xy)x\,dy = 0$ — (1)

$M = (1-xy)y$ & $N = (-1-xy)x$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so (1) is not exact.

$IF = \frac{1}{Mx - Ny} = \frac{1}{(1-xy)xy - (-1-xy)xy}$

$= \frac{1}{2xy}$

So Multiplying (1) by IF we get

$\frac{(1-xy)y\,dx}{2xy} + \frac{(-1-xy)x\,dy}{2xy} = 0$

$\left(\frac{1}{x} - y\right)dx + \left(-\frac{1}{y} - x\right)dy = 0$

$\frac{dx}{x} - \frac{dy}{y} - (y\,dx + x\,dy) = 0$

$\int \frac{dx}{x} - \int \frac{dy}{y} - \int d(xy) = \int 0$

$\ln x - \ln y - xy = c$

$\ln\left(\frac{x}{y}\right) - xy = c$

Q16 → Multiplying the given DE by $x^\alpha y^\beta$ we have

$(x^{\alpha+7}y^{\beta+2} + 3x^\alpha y^{1+\beta})dx$

$+ (3x^{\alpha+8}y^{\beta+1} - x^{\alpha+1}y^\beta)dy = 0$

Now this equ^n must be exact, so

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$(\beta+2)y^{\beta+1} \cdot x^{\alpha+7} + 3(1+\beta)y^\beta x^\alpha$

$= 3(\alpha+8)x^{\alpha+7} \cdot y^{\beta+1} - (\alpha+1)x^\alpha y^\beta$

Compairing like coefficients.

$\beta+2 = 3(\alpha+8)$ & $3+3\beta = -\alpha-1$

on Solving; $\alpha = -7$, $\beta = 1$

Q17 → option (b), $\because$ it is in the form of $\frac{dy}{dx} + Py = Q$ where $P = -x^2$, $Q = \sin x$

i.e functions of x alone.

Q18 → $\frac{dy}{dx} + (2x)y = e^{-x^2}$; $y(0) = 1$ — (1)

$IF = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$

Sol of (1) is $y(IF) = \int Q(IF)dx + C$

$y(e^{x^2}) = \int e^{-x^2} e^{x^2} dx + C$

$y(e^{x^2}) = x + C$

using $y(0) = 1 \Rightarrow C = 1$

So sol is. $y(e^{x^2}) = x+1$

or $y = (x+1)e^{-x^2}$

Q19 → $y'(y'+y) = x(x+y)$ — (1)

$(y')^2 - x^2 + y'y - xy = 0$

$(y'-x)(y'+x) + y(y'-x) = 0$

$(y'-x)(y'+x+y) = 0$

$\because y'-x \neq 0$ (given), so $y'+x+y = 0$

or $y' + (1)y = (-x)$

it is L.D.E in y & x.

$IF = e^{\int 1 dx} = e^x$ & sol is

$y(e^x) = \int e^x \cdot (-x)dx + C$

$$y e^{x} = -(x e^{x} - e^{x}) + c$$
$$y = (1-x) + c e^{-x}$$
Using $y(0) = 2 \Rightarrow c = 1$

Hence $y = (1-x) + e^{-x}$

Q20 → $\frac{dy}{dt} + p(t).y = q(t).y^n$

$$\frac{1}{y^n}\frac{dy}{dt} + p\left(\frac{1}{y^{n-1}}\right) = q \quad \text{--- (1)}$$

Put $y^{1-n} = v$

$$\frac{d}{dt}(y^{1-n}) = \frac{dv}{dt}$$
$$(1-n)\, y^{-n}.\frac{dy}{dt} = \frac{dv}{dt}$$
$$y^{-n}\frac{dy}{dt} = \frac{1}{1-n}.\frac{dv}{dt}$$

Putting these values in (1)

$$\frac{1}{1-n}\frac{dv}{dt} + p.v = q$$
$$\frac{dv}{dt} + \underbrace{(1-n)p}.\, v = \underline{(1-n)q}$$

it is LDE in V & t.

Q21 → $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1}y}{1+y^2} \quad \text{--- (1)}$$

it is LDE in x & y

$$IF = e^{\int P dy} = e^{\int\frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

Sol of (1) is

$$x(IF) = \int Q(IF)dy + c$$
$$x\left(e^{\tan^{-1}y}\right) = \int \frac{\tan^{-1}y}{1+y^2}.e^{\tan^{-1}y}dy + c$$

Put $\tan^{-1}y = t \Rightarrow \frac{dy}{1+y^2} = dt$

$$x\left(e^{\tan^{-1}y}\right) = \int t.e^{t}.dt + c$$
$$x\left(e^{\tan^{-1}y}\right) = te^{t} - e^{t} + c$$
$$x.e^{\tan^{-1}y} = (\tan^{-1}y - 1)e^{\tan^{-1}y} + c$$
$$x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$$

Q22 → $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x.\frac{2\sin y\cos y}{\cos^2 y} = x^3$$
$$\sec^2 y\frac{dy}{dx} + x.(2\tan y) = x^3 \quad \text{--- (1)}$$

Put $2\tan y = t$

$$2.\sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$$

Putting these in (1)

$$\frac{1}{2}\frac{dt}{dx} + x.t = x^3$$
$$\frac{dt}{dx} + (2x)t = 2x^3 \quad \text{--- (2)}$$

it is LDE in t & x

$$IF = e^{\int P dx} = e^{\int 2x\,dx} = e^{x^2}$$

Sol of (2) is $y(IF) = \int Q(IF)dx + c$

$$y(e^{x^2}) = \int 2x^3.e^{x^2}dx + c$$

Putting $x^2 = \alpha \Rightarrow x\,dx = \frac{d\alpha}{2}$

$$y(e^{x^2}) = \int \alpha e^{\alpha}d\alpha + c$$
$$y(e^{\alpha}) = \alpha e^{\alpha} - e^{\alpha} + c$$
$$y = (\alpha - 1) + c e^{-\alpha}$$
$$y = (x^2 - 1) + c e^{-x^2}$$

Q23 → $(x^3y^2+xy)\frac{dx}{dy}=1$

$\frac{dy}{dx} - xy = x^3y^2$

$\frac{1}{y^2}\frac{dy}{dx} - x\left(\frac{1}{y}\right) = x^3$

Put $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$

$-\frac{dt}{dx} - x(t) = x^3$

$\frac{dt}{dx} + (x)t = -x^3$ —— (1)

It is LDE in x & t.

IF $= e^{\int P dx} = e^{\int x dx} = e^{\frac{x^2}{2}}$

Sol of (1) is $t(IF)=\int Q(IF)\,dx + c$

$t(e^{x^2/2}) = \int -x^3 \cdot e^{x^2/2}\, dx + c$

Put $\frac{x^2}{2}=\alpha \Rightarrow x\,dx = d\alpha$

$t(e^\alpha) = -\int 2\alpha e^\alpha\, d\alpha + c$

$t.e^\alpha = -2(\alpha e^\alpha - e^\alpha)+c$

$t = -2\alpha + 2 + c\,e^{-\alpha}$

$\cancel{\frac{1}{y} = 2(1-e^{x^2/2}) + c\,e^{-x}}$

$\frac{1}{y} = -2\left(\frac{x^2}{2}\right) + 2 + c\,e^{-x^2/2}$

$\frac{1}{y} = 2 - x^2 + c\,e^{-x^2/2}$

or $-\frac{1}{y} = x^2 - 2 + c\,e^{-x^2/2}$

$(\because -c \simeq c)$

Q24 → $\frac{dy}{dx} + (2\tan x)y = \sin x$ —— (1)

IF $= e^{\int 2\tan x\, dx} = e^{2\log \sec x} = \sec^2 x$

sol is $y(IF) = \int Q(IF)\,dx + c$

$y(\sec^2 x) = \int \sin x \sec^2 x\, dx + c$

Put $\tan x = t$ then $\sin x = \frac{t}{\sqrt{1+t^2}}$

$\sec^2 x\, dx = dt$.

$y(\sec^2 x) = \int \frac{t}{\sqrt{1+t^2}} \cdot dt + c$

Put $1+t^2=\alpha \Rightarrow t\,dt = \frac{d\alpha}{2}$

$y(\sec^2 x) = \int \frac{d\alpha}{2\sqrt{\alpha}} + c$

$= \frac{1}{2} \frac{\alpha^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$y(\sec^2 x) = \sqrt{\alpha} + c$

$y(\sec^2 x) = \sqrt{1+t^2}+c$

$y(\sec^2 x) = \sqrt{1+\tan^2 x} + c$

$y = \frac{\cancel{\sec^2 x}\ \sec x}{\sec^2 x} + \frac{c}{\sec^2 x}$

$y = \cos x + c\cdot\cos^2 x$ —— (2)

using $y\left(\frac{\pi}{3}\right)=0 \Rightarrow c=-2$

$y=\cos x - 2\cos^2 x$ —— (3)

For Maximum 'y', $\frac{dy}{dx}=0$

$-\sin x + 4\cos x \sin x = 0$

$\sin x(4\cos x - 1)=0$

$\sin x = 0$ or $\cos x = 1/4$.

$\left(\frac{d^2y}{dx^2}\right)_{\sin x=0} = +ve$ & $\left(\frac{d^2y}{dx^2}\right)_{\cos x=\frac{1}{4}} = -ve$

Hence maximum value of y ~~occur~~ will occur at $\cos x = 1/4$.

So $\text{Max}(y) = (y)_{\cos x = \frac{1}{4}}$

$= \left[\cos x - 2\cos^2 x\right]_{\cos x=\frac{1}{4}}$

$= 0.125$

Q25 → $y = k(x-1)$

$$\frac{dy}{dx} = k = \frac{y}{x-1}$$

i.e $m_1 = \dfrac{y}{x-1}$

Slope of O-family is;

$$\cancel{m_2 = -\frac{1}{m_1} = \frac{-1}{y/x-1} = \frac{1-x}{y}}$$

$$\cancel{\text{i.e } \frac{dy}{dx} = \frac{y}{1-x} - (\text{D.E of O-family})}$$

$$\cancel{\int \frac{dy}{y} = \int (1-x)\,dx + C}$$

~~log y~~

$$m_2 = -\frac{1}{m_1} = \frac{-1}{y/x-1} = \frac{1-x}{y}$$

Hence D.E of O-Family is

$$\frac{dy}{dx} = \frac{1-x}{y} \Rightarrow \int y\,dy = \int (1-x)\,dx$$

$$\frac{y^2}{2} = \frac{(1-x)^2}{-2} + C$$

$$\frac{(x-1)^2}{2} + \frac{y^2}{2} = c^2 \quad (\because c \simeq c^2)$$

Q26 → $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$

or $(D^2 + 6D + 8)x = 0$ ——— (1)

where $D = \dfrac{d}{dt}$

AE is $m^2 + 6m + 8 = 0 \Rightarrow m = -2, -4$

CF $= C_1 e^{-2t} + C_2 e^{-4t}$ & PI $= 0$

Sol is, $x = CF + PI$

$$x = C_1 e^{-2t} + C_2 e^{-4t} \quad \text{——— (2)}$$

$$\frac{dx}{dt} = -2C_1 e^{-2t} - 4C_2 e^{-4t} \quad \text{——— (3)}$$

using $x(0) = 1$ in (2) $\Rightarrow 1 = C_1 + C_2$

using $x'(0) = 0$ in (3) $\Rightarrow 0 = -2C_1 - 4C_2$

i.e $C_1 = -2C_2$ Hence $C_1 = 2, C_2 = -1$

So complete sol is given by (2)

$$x = 2e^{-2t} - e^{-4t}$$

Q27 → $\dfrac{d^2\eta}{dx^2} - \dfrac{\eta}{l^2} = 0$

if $\dfrac{d}{dx} = D$ then $\left(D^2 - \dfrac{1}{l^2}\right)\eta = 0$ ——— (1)

AE is $m^2 - \dfrac{1}{l^2} = 0 \Rightarrow m = \pm\dfrac{1}{l}$

CF $= C_1 e^{-\frac{x}{l}} + C_2 e^{-\frac{x}{l}}$ & PI $= 0$

Sol is $\eta = CF + PI$

i.e $\eta = C_1 e^{\frac{x}{l}} + C_2 e^{-\frac{x}{l}}$ ——— (2)

Using $\eta(0) = k \Rightarrow k = C_1 + C_2$

using $\eta(\infty) = 0 \Rightarrow 0 = C_1 e^{\infty} + C_2 e^{-\infty}$

so $C_1 = 0$ & $C_2 = k$

By (2), Sol is $\eta = k e^{-x/l}$

Q28 → $\dfrac{d^2x}{dt^2} + x = 0 \Rightarrow (D^2+1)x = 0$ ——— (1)

AE is $m^2 + 1 = 0 \Rightarrow m = \pm i = 0 \pm 1 \cdot i$

i.e Root are Imaginary so we can take $\alpha = 0$ & $\beta = 1$ Hence

$$Cf = e^{\alpha t}[C_1 \cos\beta t + C_2 \sin\beta t]$$

$= C_1 \cos t + C_2 \sin t$ & PI $= 0$

Sol is $x = CF + PI = C_1 \cos t + C_2 \sin t$

$\because (x_2)_{t=0} = 0$ & $\left(\dfrac{dx_2}{dt}\right)_{t=0} = 1$

$\Rightarrow x_1 = \cos t$ & $x_2 = \sin t$

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

Q29 → $(D^4+3D^2)y = 108x^2$ — (1)

AE is $m^4+3m^2=0 \Rightarrow m = 0, 0, \pm i\sqrt{3}$

$CF = (C_1+C_2x)e^{0x} + e^{\alpha x}[C_3\cos\beta x + C_4\sin\beta x]$

$CF = C_1 + C_2x + C_3\cos x\sqrt{3} + C_4\sin x\sqrt{3}$

$PI = \frac{1}{f(D)}Q = \frac{1}{D^4+3D^2}(108x^2)$

$= 108\left[\frac{1}{3D^2\left(1+\frac{D^2}{3}\right)}(x^2)\right]$

$= 36\left[\frac{\left(1+\frac{D^2}{3}\right)^{-1}}{D^2}(x^2)\right]$

$= 36\left[\frac{1}{D^2}\left\{1-\frac{D^2}{3}+\left(\frac{D^2}{3}\right)^2-\cdots\cdots\right\}x^2\right]$

$= 36\left[\frac{1}{D^2}\left(x^2-\frac{2}{3}\right)\right]$

$= 36\left[\iint\left(x^2-\frac{2}{3}\right)dx\,dx\right]$

$= 36\int\left(\frac{x^3}{3}-\frac{2}{3}x\right)dx$

$PI = 36\cdot\left(\frac{x^4}{12}-\frac{x^2}{3}\right) = 3x^4-12x^2$

Q30 → $\frac{d^2y}{dx^2}+16y=0 \Rightarrow (D^2+16)y=0$ — (1)

AE is $m^2+16=0 \Rightarrow m = \pm 4i = 0 \pm 4i$

Sol is $y = CF + PI$

$= e^{0x}[C_1\cos 4x + C_2\sin 4x] + 0$

$y = C_1\cos 4x + C_2\sin 4x$ — (2)

$y' = -4C_1\sin 4x + 4C_2\cos 4x$ — (3)

using $(y')_{x=0} = 1 \Rightarrow C_2 = 1/4$

using $(y')_{x=\frac{\pi}{2}} = -1 \Rightarrow C_2 = -1/4$

Absurd Result, Hence No Solution

Q31 → $y''+2y'-5y=0$

or $[D^2+2D-5]y=0$ — (1)

AE is $m^2+2m-5=0$

$m = \frac{-2\pm\sqrt{4+20}}{2} = -1\pm\sqrt{6}$

Hence G. Sol is $y = CF + PI$

$y = C_1e^{(-1+\sqrt{6})x} + C_2e^{(-1-\sqrt{6})x} + 0$

Q32 → $y^{iv}-2y'''+2y''-2y'+y=0$

or $(D^4-2D^3+2D^2-2D+1)y=0$

AE is $m^4-2m^3+2m^2-2m+1=0$

$m^3(m-1)-m^2(m-1)+m(m-1)-1(m-1)=0$

$(m-1)(m^3-m^2+m-1)=0$

$(m-1)[m^2(m-1)+1(m-1)] = 0$

$(m-1)^2(m^2+1)=0$

$m = 1, 1, \pm i$

$Cf = (C_1+C_2x)e^{1x} + e^{0x}[C_3\cos x + C_4\sin x]$

Sol is $y = CF + PI$

$y = (C_1+C_2x)e^x + C_3\cos x + C_4\sin x$

Q33 → $y''-4y'+3y = 2t-3t^2$

$[D^2-4D+3]y = 2t-3t^2$

$PI = \frac{1}{f(D)}Q = \frac{1}{D^2-4D+3}(2t-3t^2)$

$= \frac{1}{3}\left[1+\left(\frac{D^2-4D}{3}\right)\right]^{-1}(2t-3t^2)$

$= \frac{1}{3}\left[1-\left(\frac{D^2-4D}{3}\right)+\left(\frac{D^2-4D}{3}\right)^2-\cdots\cdots\right](2t-3t^2)$

