

# Optimal bound on the quantum Fisher information

*Based on few initial expectation values*

**Iagoba Apellaniz**<sup>1</sup>, Matthias Kleinmann<sup>1</sup>, Otfried Gühne<sup>2</sup>,  
& Géza Tóth<sup>1,3,4</sup>

**iagoba.apellaniz@gmail.com**

<sup>1</sup>Department of Theoretical Physics, University of the Basque Country, Spain

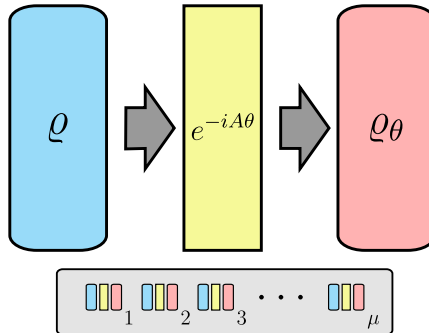
<sup>2</sup>Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Germany

<sup>3</sup>IKERBASQUE, Basque Foundation for Science, Spain

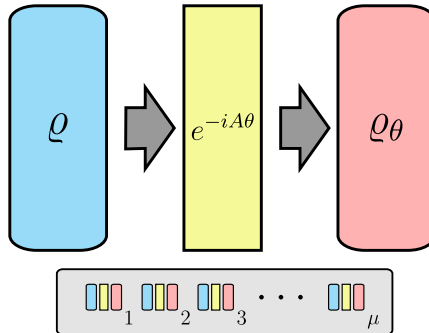
<sup>4</sup>Wigner Research Centre for Physics, Hungarian Academy of Sciences, Hungary

ICE-3 Palma de Mallorca; 2016-04-15

# Basics on quantum estimation

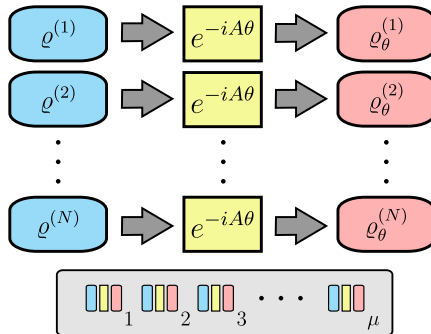


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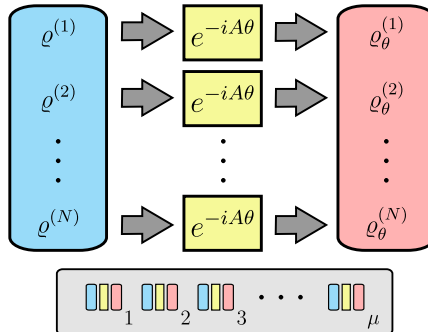
- The precision  $(\Delta\theta)^{-1} \propto \sqrt{\mu}$ .

# Basics on quantum estimation



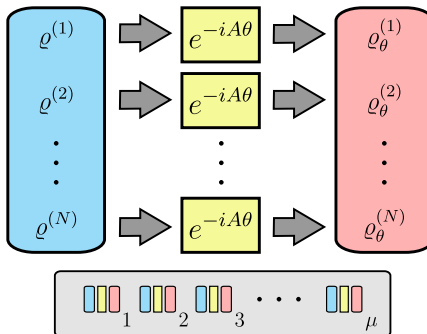
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# Basics on quantum estimation



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# Basics on quantum estimation



- The precision  $(\Delta\theta)^{-1} \propto \sqrt{\mu}$ . How it scales with  $N$ ?

# The quantum Fisher information

- The classical Cramér-Rao bound

$$(\Delta\theta)^{-1} \leq \sqrt{\mu \int dx p(x|\theta) \left( \frac{\partial \ln(p(x|\theta))}{\partial \theta} \right)^2}$$

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- The quantum CR bound

$$(\Delta\theta)^{-1} \leq \sqrt{\mu F_Q}$$

*Fisher information* maximised over all measurements.



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Best separable

$$F_Q \propto N$$

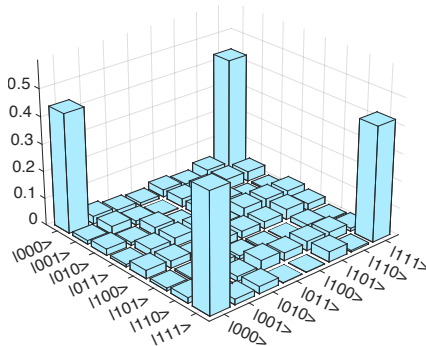
Best state

$$F_Q \propto N^2$$

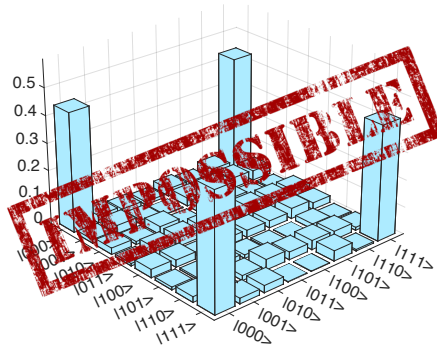
# Outline

- 1 Introduction and Motivation
- 2 QFI based on expectation values: Are they optimal?
  - Optimisation problem
- 3 Case study
  - Fidelities
  - Spin-squeezed states
  - Unpolarised Dicke states
- 4 Conclusion and outlook

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- *Many inequalities* have been proposed to lower bound the quantum Fisher Information.

