Optimal bound on the quantum Fisher information

Based on few initial expectation values

Iagoba Apellaniz [1], Matthias Kleinmann [1], Otfried Gühne [2], & Géza Tóth [1,3,4]

iagoba.apellaniz@gmail.com

[1]Department of Theoretical Physics, University of the Basque Country, Spain
[2]Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Germany
[3]IKERBASQUE, Basque Foundation for Science, Spain
[4]Wigner Research Centre for Physics, Hungarian Academy of Sciences, Hungary

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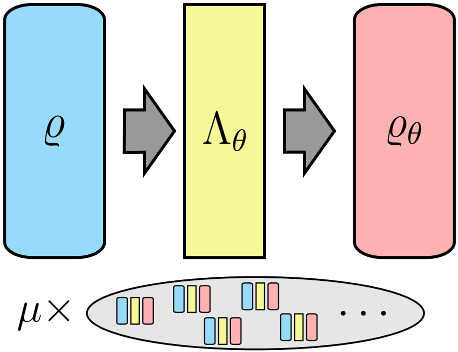
Basics on quantum estimation



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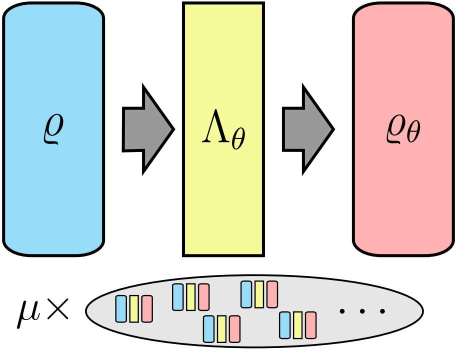

- The precision $(\Delta\theta)^{-1} \propto \sqrt{\mu}$.

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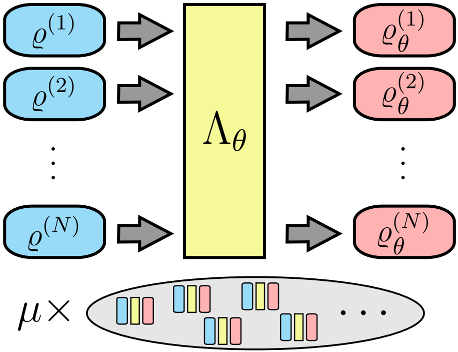

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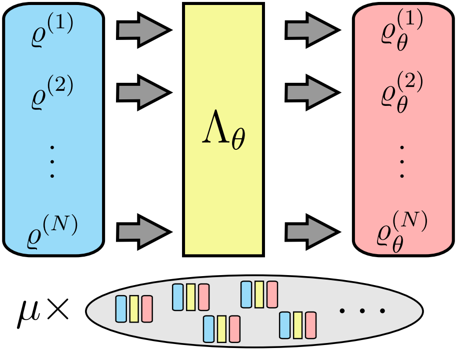

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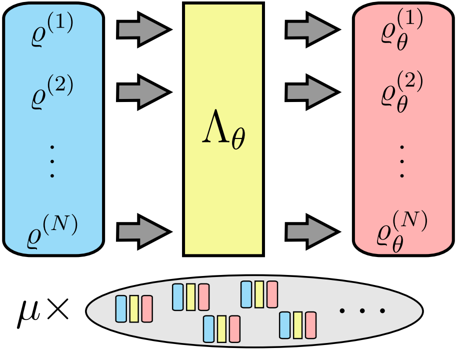

- The precision $(\Delta\theta)^{-1} \propto \sqrt{\mu}$. How it scales with N?

The quantum Fisher information

- The classical Cramér-Rao bound

$$(\Delta\theta)^{-1} \leq \sqrt{\mu \int \mathrm{d}x \, p(x|\theta) \left(\frac{\partial \ln(p(x|\theta))}{\partial\theta}\right)^2}$$

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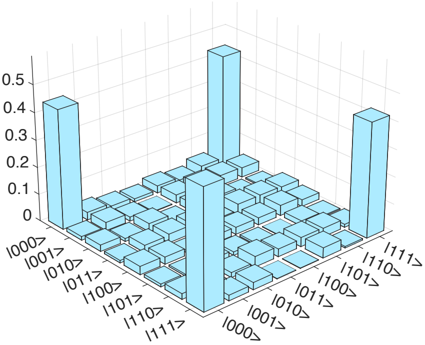
Best separable	Best state
$F_Q \propto N$	$F_Q \propto N^2$

Outline

- For large systems, we only have *a couple of expectation values* to characterise the state.



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- *Many* **inequalities** have been proposed to lower bound the quantum Fisher Information.

