# Set Theory & Algebra
## Work book - Solutions

**QNO: 01** Refer class notes

**QNO: 02**
Ans (d)

Given relation '<' is not reflexive
i.e $\forall a \in A$, $a < a$ is false

Given relation '<' is not antisymmetric

**QNO: 03** Refer class notes

**QNO: 04**
(Ans: c)

Given $S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n = S$

For 'S' to be infinite set, atleast one of set '$S_i$' must be infinite,
Suppose if all $S_i$ were finite then 'S' will also finite

**QNO: 05** Refer class notes

**QNO: 06**
(Ans: d)

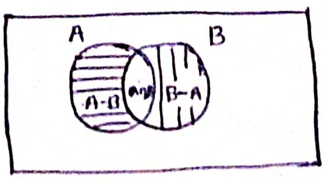
A relation which is symmetric and transitive need not be reflexive relation

Ex: $R = \{\ \}$ on the set $A = \{a, b\}$
here 'R' is empty relation which is symmetric and Transitive but not Reflexive

**QNO: 07**
Ans: c

here $|S| = 3$
$\therefore$ $|P(S)| = 2^3 = 8$

**QNO: 08**
Ans: a



QNO: 09 : Refer class notes

QNO: 10 : Refer class notes

QNO: 11 : Refer class notes

QNO: 12 : $f(x,y) = ((x+y),(x-y))$

Ans(C)

So let $z_1 = x+y$ —— (i)

$z_2 = x-y$ —— (ii)

$f(x,y) = (z_1, z_2)$

So $f^{-1}(z_1, z_2) = (x,y)$

By adding (i) and (ii)

$z_1 + z_2 = 2x$

$x = \frac{z_1+z_2}{2}$

Subtracting (i) and (ii)

$z_1 - z_2 = 2y$

$y = \frac{z_1 - z_2}{2}$

$\therefore f^{-1}(z_1, z_2) = (x,y) = \left(\frac{z_1+z_2}{2}, \frac{z_1-z_2}{2}\right)$

$\therefore f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

Alternative Method

Given $f(x,y) = ((x+y),(x-y))$

Take $x=2, y=3$

$f(2,3) = ((2+3),(2-3)) = (5,-1)$

Now the Correct inverse must map $(5,-1)$ back to $(2,3)$

By looking options one by one we find option (C).

maps $(5,-1)$ to $(2,3)$

QNO: 13 Refer class notes

QNO: 14 :- The inverse of a string does not exist under Concatenation

Ans(C)

**QNo:15 :** To do this problem, we need basics i) $\overline{x+y} = \bar{x}\cdot\bar{y}$

Ans(b) ii) $\bar{x}+1 = 1$

iii) $x+1 = 1$ $\left(\begin{matrix} + \cong \vee \\ 1 \cong \text{TRUE} \end{matrix}\right)$

Now $z * x = \bar{z}+x$ $(\because x*y = \bar{x}+y)$

$= (\overline{x*y})+x$ $(\because z = x*y)$

$= \overline{\bar{x}+y}+x$ $(\because x*y = \bar{x}+y)$

$= (\bar{\bar{x}}\cdot\bar{y})+x$ $(\because \overline{x+y} \cong \bar{x}\cdot\bar{y})$

$= x\cdot\bar{y}+x$ $(\because \bar{\bar{x}} = x)$

$= x(\bar{y}+1)$

$= x$ $(\because \bar{y}+1 = 1)$

**QNo:16**

Ans d

The given algebraic structure does not have identity element So $(\mathbb{Z}, *)$ is not a monoid, so it is not a group

**QNo:17**

Ans(b)

'e' has '3' Complements

$\bar{e} = g$, $\bar{e} = c$, $\bar{e} = d$

**QNo:18**

Refer class notes

**QNo:19**

Ans(a)

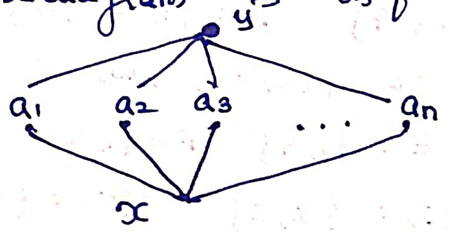
Given that $x \le a_i \ \forall i$

and $a_i \le y \ \forall i$

$\therefore x \le a_i \le y \ \forall i$

but we don't know about relationship between '$a_i$' and '$a_j$'

