MAT 216 – Linear Algebra and Fourier Analysis

Exercise Set 1.1

1. Which of the following are linear equations in x_1, x_2, and x_3 ?

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1,$ (b) $x_1 + 3x_2 + x_1x_3 = 2,$ (c) $x_1 = -7x_2 + 3x_3,$ (d) $x_1^{-2} + x_2 + 8x_3 = 5$
(e) $x_1^{3/5} - 2x_2 + x_3 = 4,$ (f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

2. Given that k is a constant, which of the following are linear equations?

(a) $x_1 - x_2 + x_3 = \sin\ k,$ (b) $kx_1 - \frac{1}{k}x_2 = 9,$ (c) $2^k x_1 + 7x_2 - x_3 = 0$

3. Find the solution set of each of the following linear equations.
(a) $7x - 5y = 3,$ (b) $3x_1 - 5x_2 + 4x_3 = 7,$ (c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1,$
(d) $3v - 8w + 2x - y + 4z = 0$

4. Find the augmented matrix for each of the following systems of linear equations.

(a) $3x_1 - 2x_2 = -1, 4x_1 + 5x_2 = 3, 7x_1 + 3x_2 = 2$
(b) $2x_1 + 2x_3 = 1, 3x_1 - x_2 + 4x_3 = 7, 6x_1 + x_2 - x_3 = 0$
(c) $x_1 + 2x_2 - x_4 + x_5 = 1, 3x_2 + x_3 - x_5 = 2, x_3 + 7x_4 = 1$

5. Find a system of linear equations corresponding to the augmented matrix.

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

6. Consider the system of equations, show that for this system to be consistent, the constants a, b, and c must satisfy $c = a + b$.

$$\begin{aligned} x + y + 2z &= a, \\ x + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

7. Show that if the linear equations $x_1 + kx_2 = c$ and $x_1 + lx_2 = d$ have the same solution set, then the equations are identical.

8. For which value(s) of the constant k does the system have no solutions? Exactly one solution? Infinitely many solutions? Explain your reasoning.

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

9. Consider the system of equations

$$
\begin{aligned}
ax + by &= k \\
cx + dy &= l \\
ex + fy &= m
\end{aligned}
$$

Indicate what we can say about the relative positions of the lines $ax + by = k, cx + dy = i$, and $ex + fy = m$ when

(a) the system has no solutions.

(b) the system has exactly one solution.

(c) the system has infinitely many solutions.

Exercise Set 1.2

1. Which of the following 3×3 matrices are in reduced row-echelon form?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

(f) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (g) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (h) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, (i) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (j) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. Which of the following 3×3 matrices are in row-echelon form?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

3. In each part determine whether the matrix is in row-echelon form, reduced row-echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given. reduced row-echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given 5. row-echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. Solve each of the following systems by Gauss-Jordan elimination.

(a) $x_1 + x_2 + 2x_3 = 8,\quad -x_1 - 2x_2 + 3x_3 = 1,\quad 3x_1 - 7x_2 + 4x_3 = 10$
(b) $2x_1 + 2x_2 + 2x_3 = 0,\quad -2x_1 + 5x_2 + 2x_3 = 1,\quad 8x_1 + x_2 + 4x_3 = -1$
(c) $x - y + 2z - w = -1,\quad 2x + y - 2z - 2w = -2,\quad -x + 2y - 4z + w = 1,\quad 3x - 3w = -3$
(d) $-2b + 3c = 1,\quad 3a + 6b - 3c = -2,\quad 6a + 6b + 3c = 5$

7. Solve each of the systems in Exercise 6 by Gaussian elimination.

8. Solve each of the following systems by Gauss-Jordan elimination.

(a) $2x_1 - 3x_2 = -2,\quad 2x_1 + x_2 = 1,\quad 3x_1 + 2x_2 = 1,$
(b) $3x_1 + 2x_2 - x_3 = -15,\quad 5x_1 + 3x_2 + 2x_3 = 0,\quad 3x_1 + x_2 + 3x_3 = 11,\quad -6x_1 - 4x_2 + 2x_3 = 30$
(c) $4x_1 - 8x_2 = 12,\quad 3x_1 - 6x_2 = 9,\quad -2x_1 + 4x_2 = -6$
(d) $10y - 4z + w = 1,\ x + 4y - z + w = 2,\quad 3x + 2y + z + 2w = 5, -2x - 8y + 2z - 2w = -4,\ x - 6y + 3z = 1$