$= \frac{1}{3}\left[(2t-3t^2)-\left(\frac{-6-4(2-6t)}{3}\right)+\right.$

$\left.\left(\frac{D^4+16D^2-8D^3}{9}\right)(2t-3t^2)+0\cdots\cdots\right]$

$$= \frac{1}{3}\left[(2t-3t^2) - \left(\frac{14+24t}{3}\right) + \frac{16(-6)}{9}\right]$$

$$= -2-2t-t^2$$

Q34 → $y'' - y' - 2y = 3e^{2x}$

or $(D^2 - D - 2)y = 3e^{2x}$ —— (1)

AE is $m^2 - m - 2 = 0 \Rightarrow m = -1, 2$

$CF = C_1 e^{-x} + C_2 e^{2x}$

$$PI = \frac{1}{f(D)} Q = \frac{1}{D^2-D-2}(3e^{2x})$$

$$= 3\left[\frac{x}{2D-1}(e^{2x})\right]$$

$$= 3\left[\frac{x}{2(2)-1} e^{2x}\right] = xe^{2x}$$

G. Sol is $y = CF + PI$

$$y = C_1 e^{-x} + C_2 e^{2x} + xe^{2x}$$ —— (2)

$\cancel{y' = -C_1 e^{-x} + 2C_2 e^{2x} + 2xe^{2x}}$ —— (3)

$\cancel{\text{using } y(0) = 0 \Rightarrow C_1 + C_2 = 0}$

$\cancel{\text{using } y'(0) = -2 \Rightarrow C_1 + 2C_2 = -2}$

using $y(0) = 0 \Rightarrow C_1 + C_2 = 0$

Now $y' = -C_1 e^{-x} + 2C_2 e^{2x} + e^{2x} + 2xe^{2x}$

using $y'(0) = -2 \Rightarrow -C_1 + 2C_2 + 1 = -2$

$C_2 = -1$ & $C_1 = 1$

Putting these values in (2)

$$y = e^{-x} - e^{2x} + xe^{2x}$$

Q35 → $y'' + y = 0 \Rightarrow (D^2+1)y = 0$ —— (1)

AE is $m^2 + 1 = 0 \Rightarrow m = \pm i = 0 \pm 1i$

$CF = e^{0t}[C_1 \cos t + C_2 \sin t]$ & $PI = 0$

Sol is $y = CF + PI = C_1 \cos t + C_2 \sin t$ —— (2)

using $y(0) = 1 \Rightarrow C_1 = 1$

$y' = -C_1 \sin t + C_2 \cos t$

using $y'(0) = 0 \Rightarrow C_2 = 0$

Hence by (2) Sol is $y = \cos t$.

Q36 → $y'' + by' + cy = 0 \Rightarrow (D^2 + bD + c)y = 0$ —— (1)

AE is $m^2 + bm + c = 0$

Let it's roots are m_1 & m_2 then

$m_1 + m_2 = -b$ & $m_1 m_2 = c$

Now given Sol is $y = xe^{-5x}$

$\Rightarrow$ Roots of AE are real & equal

i.e $m_1 = -5$ & $m_2 = -5$

Hence $m_1 + m_2 = -b$ & $m_1 m_2 = c$

$-5-5 = -b$ $\quad (-5)(-5) = c$

$b = 10$ $\quad c = 25$

i.e both b & c are +ve.

Q37 → $y'' - y = 2e^x$

$(D^2-1)y = 2e^x$ —— (1)

AE is $m^2 - 1 = 0 \Rightarrow m = 1, -1$

$CF = C_1 e^x + C_2 e^{-x}$

$$PI = \frac{1}{f(D)} Q = \frac{1}{D^2-1}(2e^x)$$

$$= 2\left[\frac{x}{2D}(e^x)\right] = xe^x$$

Sol is $y = CF + PI = C_1 e^x + C_2 e^{-x} + xe^x$ —— (2)

using $y(0) = 0 \Rightarrow C_1 + C_2 = 0$

Now $y' = C_1 e^x - C_2 e^{-x} + e^x + xe^x$

using $y'(0) = 0 \Rightarrow 0 = C_1 - C_2 + 1$

so $C_1 = -1/2$ & $C_2 = 1/2$

By (2), $y = -\frac{1}{2}e^x + \frac{1}{2}e^{-x} + xe^x$

so $y(1) = -\frac{e^1}{2} + \frac{e^{-1}}{2} + e^1 = \frac{e + e^{-1}}{2} = \cosh(1)$

$\underline{Q38} \rightarrow (D^2-4D+3)y = \cos x$ —— ①

$$PI = \frac{1}{f(D)} Q = \frac{1}{D^2-4D+3}(\cos x)$$

$$= \frac{1}{-1^2-4D+3}\cos x = \frac{1}{2}\frac{1}{(1-2D)}\cos x$$

$$= \frac{1}{2}\frac{1+2D}{1-4D^2}(\cos x) = \frac{1+2D}{2[1+4]}(\cos x)$$

$$= \frac{1}{10}(\cos x - 2\sin x)$$

$\underline{Q39} \rightarrow (D^2-9)y = e^{3x} + \sin 2x$ —— ①

$$PI = \frac{1}{f(D)} Q = \frac{1}{D^2-9}(e^{3x}+\sin 2x)$$

$$= \frac{1}{D^2-9}e^{3x} + \frac{1}{D^2-9}\sin 2x$$

$$= \frac{x}{2D}(e^{3x}) + \frac{1}{-4-9}\sin 2x$$

$$= \frac{x e^{3x}}{6} - \frac{1}{13}\sin 2x$$

$\underline{Q40} \rightarrow \frac{d^2y}{dx^2} = x$ —— ①

Integrating, $\frac{dy}{dx} = \frac{x^2}{2} + C_1$

Again integrating, $y = \frac{x^3}{6} + C_1 x + C_2$ —— ②

using $y(1)=0 \Rightarrow C_1 + C_2 = -\frac{1}{6}$

using $y(2)=0 \Rightarrow 2C_1 + C_2 = 8/6$

$C_1 = -\frac{7}{6}$, $C_2 = 1$

So by ② $y = \frac{x^3}{6} - \frac{7}{6}x + 1$

$\underline{Q41} \rightarrow (D^2-4D+4)y = x^2$

$$PI = \frac{1}{f(D)} Q = \frac{1}{D^2-4D+4}(x^2)$$

$$= \frac{1}{(D-2)^2}(x^2) = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}(x^2)$$

$$= \frac{1}{4}\left[1 + 2\left(\frac{D}{2}\right) + 3\left(\frac{D}{2}\right)^2 + \cdots\right](x^2)$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$$

$\underline{Q42} \rightarrow (D^2-4D+4)y = xe^{2x}$

$$PI = \frac{1}{D^2-4D+4}(xe^{2x}) = \frac{1}{(D-2)^2}xe^{2x}$$

$$= e^{2x}\left[\frac{1}{(D+2-2)^2}(x)\right] = e^{2x}\left[\frac{1}{D^2}x\right]$$

$$= e^{2x}\iint x\,dx\,dx = \frac{e^{2x}\cdot x^3}{6}$$

$\underline{Q43} \rightarrow (D^2-5D+6)y = e^{2x}\cos x$

$$PI = \frac{1}{(D^2-5D+6)}(e^{2x}\cos x)$$

$$= e^{2x}\left[\frac{1}{(D+2)^2-5(D+2)+6}\cos x\right]$$

$$= e^{2x}\left[\frac{1}{D^2-D}\cos x\right]$$

$$= e^{2x}\left[\frac{1}{-1-D}\cos x\right]$$

$$= -e^{2x}\left[\frac{D-1}{D^2-1}\cos x\right] = -e^{2x}\left[\frac{D-1}{-1-1}\cos x\right]$$

$$= \frac{1}{2}e^{2x}\left[-\sin x - \cos x\right]$$

$$= -\frac{1}{2}e^{2x}(\sin x + \cos x)$$

Q44 → $(D^2+9)y = x \sin 2x$

W.K.T $\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}(V)$

So $\frac{1}{D^2+9}(x\sin 2x) = x\frac{1}{D^2+9}(\sin 2x)$

$- \frac{2D}{(D^2+9)^2}\sin 2x$

$= x\frac{1}{-4+9}\sin 2x - \frac{2D}{(-4+9)^2}\sin 2x$

$= \frac{x\sin 2x}{5} - \frac{4\cos 2x}{25}$

Q45 → $(D^2+4)y = x\cos 2x$

$PI = \frac{1}{D^2+4}(x\cos 2x)$

$= x\frac{1}{D^2+4}\cos 2x - \frac{2D}{(D^2+4)^2}\cos 2x$

$= x\frac{x}{2D}\cos 2x - 2D\left[\frac{x}{2(D^2+4)2D}(\cos 2x)\right]$

$= \frac{x^2}{4}\sin 2x - \frac{x^2}{4D}(\cos 2x)$

$= \frac{x^2}{4}\sin 2x - \frac{x^2}{8}\sin 2x$

$= \frac{x^2}{8}\sin 2x$

Q46 → $PI = \frac{1}{D^2+9}(\sec 3x)$

$= \frac{1}{(D+3i)(D-3i)}(\sec 3x)$

$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right]\sec 3x$

$= \frac{1}{6i}\left[I_1 - I_2\right]$ —— (1)

$I = \frac{1}{D-3i}\sec 3x = e^{3ix}\int e^{-3ix}\sec 3x\,dx$

using $\frac{1}{D-\alpha}(Q) = e^{\alpha x}\int e^{-\alpha x}Q\,dx$

& $e^{i\theta} = \cos\theta + i\sin\theta$

$I_1 = e^{3ix}\int(\cos 3x - i\sin 3x)\sec 3x\,dx$

$= e^{3ix}\int(1 - i\tan 3x)\,dx$

$= e^{3ix}\left[x + \frac{i\log\cos 3x}{3}\right]$

Similarly $I_2 = e^{-3ix}\left[x - \frac{i\log\cos 3x}{3}\right]$

~~Putting these values in (1)~~

Now $I_1 - I_2$

$= e^{3ix}x + \frac{i}{3}e^{3ix}\log\cos 3x$

$- e^{-3ix}x + \frac{i}{3}e^{-3ix}\log\cos 3x$

$= x(e^{3ix} - e^{-3ix}) + \frac{i}{3}\log\cos 3x(e^{3ix} + e^{-3ix})$

$= x(2i\sin 3x) + \frac{i}{3}\log\cos 3x(2\cos 3x)$

Putting this value in (1)

$I = \frac{1}{6i}(I_1 - I_2)$

$= \frac{x}{3}\sin 3x + \frac{\cos 3x}{9}\log\cos 3x$

Q47 → $y'' + \lambda y' + y = 0 \Rightarrow [D^2 + \lambda D + 1]y = 0$

AE is $m^2 + \lambda m + 1 = 0$ —— (1)

Now it is given that two L.I. sol^n are $y_1 = e^{-x}$ & $y_2 = xe^{-x}$ so G. Sol will be $y = c_1e^{-x} + c_2xe^{-x}$

or $y = (c_1 + c_2x)e^{-x} = CF + PI$

So $CF = (c_1 + c_2x)e^{-x}$ & $PI = 0$

Hence Roots of AE will be $m = -1, -1$

AE is $(m+1)^2 = 0 \Rightarrow m^2 + 2m + 1 = 0$ —— (2)

Comparing (1) & (2)

$\lambda = 2$

Q48 → $y'' + \alpha y = -4\sin 2x$

or $(D^2+\alpha)y = -4\sin 2x$

$PI = \frac{1}{D^2+\alpha}(-4\sin 2x) = \frac{-4\sin 2x}{-4+\alpha}$

$\approx x\cos 2x \Rightarrow \alpha = 4.$

for, $PI = \frac{1}{D^2+4}(-4\sin 2x) = \frac{x}{2D}(-4\sin 2x)$

$= \frac{x}{2}\int(-4\sin 2x)\,dx = x\cos 2x$

Hence verified so $\boxed{\alpha=4}$.

Q49 → $D^2 y = -12x^2 + 24x - 20$ — (1)

Integrating $Dy = -4x^3 + 12x^2 - 20x + C_1$

Again Integrating

$y = -x^4 + 4x^3 - 10x^2 + C_1 x + C_2$ — (2)

using $y(0) = 5 \Rightarrow C_2 = 5$

using $y(2) = 21 \Rightarrow C_1 = 20$

So $y = -x^4 + 4x^3 - 10x^2 + 20x + 5$

$y(1) = -1^4 + 4(1)^3 - 10(1)^2 + 20(1) + 5 = 18$

Q50 → $x^2 y'' - xy' + y = \log x$ — (1)

$(x^2D^2 - xD + 1)y = \log x$

Putting $x = e^z$ or $z = \log x$ &

$xD = D_1$, $x^2D^2 = D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$

$(D_1(D_1-1) - D_1 + 1)y = z$

$(D_1-1)^2 y = z$ — (2)

AE is $(m-1)^2 = 0 \Rightarrow m = 1, 1$

$CF = (C_1 + C_2 z)e^z$

$PI = \frac{1}{f(D_1)} Q = \frac{1}{(D_1-1)^2}(z)$

$= (1-D_1)^{-2}(z)$

$= [1 + 2D_1 + 3D_1^2 + \dots\dots](z)$

$PI = z + 2 + 0$

Hence G. Sol is $y = CF + PI$

$y = (C_1 + C_2 z)e^z + (z+2)$

$= C_1 x + C_2 x\log x + \log x + 2$

Q51 → $x^2y'' - 2xy' + 2y = 0$

$[x^2D^2 - 2xD + 2]y = 0$ — (1)

using $x = e^z \Rightarrow z = \log x$ &

$xD = D_1$, $x^2D^2 = D_1(D_1-1)$, $D_1 = \frac{d}{dz}$

$[D_1(D_1-1) - 2D_1 + 2]y = 0$

$[D_1^2 - 3D_1 + 2]y = 0$ — (2)

AE is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$

$CF = C_1 e^z + C_2 e^{2z}$ & $PI = 0$.

Sol is $y = CF + PI = C_1 x + C_2 x^2$ — (3)

using $y(1) = 0 \Rightarrow C_1 + C_2 = 0$

using $y(2) = 2 \Rightarrow 2C_1 + 4C_2 = 2$

$C_1 = -1$, $C_2 = 1$

using (3), $\boxed{y = -x + x^2}$

$\Rightarrow y(3) = -3 + 3^2 = 6$

Q52 → $x^2y'' - 3xy' + 4y = 0$

$(x^2D^2 - 3xD + 4)y = 0$ — (1)

using $x = e^z$, $z = \log x$, $xD = D_1$,

$x^2D^2 = D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$

$[D_1(D_1-1) - 3D_1 + 4]y = 0$

$[D_1^2 - 4D_1 + 4]y = 0$ — (2)

AE is $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$CF = (C_1 + C_2 z)e^{2z}$ & $PI = 0$

G. Sol is $y = CF + PI = (C_1 + C_2\log x)x^2$

Hence $y_1 = x^2$ & $y_2 = x^2\log x$ are LI

T2 → $\frac{dy}{dx} = (4x+y+1)^2$ —— (1)

Put $4x+y+1 = t$

$4(1) + \frac{dy}{dx} + 0 = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$

$\frac{dt}{dx} - 4 = t^2 \Rightarrow \frac{dt}{dx} = (t^2+4)$

$\int \frac{dt}{t^2+4} = \int dx + c$

$\frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) = x + c$

$\frac{1}{2} \tan^{-1}\left[\frac{4x+y+1}{2}\right] = x + c$

T3 → ATQ, $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx$ —— (1)

$\int \frac{dx}{x} = k\int dt + \ln c$

$\ln x = kt + \ln c \Rightarrow \ln\left(\frac{x}{c}\right) = kt$

$x = ce^{kt}$ —— (2)

given $x(0)=1$, $x(2)=2$ & $x(?)=3$

Here we have assumed that original amount (at $t=0$ hrs) = 1 unit

using $x(0)=1 \Rightarrow c=1$

using $x(2)=2 \Rightarrow 2 = ce^{2k}$

$2 = 1 \cdot e^k \cdot e^k \Rightarrow e^k = \sqrt{2}$

Now using (2); $x(t) = ce^{kt}$

$\Rightarrow 3 = 1e^{kt}$

$3 = (e^k)^t = (\sqrt{2})^t$

$\log 3 = \log(\sqrt{2})^t$

$\log 3 = t\log\sqrt{2} = \frac{t}{2}\log 2$

$t = \frac{2\log 3}{\log 2}$

T4 → $\theta(0)=100$, $\theta(1)=75$, $\theta(3)=?$

ATQ, $\frac{d\theta}{dt} \propto -(\theta-25)$

$\frac{d\theta}{dt} = -k(\theta-25)$ —— (1)

