Exact methods for tardiness objectives in production scheduling: Appendix

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PROOF OF ALGORITHM 1 VALIDITY

Our notation and proofs follow the style of Hooker (2007). Let $\mathcal{P} = min\{f(\mathrm{X}) + g(\mathrm{Y}) | \mathrm{X} \in D_{\mathrm{X}}, \mathrm{Y} \in D_{\mathrm{Y}}\}$ be the formulation of the combinatorial optimization problem, as described in Section 2. X and Y are groups of variables and D_{X}, D_{Y} the respective domains. The objective function of $\mathcal{P}$ is the sum of the linear cost functions $f(\mathrm{X})$ and $g(\mathrm{Y})$.

Now, let $\mathcal{M} = \{z | z \geq f(\mathrm{X}), \mathrm{X} \in D_{\mathrm{X}}\}$ be the formulation of the master problem and $\mathcal{S} = \{\zeta | \zeta \geq g(\mathrm{Y}) + f(\hat{\mathrm{X}}), \mathrm{Y} \in D_{\mathrm{Y}}\}$, in which $\hat{\mathrm{X}}$ is a feasible solution of $\mathcal{M}$. We notice that ζ is an upper bound of $\mathcal{P}$ for all feasible solutions X and Y, as $f(\mathrm{X}) + g(\mathrm{Y}) \leq f(\hat{\mathrm{X}}) + g(\mathrm{Y}) \leq \zeta$. The validity of the decomposition will be ensured if z is a valid lower bound of $\mathcal{P}$ for all feasible solutions X.

Let $\mathrm{X} = \{Y_{im}, X_{ijm}, C_{jm}, t^d_{jm}, T_{jm}\}$ and $\mathrm{Y} = \{\sigma_{mj}, C^*_{jm}, T^*_i\}$. As $\zeta \geq f(\mathrm{X}) + g(\mathrm{Y})$ and $z \geq f(\mathrm{X})$ for all feasible solution X and Y, the cost functions are defined as $f(\mathrm{X}) = \sum_{j\in J} \sum_{m\in M} T_{jm}$ and $g(\mathrm{Y}) = \sum_{m\in \bar{M}} \sum_{j\in m}(T^*_{m_j} - T_{jm})$. The aggregation of the functions returns a lower bound of ζ: $f(\mathrm{X})+g(\mathrm{Y}) = \sum_{j\in J} \sum_{m\in M} T_{jm} + \sum_{m\in \bar{M}} \sum_{j\in m}(T^*_{m_j} - T_{jm}) \to f(\mathrm{X})+g(\mathrm{Y}) = \sum_{m\in \bar{M}} \sum_{j\in m} T^*_{m_j} = \sum_{i\in J} T^*_i \leq \zeta$.

Now that cost functions and groups of variables are defined and the validity of ζ as an upper bound is proved, we must prove that the generated optimality cuts (16) will always imply a lower bound of $\mathcal{P}$. We must construct a bounding function $B(\mathrm{X}_k)$ which adheres to the following properties:

- P_1: If $\mathrm{X}_k = \hat{\mathrm{X}}_{k-1}$, then $B(\mathrm{X}_k) = \zeta^k$, in which X_k is a feasible solution in iteration k, $\hat{\mathrm{X}}_{k-1}$ is the solution of iteration $k-1$ and ζ^k is the objective value of $\mathcal{S}$ in iteration k.
- P_2: $f(\mathrm{X}_{k-1}) + g(\mathrm{Y}_k) \geq B(\mathrm{X}_k)$ for all solutions X_{k-1}, Y_k and X_k.

Lemma 1. If properties P_1 and P_2 hold for the bounding function $B(\mathrm{X}_k)$ and the domain of variables Y is finite, then Algorithm 1 converges to the optimal solution of $\mathcal{P}$ after finitely many iterations.

Let $U_k = \zeta^k$ be the objective value of $\mathcal{S}$ in iteration k and $\hat{\mathrm{X}}_{k-1} = \{\hat{X}^{k-1}_{ijm}\}$ be the solution of $\mathcal{M}$ in iteration $k-1$, that implied U_k. We also recall that set A_{k-1} includes all assignments of iteration $k-1$ for which $\hat{X}^{k-1}_{a_0a_1a_2} = 1, a \in A_{k-1}$. Let $B(\mathrm{X}_k)$ be a bounding function, defined as follows:

$$B(X^k_{ijm}) = \begin{cases} U_k & \text{if } \{a \in A_{k-1} | X^k_{a_0a_1a_2} = 0\} = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Lemma 2. Function (1) adheres to properties P_1 and P_2 for all feasible solutions X_k.

Proof. Let X_k be the solution of $\mathcal{M}$ in iteration k.

If $\mathrm{X}_k = \hat{\mathrm{X}}_{k-1}$, then all variables X^k_{ijm} are equal with $\hat{X}^{k-1}_{ijm}$. As all assignments $a \in A_{k-1}$ for which $\hat{X}^{k-1}_{a_0a_1a_2} = 1$ are included in the solution k, then, by (1), set $\{a \in A_{k-1} | X^k_{a_0a_1a_2} = 0\} = \emptyset$ and $B_1(X^k_{ijm}) = U_k = \zeta^k \to B_1(\mathrm{X}_k) = \zeta^k$, as P_1 proposes.

