Engg. Maths

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Notes

(18 Marks)

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Syllabus

1. Linear Algebra
2. Diff calculus (utop Integration)
3. Probability & Statistics

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Linear Algebra

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$A = [a_{ij}]_{m\times n}$ → order or Size of matrix

→ element in i^{th} row & j^{th} col^n

m = no. of rows

n = no. of columns

If $m=n$, $A_{n\times n}$ is Square matrix.

if $m \neq n$, $A_{m\times n}$ is rectangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

→ $i<j$

$i>j$

$i=j$

→ Principal diagonal

If $a_{ij}=0$ for $i<j$. Then $A_{n\times n}$ is L.T.M (lower Triangular Matrix)

if $a_{ij}=0$ for $i>j$ Then $A_{n\times n}$ is U.T.M

If $a_{ij}=0$ for $i\neq j$ Then $A_{n\times n}$ is Diagonal matrix.

Operations on Matrices

① $A \pm B$ iff $O(A) = O(B)$

② $A_{m\times n} \cdot B_{p\times q}$ is possible iff $\boxed{n=p}$

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NOTE

1) $AB \neq BA$ // not commutative
2) $A[BC] = [AB]C$ // associative

Formulae

Let $A_{m \times n}$, $B_{p \times q}$, $n = p$

1. The no. of scalar multiplication required to find $(AB)_{m \times q} = mpq$.
2. The no. of additions required to find $(AB)_{m \times q} = m(p-1)q$

Q Let $P_{4\times2}$, $Q_{2\times4}$, $R_{4\times1}$ be 3 matrices. find the minimum no. of multiplication required to find PQR ?

Solⁿ we know that (W.K.T)

$$P(QR) = (PQ)R$$

Consider $P(QR)$: $Q_{2\times4} R_{4\times1} \Rightarrow$ no. of $x^n = 2\times4\times1 = 8$
$P_{4\times2} (QR)_{2\times1} \Rightarrow$ no. of $x^n = 4\times2\times1 = \underline{8}$
$\underline{16}$

Consider $(PQ)R$: $P_{4\times2} Q_{2\times4} \Rightarrow$ no. of $x^n = 4\times2\times4 = 32$
$(PQ)_{4\times4} R_{4\times1} \Rightarrow$ no. of $x^n = 4\times4\times1 = \underline{16}$
$\underline{48}$

min. no. of mult. = 16 Ans
max. no. of mult. = 48

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③ Transpose of matrix

$A = [a_{ij}]_{m \times n}$ Then $A^T = [a_{ji}]_{n \times m}$

eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 1 & 6 \end{bmatrix}$

NOTE

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(AB)^T = B^T A^T$
4) $(A^T)^n = (A^n)^T$

④ Trace $(A_{n \times n})$ $= a_{11} + a_{22} + \cdots + a_{nn}$

$= \sum_{i=1}^{n} a_{ii}$

NOTE

1) $\text{Trace}(A) = \text{Trace}(A^T)$
2) $\text{Trace}(A+B) = \text{Trace}(A) + \text{Trace}(B)$
3) $\text{Trace}(kA) = k\,\text{Trace}(A)$

Symmetric Matrix : $A = [a_{ij}]_{n \times n}$

is Sym. matrix

iff $\boxed{A^T = A}$ or $\boxed{a_{ij} = a_{ji}}$

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eg $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} = A$

Skew-Symmetric Matrix

$A_{n \times n}$ is Skew-Sym. matrix iff $\boxed{A^T = -A}$

$$\boxed{\begin{aligned} a_{ij} &= 0 \ \forall\ i = j \\ \& \ a_{ij} &= -a_{ji} \ \forall\ i \neq j \end{aligned}}$$

$$\begin{bmatrix} \diagdown & l \\ u & \diagdown \end{bmatrix} = \begin{bmatrix} \diagdown & -u \\ -l & \diagdown \end{bmatrix}$$

eg. $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

Complex Matrix

$A = [\,a_{ij}\,]_{m \times n}$

$a_{ij} \longrightarrow$ Complex number $(z = x + iy)$

Conjugate of Matrix (replace i by $-i$)

$A = [\,a_{ij}\,]_{m \times n}$ then $\bar{A} = [\,\overline{a_{ij}}\,]_{m \times n}$

eg. $A = \begin{bmatrix} i & 1+i \\ 2 & 2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} -i & 1-i \\ 2 & -2i \end{bmatrix}$

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Hermitian Matrix

$A = [a_{ij}]_{n \times n}$ is H.M iff $\boxed{\bar{A} = A^T}$

$$\begin{bmatrix} & \bar{u} \\ \bar{l} & \end{bmatrix} = \begin{bmatrix} & l \\ u & \end{bmatrix}$$

NOTE

$a_{ii} = \text{real}$

&& $\overline{a_{ij}} = a_{ji}$

eg $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

NOTE

$(\bar{A})^T = (A^T)^T$

$\boxed{A^{\theta} = A}$

θ : Transpose of conjugate

Q $A = \begin{bmatrix} 2 & x \\ 2i & 3 \end{bmatrix}$ is a H M $x = ?$

$x = a_{12} = \overline{a_{21}} = \overline{2i} = -2i$

Skew-Hermitian matrix

$A = [a_{ij}]_{n \times n}$ is Skew HM if $\boxed{\bar{A} = -A^T}$

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$$\boxed{\begin{gathered} AB = BA = I \\ \text{Then } A = B^{-1} \end{gathered}}$$

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$$\boxed{\begin{gathered} a_{ii} = \text{Zero (or) Purely Imag.} \\ \& \ \overline{a_{ij}} = -a_{ji} \end{gathered}}$$

$$\boxed{\begin{gathered} (\bar{A})^T = (-A^T)^T \\ A^\theta = -A \end{gathered}}$$

Q) $A = \begin{bmatrix} 0 & x \\ 1+i & 2i \end{bmatrix}$ is a skew-HM, $x =$ ___ ?

$$x = a_{12} = -\overline{a_{21}} = -\overline{(1+i)} = -(1-i) = i-1$$

Orthogonal Matrix

$A = [a_{ij}]_{n \times n}$ is orthogonal matrix

iff $\boxed{A \cdot A^T = A^T \cdot A = I}$

$\Downarrow$

$\boxed{A^T = A^{-1}}$

Q) $A = \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}$ for some $\alpha, \beta \in R$, if A is orthogonal $\alpha, \beta = ?$

A is orthogonal $\Rightarrow \boxed{\begin{gathered} A^T = A^{-1} \\ A \cdot A^T = I \end{gathered}}$

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Notes

$A \cdot A^T = I$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}\begin{bmatrix} 1/9 & 8/9 & \alpha/9 \\ -4/9 & 4/9 & -7/9 \\ 8/9 & 1/9 & \beta/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiply equate on both side

$$\frac{\alpha}{81} + \frac{28}{81} + \frac{8\beta}{81} = 0 \qquad \boxed{\alpha + 8\beta = -28}$$

$$\frac{8\alpha}{81} - \frac{28}{81} + \frac{\beta}{81} = 0 \qquad \boxed{8\alpha + \beta = 28}$$

$$\boxed{\begin{matrix} \alpha = 4 \\ \beta = -4 \end{matrix}}$$

NOTE The vectors in a orthogonal matrix are orthogonal to each other.
(i.e. dot product = 0)

Shortcut

$\because A = [\, X_1 \;\; X_2 \;\; X_3 \,]_{3\times 3}$ is an orthogonal matrix.

$\Rightarrow$ orthogonal to each other

$\Rightarrow \vec{X_1} \cdot \vec{X_2} = 0 \quad \& \quad \vec{X_2} \cdot \vec{X_3} = 0$

(also $\vec{X_1} \cdot \vec{X_3} = 0$)

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Notes

$$A = \begin{array}{cccc} & X_1 & X_2 & X_3 \\ & \left[\begin{array}{c} \\ \\ \\ \end{array}\right. \begin{array}{ccc} \frac{1}{9}i & -\frac{4}{9}i & \frac{8}{9}i \\ \frac{8}{9}j & \frac{4}{9}j & \frac{1}{9}j \\ \frac{\alpha}{9}k & -\frac{7}{9}k & \frac{\beta}{9}k \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right] \end{array}$$

$$\vec{X_1} \cdot \vec{X_2} = \left(\frac{1}{9} \times -\frac{4}{9}\right) + \left(\frac{8}{9} \times \frac{4}{9}\right) + \left(\frac{\alpha}{9} \times -\frac{7}{9}\right) = 0$$

$$\Rightarrow \boxed{\alpha = 4}$$

$$\vec{X_2} \cdot \vec{X_3} = \left(-\frac{4}{9} \times \frac{8}{9}\right) + \left(\frac{4}{9} \times \frac{1}{9}\right) + \left(-\frac{7}{9} \times \frac{\beta}{9}\right) = 0$$

$$\Rightarrow \boxed{\beta = -4}$$

Q $M = \begin{bmatrix} 3/4 & x \\ +1/4 & 3/4 \end{bmatrix}$ & $M^T = M^{-1}$, $x = ?$

$\Downarrow$

M is orthogonal

$$\vec{X_1} \cdot \vec{X_2} = 0$$

$$\frac{3}{4} \times x + \frac{1}{4} \times \frac{3}{4} = 0$$

$$\boxed{x = -\frac{1}{4}}$$

Unitary Matrix

$A_{n \times n}$ is unitary matrix iff $\boxed{A A^{\theta} = A^{\theta} A = I}$

$\Downarrow$

$$\boxed{A^{\theta} = A^{-1}}$$

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Notes

NOTE

1) If A, B are unitary / orthogonal matrices then $AB, BA, A^{-1}, B^{-1}, A^T, B^T$ are unitary / orthogonal matrices.

Nilpotent Matrix : $A_{n\times n}$ is nilpotent matrix of Index (or) degree (m) if $\boxed{A^m = O_{n\times n}}$, $m \in Z^+$

(+ve integer set)

Involutary Matrix : $A_{n\times n}$ is involutary matrix iff $\boxed{A^2 = I}$

$$\Downarrow$$

$$A \cdot A = I$$

$$\Downarrow$$

$$\boxed{A = A^{-1}}$$ // Self inverse

$$\boxed{A^{2n} = I}$$
$$\boxed{A^{2n-1} = A}$$

Idempotent matrix

$$\boxed{A^2 = A}$$

NOTE

1) If $A^2 = A$, Then $A^n = A$
2) If $AB = B$, $BA = A$, Then $A^2 = A$, $B^2 = B$
3) If $AB = A$, $BA = B$, Then $A^2 = A$, $B^2 = B$

[Converse is not true]

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Q1 Given $XY = Y$ & $YX = X$, Then

1) $X^2 + Y^2 = ?$

$XY = Y$ & $YX = X$

$\Rightarrow X^2 = X$ & $Y^2 = Y$

$\therefore X^2 + Y^2 = X + Y$

2) $X^3 + Y^5 = ?$

$X^2 = X \Rightarrow X^3 = X$

$Y^2 = Y \Rightarrow Y^5 = Y$

$\therefore X^3 + Y^5 = X + Y$

3) $X^n + Y^n = ?$

here $X^n = X$ & $Y^n = Y$

$\therefore X^n + Y^n = X + Y$

Q A, B are two matrices of same size such that $AB = A$ and $BA = B$, then ~~the~~ what is $(A+B)^{10} = ?$

a) 2^9 b) 2^{10} c) $2^8(A+B)$ d) $2^9[A+B]$

given $AB = A$ & $BA = B \Rightarrow A^2 = A$, $B^2 = B$

WKT : $(A+B)^2 = A^2 + B^2 + AB + BA$ $(AB \neq BA)$

$= A + B + A + B$

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$$\boxed{[A+B]^2 = 2[A+B]}$$

Sq. on both sides $\Rightarrow$ $[A+B]^4 = 2^2[A+B]^2$

$= 2^3[A+B]$

$\vdots$

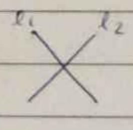
$$\boxed{[A+B]^n = 2^{n-1}[A+B]}$$

$\therefore [A+B]^{10} = 2^9[A+B]$ (d) ✓

Determinants

The factor which determines whether the given linear system has unique sol^n or not is called <u>determinant</u>.

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$



$\Downarrow$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

if $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ $\Rightarrow$ cut at only one pt.

$\Downarrow$

$a_1b_2 - a_2b_1 \neq 0$

$\Downarrow$

$|A| \neq 0$

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$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Adj : Adjoint

$A^{-1} = \frac{Adj A}{|A|}$

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$Adj A = [Cof A]^T$ cofactor

Notes

$A_{n \times n}$

fix $i = 1$

$$|A_{n\times n}| = \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \delta_{ij}$$

where δ_{ij} = minor of an element a_{ij}

eg

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\underline{i=1}$

$$= (-1)^{1+1} a_{11}\delta_{11} + (-1)^{1+2} a_{12}\delta_{12} + (-1)^{1+3} a_{13}\delta_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Properties of Determinant

1) $|A| = |A^T|$
2) $|AB| = |A||B|$
3) $|A^n| = |A|^n$
4) $|A^{-1}| = |A|^{-1}$
5) $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}$
6) $|k A_{n\times n}| = k^n |A|$
7) $|Adj A_n| = |A|^{n-1}$
8) $Adj(Adj A_n) = |A|^{n-2} A$
9) $|Adj(Adj A_n)| = |A|^{(n-1)^2}$

Notes

10) $\begin{vmatrix} a_1+k_1 & b_1+k_2 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} k_1 & k_2 \\ a_2 & b_2 \end{vmatrix}$

11) $\begin{vmatrix} a_1+k_1 & b_1 \\ a_2+k_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$

Q $|A_{3\times3}| = 6$ find

1) $|(A^T)^{-1}|$
2) $|2A|$
3) $|(2A)^{-1}|$
4) $|\text{Adj} A|$
5) $\text{Adj}(\text{Adj} A)$
6) $|\text{Adj}(\text{Adj} A)|$
7) $|A^3|$

1) $|(A^T)^{-1}| = |A^T|^{-1} = |A|^{-1} = 6^{-1} = \frac{1}{6}$

2) $|2A_{3\times3}| = 2^3|A| = 8\times6 = 48$

3) $|(2A)^{-1}| = |2A|^{-1} = \frac{1}{48}$

4) $|\text{Adj} A| = |A|^{3-1} = 6^2 = 36$

5) $\text{Adj}(\text{Adj} A) = |A|^{3-2} A = 6A$

6) $|\text{Adj}(\text{Adj} A)| = |A|^{(3-1)^2} = 6^4$

7) $|A^3| = |A|^3 = 6^3$

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Property continue ...

12) The determinant of lower Triangular Matrix, Upper Triangular Matrix and Diagonal Matrix is the product of its principal diagonal elements.

eg. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 2 \end{vmatrix}_{3\times 3} \rightarrow$ U.T.M

$$|A| = 1 \times 5 \times 2 = 10$$

13) The det. of a skew-Sym matrix of odd order $= 0$.

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{vmatrix}_{3\times 3} = 0$$

$\Downarrow$

Skew-Symm matrix of odd order $\quad [\because a_{ii} = 0 \text{ ** } a_{ij} = -a_{ji}]$

14) The det. of a orthogonal matrix / unitary matrix is ± 1.

15) The det of Involutary matrix is ± 1.

$A^2 = I \Rightarrow A = A^{-1} \Rightarrow |A| = |A^{-1}|$

$\Rightarrow |A| = \dfrac{1}{|A|} \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

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16) The det. of Idempotent matrix is 0 or 1

$$A^2 = A$$
$$\Rightarrow |A^2| = |A|$$
$$\Rightarrow |A|^2 - |A| = 0$$
$$\Rightarrow |A|(|A| - 1) = 0$$
$$\Rightarrow |A| = 0 \text{ or } |A| = 1$$

Linear Dependency

A set of vectors $x_1, x_2, x_3, \dots, x_n$ are linearly dependent if their linear combination

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = 0 \text{ for not all } a_i = 0.$$

$a_i \in R$

Linear Independency

A set of vectors $x_1, x_2, x_3, \dots, x_n$ are linearly independent if their linear combination

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = 0, \text{ only for all } a_i = 0.$$

$a_i \in R$

$\Rightarrow$ Let x_1, x_2, x_3 are L.D.

Then $a_1x_1 + a_2x_2 + a_3x_3 = 0$
$\forall a_i \neq 0$

Notes

$$\Rightarrow x_1 = -\frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3 ,$$

$$\boxed{x_1 = c_1 x_2 + c_2 x_3}$$

LD

(eg) $x_1 = [1 \ 1]$ $x_2 = [2 \ 2]$

$x_2 = 2[1 \ 1]$

$= 2x_1$

$\therefore$ LD

— or —

$a_1 x_1 + a_2 x_2 = 0$

$a_1[1 \ 1] + a_2[2 \ 2] = [0 \ 0]$

choosing $a_1 = 2, \ a_2 = 1 \neq 0$

$-2[1 \ 1] + 1[2 \ 2] = [0 \ 0]$

$[0 \ 0] = [0 \ 0]$

(eg) $x_1 = [1 \ 2 \ 1]$ $x_2 = [2 \ 4 \ 2]$

$x_2 = 2[1 \ 2 \ 1]$

$= 2x_1$

$\therefore$ LD

— or —

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{1}{2} \Rightarrow x_1, x_2$ are L.D

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— or —

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}_{2\times 3}$$

(determinants) minors 2×2

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \rightarrow \text{L.D}$$

Note: If there exists atleast one minor of 2×2 is non zero then those two vectors are linearly independent.

Property cont...

17) If any ~~the~~ rows or ~~or~~ columns are L.D. Then det. of the matrix is zero.

Q vectors $(1\ -1\ 2)(7\ 3\ x)(2\ 3\ 1)$ in R^3 are L.D, $x = ?$

3 – Dim space of vectors

Sol^n

$\because X_1, X_2, X_3$ are LD $\Rightarrow |A_{3\times 3}| = 0$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 7 & 3 & x \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1[3-3x] + 1[7-2x] + 2[15] = 0$

$\Rightarrow -5x + 40 = 0$

$\Rightarrow \boxed{x = 8}$

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NOTE:
The above def. of L.I & L.D remain same for set of rows, col^n, vectors & functions etc.

(eg) $A = \{ \cos^2 x, \sin^2 x, 1 \}$ are L.D or L.I?
$\quad\quad f_1 \quad\quad f_2 \quad f_3$

W.K.T $\quad \cos^2 x + \sin^2 x = 1$
$\quad\quad f_1 + f_2 = f_3$

$\therefore$ L.D

(eg) $A = \{ \sin^2 x, \cos^2 x, \cos 2x \}$ are L I or L D?
$\quad\quad f_1 \quad\quad f_2 \quad f_3$

W.K.T $\quad \cos 2x = \cos^2 x - \sin^2 x$
$\quad\quad f_3 = f_2 - f_1$

$\therefore$ L.D.

(eg) $\{ \tan x, \sin x, \cos x \}$ are L I or L D?
$\quad\quad f_1 \quad\quad f_2 \quad f_3$

If $\quad f_1 = a_1 f_2 + a_2 f_3$

$\tan x = a_1 \sin x + a_2 \cos x$

$\infty \neq$ finite for any a_1, a_2

$\therefore$ They are L.I.

(eg) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = ?$

by observing

$R_3 = R_2 + R_1$

$\therefore$ LD $\Rightarrow |A| = 0$

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eg $\begin{vmatrix} 2 & 1 & 2 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & -1 & 4 & 3 \end{vmatrix}_{4\times 4} = ?$

$R_3 = R_1 + R_2$

LD $\therefore |A| = 0$

(also $R_4 = R_2 + R_3$,
$C_1 + C_4 = C_2 + C_3$
$\Rightarrow C_1 + C_4 - C_2 = C_3$)

(atleast one enough to prove LD)

By Elementary Transformations

Row Trans
Col^n Transf.

$|A| \underset{E \cdot T}{\cong} |B|$

Rule 1 :- If $R_i \leftrightarrow R_j$ Then $|B| = -|A|$

Rule 2 :- If $R_i \rightarrow KR_i$ Then $|B| = K|A|$

Rule 3 :- If $R_i \rightarrow R_i + KR_j$ Then $|B| = |A|$

Similarly for column Transformations.

Q Let $A = \begin{bmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ and by

applying the following transformations

$C_2 \rightarrow C_2 + C_1$ (1) & $R_2 \rightarrow R_2 + R_3$ (2)

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Notes

Then the determinant of resultant matrix is ___?

Sol^n $\because$ ① x ② are of Rule ③

$$\Rightarrow |B| = |A| = \begin{vmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{vmatrix}$$

$$= 15 \begin{vmatrix} 3 & 4 & 3 \\ 7 & 8 & 7 \\ 13 & 2 & 13 \end{vmatrix} = 0 \quad [\because C_1 = C_2]$$

Q

$$\begin{array}{c} + \quad - \quad + \quad - \\ \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \end{vmatrix} \end{array} \to \text{fix}$$

[Select the one row / column which has max. no. of 0's]

$$= 2 \times \begin{array}{c} + \quad - \quad + \\ \begin{vmatrix} 3 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} \end{array}$$

$$= 2 \times [3(-5) + 2(1)] = -26$$

Q

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}_{4 \times 4}$$

[Whenever higher order det. is given by applying Transformation reduce it to LTM, UTM or DM or get max no. of zeros in any row or col^n]

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by E.T. :- $C_3 \to C_3 - 3C_1$

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & -6 & 1 \\ 3 & 0 & -8 & 2 \end{vmatrix} \leftarrow \text{fixed}$$

$$= -1 \times \begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 0 & -8 & 2 \end{vmatrix}$$

$$= -1 \times \left[+8 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 3 & -6 \end{vmatrix} \right]$$

$$= -1 \left[8 \times [-8] + 2 \times [-12] \right]$$

$$= 88$$

(9)
$$\begin{array}{cc} D_1 & U.T.B \end{array}$$
$$\left|\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right|_{4\times4} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = (-2)(-2) = 4$$
$$\begin{array}{cc} LTB & D_2 \end{array}$$

$\boxed{\text{If atleast one of triangular block is a zero block, Then it is called Triangular block matrix. Then } |A| = |D_1||D_2|}$

Notes

(Q)

$$\begin{array}{c} \begin{array}{cc|ccc} D_1 & & & UTB & \\ \end{array} \\ \begin{vmatrix} 2 & 3 & 4 & 7 & 8 \\ -1 & 5 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & -1 & 5 \\ 0 & 0 & 5 & 2 & 6 \end{vmatrix}_{5\times 5} \\ \begin{array}{cc|ccc} LTB & & & D_2 & \end{array} \end{array}$$

$\Rightarrow$ Triangular block matrix

$$\therefore \; |A| = |D_1|\,|D_2|$$

$$= \begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} \begin{vmatrix} 2 & 1 & 5 \\ 3 & -1 & 5 \\ 5 & 2 & 6 \end{vmatrix}$$

$$\quad\quad\quad\quad\quad\quad\quad\quad + \;\; - \;\; +$$

$$= (13)\left[5(10) - 2(-5) + 6(-5)\right]$$

$$= 13\,[50 + 10 - 30]$$

$$= 390$$

HW (Q)

$$\begin{vmatrix} 3 & 4 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 0 \\ 0 & 5 & 0 & 6 & 7 \\ 0 & 0 & 4 & 3 & 4 \end{vmatrix}$$

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Shortcuts

1)

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left[1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]$$

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2) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

3) $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$

4) $\begin{vmatrix} 1=a^0 & a & a^2 \\ 1=b^0 & b & b^2 \\ 1=c^0 & c & c^2 \end{vmatrix} = (c-b)(c-a)(b-a)$

(vander mode det.)

$\begin{vmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{vmatrix} = (5-4)(5-3)(5-2)(4-3)(4-2)(3-2)$

$= 1\cdot2\cdot3\cdot1\cdot2\cdot1$

$= 12$

(vander mode det.)

Q) $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$ (by shortcut no 1)

$= 1\cdot1\cdot1\cdot1\left[1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}\right]$

$= 5$

Q) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3(1)(2)(3) - 1^3 - 2^3 - 3^3$

$= -18$

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Notes

Q $\Delta = \cancel{\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & \sin^2 x & 1+\cos^2 x \end{vmatrix}}$

Q $\Delta = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$

If α, β are max. & min. values of Δ, then
1) $\alpha + \beta^4$ 2) $\alpha\beta^5$?

by shortcut 3

$\Delta = 1 + \sin^2 x + \cos^2 x + \sin 2x$
$= 1 + 1 + \sin 2x$
$= 2 + \sin 2x$

Range $[-1, 1]$

$\Delta_{max} = \alpha = \max[2 + \sin 2x]$
$= 2 + \max[\sin 2x]$
$= 2 + 1$
$= 3$

$\Delta_{min} = \beta = \min[2 + \sin 2x]$
$= 2 + \min[\sin 2x]$
$= 2 - 1$
$= 1$

1) $\alpha + \beta^4 = 3 + 1^4 = 4$
2) $\alpha\beta^5 = 3 \cdot 1^4 = 3$

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(2) $a \neq b \neq c$,

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = 0 \quad , \quad abc = ?$$

by prop (1)

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \underset{(-1)(-1)}{+} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$(abc+1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$abc + 1 = 0$ (or) $(c-b)(c-a)(b-a) \neq 0$

$(\because a \neq b \neq c$ given$)$

$\therefore abc + 1 = 0$

$$\boxed{abc = -1}$$

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$$\boxed{\text{Cofactor of } a_{ij} = (-1)^{i+j}\delta_{ij}}$$

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Notes

Inverse of Matrix

$$A^{-1} = \frac{\text{adj } A}{|A|}, \quad |A| \neq 0$$

If $|A| = 0$, Then A^{-1} doesnot exist.

where $\text{adj } A = (\text{cofactor matrix})^T$

$$= \left[(-1)^{i+j}\delta_{ij}\right]^T$$

eg $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = ?$

W.K.T $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{ad-bc}\begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}^T$

$$= \frac{1}{ad-bc}\begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix}$$

⑥ $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, $R^{-1} = ?$

Solⁿ W.K.T $R^{-1} = \frac{1}{|R|}\text{adj } R$, $|R| = 1(5) - 1(4)$
$= 1 \neq 0$
$\therefore R^{-1}$ exists

$$\text{Adj } R = \left[(-1)^{i+j}\delta_{ij}\right]^T$$

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$$\text{adj } R = \begin{bmatrix} +5 & -6 & +4 \\ -3 & +4 & -3 \\ +1 & -1 & +1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

$$R^{-1} = \frac{1}{1} \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

Q12 | WB

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix} \quad k = ?$$

$$\text{Adj } A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix} = \left[(-1)^{i+j} \delta_{ij}\right]^T$$

Transpose both sides

$$\left[(-1)^{i+j} \delta_{ij}\right] = \begin{bmatrix} -11 & 4 & 10 \\ -9 & -2 & k \\ 1 & -3 & 7 \end{bmatrix} \rightarrow \text{cofactor matrix of } A$$

$\therefore$ k = cofactor of elem a_{23} in "A"

$$k = (-1)^{2+3} \delta_{23} = -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = -5$$

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Notes

Q⑦ $A = [a_{ij}]_{3\times 3}$ where $a_{ij} = i^2 - j^2 \;\forall i, j$, $A^{-1} = ?$

Sol^n

$A^{-1} = \dfrac{\text{adj} A}{|A|}$ | $|A_{3\times3}| = \begin{vmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{vmatrix}$

Skew-sym. of odd order

$\therefore |A| = 0$

$\Rightarrow A^{-1}$ doesnot exist.

Q② given A, B, C, D, E are ~~non-signguis~~ non-singular matrices

$\Rightarrow A^{-1}, B^{-1}, C^{-1}, D^{-1}, E^{-1}$ exists

given $DABEC = I$, $B^{-1} = ?$

1) $B^{-1} = ECDA$ (start after B & end before B)

2) $A^{-1} = BECD$

Q $|A_{3\times3}| = 6$, then $A(\text{Adj} A) =$ ____ ?

Sol^n WKT $A^{-1} = \dfrac{\text{adj} A}{|A|}$, $|A| \neq 0$

$\Rightarrow \text{adj} A = |A| A^{-1}$

Pre x'ly by A on b.s

$A(\text{Adj} A) = |A| A A^{-1}$

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$$\boxed{A(\text{Adj } A) = |A| I}$$

$$A(\text{Adj } A) = |A| I$$
$$= 6I$$
$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Rank of Matrix

The no. of Ind. (independent) rows / col^{n}s in a matrix is rank of matrix.

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1 & 2 & \boxed{3} \\ 2 & -1 & \boxed{1} \\ 3 & 1 & \boxed{4} \end{bmatrix} \end{array} \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}$$

By observation :-

$C_3 = C_1 + C_2$ (only make any one of C_1, C_2, C_3 dep. bcoz of this similar eq).

dep LD

$C_3 - C_2 = C_1$

dep only

ratios from C_1, C_2

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow C_1, C_2 \text{ are linearly independent}$$

$\therefore \rho(A) = 2$ (No. of Independent rows in A (or) no. of ind. columns in A)

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Q) Find the rank of

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & -1 & 4 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \text{ dep} \\ R_4 \text{ dep} \end{matrix}$$

sol^n:

$R_3 = R_1 + R_2$ → dep

$R_4 = R_2 + R_3$ → dep

2×2 minor form

(If all minors from R_1 & R_2 are zero then R_1 & R_2 dep)

$$\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \neq 0$$

(If atleast one non-zero minor then independent)

∴ R_1, R_2 linearly independent

∴ $\rho(A) = 2$

NOTE [The no of independent rows = The no. of ind. col^ms]

Q) find rank of

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{bmatrix}_{4\times5}$$

dep dep

sol^n

by observation

$C_2 = 2C_1$ → dep.

$C_1 + C_3 = C_4$ → dep

Notes

Consider a minor from C_1, C_3, C_5

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{vmatrix}_{3\times3} = 1(1) + 1(2+4) \neq 0$$

$\therefore C_1, C_3$ & C_5 are linearly independent.

$\therefore \rho(A) = $ no. of ind. col^m
$= 3$

Method ②

Rank of matrix :- The highest order of non-zero minor in a matrix A is called $\rho(A)$.

$\boxed{Q21 \mid WB}$ $\boxed{\text{Rank of } (P+Q) \leq \text{Rank of } (P) + \text{Rank } (Q)}$

$$P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}_{3\times3}$$

method ②

$|P+Q| = 0$ $\quad [\because R_3 - R_2 = R_1$
$\Rightarrow$ L.D exist
$\Rightarrow |P+Q| = 0]$

$\therefore \rho(P+Q) < 3$

Now consider all minors of 2×2

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if $|A_{n\times n}| = 0 \Rightarrow \rho(A) <$ order of A
if $|A_{n\times n}| \neq 0 \Rightarrow \rho(A) =$ order of A

Notes

$\begin{vmatrix} 8 & 9 \\ 8 & 8 \end{vmatrix}_{2\times 2} = -8 \neq 0 \Rightarrow \exists$ a non-zero minor of 2×2

$\therefore$ highest order of non-zero minor of ~~2×2~~ $(P+Q)$ is 2×2

$\therefore \rho(P+Q) = 2$

Q) find Rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3}$

Sol^n

$|A_3| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 0 \quad [\because R_2 = 2R_1]$

$\Rightarrow$ 3×3 minor is zero minor

$|A| = 0 \Rightarrow \rho(A_{3\times 3}) < 3$

now consider 2×2 minors

$\because$ all 2×2 minors are zeros

$\Rightarrow \rho(A) < 2$

now consider 1×1 minor (i.e. an element of A matrix)

$\because$ there exist a non zero element in A

$\Rightarrow \exists$ a non-zero minor of 1×1 order.

[Hence, highest order of non-zero minor in A $= 1 = \rho(A)$.]

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Notes

(OR) By method - 1

$\because$ $R_2 = 2R_1$

& $R_3 = 3R_1$

$\because$ All 2 vectors are scalar multiple of ~~one vector~~ only one vector (R_1) $\therefore$ The no. of independent vectors $= \rho(A) = 1$

Method (3)

By Elementary Transf^{n}s

$$A \underset{ET}{\cong} B \Rightarrow \rho(A) = \rho(B)$$

Q16 WB

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \quad \rho(A) = ?$$

By ET $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix} \begin{matrix} \to \text{L I} \ (\because \text{ratio not same}) \\ \\ \to \text{dep} \\ \to \text{dep} \end{matrix}$$

$\Rightarrow \rho(A) = 2$

$R_3 = R_2$, $R_4 = R_2$

dep dep

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Notes

$\propto$ minor from R_1 & R_2 only

$$\begin{vmatrix} 11 & 12 \\ 10 & 10 \end{vmatrix} \neq 0 \Rightarrow R_1 \text{ \& } R_2 \text{ are independent.}$$

$\therefore \rho(A) = $ no. of Ind rows / colms in A
$= 2$

$\boxed{Q17}\boxed{WB}$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times5}$$

$\rho(A) = ?$

(every row have same elements)

$R_1 \to R_1 + R_2 + R_3 + R_4 + R_5$

$$\cong \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

(zero row/col^m is always dep.)