### Bounds for qFI

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \quad F_Q[\varrho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},$$

$$F_Q[\varrho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 (\Delta J_y^2)^2} + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle(1 + \langle J_x^2 \rangle) + 6\langle J_y J_x^2 J_y \rangle}$$

[ L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009) ]

[ Z. Zhang & L.-M. Duan, NJP **16**, 103037 (2014) ]

[ I.A., B. Lücke, J. Peise, C. Klempt & G. Toth, NJP **17**, 083027 (2015) ]

- For large systems, we only have *a couple of expectation values* to characterise the state.
- *Many inequalities* have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows *metrologically useful entanglement*.

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle}{(\Delta J_z)^2}$$

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- For large systems, we only have *a couple of expectation values* to characterise the state.
- *Many inequalities* have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows *metrologically useful entanglement*.
- It is essential either to *verify them or to find new ones* for different set of expectation values.



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# The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

Alternatively, as convex roof

$$F_Q[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k (\Delta J_z)_{|\Psi_k\rangle}^2$$

[ M.G.A. Paris, Int. J. Quant. Inf. **7**, 125 (2009) ]

[ G. Tóth & D. Petz, PRA **87**, 032324 (2013) ]

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- For pure states it's *extremely simple*

$$F_Q[\varrho, J_z] = 4 (\Delta J_z)^2$$

# Optimisation based on the Legendre Transform

- When  $g(\varrho)$  is a *convex roof*

$$g(\varrho) \geq \mathcal{B}(w := \text{Tr}[\varrho W]) = \sup_r (rw - \sup_{|\psi\rangle} [r\langle W \rangle - g(|\psi\rangle)]).$$

[ O. Gühne, M. Reimpell & R.F. Werner, PRL **98**, 110502 (2007) ]

[ J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007) ]

## Optimisation for the qFI

The *simplicity* of qFI for pure states leads to

$$\mathcal{F}(w) = \sup_r (rw - \sup_{\mu} [\lambda_{\max}(rW - 4(J_z - \mu)^2)]).$$

For more parameters

$$\mathcal{F}(\mathbf{w}) = \sup_{\mathbf{r}} (\mathbf{r} \cdot \mathbf{w} - \sup_{\mu} [\lambda_{\max}(\mathbf{r} \cdot \mathbf{W} - 4(J_z - \mu)^2)]).$$

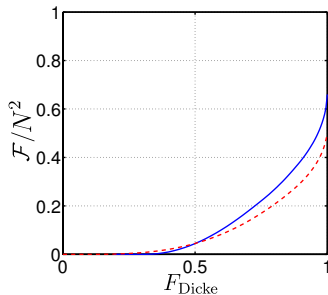
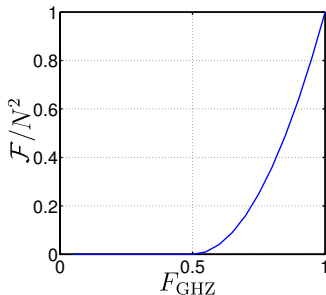
[ I.A., M. Kleinmann, O. Gühne & G. Tóth, arXiv:1511.05203 ]

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[ R. Augusiak *et al.*, arXiv:1506.08837 (2015) ]

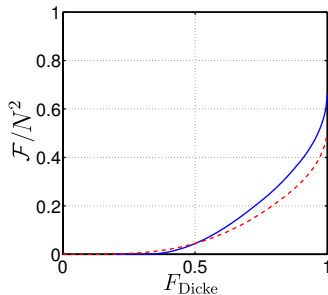
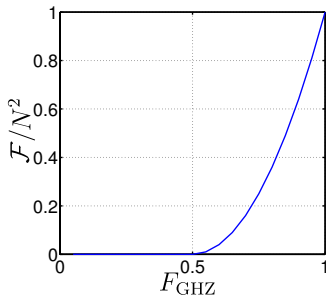
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- For fidelity of GHZ  $\implies$  *analytic solution*

$$\mathcal{F} = \Theta(F_{\text{GHZ}} - 0.5)(2F_{\text{GHZ}} - 1)^2 N^2$$



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- Reducing one dimension of  $\mathcal{F}$  on the  $\langle J_x \rangle$  direction

$$\mathcal{F} \geq \mathcal{F}(\langle J_x \rangle = 0)$$

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$$\mathcal{F}(\langle J_z \rangle, (\Delta J_x)^2) := \mathcal{F}(\langle J_z \rangle, \langle J_x^2 \rangle)$$