On the contrary, if $\mathrm{X}_k \neq \hat{\mathrm{X}}_{k-1}$, there is at least one assignment $a \in A_{k-1}$ for which $X^k_{a_0a_1a_2} = 0 \neq \hat{X}^{k-1}_{a_0a_1a_2}$. By (1), as $\{a \in A_{k-1} | X^k_{a_0a_1a_2} = 0\} \neq \emptyset$, $B_1(X^k_{ijm}) = 0$, which is an obvious lower bound of $f(\mathrm{X}_{k-1}) + g(\mathrm{Y}_k)$ for all solutions X_{k-1} and Y_k, satisfying property P_2. ■

By Lemma 1, function (1) generates cuts that ensure a convergence to optimality.

The linear expression of function (1) is inequality (16) for all iterations k. Indeed, if all assignments $a \in A_{k-1}$ are repeated in k, then $|A_{k-1}| - \sum_{a\in A_{k-1}} X_{a_0a_1a_2} = 0 \to z \geq U_k$. If any assignment is neglected, though, $|A_{k-1}| > \sum_{a\in A_{k-1}} X_{a_0a_1a_2} \to U_k - U_k \cdot (|A_{k-1}| - \sum_{a\in A_{k-1}} X_{a_0a_1a_2}) \leq 0 \to z \geq 0$.

COMPUTATIONAL RESULTS

D : L for *Loose* and T for *Tight* deadlines
LB : Best Lower Bound
UB : Best Upper Bound
Time (s) : Duration of the algorithm in seconds
Gap (%) : $100 \cdot \frac{UB-LB}{UB}$

$\lvert J\rvert$	$\lvert M\rvert$	R	D	LB	UB	Time(s)	Gap(%)
10	2	2	L	504	519	2	2.89
			T	341	357	2	4.48
	5	2	L	229	229	0	0.00
			T	132	132	1	0.00
		5	L	227	227	0	0.00
			T	130	130	0	0.00
20	2	2	L	2148	2203	4	2.50
			T	1762	1818	4	3.08
	5	2	L	738	770	8	4.16
			T	520	552	9	5.80
		5	L	738	760	8	2.89
			T	520	542	8	4.06
	10	2	L	326	354	34	7.91
			T	192	221	34	13.12
		5	L	326	339	19	3.83
			T	192	206	20	6.80
		10	L	326	336	11	2.98
			T	192	203	11	5.42
50	2	2	L	10380	10826	13	4.12
			T	9544	9982	13	4.39
	5	2	L	3528	3722	77	5.21
			T	3039	3230	70	5.91
		5	L	3528	3656	32	3.50
			T	3039	3172	30	4.19
	10	2	L	1590	1751	2468	9.19
			T	1216	1381	2459	11.95
		5	L	1590	1659	89	4.16
			T	1216	1287	81	5.52
		10	L	1590	1643	68	3.23
			T	1216	1269	59	4.18
	20	2	L	817	1162	2539	29.69
			T	495	817	2545	39.41
		5	L	817	883	2538	7.47
			T	495	559	2542	11.45
		10	L	817	855	877	4.44
			T	495	532	899	6.95
		20	L	817	843	139	3.08
			T	495	523	143	5.35
100	2	2	L	48093	49813	55	3.45
			T	46376	48012	55	3.41
	5	2	L	14210	14748	171	3.65
			T	13150	13692	179	3.96
		5	L	14210	14676	147	3.18
			T	13150	13593	139	3.26
	10	2	L	5564	6120	2721	9.08
			T	4770	5292	2709	9.86
		5	L	5564	5836	502	4.66
			T	4770	5033	517	5.23
		10	L	5564	5794	320	3.97
			T	4770	4991	302	4.43
	20	2	L	2392	3845	3221	37.79
			T	1814	3264	3143	44.42
		5	L	2392	2581	3213	7.32
			T	1814	2004	3131	9.48
		10	L	2392	2505	3208	4.51
			T	1814	1931	3134	6.06
		20	L	2392	2472	810	3.24
			T	1814	1892	735	4.12

150	2	2	L	110060	114262	68	3.68
			T	107521	111757	69	3.79
	5	2	L	29935	31163	338	3.94
			T	28324	29620	310	4.38
		5	L	29935	30981	188	3.38
			T	28324	29433	184	3.77
	10	2	L	11528	12423	1629	7.20
			T	10448	11366	1574	8.08
		5	L	11528	12003	646	3.96
			T	10448	10912	587	4.25
		10	L	11528	11938	430	3.43
			T	10448	10847	376	3.68
	20	2	L	5479	9409	2346	41.77
			T	4563	8579	2059	46.81
		5	L	5479	5879	2336	6.80
			T	4563	4994	2058	8.63
		10	L	5479	5776	2339	5.14
			T	4563	4862	2057	6.15
		20	L	5479	5714	1134	4.11
			T	4563	4806	857	5.06
200	2	2	L	188948	196610	137	3.90
			T	185590	193114	125	3.90
	5	2	L	52528	54850	759	4.23
			T	50577	52860	729	4.32
		5	L	52528	54581	395	3.76
			T	50577	52573	380	3.80
	10	2	L	21696	23566	2115	7.94
			T	20193	22016	1989	8.28
		5	L	21696	22726	1207	4.53
			T	20193	21150	1011	4.52
		10	L	21696	22635	915	4.15
			T	20193	21063	790	4.13
	20	2	L	9302	16463	3675	43.50
			T	8100	15141	3121	46.50
		5	L	9302	9942	3675	6.44
			T	8100	8722	3109	7.13
		10	L	9302	9761	3662	4.70
			T	8100	8545	3104	5.21
		20	L	9302	9679	2448	3.90
			T	8100	8463	1916	4.29