$\int \frac{d\theta}{\theta-25} = -k\int dt + \log c$

$\log(\theta-25) = -kt + \log c$

$\theta = 25 + ce^{-kt}$ —— (2)

using $\theta(0)=100 \Rightarrow c=75$

using $\theta(1)=75 \Rightarrow 75 = 25 + 75e^{-k}$

i.e $e^{-k} = \frac{2}{3}$

$\theta(3) = 25 + 75e^{-3k}$

$= 25 + 75(e^{-k})^3 = 25 + 75\left(\frac{2}{3}\right)^3$

$= 47.22°C$

T8 → $\frac{dy}{dx}\cos(x+y) + \frac{\sin(x+y)}{x} = e^x - \cos(x+y)$

Put $x+y=t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\left(\frac{dt}{dx}-1\right)\cos t + \frac{\sin t}{x} = e^x - \cos t$

$x\,dt\cos t - \cancel{x\,dx\cos t} + \sin t\,dx$

$= xe^x dx - \cancel{x\,dx\cos t}$

$x\cos t\,dt + \sin t\,dx = xe^x dx$

$\int d(x\sin t) = \int xe^x . dx + c$

$x\sin t = xe^x - e^x + c$

$x\sin(x+y) - xe^x + e^x = c$

PARTIAL DIFF EQUATION

Q8→ $y^2p - xyq = x(z-2y)$ —(1)

on Comparision with $Pp+Qq=R$

$P=y^2,\ Q=-xy,\ R=x(z-2y)$

Lagrange's subsidiary equ^n's are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \boxed{\underset{\mathrm{I}}{\frac{dx}{y^2}} = \underset{\mathrm{II}}{\frac{dy}{-xy}} = \underset{\mathrm{III}}{\frac{dz}{x(z-2y)}}}$$

Taking I & II :→ $\frac{dx}{y^2}=\frac{dy}{-xy}$

$\int x\,dx+\int y\,dy=0 \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}=C$

or $x^2+y^2=C_1$

Taking II & III :→ $\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$

$z\,dy-2y\,dy=-y\,dz$

$z\,dy+y\,dz=2y\,dy$

$\int d(yz)=\int 2y\,dy$

$yz=y^2+C$

or $y^2-yz=C_2$

Hence G. Sol is $\phi(C_1,C_2)=0$

$$\phi\left[x^2+y^2,\ y^2-yz\right]=0$$

Q9→ $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$

$x^2p+y^2q=(x+y)z$ —(1)

$\Rightarrow P=x^2,\ Q=y^2,\ R=xz+yz$

Lagrange's Subsidiary equations are

$$\boxed{\underset{(\mathrm{I})}{\frac{dx}{x^2}}=\underset{(\mathrm{II})}{\frac{dy}{y^2}}=\underset{(\mathrm{III})}{\frac{dz}{(x+y)z}}}$$

$$\simeq \frac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{x+y-(x+y)} = \underset{(\mathrm{IV})}{\frac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{0}}$$

Taking I & II; $\int\frac{dx}{x^2}=\int\frac{dy}{y^2}$

$-\frac{1}{x}=-\frac{1}{y}+C \Rightarrow \frac{1}{x}-\frac{1}{y}=C_1$

Taking I & IV; $\frac{dx}{x^2}=\frac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{0}$

$\Rightarrow \int\frac{dx}{x}+\int\frac{dy}{y}-\int\frac{dz}{z}=0$

$\ln x+\ln y-\ln z=\ln C_2$

$\frac{xy}{z}=C_2$

Hence G. Sol. is $\phi(C_1,C_2)=0$

$$\phi\left[\frac{1}{x}-\frac{1}{y},\ \frac{xy}{z}\right]=0$$

Q14→ $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$ —(1)

Let $u=XT$ —(2) is the sol of (1) where $X=f(x)$ & $T=f(t)$ so.

$X'T=2XT'+XT$

$\frac{X'}{X}=\frac{2T'+T}{T}=k$ (let)

So $\frac{X'}{X}=k$ & $\frac{2T'+T}{T}=k$

$\int\frac{X'}{X}dx=\int k\,dx+\log C_1$ & $\frac{T'}{T}=\frac{k-1}{2}$

$\ln X=kx+\ln C_1$ & $\int\frac{T'}{T}dt=\int\frac{k-1}{2}dt+\ln C_2$

$X=C_1e^{kx}$ & $\ln T=\left(\frac{k-1}{2}\right)t+\ln C_2$

$X=C_1e^{kx}$ & $T=C_2e^{\left(\frac{k-1}{2}\right)t}$

Putting these values in (2)

$u=XT=C_1C_2e^{kx+\left(\frac{k-1}{2}\right)t}$ —(3)

$\because u(x,0)=6e^{-3x}$ (given) so.

$6e^{-3x}=C_1C_2e^{kx+0} \Rightarrow C_1C_2=6$ & $k=-3$

$$u=6e^{-3x-2t}$$

//

Q15 → $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ —— (1)

Let $u = XT$ — (2) is the sol of (1) then

$XT' = \alpha X''T \Rightarrow \boxed{\underset{\text{I}}{\frac{X''}{X}} = \underset{\text{II}}{\frac{T'}{\alpha T}} = \underset{\text{III}}{k\,(\text{let})}}$

Taking I & III; $\frac{X''}{X} = k \Rightarrow X'' - kX = 0$

$(D^2 - k)X = 0$; AE's $m^2 - k = 0 \Rightarrow m = \pm k$

$CF = c_1 e^{kx} + c_2 e^{-kx}$ & $PI = 0$.

So $X = CF + PI = c_1 e^{kx} + c_2 e^{-kx}$

Taking II & III; $\frac{T'}{\alpha T} = k$

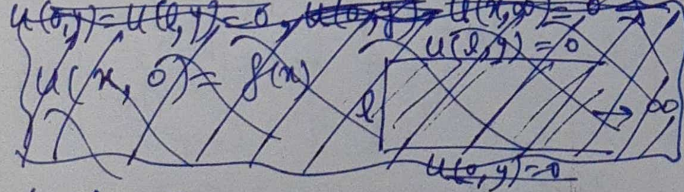
$\int \frac{T'}{T} dt = \int \alpha k \, dt + \log c_3$

$\log T = \alpha k t + \log c_3 \Rightarrow T = c_3 e^{\alpha k t}$

Putting these values in (2)

$u = XT = \left(c_1 e^{kx} + c_2 e^{-kx}\right) c_3 e^{\alpha k t}$ (b)

Q16 :→ For infinitely long plate having width 'l', temp distribution at any Random point is given by the sol. of following Laplace eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ where

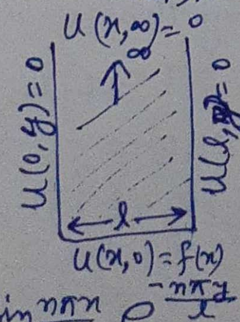


$u(0,y) = 0$

$u(l,y) = 0$

$u(x,\infty) = 0$

$u(x,0) = f(x)$



& it's sol is

$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{n\pi y}{l}}$ where

$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx.$

In this question, $l = \pi$ & $f(x) = 2$

So $u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-ny}$

where $b_n = \frac{2}{\pi}\int_0^{\pi} (2) . \sin nx \, dx$

$= \frac{4}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{4}{n\pi}\left[1 - \cos n\pi\right]$

$= \frac{4}{n\pi}\left[1 - (-1)^n\right]$

Q17 → w.k.T, De-Alembert's solution of wave eqn is

$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy$ —— (1)

where $u(x,0) = f(x)$ & $\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = g(x)$

In this question; $\frac{\partial^2 u}{\partial t^2} = \frac{1}{25}\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow c = \frac{1}{5}$, $f(x) = 3x$ & $g(y) = 3$

Putting these values in (1)

$u(x,t) = \frac{1}{2}\left[3(x+ct) + 3(x-ct)\right] + \frac{1}{2\left(\frac{1}{5}\right)}\int_{x-ct}^{x+ct} (3)\, dy$

$= 3x + \frac{5}{2}\cdot(3y)_{x-ct}^{x+ct}$

$= 3x + \frac{5}{2}\left[6ct\right] = 3x + 3t$

So $\left[u(x,t)\right]_{\binom{x=1}{t=1}} = 3(1) + 3(1) = 6$

Q18 → W.K.T the sol of Heat eqⁿ

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—(1)} \text{ is given as}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{\frac{-n^2\pi^2c^2t}{l^2}} \cdot \sin\left(\frac{n\pi x}{l}\right)$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

& Boundary conditions are;

$u(0,t) = 0$, $u(l,t) = 0$, $u(x,0) = f(x)$

Here 'l' is the length of Rod and initial temparature is $f(x)$.

in this question $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$c = 1$ & $l = \pi$ Hence sol is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cdot \sin nx$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$

T1 → $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ —(1)

Let $u = XY$ —(2) is the sol of (1) then

$X'Y = 4XY' \Rightarrow \boxed{\frac{X'}{4X} = \frac{Y'}{Y} = k \text{ (let)}}$

I II III

Taking I & III :→ $\frac{X'}{X} = 4k$

$$\int \frac{X'}{X}dx = \int 4k\,dx + \log c_1$$

$$\log X = 4kx + \log c_1 \Rightarrow X = c_1 e^{4kx}$$

Taking II & III :→ $\frac{Y'}{Y} = k$

$$\int \frac{Y'}{Y}dy = \int k\,dy + \log c_2$$

$$\log Y = ky + \log c_2 \Rightarrow Y = c_2 e^{ky}$$

By (2), $u = XY = c_1c_2 e^{4kx+ky}$ —(3)

using $u(0,y) = 8e^{-3y}$ in (3)

$$8e^{-3y} = c_1c_2 e^{0+ky}$$

$\Rightarrow c_1c_2 = 8$ & $k = -3$.

Putting these values in (3)

$$u = 8e^{-12x-3y}$$

LINEAR ALGEBRA

Q1→ Given $XY = Y$ & $YX = X$

So $X^2 + Y^2 = XX + YY$

$= X(YX) + Y(XY)$

$= (XY)X + (YX)Y$

$= YX + XY$ ⓐ

$= X + Y$

Q2→ $DABEC = I$

$D^{-1}DABECC^{-1} = D^{-1}IC^{-1}$

$ABE = D^{-1}C^{-1}$

$A^{-1}ABEE^{-1} = A^{-1}D^{-1}C^{-1}E^{-1}$

$B = A^{-1}D^{-1}C^{-1}E^{-1}$

Taking inverse both sides

$B^{-1} = (A^{-1}D^{-1}C^{-1}E^{-1})^{-1}$

$B^{-1} = (E^{-1})^{-1}(C^{-1})^{-1}(D^{-1})^{-1}(A^{-1})^{-1}$

using Reversal Law.

$B^{-1} = ECDA$ i.e ⓒ

Q3→ ⓒ Given $M^4 = I$ ——— ①

$\Rightarrow M^8 = M^{12} = M^{16} = \ldots\ldots = I$

i.e $M^{4K} = I$, $K \in N$

Again, $M^4 = I$ ——— ②

$\Rightarrow M.M^3 = I$

on Comparision with $MM^{-1} = I$

$\Rightarrow M^{-1} = M^3$

Now $M^{-1} = M^3 . I = M^3 M^{4K} = M^{4K+3}$

Q4→ $\because |A| = 6 \Rightarrow |(2A)^{-1}| = \frac{1}{|2A|}$

i.e $|(2A)^{-1}| = \frac{1}{2^3 |A|} = \frac{1}{8\times 6} = \frac{1}{48}$

Q5→ (c) $A = \frac{1}{9}\begin{bmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ \alpha & -7 & \beta \end{bmatrix}$

$\because$ A is Orthogonal,

$X_1 \cdot X_2 = 0$ & $X_2 \cdot X_3 = 0$

$\begin{bmatrix} 1 \\ 8 \\ \alpha \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} = 0$ & $\begin{bmatrix} -4 \\ 4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 1 \\ \beta \end{bmatrix} = 0$

$-4 + 32 + 7\alpha = 0$ & $-32 + 4 - 7\beta = 0$

$\alpha = 4$ & $\beta = -4$

Q6→ $P_{4\times 2}$, $Q_{2\times 4}$, $R_{4\times 1}$

There are two cases to find PQR

Case I: (PQ)R

For $P_{4\times 2} Q_{2\times 4}$,

No of Multiplication $= 4\times 2\times 4 = 32$

For $(PQ)_{4\times 4} R_{4\times 1}$,

No. of Multiplication $= 4\times 4\times 1 = 16$

i.e Total Multi $= 32 + 16 = 48$

Case II: P(QR)

For $Q_{2\times 4} R_{4\times 1}$,

No. of Multi $= 2\times 4\times 1 = 8$

For $P_{4\times 2}(QR)_{2\times 1}$,

No. of Multi $= 4\times 2\times 1 = 8$

i.e Total Multi $= 8 + 8 = 16$

So Min Multi $= \cancel{84}$ 16

Q7→ ⓓ $a_{ij} = i^2 - j^2 \ \forall\, i \,\&\, j$

$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$

it is skew symm Matrix of 3×3

$\Rightarrow |A| = 0 \Rightarrow A^{-1} = DNE.$

Q8→ $|A| = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Expanding along R_2

$|A| = (-1)\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$

$= - - - - -$

$= 88$

Q9→ $|AB| = |A|\cdot|B| = 5\times 40 = 200$

Q10→ $|A| = 100$ & Trace$(A) = 14$

$b[2\{5a-0\}] = 100$ & $a+5+2+b = 14$

$10ab = 100$ & $a+b=7$

$ab = 10$ & $a+b=7$

$\Rightarrow a=5$ & $b=2 \Rightarrow |a-b| = 3$

Q11→ $|A| = \alpha^2 - 4$

and $|A^3| = 125 \Rightarrow |A|^3 = 125$

$\Rightarrow |A| = 5 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$

Q12→ (a) $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$

K = Cofactor of a_{23}

$= (-1)^{2+3}\cdot$ Minor of a_{23}

$= -[1\times 5 - 0] = -5$

Q13→ (b) $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow |R| = 1$

$C_{11} = +M_{11} = +(2+3) = 5$

$C_{21} = -M_{21} = -(0+3) = -3$

$C_{31} = +M_{31} = +(0+1) = 1$

So Top Row of $R^{-1} = \frac{1}{|A|}[C_{11}\ C_{21}\ C_{31}]$

$= [5\ -3\ 1]$

Q14 (c)→ $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so $\rho(A) = 2$

Q15 (b)→ $a_{ij} = i\cdot j$, Let $n=3$

$A = [a_{ij}]_{3\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $\rho(A) = 1$

Q16 (b)→

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_3$

$A \sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix}$

Now $R_3 \to R_3 - R_2$ & $R_4 \to R_4 - R_2$

$A \sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 2$

Q17→ $R_2 \leftrightarrow R_3$, $R_4 \leftrightarrow R_5$

$A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$R_5 \to R_5 + (R_1 + R_2 + R_3 + R_4)$

$A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 4$

Q18→ $\rho(A)_{3\times 3} = 2 \Rightarrow |A| = 0$

$\begin{vmatrix} 4 & 7 & 2 \\ 3 & -1 & 5 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow x = 7$

Q19→ $\rho(X_{n\times 1}) = 1$ & $\rho(X^T_{1\times n}) = 1$

$\rho(XX^T) = \rho(X\cdot X^T)_{n\times n} = 1$

$\because$ Rank of product can never exceed their individual Ranks.

Q20→ $\rho(V_{3\times 1}) = 1$ & $\rho(V^T_{1\times 3}) = 1$

so $\rho(VV^T)_{3\times 3} = 1$

Reason: Rank of product can never exceed their individual Rank.

Q21 → $P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$

$R_3 \to \frac{1}{8} R_3$

$P+Q \sim \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_3$, $P+Q \sim \begin{bmatrix} 1 & 1 & 1 \\ 8 & 9 & 10 \\ 0 & -1 & -2 \end{bmatrix}$

$R_2 \to R_2 - 8R_1$,

$P+Q \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$P+Q \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(P+Q) = 2$

Q22 (b) →

$[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right]$

$R_1 \leftrightarrow R_3$; $[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & -8 & 7 \\ 4 & 3 & -12 & 5 \\ 2 & 1 & -4 & \alpha \end{array}\right]$

$R_2 \to R_2 - 4R_1$ & $R_3 \to R_3 - 2R_1$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -8 & 7 \\ 0 & -5 & 20 & -23 \\ 0 & -3 & 12 & \alpha-14 \end{array}\right]$

$R_2 \to -\frac{1}{5}R_2$, $R_3 \to -\frac{1}{3}R_3$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -8 & 7 \\ 0 & 1 & -4 & 23/5 \\ 0 & 1 & -4 & \frac{-\alpha+14}{3} \end{array}\right]$

$R_3 \to R_3 - R_2$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -8 & 7 \\ 0 & 1 & -4 & 23/5 \\ 0 & 0 & 0 & \left(\frac{-5\alpha+1}{15}\right) \end{array}\right]$

for ∞ solⁿ; $\rho(A) = \rho(A:B) = 2$

$\Rightarrow$ $-\frac{5\alpha+1}{15} = 0 \Rightarrow \alpha = \frac{1}{5}$

i.e only one value of α exist.

