Time Series Analysis
DSC534

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Notes based on notes of Prof. Konstantinos Fokianos

Lecture 2

Mathematical concepts

Definition A time series process is a set of random variables $\{X_t, t \in T\}$ where T is the set of times the process is observed.

We assume that each r.v. X_t is distributed according to some univariate distribution F_t. We will consider equidistant time time intervals and X_t real valued (but it can also take integer values). So we can write $T = \{1, 2, 3, \ldots\}$

An observed time series is a realization of the random vector $X = (X_1, \ldots, X_n)^T$, and it is denoted with small letters $(x_1, \ldots, x_n)^T$. So, a time series is only one single realization of the n-dimensional random vector X, which has multivariate c.d.f. $F_{1:n}$ ($\mathbb{R}^n \to [0,1]$). It is impossible to do Statistics with just one observation so we impose conditions on $F_{1:n}$.

Examples of Time Series

1) White noise

A collection of uncorrelated random variables with mean 0 and variance σ^2,

$$W_t \sim WN(0, \sigma^2)$$

White light $\Leftrightarrow$ all possible periodic oscillations are present with equal strength.

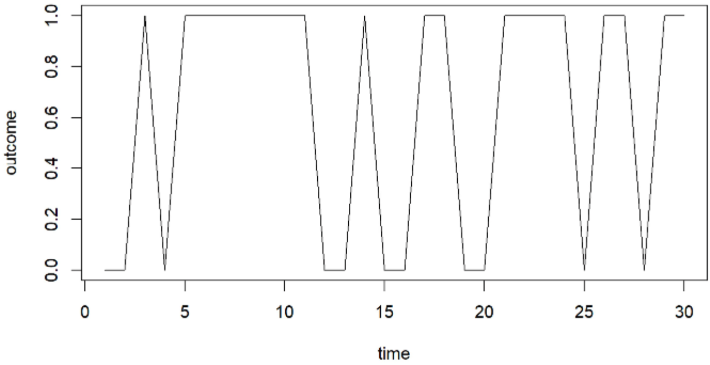
Sometimes, we want W_t to be independent and identically distributed, $W_t \sim$ i.i.d $(0, \sigma^2)$.

Bernoulli sequence (i.i.d example)

A sequence of independent Bernoulli random variables, say W_t, with probability of success p is a trivial example of a time series. $W_t = 1$ or 0 $\quad F_t = \text{Bernoulli}(p)$

Goals of Time Series analysis

```
set.seed(1245)
p=6/10
n=30
x=rbinom(n,1,p)
ts.plot(x, xlab="time",ylab="outcome") ##not that informative
```



More important examples that take into account dependence

Otherwise if W_t is i.i.d $\Rightarrow$ classical statistical theory.

2) Moving Average and filtering

moving average

Let $W_t \sim \text{i.i.d}(0, \sigma^2)$.

$$X_t = \frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right) \rightarrow \text{smooth version}$$

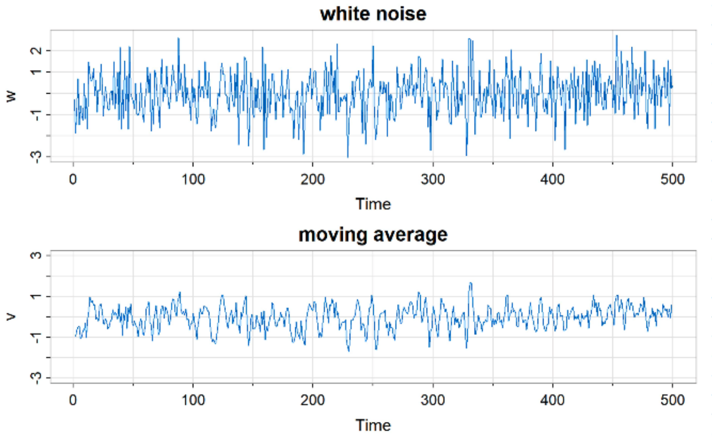
Generally, we call this process filtering

Normal sequence + filtering

A sequence W_t of independent normal random variables, say W_t, with mean μ and variance σ^2 is a trivial example of a white noise series. In this case $W_t \in \mathbb{R}$, $F_t = N(\mu, \sigma^2)$.

Below, see the effect of filtering.

```
par(mfrow=2:1)
w=rnorm(500) #500 variates from N(0,1)
v=filter(w,sides=2,filter=rep(1/3,3)) #moving average
ts.plot(w,col=4, main="white noise")
ts.plot(v,ylim=c(-3,3),col=4, main="moving average")
```



Normal sequence (slowly varying trend)