Consider 4x4 minor from R_2, R_3, R_4, R_5

D_1

$$\begin{vmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{vmatrix}$$

$+1 \quad +1 \quad +1 \quad +1 \quad D_2$

$= 1 \times 1 \times 1 \neq 0$

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$\Rightarrow R_2, R_3, R_4, R_5$ are L I

$\therefore \rho(A) =$ no. of Ind. rows

$= 4$

Q HW find Rank of

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$R_2 \to R_2 \times \frac{1}{6}$, $R_3 \to R_3 \times \frac{1}{3}$

$$\cong \begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 7 & 4 & 9 \\ 7 & -7 & 0 & -5 \end{bmatrix} \to \text{dep}$$

By observation

$R_3 = 2R_1 - R_2$

dep

consider 2x2 minor from R_1, R_2

$\begin{vmatrix} 3 & 0 \\ -1 & 7 \end{vmatrix} \neq 0 \Rightarrow R_1$ & R_2 are LI

$\therefore \rho(A) =$ no. of Ind. Row

$= 2$

Method (4)

Echelon form ☆☆☆

$$A_{m\times n} \underset{R.T}{\cong} U$$

☆☆ → (Row Trans^n only)

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Notes

Echelon Form

$$A \underset{(R.T)}{\cong} U$$

[Reduce A in such a matrix that no. of 0's before first non zero elem. should increase from first row to last row]

U =

$\rho(A)$ = no. of non-zero rows in the Echelon form of "A" (U)

Q) find Rank of
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{bmatrix}$$

by Echelon Form

[make elements below first diagonal elem as 0] (using first row only)

$R_2 \to R_2 - 2R_1$,
$R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - 3R_1$

$$\cong \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -5 & -3 \end{bmatrix}$$

[make elem. below 2nd diagonal elem. as zero] (using R_2 only)

$R_3 \to R_3 - R_2$, $R_4 \to R_4 + 2R_2$

$$\cong \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

[make elem. below 3rd diag. elem as zero]

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$R_4 \to R_4 + R_3$

$$\cong \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cong U$$

it is in Echlon Form

$\rho(A)$ = no. of non-zero rows in Echelon form
$= 4$

Q) Find the rank of $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & +1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Solⁿ

by Echelon form

make first elem. as '1'

$R_1 \leftrightarrow R_2$

$$\cong \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

[we are finding rank of matrix (don't take -ve) (-ve comes in determinants only)]

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$

$$\cong \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

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Notes

$R_2 \to R_2 \times \frac{1}{3}$ $\quad R_3 \to R_3 \times \frac{1}{2}$

$$\cong \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$, $R_4 \to R_4 - R_2$

$$\cong \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in Echelon form

$\therefore \rho(A)$ = no. of non zero rows in Echelon form of A

$= 2$

Properties of Rank of Matrix

1) $\rho(A) = \rho(A^T)$

2) $\rho(A A^T) = \rho(A)$

3) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

4) $\rho(A+B) \leq \rho(A) + \rho(B)$

5) $\rho(A-B) \geq \rho(A) - \rho(B)$

6) $\rho(A_{m \times n}) \leq \min\{m, n\}$

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Notes

7) $\rho(I_n) = n$

8) $\rho(O_{n\times n}) = 0$

9) ρ(Diagonal matrix) = no. of non zero principal diagonal elements.

Q 19 $X_{n\times 1} \neq 0$, $\rho(XX^T) = ?$

by prop ②

$$\rho(XX^T) = \rho(X_{n\times 1})$$

$$\leq \min\{1, n\} \quad [\text{by prop ⑥}]$$

$$\leq 1$$

$$\rho(X_{n\times 1}) \neq 0 \quad [\because X_{n\times 1} \neq 0]$$

$$\therefore \boxed{\rho(X_{n\times 1}) = 1 = \rho(XX^T)}$$

Q $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ $B = \begin{bmatrix} p^2+q^2 & rp+qs \\ pr+sq & r^2+s^2 \end{bmatrix}$ if $\rho(A) = N$

Then $\rho(B) = ?$

by observation

$$B = \underbrace{\begin{bmatrix} p & q \\ r & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p & r \\ q & s \end{bmatrix}}_{A^T}$$

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Notes

WKT by prop ②

$$\rho(B) = \rho(A A^T) = \rho(A) = N$$

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NOTES

1) Every square matrix ~~satisfies its~~ can be expressed as sum of symmetric & skew-symm. matrices.

Pf:- Let $A_{n\times n}$

$$A_{n\times n} = \frac{1}{2}[A+A] = \frac{1}{2}[A+A^T+A-A^T]$$

$$A_n = \boxed{\frac{1}{2}[A+A^T]} + \boxed{\frac{1}{2}[A-A^T]}$$

(sym) P + Q (skew sym)

consider $P^T = \frac{1}{2}[A+A^T]^T = \frac{1}{2}[A^T+(A^T)^T]$

$$= \frac{1}{2}[A^T+A] = P$$

consider $Q^T = \frac{1}{2}[A-A^T]^T = \frac{1}{2}[A^T-(A^T)^T]$

$$= \frac{1}{2}[A^T-A] = -\frac{1}{2}[A-A^T]$$

$$= -Q$$

$\therefore$ $P \rightarrow$ Symm. & $Q \rightarrow$ Skew-Symm.

2) Every square matrix can be expressed as sum of Herm. matrix & Skew-Hermitian matrices

(same proof "replace T by θ")

$$A_n = \frac{1}{2}[A+A^\theta] + \frac{1}{2}[A-A^\theta]$$

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Notes

3) If A is a H.M then (iA) is Skew-H.M & vice-versa

4) L U - Decomposition

Every Square matrix $A_{n\times n}$ can be written as product of LTM & UTM

i.e. $A_{n\times n} = L_{n\times n} \cdot U_{n\times n}$

where either $l_{ii}=1$ (or) $u_{ii}=1$, $\forall i$

eg. by LU-Decomposition, find L, U for $A=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

sol^n by LU-Decomposition, $A_{2\times2} = L_{2\times2} \cdot U_{2\times2}$

$$\underset{A}{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}}$$

x^{ly} & equate on b.s

$\boxed{2 = u_{11}}$, $-1 = u_{12} + 0 \times u_{22}$

$\Rightarrow \boxed{u_{12} = -1}$

$l_{21}u_{11} + 1\times 0 = 1 \Rightarrow \boxed{l_{21} = \frac{1}{2}}$

$l_{21}u_{12} + 1\times u_{22} = 2 \Rightarrow \boxed{u_{22} = 5/2}$

Sub in L & U

$L = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 2 & -1 \\ 0 & 5/2 \end{bmatrix}$

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Notes

$\boxed{WB | Q65}$ (d)

Linear Systems

Non - Homogeneous eq.

degree ⇒ 1 1 0

$$\left.\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right\}$$ represents a mathematical model is called linear system.

Can be written into matrix form.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A \quad X \quad = B$$

Homogeneous Eq^n (degree of every term must same)

$$a_1x + b_1y = 0 \rightarrow \text{Homog. eq}^n$$

If $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Then $AX = 0$ is Homogenous system.

If $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $AX = B$ is a non-Homogeneous system.

$$AX = B$$

$B = 0$ → $AX = 0$ (Homog Sys.)

$B \neq 0$ → $AX = B$ (Non-Homogen. Sys.)

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[for every non ... ie. zero vector]

Notes

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$AX = B$

$B = 0$ → $AX = 0$ (Homog. Sys.)

- unique Sol^n (or) $X = 0$ (or) zero sol^n (or) Trivial Sol^n — $\boxed{|A| \neq 0}$
- ∞^{ly} many sol^n $(X \neq 0)$ (or) non zero sol^n (or) Non-Trivial Sol^n — $\boxed{|A| = 0}$

intersect $l_1 \times l_2$

l_2, l_1

Style

$l_1 \times l_2$ overlap

$B \neq 0$ → $AX = B$ (Non-Homog Sys.)

- unique Sol^n — $\boxed{|A| \neq 0}$
- No Sol^n — $|A| = 0$
- ∞^{ly} many Sol^n — $|A| = 0$

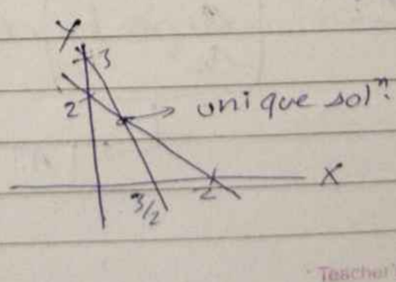
Types of Solutions

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$\boxed{\text{Case 1}}$:

$$x + y = 2 \Rightarrow \frac{x}{2} + \frac{y}{2} = 1$$
$$2x + y = 3 \Rightarrow \frac{x}{3/2} + \frac{y}{3} = 1$$

Graph



Consistent → atleast one sol^n

Inconsistent → No sol^n

Notes

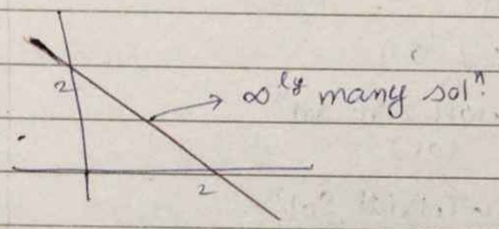
$cond^n$:- $\frac{a_1}{a_2} = \frac{1}{2} \neq \frac{b_1}{b_2} = \frac{1}{1}$

↳ $|A| \neq 0$

Case 2 :-

$X + Y = 2 \rightarrow \frac{X}{2} + \frac{Y}{2} = 1$

$2X + 2Y = 4 \rightarrow \frac{X}{2} + \frac{Y}{2} = 1$



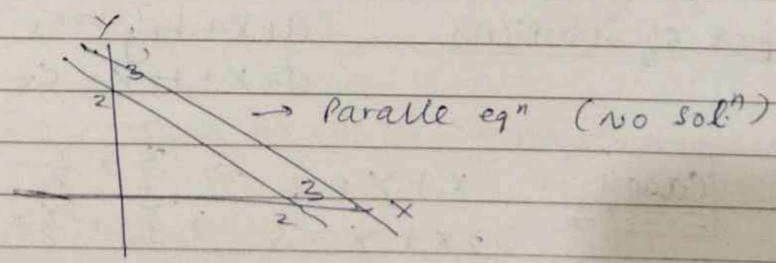
$cond^n$:- $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}} = \frac{c_1}{c_2} = \frac{1}{2}$

↳ $|A| = 0$

Case 3 :- Inconsistent (or) no sol^n

$X + Y = 2$

$2X + 2Y = 6 \Rightarrow \frac{X}{3} + \frac{Y}{3} = 1$



$cond^n$:- $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}} \neq \frac{c_1}{c_2}$

↳ $|A| = 0$

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Notes

NOTE

① $AX = B$ has unique solⁿ ~~if A 0At~~. $\Leftrightarrow$ iff $|A| \neq 0$

② $AX = 0$ has unique solⁿ iff $|A| \neq 0$

③ $AX = 0$ has Non-Trivial solⁿ (∞'ly many solⁿ) iff $|A| = 0$

④ If $AX = B$ has ∞'ly many solⁿ then $|A| = 0$

⑤ If $AX = B$ has no solⁿ then $|A| = 0$

Solving $AX = B$

Augmented matrix

1) reduce $C = [A \,\vdots\, B]$ into Echelon form
we get $\rho(A)$, $\rho(C)$

2) If $\rho(A) < \rho(C)$, then system is inconsistent

3) If $\rho(A) = \rho(C) =$ no. of unknowns, then system has unique solⁿ

4) If $\rho(A) = \rho(C) <$ no. of unknowns, then system has ∞'ly many solⁿ.

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Notes

$$C = \begin{bmatrix} A & \vdots & B \end{bmatrix} = \begin{array}{c} \begin{array}{cc|c} & A & B \end{array} \\ \left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right] \end{array}$$

$\rho(C)$ = no of independent eq^n's in the system

Solving $AX = 0$

1. reduce 'A' into Echelon Form
   we get $\rho(A)$

2) If $\rho(A)$ = no. of unknowns, then system has unique sol^n.

3) If $\rho(A)$ < no. of unknowns, System has ∞'ly many sol^n.

$\rho(A)$ = no. of Ind. eq^n's in the system.

$\boxed{Q25 \mid WB}$

Sys. don't have unique sol^n $\Rightarrow |A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & K \end{vmatrix} = 0 \Rightarrow \exists \text{ a linear dependency}$$

by observation: $2C_2 - C_1 = C_3$

$2(4) - 1 = K$

$\boxed{K = 7}$

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Notes

(or) by expanding $\boxed{k=7}$

(WB) (28) given sys. has ∞ many soln $\Rightarrow |A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & -k \end{vmatrix} = 0 \Rightarrow \exists \text{ a Linear dependency}$$

by observation

$$\boxed{K=2} \Leftarrow (R_3 = R_2 - R_1)$$

(29) given ∞^{ly} many $sol^n \Rightarrow |A| = 0$

$$|A| = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

(all rows have same elements)

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

$$\Rightarrow (a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

$\Rightarrow a+2 = 0$ (or) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$

$\boxed{a=-2}$ $\Rightarrow \boxed{a=1}$

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$\Rightarrow a = \{-2, 1\}$ (d)

Notes

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T16 Sys has non-trivial solⁿ $\Rightarrow |A| = 0$

$$|A| = \begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$ (every col/row all elements are same)

$$\begin{vmatrix} 3k-2 & 3k-2 & 3k-2 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$(3k-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$3k-2=0$ (or) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$

$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$

$$(3k-2)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 3k-11 & 0 \\ 3 & 0 & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)\left[(3k-11)^2\right] = 0$$

$$\boxed{k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}}$$

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Notes

(Q30) $A_{3\times4} X = 0$

by matrix multiplication condⁿ

order $X = 4\times1$

$\Rightarrow$ The no. of unknowns are $= 4$

but $\rho(A_{3\times4}) \le \min(3,4)$

≤ 3

here, $\rho(A_{3\times4}) <$ no of unknowns

$\Rightarrow$ System has ∞^{ly} many solutions. (b)

(Q24) $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & -2 & 5 \\ -1 & -4 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} -2 & 5 \\ -4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 5 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ -1 & -4 \end{vmatrix}$

$= 2(18) + 1(8) + 3(-14)$

$= 2$

$\neq 0 \Rightarrow$ unique solⁿ. (b)

(Q26) given non-~~Hog~~ Homogeneous ~~eqⁿ~~ system

$C = [A : B]$

$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$

reduce 'C' into Echelon form

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Notes

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right]$

~~$\rho(A) =$~~

To say system is inconsistent (No sol^n)

$\rho(A) < \rho(C)$.

If $\lambda - 6 = 0 \Rightarrow \rho(A) = 2$

for $\mu - 20 \neq 0$ ~~$\Rightarrow \mu \neq 20$~~ $\Rightarrow \rho(C) = 3$

for $\lambda = 6$ & $\mu \neq 20$ the sys. has No sol^n.

(b) ✓

shortcut

for $\lambda = 6$ & $\mu \neq 2$ eq^n (2) & (3) are inconsistent

Hence, NO Solution

Case II [∞^{ly} many sol^n]

for $\lambda = 6$, $\mu = 20$, $\Rightarrow eq^n$ (2) = (3)

$\Rightarrow$ no. of Ind. eq^n < no. of unknowns (ie $2 < 3$)

$\Rightarrow$ sys. has ∞^{ly} many $sol^n s$

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Notes

Case III [unique sol^n.] $\Rightarrow |A| \neq 0$

$$\lambda - 6 \neq 0$$

$\boxed{\lambda \neq 6}$ (μ can be anything) ($\mu = 20$ or $\mu \neq 20$)

$\rho(A) = 3 = \rho(C)$ = no of unknowns

$\Rightarrow$ unique sol^n.

Q (27) $C = [A \vdots B]$

$$C = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$$

Reduce 'C' into Echelon form

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$$

$$R_3 \to R_3 - R_2 \cong \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-3a-b \end{array}\right]$$

$\rho(A) = 2$

To say the system is consistent $\rho(A) = \rho(C) = 2$

(only if $c - 3a - b = 0$)

$\Rightarrow$ $3a + b = c$

(b) ✓

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No. of col.m in coeff. matrix A = no. of unknowns in the system

Notes

Shortcut

$$C = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & a & -6 & c \end{array}\right]$$

↳ is consistent $\Rightarrow \rho(A) = \rho(C)$

$\Downarrow$

(if there exist a L.D on A then it should also exist on C as well.)

by observation

$3R_1 + R_2 = R_3$ on 'A'

$\Rightarrow 3R_1 + R_2 = R_3$ on 'C'

$\Rightarrow 3a + b = c$

$\Rightarrow 3a + b - c = 0$ (b) ✓

Q T12

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \rightarrow \infty^{ly} \text{ many sol}^n$$

(Given)

$\Downarrow$

$\exists$ a L.D

$(|A| = 0)$

by obser. [only row dependency]

$\Longrightarrow 2R_3 - R_2 = R_1$ on A & 'C'

$2\alpha - 3 = 1$

$\boxed{\alpha = 2}$

$2\beta - 9 = 5$

$\boxed{\beta = 7}$

(a) ✓

T14 Similarly (b) ✓ $\boxed{a = 8}$ & $\boxed{b = 6}$

T15

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Tomorrow Timing (Maths) $\Rightarrow$ 2Pm to 6Pm

~~Quality x Standard~~
~~6:30pm to 9:00pm~~

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Notes

Nullity of $AX=0$

(space = set of vectors)

$AX = 0$ — X : Set of all solutions of homogeneous system (= Solution space of $AX=0$) (= Null Space of $AX=0$)

- $X=0$: dep. sol^n of $AX=0$
- $X \neq 0$: ∞^{ly} many sol^n
  - dep. sol^n
  - Indep sol^n $\downarrow$

$$\boxed{\text{dimension of set of vectors} = \text{no. of Indep. vectors in it}}$$

Nullity of $(AX=0)$ = dimension of null space of $AX=0$
= dimension of sol^n space of $AX=0$
= no. of independent sol^n of $AX=0$
= no. of unknowns $- \rho(A)$
= no. of col^n of $A - \rho(A)$

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Notes

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Q 34 nullity = n - r
(b) ✓

(35) Dim. of null space = nullity
= no. of col^n of A $- \rho(A)$
= 3 - 2
= 1

$$A = \begin{array}{c} \text{dep.} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \end{array}$$

$C_3 - C_2 = \underset{\text{dep}}{C_1}$

Since ratios are not same in C_2 & C_3
$\therefore \rho(A) = 2$

(32)
$$A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}_{4\times4}$$

$\underset{\text{dep}}{C_1} = \underset{\text{dep}}{C_2} = \underset{\text{dep}}{C_3} = 1 \times \underset{\text{indep.}}{C_4}$

→ C_1, C_2, C_3 are scalar multiple of C_4.
$\therefore \rho(A) = 1$

nullity = no. of col^n of A $- \rho(A)$
= 4 - 1
= 3

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Notes

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Q) GATE 2021 $P_{4\times5}$ matrix such that every solⁿ of $PX=0$ is a scalar multiple of $(2\ 5\ 4\ 3\ 1)^T$

Then $\rho(P) = ?$

$$P_{4\times5} X = 0$$

$$X = k \begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

only one Indep. solⁿ of $PX=0$

$\Rightarrow \boxed{\text{nullity} = 1}$

WKT, nullity = no of colⁿ of $P - \rho(P)$

$1 = 5 - \rho(P)$

$\boxed{\rho(P) = 4}$

~~Eigen values & Eigen vectors~~

NOTE

(dot product = 0)

$\Rightarrow$ All the orthogonal set of vectors are linearly independent. (Convers not True)

Basis of vector Space

let $S \subseteq V$, is a basis of V

if

1) all the vectors in S are linearly Ind.

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Notes

2) Linear combinations of vectors in 'S' generates all the vectors in 'V'.

eg. $X = \{(1\ 0\ 0)(0\ 1\ 0)(0\ 0\ 1)\}$ is a basis of R^3?
(set of all 3-dim real space of vectors)

① Condⁿ:- consider the minor of three vectors

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ all vectors are L.I.

② Condⁿ:- Linear combination of 3 vectors in S is ~~$a_1x_1 + a$~~ $a_1(\overset{i}{1}\ \overset{j}{0}\ \overset{k}{0}) + a_2(\overset{i}{0}\ \overset{j}{1}\ \overset{k}{0}) + a_3(\overset{i}{0}\ \overset{j}{0}\ \overset{k}{1})$

$= (a_1, a_2, a_3) \in R^3$, where $a_1, a_2, a_3 \in R$.

$\Rightarrow$ for different values of $a_1, a_2, a_3 \in R$ this Linear Combination generating all 3 dim. real vectors of R^3

$\Rightarrow$ 'S' spans 'R^3'

Hence 'S' is a basis of R^3.

Q33) $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

from the options $S = \{[1, -1, 0]^T, [1, 0, -1]^T\}$

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Notes

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Independent $\left[\because \frac{1}{1} \neq \frac{-1}{0} \right] \Rightarrow$ cond^n ① ✓

* Linear combinations

$$a_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ -a_1 \\ -a_2 \end{bmatrix} \quad \forall a_1, a_2 \in \mathbb{R}$$

$\Downarrow$

Since $a_1 + a_2 - a_1 - a_2 = x_1 + x_2 + x_3 = 0$

$\Rightarrow$ L. Comb of these two vectors generates all the vectors of X whose sum of coordinates is zero.

Hence, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of X.

(a) ✓

NOTE: The dimension of any vector space V = The no. of vectors in its basis

Basis of any vector space consist of all the independent vectors of vector space.

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Notes

— or —

$x_1 + x_2 + x_3 = 0$ is a Homogeneous System.

nullity = no. of Ind. Sol^n of Homog. system
= no. of unknows − $\rho(A)$
= 3 − 1
= 2

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$x_1 + x_2 + x_3 = 0$

let $x_2 = -k_1$

$x_3 = -k_2$

$x_1 = -x_2 - x_3$

$x_1 = k_1 + k_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_1 \\ -k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ for $k_1, k_2 \in \mathbb{R}$, the linear comb. of these set of two Ind. vectors generating "X".

hence The set of two Ind vectors of X forms a basis of X.

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Notes

NOTE
$$\boxed{\dim[R^n] = n}$$

Eigen values & Eigen vectors ☆☆☆ (2 marks) 100%

Let $A_{n\times n}$, for any scalar λ, $\exists$ $x \neq 0$ such that $\boxed{AX = \lambda X}$

Then λ is Eigen value (characteristic value) of $A_{n\times n}$ and $X \neq 0$ is Eigen vector correspond to an Eigen value (λ) of A.

find λ :- Let λ be Eigen value of A

Then $\boxed{AX = \lambda X}$, $X \neq 0$

$\Rightarrow AX - \lambda IX = 0$

$\Rightarrow \boxed{(A - \lambda I)X = 0}$ → homogenous system

it should posses non zero sol^n $(x \neq 0)$

$\Rightarrow \boxed{|A - \lambda I| = 0}$ → characteristic eq^n of A.

☆☆☆

solving we get (λ)

find $(x \neq 0)$:- WKT, x is a non-zero sol^n of $(A - \lambda I)x = 0$.

Sub λ in $(A - \lambda I)X = 0$.

we get equations.

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Notes

Solving the Ind. eqn, we get $X \neq 0$.

Q) find the eigen values & eigen vectors of

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Soln find λ

Solving $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$+ \quad - \quad +$

$$\Rightarrow \quad (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\left[(2-\lambda)^2 - 1^2\right] = 0$$

$\Rightarrow \quad 1-\lambda = 0$ | $\cancel{(2-\lambda)(-1)}$

$\lambda = 1$ | $(2-\lambda-1)(2-\lambda+1) = 0$

$\lambda = 1$, $\lambda = 3$

find $X \neq 0$:

1) Let $\lambda_1 = 3$

substitute $\lambda = 3$ in $(A - \lambda_1 I)X = 0$

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Notes

$$\begin{array}{l} \text{Ind.} \\ \\ \text{dep.} \end{array} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ Solving Ind Eq^{n}s :-

$$-X + Y + Z = 0 \quad — (1)$$
$$X - Y + Z = 0 \quad — (2)$$

$$\begin{array}{cccc} X & & Y & & Z \\ \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & & \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \end{array}$$

$$\frac{X}{\cancel{2}_1} = \frac{Y}{\cancel{2}_1} = \frac{Z}{\cancel{0}_0} = k \text{ (say)}$$

$$\Rightarrow \left.\begin{array}{l} \frac{X}{1} = k \Rightarrow X = k \\ \frac{Y}{1} = k \Rightarrow Y = k \\ \frac{Z}{0} = k \Rightarrow Z = 0 \end{array}\right\} X_1 = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ k \in R$$

2) find $X \neq 0$ for $\lambda_2 = 1$

Sub $\lambda = 1$ in $(A - \lambda I)X = 0$

$$\begin{array}{l} \text{Indep.} \\ \text{dep.} \\ \text{dep.} \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Notes

only ① ind. eq^n : $x+y+z=0$ (when one eq^n & three variables are given)

[by previous method in Q33]

let $y = k_1$, $z = k_2$ $\Rightarrow$ $x = -y-z$
$= -k_1 - k_2$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Ind. eigen vectors

The eigen vectors correspoing to $(\lambda = 1)$ are given by linear comb. of two independent eigen vectors

General Characteristic Eq^n:

$$|A_{n\times n} - \lambda I| = 0$$

$$|A_{n\times n} - \lambda I| = (-1)^n \lambda^n + (-1)^{n-1} \text{Trace}(A) \lambda^{n-1} + \cdots + |A| = 0$$

Coeff. of highest degree term $= (-1)^n$

& constant term of char. poly. of $A = |A|$

Notes

(24) we know that, constant term of char. poly of $A_{4\times4} = |A_{4\times4}| = 0$

$$[\because R_3 = 2R_1]$$

Q (41) WKT, $\boxed{[\text{product of eigen values}] = |A|}$

$\therefore$ $|\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4| = ||A|| = 12$

vandermode det.

Q (63) $f(t) = t^n + C_{n-1}t^{n-1} + \dots + C_1 t + C_0$

X^{ly} $(-1)^n$ to $f(t)$ to get general char eq^n of $A_{n\times n}$

$\therefore$ char. ~~pl~~ polynomial will be

$(-1)^n t^n + (-1)^n C_{n-1} t^{n-1} + \dots + (-1)^n C_0$

$\therefore$ ~~cha.~~ Constant term of char poly $= |A| = (-1)^n C_0$

(d) ✓

(Q) $\lambda^3 + \lambda^2 - 21\lambda - 45$ be the char poly of 3x3 matrix A. Then find
1) adj (adj A)
2) |adj A|
3) A (adj A)

sol^n by property (8) in determinants.

1) $adj(adj\, A_{3\times3}) = |A|^{3-2} A$

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Notes

Since char eq^n not in general form we need to multiply $(-1)^3$ to given poly.

$\cancel{-\lambda^3 + \lambda^2 + 21}$

$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 \Rightarrow$ Const term $= |A| = 45$

1) $45A$

2) $|adj\ A| = |A|^{3-1} = 45^2$

3) $A(adj\ A) = |A|\ I = 45I$

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Notes

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Geometric multiplicity of λ

= no. of Ind. Eigen vectors for a λ
= no. of unk $- \rho(A - \lambda I)$

$\lambda = 3$

$$X = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; k \in R \longrightarrow \infty \text{ Eigen vectors corr to } \lambda = 3$$

$\downarrow$

only one Ind. eigen vector of $\lambda = 3$

$X \neq 0$ is eigen vector = sol^n of homog system $(A - \lambda I)X = 0$

$\downarrow$

no. of Ind. sol^n = nullity
= no of unk $- \rho(A - \lambda I)$
= no. of Ind. Eig. vectors corr to λ

$1 = 3 - 2$

for $\lambda = 1$

$$X = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; k_1, k_2 \in R$$

Ind eig vectors for $\lambda = 1$

$= 3 - 1 = 2$

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Properties of Eigen values

1) $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

2) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = |A_{n\times n}|$

3) A & A^T have same eigen values

4) A is singular matrix ($|A| = 0$) iff atleast one of the $\lambda = 0$.

5) A is non singular matrix ($|A| \neq 0$) iff $\lambda_i \neq 0, \forall i$

6) The eigen values of real symmetric / Hermitian matrix are real.

7) Eigen values of LTM / UTM / Diagonal matrix are its Principal Diagonal elements.

8) Eigen values of Skew-Sym / Skew-Hermitian matrix are zero (or) purely Imagenary.

9) $\rho(A)$ = no. of non zero eigen values of 'A'

10) Let λ be an Eigen value of 'A'

$\Rightarrow \boxed{A X = \lambda X}$

Then 1) λ^n is Eig. value of A^n

2) $(k\lambda)$ is Eig value of (kA)

3) $(\lambda + k)$ is Eig. value of $(A + kI)$

eigen value of $I = 1$

Notes

4) $(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)$ is Eig. value of $(a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)$

11) Let λ be the Eig. value of a non-singular matrix.

$|A| \neq 0$
$\Rightarrow A^{-1}$ exists

Then,

1) $\frac{1}{\lambda}$ is Eig. value of A^{-1}

2) $\frac{|A|}{\lambda}$ is Eig. value of $(\text{Adj } A)$

NOTE: The eigen vector $X \neq 0$ corresponding to λ of a matrix A remains same for the matrices A^2, A^3, A^n, A^{-1}, Adj A, KA, and polynomial in A etc.

12) Eigen values of Involutary Matrix are ± 1

$A^2 = I$

13) Eigen values of Idempotent matrix are 0, 1.

$A^2 = A$

14) Trace of an Idempotent matrix (A) = Rank of A

Q) Let $P_{3\times3}$ real matrix such that $P^3 = P$, then $\lambda_P = ?$.

$P^3 = P \Rightarrow \lambda^3 = \lambda \Rightarrow \lambda^3 - \lambda = 0 \Rightarrow \lambda = 0, +1, -1$

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Shortcuts

1) In any row / any col^m all the non-diagonal elements are zero then the diagonal element is eigen value.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\lambda_1 = 1}$

2) Q44 $\because$ Row sum $= 15$ (for each row)

$\Rightarrow$ one of the $\lambda = 15$

$\because$ Column Sum $= 3$

$\Rightarrow \boxed{\lambda_2 = 3}$ (C)

WKT $\lambda_1 + \lambda_2 + \lambda_3 = Tr(A)$

$1 + 3 + \lambda_3 = 5$

$\boxed{\lambda_3 = 1}$

Q48 $A_{3\times3}$

given $|A - I| = 0 \Rightarrow \boxed{\lambda_A = 1}$

& Trace$(A) = 13$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 13$

$\Rightarrow \boxed{\lambda_2 + \lambda_3 = 12}$

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Notes

given $|A| = 32$

by prop ② $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 32$

$1 \cdot \lambda_2 \cdot \lambda_3 = 32$

$\boxed{\lambda_2 \cdot \lambda_3 = 32}$

$\boxed{\lambda_2 + \lambda_3 = 12}$ $\Rightarrow$ $\lambda_3 = 8$, $\lambda_2 = 4$

$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1^2 + 8^2 + 4^2 = 81$

Q(41) prop ②

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = |A|$

$= 12$ (vander Mode Det.)

Q(51) given Trace $(A_{2\times 2}) = 4$

$\Rightarrow \lambda_1 + \lambda_2 = 4$ (by propty ①)

given Trace $(A^2) = \lambda_1^2 + \lambda_2^2 = 5$

Then $|A| = \lambda_1 \cdot \lambda_2 = ?$

WKT $(a+b)^2 = a^2 + b^2 + 2ab$

$(\lambda_1 + \lambda_2)^2 = \underline{\lambda_1^2 + \lambda_2^2} + 2\lambda_1 \cdot \lambda_2$

$4^2 = 5 + 2|A|$

$\Rightarrow |A| = \frac{11}{2} = 5.5$

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Notes

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Q (46) $\lambda_1 = \lambda \neq 0 \rightarrow$ complex

$A_{3\times3}$

by observation Col^m sum $\neq$ COnstant

$\because$ row sum = 0

$\Rightarrow \lambda_2 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A_{3\times3})$

$\lambda_3 = Tr(A) - \lambda_1 - \lambda_2$

$= 20 - \lambda - 0$

$= 20 - \lambda$

(c) ✓

Q (47)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix} \rightarrow LTM$$

$\Downarrow$

$\lambda_A = 1, \omega, \omega^2$

$\lambda_{A^{102}} = 1^{102}, \omega^{102}, \omega^{204}$

WKT Trace $(A^{102}) = \Sigma \lambda_{A^{102}}$ (Prop. 1)

$= 1 + \omega^{102} + \omega^{204}$

ω = cubeth root unity
$\omega = 1^{1/3} \rightarrow \boxed{\omega^3 = 1}$

$= 1 + (\omega^3)^{34} + (\omega^3)^{68}$

$= 1 + 1 + 1$

$= 3$

(d) ✓

Teacher's Signature ______

Notes

$\boxed{\omega^3 = 1}$

$\Downarrow$

$\omega^3 - 1^3 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$

$1 + \omega + \omega^2 = 0$

$\omega = \frac{-1 \pm i\sqrt{3}}{2}$

$\boxed{\omega^2 = \overline{\omega}}$

$\alpha = 1^{1/4}$ (4th root of unity)

$\Rightarrow \alpha^4 = 1$

$\Rightarrow 1 + \alpha + \alpha^2 + \alpha^3 = 0$

$\alpha = 1^{1/5}$ (5th root of unity)

$\Rightarrow \alpha^5 = 1$

$\Rightarrow 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

Q Eigen values of $A_{3\times3}$

$(\lambda_A = 3, 2, -1)$

and $B = A^2 - A$, then find

1) $|B|$

2) Trace $[A + B]$

sol^n: WKT $|B|$ = product of λ_B

by prop 10 $\lambda_B = \lambda_A^2 - \lambda_A$

$\lambda_A = 3 \Rightarrow \lambda_B = 3^2 - 3 = 6$

$\lambda_A = 2 \Rightarrow \lambda_B = 2^2 - 2 = 2$

$\lambda_A = -1 \Rightarrow \lambda_B = (-1)^2 - (-1) = 2$

1) $|B|$ = product of λ_B

$= 6 \cdot 2 \cdot 2 = 24$

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2) $\text{Trace}[A+B] = \text{Trace}(A) + \text{Trace}(B)$

$= \Sigma \lambda_A + \Sigma \lambda_B$

$= (3+2-1) + (6+2+2)$

$= 14$

Q) 2021-EC A real 2x2 matrix A with repeated Eigen values is given as $\begin{bmatrix} x & -3 \\ 3 & 4 \end{bmatrix}$, x is +ve, x = —?