The hasse diagram is as follows



To make it as Totally order we have to arrange them as

$x \le \underbrace{\_ \le \_ \le \_ \le \_ \le \cdots \_}_{\text{These 'n' blanks can be filled with } n! \text{ ways}} \le y$

$\therefore$

QNo: 20 Refer class notes

QNo: 21 Refer class notes

QNo: 22 Refer class notes

QNo: 23 By using lattice definition. Note that option (a) also True

Ans (c)
Ans (a)

QNo: 24
QNo: 25 } Refer class notes
QNo: 26

QNo: 27
Option (c)

Let A = Set of integers
B = Set of odd integers
C = Set of even integers

Now $A \cap (B \cap C) = A \cap \phi$
$= \phi$ which is finite

$\therefore$ S1 is True (here A, B, C are infinite but $A \cap (B \cap C)$ is finite

Now let $x = 1+\sqrt{2}$, $y = 1-\sqrt{2}$ are two irrational numbers

now $x+y = (1+\sqrt{2})+(1-\sqrt{2}) = 2$ rational number

$\therefore$ $S_2$ is True



QNo: 28
Ans (a)

Given a function $f: A \to B$ and Subsets E and F of A then we have

$f(E \cup F) = f(E) \cup f(F)$ is True

let $A = \{1, 2, 3, 4\}$ $B = \{p, q, r, s\}$

$f(1) = q$, $f(2) = p$, $f(3) = r$, $f(4) = s$

now let $E = \{1, 4\}$ $F = \{1, 2\}$

now $f(E) = \{q, s\}$, $f(F) = \{q, p\}$

now $E \cup F = \{1, 2, 4\}$

$f(E \cup F) = \{q, p, s\} = f(E) \cup f(F)$

where as $f(E \cap F) \subseteq f(E) \cap f(F)$ but need not be $f(E \cap F) = f(E) \cap f(F)$

QNO: 29
QNO: 30
QNO: 31
QNO: 32 } Refer class notes

QNO: 33
Ans (c)

$R_1 =$ { (An element $x$ in A, An element $y$ in B) if $x+y$ divisible by 3}

$R_1 =$ { (1,2)(1,8) (3,6)(5,4) (7,2) (7,8)}

$R_2 =$ { (An element $x$ in B, An element $y$ in B) if $x+y$ is even ~~even~~ but not divisible by 3}

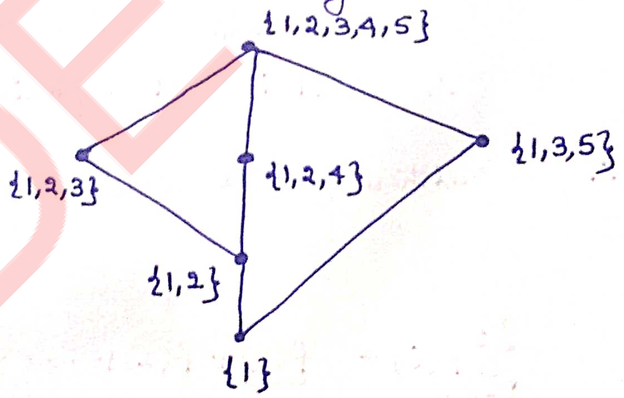
= { (2,2) (4,4) (6,2) (6,4) (8,2)}

$\therefore$ $R_1 o R_2 =$ { (1,2) (3,2) (3,4)(5,4) (7,2)}

QNO: 34
Ans(a)

In Complete lattice L, every non empty subset of 'L' has both LUB and GLB

Now it is necessary to add {1} since GLB of {1,2} and {1,3,5} is {1}




QNO: 35
QNO: 36 } Refer Class notes

QNO: 37
Ans(b)

Let us take A = {1,2}

Subsets of A are $\phi$, {1}, {2} and {1,2}

Maximum Cardinality of collection of distinct subsets is {$\phi$, {1}, {1,2}} is '3'

So with 'n' elements we have maximum cardinality is n+1

QNO:38

Option (a)

I. $f(x,y) = x+y-3$

now $f(x,e) = x$

$\Rightarrow x+e-3 = x$

$\Rightarrow e = 3 \in \mathbb{N}$

II $f(x,y) = \max(x,y)$

$f(x,e) = x$

$\Rightarrow \max(x,e) = x$

then $e = 1 \in \mathbb{N}$.