9. Solve each of the systems in Exercise 8 by Gaussian elimination.

10. Solve each of the following systems by Gauss-Jordan elimination.

(a) $5x_1 - 2x_2 + 6x_3 = 0,\quad -2x_1 + x_2 + 3x_3 = 1$
(b) $x_1 - 2x_2 + x_3 - 4x_4 = 1,\quad x_1 + 3x_2 + 7x_3 + 2x_4 = 2$
(c) $x_1 - 12x_2 - 11x_3 - 16x_4 = 5$
(d)
$$\begin{aligned} w + 2x - y &= 4 \\ x - y &= 3 \\ w + 3x - 2y &= 7 \\ 2u + 4v + w + 7x &= 7 \end{aligned}$$

11. Solve each of the systems in Exercise 10 by Gaussian elimination.

12. Without using pencil and paper, determine which of the following homogeneous systems have nontrivial solutions.

(a) $2x_1 - 3x_2 + 4x_3 - x_4 = 0, 7x_1 + x_2 - 8x_3 + 9x_4 = 0,\quad 2x_1 + 8x_2 + x_3 - x_4 = 0$
(b) $x_1 + 3x_2 - x_3 = 0,\ x_2 - 8x_3 = 0,\quad 4x_3 = 0$
(c) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0,\quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0,$
(d) $3x_1 - 2x_2 = 0,\quad 6x_1 - 4x_2 = 0$

13. Solve the following homogeneous systems of linear equations by any method.

(a) $2x_1 + x_2 + 3x_3 = 0,\quad x_1 + 2x_2 = 0,\quad x_2 + x_3 = 0$

(b) $3x_1 + x_2 + x_3 + x_4 = 0,\quad 5x_1 - x_2 + x_3 - x_4 = 0,$

(c) $2x + 2y + 4z = 0,\quad w - y - 3z = 0,\quad 2w + 3x + y + z = 0,\quad -2w + x + 3y - 2z = 0$

14. Solve the following homogeneous systems of linear equations by any method.

(a) $2x - y - 3z = 0, \quad -x + 2y - 3z = 0, \quad x + y + 4z = 0$

(b) $v + 3w - 2x = 0, \quad 2u + v - 4w + 3x = 0, \quad 2u + 3v + 2w - x = 0, -4u - 3v + 5w - 4x = 0$

(c) $x_1 + 3x_2 + x_4 = 0, \ x_1 + 4x_2 + 2x_3 = 0, \quad -2x_2 - 2x_3 - x_4 = 0, \ 2x_1 - 4x_2 + x_3 + x_4 = 0$
$x_1 - 2x_2 - x_3 + x_4 = 0$

15. Solve the following systems, where a, b, and c are constants.

(a) $2x + y = a, \quad 3x + 6y = b$

(b) $x_1 + x_2 + x_3 = a, \quad 2x_1 + 2x_3 = b, \quad 3x_2 + 3x_3 = c$

16. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

17. Reduce to reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

18. Find two different row-echelon forms of the following matrix.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

19. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi$, and $0 \leq \gamma < \pi$

$$\begin{aligned} 2\sin\ \alpha - \cos\ \beta + 3\tan\ \gamma &= 3 \\ 4\sin\ \alpha + 2\cos\ \beta - 2\tan\ \gamma &= 2 \\ 6\sin\ \alpha - 3\cos\ \beta + \tan\ \gamma &= 9 \end{aligned}$$

20. Show that the following nonlinear system has 18 solutions if $0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi$, and $0 \leq \gamma < 2\pi$.

$$\begin{aligned} \sin\ \alpha + 2\cos\ \beta + 3\tan\ \gamma &= 0 \\ 2\sin\ \alpha + 5\cos\ \beta + 3\tan\ \gamma &= 0 \\ -\sin\ \alpha - 5\cos\ \beta + 5\tan\ \gamma &= 0 \end{aligned}$$

21. For which value(s) of λ does the system of equations have nontrivial solutions?

$$\begin{aligned} (\lambda - 3)x + y &= 0 \\ x + (\lambda - 3)y &= 0 \end{aligned}$$

22. Solve the system for x_1, x_2, and x_3 in the two cases $\lambda = 1, \lambda = 2$.

$$\begin{aligned} 2x_1 - x_2 &= \lambda x_1 \\ 2x_1 - x_2 + x_3 &= \lambda x_2 \\ -2x_1 + 2x_2 + x_3 &= \lambda x_3 \end{aligned}$$

24. Solve the following system for x, y, and z.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$