Sol^n

WKT, Eigen values = (roots of char eq^n $|A-\lambda I| = 0$)

~~char eq~~ $|A-\lambda I| = \begin{vmatrix} x-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \underbrace{\lambda^2}_{ax^2} \underbrace{- \lambda(4+x)}_{+bx} \underbrace{+ (4x+9)}_{+c} \underbrace{= 0}_{=0}$

It's a quadratic eq^n in 'λ'.

WKT roots are real & equal (repeated).

$\Rightarrow b^2 - 4ac = 0$

$[-(4+x)]^2 - 4[4x+9] = 0$

$16 + x^2 + 8x - 16x - 36 = 0$

$x^2 - 8x - 20 = 0$

$\Rightarrow x = 10, -2$

since $x > 0$ (+ve)

$\therefore \boxed{x = 10}$

$\Rightarrow$ (it can be solved using prop ① & ② also)

$\lambda_1 = \lambda_2 = \lambda$

$\lambda_1 + \lambda_2 = x + 4$

$2\lambda = x + 4$

$4x + 9 = \lambda^2$

$\therefore x = -2$

$x = 10$ ✓

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Notes

Q) find λ_{max} of $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4\times4}$

GATE 2021

Sum of rows = 3 (for all)

$\therefore \lambda_1 = 3$

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 1 & 1 & 1 \\ 1 & 0-\lambda & 1 & 1 \\ 1 & 1 & 0-\lambda & 1 \\ 1 & 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

NOTE: Don't apply elementary transformation on the given matrix to find eigen values (otherwise λ will change) (Rank remain same)

(apply only on Determinant)

E T :- $R_1 \rightarrow R_1 + R_2 + R_3 + R_4$

$$(3-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0-\lambda & 1 & 1 \\ 1 & 1 & 0-\lambda & 1 \\ 1 & 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

$C_2 \rightarrow C_2 - C_1$

$C_3 \rightarrow C_3 - C_1$

$C_4 \rightarrow C_4 - C_1$

$$(3-\lambda) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -\lambda-1 & 0 & 0 \\ 1 & 0 & -\lambda-1 & 0 \\ 1 & 0 & 0 & -\lambda-1 \end{vmatrix} = 0$$

LTM

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Notes

It's LTM, $\Rightarrow (3-\lambda) \times 1 \times (-\lambda-1)^3 = 0$

$\Rightarrow \lambda = 3, -1, -1, -1$

$\boxed{\lambda_{max} = 3}$

Q find λ_{max} of

$$\begin{array}{cc} D_1 & UTB \\ \end{array}$$
$$\left[\begin{array}{cc|cc} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ \hline 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{array}\right]$$
$$\begin{array}{cc} LTB & D_2 \\ \end{array}$$

$\therefore \lambda_A = \lambda_{D_1} \,\&\, \lambda_{D_2}$

λ_{D_1} : given by $\lambda_1 + \lambda_2 = Tr(A) = 3$, $\lambda_1 \cdot \lambda_2 = |D_1| = 2$ $\Rightarrow \lambda_{D_1} = 2, 1$

λ_{D_2} : given by $\lambda_1 + \lambda_2 = Tr(D_2) = 12$, $\lambda_1 \cdot \lambda_2 = |D_2| = 35$ $\Rightarrow \lambda_{D_2} = 7, 5$

$\therefore \lambda_A = 2, 1, 7, 5$

$\boxed{\lambda_{max} = 7}$

Properties of Eigen Vectors

1) Eigen vector corresponding to an eigen value λ is not unique (bcoz they are ∞)

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Notes

2) The geometric multiplicity of an Eigen value λ
= no. of independent Eigen vectors corresponding to a λ
= no. of unknowns $- \rho(A - \lambda I)$

3) no. of non-repeated λ = no. of Indep Eig vectors

4) no. of distinct $\lambda \Rightarrow$ no. of Indep. Eigen vectors
(converse is not true)

$A_{n \times n}$

- $\lambda_i \neq \lambda_j$ $\Downarrow$ all are (Ind. Eig. vectors)
- $\lambda_i = \lambda_j$
  - dep. Eig. vectors
  - Ind. Eig vectors

5) Eigen vectors of a real symmetric matrix corresponding to distinct (λ) are orthogonal to each other.
$(\bar{X} \cdot \bar{Y} = X^T Y = 0)$

NOTE
norm of a vector 'x' $= \|X\| = \sqrt{X^T X} = \sqrt{\bar{X} \cdot \bar{X}}$
= length of vector
$= \sqrt{x^2 + y^2 + z^2}$

eg. if $X = [1\ 2\ 3]^T$
Norm of $X = \|X\| = \sqrt{X^T X}$
$= \sqrt{[1\ 2\ 3]\begin{bmatrix}1\\2\\3\end{bmatrix}}$

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Notes

$$= \sqrt{1^2+2^2+3^2}$$
$$= \sqrt{14}$$

$$\boxed{\text{Normalized vector of } x = \frac{x}{\|x\|} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{unit vector of 'x'}}$$

Q 55 $A_{2\times 2}$ is Symmetric

given x_1, x_2 are Eigen vectors of Symmetric matrix

by propty 5 $\vec{X_1} \cdot \vec{X_2} = X_1^T X_2 = 0$

Q $A = \begin{bmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be an

Eig. vector of A, Then another Eigen vector of A is ____

a) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Ans (b) ✓

A is sym-matrix

by prop 5 $\vec{X_1} \cdot \vec{X_2} = 0$ (by checking options)

Teacher's Signature (b) ✓

Notes

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (1 \times 1) + (1 \times -1) + (1 \times 0)$$
$$= 0$$
$$\Rightarrow \underline{X \perp^r Y}$$

$\therefore$ (b) ✓

Q (37) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ & $X = \begin{bmatrix} 101 \\ 101 \end{bmatrix}$, $\lambda = ?$

<u>WKT</u>: $AX = \lambda X$

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\begin{bmatrix} 404 + 202 \\ 202 + 404 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$6 \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\Rightarrow \boxed{\lambda = 6}$$

Q (53) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find $X \neq 0$ corr to λ_{max}?

$\lambda_1 = 1$

& col^m sum $= 3 \Rightarrow \boxed{\lambda_2 = 3}$

<u>WKT</u> $\lambda_1 + \lambda_2 + \lambda_3 = Tr(A)$

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Notes

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$\lambda_3 = 5-3-1=1$

$\boxed{\lambda max = 3}$

finding $X \neq 0$ for $\lambda = 3$

Sub $\lambda = 3$ in $(A-\lambda I)X = 0$

Solving we get. $X = k\begin{bmatrix}1\\1\\0\end{bmatrix}$; $k \in R$

(or) verify $[AX = 3X]$ with options

$$\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 3\begin{bmatrix}x\\y\\z\end{bmatrix}$$

option (b) ✓ $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

⇒ Q54 HW ☆☆☆

Q58 $A = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 1\\0 & 0 & 3\end{bmatrix}$ ⇒ UTM

$\therefore \lambda_A = (2, 2)(3) \rightarrow \lambda_i \neq \lambda_j$

$\lambda_i = \lambda_j$ (repeated λ)

1 Ind Eig vector

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Notes

By prop ② geometric Multiplicity $(\lambda=2)$ = no. of unk − $\rho(A-2I)$

$= 3 - \rho(A-2I)$

$= 3 - 1 = 2$

$$A - 2I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ Rank=1

∴ no. of Ind. Eig vectors of $A_{3\times3}$ $= 2+1$
$= 3$

Q 59

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{U.T.M}$$

$\Downarrow$

$\lambda_A = 2, 2, 3$

repeated λ
(∴ use Formula)
→ 1 Ind

1 Ind
(b/c 3 not repeated)

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \text{Ind} \\ \rightarrow \text{dep} \\ \rightarrow \text{Ind} \end{array}$$

GM = no. of unk − $\rho(A-2I)$

$= 3 - 2$

$= 1$

Ans = 1+1 = 2

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Notes

Diagonalization

$$D_n = P^{-1} A_n P$$

where $P = [x_1 \; x_2 \; x_3 \cdots X_n]_{n\times n}$ (model matrix)

Eigen vectors of $A_{n\times n}$

(P^{-1} should exist) $|P| \neq 0$

(All eigen vectors must be L I)

→ L I ⇒ P^{-1} exists

⇒ $A_{n\times n}$ can be diagonalized

NOTE: Necessary & Sufficient (iff) condition for $A_{n\times n}$ to be diagonalized is " all the n eigen vectors of $A_{n\times n}$ are linearly Independent "

58 → diagonalized

59 → not diagonalized (∵ only 2 eigen vectors are independent)

Q 56
$$A = \left[\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]_{4\times4}$$

D_1, D_2

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Notes

$\lambda_A = \lambda_{D1}$ & λ_{D2}

λ_{D1} :- $\begin{vmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 4$

$= \lambda^2 - 3\lambda - 2 = 0$

$\Downarrow$

$\lambda = \dfrac{3 \pm \sqrt{17}}{2}$

λ_{D2} :- It is LTM

$\therefore \lambda_{D2} = 3, 4$

$\because$ all the 4 are non repeating

$\Rightarrow \exists$ 4 no. of Ind. eig vectors of $A_{4\times4}$

Hence, $A_{4\times4}$ can be diagonalized.

(ii) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\lambda_A = 2, 2$

GM of $(\lambda = 2)$ = no. of col^n of A $- \rho(A - 2I)$

$= 2 - 1$

$= 1$

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\downarrow$

$\rho = 1$

Only 1 Ind. Eig vector for $A_{2\times2}$

Hence A cannot be diagontized.

Caleey - Hamilton Theorm

"Every Square matrix satisfies its characterstic equation"

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Notes

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Char eqⁿ of $A_{n\times n}$

$$|A_{n\times n} - \lambda I| = 0$$

$$\Rightarrow f(\lambda) = 0$$

replace λ by 'A'

$$\Rightarrow f(A) = 0$$

Q (62) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, given $\underbrace{a + d}_{Trace(A)} = \underbrace{ad - bc}_{|A|} = 1$

Trace(A) = |A| = 1

1) A^3
2) A^9
3) A^{-1}

(whenever options are given in Matrix form)
use Cayley - Hamilton Theorem

Char eqⁿ of $A_{2\times 2} = |A_{2\times 2} - \lambda I| = 0$

$$= (-1)^2 \lambda^2 + (-1)^1 Tr(A)\lambda^1 + |A| = 0$$

$$= +\lambda^2 - \lambda + 1 = 0$$

by Cayley-Hamilton Theorm : replace λ by A

$$\Rightarrow A^2 - A + I = 0$$

$$\Rightarrow \boxed{A^2 = A - I}$$

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Notes

$x^{l}y$ by A on B.S

1) $A^3 = A^2 - A$

$A^3 = A - I - A \quad (\because A^2 = A - I)$

$\boxed{A^3 = -I}$

2) $A^9 = (A^3)^3 = (-I)^3 = -I$

3) $x^{l}y$ by A^{-1} on $(A^2 = A - I)$

$A^{-1}A^2 = A^{-1}A - A^{-1}I$

$A = I - A^{-1}$

$\boxed{A^{-1} = I - A}$

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Notes

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Q) GATE 2021

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}_{2\times 2} \qquad \overbrace{|A^4 - 5A^3 + 6A^2 + 2I|}^{B} = ?$$

(whenever we see matrix polynomial we use caley H. Theorem by replacing "λ" by "A")

char eq^n of $A_{2\times 2}$: Put $n = 2$ in gen-char eq^n

$$|A_{2\times 2} - \lambda I| = (-1)^2 \lambda^2 + (-1)^1 Tr(A)\lambda^1 + |A| = 0$$

Trace $(A) = 2 + 3 = 5$

$|A| = 6$

$$|A - \lambda I| = \lambda^2 - 5\lambda + 6 = 0$$

by C-H Theorem

$$\Rightarrow A^2 - 5A + 6I = 0 \quad -(1)$$

from (Q) $|A^4 - 5A^3 + 6A^2 + 2I|$

$$= |A^2[A^2 - 5A + 6I] + 2I|$$

$$= |2 I_{2\times 2}| \qquad (\because \text{of } (1))$$

$$= 2^2 |I_2|$$

$$= 4 \times 1$$

$$= 4$$

Method (2) find λ_B

Then product of $\lambda_B = |B|$

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Notes

Q60) $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, $A^9 = ?$

Char. Eq^n : $(-1)^2\lambda^2 + (-1)^1(-3+0)\lambda^1 + 2 = 0$

$\Rightarrow \lambda = -1, -2$

[verify Eigen values of A^9 with eigen values of options]

(a) $A^9 = 511A + 510I$

$\Rightarrow \lambda^9 = 511\lambda + 510$

for $\lambda = -1$ $\Rightarrow (-1)^9 = 511(-1) + 510 \Rightarrow \boxed{-1 = -1}$

for $\lambda = -2$ $\Rightarrow (-2)^9 = 511(-2) + 510 \Rightarrow \boxed{-512 = -512}$

(a) ✓

Standard Limits

1) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} \cdot a^{m-n}$

2) $\lim_{\theta \to 0} \frac{\sin k\theta}{\theta} = \lim_{\theta \to 0} \frac{\tan k\theta}{\theta} = k$

9) If $\lim_{x \to a} (f(x))^{g(x)} = 1^\infty$

Then $\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} g(x)[f(x) - 1]}$

10) If $\lim_{x \to a} (f(x))^{g(x)} = 0^0$

Then $\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} g(x) \log f(x)}$

3) $\lim_{n \to \infty} \left(1 + \frac{p}{n}\right)^n = e^p$

4) $\lim_{n \to 0} [1 + pn]^{1/n} = e^p$

5) $\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$

6) $\lim_{x \to 0} \frac{a^x - 1}{p^x} = \log a$

7) $\lim_{x \to 0} \frac{a^x - b^x}{x} = \log\left(\frac{a}{b}\right)$

8) $\lim_{x \to 0} \frac{1 - \cos ax}{x^2} = \frac{a^2}{2}$

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Calculus

bit.ly/stdlimits

Theory book (first 4 pages) Basics

Notes

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Limit of function f(x) at x = a

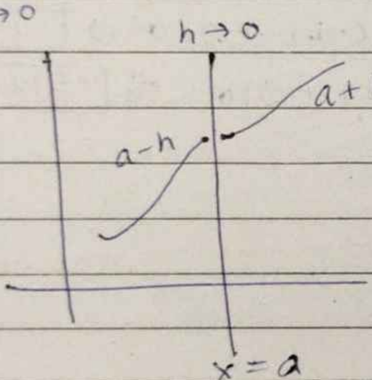
$\lim_{x\to a} f(x) = l$ exists

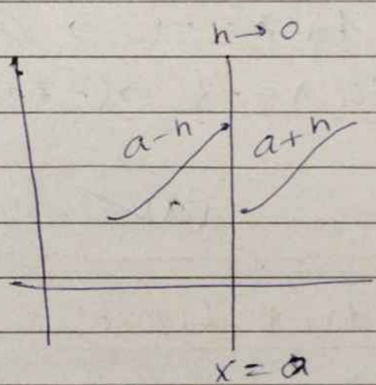
iff

$$\boxed{\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h)}$$

Graph

$\delta x \to 0$



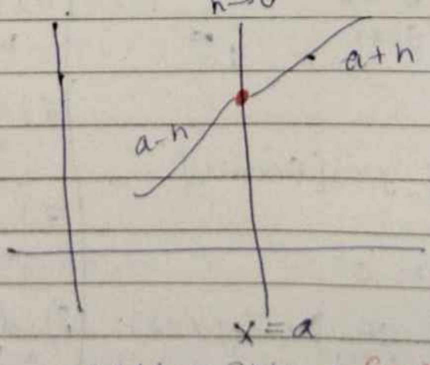


LHL = RHL

limit exists at x = a

LHL ≠ RHL

limit does not exists at x = a



LHL = RHL = f(a)

here f(x) is become cont. at x = a

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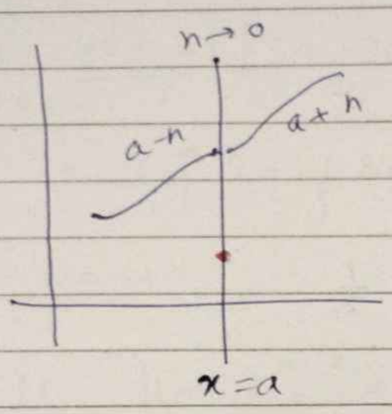
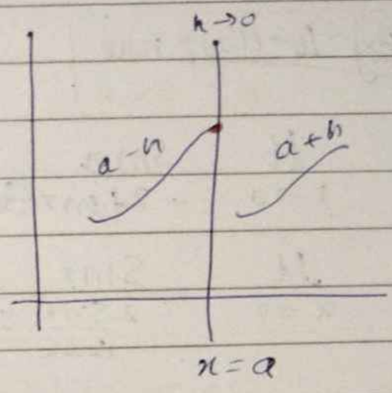
Notes





LHL = RHL ≠ f(a)

f(x) is discontinuous (removable) at x=a

LHL = f(a) ≠ RHL

f(x) is Jump disc. at x=a.

⑥ $\lim_{x\to 0} \frac{e^{4x}-1}{\sin 2x} \times \frac{x}{x} = \frac{4}{2} = 2$

—(OR)—

$\lim_{x\to 0} \frac{e^{4x}-1}{\sin 2x} = \frac{0}{0}$

apply (L-hospital rule) (whenever $\frac{0}{0}$ or $\frac{\infty}{\infty}$ comes it's indeterminate form)

L-Hosp

$\lim_{x\to 0} \frac{e^{4x}\times 4 - 0}{2\cos 2x} = \frac{4}{2} = 2$

Q⑧ $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x} = \frac{0}{0}$

$\boxed{\sec^2 x - 1 = \tan^2 x}$

by L-Hospital rule

$\lim_{x\to 0} \frac{1-\cos x}{1-\sec^2 x} = \lim_{x\to 0} \frac{1-\cos x}{-\tan^2 x} = \frac{0}{0}$

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by L-Hosp rule

$$\lim_{x \to 0} \frac{\sin x}{-2\tan x \sec^2 x \times 1}$$

$$\lim_{x \to 0} \frac{\sin x}{-2 \frac{\sin x}{\cos x} . \sec^2 x} = -\frac{1}{2}$$

Q $\lim_{x \to 0} \frac{1-\cos x^2}{2x^4} = \frac{0}{0}$

$\boxed{\lim_{n \to \infty} r^n = 0, \; r < 1}$

let $Y = x^2 \to 0$

☆ (x is in form of x^2)

$$\lim_{Y = x^2 \to 0} \frac{1-\cos Y}{2Y^2} = \frac{0}{0}$$

$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$

L-Hosp

$$\lim_{Y \to 0} \frac{0+\sin Y}{4Y}$$

$\boxed{\lim_{\theta \to 0} \frac{\sin k\theta}{\theta} = k}$

$$\lim_{Y \to 0} \frac{\sin Y}{4 \cdot Y} = \frac{1}{4}$$

Q $\lim_{x \to \infty} \left[e^{\frac{1}{5x}} - 1\right]\left[5x + \frac{x}{5} \sin \frac{1}{x}\right]$

☆ (x is in form of $\frac{1}{x}$)

Let $y = \frac{1}{x} \to 0$

$$\lim_{Y \to 0} \left[e^{\frac{1}{5}Y} - 1\right]\left[\frac{5}{Y} + \frac{1}{5Y} \sin Y\right]$$

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Notes

$$\lim_{y \to 0} \frac{e^{\frac{1}{5}y} - 1}{y} \left[5 + \frac{1}{5}\sin y\right]$$

$$= \frac{1}{5}\left[5 + \frac{1}{5}\sin(0)\right]$$

$$= 1$$

$$\boxed{\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a}$$

Q $\lim_{x \to \infty} \left[1 - \frac{2}{x}\right]^{-x} = ?$

Std Limit 3

$$\boxed{\lim_{x \to \infty} \left[1 + \frac{p}{x}\right]^{x} = e^{p}}$$

$$= \lim_{x \to \infty} \frac{1}{\left(1 + \frac{-2}{x}\right)^{x}}$$

$$= \frac{1}{e^{-2}} = e^{2}$$

Q $\lim_{x \to \infty} \left[\frac{x+1}{x-1}\right]^{x+4}$

~~Ans~~

Q $\lim_{x \to \infty} \frac{5x^3 + 3x^2 - x + 1}{2x^3 - 3x + 1}$

Method 1

$= \frac{\infty}{\infty}$ By L-Hosp → Ans.

Method 2 [Whenever $x \to \infty$
num & dep are polynomial
take max degree common from both]

$$\lim_{x \to \infty} \frac{x^3\left[5 + \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right]}{x^3\left[2 - \frac{3}{x^2} + \frac{1}{x^3}\right]}$$

$$= \frac{5 + \frac{3}{\infty} - \frac{1}{\infty} + \frac{1}{\infty}}{2 - \frac{3}{\infty} + \frac{1}{\infty}} = 5/2$$

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(Q) $\lim_{x\to\infty} \left[\frac{x+1}{x-1}\right]^{x+4}$

$$= \lim_{x\to\infty} \left[\frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left(1+\frac{-1}{x}\right)}\right]^{x+4}$$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{\ell}{x}\right)^{x} = e^{\ell}}$$

$$= \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^{x}\cdot\left(1+\frac{1}{x}\right)^{4}}{\left(1+\frac{-1}{x}\right)^{x}\cdot\left(1-\frac{1}{x}\right)^{4}}$$

$$= \frac{e^{1}\cdot\left(1+1/\infty\right)^{4}}{e^{-1}\cdot\left(1-1/\infty\right)^{4}}$$

$$= e^{2}$$

Q(12) $\lim_{x\to\infty} \sqrt{x^2+x-1} - x = \infty - \infty$ (Indeterminate form)

Rationalize

$$\lim_{x\to\infty} \sqrt{x^2+x-1} - \sqrt{x^2}$$

Indeterminate form

$\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}, 0^{0}, \infty^{0}, \infty^{\infty}, \infty-\infty$

Rationalize Factor

R.F of $\sqrt{a}-\sqrt{b} = \sqrt{a}+\sqrt{b}$

$$\lim_{x\to\infty} \left[\sqrt{x^2+x-1} - \sqrt{x^2}\right] \times \left[\frac{\sqrt{x^2+x-1}+\sqrt{x^2}}{\sqrt{x^2+x-1}+\sqrt{x^2}}\right]$$

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Notes

$$= \lim_{x\to\infty} \frac{x^2 + x - 1 - x^2}{\sqrt{x^2+x-1} + \sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{x[1 - 1/x]}{x[\sqrt{1 + 1/x - 1/x^2} + 1]}$$

$$= \lim_{x\to\infty} \frac{1 - 1/\infty}{\sqrt{1 + 1/\infty - 1/\infty} + 1} = \frac{1-0}{\sqrt{1}+1} = \frac{1}{2}$$

Q) $\lim_{x\to 0} e^x (\cos x)^{\frac{1}{\sin^2 x}}$ $= e^0 \cdot 1^{1/0}$

$= 1 \cdot 1^\infty \rightarrow$ Indeterminate

$$\boxed{\text{If } \lim_{x\to a} (f(x))^{g(x)} = 1^\infty \text{ Then } \lim = e^{\lim_{x\to a} g(x)[f(x)-1]}}$$

$$= e^0 \cdot e^{\lim_{x\to 0} \frac{1}{\sin^2 x}[\cos x - 1]} = \frac{0}{0}$$

by L-Hosp.

$$= 1 \cdot e^{\lim_{x\to 0} \frac{-\sin x}{2\sin x \cos x}} = e^{-1/2}$$

Q) $\lim_{x\to 0} x^x = ?$

$$\boxed{\text{If } \lim_{x\to a} (f(x))^{g(x)} = 0^0 \text{ then limit} = e^{\lim_{x\to a} g(x)\log f(x)}}$$

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$$\boxed{\log 0 = -\infty}$$

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Notes

$$\text{limit} = e^{\lim_{x \to 0} x \log x}$$

$$= e^{0 \cdot \log 0}$$

$$= e^{0(-\infty)} = e^{-\infty / 1/0}$$

$$= e^{-\infty/\infty}$$

$$0 \cdot \infty$$

$$= \frac{\infty}{1/0} = \frac{\infty}{\infty}$$

$\therefore$ Apply L-Hosp Rule

(Apply L. Hosp. Rule)

$$e^{\lim_{x \to 0} \frac{\log x}{1/x}} = \frac{\log 0}{1/0} = \frac{\infty}{\infty}$$

$$e^{\lim_{x \to 0} \frac{\frac{1}{x}}{-1/x^2}} = e^{-0} = 1$$

Q Leibnitz rule

$$\frac{d}{dx} \int_{\phi(x)}^{\psi(x)} f(t)\,dt = f(\psi(x))\,\psi'(x) - f(\phi(x))\,\phi'(x)$$

Q 5
$$\lim_{x \to 0} \frac{\int_0^{x^2} \sqrt{4+t^3}\,dt}{x^2}$$

$$\int_a^a f(x)\,dx = 0$$

$$\lim_{x \to 0} = \frac{\int_0^0 \sqrt{4+t^3}\,dt}{0} = \frac{0}{0}$$

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$f(1^-) \Rightarrow 1-h < 1$

Leibniz Rule

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\cdot dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Notes

L Hosp rule

$$\lim_{x \to 0} \frac{\frac{d}{dx}\int_0^{x^2} \sqrt{4+t^3}\, dt}{2x}$$

(apply Leibnitz rule)

$$\lim_{x \to 0} \frac{\sqrt{4+x^6}\,(2x) - \sqrt{4+0}\,(0)}{2x}$$

$$= 2$$

HW

Q) $$\lim_{x \to 0} \frac{x - \int_0^x \cos t^2\, dt}{x^3 - 6x}$$ Ans 0

Q1) Let $f(x)$ is continuous Everywhere

$\Rightarrow f(x)$ is cont. at $x = 1, 3$ also

cont. at $x=1$: $f(1^-) = f(1^+) = f(1)$, where $1^- < 1$ and $1^+ > 1$

$$(2x+1)_{x=1} = (ax^2+b)_{x=1}$$

$$\boxed{3 = a+b} \text{ — (1)}$$

Cont. at $x=3$: $f(3^-) = f(3^+) = f(3)$, where $3^- < 3$ and $3^+ > 3$

$$(ax^2+b)_{x=3} = (5x+2a)_{x=3}$$

$$\boxed{7a+b=15} \text{ — (2)}$$

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$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

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<u>Solving ①, ②</u>

we get $\underline{a = 2,\ b = 1}$

Differentiability of $f(x)$ at $x = a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{Exists}$$

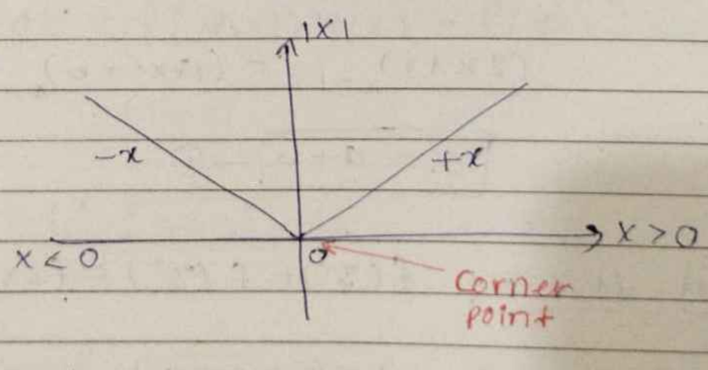
iff

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\underline{\underline{LHD}} = \underline{\underline{RHD}}$$

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Graph



1) $f(x)$ is not differentiable at the points where it has sharp edges

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	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$\frac{3\pi}{2}$	2π
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1	0	$+1$
Tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

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2) The point at which $f(x)$ is not differentable is called corner point.

3) At the corner points, $f(x)$ can have local maxima or local minima.

4) Every differentiable function is continuous. But, every continuous function need not be differentiable.

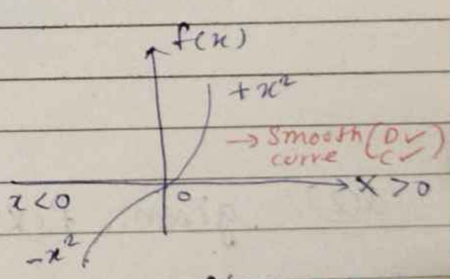
ex :⇒ $f(x) = |x|$

(4) (Q) $f(x) = x|x| \quad , \forall x \in R$

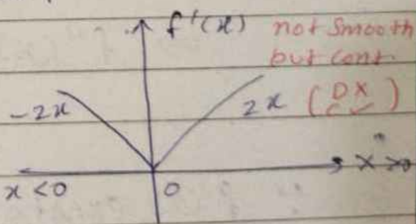
D: Differentiable?
C: continuous?

$$= \begin{cases} x(-x) & , x < 0 \\ x(x) & , x \geq 0 \end{cases}$$

$$= \begin{cases} -x^2 & , x < 0 \\ x^2 & , x \geq 0 \end{cases} \Rightarrow$$



$$f'(x) = \begin{cases} -2x & , x < 0 \\ 2x & , x \geq 0 \end{cases} \Rightarrow$$



(d) ✓

(Q) $y = \sin|x| \quad , \left.\frac{dy}{dx}\right|_{x=-\frac{\pi}{4} < 0} = ?$

⇓

$$y = \begin{cases} \sin(-x) & , x < 0 \\ \sin(x) & , x \geq 0 \end{cases}$$

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$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

Notes

$$Y = \begin{cases} \sin(-x), & x < 0 \\ \sin(x), & x \geq 0 \end{cases} \Rightarrow y = \begin{cases} -\sin x, & x < 0 \\ \sin x, & x \geq 0 \end{cases}$$

$$Y' = \begin{cases} -\cos x, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

$$\left.\frac{dY}{dx}\right|_{x = -\frac{\pi}{4} < 0} = (-\cos)_{x = -\pi/4}$$

$$= -\cos\left[-\frac{\pi}{4}\right]$$

$$= -\cos\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}}$$

Q2 given $f(x)$ is twice differentiable

$\Rightarrow f''(x)$ exists $\Rightarrow f'(x)$ exists
$\Rightarrow f(x)$ is continuous

$\because$ $f'(x)$ exists at $x = 0$

$$f'(0^-) = f'(0^+)$$

<0 $\quad$ >0

$$(2\alpha x + \beta)_{x=0} = (3\alpha x^2 + 2\beta x + 5\cos x)_{x=0}$$

$$\boxed{\beta = 5}$$

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Notes

and $\because f''(x)$ Exists at $x = 0$

$$f''(0^-) = f''(0^+)$$
$<0 \qquad >0$

$$2\alpha = (6\alpha x + 2\beta - 5\sin x)_{x=0}$$

$$\cancel{2}\alpha = \cancel{2}\beta \Rightarrow \alpha = \beta$$

$$\because \beta = 5 \Rightarrow \boxed{\alpha = \beta = 5}$$

(d) ✓

Q ③ To say $f(x)$ is diff at $x = 1$
it must be cont at $x = 1$
and should satisfy differentiability condition at $x = 1$.

cont at $x = 1$

$$f(1^-) = f(1^+) = f(1)$$

$$(e^x)_{x=1} = (\ln x + ax^2 + bx)_{x=1}$$

$$\Rightarrow \boxed{e = a + b} \quad — ①$$

for all a, b such that $a + b = e$,
$f(x)$ is Continuous at $x = 1$.

diff at $x = 1$:

$$f'(1^-) = f'(1^+)$$

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Notes

$$(e^x)_{x=1} = \left(\frac{1}{x} + 2ax + b\right)_{x=1}$$

$$\boxed{2a+b = e-1} - (2)$$

To say $f(x)$ is diff at $x=1$

$a+b=e$

$\times$ $2a+b = e-1$ both cond. should be true.

Solving ~~the~~ these independent eq^{n}s, we get unique values of a, b.

Hence, $f(x)$ is diff at $x=1$ for unique values of a, b.

