

Optimal bound on the quantum Fisher Information

Based on few initial expectation values of the prove state.

Iagoba Apellaniz¹, Matthias Kleinmann¹, Otfried Gühne²,
& Géza Tóth^{1,3,4}

iagoba.apellaniz@gmail.com

¹Department of Theoretical Physics, University of the Basque Country, Spain

²Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Germany

³IKERBASQUE, Basque Foundation for Science, Spain

⁴Wigner Research Centre for Physics, Hungarian Academy of Sciences, Hungary

Recent Advances in Quantum Metrology; Warsaw - 2016

Outline

- 1 Introduction and Motivation
- 2 QFI based on expectation values: Are they optimal?
 - Optimization problem
- 3 Case study
 - Spin squeezed states
 - Unpolarized Dicke states
- 4 Conclusion and outlook

- ① **Many inequalities** have been proposed to lower bound the quantum Fisher Information.

Bounds for qFI

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \quad F_Q[\varrho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},$$

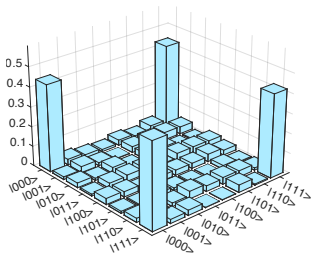
$$F_Q[\varrho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 (\Delta J_y^2)^2 + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle (1 + \langle J_x^2 \rangle)} + 6}$$

[I.A., B. Lücke, J. Peise, C. Klempt & G. Toth, NJP **17**, 083027 (2015)]

[L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009)]

[Z. Zhang & L.-M. Duan, NJP **16**, 103037 (2014)]

- 1 **Many inequalities** have been proposed to lower bound the quantum Fisher Information.
- 2 Typically, we only have *a couple of expectation values* to characterize the state.



- 1 **Many inequalities** have been proposed to lower bound the quantum Fisher Information.
- 2 Typically, we only have *a couple of expectation values* to characterize the state.



- 1 *Many inequalities* have been proposed to lower bound the quantum Fisher Information.
- 2 Typically, we only have *a couple of expectation values* to characterize the state.
- 3 The archetypical criteria that demonstrates *useful entanglement* on the state.

$$F_Q[\varrho, J_z] \geq \frac{\langle J_x \rangle}{(\Delta J_z)^2}$$

[L. Pezzé & A. Smerzi, PRL **102**, 100401 (2009)]

- ① *Many inequalities* have been proposed to lower bound the quantum Fisher Information.
- ② Typically, we only have *a couple of expectation values* to characterize the state.
- ③ The archetypical criteria that demonstrates *useful entanglement* on the state.
- ④ It is essential either to *verify them or find new ones* for different set of expectation values.

- 1 Introduction and Motivation
- 2 QFI based on expectation values: Are they optimal?
 - Optimization problem
- 3 Case study
 - Spin squeezed states
 - Unpolarized Dicke states
- 4 Conclusion and outlook

The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

$$F_Q[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k (\Delta J_z)_{|\Psi_k\rangle}^2$$

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

[S. Yu, arXiv:1302.5311]

The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

$$F_Q[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k (\Delta J_z)_{|\Psi_k\rangle}^2$$

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

[S. Yu, arXiv:1302.5311]

- For pure states it's extremely simple

$$F_Q[\varrho, J_z] = 4 (\Delta J_z)^2$$

The non-trivial exercise of computing the qFI

- Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

$$F_Q[\varrho, J_z] = \min_{\{p_k, |\Psi_k\rangle\}} 4 \sum_k p_k (\Delta J_z)_{|\Psi_k\rangle}^2$$

[G. Tóth & D. Petz, PRA **87**, 032324 (2013)]

[S. Yu, arXiv:1302.5311]

- For pure states it's extremely simple

$$F_Q[\varrho, J_z] = 4 (\Delta J_z)^2$$

- In the general case, usually *lower bounded* by its "classical" counterparts.

Optimization: Legendre Transform

- For a convex function of the state, we construct a *tight lower bound* as follows,

$$g(\varrho) \geq \mathcal{B}(\{w_k := \langle W_k \rangle\}) = \sup_{\{r_k\}} (r \cdot w - \sup_{\varrho} [r \cdot \langle W \rangle - g(\varrho)]).$$

- When $g(\varrho)$ is defined as infimum over the convex roof, the 2nd optimization simplified to pure states only,

$$\mathcal{B}(\{w_k\}) = \sup_{\{r_k\}} (r \cdot w - \sup_{|\psi\rangle} [r \cdot \langle W \rangle - g(|\psi\rangle)]).$$

[O. Gühne, M. Reimpell & R.F. Werner, PRL **98**, 110502 (2007)]

[J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007)]