III. $f(x,y) = x^y$

$f(x,e) = x = f(e,x)$

$\Rightarrow x^e = x$ and $e^x = x$

$\therefore e = 1$ and $e = x^{1/x} \notin \mathbb{N}$

$\therefore$ 'e' does not exist as left identity does not exist

QNo: 39 }
QNo: 40 } Refer Class notes

QNO:41

Ans (a)

$(x,y)\, R\, (u,v)$ if $x<u$ and $y>v$

Reflexive : we have $(x,y) \not R (x,y)$

Since neither $x<x$ nor $y>y$

$\therefore$ so it is not Reflexive

$\therefore$ neither partial order nor an equivalence Relation.

QNO: 42

Option (b)

$$\sum_{i=1}^{m} f(i) = f(1)+f(2)+\cdots+f(m)$$

$$= {}^{(m-1)}C_2 + {}^{(m-1)}C_2 + \cdots + {}^{(m-1)}C_2$$

$$= m \cdot {}^{(m-1)}C_2$$

$$= \frac{m(m-1)(m-2)}{2!}$$

$$= 3\left[\frac{m(m-1)(m-2)}{2! \times 3}\right]$$

$$= 3\left[{}^{m}C_3\right]$$

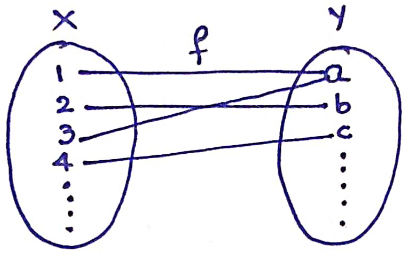
$$= 3n.$$

$\left[\because {}^{m}C_3 = n \text{ Given}\right]$

QNO: 43
QNO: 44
QNO: 45
} Refer class notes

QNO: 46
Ans (d)

Let $A = \{1, 2\}$ $B = \{3, 4\}$




Now $A \cup B = \{1, 2, 3, 4\}$ , $A \cap B = \phi$

$f(A) : f(1) = \{a\}, f(2) = \{b\}$

$\therefore f(A) = \{a, b\} \Rightarrow |f(A)| = 2$

$f(B) : f(3) = \{a\}, f(4) = \{c\}$

$\therefore f(B) = \{a, c\} \Rightarrow |f(B)| = 2$

Now $f(A \cup B) : \{a, b, c\}$

i) Now $|f(A \cup B)| = |f(A)| + |f(B)|$
$3 = 2 + 2$ is false.

ii) $f(A \cap B) = f(A) \cap f(B)$
$\Rightarrow \phi = \{a\}$ is false

iii) $|f(A \cap B)| = \min\{|f(A)|, |f(B)|$
$0 = \min\{2, 2\}$
$0 = 2$ false

iv) $S = \{a, b\}$, $T = \{a, c\}$

$S \cap T = \{a\}$,

$f^{-1}(S \cap T) = f^{-1}(a) = \{1, 3\}$

$f^{-1}(S) = \{1, \cancel{2}3\}$ $f^{-1}(T) = \{1, 3, 4\}$

$\therefore f^{-1}(S) \cap f^{-1}(T) = \{1, 3\}$

$\therefore f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

QNO: 47
Ans (b)

Suppose $f(1) = 2$
$f(2) = 1$

In that case $f o f(1) = f(f(1))$
$= f(2) \quad (\because f(1)=2)$
$= 1 \quad (\because f(2)=1)$

So clearly $f o f(1) = 1$ for $f(1) = 2$, and $f(2)=1$

So Statement 'P' is false

but statement 'Q' is True, because Total no of elements in domain and coDomain both contain odd no of elements, i.e Domain has 2015 elements and CoDomain has 2015 elements so there exist atleast one element 'i' such that $f(i)=i$

Given function is symmetric i.e $f: \{(1,2)(2,1)\}$ and every symmetric function is onto