Ans. (b) ✓

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Q) $Y = \sqrt{f(x) + \underbrace{\sqrt{f(x) + \sqrt{f(x) + \sqrt{\cdots\infty}}}}_{Y}}$, $\frac{dY}{dx} = ?$

$Y = \sqrt{f(x) + y}$ → implicit form

$Y^2 = f(x) + y$

$y^2 - Y = f(x)$ → explicit

diff. wrt "x"

$2y \frac{dY}{dx} - \frac{dY}{dx} = f'(x)$

$\frac{dY}{dx}[2y - 1] = f'(x)$

$$\boxed{\frac{dY}{dx} = \frac{f'(x)}{2y-1}}$$

Q(39) $\frac{dY}{dx} = \frac{f'(x)}{2Y-1} = \frac{\sec^2 x}{2Y-1}$ (d)

Partial Differentiation

let $\underset{\text{dep}}{u} = f(x, \overset{\text{Ind.}}{y})$

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$ → Total derivative of u

Partial derivatives

$\partial \leftarrow$ "dao"

$\boxed{\frac{d}{dx}(a^x) = a^x \log a}$

Notes

$$\left[\begin{array}{l} p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y} \\ r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2} \end{array}\right]$$

Q) $f(x, y) = y^x$, $\frac{\partial^2 f}{\partial x \partial y} = ?$

$\boxed{\frac{d}{dx}(a^x) = a^x \log a}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^x)$$

$$= y^x \cdot \log y$$

$\boxed{d[uv] = uv' + vu'}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[y^x \cdot \log y\right]$$

$\boxed{\frac{d}{dy}(y^n) = ny^{n-1}}$

$$= y^x \cdot \frac{1}{y} + (\log y) x y^{x-1}$$

$$= y^{x-1} \cdot [1 + x \log y]$$

Q) $z = e^{ax+by} \cdot F(ax - by)$, $bz_x + az_y = \_\_\_ ?$

a) az b) bz c) abz d) $2abz$

Sol^n $z_x = \frac{\partial z}{\partial x} = e^{ax+by} \cdot F'(ax - by) \times a +$

$\underbrace{F(ax - by) e^{ax+by}}_{= z} \times a$

$\boxed{d(uv) = uv' + u'v}$

$\boxed{d(e^{ax}) = e^{ax} \cdot a}$

$$z_y = \frac{\partial z}{\partial y} = e^{ax+by} \cdot F'(ax - by) \times (-b) +$$

$$\underbrace{F(ax - by) \cdot e^{ax+by}}_{= z} (b)$$

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Notes

$\therefore\ bz_x + az_y = 2abz$

(d) ✓

Q(40) $u = f(r, s)$ where $r = x+y$, $S = x-y$

$u_x + u_y = \_\_\_\_\ ?$

Sol^n chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial x}$$

$$= u_r(1) + u_s(1)$$

$$\boxed{\frac{\partial r}{\partial x} = 1,\ \frac{\partial S}{\partial x} = 1 \qquad \frac{\partial r}{\partial y} = 1,\ \frac{\partial S}{\partial y} = -1}$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r} \times \dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s} \times \dfrac{\partial s}{\partial y}$

$$= u_r(1) + u_s(-1)$$

$$\boxed{u_x + u_y = 2u_r}$$

(a) ✓

Q(41) $r = x^2 + y - z$ & $z^3 - xy + yz + y^3 = 1$, given x, y are Ind.

z is dep on x, y.

$$\frac{\partial r}{\partial x} = 2x + 0 - \frac{\partial z}{\partial x}$$

diff. w.r.t x partially

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} + 0 = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}}$$

$$\frac{\partial r}{\partial x} = 2x - \frac{y}{3z^2 + y}$$

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Notes

$$\left(\frac{\partial r}{\partial x}\right)_{(2,-1,1)} = 2(2) - \frac{(-1)}{3(1)^2-1} = 4.5$$

Homogeneous functions

$u = f(x,y)$ be a Homogeneous f^n of degree 'n'

iff $\boxed{f(kx, ky) = k^n f(x,y)}$

eg:- $f(x,y) = x \cdot \sin\left[\frac{x}{y}\right]$

$$f(kx, ky) = kx \cdot \sin\left[\frac{\cancel{k}x}{\cancel{k}y}\right]$$

$$= k\left(x \sin\left[\frac{x}{y}\right]\right)$$

$$= k^1 \cdot f(x,y)$$

$\therefore$ $f(x,y)$ is homogeneous of degree $n=1$.

eg $u = \dfrac{x^3+y^3}{x-y}$ is homogenous f^n of

degree n = deg(Numerator) − deg(Denom)
$= 3-1$
$= 2$

Euler Thms:-

① If $u = f(x,y)$ is homogeneous f^n of degree (n)
Then

(i) $x u_x + y u_y = n \cdot u$

(ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n(n-1)u$

② If $u = f(x,y) + g(x,y)$, where both f, g are homogeneous f^n of degree n_1 x n_2 respectively.

Then

i) $x u_x + y u_y = n_1 f + n_2 g$

ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n_1(n_1-1)f + n_2(n_2-1)g$

③ If $\phi(u) = F(x,y)$ is homog f^n of degree 'n',
Then

(i) $x u_x + y u_y = n \dfrac{\phi(u)}{\phi'(u)} = F(u)$

(ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy}$

$$= F(u)\left[F'(u) - 1\right]$$

(Q) $u = \log_e\left[\dfrac{x^2+y^2}{x+y}\right]$, $x u_x + y u_y = ?$ → not homog. ∴ changed.

$e^u = \dfrac{x^2+y^2}{x+y}$ → is homog f^n of degree $n = 2-1 = 1$

$\phi(u)$

Notes

by Eular Thm ③

$$xu_x + yu_y = n\frac{\phi(u)}{\phi'(u)} = 1\cdot\frac{e^u}{e^u} = 1$$

Q ⑬ $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$; $xu_x + yu_y = ?$

$\underset{\phi(u)}{\tan u} = \frac{x^3+y^3}{x-y}$ → homog of degree = 2

by Euler Theorm ③

$$xu_x + yu_y = \frac{n\cdot\phi(u)}{\phi'(u)} = \frac{2\cdot\tan u}{\sec^2 u}$$

$$= \sin 2u \qquad (\sin 2u = 2\sin u\cos u)$$

Q ⑮ $u = \boxed{\frac{x^3+y^3}{x-y}} + \boxed{x\sin\left(\frac{x}{y}\right)}$, $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = ?$

Homo f^n of degree $\boxed{n_1=2}$

Homog f^n of degree $\boxed{n_2=1}$

by Euler Theorem ②

$$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = n_1(n_1-1)f + n_2(n_2-1)g$$

$$= 2(2-1)f + 1(1-1)g$$

$$= 2f$$

(b) ✓

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Notes

Q (44) (b) ✓

Mean Value Theorems

Rolle's Thm :- If $f : [a, b] \rightarrow R$

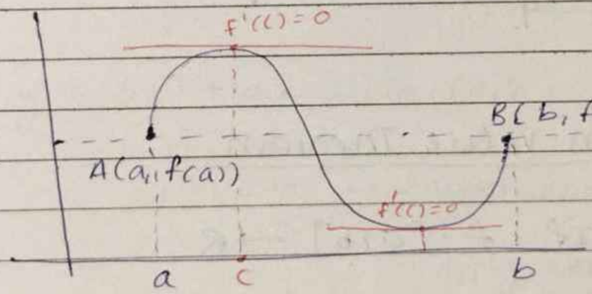
such that $\rightarrow a \le x \le b$

1) $f(x)$ is continuous in $[a, b]$ $\rightarrow a < x < b$
2) $f(x)$ is differentiable in (a, b)
3) $f(a) = f(b)$

Then $\exists$ a meanvalue $c \in (a, b)$

~~such~~ such that $\boxed{f'(c) = 0}$

$\Rightarrow$



draw a Tangent $\parallel^{el}$ to line $\overline{AB}$

Slope of Tangent at $(x=c)$ = Slope of $\overline{AB}$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\boxed{f'(c) = 0}$ where $c \in (a, b)$

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$y = f(x) \Rightarrow$ curve

$f'(x) = \frac{dy}{dx} \Rightarrow$ slope of tangent

If any tangent is $\parallel$ to x-axis then slop of tangent $= 0$

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Notes

Q (20) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

by Roll's Theorem $f'(c) = 0$

$$d\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$f'(c) = \left| \frac{e^x(\cos x) - \sin x(e^x)}{e^{2x}} \right|_{x=c} = 0$$

$$\Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

$$\Rightarrow \cos c = \sin c$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

Lagranges Mean Value Theorem

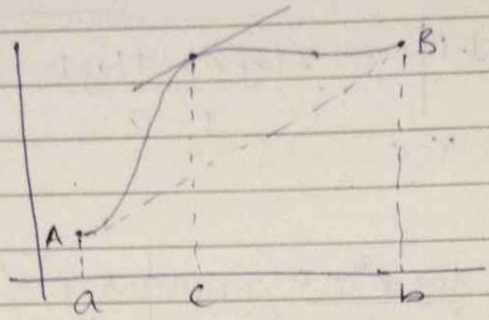
Statement : If $f : [a, b] \rightarrow R$
such that

1) $f(x)$ is continuous in $[a, b]$
2) $f(x)$ is differentiable in (a, b)

then $\exists$ a mean value $c \in (a, b)$
such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Teacher's Signature

Notes



draw a Tangent $\parallel^{el}$ to line $\overline{AB}$

$\Rightarrow$ Slope of Tangent at $(x=c)$ = Slope of $\overline{AB}$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

where $c \in (a,b)$.

Q7 $y = 5x^2 + 10x$ in $(1,2)$

Parabola → Smooth curve

Apply LMVT

∴ By LMVT

slope of y at atleast one pt of $x \in (1,2)$ is given by $\left.\frac{dy}{dx}\right|_{x=c} = \frac{y(2)-y(1)}{2-1}$

$= 25$

Q16 $f(x) = (1+x)\log(1+x)$ in $(0,1)$

by LMVT $f'(c) = \frac{f(1)-f(0)}{1-0}$

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Notes

$$(1+x)\cdot\frac{1}{(1+x)} + \log(1+x)\cdot 1 \Big|_{x=c} = \frac{2\log 2 - 1\log 1}{1-0}$$

$$1 + \log[1+c] = \frac{\log 2^2 - 0}{1}$$

$\boxed{\log a^n = n\log a \quad \log 1 = 0}$

$$\log[1+c] = \log 4 - 1$$
$$= \log 4 - \log e$$

$\boxed{\log_e e = 1}$

$$= \log\left[\frac{4}{e}\right]$$

$\boxed{\log a - \log b = \log\frac{a}{b}}$

$$1+c = \frac{4}{e}$$

$$c = \frac{4}{e} - 1$$

$$= \frac{4-e}{e} \in (0,1)$$

(c) ✓

Q 18 $f'(x) = \frac{1}{3-x^2}$, $f(0) = 1$. l.b & u.b of $f(1) = ?$

By LMVT

let $[a,b] = [0,1]$

$$f'(c) = \frac{f(1) - f(0)}{1-0}$$

$$\frac{1}{3-c^2} = \frac{f(1) - 1}{1} \Rightarrow \boxed{f(1) = 1 + \frac{1}{3-c^2}, \; c \in (0,1)}$$

Teacher's Signature ______

Notes

$\boxed{0 < c < 1}$

lb of c ub of c

Substituting l.b of 'c' = 0

$\Rightarrow f(1) = 1 + \frac{1}{3} = 1.33$

Subs u.b of 'c' = 1

$\Rightarrow f(1) = 1 + \frac{1}{3-1^2} = 1.5$

$\Rightarrow 1.33 < f(1) < 1.5$

$\therefore$ lb of $f(1) = 1.33$ & ub of $f(1) = 1.5$

(19) (i) $f(\theta) = \sin\theta(\quad) - \cos\theta(\quad) + \tan\theta(\quad)$

cont. cont cont in $[\frac{\pi}{6}, \frac{\pi}{3}]$

$\boxed{\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]}$

$\Rightarrow f(\theta)$ is cont for all $\theta \in [\pi/6, \pi/3]$

(ii) $f'(\theta) = \cos\theta(\quad) + \sin\theta(\quad) + \sec^2\theta(\quad)$

cont. cont. cont in $(\frac{\pi}{6}, \frac{\pi}{3})$

$\Rightarrow f'(\theta)$ exists for all $\theta \in (\frac{\pi}{6}, \frac{\pi}{3})$

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Notes

$\Rightarrow f(\theta)$ is diff $\forall\ \theta \in (\frac{\pi}{6}, \frac{\pi}{3})$

(iii) $f(\pi/6) = 0$ $(\because R_1 = R_2)$ and

$f(\pi/3) = 0$ $(\because R_1 = R_3)$

$\Rightarrow f(a) = f(b)$

Since all ③ conditions of Roll's Theorem are satisfied.

$\therefore$ we can say that $\exists\ \theta \in (\frac{\pi}{6}, \frac{\pi}{3})$ where

$f'(\theta) = 0$.

& also we can say $\exists\ \theta \in (\frac{\pi}{6}, \frac{\pi}{3})$ where

$f'(\theta) \neq 0$ (other than peak points)

Hence, both statements are TRUE.

(C) ✓

Cauchy's Mean Value Theorem

Statement : If $f, g : [a,b] \to R$ such that

1) f, g are continuous in $[a,b]$
2) f, g are differentiable in (a,b)
3) $g'(x) \neq 0$, $\forall x \in (a,b)$

Then $\exists$ a mean value $c \in (a,b)$

such that $$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

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Notes

Q (21) $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$

by CMVT $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{2\cancel{c}}{3c^{\cancel{2}}} = \frac{2^2 - 1^2}{2^3 - 1^3}$$

$$\boxed{c = \frac{14}{9} \in (1, 2)}$$

Taylor's Series

bit.ly / ENGGMATHNOTES CS

~~bit.ly / CorreG~~

caps

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$\boxed{\lfloor n = n!}$

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Notes

Taylor's Series

Expansion of $f(x)$ at $x = a$

(or) In the powers of $(x-a)$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^n(a) + \cdots$$

$$\boxed{\text{coeff of } (x-a)^n = \frac{f^n(a)}{\lfloor n}}$$

Mc-Laurence Series

Put $a=0$ in Taylor's

we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(0) + \cdots$$

Applications

① Taylor Series is used to express any differentiable function as a polynomial.

② Taylor Series is used to approximate the given function upto certain degree.

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Notes

(3) Taylor series ~~solution~~ is a solution of ordinary differential eq. $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.

$$y(x) = y_0 + x y_0' + \frac{x^2}{\lfloor 2} y_0'' + \cdots + \frac{x^n}{\lfloor n} y_0^{(n)} + \cdots$$

(Q) Expand $\sin x$ at $x = \pi/6$?

$$f(x) = \sin x \Rightarrow f(\pi/6) = \sin \pi/6 = \frac{1}{2}$$

$$f'(x) = \cos x \Rightarrow f'(\pi/6) = \cos \pi/6 = \sqrt{3}/2$$

$$f''(x) = -\sin x \Rightarrow f''(\pi/6) = -\sin \pi/6 = -\frac{1}{2}$$

$$f'''(x) = -\cos x \Rightarrow f'''(\pi/6) = -\cos \pi/6 = -\frac{\sqrt{3}}{2}$$

$$f^{iv}(x) = \sin x \Rightarrow f^{iv}(\pi/6) = \sin \frac{\pi}{6} = \frac{1}{2}$$

$\vdots$

Substitute in taylor's

$$\sin x = \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2\left(-\frac{1}{2}\right) + \frac{(x - \pi/6)^3}{\lfloor 3}\left(-\frac{\sqrt{3}}{2}\right) + \cdots$$

Q (22) $f(x) = \frac{\sin x}{x - \pi}$ at $x = \pi$

(we are getting $\frac{0}{0}$ as first term)
so avoid it →

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(get expansion of $\sin x$ at $x=\pi$
divide $(x-\pi)$ on both sides)

$$\sin x = 0 + (x-\pi)(-1) + \frac{(x-\pi)^2}{2}(0) + \frac{(x-\pi)^3}{\lfloor 3}(+1) + \cdots$$

$\div$ by $(x-\pi)$ on L.S

$$\frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{\lfloor 3} + \cdots$$

(b)

Q find coeff of $(x-\pi)^2$ in the exp of $e^x + \sin x$?

Sol^n $f(x) = e^x + \sin x$, $a = \pi$

$$\text{coeff of } (x-\pi)^2 = \frac{f''(\pi)}{\lfloor 2} = \frac{1}{2}\left[e^x - \sin x\right]_{x=\pi}$$
$$= \frac{1}{2}\left[e^\pi\right]$$

Q64 $f(x) = \int_0^x e^{-t^2/2}\, dt$ at $x=0$

$$a_2 = \text{coeff. of } (x-0)^2 = \frac{f''(0)}{\lfloor 2}$$

$$f'(x) = \frac{d}{dx}\int_0^x e^{-t^2/2}\, dt = e^{-x^2/2}(1) - e^{-0^2/2} \times (0)$$
$$= e^{-x^2/2}$$

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Notes

$$f''(x) = e^{-x^2/2} \times \left(\frac{-\cancel{2}x}{\cancel{2}}\right)$$

$$\frac{f''(0)}{\lfloor 2} = \frac{0}{\lfloor 2} = 0$$

(Q) Find the Linear Approximation of e^{-x} at $x = 2$?

Sol^n To get linear approx of $f(x)$ at $x = a$ neglect higher powers ~~of~~ from $(x-a)^2$ onwards

linear approx of $f(x) = f(a) + (x-a) f'(a)$

" " " $e^{-x} = e^{-2} + (x-2)(-e^{-2})$

$= (3-x)e^{-2}$

(Q) Find quadratic approx. of $f(x) = e^{-x}$ at $x = \pi$?

HW

Q25) Find quadratic approx of $f(x) = x^3 - 3x^2 - 5$ at $x = 0$?

$= \cancel{x^3} - 3x^2 - 5$

~~we use Ta~~ The expansion of a polynomial at $x = 0$ gives itself. $\therefore$ To get the quadratic approximation neglect higher powers of x, we get $-3x^2 - 5$.

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Notes

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[expansion of some standard functions]

1) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n}$

2) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} \cdots$

3) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

4) $\log[1+x] = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$

6) $\log\left[\frac{1+x}{1-x}\right] = \log(1+x) - \log(1-x)$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$$

Q1 Expansion of $\sin(x^3)$

Q2 $\sum_{n=0}^{\infty} \frac{1}{\lfloor n}$ converges to ____

Q3 Exp of e^{x+x^2}

Sol^n ① replace x by x^3 in $\sin x$

Sol^n ② Put $x = 1$ in e^x expansion. then we

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Notes

get $e^1 = \sum_{n=0}^{\infty} \frac{1}{\lfloor n}$

sol^n (3) $= e^x . e^{x^2}$

$$= \left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\right)\left(1 + x^2 + \frac{x^4}{\lfloor 2} + \frac{x^6}{\lfloor 3} + \cdots\right)$$

$$= 1 + x + x^2\left[1 + \frac{1}{2}\right] + x^3\left[1 + \frac{1}{3!}\right] + \cdots$$

~~(Q) Exp of log(sin x)~~ =

~~$f(x) = \log \sin x$~~

(Q) Exp of $\log(1 + \sin x)$

sol^n:- $f(x) = \log(1 + \sin x) \Rightarrow f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1 + \sin x} (\cos x) \Rightarrow f'(0) = \frac{1}{1} = 1$

$f''(x) = \qquad \Rightarrow f''(0) =$

Sub in mc-Laurence series.

(Q) Exp of $\log[1 + e^x]$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Notes

Increasing / Decreasing Function

⇒ $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$.

⇒ $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$.

⇒ $y = f(x)$ is strictly / monotonically increasing if $\frac{dy}{dx} > 0$, $\forall x \in R$

⇒ $y = f(x)$ is strictly / monotonically decreasing if $\frac{dy}{dx} < 0$, $\forall x \in R$.

Q) $y = ax + b$

a → slope (ie $\frac{dy}{dx}$)

If $a > 0 \Rightarrow y$ is Strictly Inc. f^n

if $a < 0 \Rightarrow y$ is Strictly Dec. f^n

Q) $f(x) = \frac{e^x}{e^x + 1}$, $\forall x \in R$

$$f'(x) = \frac{(e^x + 1)e^x - e^x(e^x)}{(e^x + 1)^2}$$

$$= \frac{e^x}{(e^x + 1)^2}, \forall \in R$$

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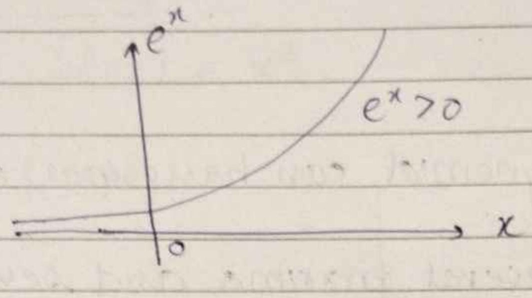
Notes

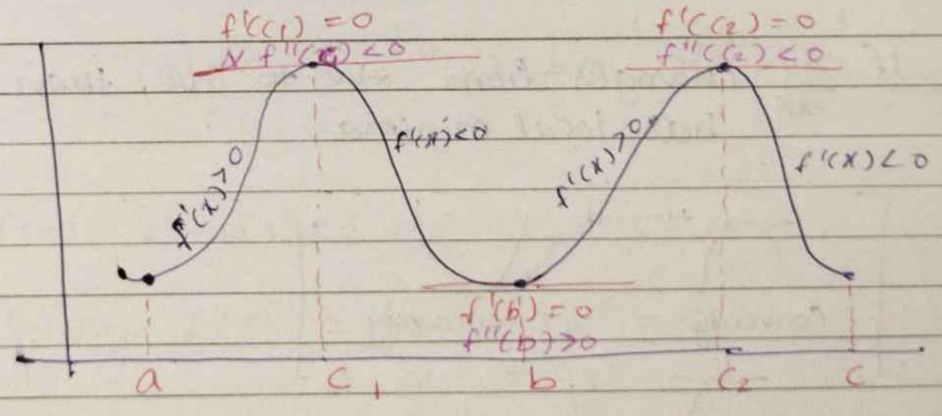


$\therefore f'(x) = \dfrac{e^x}{(e^x+1)^2} > 0, \forall x \in R$

$\therefore$ monotonically increasing function.

If $f'(x) = 0$, then function is neither ↑ nor ↓

then x is called stationary point or critical point.



Interval of Inc : $(a, c_1) \cup (b, c_2)$

Interval of Dec : $(c_1, b) \cup (c_2, c)$

Notes

Maxima & Minima

① An n^{th} degree polynomial can have $(n-1)$ extremas.

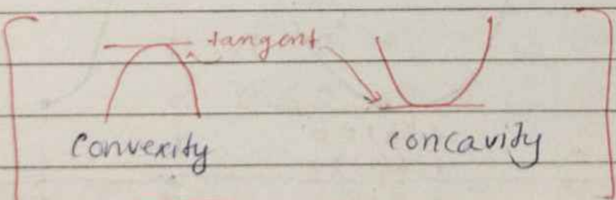
② $f(x)$ can have several maxima and several minima.

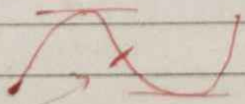
③ The greatest among all local maxima is global maxima.

④ The least among all local minima is global minima.

⑤ If $\frac{dy}{dx}$ changes from +ve to -ve, then $f(x)$ has local maxima.

⑥ If $\frac{dy}{dx}$ changes from -ve to +ve, then $f(x)$ has local minima.

⑦ 



The point at which convexity ~~cas~~ changes to concavity or concavity ~~to~~ changes to convexity is called point of Inflextion (or) Saddle pts.

If $f''(a) = 0$ & $f'''(a) \neq 0$
Then $x = a$ is saddle pt.

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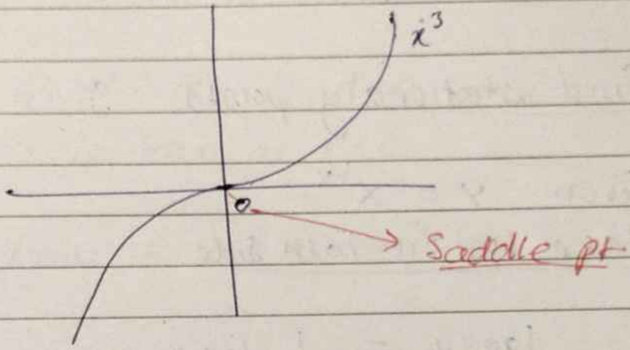
Notes

eg $f(x) = x^3$

Graph



or

$f''(x) = 6x = 0 \Rightarrow \boxed{x=0}$ ~~is saddle pt~~

$f'''(x) = 6 \Rightarrow f'''(0) = 6 \neq 0$

$\because f''(0) = 0$ & $f'''(0) \neq 0 \Rightarrow \boxed{x=0}$ is saddle pt.

Q 35 $f(x) = x^4 - 18x^2 + 9$

Solving $f''(x) = 12x^2 - 36 = 0$

$\Rightarrow x = \pm\sqrt{3}$

and $f'''(x) = 24x$

at $x = \sqrt{3} \Rightarrow f'''(\sqrt{3}) = 24\sqrt{3} \neq 0$

at $x = -\sqrt{3} \Rightarrow f'''(-\sqrt{3}) = -24\sqrt{3} \neq 0$

at $x = \pm\sqrt{3}$, $f'' = 0$ & $f''' \neq 0$

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Notes

$\therefore$ no. of Inflexion pts are "2"

Q find maximum value of $x^{1/x}$?

sol.^{n} find stationary points given by $\frac{dy}{dx} = 0$

given $y = x^{1/x}$

$\rightarrow$ take log on both side

$$\log y = \frac{1}{x} \log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \left(-\frac{1}{x^2}\right)\log x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = y\left[\frac{1-\log x}{x^2}\right]} \quad (1)$$

Now, stationary pts are given by

$$\frac{dy}{dx} = x^{1/x}\left[\frac{1-\log x}{x^2}\right] = 0$$

$$\Rightarrow 1 - \log_e x = 0$$

$$\Rightarrow \log_e x = 1$$

$$\Rightarrow \boxed{x = e} \leftarrow \text{Stationary pt.}$$

diff (1) w.r.t x

$$\frac{d^2y}{dx^2} = y\left[\frac{x^2\left[-\frac{1}{x}\right] - (1-\log x)2x}{x^4}\right]$$

$$+ \left[\frac{1-\log x}{x^2}\right]y'$$

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Notes

$$\left.\frac{d^2Y}{dx^2}\right|_{x=e} = e^{1/e}\left[-\frac{1}{e^3}\right] < 0$$

$\because Y' = 0$ & $Y'' < 0$ at $x = e$

$\therefore Y$ is max. at $x = e$.

$\therefore Y_{max} = Y(e) = e^{1/e}$

HW

Q 1) min^m value of x^x?
2) max^m value of $\left(\frac{1}{x}\right)^x$?

Q26 $f(x) = (x^2-4)^2 \quad \forall x \in R$

Sol^n finding stat points

$$f'(x) = 2[x^2-4] \times 2x = 0$$

$$\Rightarrow x = 0, \pm 2$$

$$f''(x) = 4[3x^2-4]$$

$f''(0) = -16 < 0 \Rightarrow f(x)$ is max^m at $x = 0$

$f''(+2) = 32 > 0 \Rightarrow f(x)$ is min^m at $x = 2$

$f''(-2) = 32 > 0 \Rightarrow f(x)$ is min^m at $x = -2$

$\therefore$ 2 local min & 1 maxima

(b) ✓

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e^x can never be zero.

Notes

Q. for $0 \le t \le \infty$, max^m value of $f(t) = e^{-t} - 2e^{-2t}$ occurs at $t =$ ____ ?

sol^n:

find st. pts : by $f'(t) = 0$

$$f'(t) = -e^{-t} + 4e^{-2t} = 0$$

$$\Rightarrow e^{-t}[-1 + 4e^{-t}] = 0$$

$$\Rightarrow \boxed{e^{-t} \neq 0}, \quad -1 + 4e^{-t} = 0$$

$$e^{-t} = \frac{1}{4} = 4^{-1}$$

(taking log b.s)

$$-t = -1\log 4$$

$$\boxed{t = \log 4}$$

$$f''(t) = e^{-t} - 8e^{-2t}$$

$$f''(\log 4) = e^{-\log 4} - 8e^{-2\log 4}$$

$$= \frac{1}{4} - \frac{8}{16} < 0$$

$\therefore$ maximum at $\log 4$.

~~If $f'(a)$ is zero~~

If $f'(a) = f''(a) = f'''(a) = \dots = f^{2n-1}(a) = 0$ & $f^{2n}(a) < 0$, then $f(x)$ has local maxima at $x = a$.

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Notes

If $f'(a) = f''(a) = f'''(a) = \dots = f^{2n-1}(a) = 0$ & $f^{2n}(a) > 0$. Then $f(x)$ has local minima at $x = a$.

If $f''(a) = f'''(a) = \dots = f^{2n-2}(a) = 0$ & $f^{2n-1}(a) \neq 0$
Then $x = a$ is saddle point.

Shortcuts

WKT AM $\geq$ GM

$\frac{a\cot\theta + b\tan\theta}{2} \geq \sqrt{a\cot\theta\, b\tan\theta}$

$a\cot\theta + b\tan\theta \geq 2\sqrt{ab}$

① If $Y = a\cos\theta + b\sin\theta + c$
Then

$$Y_{max} = c + \sqrt{a^2+b^2}$$
$$Y_{min} = c - \sqrt{a^2+b^2}$$

reciprocal

② If $Y = a\cot\theta + b\tan\theta$
$= a\,\mathrm{cosec}\theta + b\sin\theta$
$= a\sec\theta + b\cos\theta$
Then $Y_{min} = 2\sqrt{ab}$

③ If $Y = a\cos^2\theta + b\sin^2\theta$ $(a > b)$
Then

$$Y_{max} = a, \quad Y_{min} = b$$

④ If $f(x) = x^n e^{-x}$, $f(x)$ is max^m at $x = n$.

⑤ $f(x) = \frac{x}{(x+a)(x+b)}$ is max^m at $x = \sqrt{ab}$

~~⑥ $y = (f(x))^2$, $y_{min} = 0$~~

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Notes

1 Q $f(x) = x^2 \cdot e^{-x}$, $\forall x \in R$, $f_{max} = ?$

2 Q $f(x) = (x-1)^{2/3}$, $\forall x \in R$, $f_{min} = ?$

3 Q $f(x) = \dfrac{x}{(x+1)^2}$, $\forall x \in R$, find $f_{max} = ?$

sol 1 by shortcut 4 $f(x)$ is max^m at $x = n = 2$

$$\therefore f_{max} = f(2) = 2^2 e^{-2}.$$

sol 2

$$f(x) = \left[(x-1)^{1/3}\right]^2 \geq 0 \quad \forall x \in R$$

$\therefore f_{min} = 0$, occurs at $x = 1$.

sol 3 The given $f(x)$ can be written as

$$f(x) = \frac{x}{(x+1)(x+1)} \quad \forall x \in R$$

by shortcut 5

$f(x)$ is max^m at $x = \sqrt{ab}$
$= \sqrt{1 \times 1}$
$= 1$

$$\therefore f_{max} = f(1) = \frac{1}{(1+1)^2} = \frac{1}{4}$$

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$\cos(a+b) = \cos a \cos b - \sin a \sin b$

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Notes

Q $y = 5\cos\theta + 3\cos\left[\frac{\pi}{3}+\theta\right] + 3$, $f_{max} = ?$
$f_{min} = ?$

Sol^n $y = 5\cos\theta + 3\left[\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta\right] + 3$

$= \underbrace{\left[5+\frac{3}{2}\right]}_{a}\cos\theta - \underbrace{\frac{3\sqrt{3}}{2}}_{b}\sin\theta + \underbrace{3}_{c}$

~~Ymax~~ by Shortcut 1

$Y_{max} = 3 + \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$= 3 + 7$
$= 10$

$Y_{min} = 3 - 7$
$= -4$

Q $f(x) = \frac{e^{\sin x}}{e^{\cos x}}$, $\forall x \in R$

$f(x) = e^{\sin x - \cos x}$, $\forall x \in R$

NOTE finding max / min of exponent fun.

maximize the power to get maximum
minimize " " minimum

$f_{max} = e^{\max(\sin x - \cos x)}$
$= e^{0 + \sqrt{(-1)^2 + 1^2}} = e^{\sqrt{2}}$ (by shortcut 2)

Teacher's Signature

Notes

$$f_{min} = e^{min(\sin x - \cos x)}$$

$$= e^{0 - \sqrt{(-1)^2+1^2}}$$

$$= e^{-\sqrt{2}}$$

Ⓠ $Y = 2\cos\theta + 3\sec\theta$, $Y_{min} = ?$

reciprocal

by S.C ②

$$Y_{min} = 2\sqrt{ab}$$
$$= 2\sqrt{2\times 3}$$
$$= 2\sqrt{6}$$

reciprocal

Ⓠ $y = x^2 + \frac{1}{x^2}$, $y_{min} = ?$

$Y_{min} = 2\sqrt{1\times 1} = 2$ (by S.C.②)

Ⓠ $Y = 2e^{-x} + 3e^{x}$, $Y_{min} = ?$

reciprocal

$Y_{min} = 2\sqrt{2\times 3} = 2\sqrt{6}$ (by S.C②)

reciprocal

Ⓠ $Y = 2^{x} + 3(2^{-x})$, $Y_{min} = ?$

$Y_{min} = 2\sqrt{1\times 3} = 2\sqrt{3}$ (by S.C②)

Ⓠ $Y = \underset{a}{25}\cos^2\theta + \underset{b}{5}\sin^2\theta$ is max^{m} at $\theta = ?$

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Notes

Sol^n $(a>b)$

by S.C (3)

$Y_{max} = a = 25$ is possible when $\boxed{\theta = 0}$

Global (absolute) max^m / min of $f(x)$ in $[a,b]$

Global max^m of $f(x)$ in $[a,b]$ = max$\{f(a), f(b), \text{local max}^m\}$

Global min^m of $f(x)$ in $[a,b]$ = min$\{f(a), f(b), \text{local min}^m\}$

Q find global minima of $2x^3 - 3x^2$ in $[-1, 2]$

Sol^n:

$f(a) = f(-1) = 2(-1)^3 - 3(-1)^2$
$= -2-3$
$= -5$

$f(b) = f(2) = 2(2)^3 - 3(2)^2$
$= 16 - 12$
$= 4$

find local minima : ~~given by f'~~

find S.pts : given by $f'(x) = 6x^2 - 6x = 0$
$\Rightarrow \boxed{x = 0, 1} \in [-1, 2]$

$f''(x) = 12x - 6$

$f''(0) = -6 < 0 \Rightarrow f(x)$ is max^m at $x = 0$

$f''(1) = 6 > 0 \Rightarrow f(x)$ is min^m at $x = 1$

$\therefore$ local min $= f(1) = 2(1)^3 - 3(1)^2 = -1$

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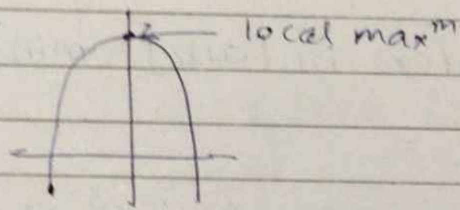
Notes

$\therefore$ Absolute min. of $f(x)$ in $[-1, 2)$
$= \min\{-5, 4, -1\}$
$= -5$

Q. find minimum value of $-x^2+10x+100$ in $[5, 10]$?

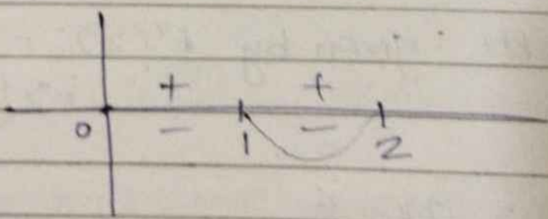
Sol^n 2^{nd} degree polynomial & coeff of x^2 is $-ve$
$\therefore$ it's Inverted parabola.



