Time Series Analysis
DSC534

Dr Yiolanda Englezou

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Notes based on notes of Prof. Konstantinos Fokianos

Lecture 3

Measures of Dependence

When we observe $X = (X_1, \dots, X_n)^T$ from a time series $(X_t, t \in T)$ we deal only with a single observation. Hence, we need to do the joint c.d.f

$$F_{1:n}(x_1, \dots, x_n) = \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n)$$

It is impossible to know this c.d.f unless we assume joint normality.

Consider marginals

$$F_t(x) = \mathbb{P}(X_t \leq x), \quad t = 1, \dots, n$$

$$f_t(x) = \frac{dF_t(x)}{dx}$$

Mean function $\mu_t = \mu_{X_t} = E(X_t) = \int x \, f_t(x) \, dx$, αν υπάρχει

Autocovariance Function

Definition Assume that (X_t) is a time series with $E(X_t^2)$ exists.

The autocovariance function is defined

$$\gamma_X(s,t) = Cov(X_s, X_t), \quad \text{for all } s,t.$$
$$= E\{(X_s - \mu_s)(X_t - \mu_t)\}$$

Note $\gamma_X(s,t) = \gamma_X(t,s)$

The autocovariance function measures the linear dependence between two measurements observed at different time points.

Recall that $\gamma_X(t,s) = 0 \Rightarrow E(X_s X_t) = E(X_s)E(X_t) = \mu_s \cdot \mu_t$

$\Rightarrow X_s, X_t$ are uncorrelated but not independent.

But if (X_s, X_t) are normal then we have that X_s, X_t are independent.

If $s=t$ then we have the variance of X_t

$$\gamma_X(t,t) = E(X_t - \mu_t)^2 = Var(X_t)$$

Example #1 Autocovariance of white noise, $W_t \sim WN(0,\sigma^2)$

Recall that $E(W_t)=0$

$$\gamma_W(s,t) = Cov(W_s, W_t) = E(W_s W_t)$$
$$= \begin{cases} \sigma^2, & s=t \\ 0 & s\neq t \end{cases}$$

Property (covariance of linear combinations)

If the random variables $U = \sum_{j=1}^{m} a_j X_j$, $V = \sum_{k=1}^{r} b_k Y_k$

are linear combinations of (finite variance) r.v. (X_j), (Y_k)

$$Cov(U, V) = \sum_{j=1}^{m} \sum_{k=1}^{r} a_j b_k Cov(X_j, Y_k)$$

Example #2 We will calculate the autocovariance function of a moving average

$$X_t = \frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right), \quad W_t \sim WN(0, \sigma_w^2)$$

$$\gamma_X(s,t) = Cov(X_s, X_t)$$

$$= Cov\left\{ \frac{1}{3}\left(W_{s-1} + W_s + W_{s+1}\right), \frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right) \right\}$$

When $s = t$

$$\gamma_X(t,t) = \frac{1}{9} Cov\left\{ \left(w_{t-1} + w_t + w_{t+1}\right); \left(w_{t-1} + w_t + w_{t+1}\right) \right\}$$

$$= \frac{1}{9}\left\{ Cov(w_{t-1}, w_{t-1}) + Cov(w_t, w_t) + Cov(w_{t+1}; w_{t+1}) \right\}$$

↳ because everything else is 0

$$= \frac{1}{9}\left\{ \sigma_w^2 + \sigma_w^2 + \sigma_w^2 \right\}$$

$$= 3/9 \; \sigma_w^2$$

When $s = t+1$

$$\gamma_x(t+1, t) = \frac{1}{9} \operatorname{Cov}\left\{ (w_t + w_{t+1} + w_{t+2}) ; (w_{t-1} + w_t + w_{t+1}) \right\}$$

$$= \frac{1}{9} \left\{ \operatorname{Cov}(w_t, w_t) + \operatorname{Cov}(w_{t+1}, w_{t+1}) \right\}$$

$$= \frac{2}{9} \sigma_w^2$$

When $s = t-1$

$$\gamma_x(t-1, t) = \frac{2}{9} \sigma_w^2$$

When $s = t+2$

$$\gamma_x(t+2, t) = \frac{1}{9} \operatorname{Cov}\left[(\underline{w_{t+1}} + w_{t+2} + w_{t+3}), (w_{t-1} + w_t + \underline{w_{t+1}}) \right]$$

$$= \sigma_w^2 / 9$$

When $s = t-2$

$$\gamma_x(t-2, t) = \sigma_w^2 / 9$$

when $|s-t| \geq 2$ $\quad \gamma_x(s,t) = 0$

$$\gamma_x(s,t) = \begin{cases} 3/9\,\sigma_w^2 & s = t \\ 2/9\,\sigma_w^2 & |s-t| = 1 \\ 1/9\,\sigma_w^2 & |s-t| = 2 \\ 0 & |s-t| > 2 \end{cases}$$

Covariance function that decreases as the seperation between time points increases and becomes 0 if the seperation is three or more.

