Optimal bound on the quantum Fisher Information

Based on few initial expectation values of the prove state.

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Outline

1. *Many* **inequalities** have been proposed to lower bound the quantum Fisher Information.

Bounds for qFI

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \qquad F_Q[\varrho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},$$

$$F_Q[\varrho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 \, (\Delta J_y^2)^2} + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle (1 + \langle J_x^2 \rangle) + 6}$$

[**I.A.**, B. Lücke, J. Peise, C. Klempt & G. Toth, NJP **17**, 083027 (2015)]

[L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009)]

[Z. Zhang & L.-M. Duan, NJP **16**, 103037 (2014)]

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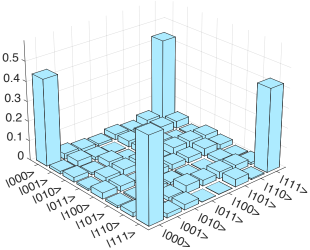

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2. Typically, we only have *a couple of expectation values* to characterise the state.
3. The archetypical criteria that demonstrates *metrologicaly useful entanglement* on the state.
4. It is essential either to *verify them or find new ones* for different set of expectation values.

The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda,\gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

$$F_Q[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k \, (\Delta J_z)^2_{|\Psi_k\rangle}$$

[M.G.A. Paris, Int. J. Quant. Inf. **7**, 125 (2009)]

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

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- In the general case, usually *lower bounded* by its *"classical"* counterparts.

Optimisation: Legendre Transform

- For a convex function of the state, we construct a *thight lower bound* as follows,

$$g(\varrho) \geq \mathcal{B}\big(\{w_k := \langle W_k \rangle\}\big) = \sup_{\{r_k\}} \Big(r \cdot w - \sup_{\varrho} [r \cdot \langle W \rangle_\varrho - g(\varrho)] \Big).$$

The scalar product reads as follows $r \cdot w \equiv \sum_k r_k w_k$
and $r \cdot \langle W \rangle_\varrho \equiv \sum_k r_k \langle W_k \rangle_\varrho$.

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- When $g(\varrho)$ is defined as *infimum over the convex roof*, the 2^{nd} optimisation simplified to pure states only,

$$\mathcal{B}(\{w_k\}) = \sup_{\{r_k\}} \big(r \cdot w - \sup_{|\psi\rangle}[r \cdot \langle W \rangle - g(|\psi\rangle)]\big).$$

[O. Gühne, M. Reimpell & R.F. Werner, PRL **98**, 110502 (2007)]

[J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007)]

Optimisation for the qFI

The *simplicity* of qFI for pure states allows to rewrite the problem:

$$\mathcal{F}(\{w_k\}) = \sup_{\{r_k\}} \left(r \cdot w - \sup_{\mu} [\lambda_{\max}(r \cdot W - 4(J_z - \mu)^2)] \right).$$

- Therefore, we have parametrised the optimisation, which leads to a *more efficient finding* of the solution.

[**I.A.**, M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

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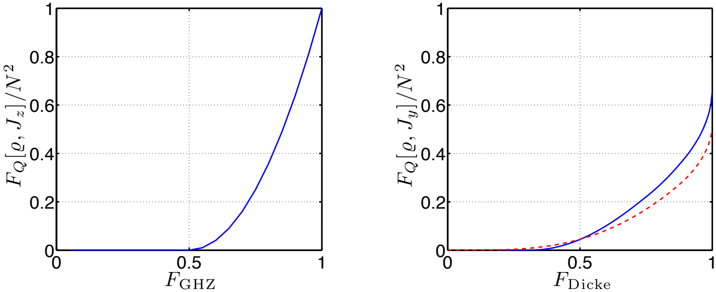

- One of the cases using projector operators, *i.e.*, using the *fidelity*, leads to *analytic soulution*!

Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

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3. Hence, the optimisation can be accomplished *only using 2 operators* $\{J_z, J_x^2\}$ while the resulting bound is mapped directly to $\langle J_z \rangle, (\Delta J_x)^2$.
4. Since $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$ is a valid lower bound, we compare it with our numerical result.