Optimization for the qFI

The *simplicity* of qFI for pure states allows to rewrite the optimization

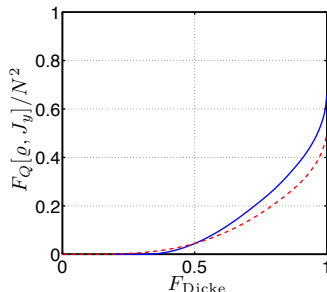
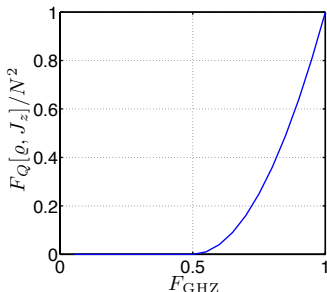
$$\mathcal{F}(\{w_k\}) = \sup_{\{r_k\}} (r \cdot w - \sup_{\mu} [\lambda_{\max}(r \cdot W - 4(J_z - \mu)^2)]).$$

- Therefore, we have parametrised the optimization, which leads to a *more efficient finding* of the solution.

[I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

- 1 Introduction and Motivation
- 2 QFI based on expectation values: Are they optimal?
 - Optimization problem
- 3 Case study
 - Spin squeezed states
 - Unpolarized Dicke states
- 4 Conclusion and outlook

- We'll present 2 main cases, spin-squeezed states and unpolarized Dicke states.
- Though, we apply our method to projectors with great success, we will focus on collective J_n operators.
- One of the cases using projector operators, *i.e.*, using the fidelity leads to *analytic solution*!



Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 1 We use the following 3 operators $\{J_z, J_x, J_x^2\}$ to characterize the input state with their respective expectation values.

Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 1 We use the following 3 operators $\{J_z, J_x, J_x^2\}$ to characterize the input state with their respective expectation values.
- 2 In the direction of $\langle J_x \rangle$ the worst case is when $\langle J_x \rangle$ takes the *value zero*.

Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 1 We use the following 3 operators $\{J_z, J_x, J_x^2\}$ to characterize the input state with their respective expectation values.
- 2 In the direction of $\langle J_x \rangle$ the worst case is when $\langle J_x \rangle$ takes the *value zero*.
- 3 Hence, the optimisation can be accomplished *only using 2 operators* $\{J_z, J_x^2\}$ while the resulting bound is mapped directly to $\langle J_z \rangle, (\Delta J_x)^2$.

Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 1 We use the following 3 operators $\{J_z, J_x, J_x^2\}$ to characterize the input state with their respective expectation values.
- 2 In the direction of $\langle J_x \rangle$ the worst case is when $\langle J_x \rangle$ takes the *value zero*.
- 3 Hence, the optimisation can be accomplished *only using 2 operators* $\{J_z, J_x^2\}$ while the resulting bound is mapped directly to $\langle J_z \rangle, (\Delta J_x)^2$.
- 4 Since $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$ can be used too, we compare it with our numerical result.

- We numerically optimise the lower bound of qFI for a 4 particle system for all possible values of $\langle J_z \rangle$ and $(\Delta J_x)^2$.

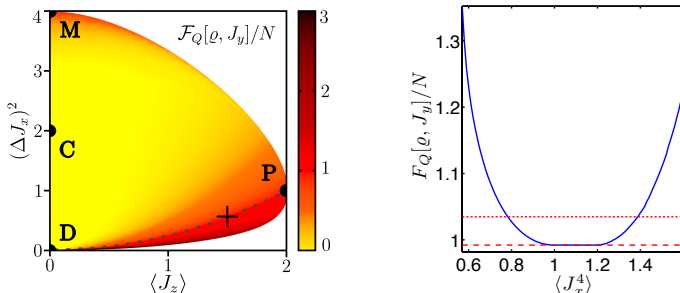


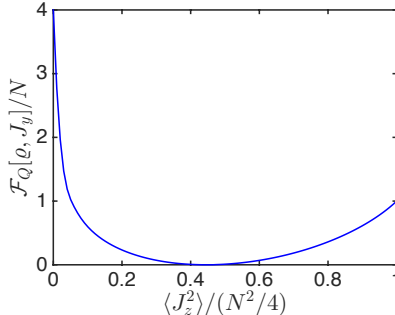
Figure: Below the dashed line, F_Q surpasses the shot noise limit. Cross point enhanced with extra parameter. The result shows an extreme similarity with respect to $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$.

Metrology with unpolarised Dicke states

- 1 In order to characterize the unpolarized Dicke state, we use these 3 operators, $\{J_x^2, J_y^2, J_z^2\}$, with the following constraint, $\langle J_x^2 \rangle = \langle J_y^2 \rangle$.

Metrology with unpolarised Dicke states

- 1 In order to characterize the unpolarized Dicke state, we use these 3 operators, $\{J_x^2, J_y^2, J_z^2\}$, with the following constraint, $\langle J_x^2 \rangle = \langle J_y^2 \rangle$.
- 2 For $\sum_I \langle J_I^2 \rangle = \frac{N}{2}(\frac{N}{2} + 1)$, *i.e.* maximal, the following figure shows an illustrative result for 6 particles.