$\therefore$ Statement R is TRUE

$\therefore$ Ans (b)

QNO: 48
Ans (a)

If $g(x) = 1-x \quad h(x) = \frac{x}{x-1}$

Now $\frac{g(h(x))}{h(g(x))} = \frac{g\left(\frac{x}{x-1}\right)}{h(1-x)} = \frac{1-\frac{x}{x-1}}{\frac{1-x}{1-x-1}} = \frac{-x}{(1-x)^2} = \frac{h(x)}{g(x)}$

QNO: 49 Refer class notes

QNO: 50 Refer class Notes

QNO: 51
ans (c)

$R = \{(p,q),(r,s) \mid p-s = q-r\}$

$\forall (p,q) \in Z^+ \times Z^+$, $((p,q),(p,q)) \in R$ is TRUE iff $p-q = q-p$ which is false.

$\therefore$ not Reflexive.

but it is symmetric

QNo: 52 Refer class notes

QNo: 53 (Ans d) : By using Countable set definition

QNo: 54 Ans (b).

Given $\forall a, b \in G$, $aR_1b$ iff $\exists g \in G$ such that $a = g^{-1}bg$

Reflexive: $a = g^{-1}ag$ is True if we take $g = e$ where 'e' is identity in group

Symmetric: Let $aRb$

$\Rightarrow a = g^{-1}bg$ for some 'g'

now $b = gag^{-1}$

$= (g^{-1})^{-1}ag^{-1}$ ($\because g \in G$ and 'G' is group $\therefore g^{-1}$ also exist)

So symmetric

Transitive let $aRb$, and $bRc$

$\Rightarrow a = g_1^{-1}bg_1$ and $b = g_2^{-1}cg_2$ for some $g_1, g_2 \in G$

Now $a = g_1^{-1}g_2^{-1}cg_2g_1$

$= (g_2g_1)^{-1}cg_2g_1$ ($\because g_1 \in G, g_2 \in G$ then $g_2g_1 \in G$)

$\Rightarrow aRc$

$\therefore R_1$ is equivalence Relation.

$R_2$ is not Reflexive so not an equivalence Relation.

QNo: 55 Ans (a)

Note $[D_n, |]$ is boolean algebra if 'n' can be expressed as square free integers (i.e none of divisors of 'n' has perfect squares)

**QNO:56** (Ans d)

**Note:** Topological Sort is related to algorithm subject

**QNO:57**

Refer algorithm subject (All pairs shortest path Algorithm)

**QNO:58**

Ans c

Every group $\mathbb{Z}_p = \{1, 2, 3, \cdots p-1\}_{\times p}$ is cyclic here $p = 7$. So it is cyclic and $\langle 3 \rangle$ $\langle 5 \rangle$ are generators

**QNO:59**

Ans c

Refer class notes

**QNO:60**

Ans (b)

$$= \frac{1}{1!\,9!} + \frac{1}{3!\,7!} + \frac{1}{5!\,5!} + \frac{1}{7!\,3!} + \frac{1}{9!\,1!}$$

$$= \frac{1}{10!}\left[\frac{10!}{9!} + \frac{10!}{3!\,7!} + \frac{10!}{5!\,5!} + \frac{10!}{7!\,3!} \times \frac{10!}{9!}\right]$$

$$= \frac{252}{10!}$$

$$= \frac{2^9}{10!} = \frac{2^a}{b!}$$

$\therefore$ $a = 9$, $b = 10$

$\therefore$ Ans (b)

**QNO:61** : Refer algorithm subject (All pairs shortest path Algorithm)

**QNO:62**

Ans: d

$A = \{6, 7, 8, 9, 10\}$

$R = \{(6,6)\ (7,7)\ (8,8)(9,9)\ (10,10)(6,7)\ (7,6)\ (8,9)(9,8)$
$(9,10)\ (10,9)\ (8,10)\ (10,8)\}$

$[6] = \{6, 7\}$, $[7] = \{7, 6\}$ $[8] = \{8, 9, 10\}$
$[9] = \{8, 9, 10\}$
$[10] = \{8, 9, 10\}$

