Introduction and Motivation QFI based on expectation values Case study Conclusion and outlook

# Optimal bound on the quantum Fisher information

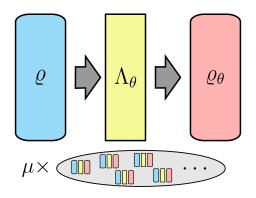
Based on few initial expectation values

**lagoba Apellaniz** <sup>1</sup>, Matthias Kleinmann <sup>1</sup>, Otfried Gühne <sup>2</sup>, & Géza Tóth <sup>1,3,4</sup>

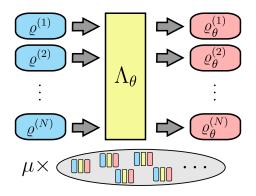
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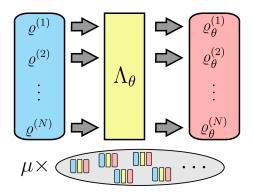
ICE-3 Palma de Mallorca; 2016-04-15



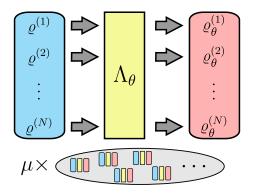
• The precision  $(\Delta \theta)$  is proportional to  $\sqrt{\mu}$ .



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• The precision  $(\Delta \theta)$  is proportional to  $\sqrt{\mu}$ . But for N?

# The quantum Fisher information

The classical Cramér-Rao bound

$$(\Delta \theta)^2 \ge$$

The quantum CR bound

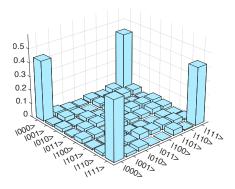
$$(\Delta \theta)^2$$

Fisher information maximised over all possible measurements.

### Outline

- Introduction and Motivation
- 2 QFI based on expectation values: Are they optimal?
  - Optimisation problem
- Case study
  - Fidelities
  - Spin-squeezed states
  - Unpolarised Dicke states
- Conclusion and outlook

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- Many inequalities have been proposed to lower bound the quantum Fisher Information.

#### Bounds for qFI

$$F_{Q}[\varrho, J_{z}] \geq \frac{\langle J_{x} \rangle^{2}}{(\Delta J_{y})^{2}}, \qquad F_{Q}[\varrho, J_{y}] \geq \beta^{-2} \frac{\langle J_{x}^{2} + J_{z}^{2} \rangle}{(\Delta J_{z})^{2} + \frac{1}{4}},$$

$$F_{Q}[\varrho, J_{z}] \geq \frac{4(\langle J_{x}^{2} + J_{y}^{2} \rangle)^{2}}{2\sqrt{(\Delta J_{x}^{2})^{2} (\Delta J_{y}^{2})^{2} + \langle J_{x}^{2} \rangle - 2\langle J_{y}^{2} \rangle(1 + \langle J_{x}^{2} \rangle) + 6\langle J_{y}J_{x}^{2}J_{y} \rangle}}$$

- [ L. Pezzé & A. Smerzi, PRL 102, 100401 (2009) ]
- [ Z. Zhang & L.-M. Duan, NJP 16, 103037 (2014) ]
- [ I.A., B. Lücke, J. Peise, C. Klempt & G. Toth, NJP 17, 083027 (2015) ]

- For large systems, we only have a couple of expectation values to characterise the state.
- Many inequalities have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows metrologically useful entanglement.

$$F_Q[\varrho,J_z] \geq \frac{\langle J_x \rangle}{\left(\Delta J_z\right)^2}$$

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- For large systems, we only have a couple of expectation values to characterise the state.
- Many inequalities have been proposed to lower bound the quantum Fisher Information.
- The archetypical criteria that shows metrologically useful entanglement.
- It is essential either to verify them or to find new ones for different set of expectation values.



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### The non-trivial exercise of computing the qFI

Different forms of the qFI

$$F_Q[\varrho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2$$

Alternatively, as convex roof

$$F_{Q}[\varrho, J_{z}] = \min_{\{p_{k}, |\Psi_{k}\rangle\}} 4 \sum_{k} p_{k} (\Delta J_{z})^{2}_{|\Psi_{k}\rangle}$$

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[ M.G.A. Paris, Int. J. Quant. Inf. 7, 125 (2009) ] [ G. Tóth & D. Petz, PRA 87, 032324 (2013) ]
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• For pure states it's extremely simple

$$F_Q[\varrho, J_z] = 4 (\Delta J_z)^2$$

# Optimisation based on the Legendre Transform

• When  $g(\varrho)$  is a *convex roof* 

$$g(\varrho) \geq \mathcal{B}(w := \operatorname{Tr}\left[\varrho W
brack]
ight) = \sup_{r} \left(rw - \sup_{|\psi
angle}[r\langle W
angle - g(|\psi
angle)]
ight).$$

- [ O. Gühne, M. Reimpell & R.F. Werner, PRL 98, 110502 (2007) ]
- [ J. Eisert, F.G.S.L. Brandão & K.M.R. Audenaert, NJP **9**, 46 (2007) ]

#### Optimisation for the qFI

The *simplicity* of qFI for pure states leads to

$$\mathcal{F}(w) = \sup_{r} \big( rw - \sup_{\mu} [\lambda_{\max}(rW - 4(J_z - \mu)^2)] \big).$$