$\therefore$ no local minima
$\therefore$ min of $f(x)$ in $[5, 10] = \min\{f(5), f(10), \cancel{\text{local min}}\}$
$= \min\{125, 100\}$
$= 100$

Q. $f(x) = x(x-1)(x-2)$, $\forall x \in R$
find max^m in $[1, 2]$?

Sol^n



at $x = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right) < 0$

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Notes

$\Rightarrow f(x) \leq \quad \forall x \in [1, 2]$

$\Rightarrow$ max value of $f(x) = 0$
at $x = 1, 2$

(Q) find max^m value of $f(x) = 1 + 2\sin x + 2\cos^2 x$
in $[0, \frac{\pi}{2}]$?

(approach is important)

a) 7/2 b) 5/2 c) 3/2 d) 1/2

$f(x) = 1 + 2\sin x + 2[1 - \sin^2 x]$
$f(x) = 3 + 2\sin x - 2\sin^2 x$ in $[0, \frac{\pi}{2}]$

Let $\sin x = t$,
$\Rightarrow \boxed{f(x) = 3 + 2t - 2t^2 \text{ in } [0, 1]}$

Inverted parabola
⇓
local max. exists

$f(a) = f(0) = 3$
$f(b) = f(1) = 3$

local maxima

find S.t pts

$f'(t) = 2 - 4t = 0 \Rightarrow t = 1/2$

$f''(t) = -4 < 0$

$\therefore f(t)$ is max^m at St-pt $t = 1/2$

local max $= f(\frac{1}{2}) + 3 + 2(\frac{1}{2}) - 2(\frac{1}{2})^2 = \frac{7}{2}$

$\therefore$ max of $f(t)$ in $[0, 1] = \max\{f(0), f(1), \text{local max}\}$
$= \max\{3, 3, \frac{7}{2}\} = \frac{7}{2}$ (a)

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$Y = X^2 \rightarrow$ parabola
$Y = -X^2 \rightarrow$ inverted parabola

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Extremize $u = f(x, y)$

1) Solving $p = 0, q = 0$ weget stationary pts.
2) find r, s, t at stationary pts
3) If $rt - s^2 > 0$ & $r > 0 \Rightarrow u$ is minimum at St pt.

4) If $rt - s^2 > 0$ & $r < 0 \Rightarrow u$ is maximum at St. pt.

5) If $rt - s^2 < 0$ Then (x, y) is saddle pt where u is neither max^m nor min^m.

Q 46 $u = 2x^4 + y^2 - x^2 - 2y$

Find St. pt :- $du = 0$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = p = 0 \;, \; \frac{\partial u}{\partial y} = q = 0$$

$$\Rightarrow p = 8x^3 - 2x = 0 \quad \Big| \quad q = 2y - 2 = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x = 0, \pm \frac{1}{2} \quad \Big| \quad \Rightarrow \boxed{Y = 1}$$

$\therefore$ St pts $(0, 1), \left(\frac{1}{2}, 1\right), \left(-\frac{1}{2}, 1\right)$

Find r, s, t

$$r = u_{xx} = 24x^2 - 2$$
$$s = u_{xy} = 0$$
$$t = u_{yy} = 2$$

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$$\boxed{\begin{gathered} u(x,y,z) \\ du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \end{gathered}}$$

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$(rt - s^2)_{(0,1)} = (-2)(2) - 0^2 = -4 < 0$

$\Rightarrow (0,1)$ is saddle pt

$(rt - s^2)_{(\frac{1}{2},1)} = (4)(2) - 0^2 = 8 > 0$ & $r = 4 > 0$

$\Rightarrow u$ is min at $(\frac{1}{2}, 1)$

(b) ✓

Q47 HW (optimal means max^m or min^m)

TIP: Since x^2, y^2 coeffs are > 0
$\Rightarrow$ u is minimum only at St pts.

Q) The minimum value of ~~$f(x,y,z) = x^2 + 5y^2 +$~~
$f(x,y,z) = x^2 + 5y^2 + 5z^2 - 4x + 10y - 40z + 300$.

Find St pts.

$$\frac{\partial u}{\partial x} = 2x - 4 = 0 \Rightarrow \boxed{x = 2}$$

$$\frac{\partial u}{\partial y} = 10y + 10 = 0 \quad \boxed{y = -1}$$

$$\frac{\partial u}{\partial z} = 10z - 40 = 0 \Rightarrow \boxed{z = 4}$$

$\therefore$ St pt $P(x,y,z) = P(2,-1,4)$

$\therefore$ f_{min} is given by $f(2,-1,4)$.

$f_{min} = f(2,-1,4) = 4 + 5 + 80 - 8 - 10 - 160 + 300$
$= 211$

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Notes

Extremize $u = f(x, y)$ wrt condition $\phi(x, y) = c_1$

$\Downarrow$

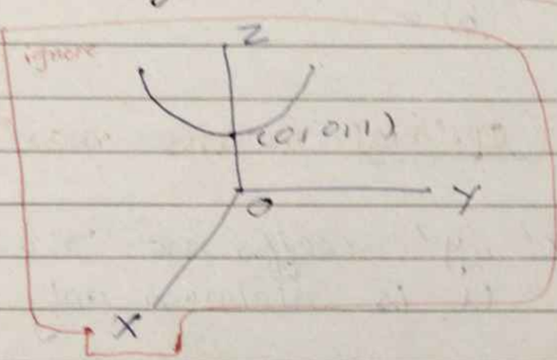
$\boxed{y = F(x)}$

$u = f(x, F(x))$ to be extremized

Q (48) $z^2 = 1 + xy$

sol^n

ignore



let nearest pt be $P(x, y, z)$

$\therefore$ the distance b/w origin & $P(x, y, z)$ lies on the surface is to be minimum.

$\therefore$ distance $= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$ to be min^m

$\therefore u = d^2 = x^2 + y^2 + 1 + xy$ to be minimum.

find St pt: $\frac{\partial u}{\partial x} = 2x + y = 0$ → Homog eq^n

$\frac{\partial u}{\partial y} = 2y + x = 0$

$\Downarrow$

Solving $\Rightarrow$ $x = 0, y = 0 \Rightarrow z^2 = 1 + 0 = 1$

$z = \pm 1$

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Notes

$\therefore$ St pt is $P(0, 0, \pm 1)$

$D_{min} = \sqrt{0^2+0^2+1^2}$
$= 1$

Since x^2, y^2 coeff > 0 in d. ($\therefore$ Dist is min^m)

Shortcuts

① If $x+y=k$, $\forall\, x, y \in R$, Then (i) xy is max^m when $x=y=\frac{k}{2}$.

(ii) x^2+y^2 is minimum at $x=y=\frac{k}{2}$
(iii) x^3+y^3 is minimum at $x=y=\frac{k}{2}$

② If product of two +ve real numbers is a const (k). Then their sum will be least when they are equal ($\sqrt{k}, \sqrt{k}$).

③ The max^m Area of rectangle inscribed in an Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is ($2ab$).

④ The max^m area of rectangle inscribed in a circle of radius (r) is $2r^2$

⑤ The max^m height of cone having largest volume inscribed in a sphere of radius 'r' is $\frac{4r}{3}$.

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(6) The max^m height of cylinder having largest volume inscribed in a sphere of radius 'r' is $\frac{2r}{\sqrt{3}}$.

(7) If Sum of ~~hyp~~ hypotenous & a side in a right angled triangle is kept constant. In order to have max^m area of Δ^{le}, angle b/w hypotenous & side is $\left(\frac{\pi}{3} = 60^\circ\right)$.

(Q) find min. value of x^2+y^2 wrt Straight line $x+y=1$?

by S.C (1.2)

$$x^2+y^2 \text{ is min. at } x=y=\frac{k}{2}=\frac{1}{2}$$

$$\therefore \text{min value of } x^2+y^2 = \left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$$

$$= \frac{2}{4} = \frac{1}{2}$$

(Q) If Trace of sym. matrix $A_{2\times2}$ is 14, Then $|A|_{max} = ?$

Sol^n.

$$\text{Trace } (A_{2\times2}) = \lambda_1+\lambda_2 = 14 \quad [\lambda_1, \lambda_2 \in R]$$

wkt

$$|A| = \lambda_1 \cdot \lambda_2$$

$$\therefore |A| = \lambda_1\lambda_2 \text{ is max when } \lambda_1=\lambda_2=\frac{k}{2}=\frac{14}{2}=7$$

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Tomorrow
2 to 6:30pm

Notes

$\therefore |A|_{max} = 7 \times 7 = 49$ (by SC 1.1)

Q 37 Ellipse $\frac{x^2}{1^2} + \frac{y^2}{(1/2)^2} = 1$

$a = 1, \quad b = \frac{1}{2}$

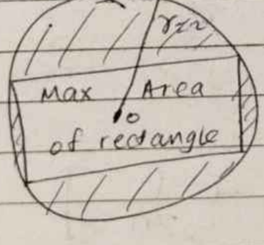
$\therefore$ by S.C 3

$\Delta_{max} = 2ab = 2 \times 1 \times \frac{1}{2} = 1$

Q 31 by SC 5

$H_{max} = \frac{4r}{3} = \frac{4 \times 1}{3} = \frac{4}{3}$ (d) ✓

Q



Area of shaded = Area of $\bigcirc^{le}$ − max^m of rect in circle

$= \pi(2)^2 - 2(2)^2$ (by SC 4)

$= 4[\pi - 2]$

HW

Q find max^m value of $x+y$, given that $x^2+y^2=1$?

(A) $\sqrt{2}$ ✓

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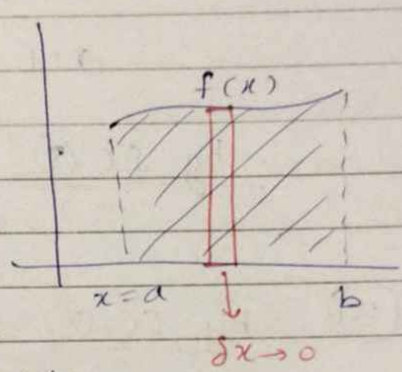
Notes

Integration

Definite Integrals

$$I = \int_{x=a}^{b} f(x)\,dx$$



= Area under the curve $y=f(x)$ between $x=a,b$ bounded by x-axis.

Fundamental Theorem of Integ. calc.

$$\lim_{\substack{n\to\infty \\ \delta x\to 0}} \sum_{i=1}^{n} f(x_i)\,\delta x_i = \int_a^b f(x)\,dx$$

Q) $$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\dots+\underset{(n+n)}{\frac{1}{2n}}\right]$$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\left[1+\frac{r}{n}\right]}$$

$$\left[\begin{array}{l} \text{let } \frac{r}{n}=x \\ \delta x = \frac{\delta r}{n} = \frac{1}{n} \to 0 \end{array}\right]$$

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$$\boxed{\int \frac{1}{1+x^2}\,dx = \tan^{-1}x}$$

Notes

$$\lim_{\substack{n\to\infty \\ \delta x\to 0}} \sum \underbrace{\left(\frac{1}{1+x}\right)\delta x}_{f(x)\,\delta x} = \int_{x=0}^{1} \frac{1}{1+x}\,dx$$

$$= \log(1+x]\Big|_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2$$

② $$\lim_{n\to\infty}\left[\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+(n-1)^2}\right]$$

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n}{n^2+r^2}$$

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{\cancel{n}}{n^{\cancel{2}}\left(1+\left(\frac{r}{n}\right)^2\right)}$$

δx $\quad$ x

$$\left[\begin{array}{l} x = \frac{r}{n} \\ r=0 \Rightarrow x = \frac{0}{n} = 0. \\ r = n-1 \Rightarrow x = \frac{n-1}{n} = 1-\frac{1}{n} = 1-0 = 1 \end{array}\right]$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}x\Big]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

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$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

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~~# Intermediate~~

Mean Value Theorem (on Integrals)

$f(x)$ is cont in $[a,b]$

Then $\exists\ c \in (a,b)$ such that

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)}$$

Properties of Integrals

1) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

2) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

3) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

4) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \text{ (even } f^n) \\ 0, & \text{if } f(-x) = -f(x) \text{ (odd } f^n) \end{cases}$

5) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \dots \times \frac{3}{4} \times \frac{1}{2}\right] k$$

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$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \quad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

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where $k = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

6) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ = ~~[(m-1)(m-3)... 2(or 1)][(n-1)(n-3)... 2(or 1)]~~

$$= \frac{[(m-1)(m-3)\cdots 2(\text{or})1][(n-1)(n-3)\cdots 2(\text{or})1]}{[(m+n)(m+n-2)\cdots 2(\text{or})1]} \times k$$

where $k = \begin{cases} \pi/2, & \text{if } m, n \text{ are Even} \\ 1, & \text{else} \end{cases}$

7) $\int e^x [f(x) + f'(x)] \, dx = e^x f(x)$

8) $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

9) $\int uv = uv_1 - u'v_2 + u''v_3 \cdots$ (by ILATE)

$\boxed{\int uv = u\int v - \int(u'\int v)}$ ★★★

10) $\int \frac{f'(x)}{f(x)} \, dx = \log f(x) + c$

11) $\int (f(x))^n f'(x) \, dx = \frac{(f(x))^{n+1}}{n+1} + c$

Q) $\int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$

$$\int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta} \, d\theta$$

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Notes

$$\int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \cos\left[0+\frac{\pi}{2}-\theta\right]}\,d\theta$$

$$= \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4} \quad \text{(by property ③)}$$

Q ① $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\cot\theta}$

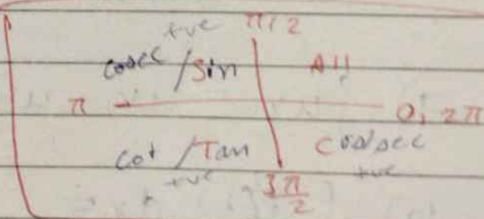
Ans: $\dfrac{\frac{\pi}{2}-0}{2}$

$= \dfrac{\pi}{4}$ (prop ③)

Q ② $\displaystyle\int_2^{16} \frac{\underbrace{x^n}_{f(x)}}{\underbrace{x^n}_{f(x)} + \underbrace{(18-x)^n}_{f(2+16-x)}}\,dx$

$= \dfrac{16-2}{2} = 7$ (prop ③)

Q $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\tan^3\theta}$ Ans. $\dfrac{\pi}{4}$ (prop ③)



Q $I = \displaystyle\int_0^{\pi} \underbrace{\boxed{\sin^3\theta \cos^5\theta}}_{f(\theta)}\,d\theta = ?$

$f(\pi-\theta) = \sin^3(\pi-\theta)\cos^5(\pi-\theta)$

$= \sin^3\theta\,(-\cos\theta)^5$

$= \sin^3\theta\,(-\cos^5\theta)$

$= -f(\theta)$

(by prop ②)

$I = 0$

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Notes

(Q) $I = \int_0^{\pi} x \boxed{\underbrace{\sin^4 x \cos^6 x}_{f(x)}} dx = ?$

Prop (12) ☆☆☆ $\int_a^b x f(x) dx = \frac{a+b}{2} \int_a^b f(x) d(x)$

if $f(a+b-x) = f(x)$

$f(0+\pi-x) = \sin^4(0+\pi-x)\cos^6(0+\pi-x)$
$= \sin^4(x)(-\cos x)^6$
$= \sin^4 x \cos^6 x$
$= f(x)$

∴ by Property (12)

$\int_0^{\pi} x f(x) dx = \frac{0+\pi}{2} \int_0^{\pi} \sin^4 x \cos^6 x \, dx$

even

$= \frac{\pi}{\cancel{2}} \times \cancel{2} \int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ (by Prop (2))

$= \frac{\pi \times \cancel{2}}{\cancel{2}} \left[\frac{(4-1)(4-3)(6-1)(6-3)(6-5)}{(10)(8)(6)(4)(2)} \right] \times \frac{\pi}{2}$

$= \frac{\pi^2}{2} \left[\frac{3 \times 1 \times \cancel{5} \times \cancel{3} \times 1}{\underset{2}{\cancel{10}} \times 8 \times \underset{2}{\cancel{6}} \times 4 \times 2} \right]$

$= \frac{3\pi^2}{512}$

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Q) $\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^4 + \cos^2(x-1)}\, dx = ?$

let $x-1 = t \Rightarrow dx = dt$

$x = 0 \Rightarrow t = -1$
$x = 2 \Rightarrow t = 1$

$$= \int_{-1}^{1} \frac{t^2 \sin t}{t^4 + \cos^2 t}\, dt \rightarrow \text{odd } f^n$$

$= 0$ (by ppty ④)

Q) $I = \int_{-1}^{1} \log\left[\frac{2-x}{2+x}\right] dx = 0$ (by prop ④)

odd f^n

$f(-x) = \log\left[\frac{2-(-x)}{2+(-x)}\right] = \log\left[\frac{2+x}{2-x}\right]$

$= \log\left[\frac{2-x}{2+x}\right]^{-1} = -\log\left[\frac{2-x}{2+x}\right]$

$= -f(x)$

Q) $I = \int_{-2}^{3} |x|\, dx = ?$

WKT $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

$I = \int_{-2}^{0} -x\, dx + \int_{0}^{3} +x\, dx$

$(x < 0)$ $\quad (x \geq 0)$

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$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

Notes

$$= -\frac{x^2}{2}\Big]_{-2}^{0} + \frac{x^2}{2}\Big]_{0}^{3}$$

$$= (0+2) + \frac{9}{2} = \frac{13}{2}$$

Q) $\int_{-1}^{1} x\, e^{-|x|}\, dx = ?$

$$f(-x) = (-x)e^{-|-x|} = (-x)e^{-x}$$
$$= -[x e^{-x}]$$
$$= -f(x)$$

$\therefore f(x)$ is odd f^n in $[-1, 1]$

$\therefore$ $\boxed{I = 0}$

Q) $I = \int_{-2}^{5} |5 - 3x|\, dx = ?$

let $5 - 3x = t$
$-3\,dx = dt$
$x = -2 \Rightarrow t = 11$
$x = 5 \Rightarrow t = -10$

$$\boxed{\int_a^b f\,d(x) = -\int_b^a f\,d(x)}$$

$$I = \int_{11}^{-10} |t| \left(-\frac{dt}{3}\right)$$

$$= \frac{1}{3}(+)\int_{-10}^{11} |t|\, dt = \frac{1}{3}\int_{-10}^{11} |t|\, dt$$

$$= \frac{1}{3}\left[\int_{-10}^{0} -t\, dt + \int_{0}^{11} t\, dt\right]$$

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$I = \int u \cdot v$

$u\int v - u'\int v_1 + u''\int v_2 \ldots$

(u is taken the one which get become zero faster on differentiation)

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$$= \frac{1}{3}\left[\left[\frac{-t^2}{2}\right]_{10}^{0} + \left[\frac{t^2}{2}\right]_{0}^{11}\right]$$

$$= \frac{1}{3}\left[\frac{100}{2} + \frac{121}{2}\right]$$

$$= \frac{221}{3\times 2} = \frac{221}{6}$$

Q) $\int_{0\,(x>0)}^{2\pi} |x \sin x|\, dx = ?$

$|x \sin x| = |x||\sin x|$

$$= \begin{cases} x(\sin x) & , 0 \le x < \pi \\ x(-\sin x) & , \pi \le x \le 2\pi \end{cases}$$

WKT $|\sin x| = \begin{cases} \sin x & 0 \le x \le \pi \\ -\sin x & \pi < x \le 2\pi \end{cases}$

$$I = \int_0^{\pi} \underset{u \cdot v}{x \sin x\, dx} + \int_{\pi}^{2\pi} -x \sin x\, dx$$

$$= (x)(-\cos x) - 1\cdot(-\sin x) + 0\Big]_0^{\pi} - \Big[\qquad\Big]_{\pi}^{2\pi}$$

$$= [\pi - 0] - [-2\pi - \pi]$$

$$\boxed{I = 4\pi}$$

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$$\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$$

ILATE

Notes

⑦ $\int_0^{\pi} |\cos x|\,dx = ?$ Ans ②

⑧ $I = \int_0^{\pi/4} \log[1+\tan\theta]\,d\theta$

By ppty ①

$$I = \int_0^{\pi/4} \log\left[1+\tan\left[\frac{\pi}{4}-\theta\right]\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{1}{1} + \frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log 2\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \log 2\cdot(\theta)_0^{\pi/4} - I$$

$$2I = \log 2\left(\frac{\pi}{4}\right)$$

$$\boxed{I = \frac{\pi}{8}\log 2}$$

⑨ $I = \int_0^{\pi/2} \log \tan\theta\,d\theta = ?$

by propty ①

$$I = \int_0^{\pi/2} \log\tan\left[\frac{\pi}{2}-\theta\right]d\theta$$

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Notes

$$I = \int_0^{\pi/2} \log \cot\theta \, d\theta$$

$$I + I = \int_0^{\pi/2} (\log \tan\theta + \log \cot\theta)\, d\theta$$

$$2I = \int_0^{\pi/2} \log [\cancel{\tan\theta} \times \cancel{\cot\theta}]\, d\theta$$

$$\boxed{I = 0}$$

Q $I = \int x \log x \, dx$

let $\log_e x = t$

$\Downarrow$

$x = e^t$

$dx = e^t dt$

$$I = \int t \cdot e^t \cdot e^t \, dt$$

$$I = \int \underset{u}{t} \cdot \underset{v}{e^{2t}} \, dt$$

$$= t \cdot \frac{e^{2t}}{2} - 1 \cdot \frac{e^{2t}}{4}$$

$$= x^2 \left[\frac{\log x}{2} - \frac{1}{4}\right]$$

Improper Integrals

Type-I : $\int_{-\infty}^{\infty} f(x)\, dx$ (or) $\int_0^{\infty} f(x)\, dx$

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Notes

Type-2 :- $I = \int_{x=a}^{b} f(x)\,dx$ → disc (or) undefined in $[a,b]$

eg. $I = \int_{-1}^{1} \frac{1}{x^2}\,dx$ → disc at $x=0 \in [-1,1]$

$= \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$

$= \left[\frac{-1}{x}\right]_{-1}^{0} + \left[\frac{-1}{x}\right]_{0}^{1}$

$= \infty$

∴ I doesnot exist (or) divergent.

eg. $I = \int_{0}^{1} \frac{1}{1-x}\,dx$ → disc at $x=1 \in [0,1]$

$= \left.\frac{\log(1-x)}{-1}\right]_{0}^{1} = \infty$

∴ I is divergent (or) doesnot exist.

(Q) $I = \int_{0}^{\infty} \frac{x\,dx}{(x^2+1)^2}$ // Improper Integral Type-I

let $x^2+1 = t$
$\Rightarrow 2x\,dx = dt$
$x=0 \Rightarrow t=1$
$x=\infty \Rightarrow t=\infty$

$I = \int_{1}^{\infty} \frac{dt}{2t^2}$

$= \frac{1}{2}\left[-\frac{1}{t}\right]_{1}^{\infty}$

$= \frac{1}{2}[0+1] = \frac{1}{2}$ → finite ∴ convergent

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$\Sigma n = \frac{n(n+1)}{2}$

$\Sigma (n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$

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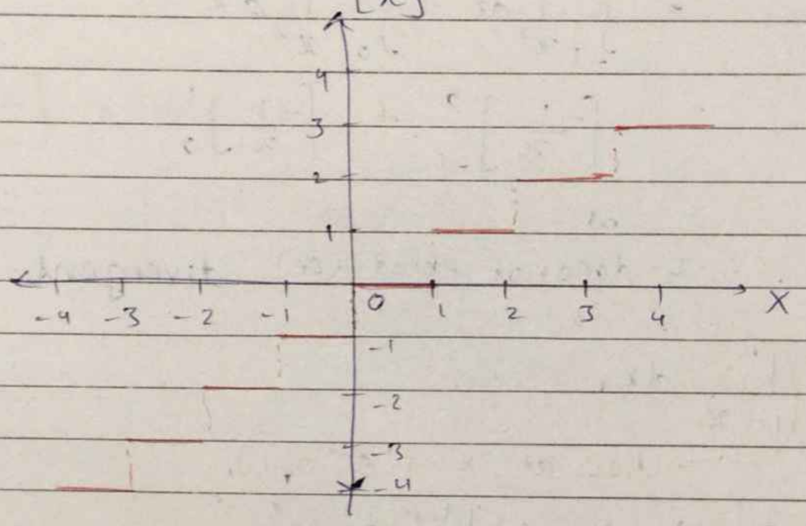
Step function

$[x]$ = (Greatest Integer Function)
= greatest Integer $\le x$

$[1.96] = 1$



$[15.86] = 15$

$f(x) = [x]$ is disc (discontinuous) at every integer

Q) $I = \int_0^n [x]\,dx, \quad n \in Z^+$

$[x]$ → disc f^n

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 (3)\,dx + \cdots + \int_{n-1}^{n} (n-1)\,dx$

$= 1 + 2[x]_2^3 + \cdots + (n-1)(x)_{n-1}^{n}$

$= 1 + 2 + 3 + 4 + \cdots + (n-1)$

$= \frac{n(n-1)}{2}$

Teacher's Signature ____________

Notes

Q $I = \int_0^1 [3x]\, dx$

let $3x = t$
$3\,dx = dt$
$x = 0 \Rightarrow t = 0$
$x = 1 \Rightarrow t = 3$

$$I = \int_0^3 [t] \frac{dt}{3}$$

$$= \frac{1}{3}\left[\int_0^1 0\,dt + \int_1^2 1\,dt + \int_2^3 2\,dt \right]$$

$$= \frac{1}{3}[3]$$

$$= 1$$

Gamma function

By defn $\quad \Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

NOTE
1) $\Gamma n+1 = n \Gamma n = n!$
2) $\Gamma 2 = \Gamma 1 = 1$
3) $\Gamma 1/2 = \sqrt{\pi}$

Q $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = ?$

$$= 2\int_0^\infty e^{-ax^2}\, dx$$

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$$\left[\begin{array}{l} \text{let } ax^2 = t \\ 2ax\,dx = dt \\ dx = \frac{1}{2a}\frac{\sqrt{a}}{\sqrt{t}}\,dt \end{array}\right]$$

Rule $\boxed{\text{Property}}$

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}}$$

$$= \frac{\cancel{2}}{2\sqrt{a}}\int_0^{\infty} e^{-t}\,t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{\sqrt{a}}\int_0^{\infty} e^{-t}\,t^{\left(-\frac{1}{2}+1\right)-1}\,dt$$

$$= \cancel{\frac{1}{\sqrt{a}}\int_0^{\infty} e} = \frac{1}{\sqrt{a}}\Gamma\tfrac{1}{2} = \sqrt{\frac{\pi}{a}}$$

Q① $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{1/2}} = \sqrt{2\pi}$

Q② $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\frac{\pi}{1}} = \sqrt{\pi}$

Q③ $\displaystyle\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,\underbrace{dx\,dy}_{\text{Ind.}} = \int_0^{\infty} e^{-x^2}dx\int_0^{\infty} e^{-y^2}dy = \frac{1}{2}\sqrt{\frac{\pi}{1}}\times\frac{\sqrt{\pi}}{2\sqrt{1}} = \frac{\pi}{4}$

Q④ $\displaystyle\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$

$\because$ limits are constant for x, y, z $\therefore$ Ind. ($\therefore$ we can do integ simultaneously)

$$= \frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}$$

$$= \frac{\pi\sqrt{\pi}}{8}$$

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Probability & Statistics

Notes

Random Experiment:

An experiment whose result is unpredictable is called Random experiment.

Ex: 1) Tossing a coin
2) Throwing a dice

Event :- Each outcome of Random experiment is called an event

Types of Events :-

① Exhaustive Events :- All possible outcomes of experiment are called exhaustive events.

Ex ⓐ In Tossing a coin
Exhaustive Events are {H, T}

ⓑ Throwing Dice : Exhaustive Events = {1,2,3,4,5,6}

→ getting '7' is a non-exhaustive (or) Impossible event.

② Equally likely Events :- The events whose chance of occurance is same (no prefence) are called equally likely.

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Ex ① H, T are equi probable

$$P(H) = P(T) = \frac{1}{2}$$

② getting 1, 2, 3, 4, 5, 6 are equi probable

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$$

③ Mutually Exclusive Events :- Happening of one event prevents the happening of another event, Then those events are mutually exclusive.

Ex :- ① In Tossing a coin H, T do not occurs once.

If head appears then Tail not & vice versa.

i.e. $H \cap T = \phi$

② getting a even numbers & odd numbers in throwing dice are mutually exclusive.

③ King & diamond are possible in a single draw from a pack of cards

i.e. $K \cap \diamond = 1$

$\therefore K, \diamond$ are not mutually Exclusive.

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Notes

(4) <u>Independent Events</u> : Happening of an event do not influence the happening of another event then those events are called Independent.

If A, B are independent events, Then P(A) does not alter P(B)

<u>Ex</u> : A person Apply for Jobs A, B, C. Then getting Job 'A', Job 'B' & Job 'C' are Independent events

→ <u>Odds Infavour of an events</u>

no. of ways favourable to event 'A' is 'm'
&
no. of ways ~~fa~~ unfavourable to event 'B' is 'n'
then

$$\text{odds in favour} = \frac{m}{n}$$

$$\text{odds against} = \frac{n}{m}$$

<u>Definition of Probability</u>

$$P(A) = \frac{\text{no. of favourable to event 'A'}}{\text{no. of exacustive events of experiment}}$$

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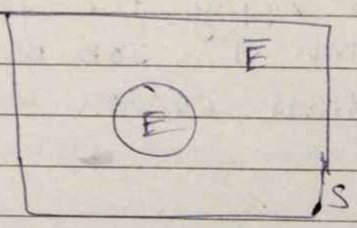
$$\boxed{P(A) = \frac{m}{n}}$$

Set definitions

Sample Space :- Set of all exhaustive events of an experiment is called Sample space (S).



Let 'E' be an event in sample space 'S'.

$$P(E) = \frac{\text{no. of favourable to 'E'}}{\text{no. of exhaustive events}}$$

$$\boxed{P(E) = \frac{n(E)}{n(S)}}$$

E, $\bar{E}$ are mutually exclusive

& $n(E) + n(\bar{E}) = n(S)$

÷ by n(S) both side

$$\frac{n(E)}{n(S)} + \frac{n(\bar{E})}{n(S)} = \frac{n(S)}{n(S)}$$

$$P(E) + P(\bar{E}) = P(S) = 1$$

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Notes

$$\boxed{P(\overline{E}) = 1 - P(E)}$$

NOTE

1.) $P(S) = 1$

2.) $P(\phi) = 0$

3.) $0 \leq P(E) \leq 1$

4.) $P(E) + P(\overline{E}) = 1$

5.) If $A \subseteq B$ Then $P(A) \leq P(B)$

6.) If A, B are mutually exclusive Then $P(A \cap B) = 0$.