Q23 (b) → $A_{3\times4}$ & $[A:B]_{3\times5}$

$\because$ $AX = B$ is an inconsistent system $\Rightarrow \rho(A) \neq \rho(A:B)$

or $\rho(A) < \rho(A:B)$

$\because$ Max $\rho(A:B) \leq 3$

$\Rightarrow$ Max $\rho(A) \leq 2$

Q24 (b) → on comparision with

$AX = B \Rightarrow A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & -2 & 5 \\ -1 & -4 & 1 \end{bmatrix}$

$\because |A| \neq 0 \Rightarrow$ unique sol. exist.

Q25 (b) → $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}$

if $AX = B$ does not have unique sol then we have $|A| = 0$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 0 \Rightarrow k = 5$

Q26 (b) → $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{array}\right]$

$R_3 \to R_3 - R_2$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right]$

for No Solⁿ, $\rho(A) \neq \rho(A:B)$

let $\rho(A) = 2$ & $\rho(A:B) = 3$

$\Rightarrow \lambda = 6$ & $\mu \neq 20$

Q27 (b) → $[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$

$R_3 \to R_3 - R_2$

$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-b-3a \end{array}\right]$

For Consistency, $\rho(A) = \rho(A:B)$

$\Rightarrow c-b-3a = 0 \Rightarrow 3a+b-c=0$

Q28 → $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 5 \\ 1 & 2 & -k & 4 \end{array}\right]$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 3 \\ 0 & 1 & -k-1 & 3 \end{array}\right]$

$R_3 \to R_3 - R_2$

$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -k+2 & 0 \end{array}\right]$

For ∞ Sol^n, $\rho(A) = \rho(A:B) < 3$

$\Rightarrow k = 2$

Q29 (d) → for ∞ Sol^n, $|A| = 0$

$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0 \Rightarrow a = -2, 1$

Q30 (b) → $\because A_{3\times4}$ & $AX = 0$

$\Rightarrow$ it is underdetermined Homogeneous system, which has always infinite Sol^n.

Q31 (c) → System will Trivial Sol

if $|A| = 0 \Rightarrow \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$p^3 + q^3 + r^3 - 3pqr = 0$

$\Rightarrow (p+q+r)[p^2+q^2+r^2-pq-qr-rp] = 0$

$(p+q+r)[(p-q)^2 + (q-r)^2 + (r-p)^2] = 0$

$\rightarrow p+q+r = 0$ & $p = q = r$

Q32 → $A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}_{4\times4}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$A \sim \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$ i.e $\rho(A) = 1$

So Nullity = order − Rank = 4−1 = 3

Q33 (a) → $X = \{x \in \mathbb{R}^3 ;\ x_1 + x_2 + x_3 = 0\}$

i.e X is a set of vectors in which sum of components = 0

So Any vector having above property will be a member of X.

Now let us take $A = \{(1,-1,0)^T, (1,0,-1)^T\}$

where $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\because$ any member of X can be written as a linear combination of X_1 & X_2

So we can say that "A spans X"

Also X_1 & X_2 are L.I

So A is basis for X

Q34 (b) → $A_{m\times n} X_{n\times1} = O_{m\times1}$

So Dimension of Sol^n space $= n - r$

Q35 (b) → $A=\begin{bmatrix}0&1&1\\1&-1&0\\-1&0&-1\end{bmatrix}$

$R_1 \leftrightarrow R_2$, $A \sim \begin{bmatrix}1&-1&0\\0&1&1\\-1&0&-1\end{bmatrix}$

$R_3 \to R_3+R_1$, $A \sim \begin{bmatrix}1&-1&0\\0&1&1\\0&-1&-1\end{bmatrix}$

$R_3 \to R_3+R_2$, $A \sim \begin{bmatrix}1&-1&0\\0&1&1\\0&0&0\end{bmatrix}$

So $\rho(A)=2$ & Nullity $=1$

i.e Dimension of Null space $=1$

Q36 (a) using concept of Diagonalisation

$P^{-1}AP = D \Rightarrow A = PDP^{-1}$

where $P=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$, $D=\begin{bmatrix}8&0\\0&4\end{bmatrix}$

so $A=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}8&0\\0&4\end{bmatrix}\cdot\frac{1}{-2}\begin{bmatrix}-1&-1\\-1&1\end{bmatrix}$

$=\begin{bmatrix}6&2\\2&6\end{bmatrix}$

Q37 (C) → W.K.T. $AX=\lambda X$

$\begin{bmatrix}4&2\\2&4\end{bmatrix}\begin{bmatrix}101\\101\end{bmatrix}=\lambda\begin{bmatrix}101\\101\end{bmatrix}$

$\begin{bmatrix}606\\606\end{bmatrix}=\begin{bmatrix}101\lambda\\101\lambda\end{bmatrix} \Rightarrow \lambda=6$

Q38 (d) → $L(x)=M.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$\vec{b}\times\vec{x}=M\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&1&0\\x_1&x_2&x_3\end{vmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$x_3\hat{i}+0\hat{j}-x_1\hat{k}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$\begin{bmatrix}x_3\\0\\-x_1\end{bmatrix}=\begin{bmatrix}ax_1+bx_2+cx_3\\dx_1+ex_2+fx_3\\gx_1+hx_2+ix_3\end{bmatrix}$

On Comparision, $a=b=0$ & $c=1$, $d=e=f=0$ & $g=-1$, $h=i=0$

So $M=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$

C. Equⁿ of M is $|M-\lambda I|=0$

$\begin{vmatrix}-\lambda&0&1\\0&-\lambda&0\\-1&0&-\lambda\end{vmatrix}=0 \Rightarrow \lambda=0, i, -i$

Q39 (c) → $\lambda_1+\lambda_2+\lambda_3=$ Trace

$3+\lambda_2+\lambda_3=1+P$

So $\lambda_2+\lambda_3=P-2$

Q40 (c) → E Value of X are $-2, -3$

E. Value of $(X+I) = (-2+1)$ & $(-3+1)$
$= -1$ & -2

E. Value of $(X+I)^{-1} = \frac{1}{-1}$ & $\frac{1}{-2}$
$= -1$ & $-1/2$

E Value of $(X+5I) = (-2+5)$ & $(-3+5)$
$= 3$ & 2

E. Values of $(X+I)^{-1}(X+5I)$
$= (-1)(3)$ & $(-\frac{1}{2})(2)$
$= -3$ & -1

Q41 → Product of E Values $= |A|$
$= 12$

Q42 (a) → $A=\begin{bmatrix}2&1\\1&k\end{bmatrix}$

∵ E Values are +ve so

$\lambda_1+\lambda_2>0$ & $\lambda_1\lambda_2>0$

$Tr(A)>0$ & $|A|>0$

$2+k>0$ & $2k-1>0$

$k>-2$ & $k>1/2$

$\Rightarrow k>\frac{1}{2}$

By taking intersection of above two inequalities.

Q43 → (c) Let E.values are: $\lambda_1, \lambda_2, \lambda_3$

then $\lambda_1+\lambda_2+\lambda_3 = \text{Trace}(A) = 11$ —— ①

& $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A| = 61$ —— ②

Now only option ⓒ is satisfying ① & ② Hence

$\lambda_1 = 1, \lambda_2 = 5+6i, \lambda_3 = 5-6i$

Q44(c) → C. Equ^n of A is

$|A-\lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 & 4 & 5 \\ 5 & (1-\lambda) & 2 & 3 & 4 \\ 4 & 5 & (1-\lambda) & 2 & 3 \\ 3 & 4 & 5 & (1-\lambda) & 2 \\ 2 & 3 & 4 & 5 & (1-\lambda) \end{vmatrix} = 0$$

$R_1 \to R_1 + (R_2+R_3+R_4+R_5)$

$$\begin{vmatrix} (15-\lambda) & (15-\lambda) & (15-\lambda) & (15-\lambda) & (15-\lambda) \\ 5 & (1-\lambda) & 2 & 3 & 4 \\ 4 & 5 & (1-\lambda) & 2 & 3 \\ 3 & 4 & 5 & (1-\lambda) & 2 \\ 2 & 3 & 4 & 5 & (1-\lambda) \end{vmatrix} = 0$$

$$(15-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & (1-\lambda) & 2 & 3 & 4 \\ 4 & 5 & (1-\lambda) & 2 & 3 \\ 3 & 4 & 5 & (1-\lambda) & 2 \\ 2 & 3 & 4 & 5 & (1-\lambda) \end{vmatrix} = 0$$

$\Rightarrow \lambda = 15$ which is real E Value so No Need to find other Eigen values as is done.

Q45(a) → $A = [a_{ij}] = \begin{cases} i, & i=j \\ 0, & i \neq j \end{cases}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & n \end{bmatrix}_{n\times n}$$

Now, Sum of E Values = Trace(A)

so Req. Sum $= 1+2+3+\cdots+n$

$= \frac{n(n+1)}{2}$

Q46 (c) →
$$A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$$

$\because |A| = \begin{vmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{vmatrix} = \cdots = 0$

So one E. Value is $\lambda_1 = 0$

& ATQ, another E Value $\lambda_2 = \lambda$

and we know that

$\lambda_1+\lambda_2+\lambda_3 = \text{Trace}(A)$

$0 + \lambda + \lambda_3 = 40+30-50$

i.e $\lambda_3 = (20-\lambda)$

Q47(d) → W.K.T. Cube Root of unity is defined as

$\omega = \frac{-1+i\sqrt{3}}{2}$ & $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

where $\omega^{3n} = 1$ & $1+\omega+\omega^2 = 0$

i.e $\omega^3 = \omega^6 = \omega^9 = \omega^{12} = \cdots = 1$

Now $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix}$

which is L.T.M, So

E. Values of A are $1, \omega, \omega^2$

" " " A^{102} are $1^{102}, \omega^{102}, \omega^{204}$

$= 1, 1, 1$

Hence $\text{Trace}(A^{102})$ = Sum of E Value

$= 1+1+1 = 3$

Q48 (d) → $A_{3\times3}$ s.t

$|A-I|=0$, $Trace(A)=13$ & $|A|=32$

on comparision with

$|A-\lambda_1 I|=0$, $\lambda_1+\lambda_2+\lambda_3=13$, $\lambda_1\lambda_2\lambda_3=32$

$\Rightarrow \lambda_1=1$ so $1+\lambda_2+\lambda_3=13$, $1\cdot\lambda_2\lambda_3=32$

$\Rightarrow \lambda_2+\lambda_3=12$ & $\lambda_2\lambda_3=32$

On solving $\lambda_2=8$, $\lambda_3=4$

i.e E.Values of A are ;

$\lambda_1=1$, $\lambda_2=8$, $\lambda_3=4$

Hence Req. Sum $=\lambda_1^2+\lambda_2^2+\lambda_3^2$

$=1^2+8^2+4^2=81$

Q49 (b) →

$$M=\begin{bmatrix}2 & 3+2i & -4\\ 3-2i & 5 & 6i\\ -4 & -6i & 3\end{bmatrix}$$

$\because M^\theta=M \Rightarrow M$ is Hermitian

Now, $(iM)^\theta = i^\theta M^\theta = (\bar{i})^T(M)$

$=(-i)^T M = -iM$

Hence iM is skew Hermitian

so statement Q is correct.

Also w.K.T, E Values of Hermitian Matrix lies on Real axis so statement R also correct

So Ans $=$ (b)

Q50 → W- $A=\begin{bmatrix}1&0&0&0&1\\0&1&1&1&0\\0&1&1&1&0\\0&1&1&1&0\\1&0&0&0&1\end{bmatrix}_{5\times5}$

$R_3\to R_3-R_2$, $R_4\to R_4-R_2$, $R_5\to R_5-R_1$

$A\sim\begin{bmatrix}1&0&0&0&1\\0&1&1&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} \Rightarrow \rho(A)=2$

$\because$ No. of Non Zero E Values $\le$ Rank

$\Rightarrow$ out of 5 E.Values of A, three E.Values are $\lambda=0,0,0$

Now C Equ^n of A is

$|A-\lambda I|=0$

$$\begin{vmatrix}(1-\lambda)&0&0&0&1\\0&1-\lambda&1&1&0\\0&1&1-\lambda&1&0\\0&1&1&1-\lambda&0\\1&0&0&0&1-\lambda\end{vmatrix}=0$$

$R_1\to R_1+R_5$ & $R_2\to R_3+R_3+R_4$

$$\begin{vmatrix}(2-\lambda)&0&0&0&(2-\lambda)\\0&(3-\lambda)&(3-\lambda)&(3-\lambda)&0\\0&1&(1-\lambda)&1&0\\0&1&1&(1-\lambda)&0\\1&0&0&0&(1-\lambda)\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)(3-\lambda)\begin{vmatrix}1&0&0&0&1\\0&1&1&1&0\\0&1&(1-\lambda)&1&0\\0&1&1&(1-\lambda)&0\\1&0&0&0&(1-\lambda)\end{vmatrix}=0$$

So $\lambda=2$ & 3

& there is No Need to Solve Rest E. Values because these are obviously 0, 0, 0 as discussed above.

Hence E.Values of $A=2,3,0,0,0$

So Product of Non Zero E Values $=2\times3=6$

Q51 → ∵ $A_{2\times2}$ and given,
Trace(A) = 4 & Trace(A^2) = 5
$\Rightarrow \lambda_1+\lambda_2=4$ & $\lambda_1^2+\lambda_2^2=5$
So $(\lambda_1+\lambda_2)^2=\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2$
$(4)^2=5+2\lambda_1\lambda_2$
i.e $\lambda_1\cdot\lambda_2=\frac{11}{2}$
$\Rightarrow |A|=\frac{11}{2}=5.5$

Q52 → (b) $A=\begin{bmatrix}1&-1&2\\0&1&0\\1&2&1\end{bmatrix}$

Given $\lambda=1$
Let us take option (b)
$AX=\lambda X$
$$\begin{bmatrix}1&-1&2\\0&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}-4\\2\\1\end{bmatrix}=\lambda\begin{bmatrix}-4\\2\\1\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}-4\\2\\1\end{bmatrix}=\lambda\begin{bmatrix}-4\\2\\1\end{bmatrix}$$
$\Rightarrow \lambda=1$ Hence verified
i.e our assumption is Correct

Q53 : → C. Equⁿ of A is $|A-\lambda I|=0$
$$\begin{vmatrix}(2-\lambda)&1&1\\1&(2-\lambda)&1\\0&0&(1-\lambda)\end{vmatrix}=0$$
After solving we get $(\lambda-1)^2(\lambda-3)=0$
ie $\lambda=3,1,1$
so Largest E. Value is $\lambda=3$
Now using options (b)
$AX=\lambda X$
$$\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\1\\0\end{bmatrix}$$
$$\begin{bmatrix}3\\3\\0\end{bmatrix}=\begin{bmatrix}\lambda\\\lambda\\0\end{bmatrix}\Rightarrow\lambda=3$$
Hence verified i.e our assumption is correct.

Q54 (d) : Given $P\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}3\\0\\-3\end{bmatrix}=3\begin{bmatrix}1\\0\\-1\end{bmatrix}$
so $\lambda=3$ is an E. Value of P
and remaining E Values are 1 & 2
So $\lambda_1=1,\lambda_2=2,\lambda_3=3$
Now $\lambda_1+\lambda_2+\lambda_3=\text{Trace}(P)$
$1+2+3=0+1+b\Rightarrow b=5$
& $\lambda_1\lambda_2\lambda_3=|P|$
$$1\times2\times3=\begin{vmatrix}0&-2&-3\\-1&1&-1\\a&2&b\end{vmatrix}$$
$6=0[b+2]-(-1)[-2b+6]+a[2+3]$
$6=0-2b+6+5a$
$5a=2b\Rightarrow a=\frac{2b}{5}=\frac{2\times5}{5}=2$
i.e $P=\begin{bmatrix}0&-2&-3\\-1&1&-1\\2&2&5\end{bmatrix}$
∵ E Values of P = 1, 2, 3
So E Values of $P^4=1^4,2^4,3^4$
$=1,16,81$
Again we know that E Vectors of P & P^4 are same.
Now E. Value Corresponding to vector $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ can be calculated as follows; $PX=\lambda X$
$$\begin{bmatrix}0&-2&-3\\-1&1&-1\\2&2&5\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\-1\\0\end{bmatrix}$$
$$\begin{bmatrix}2\\-2\\0\end{bmatrix}=\begin{bmatrix}2\lambda\\-2\lambda\\0\end{bmatrix}\Rightarrow\lambda=2$$
so For Matrix P, E Vector for $\lambda=2$ is $\begin{bmatrix}1\\-1\\0\end{bmatrix}$
Hence " " P^4, " " $\lambda=2^4$ will be $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ so we have,
$$P^4\begin{bmatrix}1\\-1\\0\end{bmatrix}=2^4\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}16\\-16\\0\end{bmatrix}$$

Q55 → $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ ⟨ $\lambda_1+\lambda_2 = 130$, $\lambda_1\lambda_2 = -900$

$X_1^T X_2 = [70 \ (\lambda_1 - 50)] \cdot \begin{bmatrix} (\lambda_2 - 80) \\ 70 \end{bmatrix}$

$= [(70\lambda_2 - 5600) + (70\lambda_1 - 3500)]$

$= [70(\lambda_1+\lambda_2) - 4900]$

$= [70 \times 130 - 4900] = [0]$

Q56 (a) → (I) C. Equⁿ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (2-\lambda) & 2 & 0 & 0 \\ 2 & (1-\lambda) & 0 & 0 \\ 0 & 0 & (3-\lambda) & 0 \\ 0 & 0 & 1 & (4-\lambda) \end{vmatrix} = 0$$

After solving; $\lambda = 3, 4, \frac{3 \pm \sqrt{17}}{2}$

∵ we have Four different E. Values so Corresponding E. Vectors will be LI i.e A is diagonalizable

∵ Necessary condition Holds as $|A| \neq 0$ and Sufficient Condition also Holds as Number of LI E. Vectors = order

(II) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ ⇒ E Values are $\lambda = 2, 2$

∵ $|A| \neq 0$ so Necessary Condition Holds

Now Number of LI E. Vectors for $(\lambda = 2)$

$= \text{order} - \rho(A - 2I) = 2 - \text{Rank of} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= 2 - 1 = 1$

∵ Number of LI E. Vectors ≠ order so Sufficient condition not Holds Hence it is Not Diagonalizable.

