Optimal bound on the quantum Fisher Information

Based on few initial expectation values of the prove state.

Iagoba Apellaniz [1], Matthias Kleinmann [1], Otfried Ghüne [2], & Géza Tóth [1,3,4]

iagoba.apellaniz@gmail.com

[1]Department of Theoretical Physics, University of the Basque Country, Spain
[2]Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Germany
[3]IKERBASQUE, Basque Foundation for Science, Spain
[4]Wigner Research Centre for Physics, Hungarian Academy of Sciences, Hungary

Recent Advances in Quantum Metrology; Warsaw - 2016

Outline

1. Introduction and Motivation
2. QFI based on expectation values: Are they optimal?
   - Optimization problem
3. Case study
   - Spin squeezed states
   - Unpolarized Dicke states
4. Conclusion and outlook

1. *Many* **inequalities** have been proposed to lower bound the quantum Fisher Information.

Bounds for qFI

$$\mathcal{F}[\varrho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \qquad \mathcal{F}[\varrho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},$$

$$\mathcal{F}[\varrho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 \, (\Delta J_y^2)^2} + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle (1 + \langle J_x^2 \rangle) + 6}$$

[**I.A.**, B. Lücke, J. Peise, C. Klempt & G. Toth, NJP **17**, 083027 (2015)]
[L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009)]
[Z. Zhang & L.-M. Duan, NJP **16**, 103037 (2014)]

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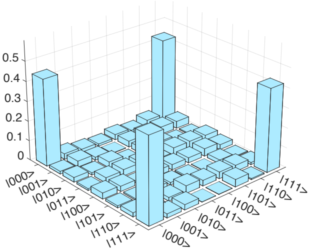
2. Typically, we only have *a couple of expectation values* to characterize the state.



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2. Typically, we only have *a couple of expectation values* to characterize the state.
3. The archetypical criteria that demonstrates *useful entanglement* on the state.
4. It is essential either to *verify them or find new ones* for different set of expectation values.

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The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$\mathcal{F}[\varrho, J_z] = 2 \sum_{\lambda,\gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

$$\mathcal{F}[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k \left(\Delta J_z\right)^2_{|\Psi_k\rangle}$$

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

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- In the general case, usually *lower bounded* by its *"classical"* counterparts.

Optimization: Legendre Transform

- For a convex function of the state, we construct a *thight lower bound* as follows,

$$g(\varrho) \geq \mathcal{B}\big(\{w_k := \langle W_k \rangle\}\big) = \sup_{\{r_k\}} \Big(r \cdot w - \sup_{\varrho}[r \cdot \langle W \rangle - g(\varrho)]\Big).$$

- When $g(\varrho)$ is deffined as infimum over the convex roof, the 2^{nd} optimization simplified to pure states only,

$$\mathcal{B}(\{w_k\}) = \sup_{\{r_k\}} \Big(r \cdot w - \sup_{|\psi\rangle}[r \cdot \langle W \rangle - g(|\psi\rangle)]\Big).$$

[O. Gühne, M. Reimpell & R.F. Werner, PRL **98**, 110502 (2007)]

[J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007)]

Optimization for the qFI

Different because of *simplicity of the qFI for pure states*.

$$\mathcal{F}(\{w_k\}) = \sup_{\{r_k\}} \left(r \cdot w - \sup_{\mu} [\lambda_{\max}(r \cdot W - 4(J_z - \mu)^2)] \right).$$

- Therefore, we have parametrised the optimization, which leads to a *more efficient finding* of the solution.

[**I.A.**, M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

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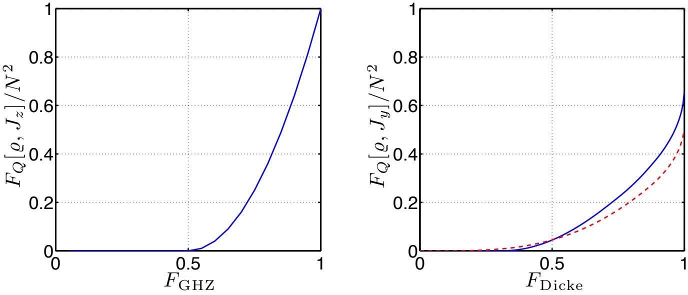
- We'll present 2 main cases, spin-squeezed states and unpolarized Dicke states.
- Though, we apply our method to projectors with great success, we will focus on collective $J_{\mathbf{n}}$ operators.
- One of the cases using projector operators, *i.e.*, using the *fidelity* leads to *analytic soulution*!



Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

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2. In the direction of $\langle J_x \rangle$ the worst case is when $\langle J_x \rangle$ takes the *value zero*.
3. Hence, the optimisation can be accomplished *only using 2 operators* $\{J_z, J_x^2\}$ while the resulting bound is mapped directly to $\langle J_z \rangle, (\Delta J_x)^2$.

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3. Hence, the optimisation can be accomplished *only using 2 operators* $\{J_z, J_x^2\}$ while the resulting bound is mapped directly to $\langle J_z \rangle, (\Delta J_x)^2$.
4. Since $\mathcal{F} \geq \langle J_z \rangle^2 / (\Delta J_x)^2$ can be used too, we compare it with our numerical result.

- We numerically optimise the lower bound of qFI for a 4 particle system for all possible values of $\langle J_z \rangle$ and $(\Delta J_x)^2$.

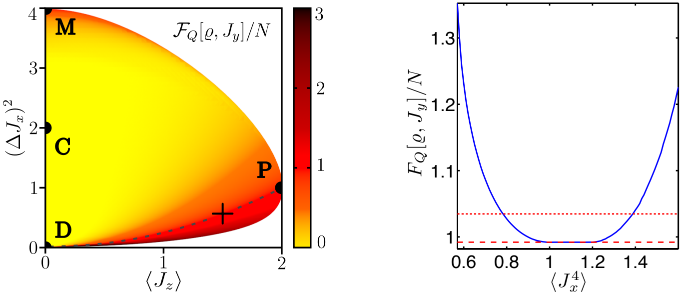

Figure: Below the dashed line, $\mathcal{F}$ surpasses the shot noise limit. Cross point enhanced with extra parameter. The result shows an extreme similarity with respect to $\mathcal{F} \geq \langle J_z \rangle^2 / (\Delta J_x)^2$.

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Metrology with unpolarised Dicke states

1. In order to characterize the unpolarized Dicke state, we use these 3 operators, $\{J_x^2, J_y^2, J_z^2\}$, with the following constraint, $\langle J_x^2 \rangle = \langle J_y^2 \rangle$.

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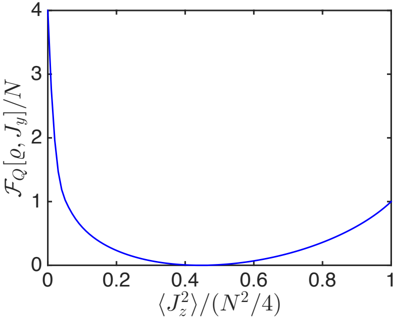
2. For $\sum_l \langle J_l^2 \rangle = \frac{N}{2}(\frac{N}{2}+1)$, *i.e.* maximal, the following figure shows an illustrative result for 6 particles.



Realistic characterization of Dicke state

Experiment → [B. Lücke *et al.*, PRL **112**, 155304 (2014)]

- We now consider an interesting experimental case with BECs where the following initial values are measured.

Experimental details of unpolarized Dicke state with BEC

$$N = 7900 \qquad \langle J_z^2 \rangle = 112 \pm 31$$
$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6$$

- For that number of particles we developed a *powerful extrapolation* tool.

Extrapolation

Procedure:

1. First, we proportionally extrapolate the expectation values to the symmetric subspace. Easier to handle for larger systems.

$$\langle J_z^2 \rangle_{\mathrm{sym},N} = 145.69 \qquad \langle J_x^2 \rangle_{\mathrm{sym},N} = 7.8 \times 10^6$$

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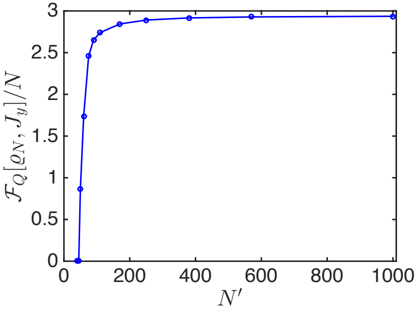
3. The extrapolation is directly obtained by

$$\mathcal{F}_N \approx \frac{N^2}{N'^2}\mathcal{F}_{N'}$$


3
2.5
2
1.5
1
0.5
0
0
200
400
600
800
1000
$\mathcal{F}_Q[\varrho_N, J_y]/N$
N'

Conclusion and Outlook

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Thank you for your attention!

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