Delegation with Endogenous States

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Delegation

- Delegation problems are widespread:
  - A party with authority to make a decision (**Principal**)
  - must rely on a better informed party (**Agent**)
  - Should the principal give flexibility to the agent, or instead restrict what the agent can choose?
- Some examples:
  - CEO selects feasible projects
    Manager (better informed about their profitability) chooses one
  - Regulator restricts the prices that a monopolist (better informed about costs) can charge

Moral hazard

- Before choosing an action, agent can exert effort and affect outcomes
  - Effort is typically unobservable
  - Agent cannot fully control outcomes
- Examples:
  - Manager's effort affects potential profits of various projects
  - Monopolist can adopt practices that reduce production costs

Results

- We characterize the optimal delegation set when the principal cares about the agent's effort and action
- The agent's effort does not affect the variance of the state (both with aligned and misaligned preferences)
  - Optimal delegation set has a floor: actions below a threshold are excluded
- Variance increasing in the effort level (with aligned preferences)
  - The optimal delegation set has a gap: intermediate actions are excluded
- Variance decreasing in the effort level (with aligned preference)
  - The optimal delegation set has a floor and a ceiling

Closely related literature

- Delegation with misaligned preferences, no moral hazard:
  - Holmström (1977, 1984)
  - Alonso and Matouschek (2008)
  - Amador and Bagwell (2013)
- Delegation with moral hazard:
  - Armstrong and Vickers (2010)
- Delegation with information acquisition:
  - Szalay (2005)
  - Deimen and Szalay (2018)

Timing of the game

Principal selects a delegation set $A \subseteq \mathbb{R}$ (A closed)

Agent exerts costly effort $\gamma \geq 0$

The effort γ affects the distribution of the state ω

Agent observes the state and chooses an action $a \in A$

Uniform distribution and quadratic payoff function

Uniform distribution with shifting support

When the effort is $\gamma \geqslant 0$ the state ω is uniformly distributed in the unit interval $[\gamma, \gamma + 1]$

Payoffs

The parties' payoffs are:

$$U_P(a, \omega, \gamma) = u_P(a, \omega) + \tilde{v}(\omega)$$

$$U_A(a, \omega, \gamma) = u_A(a, \omega) - c(\gamma)$$

We assume

$$u_P(a,\omega) = -(\omega+\beta-a)^2$$
$$u_A(a,\omega) = -(\omega-a)^2$$

Misaligned preferences: if $\beta > 0$ ($\beta < 0$) the agent prefers lower (higher) actions than principal

Our analysis also covers the case of aligned preferences ($\beta = 0$)

Cost function is quadratic: $c(\gamma) = \frac{\gamma^2}{2}$

The benefit function $\tilde{v}(\cdot)$ is increasing and concave

We let $v(\gamma)$ denote the expected value of $\tilde{v}(\cdot)$ when the effort level is γ :

$$v(\gamma) = \int_{\gamma}^{\gamma+1} \tilde{v}(\omega)\, d\omega$$

The delegation set A and the effort level γ induce expected payoffs:

$$V_P(A,\gamma) = -\int_{\gamma}^{\gamma+1} (\omega + \beta - \hat{a}(\omega, A))^2 \, d\omega + v(\gamma)$$

$$V_A(A,\gamma) = -\int_{\gamma}^{\gamma+1} (\omega - \hat{a}(\omega, A))^2 \, d\omega - \frac{\gamma^2}{2}$$

where $\hat{a}(\omega, A) = \arg\max_{a \in A} -(\omega - a)^2$

Necessary conditions for optimal effort

Given a delegation set A, the agent solves the following problem:

$$\max_{\gamma \geqslant 0} \int_{\gamma}^{\gamma+1} \left[\max_{a \in \tilde{A}} - (\omega - a)^2 \right] d\omega - \frac{\gamma^2}{2} =$$

$$\max_{\gamma \geqslant 0} \int_{\gamma}^{\gamma+1} - (\omega - \hat{a}(\omega, A))^2 \, d\omega - \frac{\gamma^2}{2}$$