A sequence of independent random variables W_t with mean

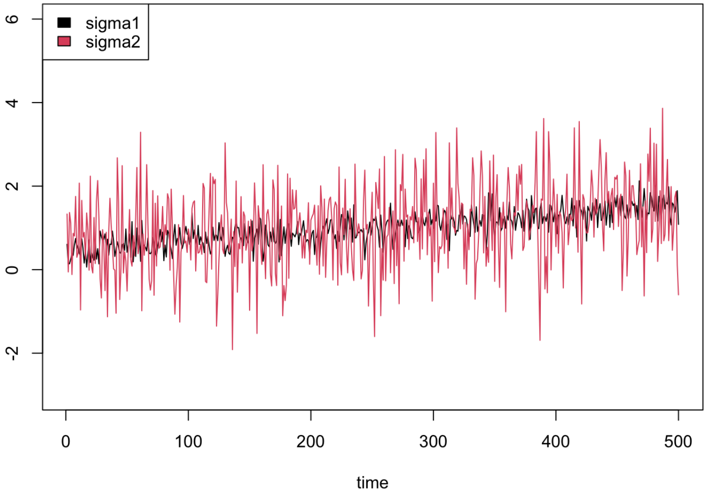
$$\mu_t = \alpha + \beta t / n$$

and variance σ^2, is another example of time series.

In this case, if we assume normality, $W_t \in \mathbb{R}$, $F_t = N(\mu_t, \sigma^2)$.

The following plot show the effect of σ^2 to the observed data.

```
set.seed(1245)
alpha=0.5
beta=1
n=500
mu.t=alpha+beta*(1:n)/n
sigma1=0.25
sigma2=1
x1=rnorm(n,mu.t,sigma1)
x2=rnorm(n,mu.t,sigma2)
ts.plot(cbind(x1,x2),col=1:2, xlab="time",ylim=c(-3,6))
leg.txt<-c("sigma1","sigma2")
legend("topleft",leg.txt,fill=1:2)
```


sigma1
sigma2
6
4
2
0
-2
0
100
200
300
400
500
time

Autoregressions

$$X_t = X_{t-1} - 0.8\, X_{t-2} + W_t, \quad t = 1, \dots, 500$$

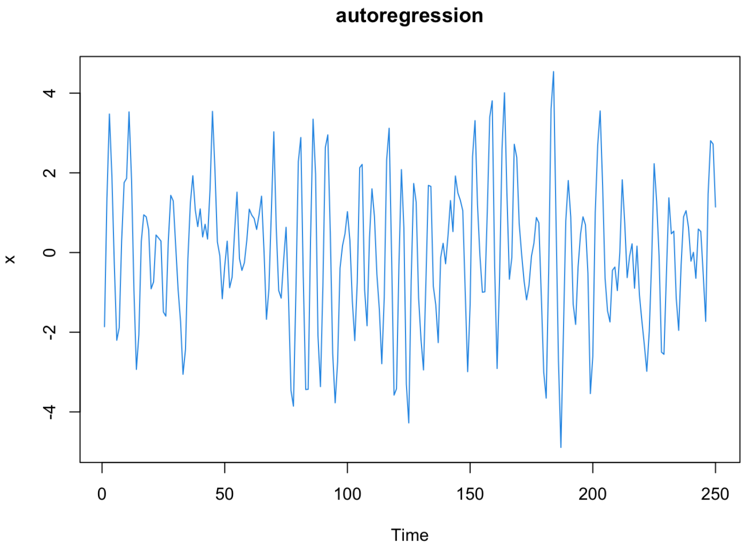
Regression or prediction of the current value X_t as a function of the past two values of the series (autoregression).

There exists a problem with starting values x_0, x_{-1} but if we know them then the process can run.

The function filter() is using $x_0 = x_{-1} = 0$.

Run longer than needed and remove initial values

```
w=rnorm(250+50) #50 extra to avoid startup problems
##remove the first 50 observations
x=filter(w,filter=c(1,-.80),method="recursive")[-(1:50)]
ts.plot(x,main="autoregression",col=4)
```


autoregression
x
4
2
0
-2
-4
0
50
100
150
200
250
Time

Random walk with drift

A method to analyze trend is the random walk with drift model

$$X_t = \delta + X_{t-1} + W_t, \qquad X_0 = 0$$
$$W_t \sim WN(0, \sigma^2)$$

$\delta \to$ drift If $\delta = 0 \Rightarrow$ simple random walk.

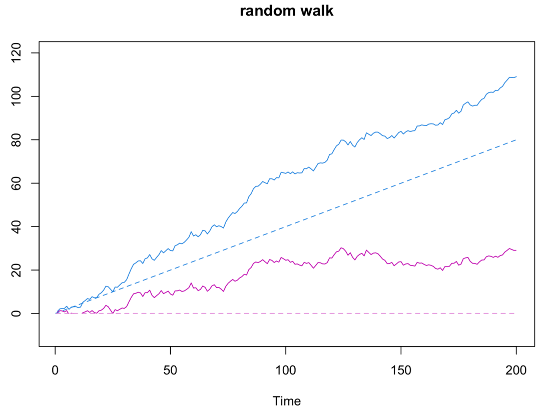
If $\delta = 0$, then the value of X_t is the value of X_{t-1} plus noise.

$$\begin{aligned} X_t &= \delta + X_{t-1} + W_t \\ &= \delta + (\delta + X_{t-2} + W_{t-1}) + W_t \\ &= 2\delta + X_{t-2} + W_{t-1} + W_t \\ &= 3\delta + X_{t-3} + W_{t-2} + W_{t-1} + W_t \\ &= t\delta + \cancel{X_0}^{\,0} + \sum_{j=1}^{t} W_j \end{aligned}$$

