Time Series Analysis
DSC534

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Notes based on notes of Prof. Konstantinos Fokianos

Lecture 4

Weak Stationarity

Strict stationarity is really strong assumption.

You cannot assess strict stationarity for a single data set.

Definition (Weak Stationarity)

A weakly **stationary** time series (X_t) is a finite variance process that satisfies

1.) The mean function μ_t, is constant (does not depend on t)

2) The autocovariance function $\gamma_X(s,t)$ depends on s and t through their difference $|s-t|$.

Remark: From previous lecture strict stationarity $\Rightarrow$ weak stationarity

The converse is true for Gaussian processes

We write $E(X_t) = \mu_t = \mu \quad \forall \ t$

If $s = t+h$ $\quad \gamma_X(s,t) = \gamma_X(t+h, t) = Cov(X_{t+h}, X_t)$

$h =$ lag $\quad = Cov(X_h, X_0) = \gamma_X(h,0) = \gamma_X(h)$

So we can define $\gamma_X(h) = Cov(X_{t+h}, X_t) = E\left[(X_{t+h} - \mu)(X_t - \mu)\right]$

ACF $\quad \rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)}$

Example 1 (White Noise)

$W_t \sim WN(0, \sigma_w^2)$, it is a stationary process since $E(W_t) = 0$

$$\gamma_W(h) = \begin{cases} \sigma_w^2, & h = 0 \\ 0; & h \neq 0 \end{cases} \qquad h = 0, \pm 1, \pm 2, \ldots$$

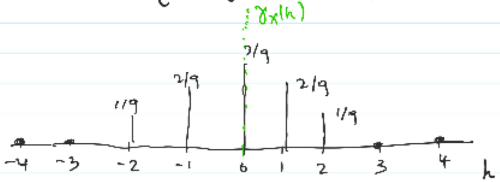
$\hookrightarrow Cov(X_t, X_{t+h})$

Example 2 (Moving Average)

Recall that $X_t = \frac{1}{3}(W_{t-1} + W_t + W_{t+1})$, $W_t \sim WN(0, \sigma_w^2)$

$E(X_t) = 0$

$$\gamma_X(h) = \begin{cases} 3/9\, \sigma_W^2 & h = 0 \\ 2/9\, \sigma_W^2 & h = \pm 1 \ (|h| = 1) \\ 1/9\, \sigma_W^2 & h = \pm 2 \ (|h| = 2) \\ 0 & |h| > 2 \end{cases}$$



Why $\gamma_X(h)$ is symmetric? i.e $\gamma_X(-h) = \gamma_X(h)$ for all $h > 0$

$\gamma_X(-h) = Cov(X_{t-h}, X_t) = Cov(X_t, X_{t+h}) = \gamma_X(h)$

Example 3 (Random Walk)

$W_t \sim WN(0, \sigma_w^2)$ $\qquad X_t = \sum_{j=1}^{t} W_j$

$E(X_t) = 0$ (constant), $\qquad \gamma_X(s,t) = \min(s,t)\,\sigma_w^2$

so it is not stationary!

Random walk with drift $\quad X_t = \delta t + \sum_{j=1}^{t} W_j$, $\quad E(X_t) = \delta t$

not stationary.

Example 4 (non-linear moving average)

$W_t \sim WN(0, \sigma_w^2)$ $\qquad X_t = W_t W_{t-1}$ (prove that X_t is stationary)

Example 5 (trend stationarity)

Suppose that $X_t = a + bt + Y_t$, (Y_t) stationary

$E(Y_t) = \mu_y$, $\gamma_Y(h)$

$E(X_t) = a + bt + \mu_y$ (so the mean is not constant)

$\gamma_X(h) = Cov(X_{t+h}, X_t) = E\left\{ (X_{t+h} - E(X_{t+h}))(X_t - E(X_t)) \right\}$

$= E(Y_{t+h} - \mu_y)(Y_t - \mu_y) = \gamma_Y(h)$

Such models may be considered as having stationary behavior around a linear trend (trend stationarity).

Example 6 (Random Cosine Wave)

$$Y_t = \cos\left[2\pi\left(\frac{t}{12} + \phi\right)\right], \quad t = 0, \pm 1, \ldots$$

$\phi \sim U(0,1)$

$$E(Y_t) = E\left\{\cos\left[2\pi\left(\frac{t}{12} + \phi\right)\right]\right\}$$

$$= \int_0^1 \cos\left(2\pi\left(\frac{t}{12} + \phi\right)\right) d\phi$$

$$= \frac{1}{2\pi} \sin\left[2\pi\left(\frac{t}{12} + \phi\right)\right]\Big|_0^1 = 0$$

$$\gamma_Y(h) = Cov(Y_t, Y_{t+h})$$

$$= E(Y_t \cdot Y_{t+h})$$

$$= E\left\{\cos\left[2\pi\left(\frac{t}{12} + \phi\right)\right] \cos\left[2\pi\left(\frac{t+h}{12} + \phi\right)\right]\right\}$$

$$= \int_0^1 \cos\left[2\pi\left(\frac{t}{12}+\phi\right)\right]\cos\left[2\pi\left(\frac{t+h}{12}+\phi\right)\right]$$

$$= \frac{1}{2}\int_0^1 \left\{ \cos\left(2\pi\,\frac{(-h)}{12}\right) + \cos\left[2\pi\left(\frac{2t+h}{12}+2\phi\right)\right] d\phi\right.$$

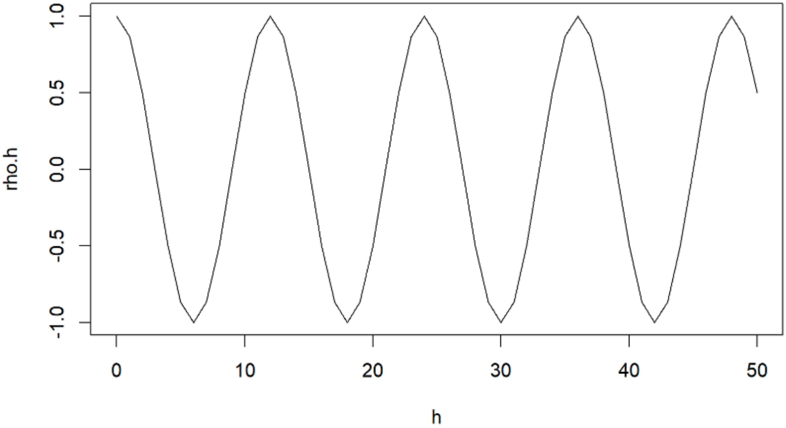
$$= \frac{1}{2}\cos\left[2\pi\,\frac{(-h)}{12}\right] + \frac{1}{4\pi}\sin\left[2\pi\left(\frac{2t+h}{12}+2\phi\right)\right]\Bigg|_0^1$$

$$= \frac{1}{2}\cos\left(2\pi\,\frac{|h|}{12}\right)$$

$$\rho_x(h) = \frac{\gamma_y(h)}{\gamma_y(0)} = \cos\left(2\pi\,\frac{|h|}{12}\right), \quad h = 0, \pm 1, \ldots \quad \text{Stationarity?}$$

```
t<-1:500
Phi<-runif(500)
Y<-cos(2*pi((t/12)+Phi))
ts.plot((Y))
acf(Y)

h<-0:50
rho.h<-cos((2*pi*h/12))
plot(h,rho.h,type="l")
```



Open Lab3 in Rstudio.