- We numerically optimise the lower bound of qFI for a 4 particle system for all possible values of $\langle J_z \rangle$ and $(\Delta J_x)^2$.

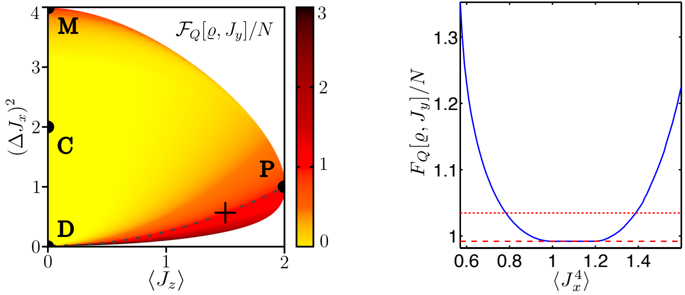

Figure: Below the dashed line, $\mathcal{F}$ surpasses the shot noise limit. Cross point enhanced with extra parameter. The result shows an extreme similarity with respect to $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$.

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Metrology with unpolarised Dicke states

1. In order to characterize the unpolarised Dicke state, we use these 3 operators, $\{J_x^2, J_y^2, J_z^2\}$, with the following constraint, $\langle J_x^2\rangle = \langle J_y^2\rangle$.

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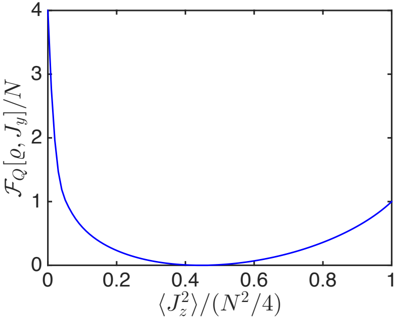
2. For $\sum_l \langle J_l^2 \rangle = \frac{N}{2}(\frac{N}{2}+1)$, *i.e.* maximal, the following figure shows an illustrative result for 6 particles.



Realistic characterisation of Dicke state

Experiment → [B. Lücke *et al.*, PRL **112**, 155304 (2014)]

- We now consider an interesting experimental case with BECs where the following initial values are measured.

Experimental details of unpolarized Dicke state with BEC

$$N = 7900 \qquad \langle J_z^2 \rangle = 112 \pm 31$$
$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6$$

- For that number of particles we developed a *powerful extrapolation* tool.

Extrapolation

Procedure:

1. First, we equivalently increase the expectation values and the unknown bound *to the symmetric subspace*.

$$\langle J_z^2 \rangle_{\mathrm{sym},N} = 145.69 \qquad \langle J_x^2 \rangle_{\mathrm{sym},N} = 7.8 \times 10^6$$

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3. The extrapolation is directly obtained by

$$\frac{\mathcal{F}}{\sum_l\langle J_l^2\rangle} \approx \frac{\mathcal{F}_{\mathrm{sym},N}}{N^2/4} \approx \frac{\mathcal{F}_{\mathrm{sym},N'}}{N'^2/4}$$

- We are ready now to find numerically the lower bound for the experimental case.

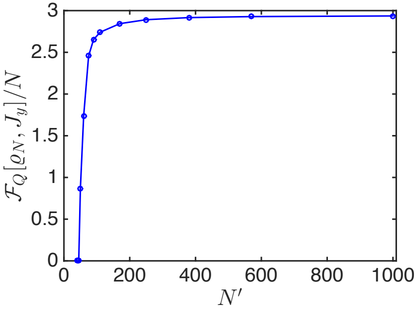

Figure: The extrapolated lower bound approaches rapidly to the desired value. The points over the line indicate the systems used for optimisation.

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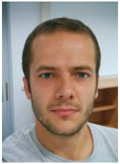
4. It has been shown that the *optimisation method can be complemented* with scaling considerations such as the extrapolation for the Dicke states case.
5. This *method is constructed to be universal* and it could be used in many other situations.

Thank you for your attention!

Groups' home pages

→ *https://sites.google.com/site/gedentqopt*

→ *http://www.physik.uni-siegen.de/tqo/*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