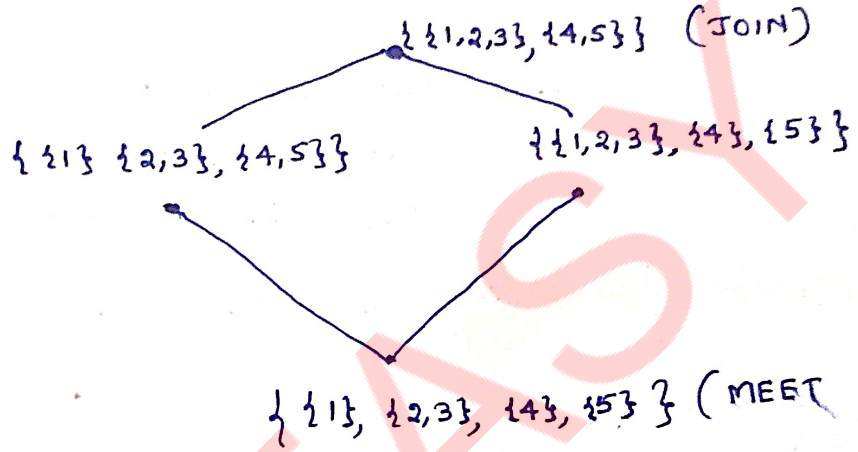
$\therefore$ Partion $= \{\{6,7\}, \{8,9,10\}\}$

**QNO: 63**
Ans(a)

$A = \{1, 2, 3, 4, 5\}$

$a = \{\{1,2,3\}, \{4\}, \{5\}\}$

$b = \{\{1\}, \{2,3\}, \{4,5\}\}$




(Kindly Refer class notes for refinement definition)

**QNO: 64**
Ans(a)

Chain is a another name of TOSET

**QNO: 65**
Ans(d)

Refer class notes

**QNO: 66**
(Ans(d)

Let $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} ae & ae \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\therefore ae = a; \quad ae = b$

$\therefore e = 1; \quad e = \frac{b}{a}$ identity does not exist.

**QNO: 67**
Ans(b)

The duality principle ensures that "if we exchange every symbol by its dual in a formula we get the dual result

→ 1 change to 0
→ 0 change to 1
→ + change to • and • change to +

QNO: 68

Ans (C)

$x R y$ iff $y = x + 2$

$x S y$ iff $x \le y$

Given set $\{1, 3, 5\}$

Now $1 \le y$, $3 \le y$, $5 \le y$ when we apply 'S'

i.e $(1, y)$ $(3, y)$ $(5, y)$

Now $1 R y$ if $y = 1+2$ when we apply 'R'

$3 R y$ if $y = 3+2$

$5 R y$ if $y = 5+2$

$\therefore$ $(1,3)$ $(3,5)$ $(5,7)$

QNO: 69

Ans C

Since $S = \{a, b, d\} \subseteq A = \{a, b, c, d\}$

$$\left.\begin{array}{l} C_S(a) = 0 \\ C_S(b) = 0 \\ C_S(d) = 0 \end{array}\right\} \text{ here } a, b, d \in S$$

$$C_S(c) = 1 \; \} \text{ here } c \notin S$$

$\therefore$ $C_S = \{(a,0)(b,0)(c,1)(d,0)\}$

QNO: 70

Ans: 4

We have find least positive integer 'n' such that

$\left(\frac{1}{\sqrt{2}}[1+i]\right)^n = 1$ (here $e = 1$ is identity and 'n' is oder of the element.

If we take $n = 4$ then

$\left[\frac{1}{\sqrt{2}}(1+i)\right]^4 = 1$

$\therefore$ Order is 4 ]

QNo: 71

Ans: 2

Upperbounds of $\{a, c, d, f\} = \{h, i\}$

QNo: 72

atmost 4

G is a finite group

A and B are subgroups of G

and $O(A) = 4$, $O(B) = 5$

then $O(A)$ divides $O(G)$, $O(B)$ divides $O(G)$

$A \cap B$ is also subgroup of G

and $O(A \cap B)$ divides $O(G)$

and $O(A \cap B) \leq O(A)$ ~~and $O(A \cap B) \leq O(B)$~~

$\therefore$ $O(A \cap B)$ atmost 4

QNo: 73

Refer Class Notes

I wish you all the best

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