Q57 (d) → Let P is of order 2×2

∵ P has only one E. Vector

⇒ E Values of P are necessarily repeated.

Again, order = 2 & No. of LI E Vectors = 1 So P can not be diagonalized.

Q58 → (C) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

∵ it is U.T.M so $\lambda = 2, 2, 3$

Now, Number of LI E. Vectors for $(\lambda = 2) = \text{order} - \rho(A - 2I)$

$= 3 - \text{Rank of} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$= 3 - 1 = 2$

Also W.K.T, E. Vectors for different E values are LI.

So over all we have 3 LI E Vectors for given Mat A.

Two for $(\lambda = 2)$ & one for $(\lambda = 3)$

Q59 → $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

∵ it is U.T.M so $\lambda = 2, 2, 3$

Now, Number of LI E. Vector for $(\lambda = 2) = \text{order} - \rho(A - 2I)$

$= 3 - \text{Rank of} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= 3 - 2 = 1$

So over all we have only two LI Eigen vectors. one for $(\lambda = 2)$ & one for $(\lambda = 3)$

Q60 (a) → C. Equⁿ of A is

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -3-\lambda & -2 \\ 1 & 0-\lambda \end{vmatrix} = 0$

$\lambda^2 + 3\lambda + 2 = 0$

By Cayley Hamilton theorem, we can replace $\lambda \to A$ so

$A^2 + 3A + 2I = 0$

i.e $A^2 = (-3A - 2I)$

$A^4 = A^2 \cdot A^2 = (-3A - 2I)(-3A - 2I)$

$= -15A - 14I$

$A^8 = A^4 \cdot A^4 = (-15A - 14I)(-15A - 14I)$

$= -255A - 254I$

$A^9 = A \cdot A^8 = (\overset{A}{\cancel{-3A-2I}})(-255A - 254I)$

$= -255A^2 - 254A$

$= -255(-3A - 2I) - 254A$

$= 511A + 510I$

Q61 (d) → C. Equ^n of P is

$\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

By C.H.T we can replace $\lambda \to P$

ie $P^3 + P^2 + 2P + I = 0$

Pre Multiplying by P^{-1} both sides

$P^{-1}P^3 + P^{-1}P^2 + 2P^{-1}P + P^{-1}I = P^{-1}.0$

$P^2 + P + 2I + P^{-1} = 0$

or $P^{-1} = -(P^2 + P + 2I)$

Q62 → (c) Characteristic equ^n of A is, $|A - \lambda I| = 0$

or $\lambda^2 - (\text{Trace } A)\lambda + |A| = 0$

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

$\cancel{\lambda^2 - \lambda + 1}$ $\lambda^2 - \lambda + 1 = 0$

By C.H.T, we can replace $\lambda \to A$

so $A^2 - A + I = 0$ i.e $A^2 = A - I$

& $A^3 = AA^2 = A(A - I) = A^2 - AI$

$= (A - I) - A = -I$

Q63 (d) → Consider $A_{n\times n}$ then standard form of Char. Equ^n is

$|A - \lambda I| = 0$

$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0$

where $|A| = (-1)^n \dfrac{a_n}{a_0}$

Now given char. equ^n is

$t^n + c_{n-1}t^{n-1} + \dots + c_1 t + c_0 = 0$

on comparision,

$\det(A) = (-1)^n c_0$

Q64 → C. Equ^n of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)[(1-\lambda)(2-\lambda) - 0] = 0$

$(\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$

$(\lambda - 1)(\lambda^2 - 4) = 0$

$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By C.H.T, $A^3 - A^2 - 4A + 4I = 0$ — ①

Now $B = A^3 - A^2 - 4A + 5I$

$= (A^3 - A^2 - 4A + 4I) + I$

$B = 0 + I = I$

So $|B| = |I| = 1$

Q65 → (d) By using options, Let us take (d).

$\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

Hence verified.

Rest of the options are wrong.

Note: we can also solve above Question by using CROUT method

Q66 → using Crout Method.

$A = LU$

$= \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}\begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{bmatrix}$

$l_{11} = 2,\ l_{21} = 4,\ l_{11}u_{12} = 2 \Rightarrow u_{12} = 1$

& $l_{21}u_{12} + l_{22} = 9$

$4(1) + l_{22} = 9 \Rightarrow l_{22} = 5$

T2 → we have three types of Mat

$[- -], [- - - -], [\vdots]$

& each Place can be filled by 4 ways so Ans = 3×4^4

T3 → $A\begin{bmatrix}2\\-1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\\0\end{bmatrix}$, $A\begin{bmatrix}-1\\2\\1\end{bmatrix}=2\begin{bmatrix}-1\\2\\1\end{bmatrix}$,

& $A\begin{bmatrix}0\\-1\\2\end{bmatrix}=3\begin{bmatrix}0\\-1\\2\end{bmatrix} \Rightarrow \lambda=1,2,3.$

& Modal Mat $= Q = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & 1 & 2\end{bmatrix}$

Now using concept of diagonalisation

$Q^{-1}AQ = D \Rightarrow AQ = QD$

i.e $AQ = QD = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$

$= \begin{bmatrix}2 & -2 & 0\\ -1 & 4 & -3\\ 0 & -2 & 6\end{bmatrix}$

T8 → Area $= \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 2 & 2 & 1\\ 4 & 3 & 1\end{vmatrix}$

$= \frac{1}{2}[1(2-3)-0(2-4)+1(6-8)]$

$= -3/2$

But Area can not be –ve so R. Area $= 3/2$

T-17 → option (b) is wrong, ∵ we cannot find the power of E-vector.

i.e $X^2 = \begin{bmatrix}\vdots\end{bmatrix}_{n\times1}\begin{bmatrix}\vdots\end{bmatrix}_{n\times1}$ = Not defined

so X^m also not defined.

T-19 → Sum of E. Values = Trace (A)

$= \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots\cdots+\frac{1}{n(n+1)}$

$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$= 1-\frac{1}{n+1}$

T-20 → Product of E Values = |A|

$= 1\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdots\cdots\frac{1}{n} = \frac{1}{n!}$

T-22 → E Values of A are 3, 2, –1

" " " A^2 are 9, 4, 1

∵ $B = A^2 - A$ So E Values of B are

$9-3=6$
$4-2=2$
$1-(-1)=2$

So $|B| = (6)(2)(2) = 24$

T-23 → ∵ $\alpha = e^{2\pi i/5}$

$\Rightarrow \alpha^5 = e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$

& α is 5^{th} Root of unity so.

$\alpha^5 = 1$ & $1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$

Now,

E Values of $M = 1, \alpha, \alpha^2, \alpha^3, \alpha^4$

" " " $M^2 = 1, \alpha^2, \alpha^4, \alpha^6, \alpha^8$

$= 1, \alpha^2, \alpha^4, \alpha, \alpha^3$

& " " " $I = 1, 1, 1, 1, 1$

~~So E Values of (I+M+M²)~~

$Tr(M) = 1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$

$Tr(M^2) = 1+\alpha^2+\alpha^4+\alpha+\alpha^3 = 0$

$Tr(I) = 1+1+1+1+1 = 5$

So $Tr(I+M+M^2)$

$= Tr(I)+Tr(M)+Tr(M^2)$

$= 5+0+0 = 5$

T24 → if $A_{4\times4}$ then Constant term in the Char poly of A

$= (-1)^4|A| = |A| = 0$

∵ $|A| = \begin{vmatrix}1 & 2 & 6 & 5\\ -1 & 3 & 2 & -5\\ 2 & 4 & 12 & 10\\ 3 & -2 & 1 & -4\end{vmatrix} = \cdots = 0$

(here we have used the application of C-H-T).

CALCULUS

Q1 → (a) $f(x)=\begin{cases} 2x+1, & x\le 1 \\ ax^2+b, & 1<x<3 \\ 5x+2a, & x\geq 3\end{cases}$

for continuity at $x=1$:→

LHL = RHL = $f(1)$

$\lim_{x\to 1^-}(2x+1)=\lim_{x\to 1^+}(ax^2+b)=(2x+1)_{x=1}$

$3=a+b$ —— ①

for continuity at $x=3$ →

LHL = RHL = $f(3)$

$\lim_{x\to 3^-}(ax^2+b)=\lim_{x\to 3^+}(5x+2a)=\cancel{(5x+2a)}$
$=f(3)$

$9a+b=15+2a=15+2a$

$7a+b=15$ —— ②

Solving ① & ②, $a=2,\ b=1$

Q2 → (d) $f(x)=\begin{cases}\alpha x^2+\beta x, & x<0\\ \alpha x^3+\beta x^2+5\sin x, & x\geq 0\end{cases}$

$f'(x)=\begin{cases}2\alpha x+\beta, & x<0\\ 3\alpha x^2+2\beta x+5\cos x, & x\geq 0\end{cases}$

so LHD = RHD

$\Rightarrow f'(0^-)=f'(0^+)$

or $\beta=5$

Again, $f''(x)=\begin{cases}2\alpha, & x<0\\ 6\alpha x+2\beta-5\sin x, & x\geq 0\end{cases}$

∵ $f(x)$ is twice diff so

LHD = RHD

$\Rightarrow f''(0^-)=f''(0^+)$

$2\alpha=2\beta\Rightarrow\alpha=\beta=5$

Q3 → (b) $f(x)=\begin{cases}e^x, & x<1\\ \ln x+ax^2+bx, & x\geq 1\end{cases}$

for continuity at $x=1$,

LHL = RHL at $(x=1)$

$f(1^-)=f(1^+)$

$e^1=\log 1+a+b$

or $a+b=e$ —— ①

Now $f'(x)=\begin{cases}e^x, & x<1\\ \frac{1}{x}+2ax+b, & x\geq 1\end{cases}$

for differentiability at $x=1$,

LHD = RHD

$f'(1^-)=f'(1^+)$

$e^1=\frac{1}{(1)}+2a(1)+b$

or $2a+b=e-1$ —— ②

Solving ① & ② we will get unique values of a & b.

Q4 → (d) $f(x)=\begin{cases}-x^2, & x<0\\ x^2, & x\geq 0\end{cases}$

∵ LHL = RHL = $f(0)\Rightarrow f(x)$ is continuous at $x=0$

$f'(x)=\begin{cases}-2x, & x<0\\ 2x, & x>0\end{cases}$

∵ LHD = RHD at $(x=0)$ so $f(x)$ is differentiable at $x=0$.

$f''(x)=\begin{cases}-2, & x<0\\ 2, & x>0\end{cases}$

∵ LHD $=-2$ & RHD $=2$ i.e. LHD ≠ RHD

so $f''(x)$ does not exist.

Hence $f(x)$ is diff ∵ $f'(x)$ exist.

But $f'(x)$ is not diff ∵ $f''(x)=$ DNE

Q5 → $y = x^2 + 2x + 10$

$y' = 2x + 2$

or $\left(\frac{dy}{dx}\right)_{x=1} = 2(1)+2 = 4$

Q6 → $f(x) = \sin|x|$

$= \begin{cases} \sin(-x), & x<0 \\ \sin(+x), & x>0 \end{cases}$

$f(x) = \begin{cases} -\sin x, & x<0 \\ +\sin x, & x>0 \end{cases}$

$f'(x) = \begin{cases} -\cos x, & x<0 \\ +\cos x, & x>0 \end{cases}$

$f'\left(-\frac{\pi}{4}\right) = (-\cos x)_{x=-\frac{\pi}{4}} = -\cos\left(-\frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right) = -1/\sqrt{2}$

Q7 → $f(x) = 5x^2 + 10x$ & $f'(x) = 10x+10$

Here $a=1$, $f(a)=15$, $b=2$, $f(b)=40$

By L.M.V.Th, $\frac{f(b)-f(a)}{b-a} = f'(c)$

$\frac{40-15}{2-1} = \cancel{10c+10}\ f'(c)$

or $\left(\frac{dy}{dx}\right)_{x=c} = 25$

Q8 → (b) $\lim_{x\to 0} \frac{x - \sin x}{x - \tan x} \approx \frac{0}{0}$ form

$= \lim_{x\to 0} \left(\frac{1-\cos x}{1-\sec^2 x}\right) \approx \frac{0}{0}$ form

$= \lim_{x\to 0} \frac{\sin x}{-2\sec^2 x . \tan x} = \lim_{x\to 0}\left(\frac{-1}{2}\cos^3 x\right)$

$= -1/2$

Q9 → (c) $\lim_{x\to 0} \left(\frac{\tan x - x}{x^3}\right)$

$= \lim_{x\to 0} \left[\frac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\right) - x}{x^3}\right]$

$= \lim_{x\to 0} \left(\frac{1}{3} - \frac{2}{15}x^2 + \dots\right) = \frac{1}{3}$

Q10 → Let $K = \lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^{x+4}$ ($\approx 1^\infty$ form)

$\log K = \lim_{x\to\infty} \log\left(\frac{x+1}{x-1}\right)^{x+4}$

$= \lim_{x\to\infty} (x+4)\log\left(\frac{x+1}{x-1}\right)$

$= \lim_{x\to\infty} \left[\frac{\log\left(\frac{x+1}{x-1}\right)}{(1/x+4)}\right] \approx \left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to\infty} \frac{\frac{1}{\left(\frac{x+1}{x-1}\right)}\left(\frac{(x-1)-(x+1)}{(x-1)^2}\right)}{-\frac{1}{(x+4)^2}}$

$= \lim_{x\to\infty} \frac{2(x+4)^2}{(x^2-1)} = 2$

So $K = e^2$ (b)

Q11 → $\lim_{\alpha\to 0} \left(\frac{x^\alpha - 1}{\alpha}\right) \approx \frac{0}{0}$ form

Differentiating Numerator and Denominator w.r.to 'α'.

$= \lim_{\alpha\to 0} \left(\frac{x^\alpha \log x}{1}\right) = \log x$. (a)

Q12→ c $\lim_{x\to\infty}\left(\sqrt{x^2+x-1}-x\right)$

$= \lim_{x\to\infty} \dfrac{\left(\sqrt{x^2+x-1}-x\right)\left(\sqrt{x^2+x-1}+x\right)}{\left(\sqrt{x^2+x-1}+x\right)}$

$= \lim_{x\to\infty}\left(\dfrac{(x^2+x-1)-x^2}{\sqrt{x^2+x+1}+x}\right)$

$= \lim_{x\to\infty}\left[\dfrac{\left(1-\frac{1}{x}\right)}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}\right] = \dfrac{1}{2}$

Q13→ Put $\frac{1}{5x}=y$, when $x\to\infty$, $y\to 0$

$\lim_{x\to\infty}\left(e^{\frac{1}{5x}}-1\right)\left(5x+\frac{x}{5}\sin\frac{1}{x}\right)$

$= \lim_{x\to\infty}\left(e^y-1\right)\left[\frac{1}{y}+\frac{1}{25y}\sin 5y\right]$

$= \lim_{x\to\infty}\left(\dfrac{e^y-1}{y}\right)\left(\dfrac{25+\sin 5y}{25}\right)$

$= 1\times\left(\dfrac{25+0}{25}\right) = 1$

Q14→ d $\lim_{x\to 0}\left(\dfrac{x\sin x}{1-\cos x}\right) \approx \left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\dfrac{x\cos x+\sin x}{0+\sin x} \quad \left(\approx \frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\left(\dfrac{-x\sin x+\cos x+\cos x}{\cos x}\right) = 2$

Q15→ $\lim_{x\to 0}\dfrac{\int_0^{x^2}\sqrt{4+t^3}\,dt}{x^2} \quad \left(\approx\frac{0}{0}\right)$

$= \lim_{x\to 0}\left[\dfrac{\frac{d}{dx}\int_0^{x^2}\sqrt{4+t^3}\,dt}{\frac{d}{dx}(x^2)}\right]$

Now using Leibnitz Rule of Diff under the sign of integration,

$= \lim_{x\to 0}\left[\dfrac{2x\sqrt{4+x^6}-0}{2x}\right] = 2$

Q16→ c $f(x)=(1+x)\log(1+x)$

$f'(x)=(1+x)\dfrac{1}{1+x}+\log(1+x)$

$= 1+\log(1+x)$

if $a=0$, $f(a)=f(0)=0$

if $b=1$, $f(b)=f(1)=2\log 2$

By L-M-V.Th, $\dfrac{f(b)-f(a)}{b-a}=f'(c)$

$\dfrac{2\log 2-0}{1-0}=\left[1+\log(1+x)\right]_{x=c}$

$\log(1+c)=2\log 2-1$

$\log(1+c)=\log 2^2-\log e$

$\log(1+c)=\log\left(\frac{4}{e}\right)$

$c=\frac{4}{e}-1=\frac{4-e}{e}$

Q17→ $x=a\cos^3\theta \Rightarrow \dfrac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$

$y=a\sin^3\theta \Rightarrow \dfrac{dy}{d\theta}=3a\sin^2\theta.\cos\theta$

$\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}=-\tan\theta$

By L-M.V.Th, $\dfrac{f(B)-f(A)}{B-A}=f'(C)$

$\dfrac{a-0}{0-a}=f'(x)\Rightarrow f'(x)=-1$

or $(-\tan\theta)=-1\Rightarrow\theta=\pi/4$

So $x=\left(a\cos^3\theta\right)_{\pi/4}=a/2\sqrt{2}$

$y=\left(a\sin^3\theta\right)_{\pi/4}=a/2\sqrt{2}$ d

Q18 → Let $x \in (0,1)$ i.e $a=0,\ b=1$

Now, $f'(x) = \frac{1}{3-x^2} \Rightarrow f''(x) = \frac{2x}{(3-x^2)^2}$

for $x \in (0,1)$ we have $f''(x) > 0$

$\Rightarrow f'(x)$ is increasing in $(0,1)$

i.e $\text{Min } f'(x) = f'(0) = 1/3$

$\text{Max } f'(x) = f'(1) = 1/2$

Now by common sense, we can write

$\text{Min } f'(x) < f'(x) < \text{Max } f'(x)$

$$\frac{1}{3} < \frac{f(1)-f(0)}{1-0} < \frac{1}{2}$$

$$\frac{1}{3} < f(1) - 1 < \frac{1}{2}$$

$$\boxed{1.33 < f(1) < 1.5}$$

(b)

Q19 → (c)

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & \tan\frac{\pi}{6} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & \tan\frac{\pi}{3} \end{vmatrix}$$

$\because \theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 0 = f\left(\frac{\pi}{3}\right)$

(I) So by Rolle's theorem, $f'(\theta) = 0$

Hence (I) is True.