7.) If A, B are Independent $P(A \cap B) = P(A) \cdot P(B)$

8.) Addition Theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

(9) odds Infavour of event 'E' $= P(E) : P(\overline{E})$

10) odds against of event 'E' $= P(\overline{E}) : P(E)$

12) $P(\overline{A \cup B}) = 1 - P(A \cup B) = P(\overline{A} \cap \overline{B})$

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Notes

Q(12) Experiment : drawing 4 numbers from 14 numbers without replacement.

$\therefore n(S) = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001$

let E = product of 4 numbers to be positive

E = { All 4 +ve, All 4 -ve, (2 +ve × 2 -ve)}

$n(E) = {}^6C_4 + {}^8C_4 + ({}^6C_2 \times {}^8C_2)$
$= 505$

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{505}{1001}$

Q(6) Box 1 Box 2

Box 1	Box 2
3, 6, 9, 12, 15	6, 11, 16, 21, 26

Exp: Drawing a chip from each box & multiplying the numbers on them.

$n(S) = {}^5C_1 \times {}^5C_1 = 25$ ways

let E = products of numbers to be even
= { (even, even), (even, odd), (odd, even)}

$\bar{E}$ = { (odd, odd)}

$n(\bar{E}) = {}^3C_1 \times {}^2C_1 = 6$

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Notes

$\therefore P(E) = 1 - P(\bar{E})$

$= 1 - \frac{n(\bar{E})}{n(S)} = 1 - \frac{6}{25} = \frac{19}{25}$

Q(7) Exp:- drawing one ball from each container

$\therefore n(S) = {}^7C_1 \times {}^7C_1 = 49$

Let E = drawing 1 blue & 1 Red

$E = \{(\text{Red, blue})\}$

$n(E) = {}^4C_1 \times {}^3C_1 = 12$

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{12}{49}$

E_2 : 1 Red & 1 Green

$n(E_2) = {}^4C_1 \times {}^4C_1 = 16$

E_3 : 2 Green

$n(E_3) = {}^3C_1 \times {}^4C_1 = 12$

E_4 : 1 Blue & 1 Green

$n(E_4) = {}^3C_1 \times {}^3C_1 = 9$

Q(9) Exp: Selection of 4 students from 9 students

$n(S) = {}^9C_4 = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$

Let E = In a selection no. of Girls > no. of boys.

Notes

5	4
B	G
1	3
0	4

$E = \{(4G, 0B), (3G, 1B)\}$

$n(E) = {}^4C_4 \times {}^5C_0 + {}^4C_3 \times {}^5C_1$

$= 1 \times 1 + 4 \times 5$

$= 1 + 20$

$= 21$

$P(E) = \frac{n(E)}{n(S)} = \frac{21}{126} = \frac{1}{6}$

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Notes

Q(14)

125 bolts $\rightarrow \frac{1}{5} \times 125 = 25$ = no. of def bolts
200 nuts $\rightarrow \frac{3}{4} \times 200 = 150$ = no. of def nuts

175 = no of def items

Exp :- drawing an item from bin

$\therefore n(S) = {}^{325}C_1 = 325$

let E = drawn item is ~~either~~ defective (or) nut

$E = n \cup def$

$P(E) = P(n \cup def) = P(nut) + P(def\ item) - P(n \cap def)$

$= \frac{{}^{200}C_1}{325} + \frac{{}^{175}C_1}{325} - \frac{{}^{150}C_1}{325}$

$= \frac{225}{325}$

$= \frac{9}{13}$

(or)

Short cut

$P(E) = 1 - P(\text{non def bolts})$

$= 1 - \frac{100}{325} = \frac{225}{325}$

$= \frac{9}{13}$

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Notes

19 given X, Y are Ind events

$\Rightarrow \bar{X}, \bar{Y}$ are also Ind events

$P[X \cup \bar{Y}] = 0.7$

$P(X) = 0.4$

$P(X \cup Y) = ?$

$P(X \cap Y) = P(X) \cdot P(Y)$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$
$$= P(X) + P(Y) - P(X) \cdot P(Y)$$

$$P(X \cup \bar{Y}) = P(X) + P(\bar{Y}) - P(X \cap \bar{Y})$$
$$= P(X) + P(\bar{Y}) - P(X) P(\bar{Y})$$
$$= P(X) + P(\bar{Y})(1 - P(X))$$
$$0.7 = 0.4 + P(\bar{Y})(1 - 0.4)$$
$$0.3 = P(\bar{Y})(0.6)$$

$$P(\bar{Y}) = 0.5$$
$$\Rightarrow P(Y) = 1 - P(\bar{Y}) = 0.5$$

$$\therefore P(X \cup Y) = 0.4 + 0.5 - 0.4 \times 0.5$$
$$= 0.7$$

Q A mathematics problem is given to 3 students A, B, C in a class. And their respective probabilities of solving are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Then find the prob. of that the problem is solved.

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Notes

Sol^n $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

$P(A \cup B \cup C) \rightarrow$ problem is solved.
(atleast one of A, B, C solved the problem)

Given here A, B, C are independent events.

$$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$$
$$= 1 - P[\bar{A} \cap \bar{B} \cap \bar{C}]$$
$$= 1 - [P(\bar{A}).P(\bar{B}).P(\bar{C})]$$
$$= 1 - [(1 - P(A))(1 - P(B))(1 - P(C))]$$
$$= 1 - \left[\left(\frac{1}{2}\right).\left(\frac{2}{3}\right).\left(\frac{3}{4}\right)\right]$$
$$= \frac{3}{4}$$

(8) Exp: drawing 3 balls one after other without replacement.

let E: drawing 1 R & 2 black balls

$$E = \{(R, B, B), (B, R, B), (B, B, R)\}$$

$$P(E) = P(R, B, B) + P(B, R, B) + P(B, B, R)$$
$$= P(R)P(B)P(B) + P(B)P(R)P(B) + P(B)P(B)P(R)$$
$$= \left(\frac{{}^4C_1}{10} \times \frac{{}^6C_1}{9} \times \frac{{}^5C_1}{8}\right) + \left(\frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}\right) + \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right)$$

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Notes

$= 3\left[\frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}\right]$

$= \frac{1}{2}$

Q(15) $P(W) = 0.5 \Rightarrow P(L) = 1 - P(W) = 0.5$

given W, L are Ind. events

E = India's 2nd win in 3rd match.

(it's one after other case)

↳ (don't find sample space)

$E = \{(W, L, W), (L, W, W)\}$

2nd win 2nd win

$P(E) = P(W, L, W) + P(L, W, W)$

$= P(W).P(L).P(W) + P(L).P(W).P(W)$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$

$= \frac{1}{4}$

Q(24) let E = no. of required tosses is odd

= " " " " " to get the first head.

E = { H, TTH, TTTTH, ... }

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Notes

$P(E) = P(H) + P(TTH) + P(TTTTH) + \cdots$

$= \frac{1}{2} + \left(\frac{1}{2^3}\right) + \frac{1}{2^5} + \frac{1}{2^7} + \cdots$

$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots \infty\right]$ → GP

$= \frac{1}{2}\left[S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}}\right]$

$= \frac{2}{3}$

(27) $P(\text{win}) = p = 1/6$, $P(\text{loss}) = q = 5/6$

If A starts game

E = A wins game

$= \{A_W, A_L B_L A_W, A_L B_L A_L B_L A_W, \cdots\}$

$P(E) = P(\text{A win game})$

$P(E) = P(A_W) + P(A_L B_L A_W) + P(A_L B_L A_L B_L A_W) + \cdots \infty$

$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \cdots + \infty$

$= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots \infty\right)$

$= \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right]$

$= \frac{6}{11}$

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Notes

— Or —

Short cut

If n person playing in order.
Then ..

$$P(1^{st} \text{ person win}) = \frac{p}{1-q^n}$$

$$P(2^{nd} \text{ person win}) = \frac{pq}{1-q^n}$$

$$P(3^{rd} \text{ person win}) = \frac{pq^2}{1-q^n}$$

$$\vdots$$

$$P(n^{th} \text{ person win}) = \frac{pq^{n-1}}{1-q^n}$$

— (OR) —

$$P(A \text{ win}) = \frac{1/6}{1-\left(\frac{5}{6}\right)^2} \qquad \left(\because = \frac{p}{1-q^2}\right)$$

$$= \frac{6}{11}$$

$$P(B \text{ win}) = \frac{pq}{1-q^2} = \frac{\frac{1}{6}\times\frac{5}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{5}{11}$$

Q) If A, B, C are ~~three pl~~ playing in an order. ~~with a~~ And they are tossing a fair coin. So, what are the probabilities ~~that~~ of them (A, B, C) getting the head?

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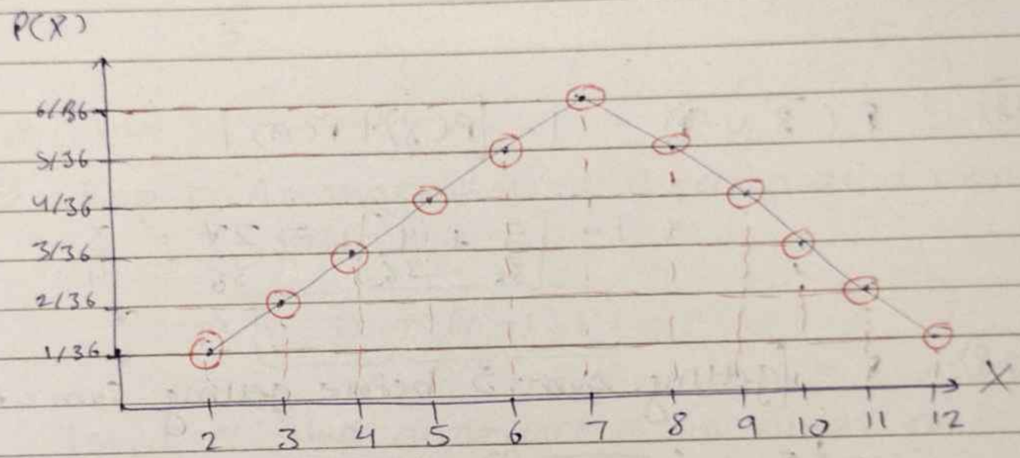
Notes

Exp: Throwing a dice 2 times

$\Rightarrow n(S) = 6 \times 6 = 36$



X = Sum of face values

$\Rightarrow$ Symmetric Prob distribution

Mean = Median = Mode

$\frac{\sum x_i}{n} = \frac{77}{11} = 7$ 7   7 (having highest Prob.)

Q③ $P[\text{Sum} > 8] = P(9) + P(10) + P(11) + P(12)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$

$= \frac{10}{36}$

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Notes

Q (17) $P(E) = P(4) + P(8) + P(6) + P(12)$

$= \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{1}{36} = \frac{14}{36} = \frac{7}{18}$

Q (18) $P(\overline{8 \cup 9}) = 1 - [P(8) + P(9)]$

$= 1 - \left[\frac{5}{36} + \frac{4}{36}\right] = \frac{27}{36} = \frac{3}{4}$

Q (28) E = {getting sum = 5 before getting sum = 7}

$= \{5, (\overline{5 \cup 7}, 5), (\overline{5 \cup 7}, \overline{5 \cup 7}, 5), \ldots\}$

$P(E) = P(5) + P(\overline{5 \cup 7}, 5) + P(\overline{5 \cup 7}, \overline{5 \cup 7}, 5) + \cdots \infty$

$$\boxed{\begin{aligned} P[\overline{5 \cup 7}] &= 1 - [P(5) + P(7)] \\ &= 1 - \left[\frac{4}{36} + \frac{6}{36}\right] \\ &= \frac{26}{36} \end{aligned}}$$

$P(E) = \frac{4}{36} + \left(\frac{26}{36} \times \frac{4}{36}\right) + \left(\left(\frac{26}{36}\right)^2 \times \frac{4}{36}\right) + \cdots \infty$

$= \frac{4}{36}\left[1 + \frac{26}{36} + \left(\frac{26}{36}\right)^2 + \cdots \infty\right]$

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Notes

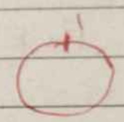
$$= \frac{4}{36} \left[S_\infty = \frac{1}{1-\frac{26}{36}} \right]$$

$$= \frac{2}{5}$$

⇒ ✮ (Prob on permutations)

Q ⑩ Exp :- Arrangement of n person at a round table

$$\Rightarrow \boxed{n(S) = (n-1)!}$$



$1 \times (n-1)!$

let E = ~~The~~ arrangements in which two specified persons don't sit together.

~~Let~~ $\bar{E}$ = The arrangements in which two specified persons sit together.

(AB = X)

$AB = X$

$n(\bar{E})$ = ~~The no. of~~ $(n-2)! \times 2!$ (arranging (n−1) persons)

AB

WKT $P(E) = 1 - P(\bar{E})$

$$= 1 - \frac{n(\bar{E})}{n(S)} = 1 - \frac{(n-2)! \times 2!}{(n-1)!}$$

$$= \frac{n-3}{n-1}$$

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Notes

Q (11) Exp: Arrangement of letters of the words

PROBABILITY $\therefore$ $\boxed{n(S) = \frac{11!}{2!\cdot 2!}}$

let E = 2 B's and also 2 I's occur together

$\boxed{n(E) = 9!}$

$P(E) = \frac{n(E)}{n(S)} = \frac{9!}{\frac{11!}{2!2!}} = \frac{9!2!2!}{11!} = \frac{2}{55}$

Q A, B are mutually exclusive such that $A \cup B = S$, then find max^m value of $P(A)\cdot P(B) = ?$

GATE

Sol^n given $A \cap B = \phi \Rightarrow P(A\cap B) = P(\phi) = 0$

& $A \cup B = S \Rightarrow P(A\cup B) = P(S)$ sample space

$P(A) + P(B) - P(A\cap B) = 1$

$\Rightarrow \boxed{P(A) + P(B) = 1} = k$

by shortcut from max & min

$P(A)\cdot P(B)$ is max^m when $P(A) = P(B) = \frac{k}{2} = \frac{1}{2}$

$\therefore P(A) P(B) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

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Conditional Probability

For two dependent events A, B.
$P(B/A)$ denotes the probability of 'B' when 'A' has already occured. It is called conditional Probability. read it as Probability of 'B' given 'A'.

If A, B are two events happening sequentially when $P(A)$ is given, Then the probability of B is denoted by $P(B/A)$ called conditional probability.

Multiplication Theorem

$$P(A \cap B) = P(A) \cdot P(B/A)$$
$$= P(B) \cdot P(A/B).$$

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Case : If A, B are Ind. events.
by x^n Thm. $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

$36 \rightarrow 30 + \boxed{6}$ equal

$30 \rightarrow \boxed{15} + \boxed{15}$

$I > II$ $I < II$

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Notes

Q(21) Exp :- Throwing a dice twice

$\Rightarrow n(S) = 6 \times 6 = 36$

$$P\left[\frac{\text{Sum} = 7}{I\text{ digit} > II\text{ digit}}\right] = \frac{P[\text{Sum} = 7 \times I\text{ digit} > II\text{ digit}]}{P(I\text{ digit} > II\text{ digit})}$$

$$\left(\because P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}\right)$$

$$= \frac{\frac{3}{36}}{\frac{15}{36}}$$

$$= \frac{1}{5}$$

Q(22) $$P\left[\frac{\text{Exactly 2H in 3 Toss}}{1^{st}\text{ Toss is H.}}\right]$$

$\{(H,H,T),(H,T,H)\}$

$$= \frac{P[\text{Exactly 2H in 3 Toss} \times 1^{st}\text{ Toss is H}]}{P[1^{st}\text{ Toss is H}]}$$

$$= \frac{P(HHT) + P(HTH)}{P(H) = \frac{1}{2}}$$

$$= \frac{\frac{1}{2^3} + \frac{1}{2^3}}{\frac{1}{2}} = \frac{1}{2}$$

Teacher's Signature

Notes

Q (23) $P[P_1] = 0.3$ $P[P_2] = 0.2$

$P\left[\frac{P_1}{P_2}\right] = 0.6$ $P(P_1 \cap P_2) = ?$

by x^n Thm

$P[P_1 \cap P_2] = P(P_1)\,P\left[\frac{P_2}{P_1}\right] = P(P_2)\,P\left(\frac{P_1}{P_2}\right)$

$= 0.2 \times 0.6$

$= 0.12$

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Notes

(25) Q

$P(P) = 1/4$

$P(P/Q) = 1/2$

$P(Q/P) = 1/3$

$P(\bar{P}/\bar{Q}) = ?$

by x^n Thm

$$P(P \cap Q) = P(P)\,P\left(\frac{Q}{P}\right) = P(Q)\cdot P\left(\frac{P}{Q}\right)$$

$$P(P \cap Q) = \frac{1}{4} \times \frac{1}{3} = P(Q)\cdot \frac{1}{2}$$

$$\boxed{P(Q) = 1/6}$$

$$P\left(\frac{\bar{P}}{\bar{Q}}\right) = \frac{P(\bar{P} \cap \bar{Q})}{P(\bar{Q})} = \frac{P(\overline{P \cup Q})}{1 - P(Q)}$$

$$= \frac{1 - P[P \cup Q]}{1 - P(Q)} = \frac{1 - [P(P) + P(Q) - P(P \cap Q)]}{1 - P(Q)}$$

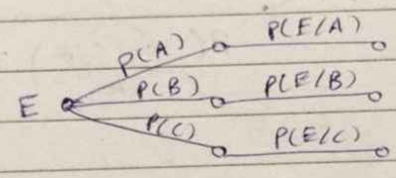
$$= \frac{4}{5}$$

Q (26) $\boxed{\text{HHTTTTTTTT}}$

(c) ✓

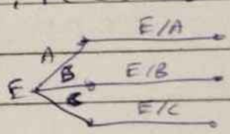
Total Probability: If 'E' be an event corresponds to mutually exclusive & exhaustive events A, B, C of an experiment

$$P(E) = P(A)\cdot P\left(\frac{E}{A}\right) + P(B)\cdot P\left(\frac{E}{B}\right) + P(C)\cdot P\left(\frac{E}{C}\right)$$



Baye's Theorem

P(E/A), P(E/B), P(E/C); P(A), P(B), P(C) are given then



conditional prob $P\left(\frac{A}{E}\right) = \frac{P(A\cap E)}{P(E)}$

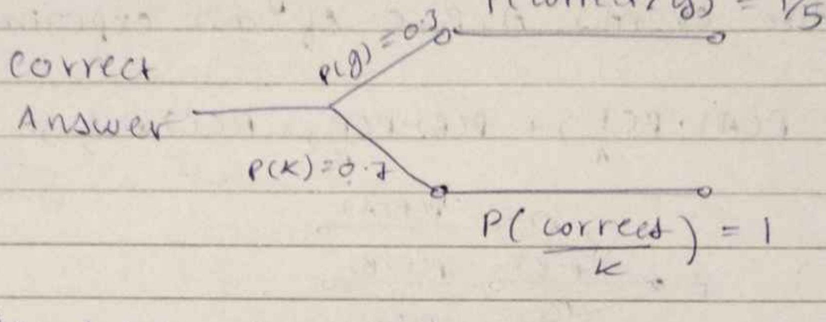
$$\Rightarrow \boxed{P(A/E) = \frac{P(A)\cdot P(E/A)}{P(E)}}$$

$$P(B/E) = \frac{P(B)\cdot P(E/B)}{P(E)}$$

$$P(C/E) = \frac{P(C)\cdot P(E/C)}{P(E)}$$

Notes

Q31) $P(K) = 0.7$, $P(G) = 0.3$



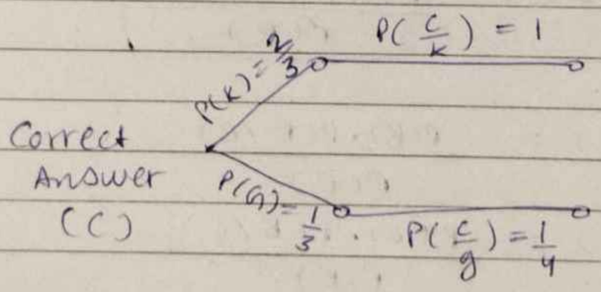
by total prob

$$P[\text{Correct Answer}] = P(g) \cdot P\left(\frac{c}{g}\right) + P(k) \cdot P\left(\frac{c}{k}\right)$$

$$= \left(0.3 \times \frac{1}{5}\right) + (0.7 \times 1)$$

$$= 0.76.$$

Q33) $P\left[\frac{k}{c}\right] = ?$



$$P\left[\frac{k}{c}\right] = \frac{P[k \cap c]}{P[C]} = \frac{P(k) \cdot P\left[\frac{c}{k}\right]}{P(k) \cdot P\left(\frac{c}{k}\right) + P(g) \cdot P\left(\frac{c}{g}\right)}$$

$$= \frac{\frac{2}{3} \times 1}{\left(\frac{2}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{1}{4}\right)} = \frac{8}{9}$$

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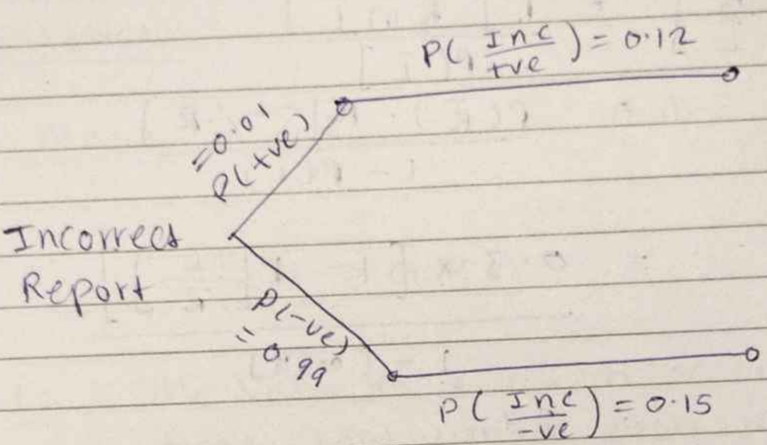
Making tree is imp. (bcoz it can be done in 2 ways)

Notes

Q 32



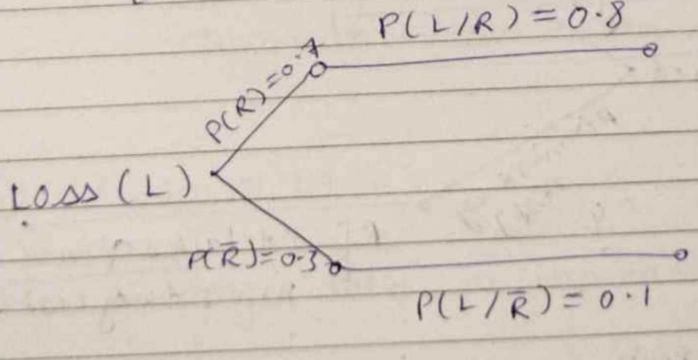
by total prob.

$$P[\text{Incorrect report}] = P(+ve)P\left[\frac{Inc}{+ve}\right] + P(-ve)P\left[\frac{Inc}{-ve}\right]$$

$$= 0.01 \times 0.12 + 0.99 \times 0.15$$

$$= 0.1497$$

Q 34 $P\left[\frac{\bar{R}}{L}\right] = ?$



by total Prob

$$P(L) = 0.7 \times 0.8 + 0.3 \times 0.1$$

$$= 0.59$$

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Notes

$$P\left[\frac{\bar{R}}{\bar{L}}\right] = \frac{P[\bar{R} \cap \bar{L}]}{P(\bar{L})}$$

$$= \frac{P(\bar{R}) \cdot P[\bar{L}/\bar{R}]}{1 - P(L)}$$

$$= \frac{0.3 \times \left[1 - P\left[\frac{L}{\bar{R}}\right]\right]}{1 - [0.59]}$$

$$= \frac{27}{41}$$

Q (30)

6 keys in Pocket → 2 Id keys; 4 non-Id keys

Id locks opened:

- $P[\text{Id key lost}] = 2/6$, $P\left[\frac{\text{Id locks opened}}{\text{Id key lost}}\right] = \frac{1}{5}$
- $P[\text{non-Id key lost}] = \frac{4}{6}$, $P\left[\frac{\text{Id locks opened}}{\text{non Id key lost}}\right] = \frac{2}{5}$

J.S. (imp)

<u>by total Prob</u>

$$P(E) = \frac{2}{6} \times \frac{1}{5} + \frac{4}{6} \times \frac{2}{5} = \frac{1}{3}$$

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Notes

Statistics

A. Mean $= \frac{\sum x_i}{n}$, G.M $= \sqrt[n]{x_1 x_2 x_3 \cdots x_n}$

$\boxed{A.M \geq G.M}$.

Mode : The value of x which occurs max^m times (or) whose prob is max^m

Median :- Arrange the data in either increasing (or) decreasing order

Then 1) If no. of observations is odd
Then Median = middle term $\left(\frac{n+1}{2}\right)^{th}$ term

2) If no. of observations is even
then $\boxed{\text{median} = \text{Avg of two midde terms}}$

Q 35 Mode = 17

Q 36 Arrange data in Increasing order

→ 32, 45, 49, 51, 53, 56, 60, 62, 66, 79

middle terms

median $= \frac{53+56}{2} = 54.5$

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Q(37) mean $= \frac{\sum x_i}{n}$

$$x = \frac{3 + x + 2 + 4}{4}$$

$\Rightarrow \boxed{x = 3}$.

$\therefore$ Mode = the value repeats max times
= 3.

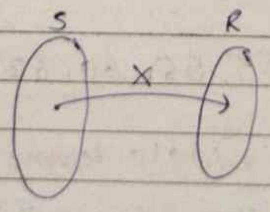
Random Variables

$X : S \to R$ number $\Rightarrow$ The value of X is a numb.
↓
outcomes

'X' is a numerical representation of outcomes of exp.

A real variable 'X' be associated with outcome of Random Experiment

$X : S \to R$



Notes

Random variable which takes a finite set of values are called <u>Discrete Random Variable (DRV)</u>

<u>Discrete Random variable</u>

① <u>Probability Mass Function</u>

$$P(X = X_k) = P_k$$

② $\sum_{i=0}^{n} P(X = X_i) = 1$

③ <u>Cumulative distribution fun</u>

$$F(X_k) = P(X \leq X_k) = \sum_{i=0}^{k} P(X = X_k)$$

④ Expectation of 'x' = Mean of Distribution

$$E(X) = \sum_{i=0}^{X} X_i P(X = X_i)$$

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Notes

Random variable which takes infinite set of values is called Continuous Random Variable.

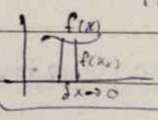
Continuous Random Variable $\Rightarrow$ Prob. mass f^n
$P(X = X_k) = 0$

① Probability Density function

$$f(x) = \begin{cases} f_1(x), & x \le a \\ f_2(x), & x > a \end{cases}$$

$$f(X_k) = \frac{P\left(X_k - \frac{\delta x}{2} < X_k < X_k + \frac{\delta x}{2}\right)}{\delta x}$$

$f(X_k)\,\delta x$ = area under the $f(x)$ in neighbourhood of X_k



② $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

③ Cumulative distribution func.

$$F(X_k) = P(X \le X_k) = \int_{-\infty}^{X_k} f(x)\,dx$$

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

④ Expectation of X = Mean of Distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \text{Moment } (1^{st})$$

⑤ $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx \quad (2^{nd} \text{ Moment})$$

⑥ r^{th} moment $= E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$

⑦ variance $V(X) = E(X^2) - (E(X))^2$

⑧ $$S.D = \sigma_X = \sqrt{var(X)} \ge 0$$

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Notes

expectation

<u>Properties</u> : ① $E(\text{const}) = \text{const}$

② $E(ax+b) = aE(X)+b$

③ $E(XY) = E(X)E(\frac{Y}{X}) = E(Y)E(\frac{X}{Y})$

④ $\text{var}(\text{const}) = 0$

⑤ $\text{var}(ax+b) = a^2\,\text{var}(X)$

⑥ $\sigma_x(ax+b) = a\sigma_x$

⑦ $\text{var}(ax+bY) = a^2\,\text{var}(X) + b^2\,\text{var}(Y) + 2ab\,\text{cov}(X,Y)$

$\text{cov}(X,Y) = \text{covariance}(X,Y)$
$= |E(X)E(Y) - E(XY)|$

$\boxed{\begin{array}{l}\text{If } X, Y \text{ are Ind. R.V : } \text{Cov}(X,Y)=0 \\ \therefore \text{var}[ax+by] = a^2\,\text{var}(X) + b^2\,\text{var}(Y)\end{array}}$

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Q) X, Y are two random variables such that $X+Y=2$ & $2X+Y=3$. Then $E(X)\cdot E(Y)=?$

Sol^n

$$X+Y=2$$
$$2X+Y=3$$

Solving we get $X=1, Y=1$

$$\therefore E(X)\cdot E(Y) = E(1)\cdot E(1)$$
$$= 1\times 1$$
$$= 1$$

Q) X, Y are two Ind. R.V with variances 1, 2 respectively.

Then $var[X-Y]=?$

Sol^n

$var(X)=1$, $var(Y)=2$

$$var(X-Y) = var[\overset{a}{1}\cdot X + \overset{b}{(-1)}Y]$$
$$= 1^2\, var(X) + (-1)^2 var(Y) + 2ab\, Cov(X,Y)$$
$$= 1(1) + 1(2) + 2(1)(-1)(0)$$
$$= 3$$

[$\because$ X, Y are Independent]

Q(39) $f(x) = e^{-x}$, $x>0$
$= 0$, else

p.d.f

Then 1) $P[x>1]$?
2) $P[|x-2|<1]$?

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$$\boxed{\begin{aligned}&\text{if } |x| < a \\ &\Rightarrow -a < x < a\end{aligned}}$$

Notes

1) $P(x>1) = \int_1^{\infty} f(x)\,dx = \int_1^{\infty} e^{-x}\,dx = \left.\frac{e^{-x}}{-1}\right]_1^{\infty} = \frac{1}{e}$

2) $|X-2| < 1$

$\Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

$\therefore P[\,|X-2| < 1\,] = P[\,1 < x < 3\,]$

$= \int_1^3 e^{-x}\,dx = \frac{1}{e} - \frac{1}{e^3}$

3) $P[\,|X-2| > 1\,] = 1 - P[\,|X-2| \leq 1\,]$

$= 1 - P[\,1 < x < 3\,]$

$= 1 - \int_1^3 e^{-x}\,dx$

$= 1 - \frac{1}{e} + \frac{1}{e^3}$

Q 42 $f(x) = k\,x^n e^{-x}$, $x \geq 0$ is a p.d.f

$k\ E(X) = 3$, $\therefore k, n = ?$

WKT : $f(x)$ is a valid p.d.f iff the area under $f(x) = 1$

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} k\,x^n e^{-x}\,dx = 1$

$\Rightarrow k \int_0^{\infty} e^{-x} x^{(n+1)-1}\,dx = 1$

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$\Rightarrow k\,\Gamma(n+1) = 1$

$$\Rightarrow \boxed{k = \frac{1}{\Gamma(n+1)} = \frac{1}{n!}} \quad —①$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = 3$$

$$= \int_0^{\infty} k e^{-x} x^{n+1}\,dx = 3$$

$$\Rightarrow k \int_0^{\infty} e^{-x} x^{(n+2)-1}\,dx = 3$$

$$\Rightarrow k\,\Gamma(n+2) = 3$$

$$\Rightarrow \boxed{k = \frac{3}{\Gamma(n+2)} = \frac{3}{(n+1)!}} \quad —②$$

solving ① & ②

$$\frac{1}{n!} = \frac{3}{(n+1)!}$$

$$\Rightarrow \boxed{n = 2} \quad \Rightarrow \boxed{k = \frac{1}{2}}$$

(c) ✓

Q(43) <u>WKT</u> f(x) is valid pdf
iff, area under the curve = 1

$\Rightarrow$ sum of areas of $\Delta^{les} = 1$

$$\Rightarrow \left(\frac{1}{2} \times 1 \times h\right) + \left(\frac{1}{2} \times 1 \times 2h\right) + \left(\frac{1}{2} \times 1 \times 3h\right) = 1$$

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Notes

$\Rightarrow \boxed{h = \frac{1}{3}}$

Q (48) ~~Exp~~ Exp: drawing a piece of paper

$\therefore \boxed{n(S) = {}^9C_1}$

Expectation of length of word drawn = ?

let define x = length of word drawn

DRV

X	3	4	5
$P(X)$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{3}{9}$

} Probability Distribution

i) $E(X) = \sum x_i p_i$

$= \left(3 \times \frac{4}{9}\right) + \left(4 \times \frac{2}{9}\right) + \left(5 \times \frac{3}{9}\right)$

$= 3.88$

ii) 2^{nd} moment of 'x' $= E(x^2) = \sum x_i^2 p_i$

$= \left(3^2 \times \frac{4}{9}\right) + \left(4^2 \times \frac{2}{9}\right) + \left(5^2 \times \frac{3}{9}\right)$

$= 15.88$

iii) variance $(x) = E(x^2) - (E(x))^2 = 15.88 - (3.88)^2$

$= 0.834$

iv) $\sigma_x = \sqrt{var(x)} = 0.913$

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Notes

(v) $E(2x+3) = 2E(x)+3 = 2(3.9)+3.$

vi) $V[2x+3] = 2^2 \text{var}(x) = 4 \times 0.884$

vii) $\sigma(2x+3) = 2\sigma_x = 2[0.913]$

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AGP

$1 + 2x + 3x^2 + 4x^3 + \cdots + n x^{n-1} = (1-x)^{-2}$

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Notes

Q Tossing a coin till we get 1st head with $P(H) = P$, find expectation of number of Tosses required?