(increasing or decreasing behavior)

For the following plot $\delta = 0$ (red) and $\delta = 4/10$ (blue)

```
set.seed(12345)
w=rnorm(200)
x=cumsum(w)
wd=w+4/10
xd=cumsum(wd)
ts.plot(xd,ylim=c(-10,120),main="random walk",ylab=" ",col=4)
clip(0,200,0,120)
abline(a=0,b=.4,lty=2,col=4) #drift
lines(x,col=6)
clip(0,200,0,120)
abline(h=0,col=6,lty=2)
```



Signal in noise model

An underlying signal with some consistent periodic variation contaminated with noise can be a useful representation for time series data.

$$X_t = 2\cos\left(2\pi\frac{(t+15)}{50}\right) + W_t, \quad t=1,\dots,500$$

sinusoidal waveform $A\cos(2\pi\omega t+\phi)$

A: amplitude $\qquad A = 2$

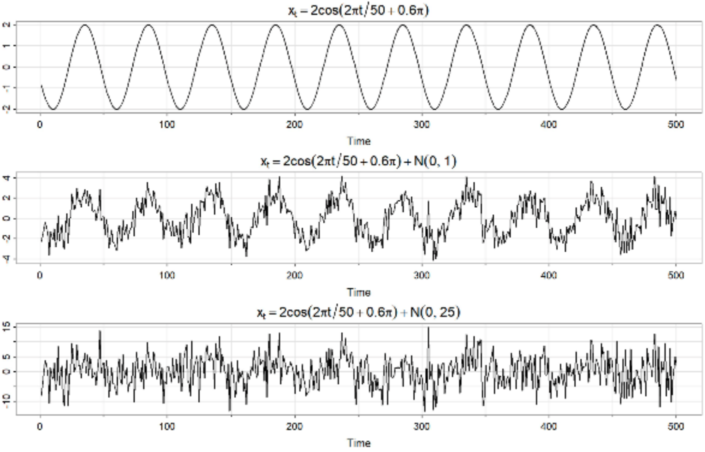
ω: frequency $\qquad \omega = 1/50$ (one cycle every 50 time points)

ϕ: phase shift $\qquad \phi = \frac{2\pi\cdot 15}{50} \approx \frac{6}{10}\pi$.

For the following plot, additive noise $\sigma_w^2 = 1$ (middle panel) $\sigma_w^2 = 5$ (bottom panel)

Ratio of the amplitude of the signal to σ_w^2 is usually called signal-to-noise ratio (SNR).

```
library(astsa)
cs=2*cos(2*pi*(1:500)/50+.6*pi)
cs=2*cos(2*pi*(1:500+15)/50) #same thing
w=rnorm(500,0,1)
par(mfrow=c(3,1))
ts.plot(cs,ylab=" ",main=expression(x[t]==2*cos(2*pi*t/50+.6*pi)))
ts.plot(cs+w,ylab=" ",main=expression(x[t]==2*cos(2*pi*t/50+.6*pi)+N(0,1)))
ts.plot(cs+5*w,ylab=" ",main=expression(x[t]==2*cos(2*pi*t/50+.6*pi)+N(0,25)))
```



$SNR = \frac{2}{1} = 2$

$SNR = \frac{2}{5}$

Measures of Dependence

As we said before if we observe a sample from a time series $(X_1, .., X_n)$ then we need to know the joint c.d.f.

$$F_{1:n}(x_1, .., x_n) = \mathbb{P}(X_1 \leq x_1, .., X_n \leq x_n).$$

It is impossible to know the form of the c.d.f. unless we assume joint normality. Plots of this c.d.f. cannot be drawn and its analytical calculation is very challenging.

Consider marginal

$$F_t(x) = \mathbb{P}(X_t \leq x) \qquad \text{c.d.f}$$

$$f_t(x) = \frac{d F_t(x)}{dx}, \qquad \text{density.}$$

Definition (mean function)

The mean function is defined by

$$\mu_{X_t} = \underset{\mu_t}{\underset{\shortparallel}{E(X_t)}} = \int x \, f_t(x) \, dx \qquad \text{if it exists.}$$

Example # 1

(W_t), white noise sequence $\mu_{W_t} = E(W_t) = 0$ for all t

Now consider filtering (moving average)

$$X_t = \frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right)$$

$$\mu_{X_t} = E(X_t) = 0$$

Example 2.

Random walk with drift $X_t = \delta t + \sum_{j=1}^{t} W_j$

$$\mu_{X_t} = E(X_t) = E\left(\delta t + \sum_{j=1}^{t} W_j\right) = \delta t$$

Example 3

$$X_t = 2\cos\left(2\pi \frac{(t+15)}{50}\right) + W_t$$

$$\mu_{X_t} = E(X_t) = E\left\{2\cos\left(\frac{2\pi(t+15)}{50}\right) + W_t\right\}$$

$$= 2\cos\left[\frac{2\pi(t+15)}{50}\right]$$