Example 3 (Random Walk)

We will calculate the autocovariance function of

$$X_t = \sum_{j=1}^{t} W_j \quad , \qquad W_j \sim WN(0, \sigma_w^2)$$

$$\gamma_X(s,t) = Cov(X_s, X_t) = Cov\left(\sum_{j=1}^{s} W_j , \sum_{j=1}^{t} W_j\right)$$

$s < t$ $$Cov\left(\sum_{j=1}^{s} W_j , \sum_{j=1}^{s} W_j + \sum_{j=s+1}^{t} W_j\right) = s\,\sigma_w^2$$

$s = t$ $$Cov\left(\sum_{j=1}^{s} W_j , \sum_{j=1}^{s} W_j\right) = s\,\sigma_w^2$$

$s > t$ $$Cov\left(\sum_{j=1}^{t} W_j + \sum_{j=t+1}^{s} W_j , \sum_{j=1}^{t} W_j\right) = t\,\sigma_w^2$$

$$\gamma_X(t,s) = \min(s,t)\,\sigma_w^2$$

This depends on particular time points

$Var(X_t) = t\,\sigma_w^2$ so the variance of the random walk increases without bound.

Example 4 (Non-linear moving average)

$$X_t = W_t \cdot W_{t-1} \qquad W_t \sim WN(0,\sigma^2)$$

$$\mu_t = E(X_t) = E(W_t W_{t-1}) = E(W_t)\, E(W_{t-1}) = 0$$

$$\gamma_X(s,t) = Cov(X_s, X_t) = E(X_s X_t)$$

$s=t$ $\quad \gamma_X(t,t) = E(X_t^2) = E(W_t^2 W_{t-1}^2) =$

$$= E(W_t^2)\, E(W_{t-1}^2) = \sigma_w^2 \cdot \sigma_w^2 = \sigma_w^4$$

$s=t+1$ $\quad \gamma_X(t+1,t) = E(X_{t+1} X_t) = E(W_{t+1} W_t W_t W_{t-1}) = 0$

$s=t-1$ $\quad \gamma_X(t-1,t) = 0$

In general for $|s-t| \geq 2$ then $\gamma_X(s,t) = 0$

$$\gamma_X(s,t) = \begin{cases} \sigma_w^4, & s=t \\ 0, & s \neq t \end{cases}$$

This is an example where the sequence (X_t) consists of dependent r.v. but still their autocorrelation function is 0.

It is more convenient to define a measure of association between $(-1, 1)$.

Definition (Autocorrelation Function)

The autocorrelation function (ACF) is defined as

$$\rho_X(s,t) = \frac{\gamma_X(s,t)}{\sqrt{\gamma_X(s,s)}\sqrt{\gamma_X(t,t)}}$$

It holds that

$$-1 \leq \rho_X(s,t) \leq 1.$$

If we can predict perfectly X_t by X_s then we can write

$$X_t = \beta_0 + \beta_1 X_s .$$

If $\rho_X(s,t) = 1 \implies \beta_1 > 0$

$\rho_X(s,t) = -1 \implies \beta_1 < 0$

For the previous examples (check!)

White noise
$$\rho_w(s,t) = \begin{cases} 1, & s=t \\ 0, & s \neq t \end{cases}$$

Moving average
$$\rho_w(s,t) = \begin{cases} 1, & s=t \\ 2/3, & |s-t|=1 \\ 1/3, & |s-t|=2 \\ 0, & |s-t|>2 \end{cases}$$

Random Walk
$$\rho_x(s,t) = \frac{\min(s,t)\,\cancel{\sigma_w^2}}{\sqrt{s\,\cancel{\sigma_w^2}}\,\sqrt{t\,\cancel{\sigma_w^2}}}$$
$$= \begin{cases} \sqrt{s/t} & s \le t \\ \sqrt{t/s} & s > t \end{cases}$$

Non-linear Moving average
$$\rho_x(s,t) = \begin{cases} 1, & s=t \\ 0, & s \neq t \end{cases}$$

Stationary Time Series

We introduce a fundamental concept in time series analysis

Definition A strictly stationary time series is defined by assuming the probabilistic behavior of every collection of variables

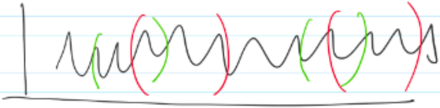
$$\{X_{t_1}, \ldots, X_{t_n}\}$$

is identical to that any time shifted set

$$\{X_{t_1+h}, \ldots, X_{t_n+h}\}$$

$$\mathbb{P}(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n) = \mathbb{P}(X_{t_1+h} \le x_1, \ldots, X_{t_n+h} \le x_n)$$

for all $n = 1, 2, \ldots$ all time points $t_1, \ldots, t_n$ all $x_1, \ldots, x_n$ and $h = 0, \pm 1, \pm 2, \ldots$



Suppose that $n=1$, $\mathbb{P}(X_t \le x) = P(X_{t+h} \le x)$ for all h

So the probability that X_t is less than 1 at 13.00 is equal to the probability that X_t is less than 1 at 17:00.

$\mu_{X_t} = \mu_{X_s}$ for all s, t $\Rightarrow$ $\mu_t = \mu$ (constant)

↳ all examples we have seen have $\mu = 0$ except random walk.

Suppose $n=2$.

$\mathbb{P}(X_t \le x_1, X_s \le x_2) = \mathbb{P}(X_{t+h} \le x_1, X_{s+h} \le x_2)$

for all s, t, h.

$\gamma_X(s+h, t+h) = \gamma_X(s, t)$

The autocovariance function depends on the time difference between time points and not on the time point.