Let X = no. of tosses required to get 1st head.

X	1	2	3	4	5	6	....
Possibilities	H	TH	TTH	TTTH	TTTTH	TTTTTH	
P(X)	P	$(1-P)P$	$(1-P)^2P$	$(1-P)^3P$	$(1-P)^4P$	$(1-P)^5P$	

$\therefore E(X) = \sum x_i p_i = P + 2(1-P)P + 3(1-P)^2P + 4(1-P)^3P + \cdots$

$= P\left[1 + 2(1-P) + 3(1-P)^2 + 4(1-P)^3 + \cdots\right]$

$= P\left[1 - (1-P)\right]^{-2}$

$= P\left[\frac{1}{P^2}\right] = \frac{1}{P}$

Q(46) Let X = marks of one question

X	1	$-1/4$
P(X)	$1/4$	$3/4$

$\therefore E[X] = \sum x_i P_i$

$= 1 \times \frac{1}{4} + \left(-\frac{1}{4}\right)\frac{3}{4} = \frac{1}{16}$

$\therefore E[\text{marks of 150 questions}] = 150 \times \frac{1}{16}$

$E[\text{marks of 1 student}] = 150 \times \frac{1}{16}$

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$\therefore E[\text{marks of 1000 Students}] = 1000 \times 150 \times \frac{1}{16}$

$= 9375$

Binomial Distribution $B(n, P)$: (with replacement)

n = no. of times experiment repeated
(or) no of trials
X = no. of success
P = Prob of Success
$q = 1 - P$ = Prob of failure

Then prob of r (no. of Success in n trials)

$$= \boxed{P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}}$$

& Mean = $\cancel{n}P$, variance = npq

When to apply Binomial D.?

In those experimens with only two possible outcome. One is success & other is failure.

eg. Toss a coin (H or T)
(Succ. or failu.)

Q) A fair coin is tossed 4 times successively. find the prob. of

1) P[Exactly 2 Heads]

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Notes

2) P[atmost 1 head]
3) P[atleast 1 head]
4) P[n(H) > n(T)]
5) P(n(H) − n(T) = 3)

Let x = no. of heads (success event)

$P(H) = p = \frac{1}{2} \Rightarrow q = \frac{1}{2}$ $\boxed{n=4}$

1) $P[x=2] = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

2) $P[x \le 1] = P(x=0) + P(x=1)$
$= {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 + {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3$

3) $P[x \ge 1] = 1 - P[x=0] = 1 - \left[{}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)\right]$

4)

H	T
4	0
3	1

$= P[x=4] + P[x=3] = {}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 + {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$

5) $P[\phi] = 0$

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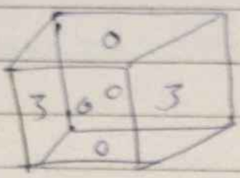
Q 53



$P(3) = \frac{2}{6}$, $P(0) = \frac{4}{6}$

$n = 5$ times

by B.D : Let x = no of success
x = no. of times '3' occurs

$\Rightarrow p \Rightarrow P(3) = 2/6$
$q = P(0) = 4/6$

Let E = (Sum = 12) in 5 throws
E = [3 occurs for 4 times in 5 throws].

$\therefore P(E) = P[X = 4] = {}^5C_4 (2/6)^4 (4/6)^1$

Q 52 $P[4^{th}$ head in 10^{th} Toss$] = P\left[\left(\text{Exactly } 3H \text{ in } 9 \text{ Tosses}\right) H\right]$ 10th Toss 4th head

$= P[3H$ in 9 Toss$] \times P(H)$
$= {}^9C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^6 \times \frac{1}{2}$

Q x follows B.D with mean '4' & variance '3'
find $P(X=0) = ?$

$$\boxed{P(X=0) = {}^nC_0 (P)^0 (q)^n}$$

Given that mean $= np = 4$
variance $= npq = 3$

Notes

$\Rightarrow q = \frac{3}{4} \Rightarrow p = \frac{1}{4} \Rightarrow n = 16$

$\therefore P(X=0) = {}^{16}C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16}$

$= \left(\frac{3}{4}\right)^{16}$

Poisson Distribution : $P(n, \lambda)$, $X \to$ DRV

If $n \to$ large & $P \to$ least so that

$\therefore \lambda = np =$ finite

Then $P[\,r = \text{no. of success}\,] = P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$

where $\boxed{\lambda = \text{mean} = \text{variance} = np}$

Q (56) let X = no. of Accidents in a month

and given that Avg no. of ACC in a month $\lambda = 5.2$

$\therefore P[X<2] = P[X=0] + P[X=1]$

$= \frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} = e^{-5.2}[1+5.2]$

Teacher's Signature

Notes

(least) (large)

Q57 $P = 0.001$ & $n = 2000$

$$\Rightarrow \boxed{\lambda = np = 2000 \times \frac{1}{1000} = 2}$$

$\Rightarrow$ x follows P.D

Let x = no. of people suffers a bad reaction

$$\Rightarrow P(x > 2) = 1 - P[X \leq 2]$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} + \frac{e^{-\lambda}\lambda^2}{\lfloor 2}\right]$$

$$= 1 - e^{-2}[5]$$

Q55 given $\lambda = 3$ per year

Let X = no. of events in 2 year duration

$\Rightarrow \boxed{\lambda \text{ for } 2 \text{ years} = 2 \times 3 = 6}$

$$P[X \leq 2] = P[X=0] + P(X=1) + P(X=2)$$

$$= \frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} + \frac{e^{-\lambda}\lambda^2}{\lfloor 2}$$

$$= e^{-6}[1 + 6 + 18] = 25 \times e^{-6}$$

Q X follows poission dis. with 2^{nd} moment '2'. then find: 1) var(X)

2) $E[X+2]^2$

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Notes

Sol^n $X \to P.D$

given, $E[X^2] = 2$, $V(X) = ?$

In PD
$E(X) = V(X)$
$= \lambda = np$

WKT $V(X) = E[X^2] - (E[X])^2$

$$\lambda = 2 - \lambda^2$$
$$\lambda^2 + \lambda - 2 = 0$$
$$\lambda = V(X) = -2, 1$$

not possible
$(\because V[X] > 0)$

(1) $\therefore \boxed{V(X) = E(X) = 1}$

(2) $E[X+2]^2 = E[X^2 + 4 + 4X]$
$= E(X^2) + E(4) + 4E(X)$
$= 2 + 4 + 4(1)$
$= 10$

Q X follows P.D with Rel^n $P(X=2) = \frac{2}{3} P(X=1)$
$P(X=0) = ?$

Sol^n:
X follows P.D $\Rightarrow P(X=0) = \frac{e^{-\lambda}\lambda^0}{\lfloor 0} = e^{-\lambda}$

From give eq^n in Q,

$$\frac{\cancel{e^{-\lambda}} \cdot \lambda^2}{\lfloor 2} = \frac{2}{3} \frac{\cancel{e^{-\lambda}} \lambda^1}{\lfloor 1}$$

$$\lambda^2 - \frac{4}{3}\lambda = 0$$

Teacher's Signature

Notes

$\therefore \lambda = 0, 4/3$

$\lambda = V(X) = 0$ only if X is const

$\therefore$ here $\lambda = V(X) \neq 0 \Rightarrow \lambda = 4/3$

$\therefore \boxed{P(X=0) = e^{-4/3}}$

Normal Distribution : $N(\mu, \sigma^2)$

if $n \to \infty$ Binomial Dist then Normal Dist

$n \to \infty$, X is C.R.V with given μ & σ^2 having p.d.f, $F(X) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$ is called Normal Distribution.

let $z = \frac{X-\mu}{\sigma}$ whose mean $E(Z) = 0$ & $V(Z) = 1$

$\sigma_z = 1$

fixed for any X, μ & σ

$\therefore$ Z is called Standard normal variable.

having P.D.F $\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz \cong e^{-x^2}$

Sym at $x=0$

$\Rightarrow$ Sym at $z=0$

Then C.D.F $F_Z(z_k) = P(Z \leq z_k)$

$$= \int_{-\infty}^{z_k} \phi(z)\, dz$$

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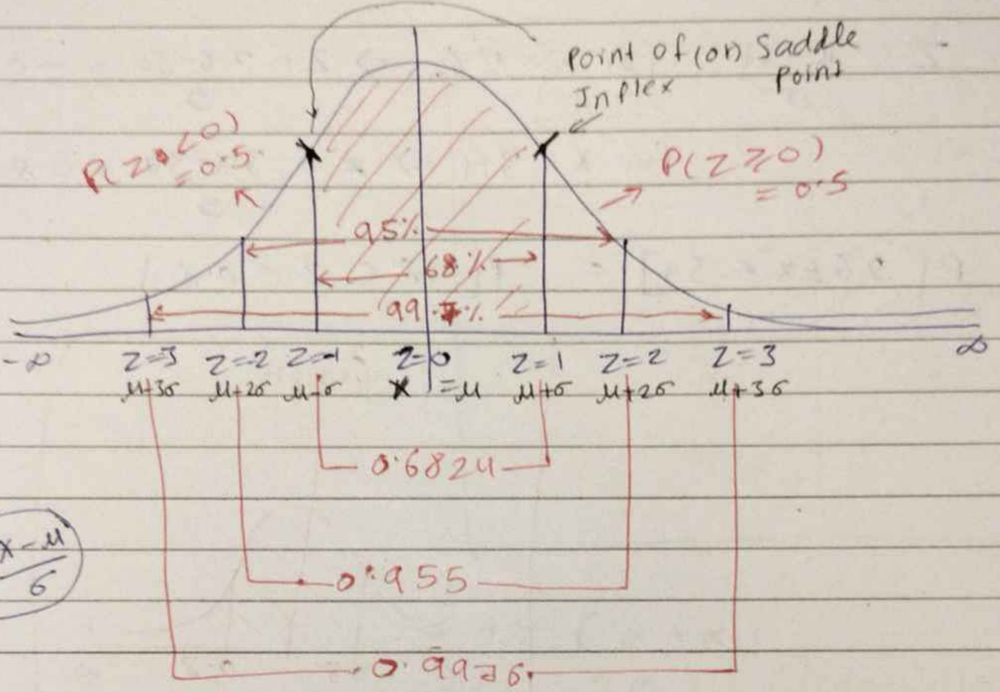
Notes

⇒ Standard Normal Curve $N(0,1)$



$z = \frac{x-\mu}{\sigma}$

$\int_{-\infty}^{\infty} \phi(z)\,dz = 1$

⇒ area under std normal curve = 1

⇒ It is sym.
Prob. Dist
⇓
mean = median = mode
($z = 0$)

Q 61 X is NRV with $\mu = 30$ & $\sigma = 5$
$P[26 \le x \le 34] = ?$

(Prob is given by area under stand. normal curve)

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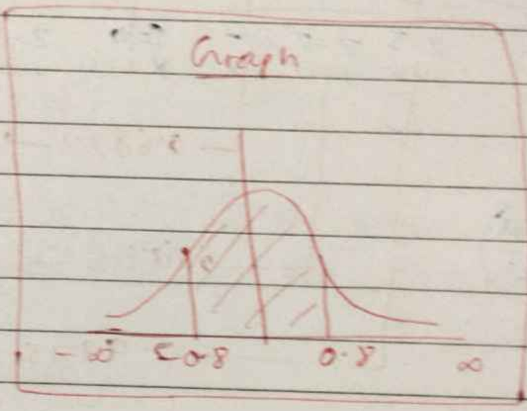
Notes

converting into std normal variable by

$Z = \frac{x-\mu}{\sigma}$, for $x = 26 \Rightarrow Z = \frac{26-30}{5} = -0.8$

for $x = 34 \Rightarrow z = \frac{34-30}{5} = 0.8$

$\therefore P[26 \le x \le 34] = P[-0.8 < Z \le 0.8]$



by sym. $= 2[P(0 \le Z \le 0.8)$

$= 2[A(Z = 0.8)]$

$= 2[0.2881]$ //(given)

$= 0.5762$

Q (58) $\mu = 100$, $P(X \ge 110) = \alpha$

$P[90 \le X \le 110] = ?$

for $x = 90 \Rightarrow Z = \frac{90-100}{\sigma} = \frac{-10}{\sigma}$

$X = 110 \Rightarrow Z = \frac{110-100}{\sigma} = \frac{10}{\sigma}$

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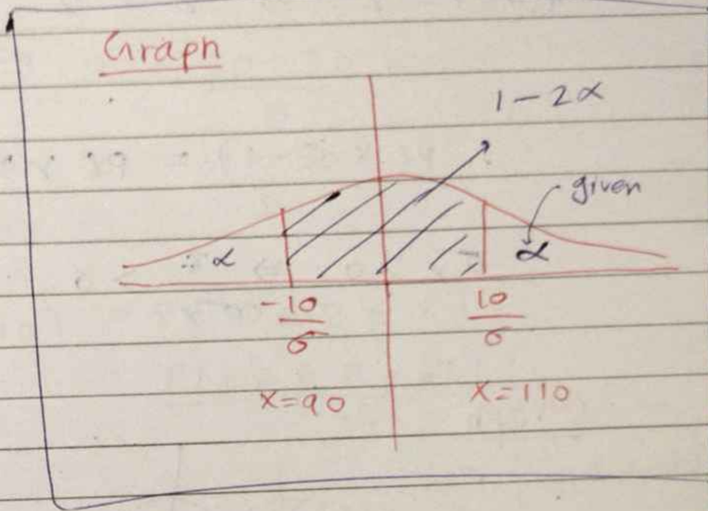
Notes

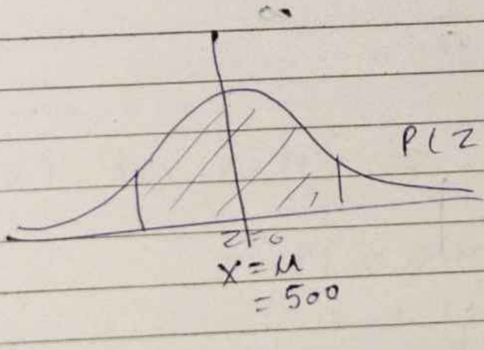
$$P[90 \le X \le 110] = P\left[\frac{-10}{\sigma} \le Z \le \frac{10}{\sigma}\right]$$

Graph



$$= 1 - [\alpha + \alpha]$$
$$= 1 - 2\alpha \quad \text{(using graph)}$$

(65)



$P(Z \ge 0) = 0.5$ $\therefore 50\%$

Q) Let X be a R.V following N.D with $\mu = 1$, $\sigma^2 = 4$ & Y be another N.R.V with $\mu = -1$, σ_Y^2 if $\boxed{P(X \le -1) = P(Y \ge 2)}$, $\sigma_Y = ?$

convert into 'Z'

Sol^n

Notes

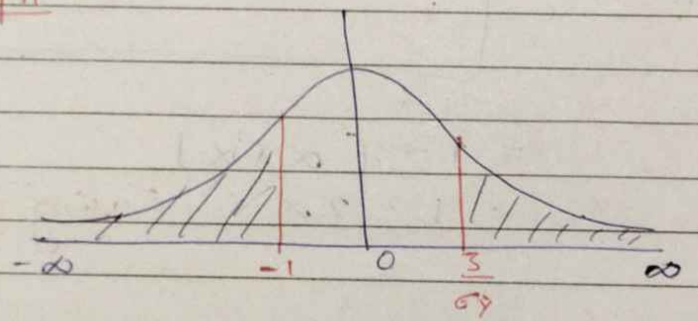
for $X = -1 \Rightarrow Z = \frac{-1-1}{2} = -1$

for $Y = 2 \Rightarrow Z = \frac{2-(-1)}{\sigma_Y} = \frac{3}{\sigma_Y}$

$\therefore P(X \le -1) = P(Y \geq 2) \Rightarrow P[Z \le -1] = P[Z \geq \frac{3}{\sigma_Y}]$

$\because \sigma_Y > 0 \Rightarrow \frac{3}{\sigma_Y} > 0$

Graph



due to symmetry.

$$|-1| = \left|\frac{3}{\sigma_Y}\right|$$

$$\Rightarrow \boxed{\sigma_Y = 3}$$

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25 11 21

Notes

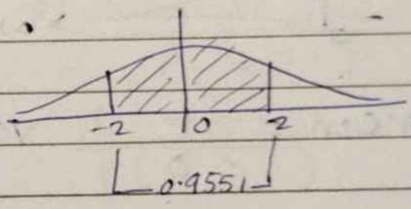
Q) X is NRV with $\mu = 30$ & $\sigma = 5$ find $P[30 \le x \le 40]$ & $P[X \ge 45]$

soln 1) for $X = 30 \Rightarrow Z = \frac{30-30}{5} = 0$

$X = 40 \Rightarrow Z = \frac{40-30}{5} = 2$

$$P[30 \le X \le 40] = P[0 \le Z \le 2]$$
$$= \frac{P[-2 \le Z \le 2]}{2}$$

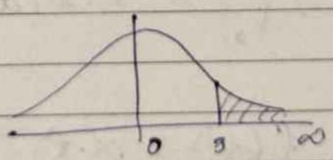
due to Symmetry



$$= \frac{0.9551}{2}$$

2) for $X = 45 \Rightarrow Z = \frac{45-30}{5} = 3$

$$\therefore P[X \ge 45] = P[Z \ge 3]$$



$$= P[Z \ge 0] - P[0 < Z < 3]$$
$$= 0.5 - \frac{P[-3 < Z \le 3]}{2}$$
$$= 0.5 - \frac{0.9976}{2}$$

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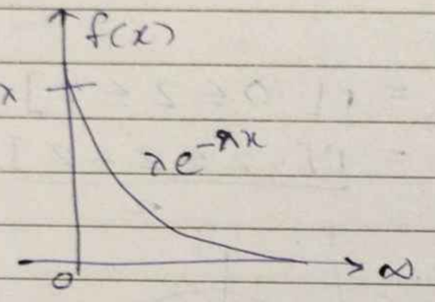
Notes

Exponential Distribution : $E(\lambda)$

$\hookrightarrow$ Uniparameter dist

X is CRV, such that

P.d.f $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$ is exp dist.



mean $= \frac{1}{\lambda}$, variance $= \frac{1}{\lambda^2}$

$$\boxed{\text{Mean} = S \cdot D(\sigma) = \frac{1}{\lambda}}$$

r^{th} moment

$$E(x^r) = \int_0^\infty x^r f(x)\, dx$$

Q66 given Pdf $f(t) = \alpha \cdot e^{-\alpha t}$, $t \geq 0$ (exp. Dist)

$$P[100 \leq t \leq 200] = \int_{100}^{200} f(t)\, dt = \int_{100}^{200} \alpha e^{-\alpha t}\, dt$$

$$= \left. \frac{e^{-\alpha t}}{-1} \right]_{100}^{200}$$

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Notes

$$= e^{-100\alpha} - e^{-200\alpha}$$

Q(67) given 'z' is exponential R.V follows exp distribu with ~~parameter~~ λ $\boxed{mean\left(\frac{1}{\lambda}\right) = 1} \Rightarrow \boxed{\lambda = 1}$

$$\Rightarrow \text{p.d.f} = f(z) = \begin{cases} \lambda e^{-\lambda z} & , z \geq 0 \\ 0 & , z < 0 \end{cases}$$

$$\Rightarrow f(z) = \begin{cases} e^{-z} & , z \geq 0 \\ 0 & , z < 0 \end{cases}$$

Then $P\left[\frac{z \geq 2}{z > 1}\right] = \frac{P[z > 1 \;\&\; z > 2]}{P(z > 1)}$

$$= \frac{P[z > 2]}{P[z > 1]}$$

$$= \frac{\int_2^{\infty} e^{-z}\,dz}{\int_1^{\infty} e^{-z}\,dz}$$

$$= \frac{\left[e^{-z}/-1\right]_2^{\infty}}{\left[e^{-z}/-1\right]_1^{\infty}} = \frac{1/e^2}{1/e} = \frac{1}{e}$$

Q(68) given $\lambda = 360$ per hour

$$\lambda \text{ per second} = \frac{360}{60 \times 60} = \frac{1}{10} = 0.1$$

as arrival time follows a CRV with poission rate λ, we use exponential dist

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Any Poiss... ↳ Discrete

use ~~...~~ Poission Dist

Notes | Any Poission process follows continuous RV
we use exponential Dist

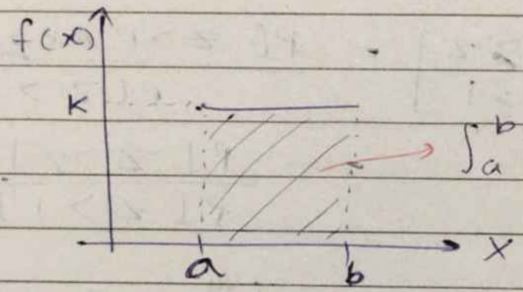
Date / /
Page No.

$$P[6 \le t \le 10] = \int_6^{10} f(t)\,dt = \int_6^{10} (0.1)e^{-(0.1)t}\,dt$$

Uniform Distribution $U[a,b]$

X is CRV follows uniform Distr. in $[a,b]$ having p.d.f

$$f(x) = \begin{cases} k = \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{else} \end{cases}$$



$$\int_a^b f(x)\,dx = \int_a^b k\,dx = 1$$

$$= k[x]_a^b = 1$$

$$\Rightarrow \boxed{k = \frac{1}{b-a}}$$

Mean $= \dfrac{a+b}{2}$, $\underset{\text{(variance)}}{V(x)} = \dfrac{(b-a)^2}{12}$

$$E[X^r] = \int_a^b X^r f(x)\,dx = \int_a^b x^r \cdot \frac{1}{b-a}\,dx$$

Q X follows U.D over $[0,1]$, Then 1) $E(x)$
2) $V(x)$
3) $E[x^3]$

↓ also k/a rectangular distribution

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Notes

Solⁿ $X : [a, b] = [0, 1]$ follows U.D
with p.d.f

$$f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{1-0} = 1 & , 0 \le x \le 1 \\ 0 & , \text{else} \end{cases}$$

1) $E(X) = \text{mean} = \frac{a+b}{2} = \frac{1}{2}$

2) $V(X) = \frac{(b-a)^2}{12} = \frac{1}{12}$

3) $E[X^3] = \int_a^b x^3 f(x)\,dx = \int_0^1 x^3 \frac{1}{1-0}\,dx = \frac{1}{4}$

Q 70

0 ———— 1 mt

Expected length of shorter stick = ?

let X = length of the shorter stick

$\Rightarrow X < 1/2$ & $X > 0$

$\Rightarrow X$ follows U.D over $(0, 1/2)$

$\therefore \text{Mean} = E(X) = \frac{a+b}{2} = \frac{0 + 1/2}{2} = \frac{1}{4}$

Q 69 X, Y follows uniform distr & independent over the interval $[-1, 1]$.

$\Rightarrow f(x) = \frac{1}{1-(-1)} = \frac{1}{2}$, $f(y) = \frac{1}{1-(-1)} = \frac{1}{2}$

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Notes

$$P[\max(X,Y) < \tfrac{1}{2}] = P[X < \tfrac{1}{2} \;\&\; Y < \tfrac{1}{2}]$$

$$= P[X < \tfrac{1}{2}] \times P(Y < \tfrac{1}{2}) \quad (\because X, Y \text{ are Ind. R.V})$$

$$= \int_{-1}^{1/2} f(x)\,dx \times \int_{-1}^{1/2} f(y)\,dy$$

$$= \frac{1}{2}[x]_{-1}^{1/2} * \frac{1}{2}[x]_{-1}^{1/2}$$

$$= \frac{9}{16}$$

Q (71) let X = arrival time at J^n in [0,5] cycle

Exp of waiting time (t) = ?

2 min – G
3 min – R

uniformly Distributed in [0,5]

let 1^{st} 2 min → Green
next 3 min → Red

$$\text{waiting time } (t) = \begin{cases} 0, & 0 \le x < 2 \\ 5 - x, & 2 \le x \le 5 \end{cases}$$

(t)

$$\therefore E(t) = \int_0^5 t \cdot f(x)\,dx$$

$$= \int_0^2 0 \cdot f(t)\,dt + \int_2^5 (5-x) f(x)\,dx$$

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bit.ly/C8wbSoln → Prob. Ch. Soln. (other workbook) along with DM

$f(x) = \frac{1}{b-a}$

Notes

$$= 0 + \int_2^5 (5-x)\frac{1}{5}\,dx$$

$$= \left[x - \frac{x^2}{10}\right]_2^5$$

$$= \frac{9}{10}$$

$$= 0.9$$

Q 2020 X follows U.D over $[-2, 10]$ → $f(x) = \frac{1}{10-(-2)} = \frac{1}{12}$

for $Y = 2X - 6$, ~~the prob. for $Y \le 7$~~ $P\left[\frac{Y \le 7}{X \ge 5}\right] = ?$

$$P\left[\frac{Y \le 7}{X \ge 5}\right] = \frac{P[X \ge 5 \ \&\ Y \le 7]}{P[X \ge 5]}$$

$$\left[\begin{array}{l} \text{for } Y = 2X - 6 \\ X = \frac{Y+6}{2} \\ \text{for } Y = 7,\ X = \frac{7+6}{2} = 6.5 \end{array}\right]$$

$$= \frac{P[X \ge 5 \ \&\ X \le 6.5]}{P[X \ge 5]}$$

$$= \frac{\int_5^{6.5} f(x)\,dx}{\int_5^{10} f(x)\,dx} = \frac{\frac{1}{12}(x)_5^{6.5}}{\frac{1}{12}(x)_5^{10}} = 0.3$$

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CH 1 Linear Algebra

→ **Array**

$A = [a_{ij}]_{m \times n}$

m = # rows, n = # columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Elements above the diagonal: $i<j$

Elements below the diagonal: $i>j$

Diagonal elements: $i=j$ (Principal diagonals)

$a_{ij} = 0, \forall i<j \Rightarrow$ LTM

$a_{ij} = 0, \forall i \neq j \Rightarrow$ Diagonal Matrix

$a_{ij} = 0, \forall i>j \Rightarrow$ UTM

→ $AB \neq BA$ (C ✗)

$A[BC] = [AB]C$ (A ✓)

→ $A_{m\times n}, B_{p\times q}, n = p$

$(AB)_{m\times q}$

→ # scalar multiplications $= m \times p \times q$

→ # additions required $= m \times (p-1) \times q$

→ **Transpose of a Matrix**

$A = [a_{ij}]_{m\times n}$

$A^T = [a_{ji}]_{n\times m}$

$(A^T)^T = A$

$(AB)^T = B^T A^T$

$(A^T)^n = (A^n)^T$

→ **Trace of Matrix**

= Sum of principle diagonal elements

$$\boxed{Tr(A_{n\times n}) = \sum_{i=1}^{n} a_{ii}}$$

$Tr(A) = Tr(A^T)$

$Tr(A+B) = Tr(A) + Tr(B)$

$Tr(KA) = K\,Tr(A)$

→ **Symmetric Matrix**

$A = [a_{ij}]_{n\times n}$

is symmetric matrix

iff $\boxed{A^T = A}$ or $\boxed{a_{ij} = a_{ji}}$

→ **Skew Symmetric Matrix**

$$\boxed{A^T = -A}$$

$$\boxed{\begin{array}{l} a_{ij} = 0 \quad , \forall i = j \\ \& \ a_{ij} = -a_{ji} \ , \forall i \neq j \end{array}}$$

$$\begin{bmatrix} \diagdown & l \\ u & \diagdown \end{bmatrix} = \begin{bmatrix} \diagdown & -u \\ -l & \diagdown \end{bmatrix}$$

eg $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

→ Complex Matrix

$A = [a_{ij}]_{m \times n}$

↳ complex no. $(z = x + iy)$

→ Conjugate of Matrix ($\bar{A}$) (replace i & $-i$)

eg $A = \begin{bmatrix} i & 1+i \\ 2 & 2i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -i & 1-i \\ 2 & -2i \end{bmatrix}$

→ Hermitian Matrix

$\boxed{\bar{A} = A^T}$

$\boxed{\begin{bmatrix} \diagdown & \bar{u} \\ \bar{l} & \diagdown \end{bmatrix} = \begin{bmatrix} \diagdown & l \\ u & \diagdown \end{bmatrix}}$

$\boxed{\begin{matrix} a_{ii} = \text{real} \\ \& \ \overline{a_{ij}} = a_{ji} \end{matrix}}$

↳ A^θ (Tranjugate of Matrix)

↳ (Transpose & conjugate)

$(\bar{A})^T = (A^T)^T = A$

$\boxed{A^\theta = A}$

if A is HM

→ Skew-Hermition Matrix

$\boxed{\bar{A} = -A^T}$

$\boxed{\begin{matrix} a_{ii} = \text{Purely Imagionary or zero} \\ \& \ \overline{a_{ij}} = -a_{ji} \end{matrix}}$

$\boxed{A^\theta = -A}$

$(\bar{A})^T = (-A^T)^T$

→ orthogonal Matrix

$\boxed{A^{-1} = A^T}$

$A^T \cdot A = A \cdot A^T = I$

Vectors in orthogonal matrix are orthogonal to each other.

(ie. dot product $= 0$)

eg $A = [x_1 \ x_2 \ x_3]_{3\times3}$ is orthogonal

$\Rightarrow \vec{x_1} \cdot \vec{x_2} = \vec{x_2} \cdot \vec{x_3} = \vec{x_1} \cdot \vec{x_3} = 0$

→ Unitary Matrix

$\boxed{A^{-1} = A^\theta}$

$A \cdot A^\theta = A^\theta \cdot A = I$

If A, B are unitary/orthogonal matrices then AB, BA, A^{-1}, B^{-1}, A^T, B^T are unitary / orthogonal matrices

→ Nilpotent Matrix

A is nilpotent matrix of degree/order 'm'

$\Rightarrow \boxed{A^m = O_{n\times n}}, m \in Z^+$

→ Involutary Matrix

$\boxed{A^{-1} = A}$ $\boxed{A^2 = I}$

ie $\begin{pmatrix} A^{2n} = I \\ A^{2n-1} = A \end{pmatrix}$

→ Idempotent Matrix

$\boxed{A^2 = A}$

★★★

- $A^2 = A \Rightarrow A^n = A$
- $AB = B,\ BA = A \Rightarrow A^2 = A, B^2 = B$
- $AB = A,\ BA = B \Rightarrow A^2 = A, B^2 = B$

→ Determinant

$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} \\ c & d \end{vmatrix} = ad - bc$

$$|A_{n\times n}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\cdot \delta_{ij}$$

where δ_{ij} = minor of an elem a_{ij}

eg $\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{vmatrix}$

$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

→ Prop. of Determinand

- $|A| = |A^T|$
- $|AB| = |A|\cdot|B|$
- $|A^n| = |A|^n$
- $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- $|k\cdot A_{n\times n}| = k^n|A|$
- $|Adj\ A_n| = |A|^{n-1}$
- $Adj(Adj\ A) = |A|^{n-2}\cdot A$
- $|Adj(Adj\ A)| = |A|^{(n-1)^2}$
- $\begin{vmatrix} a_1+k_1 & b_1 \\ a_2+k_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$
- ★ Det. of LTM, UTM, DM = Product of diagonal elems.
- Det. of Skew-Symm. matrix of odd order = 0
- Det. of orthogonal/unitary matrix = ±1
- Det. of Involutary matrix = ±1
- Det. of Idempotent matrix = 1 or 0
- If two rows or colms are same $\Rightarrow |A| = 0$

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→ Linear Dependency &
Linear Independency

- If two vectors only, check by ratio
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_2} \Rightarrow LD$$
- more then two vectors, check by determinant

all minors zero ⇒ LD

* (atleast one minor non-zero ⇒ LI)

- two rows or colms are equal or dependent ⇒ LD

In $A_{3\times3}$, if $|A| = 0$
⇒ LD, Singular matrix, non-zero Sol^n, non-trivial Sol^n

if $|A| \neq 0$
⇒ LI, non-singular matrix, zero sol^n, Trivial sol^n, unique sol^n

→ Elementary Transformation

$$|A| \underset{ET}{\cong} |B|$$

Rules

1. $R_i \leftrightarrow R_j \Rightarrow |B| = -|A|$
2. $R_i \to kR_1 \Rightarrow |B| = k\cdot|A|$
3. $R_i \to R_i + kR_j \Rightarrow |B| = |A|$

* Make max. no. of zeros before calc. of determinant
Select row/colm with max # of zeros to find det.