(II) Again, $f'(\theta) \neq 0$ means there exist a point θ between $\frac{\pi}{6}$ & $\frac{\pi}{3}$ where tangent is not horizontal which is obviously True because given function $f(\theta)$ is continuous also b/w $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$.

Q20 → $f(x) = \frac{\sin x}{e^x}$

$$f'(x) = \frac{e^x.\cos x - \sin x.\ e^x}{(e^x)^2}$$

By Rolle's theorem, $f'(c) = 0$

$$\left(\frac{\cos x - \sin x}{e^x}\right)_{x=c} = 0$$

$$\Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

or $\tan c = 1 \Rightarrow c = \pi/4$

Q21 → $f(x) = x^2,\ g(x) = x^3$.

$a=1$ & $b=2$. (given)

by Cauchy's M.V.Th,

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{4-1}{8-1} = \left(\frac{2c}{3c^2}\right) \Rightarrow c = \frac{14}{9}$$

Q22 → (b) $f(x) = \frac{\sin x}{x-\pi}$, $x = \pi$

Let $g(x) = \sin x$ then $f(x) = \frac{g(x)}{x-\pi}$

First we will find the T.S. Expansion of $g(x)$ in the Nbd of $x=\pi$,

$$g(x) = g(\pi) + (x-\pi)g'(\pi) + \frac{(x-\pi)^2}{2!}g''(\pi) + \cdots\cdots$$

$$\sin x = 0 + (x-\pi)(-1) + 0 + \frac{(x-\pi)^3}{3!}(1) + \cdots\cdots$$

$$\frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} + \cdots\cdots$$

i.e $f(x) = \frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} + \cdots\cdots$

Note: In above question, it was easy to find the T.S. Exp of $g(x)$ instead of $f(x)$ that's why we have used above Trick.

Q23 → $f(x) = e^x$, $a = 2$.

So T.S Exp of $f(x)$ in the Nbd of $x=2$ is given as

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \dots\dots$$

$$e^x = e^2 + (x-2)e^2 + \frac{(x-2)^2}{2!}e^2 + \dots\dots$$

So Coeff of $(x-2)^4 = \frac{e^2}{4!}$ (b)

Q24 → $f(x) = e^{-x}$, $a = 2$

Linear Approximation is,

$$f(x) = f(2) + (x-2)f'(2)$$

$$e^{-x} = e^{-2} + (x-2)(-e^{-2})$$

$$= \frac{(3-x)}{e^2}$$ (a)

Q25 → $f(x) = x^3 - 3x^2 - 5 \Rightarrow f(0) = -5$

$f'(x) = 3x^2 - 6x \Rightarrow f'(0) = 0$

$f''(x) = 6x - 6 \Rightarrow f''(0) = -6$

Q. Approximation is,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$

$$= -5 + x(0) + \frac{x^2}{2}(-6)$$

$$= -3x^2 - 5$$ (b)

Q26 → $f(x) = (x^2-4)^2$

$f'(x) = 4x(x+2)(x-2)$

Critical Points are; $x = -2, 0, 2$

Sign of $f'(x)$: $-$ | $+$ | $-$ | $+$ at -2, 0, 2

(min) max (min)

So $f(x)$ has only Two minima (b)

Q27 → (a) $f(x) = x^2 e^{-x}$

$f'(x) = -x^2 e^{-x} + 2x e^{-x}$

Critical points are, $f'(x) = 0$

$\Rightarrow e^{-x}(-x^2 + 2x) = 0$

or $x = 0, 2$

Now $f'(x) = -x(x-2)e^{-x}$

$-$ | $+$ | $-$ at 0, 2 Sign of $f'(x)$

(min) (max)

So Maxima occurs at $x = 2$

Q28 → $y = f(x) = x^{1/x}$

$\log y = \log x^{1/x}$.

$\log y = \frac{\log x}{x}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$$

$$\frac{dy}{dx} = \frac{x^{1/x}(1 - \log x)}{x^2}$$

Critical Point is $\frac{dy}{dx} = 0$

$$\frac{x^{\frac{1}{x}}(1 - \log x)}{x^2} = 0 \Rightarrow x = e$$

$\because \left(\frac{d^2y}{dx^2}\right)_{x=e} = -ve$ (b)

So Maxima will occur at $x = e$

Q29 → $f(x) = \frac{e^{\sin x}}{e^{\cos x}} = e^{\sin x - \cos x}$

$f'(x) = e^{\sin x - \cos x}(\cos x + \sin x)$

Now $f'(x) = 0 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}$

$\because f''\left(\frac{3\pi}{4}\right) = -ve$, so it is point of Maxima.

and Max Value $= f\left(\frac{3\pi}{4}\right)$

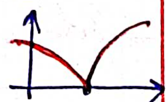
$$= \left(e^{\sin x - \cos x}\right)_{x=\frac{3\pi}{4}} = e^{\sqrt{2}}$$

i.e (a)

Q33 → (c) $f(x) = (x-1)^{2/3} \Rightarrow f'(x) = \frac{2}{3(x-1)^{1/3}}$

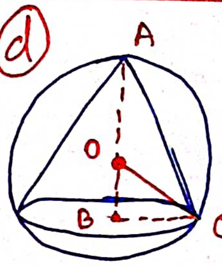
$\because f'(x) =$ undefined at $x=1$ so it will be only critical point. Hence

Min $f(x) = f(1) = (1-1)^{2/3} = 0$



& Above Minima occurs at $x=1$

Q31 → (d)



$OC = 1$ mtr

Let $AB = h$

& $BC = r$

$OB^2 + BC^2 = OC^2$

$(h-1)^2 + r^2 = 1$

Now $V = \frac{1}{3}\cdot\pi r^2 h$

$$V = \frac{\pi}{3}\left[1-(h-1)^2\right].h$$

For Maximum Volume, Necessary Condition is $\frac{dV}{dh} = 0 \Rightarrow h = 4/3$.

Q32 → $f(x) = \frac{e^x}{1+e^x}$

$$f'(x) = \frac{(1+e^x).e^x - e^x(0+e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

By Common Sense, $f'(x) > 0$ always so $f(x)$ will strictly Increasing Funcⁿ. i.e Monotonically Increases. (a)

Q33 → See above

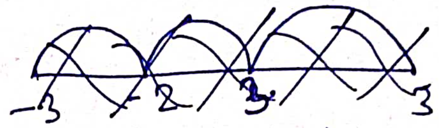
Q34 → $f(x) = x^3 - 3x^2 - 24x + 100$

$f'(x) = 3x^2 - 6x - 24$

$= 3(x^2 - 2x - 8)$

$= 3(x+2)(x-4)$

C. Points are $x = -2$ & 4



$\because x = 4$ lies outside the given interval $[-3, 3]$ so only C.P will be $x = -2$

-3 (min) -2 (max) 3 (min)

Minima will occur at Corner Points i.e at $x = \pm 3$.

$f(3) = 28$ & $f(-3) = 118$

~~f(-3)~~ = So Minimum Value = 28

i.e (b)

Q35 → $f(x) = x^4 - 18x^2 + 9$

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36 = 12(x^2 - 3)$

$f'''(x) = 24x$.

Points of Inflexions are

$f''(x) = 0 \Rightarrow x = \pm\sqrt{3}$

$\because f'''(\sqrt{3}) \neq 0$ & $f'''(-\sqrt{3}) \neq 0$

$\because$ Both $\sqrt{3}$ & $-\sqrt{3}$ are Points of Inflexion. (c)

Q36 → $f(x) = -x^3+6x^2+2x+1$

$\phi(x) = f'(x) = -3x^2+12x+2$

$\phi'(x) = f''(x) = -6x+12$

$\phi''(x) = f'''(x) = -6$

we have to find the Maximum value of $\phi(x)$.

For C. Points $\phi'(x) = 0 \Rightarrow x = +2$

& $\phi''(+2) = -6$ i.e <0

So. $x=+2$ is the point of Maxima

& Max $f'(x) = [\text{Max } \phi(x)]_{x=+2}$

$= \phi(2) = 14$

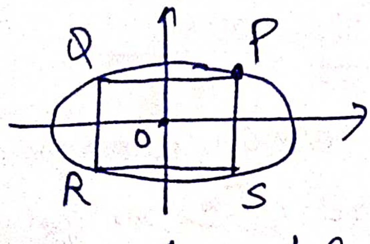
Q37 → $x^2+4y^2=1 \Rightarrow \frac{x^2}{1^2} + \frac{y^2}{(1/2)^2} = 1$

~~Let~~ W.K. That parametric coordinates of any point on the ellipse is given as $(a\cos\theta, b\sin\theta)$

$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

So In this Question, Let

$P(\cos\theta, \frac{1}{2}\sin\theta)$ be any point on the given ellipse. then



$PQ = 2\cos\theta$

$PS = \sin\theta$

So Area of Rectangle is

$A = (2\cos\theta).\sin\theta = \sin 2\theta$

$\frac{dA}{d\theta} = 2\cos 2\theta$, $\frac{d^2A}{d\theta^2} = -4\sin 2\theta$

For Maximum area,

$\frac{dA}{d\theta} = 0 \Rightarrow 2\cos 2\theta = 0$

$\cos 2\theta = 0 \Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow \theta = \pi/4$

$\because \left(\frac{d^2A}{d\theta^2}\right)_{\theta=\pi/4} = -4 < 0$

So $\theta = \frac{\pi}{4}$ will be Point of max.

& Max Area $= (A)_{\theta=\frac{\pi}{4}}$

$= (\sin 2\theta)_{\theta=\frac{\pi}{4}} = 1$

Q38 → $f(x) = x(x-1)(x-2)$

$f(x) = x^3-3x^2+2x$

$f'(x) = 3x^2-6x+2$

$f''(x) = 6x-6$

For C. Points $f'(x) = 0 \Rightarrow x = 1 \pm \frac{1}{\sqrt{3}}$

only $x = 1+\frac{1}{\sqrt{3}}$ lies in the given Domain $[1, 2]$

So only C.P is $x = 1+\frac{1}{\sqrt{3}}$

Sign of $f'(x)$:

$-$		$+$
1 (max)	$\left(1+\frac{1}{\sqrt{3}}\right)$ (min)	2 (max)

So Maxima will occur at $x = 1$ & 2

$f(1) = 0$ & $f(2) = 0$

So Maximum Value $= 0$

Q39 → d $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots}}}$

$y = \sqrt{\tan x + y}$

$y^2 = \tan x + y \Rightarrow y^2 - y = \tan x$

$\frac{d}{dx}(y^2 - y) = \frac{d}{dx}(\tan x)$

$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$

$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$

Q40 → a $u = f(r, s)$ where

$r = x+y$ & $s = x-y$.

$\because u \longrightarrow (r, s) \longrightarrow (x, y)$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$

$= u_r(1) + u_s(1)$

$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$

$= u_r(1) + u_s(-1)$

So $u_x + u_y = 2u_r$

Q41 → $z^3 - xy + yz + y^3 - 1 = 0$ —— (1)

∵ it is given that x & y are independent variables so we can assume that $z = f(x, y)$ in (1).

Now Diff. (1) partially w.r. to x

$\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(1) = 0$

$3z^2\frac{\partial z}{\partial x} - y(1) + y\frac{\partial z}{\partial x} + 0 - 0 = 0$

$\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$

Now $r = x^2y - z$

so $\frac{\partial r}{\partial x} = 2x + 0 - \frac{\partial z}{\partial x}$

ie $\frac{\partial r}{\partial x} = 2x - \frac{y}{3z^2 + y}$

$\left(\frac{\partial r}{\partial x}\right)_{(2,-1,1)} = 2(2) - \frac{-1}{3(1)^2 + (-1)}$

$= 4.5$.

Q42 → $u = \log_e\left(\frac{x^2+y^2}{x+y}\right)$

Let $\boxed{V = e^u} = \frac{x^2+y^2}{x+y}$.

∵ V is Homogeneous function of x & y of degree $n=1$

So Applying Euler theorem for 'V' i.e $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n.v$

$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = n.e^u$

$x(e^u).\frac{\partial u}{\partial x} + y(e^u).\frac{\partial u}{\partial y} = 1.e^u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Q43 → a $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

Let $\boxed{V = \tan u} = \frac{x^3+y^3}{x+y}$

Here V is Homogeneous func^n of degree $n=2$ so Apply Euler theorem for 'V'

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n.v$

$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2.\tan u$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}$$

$$=\sin 2u$$

Q44 → (b) Let $\boxed{V=\sin u}=\frac{x+2y+3z}{x^8+y^8+z^8}$

$\because$ V is Homog func^n of degree $n=-7$

so by Euler theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=-7.v$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-7\tan u.$$

Q45 → (b) $u=\frac{x^3+y^3}{x-y}+x.\sin\left(\frac{x}{y}\right)$

$u = v+w$ (let)

where V is Homog fun^n of $n_1=2$

& w " " " " $n_2=1$

$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}$

$=(x^2V_{xx}+2xyV_{xy}+y^2V_{yy})$

$+(x^2W_{xx}+2xyW_{xy}+y^2W_{yy})$

$=n_1(n_1-1)v+n_2(n_2-1)w$

$=2(2-1)v+1.(1-1)w$

$=2v=2\left(\frac{x^3+y^3}{x-y}\right)$

Q46 → (a) $f(x,y)=2x^4+y^2-x^2-2y$

For critical Points,

$\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$

$8x^3-2x=0$ & $2y-2=0$

$2x(4x^2-1)=0$ & $y=1$

$x=0,\pm\frac{1}{2}$ & $y=1$

So we have three Critical Points

$P_1(0,1)$, $P_2(\frac{1}{2},1)$ & $P_3(-\frac{1}{2},1)$

Now using Lagrange's conditions for P_1, P_2 & P_3 we can verify that only $P_1(0,1)$ is the point of Minima.

Q47 → $f(x,y)=4x^2+6y^2-8x-4y+8$

$f_x=8x-8$, $f_{xx}=8$

$f_y=12y-4$, $f_{yy}=12$, $f_{xy}=0$

Now, For C. Points

$\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$

$8x-8=0$ & $12y-4=0$

$x=1$ & $y=1/3$ i.e $P(1,\frac{1}{3})$

Now $r=(f_{xx})_P=8$,

$s=(f_{xy})_P=0$ & $t=(f_{yy})_P=12$

$\because rt-s^2=8\times12-(0)^2=96>0$

& $r=8>0$

So $P(1,\frac{1}{3})$ is point of Minima

& Min Value $=(f(x,y))_P$

$=f(1,\frac{1}{3})$

$=(4x^2+6y^2-8x-4y+8)_{P(1,\frac{1}{3})}$

$=\frac{10}{3}$ (a)

Q48 → Let $P(x,y,z)$ be any Point on the surface $z^2 = 1+xy$ then ——— (1)

$OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

$OP = \sqrt{x^2+y^2+1+xy}$

Let $u = OP^2 = x^2+y^2+1+xy$

then $OP = \sqrt{u}$ & $OP_{min} = \sqrt{u_{min}}$

Now we will try to find the minimum value of u i.e

$u = x^2+y^2+1+xy$ ——— (2)

$u_x = 2x+y$, $u_{xx} = 2$ & $u_{xy} = 1$

$u_y = 2y+x$, $u_{yy} = 2$

Now Critical Point is

$\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$

$2x+y=0$ & $2y+x=0$

$\Rightarrow x=0, y=0$.

ie $P(0,0)$ is C. Point for u.