For more parameters

$$\mathcal{F}(\mathbf{w}) = \sup_{\mathbf{r}} \left( \mathbf{r} \cdot \mathbf{w} - \sup_{\mu} [\lambda_{\max} (\mathbf{r} \cdot \mathbf{W} - 4(J_z - \mu)^2)] \right).$$

[ I.A., M. Kleinmann, O. Gühne & G. Tóth, arXiv:1511.05203 ]

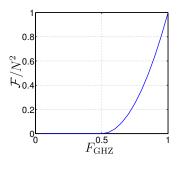
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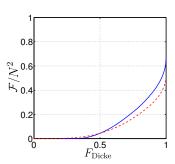
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[ R. Augusiak et al., arXiv:1506.08837 (2015) ]

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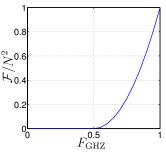
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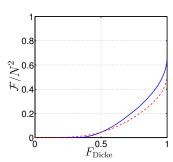




### ullet Measuring $F_{ m GHZ}$ and $F_{ m Dicke}$

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• For fidelity of GHZ  $\implies$  analytic solution

$$\mathcal{F} = \Theta(F_{\rm GHZ} - 0.5)(2F_{\rm GHZ} - 1)^2 N^2$$

# Measuring $\langle J_z angle$ and $\left(\Delta J_{\!\scriptscriptstyle X} ight)^2$ for Spin Squeezed States

• 3 operators  $\{J_z, J_x, J_x^2\}$ 

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- Reducing one dimension of  $\mathcal{F}$  on the  $\langle J_{\mathsf{x}} \rangle$  direction

$$\begin{split} \mathcal{F} \geq \mathcal{F}(\langle J_x \rangle = 0) \\ & \quad \quad \ \ \, \psi \\ \mathcal{F}(\langle J_z \rangle, (\Delta J_x)^2) := \mathcal{F}(\langle J_z \rangle, \langle J_x^2 \rangle) \end{split}$$

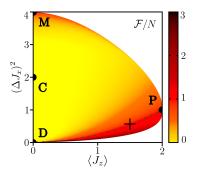
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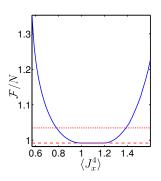
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• Pezze-Smerzi bound,  $F_Q \ge \langle J_z \rangle^2 / (\Delta J_x)^2$ , can be verified.

#### • 4-particle system





Left: For  $(\Delta J_x)^2 < 1.5$  it almost coincides with the P-S bound  $F_Q \geq \langle J_z \rangle^2/(\Delta J_x)^2$ . Right: The measurement of  $\langle J_x^4 \rangle$  improves the bound.

[I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203]

### Scaling the result for la rge systems

Experimental setup  $\rightarrow$  [ C. Gross et al., Nature 464, 1165 (2010) ]

$$N = 2300$$
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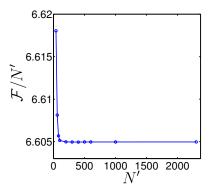
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P-S bound results is

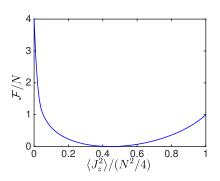
$$\frac{F_Q}{N} \ge \frac{1}{\xi_s^2} = 6.605$$

- Starting from small systems, and assuming bosonic symmetry.
- The results obtained with our method converge to P-S bound!



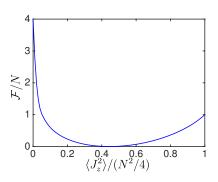
# Metrology with unpolarised Dicke states

• 3 operators  $\{J_x^2, J_y^2, J_z^2\}$ ; Experimental constraint:  $\langle J_x^2 \rangle = \langle J_y^2 \rangle$ .



# Metrology with unpolarised Dicke states

- 3 operators  $\{J_x^2, J_y^2, J_z^2\}$ ; Experimental constraint:  $\langle J_x^2 \rangle = \langle J_y^2 \rangle$ .
- For  $\sum_I \langle J_I^2 \rangle = \frac{N}{2} (\frac{N}{2} + 1)$ , i.e. bosonic symmetry, and 6-particle system:



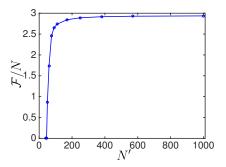
### Realistic characterisation of Dicke state

Experiment  $\rightarrow$  [ B. Lücke et al., PRL 112, 155304 (2014) ]

$$N = 7900$$
  $\langle J_z^2 \rangle = 112 \pm 31$ 

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6$$

 For that large system, we start from small ones similar to the spin-squeezed states. Numerical lower bound.



Similarly to the spin-squeezed states, the bound converges quickly.

[ I.A., M. Kleinmann, O. Güne & G. Tóth, arXiv:1511.05203 ]

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- We used our approach to verify that the P-S bound is tight.
- We have shown that the lower bounds can be improved with extra constraints.
- For large systems the optimisation method can be complemented with scaling considerations.
- The method very versatile and it can be used in many other situations.

# Thank you for your attention!

Preprint  $\rightarrow$  arXiv:1511.05203

### Groups' home pages

- → https://sites.google.com/site/gedentqopt
- → http://www.physik.uni-siegen.de/tqo/



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