* If atleast one block is zero

eg
$$A = \left|\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 6 & 7 & 4 & 3 \\ 8 & 9 & 2 & 1 \end{array}\right|$$
(D_1 = upper-left block, D_2 = lower-right block)

$\Rightarrow |A| = |D_1| * |D_2|$

→ Shortcuts

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd * \left[1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

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Notes

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-b)(c-a)(b-a)$$

vander mode determinent

→ Inverse of a matrix

$$\boxed{A^{-1} = \frac{\text{Adjoint } A}{|A|}}, \ |A| \neq 0$$

$|A| = 0 \Rightarrow A^{-1}$ doesn't exist

$\text{adj } A = (\text{cofactor } M.)^T$

$= \left[(-1)^{i+j} . \delta_{ij}\right]^T$

→ Rank of Matrix ($\rho(A)$)

↳ #f independent rows or colms in matrix

or

Highest order of non-zero minor in a matrix A.

☆☆☆

$$\boxed{\begin{aligned} |A_{n\times n}| \neq 0 &\Rightarrow \rho(A) = n \\ |A_{n\times n}| = 0 &\Rightarrow \rho(A) < n \end{aligned}}$$

→ Echelon Form

$A_{m\times n} \underset{RT}{\cong} U$

↳ only Row Transformations

reduce A in such matrix in which #f 0s before first non-zero elem. should increase from first row to last row.

eg $U = \begin{bmatrix} \circ & - & - \\ \circ\circ & & \\ \circ\circ\circ & & \end{bmatrix}$

$\rho(A) = $ #f non-zero rows in U

→ Properties of $\rho(A)$

- $\rho(A) = \rho(A^T)$
- $\rho(AA^T) = \rho(A)$
- $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
- $\rho(A+B) \leq \rho(A) + \rho(B)$
- $\rho(A-B) \geq \rho(A) - \rho(B)$
- $\rho(A_{m\times n}) \leq \min\{m, n\}$
- $\rho(I_n) = n$
- $\rho(O_{n\times n}) = 0$
- $\rho\begin{pmatrix}\text{Diagonal} \\ \text{matrix}\end{pmatrix} = $ #f non-zero elem in diagonal

→ every matrix can be rep by sum of symmetric and skew-symmetric matrices

$$A_n = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T]$$

→ Every matrix can be represented as sum of Hermition & Skew H. m.

$$A_n = \frac{1}{2}[A + A^\theta] + \frac{1}{2}[A - A^\theta]$$

→ if A is HM
(iA) is skew HM.

→ LU Decomposition

every square matrix $A_{n\times n}$ can be written as product of UTM × LTM where either $l_{ii} = 1$ or $u_{ii} = 1$, $\forall i$

$$\boxed{A_{n\times n} = L_{n\times n} \cdot U_{n\times n}}$$

→ Linear System

$a_1 x + b_1 y = c_1$
$a_2 x + b_2 y = c_2$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\boxed{A \cdot X = B}$$

$\boxed{AX = 0}$ Homogeneous System

$\boxed{AX = B}$ Non-Homogeneous System

$AX = 0$ Homog. Sys.

Unique sol^n	∞^{ly} many sol^n
$\lvert A\rvert \neq 0$	$\lvert A\rvert = 0$

$AX = B$ Non-Homog. Sys

unique Sol^n	∞^{ly} many Sol^n	No Sol^n
$\lvert A\rvert \neq 0$	$\lvert A\rvert = 0$	$\lvert A\rvert = 0$
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Consistent — atleast one sol^n

Inconsistent

$|A| \neq 0 \Rightarrow$ unique sol^n

→ Solving $AX = B$

↳ $C = [A : B]$ Augmented Matrix
↓ reduce into
Echelon Form
& get $\rho(A)$ & $\rho(C)$

↳ $\rho(A) < \rho(C)$
⇒ Inconsistent
(No sol^n)

↳ $\rho(A) = \rho(C) <$ # of unknowns
$\Rightarrow \infty$'ly many Sol^n

↳ $\rho(A) = \rho(C) =$ # of unks
$\Rightarrow$ unique Sol^n

→ Solving $AX = 0$

$A \to$ echelon form
↳ $\rho(A) =$ # of unks $(|A| \neq 0)$
$\Rightarrow$ unique sol^n
↳ $\rho(A) <$ # unk $(|A| = 0)$
$\Rightarrow \infty$'ly many sol^n

→ Nullity of $(AX = 0)$

= dimension of sol^n space (or Null space) of $(AX = 0)$
= # of unks $- \rho(A)$
(ie # of colms of A)

→ Basis of Vector Space

let $S \subseteq V$, is basis of V iff,
1) all vectors of S are LI
2) Linear comb of S generates all vectors in V.

→ Eigen value & Eigen vectors

$$\boxed{(A - \lambda I)X = 0}$$

$\lambda \to$ eigen values
$X \to$ eigen vectors

to find λ (eigen value)
$|A - \lambda I| = 0$
↳ characteristic eq. of A
solve & get λ

to find $(X \neq 0)$ (eigen vectors)
put λ values in char eq^n & get equations
Solving Indep. eq^n we get eigen vectors

→ General Char eq^n

$$|A_{n\times n} - \lambda I| = 0$$

$$(-1)^n \lambda^n + (-1)^{n-1} \cdot Tr(A) \cdot \lambda^{n-1} + \cdots + |A| = 0$$

Coeff. of highest degree term $= (-1)^n$

Constant term $= |A|$

→ **Geometric Multiplicity of λ**

= #f Indep eigen vectors for λ
= #f unks − $\rho(A-\lambda I)$

eg) $A_{n\times n}$ we get n unique λ
⇒ n vectors (Indep.)
⇒ Diagonalizable

→ **Props. of Eigen Values**

1.) $\lambda_1+\lambda_2+\cdots+\lambda_n = a_{11}+a_{22}+\cdots+a_{nn}$
2.) $\lambda_1\cdot\lambda_2\cdot\lambda_3\cdots\lambda_n = |A_{n\times n}|$
3.) A & A^T same eigen values
4.) Eigen values of LTM/UTM/DM are its principle diagonal elements.
5.) $\rho(A)$ = #f non-zero eigen values
6) If λ is eigen value of A
⇒
① A^n eigen value is λ^n
② KA " " " $K\cdot\lambda$
③ A+KI " " " $(\lambda+K)$
7) In any row/col^m all non-diagonal elems are zero ⇒ diagonal elem is eigen value
8.) If Colm sum or Row sum is same for each colm or Row ⇒ sum is eigen value

→ **Properties of Eigen vectors**

1) GM of e.value λ
= #f e.vectors of λ that are independent
= #unk − $\rho(A-\lambda I)$
2.) #f non-repeated eigen values = #f Indep eig. vectors
3.) #f distinct λ = #f Indep. eig. vectors
4.) eigen vectors of Symm matrix are orthogonal to each other ($\bar{X}\cdot\bar{Y} = X^T\cdot Y = 0$)
5) norm of vector 'X' ($\|X\|$)
$= \sqrt{X^T\cdot X} = \sqrt{\bar{X}\cdot\bar{X}}$
= length of vector
$= \sqrt{x^2+y^2+z^2}$

→ **Diagonalization**

$D_n = P^{-1}A_nP$

$P = [X_1\ X_2\ X_3 \cdots X_n]_{n\times n}$
↑
model matrix (contains all eigen vectors)
$|P|\neq 0 \Rightarrow P^{-1}$ exist
↳ all eigen vectors are linearly Independent
⇒ Diagonalizable
(ie GM = # unks)

→ Caley - Hamilton Theorem

"every Square matrix satisfies its char eqⁿ"

$$|A - \lambda I| = 0$$

$$\downarrow \text{replace } \lambda \text{ by } A$$

$$f(A) = 0$$

(use when options are in matrix form)

WB CH1 LA

1	a	11	±3	21	2	31	c	41	12	51	55	61	d
2	c	12	a	22	b	32	3	42	a	52	b	62	c
3	c	13	b	23	c	33	ai	43	c	53	b	63	d
4	1/48	14	c	24	b	34	b	44	c	54	d	64	1
5	c	15	b	25	d	35	b	45	a	55	0	65	d
6	16	16	b	26	b	36	a	46	c	56	a	66	5
7	d	17	4	27	b	37	c	47	d	57	d		
8	88	18	7	28	2	38	d	48	d	58	c		
9	200	19	1	29	d	39	c	49	b	59	2		
10	3	20	1	30	b	40	c	50	6	60	a		

good Questions {6, 23, 28, 29, 42, 48, 53 }

CH (2) Calculus

→ Standard Limits

(1) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} \cdot a^{m-n}$

(2) $\lim_{\theta \to 0} \frac{\sin k\theta}{\theta} = \lim_{\theta \to 0} \frac{\tan k\theta}{\theta} = k$ $\quad \lim_{x \to \infty} \frac{\sin x}{x} = 0$

(3) $\lim_{n \to \infty} \left(1 + \frac{p}{n}\right)^n = e^p$

(4) $\lim_{n \to 0} (1 + pn)^{1/n} = e^p$

(5) $\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$

(6) $\lim_{x \to 0} \frac{a^x - 1}{p^x} = \log a$

(7) $\lim_{x \to 0} \frac{a^x - b^x}{x} = \log\left(\frac{a}{b}\right)$

(8) $\lim_{x \to 0} \frac{1 - \cos ax}{x^2} = \frac{a^2}{2}$

(9) If $\lim_{x \to a} (f(x))^{g(x)} = 1^\infty$

Then $\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} g(x)[f(x) - 1]}$

(10) If $\lim_{x \to a} (f(x))^{g(x)} = 0^0$

Then $\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to \infty} g(x) \cdot \log(f(x))}$

→ Limit of Function f(x) at x = a

$\lim_{x \to a} f(x) = l$ exists iff

LHL = RHL = f(a)
$f(a^-)$ $\quad f(a^+)$

⇒ f(x) is continuous at x = a

L - Hospital Rule

Indeterminant form appears $\frac{0}{0}$, $\frac{\infty}{\infty}$, 1^∞, 0^0, ∞^0, ∞^∞, $\infty-\infty$

then app. LH Rule (ie differentiate Numerator & denominator & then solve it)

→ Leibniz Rule

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\cdot dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

→ Differentiability of f(x) at $x=a$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{exists iff}$$

$$\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

LHD = RHD

- f(x) is not differentiable at sharp edges
- f(x) is differentiable ⇒ f(x) is continuous
- f(x) is continuous at $x=a$
  $\Rightarrow f(a^+) = f(a^-) = f(a)$
- f(x) is differentiable at $x=a$.
  $\Rightarrow f'(a^+) = f'(a^-)$

→ Partial Differentiation

$$u = f(x,y)$$

$$\boxed{du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy}$$

$p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$, $r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$, $t = \frac{\partial^2 u}{\partial y^2}$

→ Homogeneous function

$u = f(x,y)$ is homogeneous fun^c of degree "n".

iff $\boxed{f(kx, ky) = k^n f(x,y)}$

→ Euler Theorem

① If $u = f(x,y)$ is homogeneous fun^c of degree (n) then

(i) $x \cdot u_x + y \cdot u_y = n \cdot u$

(ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n(n-1)u$

② If $u = f(x,y) + g(x,y)$ f & g → homog. fun^c of dg n_1 & n_2

Then

(i) $x u_x + y u_y = n_1 f + n_2 g$

(ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n_1(n_1-1)f + n_2(n_2-1)g$

③ If $\phi(u) = F(x,y)$ is homog. fun^c of deg 'n'

Then

(i) $x u_x + y u_y = n \cdot \frac{\phi(u)}{\phi'(u)} = F(u)$

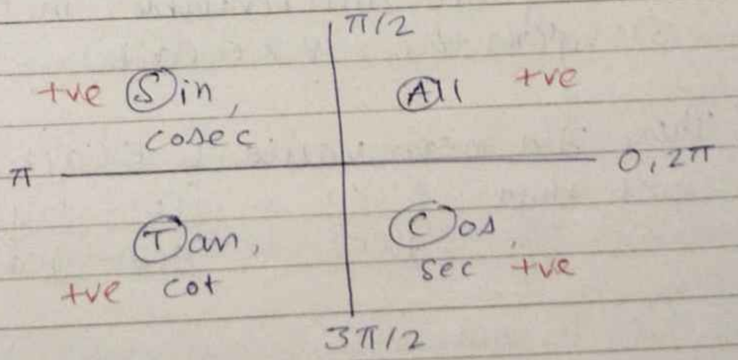
(ii) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = F(u)[F'(u) - 1]$

Imp formulas

- $(u \cdot v)' = u'v + uv'$
  $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- $\int uv = u\int v - \int (u' \int v)$
- $\sin^2 x + \cos^2 x = 1$
  $1 + \tan^2 x = \sec^2 x$
  $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\sin 2x = 2\sin x \cos x$
  $\cos 2x = \sin^2 x - \cos^2 x$
  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $(\sin x)' = \cos x$
  $(\cos x)' = -\sin x$
  $(\tan x)' = \sec^2 x$
  $(\sec x)' = \sec x \cdot \tan x$
- $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
  $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
  $\frac{1 + \tan x}{1 - \tan x} = \tan\left(x + \frac{\pi}{4}\right)$
-

	0	π/6	π/4	π/3	π/2	π	3π/2	2π
sin	0	1/2	1/√2	√3/2	+1	0	−1	0
cos	1	√3/2	1/√2	1/2	0	−1	0	+1
tan	0	1/√3	1	√3	∞	0	−∞	0

- $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$
- $\sin(-\theta) = -\sin\theta$
  $\cos(-\theta) = \cos\theta$
  $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$
- $\frac{d}{dx}(a^x) = a^x \log a$ | $\frac{d}{dx}(e^x) = e^x$
-



- $(\log x)' = \frac{1}{x}$
- $\int \frac{1}{x} = \log x$

→ Mean Value Theorem

↳ Rolle's Theorem

If $f:[a,b] \to R$ such that

1) $f(x)$ is continuous in $[a,b]$
2) $f(x)$ is differentiable in (a,b)
3) $f(a) = f(b)$

Then $\exists$ a mean value $c \in (a,b)$ such that

$$\boxed{f'(c) = 0}$$

↳ Lagrange MVT

If $f:[a,b] \to R$ such that

1) $f(x)$ is continuous in $[a,b]$
2) $f(x)$ is differentiable in (a,b)

then $\exists$ a mean value $c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

↳ Cauchy's MVT

If $f,g : [a,b] \to R$ such that

1) f,g are continuous in $[a,b]$
2) f,g are differentiable in (a,b)
3) $g'(x) \neq 0, \forall x \in (a,b)$

Then $\exists$ a mean value $c \in (a,b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

→ Taylor's Series

expansion of $f(x)$, in power of $(x-a)$ or at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n}f^n(a)$$

$$\boxed{\text{coeff of } (x-a)^n = \frac{f^n(a)}{\lfloor n}}$$

(a) Standard Expansion

(1) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n}$

(2) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} - \cdots$

(3) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

(4) $\log[1+x] = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

(5) $\log[1-x] = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$

(6) $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

→ Increasing / Decreasing function

$Y = f(x)$ is increasing if $\frac{dY}{dx} > 0$

$Y = f(x)$ is decreasing if $\frac{dY}{dx} < 0$

Strictly / monotonically increasing if $\frac{dY}{dx} > 0, \forall x \in R$

Strictly / monotonically decreasing if $\frac{dY}{dx} < 0, \forall x \in R$

→ Maxima & Minima

- n degree polynomial $\xrightarrow{\text{have}}$ (n−1) extremas
- $\frac{dy}{dx}$ change from +ve to −ve ⇒ local maxima
- $\frac{dy}{dx}$ change from −ve to +ve ⇒ local minima
- Saddle point / point of inflection
  $f''(a) = 0$ & $f'''(a) \neq 0$
  $x = a$ is point of inflection
- $f''(a) < 0$ ⇒ maxima at $x = a$
- $f''(a) > 0$ ⇒ minima at $x = a$
- $f'(x) = 0$, will give stationary points

Shortcuts

① If $y = a\cos\theta + b\sin\theta + c$

Then

$$y_{max} = c + \sqrt{a^2+b^2}$$
$$y_{min} = c - \sqrt{a^2+b^2}$$

② If $y = a \cdot \underbrace{\cot\theta + b \cdot \tan\theta}_{\text{Reciprocal}}$

$= a.\operatorname{cosec}\theta + b.\sin\theta$

$= a.\sec\theta + b.\cos\theta$

then $y_{min} = 2\sqrt{ab}$

③ If $y = a\cos^2\theta + b\sin^2\theta$ $(a > b)$

Then

$$y_{max} = a, \quad y_{min} = b$$

④ If $f(x) = x^n . e^{-x}$,

then $f(x)$ is max^m at $x = n$

⑤ $f(x) = \frac{x}{(x+a)(x+b)}$ is max^m at $x = \sqrt{ab}$

→ Absolute/Global Max^m/min of $f(x)$ in $[a,b]$

Global max^m = max { $f(a)$, $f(b)$, local max^m }
Global min^m = min { $f(a)$, $f(b)$, local min^m }

→ Extremize $u = f(x,y)$

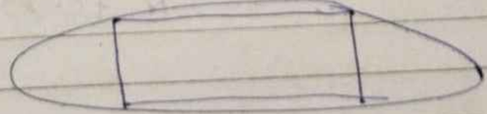
1.) Solving $p=0, q=0$ get stationary points
2.) find r, s, t at st. pts
3.) if $rt - s^2 > 0$ & $r > 0 \Rightarrow u$ is min at St pt
4.) If $rt - s^2 > 0$ & $r < 0 \Rightarrow u$ is max at St pt
5.) If $rt - s^2 < 0 \Rightarrow$ Saddle pt

Shortcut

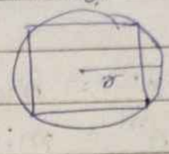
1.) If $x+y=k$, $\forall x,y \in R$. Then
↳ xy is max^m when $x = y = \frac{k}{2}$
↳ x^2+y^2 is min^m at $x=y=\frac{k}{2}$
↳ x^3+y^3 is min^m at $x = y = \frac{k}{2}$

2. If $xy = k$
Sum $(x+y)$ is minimum
when $x = y = \sqrt{k}$

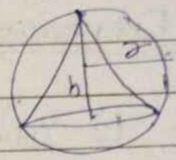
3. max^m area of rectangle inscribed in an Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $2ab$



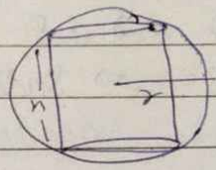
(4.) max^m area of rectangle inscribed in a circle of radius (r) is $2r^2$



(5) max^m height of cone having largest volume iscribed in a sphere of radius 'r' is $\frac{4r}{3}$



(6) max^m height of cylinder having largest volume inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$



→ Integration Properties

(1.) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

(2) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

(3) $$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

(4.) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \text{ (even fun}^c\text{)} \\ 0, & \text{if } f(-x) = -f(x) \text{ (odd fun}^c\text{)} \end{cases}$$

(5) $\int_0^{\pi/2} \sin^n x \cdot dx = \int_0^{\pi/2} \cos^n x \cdot dx$

$$= \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \right] \cdot k$$

$$k = \begin{cases} \pi/2, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

(6) $\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx$

$$= \left[\frac{[(m-1)(m-3)\cdots 2 \text{ (or) } 1][(n-1)(n-3)\cdots 2 \text{ (or) } 1]}{(m+n)(m+n-2)\cdots 2 \text{ (or) } 1} \right] \cdot k$$

$$k = \begin{cases} \pi/2, & \text{if } m, n \text{ even both} \\ 1, & \text{else} \end{cases}$$

(7) $\int e^x [f(x) + f'(x)] \, dx = e^x \cdot f(x)$

(8) $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

(9) $\int uv = u\int v - \int(u'\int v)$ use ILATE to choose u (which can becom zero fast on differentiation)

(10) $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$

(11) $\int (f(x))^n \cdot f'(x) \, dx = \frac{(f(x))^{n+1}}{n+1} + c$

(12) $\int_a^b x f(x) \, dx = \frac{a+b}{2} \int_a^b f(x) \, dx$

if $f(a+b-x) = f(x)$

(13) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

(14) $\int \tan x = \log(\sec x)$

Notes

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→ <u>Step function</u>

$[x]$ = greatest Integer function
(ie greatest int $\le x$)
eg $[1.96] = 1$

$f(x) = [x]$ is discontinuous at every integer

→ <u>Gamma function</u>

$$\boxed{\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx}$$

$$\Gamma n+1 = n\,\Gamma n = n!$$

$$\Gamma 2 = \Gamma 1 = 1$$

$$\Gamma 1/2 = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} \cdot dx = \sqrt{\frac{\pi}{a}}$$

<u>WB keys</u>

1	a	12	c	23	b	34	b	45	b	56	c
2	d	13	1	24	a	35	c	46	b	57	b
3	b	14	d	25	b	36	14	47	a	58	b
4	d	15	d	26	b	37	1	48	1	59	a
5	4	16	c	27	a	38	0	49	c	60	b
6	$-1/\sqrt{2}$	17	d	28	b	39	d	50	c	61	b
7	25	18	b	29	a	40	a	51	b	62	4
8	b	19	c	30	a	41	4.5	52	a	63	0.5
9	c	20	$\pi/4$	31	d	42	1	53	b	64	0
10	d	21	14/9	32	a	43	a	54	a		
11	a	22	b	33	a	44	b	55	c		

PYQ good Qs

1, 2, 7, 10, 11, 12, 13, 14, 17, 18, 24, 26, 30, 31, 32, 34

<u>good Q's</u>

{3, 8, 10, 13, 14, 15, 16, 17, 19, 25, 29, 33, 34, 35, 36, 38, 40, 44, 46, 52, 55, 56, 59, 61, 62, 63, 64}

CH 3 Prob. & Statistics

→ Random Experiment (unpredictable result)

→ Event (outcome of random experiment)

- ↳ Exhaustive Event (all possible outcomes)
- ↳ Equally Likely Event
- ↳ Mutually Exclusive Event
- ↳ Independent Event

Notes

→ Probability

$$P(A) = \frac{\#f \text{ favourables to event } A}{\#f \text{ exhaustive event of exp.}}$$

↳ $P(S) = P(A) + P(\bar{A}) = 1$

↳ $0 \leq P(E) \leq 1$

↳ A, B are Mutal Exclusive events

$P(A \cap B) = 0$

↳ A, B are Independent events

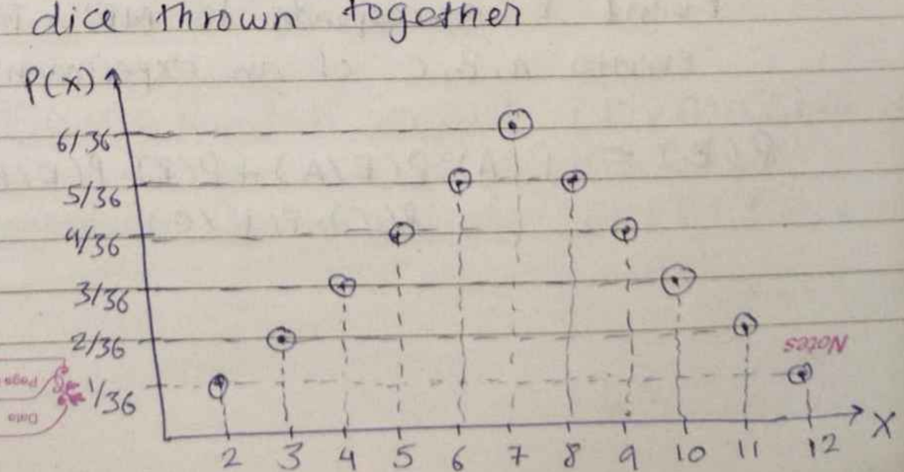
$P(A \cap B) = P(A) \cdot P(B)$

↳ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

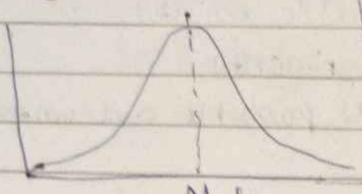
↳ $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$

$= 1 - P(A \cup B)$

↳ two dice thrown together

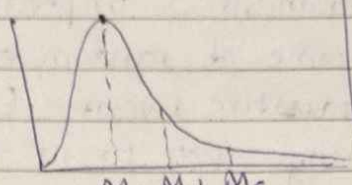


↳ Sym. Prob. Dist.



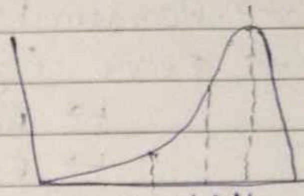
$M_o = M_d = M_e$
(Median) (Mean)

↳ +ve P.D.



$M_o < M_d < M_e$

↳ −ve P.D.



$M_o > M_d > M_e$

→ Conditional Probability

$P(A/B)$ (ie Prob. of A given B)
(A, B are dependent)

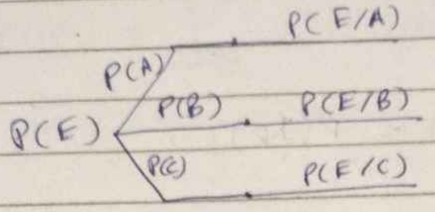
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A/B) \cdot P(B)$$
$$= P(B/A) \cdot P(A)$$

→ Total Probability

Event E corresponds to ME & Exhaustive events A, B, C of an experiment.

$$P(E) = P(A) \cdot P(E/A) + P(B) \cdot P(E/B) + P(C) \cdot P(E/C)$$



$$P(A/E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A) \cdot P(E/A)}{P(E)}$$

→ Statistics

↳ Mean

$$AM = \frac{\sum X_i}{n} \quad , \quad GM = \sqrt[n]{X_1 X_2 X_3 \cdots X_n}$$

$$A.M \geq G.M$$

↳ Mode : value of X which have highest probability (ie most repeated value)

↳ Median : (Arrange in ASC or DSC order)

$$\text{Median} = \begin{cases} \text{middle element} & , n = \text{odd} \\ \text{avg. of two middle elements} & , n = \text{even} \end{cases}$$

→ Random Variable

↳ Discrete Random Variable (DVR) (finite values)

↳ Continuous Random Variable (CVR) (∞ values)

→ DRV

↳ Prob. Mass Fun^c

$$P(X = X_k) = P(X_k)$$

↳ $\sum_{i=0}^{n} P(X_i) = 1$

↳ Commulative Distributive fun^c

$$F(X_k) = \sum_{i=0}^{k} P(X_k)$$

↳ Expectation E(x) / Mean of Distribution

$$E(X) = \sum_{i=0}^{k} x_i \cdot P(X_i)$$

→ CRV

↳ Prob. Density fun^c

$$f(x) = \begin{cases} f_1(x), & x \leq a \\ f_2(x), & x > a \end{cases}$$

↳ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

↳ Commulative Dist. fun^c.

$$F(X_k) = P(X \leq X_k) = \int_{-\infty}^{X_k} f(x) \cdot dx$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) \cdot dx$$

↳ Expectation of X / Mean of Distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \text{1st Moment}$$

$\hookrightarrow E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$ = 2nd Moment

$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$ = r^{th} Moment

$\hookrightarrow$ variance

$V(X) = E(X^2) - (E(X))^2$

$\hookrightarrow$ SD

$\sigma_x = \sqrt{var(x)} \geq 0$

→ Properties

$\hookrightarrow E(\text{constant}) = \text{constant}$

$\hookrightarrow E(ax+b) = aE(X)+b$

$\hookrightarrow E(XY) = E(X)\cdot E(\frac{Y}{X}) = E(Y)\cdot E(\frac{X}{Y})$

$\hookrightarrow var(\text{const.}) = 0$

$\hookrightarrow var(ax+b) = a^2 Var(x)$

$\hookrightarrow \sigma_x(ax+b) = a\cdot\sigma_x$

$\hookrightarrow var(ax+bY) = a^2 Var(x) + b^2 var(Y) + 2ab\,\boxed{Covar(X,Y)}$

$$\boxed{Covar(X,Y) = |E(X).E(Y) - E(XY)|}$$

If X, Y are Indep. R.V
Cov(X, Y) = 0

$$\boxed{\begin{array}{l} Cov(X,Y) = E[(X-E(X))][(Y-E(Y))] \\ (Cov(X,Y))^2 \leq Var(X).Var(Y) \end{array}}$$

→ Binomial Distribution $B(n, p)$

⊛ use when only two possible outcome; ie Success & failure, eg Toss a coin (H/T)

n = # of trails
X = no. of Success
p = prob. of Success
$q = 1 - p$ = Prob. of failure

$$\boxed{P(X=r) = {}^{n}C_{r} \cdot p^{r} \cdot q^{n-r}}$$

$$\boxed{\begin{array}{l} \text{mean} = np \\ \text{variance} = npq \end{array}}$$

→ Poission Distribution : $P(n, \lambda)$

$X \rightarrow$ DRV

$n \rightarrow$ large & $P \rightarrow$ least, So that
$\lambda = np$ = finite

$$P(r = \# f \text{ success}) = \boxed{P(X=r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}}$$

$$\boxed{\lambda = \text{mean} = \text{variance} = np}$$

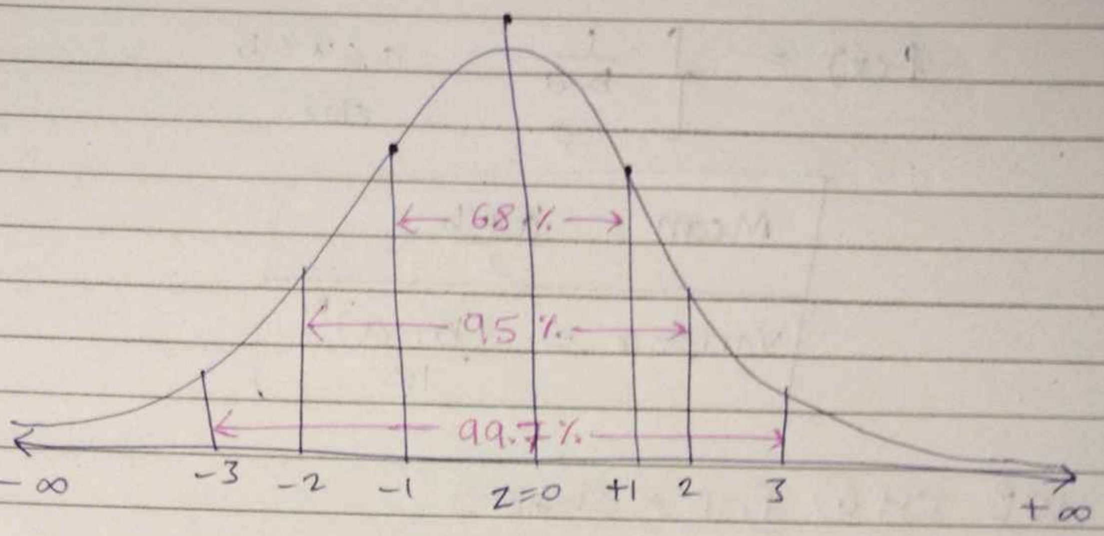
→ Normal Distribution : $N(\mu, \sigma^2)$

$X \rightarrow$ CRV

$n \rightarrow \infty$ (μ, σ^2 is given)

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

$$\boxed{F(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$



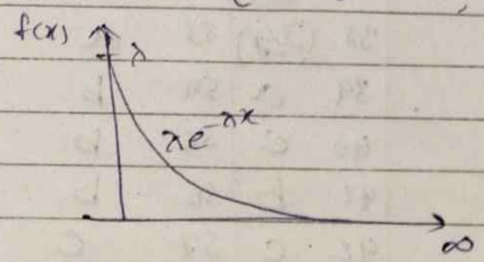
$$\int_{-\infty}^{\infty} \phi(z)\cdot dz = 1$$

→ <u>Exponential Distribution</u> $(E(\lambda))$

$X \to CRV$

PDF,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$



$$\boxed{\begin{array}{c} \text{mean} = \dfrac{1}{\lambda} \\ \hline \text{variance} = \dfrac{1}{\lambda^2} \end{array}}$$

→ <u>Uniform Distribution</u> :- $U[a,b]$

$X \to CRV$

uniformly Distributed over $[a,b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , \text{else} \end{cases}$$

$$\boxed{\text{Mean} = \frac{a+b}{2}}$$

$$\boxed{\text{Variance} = \frac{(b-a)^2}{12}}$$

WB CH 6 Prob & Statistics

1	b	16	(0.2)	31	(0.76)	46	(d)	61	b
2	b	17	d	32	(0.1497)	47	c	62	C
3	d	18	d	33	d	48	(3.89)	63	C
4	b	19	a	34	d	49	(3.5)	64	(99.7%)
5	c	20	d	35	e	50	a	65	(50%)
6	d	21	b	36	(54.5)	51	d	66	a
7	c	22	b	37	(3)	52	c	67	(1/e)
8	d	23	c	38	(3/7)	53	a	68	(0.18)
9	d	24	c	39	a	54	b	69	b
10	b	25	a	40	c	55	b	70	(1/4)
11	b	26	c	41	b	56	b	71	(0.9)
12	a	27	d	42	c	57	c		
13	c	28	a	43	a	58	a		
14	(9/13)	29	(25/56)	44	b	59	d		
15	(0.25)	30	a	45	(0.24)	60	b		

good Q's { 1, 2, 4, 10, 13, 14, 15, 16, 17, 22, 25, 26, 29,
33, 38, 40, 42, 51, 54, 59, 62, 63, 69, 70, 71 }

PYQ good Q's { 7, 13, 21, 22, 23, 27, 30, 32, 39, 40, 43, 44, 47, 48, 51, 53,
54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 65 }