Now $r = (u_{xx})_P = 2$, $s = (u_{xy})_P = 1$,

$t = (u_{yy})_P = 2$

$\because rt - s^2 > 0$ & $r > 0$

So $P(0,0)$ is the Point of Minima for u and $u_{min} = 1$ (using (2))

Hence $OP_{min} = \sqrt{u_{min}} = \sqrt{1} = 1$

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Q49 → (C)

$I = \int_0^{\pi/4} \log(1+\tan x)\,dx$

$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right].dx$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$

$I = \int_0^{\pi/4} \log 2\,dx - I$

$2I = \log 2\,(x)_0^{\pi/4} = \frac{\pi}{4}\log 2$

$I = \frac{\pi}{8}\log 2$

Q50 → (C)

$I = \int_0^{\pi/2} (a^2\cos^2 x + b^2\sin^2 x).dx$ ——— (1)

$= \int_0^{\pi/2} \left[a^2\cos^2\left(\frac{\pi}{2}-x\right)+b^2\sin^2\left(\frac{\pi}{2}-x\right)\right].dx$

$I = \int_0^{\pi/2} (a^2\sin^2 x + b^2\cos^2 x)dx$ ——— (2)

Adding (1) & (2)

$2I = \int_0^{\pi/2} (a^2+b^2)\,dx$

$= (a^2+b^2).\left(\frac{\pi}{2}-0\right)$

$I = (a^2+b^2).\frac{\pi}{4}$

Q51 → (b) $I=\int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$

$$= \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{3+1}{2}}{2\,\Gamma\frac{5+3+2}{2}} = \frac{\Gamma 3\,\Gamma 2}{2\cdot\Gamma 5}$$

$$= \frac{2!\cdot 1!}{2\cdot 4!} = \frac{1}{24}$$

Q52 → (a) $I=\int_0^1 x^6 (1-x^2)^{\frac{1}{2}} \, dx$

Put $x^2=y \Rightarrow x=\sqrt{y} \Rightarrow dx = \frac{dy}{2\sqrt{y}}$

$$I=\int_0^1 (y^{1/2})^6 (1-y)^{\frac{1}{2}} \cdot \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_0^1 y^{\frac{5}{2}} \cdot (1-y)^{\frac{1}{2}} \cdot dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{7}{2}-1} \cdot (1-y)^{\frac{3}{2}-1} \cdot dy$$

$$= \frac{1}{2} B\left(\frac{7}{2}, \frac{3}{2}\right) = \frac{1}{2} \frac{\Gamma 7/2\,\Gamma 3/2}{\Gamma\left(\frac{7}{2}+\frac{3}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)\cdot\left(\frac{1}{2}\cdot\sqrt{\pi}\right)}{4!} = \frac{5\pi}{256}$$

Q53 → (b) $I=\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$

$$= 2\int_0^{\pi/2} \frac{\sec^2 x \cdot dx}{a^2 + b^2 \tan^2 x} = \frac{2}{b^2}\int_0 \frac{\sec^2 x \, dx}{\left(\frac{a}{b}\right)^2 + \tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

At $x=0$, $t=0$ & at $x=\frac{\pi}{2}$, $t=\infty$

$$I = \frac{2}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{2}{b^2} \cdot \frac{1}{(a/b)} \left[\tan^{-1}\left(\frac{t}{a/b}\right)\right]_0^{\infty}$$

$$= \frac{2}{ab} \cdot \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$

Q54 → (a)

$$I = \int_0^{\pi} x \sin^6 x \cdot \cos^4 x \, dx$$

$$= \int_0^{\pi} (\pi-x) \sin^6(\pi-x) \cos^4(\pi-x) \, dx$$

$$I = \pi\int_0^{\pi} \sin^6 x \cos^4 x \, dx - I$$

$$2I = 2\pi\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$I = \pi\left[\frac{\Gamma\frac{6+1}{2}\,\Gamma\frac{4+1}{2}}{2\,\Gamma\frac{6+4+2}{2}}\right]$$

$$= \frac{\pi}{2}\left[\frac{\Gamma 7/2\,\Gamma 5/2}{\Gamma 6}\right]$$

$$= \frac{\pi}{2}\left[\frac{\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)\cdot\left(\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)}{5!}\right]$$

$$= \frac{3\pi^2}{256}$$

Q55 → (c) $I = \int_0^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x} dx$

$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$

$I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x} - I$

$2I = 2\pi \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} \cdot dx$

Put $\cos x = t \Rightarrow \sin x \, dx = -dt$

At $x=0$, $t=1$ & at $x=\frac{\pi}{2}$, $t=0$

$I = \pi \int_1^0 \frac{-dt}{1+t^2} = \pi . (\tan^{-1} t)_0^1$

$= \pi(\tan^{-1}(1) - 0) = \pi^2/4.$

Q56 → (c) $|1-x| = \begin{cases} -(1-x), & (1-x)<0 \\ +(1-x), & (1-x)>0 \end{cases}$

$= \begin{cases} x-1, & x>1 \\ 1-x, & x<1 \end{cases}$

$I = \int_0^2 |1-x| dx$

$= \int_0^1 (1-x) dx + \int_1^2 (x-1) dx$

$= \left(x - \frac{x^2}{2}\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^2$

$= \left(\frac{1}{2}\right) + \left[(2-2) - \left(\frac{1}{2} - 1\right)\right] = 1$

Q57 → (b)

$I = \int_0^{\pi/6} \cos^4(3\theta) . \sin^3(6\theta) d\theta$

Put $3\theta = t \Rightarrow d\theta = \frac{dt}{3}$

At $\theta = 0$, $t=0$ & at $\theta = \frac{\pi}{6}$, $t = \frac{\pi}{2}$

$I = \int_0^{\pi/2} \cos^4(t) . \sin^3(2t) \frac{dt}{3}$

$= \frac{1}{3} \int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cos t)^3 dt$

$= \frac{8}{3} \int_0^{\pi/2} \sin^3 t . \cos^7 t . dt$

$= \frac{8}{3} \cdot \frac{\Gamma\frac{3+1}{2} \cdot \Gamma\frac{7+1}{2}}{2\Gamma\frac{3+7+2}{2}}$

$= \frac{4}{3} \cdot \frac{\Gamma 2 \cdot \Gamma 4}{\Gamma 6} = \frac{4}{3} \cdot \frac{\underline{|1} \, \underline{|3}}{\underline{|5}}$

$= \frac{1}{15}.$

Q58 → (b) $I = \int_0^2 \frac{(x-1)^2 \cdot \sin(x-1)}{(x-1)^2 + \cos(x-1)} dx$

Put $x-1 = t \Rightarrow dx = dt$.

At $x=0$, $t=-1$ & At $x=2$, $t=1$

$I = \int_{-1}^1 \left(\frac{t^2 \cdot \sin t}{t^2 + \cos t}\right) dt = 0$

$\because$ Integrand is an odd function

Q59 → (a) $I=\int_0^{2\pi}\left(\frac{3}{9+\sin^2\theta}\right)d\theta$

$=2\int_0^{\pi}\frac{3.d\theta}{9+\sin^2\theta}=4\int_0^{\pi/2}\frac{3.d\theta}{9+\sin^2\theta}$

$=12\int_0^{\pi/2}\frac{\sec^2\theta\, d\theta}{9\sec^2\theta+\tan^2\theta}$

$=12\int_0^{\pi/2}\frac{\sec^2\theta.d\theta}{9+10\tan^2\theta}$

$=\frac{12}{10}\int_0^{\pi/2}\frac{\sec^2\theta.d\theta}{\left(\sqrt{\frac{9}{10}}\right)^2+\tan^2\theta}$

Put $\tan\theta=t \Rightarrow \sec^2\theta\, d\theta=dt$

At $\theta=0$, $t=0$ & At $\theta=\frac{\pi}{2}$, $t=\infty$

$=\frac{12}{10}\int_0^{\infty}\frac{dt}{\left(\sqrt{\frac{9}{10}}\right)^2+t^2}$

$=\frac{12}{10}\cdot\frac{1}{\sqrt{9/10}}\left[\tan^{-1}\left(\frac{t}{\sqrt{9/10}}\right)\right]_0^{\infty}$

$=\frac{4}{\sqrt{10}}\left[\tan^{-1}\infty-\tan^{-1}0\right]=\frac{2\pi}{\sqrt{10}}$

Q60 → (b) $I=\int_0^{\pi}x.\cos^2x\,dx$

$=\int_0^{\pi}(\pi-x)\cos^2(\pi-x)\,dx$

$I=\int_0^{\pi}\pi\cos^2x\,dx-\int_0^{\pi}x\cos^2x\,dx$

$I=2\int_0^{\pi/2}\pi.\cos^2x\,dx-I$

$2I=2\pi\int_0^{\pi/2}\sin^0x.\cos^2x\,dx$

$I=\pi\left[\frac{\Gamma\frac{0+1}{2}\,\Gamma\frac{2+1}{2}}{2\Gamma\frac{0+2+2}{2}}\right]$

$=\frac{\pi}{2}\cdot\left[\frac{\Gamma 1/2\cdot\Gamma 3/2}{\Gamma 2}\right]$

$=\frac{\pi}{2}\left[\frac{\sqrt{\pi}\cdot\left(\frac{1}{2}\cdot\sqrt{\pi}\right)}{\lfloor 1}\right]=\frac{\pi^2}{4}$

Q61 → (b) $I=\int_0^{\infty}e^{-y^3}\cdot y^{1/2}\cdot dy$

Put $y^3=x \Rightarrow y=x^{1/3}$

$dy=\frac{1}{3}x^{-2/3}\cdot dx$

$I=\int_0^{\infty}e^{-x}\cdot(x^{1/3})^{1/2}\cdot\frac{1}{3}\cdot x^{-2/3}\cdot dx$

$=\frac{1}{3}\int_0^{\infty}e^{-x}\cdot x^{-\frac{1}{2}}\cdot dx$

$=\frac{1}{3}\int_0^{\infty}e^{-x}\cdot x^{\frac{1}{2}-1}\cdot dx=\frac{\Gamma 1/2}{3}$

$=\sqrt{\pi}/3.$

Q62 → $\int_0^{2\pi}|x\sin x|\,dx=K\pi$

$\int_0^{\pi}(+x\sin x)dx+\int_{\pi}^{2\pi}(-x\sin x)\,dx=K\pi$

$(-x\cos x+\sin x)_0^{\pi}-(-x\cos x+\sin x)_{\pi}^{2\pi}=K.\pi$

$(\pi)+(2\pi-\pi(-1))=K\pi$

$4\pi=K.\pi \Rightarrow K=4$

Q63 → $I = \int_{0.25}^{1.25} f(x)\,dx = \int_{0.25}^{1.25} (x - [x])\,dx$

$= \int_{0.25}^{1} (x - [x])\,dx + \int_{1}^{1.25} (x - [x])\,dx$

$= \int_{0.25}^{1} (x - 0)\,dx + \int_{1}^{1.25} (x - 1)\,dx$

$= \left(\frac{x^2}{2}\right)_{1/4}^{1} + \left(\frac{x^2}{2} - x\right)_{1}^{1.25}$

$= 0.5$

Q64 → $f(x) = \int_0^x e^{-(t^2/2)}\,dt$

$f'(x) = \frac{d}{dx} \int_0^x e^{-\left(\frac{t^2}{2}\right)}\,dt$

using Leibnitz Rule of differentiation under the sign of integration.

$= \frac{d}{dx}(x) \cdot e^{-\left(\frac{x^2}{2}\right)} - \frac{d}{dx}(0) \cdot e^{0}$

$f'(x) = e^{-(x^2/2)} - 0$

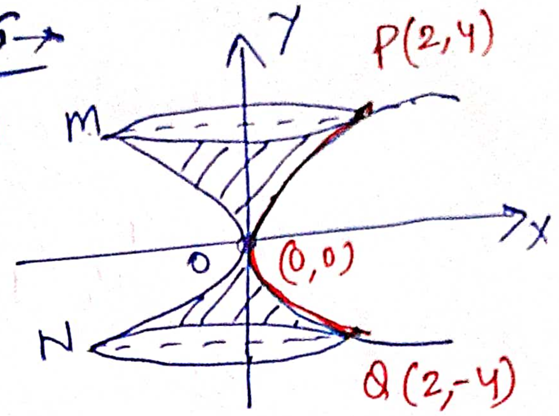
$f''(x) = e^{-(x^2/2)} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right)$

$= -x \cdot e^{-(x^2/2)}$

Now, in Taylor series expansion

$a_2 = \frac{f''(0)}{2!} = \frac{0}{2!} = 0$

Q66 → (b)



$y^2 = 8x \Rightarrow x = y^2/8$.

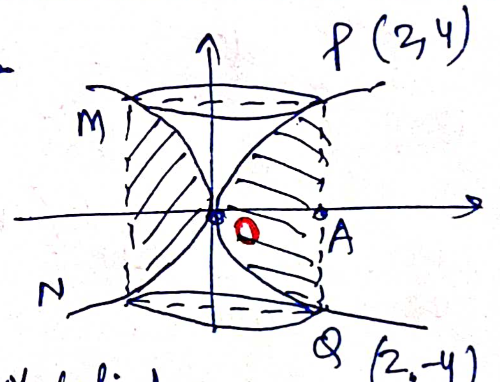
Required Volume $= 2$ Vol. of OPM

$= 2 \int_{y=0}^{y=4} \pi x^2\,dy$

$= 2\pi \int_{y=0}^{4} \left(\frac{y^2}{8}\right)^2 dy = \frac{2\pi}{64}\left(\frac{y^5}{5}\right)_0^4$

$= \frac{\pi}{32} \cdot \frac{4^5}{5} = \frac{32\pi}{5}$

Q65 → (a)



Height of cylinder $= PQ = h = 8$

radius " " $= OA = r = 2$

Volume of cylinder $= \pi r^2 h = \pi (2)^2 8$
$= 32\pi$

Volume Hollow part $= \frac{32\pi}{5}$
(using above Part)

Now Required Volume $= 32\pi - \frac{32\pi}{5}$

$= \frac{128\pi}{5}$.

Q67 → $y = \frac{2}{3}x^{3/2} \Rightarrow \frac{dy}{dx} = x^{1/2}$

Req. Length $= \int_{x=a}^{b} \sqrt{1+(y')^2}\,.dx$

$$= \int_{x=0}^{1} \sqrt{1+(\sqrt{x})^2}\,dx = \int_0^1 (1+x)^{1/2}.dx$$

$$= \frac{(1+x)^{3/2}}{3/2} = 1.22 \quad (d)$$

Q68 → $r = a(1+\cos\theta)$; $0 \le \theta \le \pi$

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,.\,d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,.d\theta$$

$$= a\int_0^{\pi} \sqrt{2+2\cos\theta}\,.d\theta$$

$$= a\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2} - 1}\,.d\theta$$

$$= 2a\int_0^{\pi} \cos^2\frac{\theta}{2}\,d\theta = 2a\left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= 4a(1-0) = 4a$$

Q69 → $x = \cos t$, $y = \sin t$, $z = \frac{2}{\pi}t$

$\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = \frac{2}{\pi}$, $0 \le t \le \frac{\pi}{2}$

Req. length $= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,.dt$

$$= \int_0^{\pi/2} \sqrt{\sin^2 t + \cos^2 t + \frac{4}{\pi^2}}\,.\,dt$$

$$= \sqrt{1+\frac{4}{\pi^2}}\int_0^{\pi/2} (1)\,dt = \frac{\sqrt{\pi^2+4}}{\pi}\cdot\frac{\pi}{2}$$

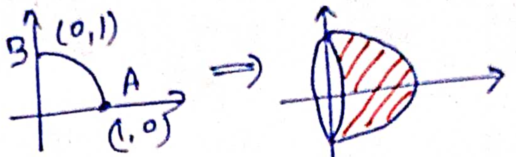
$$= 1.86$$

Q70 → $x = \cos\left(\frac{\pi u}{2}\right)$, $y = \sin\left(\frac{\pi u}{2}\right)$

At $u=0$; $x=1$, $y=0$

At $u=1$; $x=0$, $y=1$

So given curve is a part of circle of radius 1 in 1^{st} Quadrant



So after about x axis, we will get Hemisphere of radius = 1.