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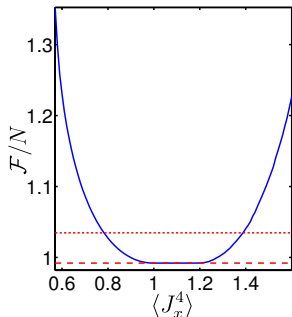
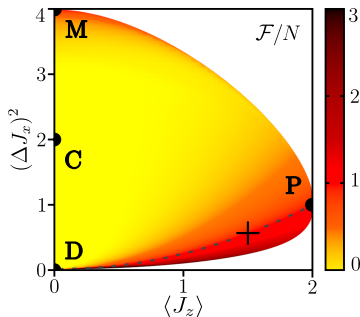
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- Pezze-Smerzi bound,  $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$ , can be verified.

• 4-particle system



**Left:** For  $(\Delta J_x)^2 < 1.5$  it almost coincides with the P-S bound  $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$ . **Right:** The measurement of  $\langle J_x^4 \rangle$  improves the bound.

[ I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203 ]

## Scaling the result for large systems

Experimental setup  $\rightarrow$  [ C. Gross *et al.*, Nature **464**, 1165 (2010) ]

$$N = 2300 \quad \xi_s^2 = -8.2\text{dB} = 0.1514$$

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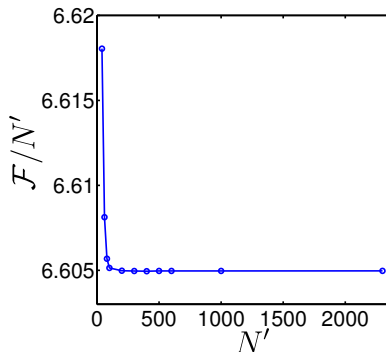
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- P-S bound results is

$$\frac{F_Q}{N} \geq \frac{1}{\xi_s^2} = 6.605$$

- Starting from *small systems*, and assuming bosonic symmetry.
- The results obtained with our method *converge* to P-S bound!



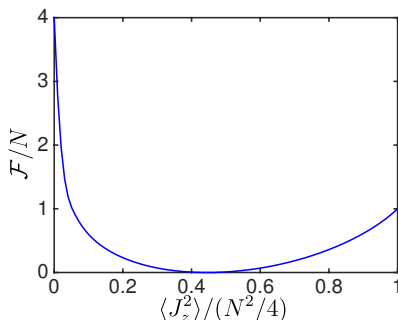


# Metrology with unpolarised Dicke states

- $\{J_x^2, J_y^2, J_z^2\}$ ; Experimental constraint:  $\langle J_x^2 \rangle = \langle J_y^2 \rangle$ .

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**Figure:** For  $\sum_I \langle J_I^2 \rangle = \frac{N}{2}(\frac{N}{2} + 1)$ , i.e. *bosonic symmetry*, and 6-particle system.

# Realistic characterisation of Dicke state

Experiment  $\rightarrow$  [ B. Lücke *et al.*, PRL **112**, 155304 (2014) ]

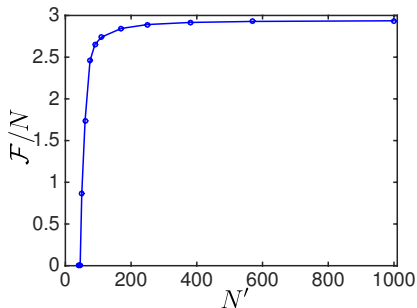
$$N = 7900$$

$$\langle J_z^2 \rangle = 112 \pm 31$$

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6$$

- For that large system, we start from **small ones** similar to the spin-squeezed states.

- Numerical lower bound.



Similarly to the spin-squeezed states, the bound *converges quickly*.

[ I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203 ]

# Conclusion and Outlook

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- 2 We used *our approach to verify* that the P-S bound is tight.
- 3 We have shown that the lower bounds can be *improved with extra constraints*.
- 4 For large systems the *optimisation method can be complemented* with scaling considerations.
- 5 The *method very versatile* and it can be used in many other situations.

# *Thank you for your attention!*

Preprint → arXiv:1511.05203

Groups' home pages

→ <https://sites.google.com/site/gedentqopt>

→ <http://www.physik.uni-siegen.de/tqo/>



iagoba



matthias



otfried



géza