Bounds for qFI

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \qquad F_Q[\varrho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},$$

$$F_Q[\varrho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 (\Delta J_y^2)^2} + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle(1 + \langle J_x^2 \rangle) + 6\langle J_y J_x^2 J_y \rangle}$$

[L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009)]

[Z. Zhang & L.-M. Duan, NJP **16**, 103037 (2014)]

[**I.A.**, B. Lücke, J. Peise, C. Klempt & G. Toth, NJP **17**, 083027 (2015)]

- For large systems, we only have *a couple of expectation values* to characterise the state.
- *Many* **inequalities** have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows *metrologically useful entanglement*.

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- *Many* **inequalities** have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows *metrologically useful entanglement*.
- It is essential either to *verify them or to find new ones* for different set of expectation values.

The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2\sum_{\lambda,\gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle\lambda|J_z|\gamma\rangle|^2$$

Alternatively, as convex roof

$$F_Q[\varrho, J_z] = \min_{\{p_k,|\Psi_k\rangle\}} 4\sum_k p_k \left(\Delta J_z\right)^2_{|\Psi_k\rangle}$$

[M.G.A. Paris, Int. J. Quant. Inf. **7**, 125 (2009)]

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

[S. Yu, arXiv:1302.5311]

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- For pure states it's *extremely simple*

$$F_Q[\varrho, J_z] = 4\left(\Delta J_z\right)^2$$

Optimisation based on the Legendre Transform

- When $g(\varrho)$ is a *convex roof*

$$g(\varrho) \geq \mathcal{B}(w := \mathrm{Tr}\,[\varrho W]) = \sup_{r} \left(rw - \sup_{|\psi\rangle} [r\langle W\rangle - g(|\psi\rangle)] \right).$$

[O. Gühne, M. Reimpell & R.F. Werner, PRL **98**, 110502 (2007)]

[J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007)]

Optimisation for the qFI

The *simplicity* of qFI for pure states leads to

$$\mathcal{F}(w) = \sup_{r} \left(rw - \sup_{\mu}[\lambda_{\max}(rW - 4(J_z - \mu)^2)] \right).$$

For more parameters

$$\mathcal{F}(\mathbf{w}) = \sup_{\mathbf{r}} \left(\mathbf{r} \cdot \mathbf{w} - \sup_{\mu}[\lambda_{\max}(\mathbf{r} \cdot \mathbf{W} - 4(J_z - \mu)^2)] \right).$$

[**I.A.**, M. Kleinmann, O. Gühne & G. Tóth, arXiv:1511.05203]

- Measuring F_{GHZ} and F_{Dicke}

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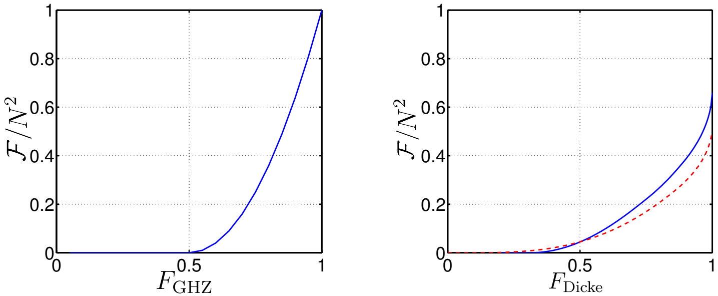

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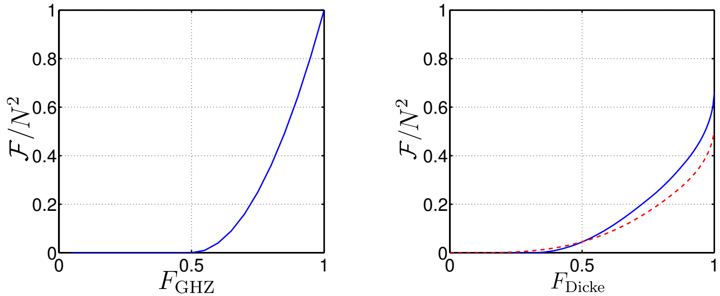

- For fidelity of GHZ $\Longrightarrow$ *analytic solution*

$$\mathcal{F} = \Theta(F_{\mathrm{GHZ}} - 0.5)(2F_{\mathrm{GHZ}} - 1)^2 N^2$$

Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 3 operators $\{J_z, J_x, J_x^2\}$

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- Reducing one dimension of $\mathcal{F}$ on the $\langle J_x \rangle$ direction

$$\mathcal{F} \geq \mathcal{F}(\langle J_x \rangle = 0)$$
$$\Downarrow$$
$$\mathcal{F}(\langle J_z \rangle, (\Delta J_x)^2) := \mathcal{F}(\langle J_z \rangle, \langle J_x^2 \rangle)$$

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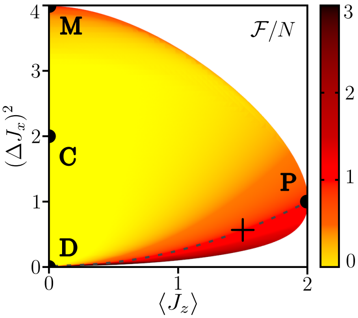
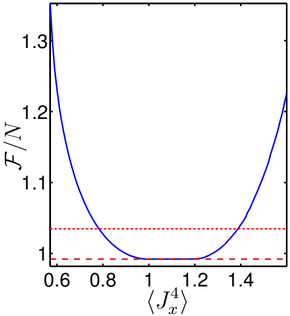
- Pezze-Smerzi bound, $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$, can be verified.