Hence Surface Area $= 2\pi r^2$

$$= 2\pi(1)^2 = 2\pi$$

Q71 → $I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}.\,dx\,dy$

$$= \int_{x=0}^{\infty} e^{-x^2}dx \times \int_0^{\infty} e^{-y^2}dy$$

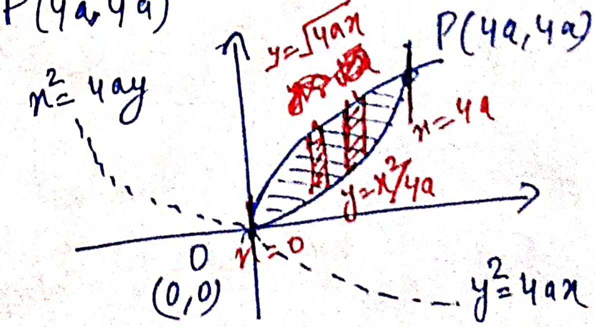
* using Gamma function Concept

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4} \quad (d)$$

Q72 → $y^2 = 4ax$ —① & $x^2 = 4ay$ ②

Solving ① & ② ; $x=0, y=0$ & $x=4a$, $y=4a$

So Intersecting Point are (0,0) & P(4a,4a)



i.e $0 \le x \le 4a$ & $\frac{x^2}{4a} \le y \le \sqrt{4ax}$

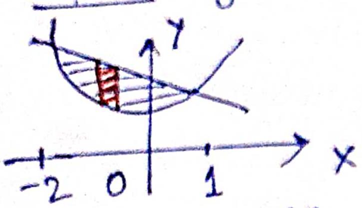
Req. Area $= \int\int (1)\,dy\,dx$

$$= \int_{x=0}^{4a}\int_{y=x^2/4a}^{\sqrt{4ax}} (1).\,dy\,dx$$ (c)

$$= \int_{x=0}^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= \left(\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right)_{x=0}^{4a} = \frac{16}{3}a^2$$

Q73 → $y = x^2+1$ & $y = 3-x$



$-2 \le x \le 1$

$x^2+1 \le y \le 3-x$

Req. Area $= \iint (1)\, dy\, dx$

$$= \int_{x=-2}^{1}\int_{y=x^2+1}^{3-x} (1).\,dy\,dx$$ (b)

$$= \int_{x=-2}^{1} \{(3-x)-(x^2+1)\}\, dx = \frac{9}{2}$$

Q74 → $I = \int_0^1\int_0^{x^2} e^{y/x}.\,dy\,dx$

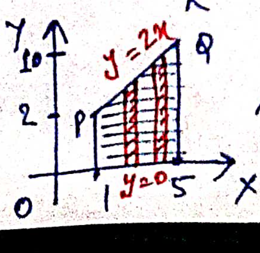
$$= \int_{x=0}^{1}\left(\int_{y=0}^{x^2} e^{y/x}\,dy\right) dx$$

$$= \int_{x=0}^{1} \left(\frac{e^{y/x}}{1/x}\right)_{y=0}^{x^2} . dx = \int_0^1 (xe^x - x)\, dx$$

$$= \left(xe^x - e^x - \frac{x^2}{2}\right)_0^1 = 0.5$$

Q75 → $C = 6\times10^{-4}$ & $I = C\iint_R xy^2\, dx\,dy$

$I = C\iint_R xy^2\, dy\, dx$ (using Vertical strip)



P(1, 2) & Q(5, 10) so equ^n of PQ is $y = 2x$

i.e $1 \le x \le 5$ & $0 \le y \le 2x$

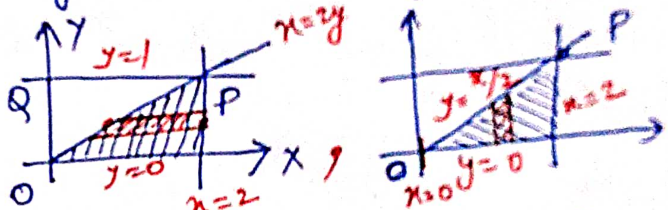
$$I = C\int_{x=1}^{5}\left(\int_{y=0}^{2x} y^2\,dy\right)(x\,dx)$$

$$= C\int_{x=1}^{5} \left(\frac{y^3}{3}\right)_{y=0}^{2x} \cdot (x\,dx)$$

$$= C\int_{x=1}^{5} \frac{8}{3} x^4\, dx = \frac{8C}{3}\left(\frac{x^5}{5}\right)_1^5$$

$$= \frac{8\times 6\times10^{-4}}{3\times5}\left(5^5 - 1^5\right) = 0.99$$

Q76 → Region of integration is bounded by $y=0, y=1, x=2y$ & $x=2$



$0 \le y \le 1$, $2y \le x < 2$

$0 \le x \le 2$, $0 \le y \le x/2$

It will be easier to solve the above Question by using Vertical strip.

$$I = \int_{y=0}^{1}\int_{x=2y}^{2} e^{x^2}.\,dx\,dy$$ (using H-strip)

$$= \int_{x=0}^{2}\int_{y=0}^{x/2} e^{x^2}.\,dy\,dx$$ (using V-strip)

$$= \int_{x=0}^{2}\left\{\int_{y=0}^{x/2} 1.\,dy\right\} e^{x^2}\,dx$$

$$= \int_{x=0}^{2} \left(\frac{x}{2}-0\right)e^{x^2}.\,dx$$

$$= \frac{1}{2}\int_{x=0}^{2} x.\,e^{x^2}\,dx$$

Put $x^2 = t$, $x\,dx = \frac{dt}{2}$

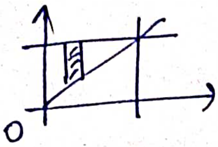
$$= \frac{1}{4}\int_0^4 e^t\,dt = \frac{e^4-1}{4}$$ Ans

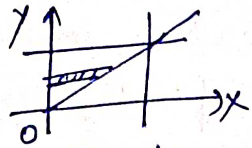
(c)

Q77 → $I=\int_{x=0}^{\pi}\int_{y=x}^{\pi}\int_{z=0}^{2}\frac{\sin y}{y}\,dz\,dy\,dx$

$=\int_{z=0}^{2}\left\{\int_{x=0}^{\pi}\int_{y=x}^{\pi}\left(\frac{\sin y}{y}\right)dy\,dx\right\}.dz$

$(y=x, y=\pi, x=0, x=\pi)$; $(x=0, x=y, y=0, y=\pi)$





By changing the order of integration,

$I=\int_{z=0}^{2}\left[\int_{y=0}^{\pi}\int_{x=0}^{y}\left(\frac{\sin y}{y}\right)dx\,dy\right].dz$

$=\int_{z=0}^{2}\left[\int_{y=0}^{\pi}\left\{\int_{x=0}^{y}(1)\,dx\right\}\frac{\sin y}{y}\,dy\right].dz$

$=\int_{z=0}^{2}\left[\left[\int_{y=0}^{\pi}y.\frac{\sin y}{y}\,dy\right]\right].dz$

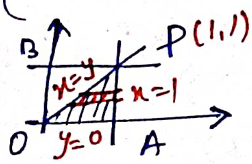
$=\int_{z=0}^{2}\left(-\cos y\right)_{y=0}^{\pi}.dz=-(-1-1)\int_{z=0}^{2}(1)\,dz$

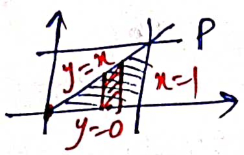
$=+2\,(2)=4$ (d)

Q78 → A.T.Q

$I=\int_0^1\int_y^1(xy)\sin(xy)\,dx\,dy=\int_0^1\int_a^b(xy)\sin(xy)\,dy\,dx$

$(y=0, y=1, x=y, x=1)$ $(x=0, x=1, y=a, y=b)$





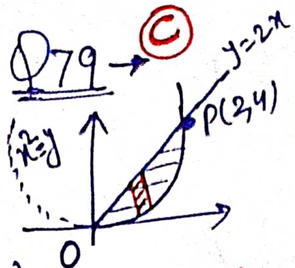
i.e $0\le x\le 1$, $0\le y\le x$

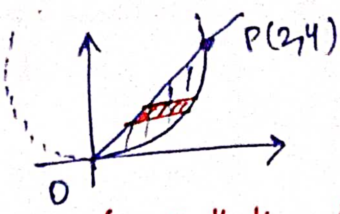
A.T. Concept;

$I=\int_0^1\int_y^1 xy\sin(xy)\,dx\,dy=\int_0^1\int_0^x xy\sin(xy)\,dy\,dx$

Hence $a=0, b=x$ (a)

Q79 → (c)



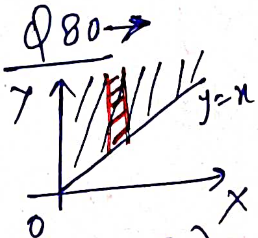


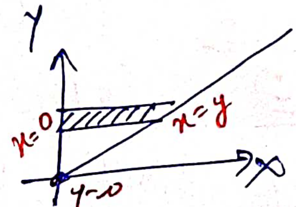
$\left(\begin{matrix}x=0, x=2\\ y=x^2, y=2x\end{matrix}\right)$ $\left(\begin{matrix}y=0, y=4\\ x=\frac{y}{2}, x=\sqrt{y}\end{matrix}\right)$

$I=\int_0^2\int_{x^2}^{2x}f(x,y)\,dy\,dx$ (using V-strip)

$=\int_{y=0}^{4}\int_{x=y/2}^{\sqrt{y}}f(x,y)\,dx\,dy$ (using H-strip)

Q80 →





$\left(\begin{matrix}y=x, y=\infty\\ x=0, x=\infty\end{matrix}\right)$ $\left(\begin{matrix}x=0, x=y\\ y=0, y=\infty\end{matrix}\right)$

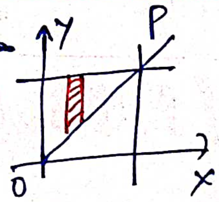
$I=\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{1}{y}e^{-y/2}.dy\,dx$

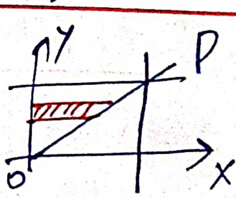
$=\int_{y=0}^{\infty}\int_{x=0}^{y}\frac{1}{y}e^{-y/2}.dx\,dy$

$=\int_{y=0}^{\infty}\left(\int_{x=0}^{y}(1)\,dx\right)\frac{1}{y}e^{-y/2}.dy$

$=\int_{y=0}^{\infty}e^{-y/2}.dy=\left(\frac{e^{-y/2}}{-1/2}\right)_0^{\infty}=2$.

Q81 →





$\left(\begin{matrix}x=0, x=\pi/2\\ y=x, y=\pi/2\end{matrix}\right)$ $\left(\begin{matrix}y=0, y=\pi/2\\ x=0, x=y\end{matrix}\right)$

$O=(0,0)$ & $P\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

OP: $y=x$.

$$I = \int_{x=0}^{\pi/2}\int_{y=x}^{\pi/2}\left(\frac{\cos y}{y}\right) dy\, dx$$

$$= \int_{y=0}^{\pi/2}\int_{x=0}^{y}\left(\frac{\cos y}{y}\right) dx\, dy$$

$$= \int_{y=0}^{\pi/2}\left(\int_{x=0}^{y}(1)\, dx\right)\frac{\cos y}{y}\, dy$$

$$= \int_{y=0}^{\pi/2}\cos y\, dy = \left(\sin y\right)_0^{\pi/2} = 1$$

Q82: $x = uv$ & $y = \frac{v}{u}$

(c) $\because (x,y) \longrightarrow (u,v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u} - \left(\frac{-v}{u}\right) = \frac{2v}{u}$$

Now, ATQ;

$$\iint f(x,y)\, dx\,dy = \iint f\left(uv, \frac{v}{u}\right) \underline{\phi(u,v)\, du\,dv}$$

and AT concept;

$$\iint f(x,y)\, dx\,dy = \iint f(u,v)\, \underline{|J|\, du\,dv}$$

on comparision; $\phi = |J| = \frac{2v}{u}$.

Q83: $V = \iiint \rho\, d\rho\, d\phi\, dz$

$$= \int_{z=3}^{4.5}\int_{\phi=\pi/8}^{\pi/4}\int_{\rho=3}^{5} \rho .\, d\rho\, d\phi\, dz$$

$$= \int_3^5 \rho\, d\rho \times \int_{\pi/8}^{\pi/4}(1)\, d\phi \times \int_3^{4.5}(1)\, dz = 4.7$$

84 $\rightarrow$ Given Region;

$$x^2+y^2 \le z^3 \quad \& \quad 0 \le z \le 1$$

Above Region is described by Infinite disc lying above each other on Z-axis.

Let us consider any Random disc of radius r and thickness dz

i.e $x^2+y^2 \le r^2$ with thickness $= dz$

then on comparision with $x^2+y^2 \le z^3$

we get $r^2 = z^3$.

Volume of this Random disc is

$$dV = \pi r^2\, dz \Rightarrow dV = \pi z^3 . dz$$

$$\int_0^V dV = \int_{z=0}^{1} \pi z^3\, dz$$

$$V = \pi\left(\frac{z^4}{4}\right)_0^1 = \pi/4.$$

Q85 (a) $\rightarrow$ Let $\left(\frac{x}{a}\right)^{2/3} = u$, $\left(\frac{y}{b}\right)^{2/3} = v$, $\left(\frac{z}{c}\right)^{2/3} = w$

$\Rightarrow x = a u^{3/2}$, $y = b v^{3/2}$, $z = c w^{3/2}$

Req. Volume $= 8\times$ Vol. in 1st octant

$$= 8\times \iiint (1)\, dx\,dy\,dz$$

$$= 8\iiint \frac{27abc}{8} . u^{1/2} . v^{1/2} . w^{1/2}\, du\,dv\,dw$$

$$= 27abc \iiint u^{\frac{3}{2}-1} . v^{\frac{3}{2}-1} . w^{\frac{3}{2}-1}\, du\,dv\,dw$$

$$= 27abc\, \frac{\Gamma 3/2\, \Gamma 3/2\, \Gamma 3/2}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1\right)} = \frac{4\pi abc}{35}$$

Note: Here we have used Diritchlet's Formula in 1st octant i.e

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$$

T-1 → $y = x + \sqrt{y}$ —— (1)

$(y-x)^2 = y \Rightarrow y^2 + x^2 - 2xy = y$

$y^2 - (2x+1)y + x^2 = 0$

At $x=2$; $y^2 - 5y + 4 = 0$

$(y-4)(y-1) = 0 \Rightarrow y = 1 \,\&\, 4$

But $y=1$ & $x=2$ is not satisfying eq.(1) so only possible value of y (at $x=2$) $= 4$

T22 → W.K.T.

Value of $f(x) = a\cos x + b\sin x$ lies b/w $\pm\sqrt{a^2+b^2}$ i.e

$$-\sqrt{a^2+b^2} \le (a\cos x + b\sin x) \le +\sqrt{a^2+b^2}$$

So In this Question,

$$f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$$

$$= 5\cos\theta + 3\left[\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right] + 3$$

$$= 5\cos\theta + 3\left[\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right] + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

So Max $f(\theta) = $ Max $\left[\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right] + 3$

$$= \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} + 3$$

$$= \sqrt{\frac{169+27}{4}} + 3 = 7 + 3 = 10$$

T24 → $x^a y^b = (x+y)^{a+b}$

$\log x^a + \log y^b = \log (x+y)^{a+b}$

$a\log x + b\log y = (a+b)\log(x+y)$

Diff. w.r.to 'x'

$$\frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = \frac{a+b}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{b}{y} - \frac{a+b}{x+y}\frac{dy}{dx} = \frac{a+b}{x+y} - \frac{a}{x}$$

$$\left(\frac{bx + by - ay - by}{y(x+y)}\right)\frac{dy}{dx} = \frac{ax + bx - ax - ay}{x(x+y)}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

T35 → $f(n) = \int_0^{\pi/4} \tan^n x$

$$f(3) + f(1) = \int_0^{\pi/4} (\tan^3 x + \tan x)\,dx$$

$$= \int_0^{\pi/4} \tan x(\tan^2 x + 1)\,dx$$

$$= \int_0^{\pi/4} \tan x \cdot \sec^2 x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

At $x=0$, $t=0$ & At $x = \frac{\pi}{4}$, $t=1$

$$= \int_0^1 t \cdot dt = \left(\frac{t^2}{2}\right)_0^1 = 1/2.$$

T41 → $I = \int_{y=0}^{\infty}\int_{x=0}^{y} y\, e^{-(x+y)}\,dx\,dy$

$$= \int_{y=0}^{\infty}\left(\int_{x=0}^{y} e^{-x}\,dx\right) y\, e^{-y}\,dy$$

$$= \int_{y=0}^{\infty} (1 - e^{-y})\, y \cdot e^{-y}\,dy$$

$$= \int_0^{\infty} y e^{-y}\,dy - \int_0^{\infty} y e^{-2y}\,dy$$

$$= \left(-y e^{-y} - e^{-y}\right)_0^{\infty} - \left[-\frac{y e^{-2y}}{2} - \frac{e^{-2y}}{4}\right]_0^{\infty}$$

$$= -[0-1] + \left[0 - \left(0 + \frac{1}{4}\right)\right]$$

$$= 1 - \frac{1}{4} = 3/4$$