Realistic characterization of Dicke state

Experiment \rightarrow [B. Lücke *et al.*, PRL **112**, 155304 (2014)]

- We now consider an interesting experimental case with BECs where the following initial values are measured.

Experimental details of unpolarized Dicke state with BEC

$$\begin{aligned} N &= 7900 & \langle J_z^2 \rangle &= 112 \pm 31 \\ \langle J_x^2 \rangle &= \langle J_y^2 \rangle &= 6 \times 10^6 \pm 0.6 \times 10^6 \end{aligned}$$

- For that number of particles we developed a *powerful extrapolation* tool.

Extrapolation

Procedure:

- 1 First, we proportionally increase the expectation values and the unknown bound *to the symmetric subspace*.

$$\langle J_z^2 \rangle_{\text{sym},N} = 145.69 \quad \langle J_x^2 \rangle_{\text{sym},N} = 7.8 \times 10^6 \quad \mathcal{F} = \frac{1}{\gamma} \mathcal{F}_{\text{sym},N}$$

Extrapolation

Procedure:

- 1 First, we proportionally increase the expectation values and the unknown bound *to the symmetric subspace*.

$$\langle J_z^2 \rangle_{\text{sym}, N} = 145.69 \quad \langle J_x^2 \rangle_{\text{sym}, N} = 7.8 \times 10^6 \quad \mathcal{F} = \frac{1}{\gamma} \mathcal{F}_{\text{sym}, N}$$

- 2 We freeze $\langle J_z^2 \rangle$ while we reduce the particle number of an *auxiliary system*. Here, we perform optimisation.

$$2\langle J_x^2 \rangle_{\text{sym}, N'} = \frac{N'}{2} \left(\frac{N'}{2} + 1 \right) - \langle J_z^2 \rangle_{\text{sym}, N'}$$

Extrapolation

Procedure:

- 1 First, we proportionally increase the expectation values and the unknown bound *to the symmetric subspace*.

$$\langle J_z^2 \rangle_{\text{sym}, N} = 145.69 \quad \langle J_x^2 \rangle_{\text{sym}, N} = 7.8 \times 10^6 \quad \mathcal{F} = \frac{1}{\gamma} \mathcal{F}_{\text{sym}, N}$$

- 2 We freeze $\langle J_z^2 \rangle$ while we reduce the particle number of an *auxiliary system*. Here, we perform optimisation.

$$2\langle J_x^2 \rangle_{\text{sym}, N'} = \frac{N'}{2} \left(\frac{N'}{2} + 1 \right) - \langle J_z^2 \rangle_{\text{sym}, N'}$$

- 3 The extrapolation is directly obtained by

$$\mathcal{F}_{\text{sym}, N} \approx \frac{N^2}{N'^2} \mathcal{F}_{\text{sym}, N'}$$

- γ is found to be TODO. We are ready to find the lower bound.

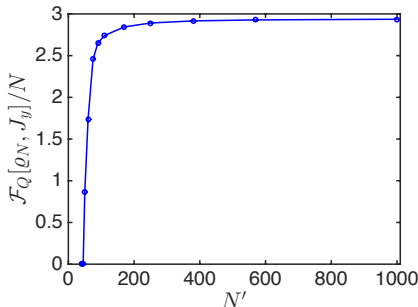


Figure: A fast approach to the real value of the bound is obtained with this method.

[I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

Conclusion and Outlook

- ① We have found that on very interesting and also realistic cases *the optimization is feasible*.

Conclusion and Outlook

- ① We have found that on very interesting and also realistic cases *the optimization is feasible*.
- ② We used *our approach to verify* the tight bounding of one of the inequalities with great success.

Conclusion and Outlook

- ① We have found that on very interesting and also realistic cases *the optimization is feasible*.
- ② We used *our approach to verify* the tight bounding of one of the inequalities with great success.
- ③ We have shown that the lower bounds can be *improved with few extra considerations*.

Conclusion and Outlook

- ① We have found that on very interesting and also realistic cases *the optimization is feasible*.
- ② We used *our approach to verify* the tight bounding of one of the inequalities with great success.
- ③ We have shown that the lower bounds can be *improved with few extra considerations*.
- ④ It has been shown also that the *optimisation method can be complemented* with scaling considerations such as the extrapolation for the Dicke state case.

Conclusion and Outlook

- ① We have found that on very interesting and also realistic cases *the optimization is feasible*.
- ② We used *our approach to verify* the tight bounding of one of the inequalities with great success.
- ③ We have shown that the lower bounds can be *improved with few extra considerations*.
- ④ It has been shown also that the *optimisation method can be complemented* with scaling considerations such as the extrapolation for the Dicke state case.
- ⑤ This *method is universal* and can be applied elsewhere.

Thank you for your attention!

Group's home page → <https://sites.google.com/site/gedentqopt>

TODO: Attach more info