- 4-particle system





Left: For $(\Delta J_x)^2 < 1.5$ it almost coincides with the P-S bound $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$. Right: The measurement of $\langle J_x^4 \rangle$ improves the bound.

[**I.A.**, M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

Scaling the result for la rge systems

Experimental setup → [C. Gross *et al.*, Nature **464**, 1165 (2010)]

$$N = 2300 \qquad \xi_{\mathrm{s}}^2 = -8.2\mathrm{dB} = 0.1514$$

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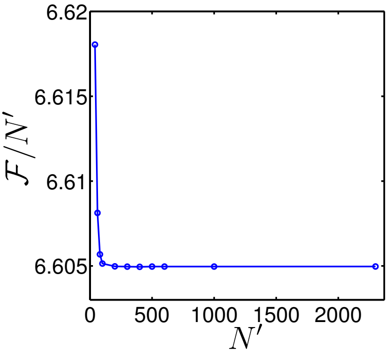
- P-S bound results is

$$\frac{F_Q}{N} \geq \frac{1}{\xi_s^2} = 6.605$$

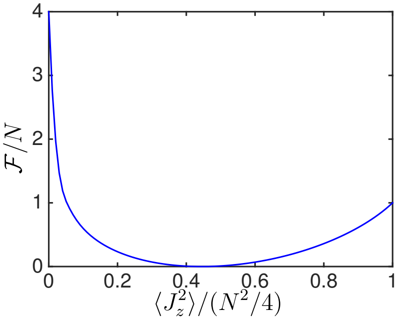
- Starting from *small systems*, and assuming bosonic symmetry.

- The results obtained with our method *converge* to P-S bound!



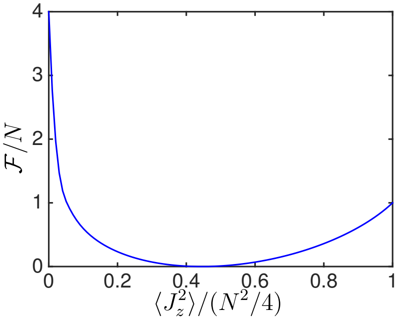
Metrology with unpolarised Dicke states

- 3 operators $\{J_x^2, J_y^2, J_z^2\}$; Experimental constraint: $\langle J_x^2 \rangle = \langle J_y^2 \rangle$.



Metrology with unpolarised Dicke states

- 3 operators $\{J_x^2, J_y^2, J_z^2\}$; Experimental constraint: $\langle J_x^2\rangle = \langle J_y^2\rangle$.
- For $\sum_l \langle J_l^2\rangle = \frac{N}{2}(\frac{N}{2}+1)$, i.e. *bosonic symmetry*, and 6-particle system:



Realistic characterisation of Dicke state

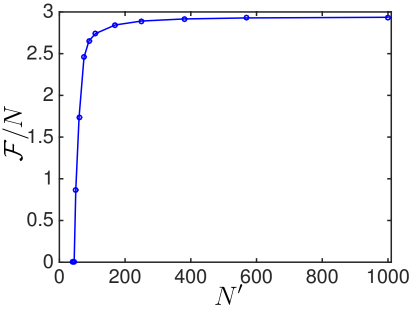
Experiment → [B. Lücke *et al.*, PRL **112**, 155304 (2014)]

$$N = 7900 \qquad \langle J_z^2 \rangle = 112 \pm 31$$

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6$$

- For that large system, we start from small ones similar to the spin-squeezed states.

- Numerical lower bound.



Similarly to the spin-squeezed states, the bound *converges quickly*.

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Conclusion and Outlook

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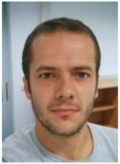
2. We used *our approach to verify* that the P-S bound is tight.
3. We have shown that the lower bounds can be *improved with extra constraints*.
4. For large systems the *optimisation method can be complemented* with scaling considerations.
5. The *method very versatile* and it can be used in many other situations.

Thank you for your attention!

Preprint → arXiv:1511.05203

Groups' home pages

→ *https://sites.google.com/site/gedentqopt*
→ *http://www.physik.uni-siegen.de/tqo/*



iagoba

matthias

otfried

géza