First-order conditions for interior γ :

$$\left(\gamma - \hat{a}\left(\gamma, \tilde{A}\right)\right)^2 - \left(\gamma + 1 - \hat{a}\left(\gamma + 1, \tilde{A}\right)\right)^2 = \gamma$$

In general, the first-order conditions are not sufficient (the problem is not necessarily concave)

Concavity under interval delegation

Lemma 1 *Suppose that the delegation set is an interval* $[\underline{a}, \bar{a}]$ *for some* $\underline{a} \leq \bar{a}$*. For every* γ*, let* $z(\gamma)$ *denote the agent's expected payoff if the effort is* γ *:*

$$z(\gamma) = -\int_{\gamma}^{\gamma+1} \left[\max_{a \in [\underline{a}, \bar{a}]} u_A(a, \omega) \right] d\omega - \frac{\gamma^2}{2}$$

The function $z(\cdot)$ *is concave.*

Optimal interval delegation

Let $\tilde{A}$ be the optimal delegation set and let γ be the optimal effort level.

We say that $\tilde{A}$ is *minimal* if for every $a \in \tilde{A}$ there exists $\omega \in [\gamma, \gamma + 1]$ such that $a = \hat{a}(\omega, A)$.

In what follows we focus on *minimal* optimal delegation sets.

Proposition 1 *Let $\gamma > 0$ be an optimal effort level and $\tilde{A}$ a (minimal) optimal delegation set. Then $\tilde{A}$ is convex. Moreover, either $\tilde{A} \subset [\gamma, \gamma + 1]$ or $\tilde{A} = \{a\}$ with $a > \gamma + 1$.*

Optimal interval delegation: sketch of the proof

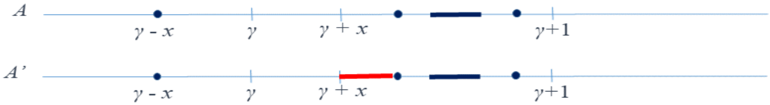
Step 1: *If $\tilde{A} \cap (\gamma, \gamma + 1) = \emptyset$, then $\tilde{A}$ is a singleton.*

The delegation set A' yields to the principal a larger payoff than A

We work with a relaxed problem: the agent's level of effort has to satisfy the first-order conditions

Step 2: *Let $\tilde{A}$ denote the optimal delegation set and let $\underline{a}$ denote the smallest element of $\tilde{A}$. Then either $\tilde{A}$ is a singleton or $\underline{a} \geqslant \gamma$.*

The delegation set A' yields to the principal a larger payoff than A



Step 3: *Let $\tilde{A}$ denote the optimal delegation set and let $\bar{a}$ denote the largest element of $\tilde{A}$. Then either $\tilde{A}$ is a singleton or* $\bar{a} \leqslant \gamma + 1$.

The delegation set A' yields to the principal a larger payoff than A

Step 4: Suppose that the optimal delegation set $\tilde{A}$ is not a singleton

$\tilde{A}$ solves the relaxed problem. Therefore, $\tilde{A} \subseteq [\gamma, \gamma + 1]$

Suppose that $\tilde{A}$ has a gap. The principal's payoff increases if the gap is filled

The interval delegation set induces the same effort level as $\tilde{A}$ (it satisfied the same first-order conditions and the problem is concave)

Step 5: If $\tilde{A} = \{a\}$, then $a \geqslant \gamma$

If $\tilde{A} = \{a\}$, with $a < \gamma$, the agent has an incentive to reduce the effort level

Floor Delegation

Proposition 2 *Let* $\gamma > 0$ *be the optimal level of effort and* $\tilde{A}$ *the optimal delegation set. If* $\tilde{A} \subseteq [\gamma, \gamma+1]$ *then* $\tilde{A} = [\underline{a}, \gamma+1]$ for some $\underline{a} > \gamma$.

Notice that in this case the *floor* delegation set $[\underline{a}, \infty]$ is also optimal

Suppose that the optimal delegation set is

$$\tilde{A} = [\underline{a}, \bar{a}] \subseteq [\gamma, \gamma + 1]$$

The first-order conditions

$$(\gamma - \underline{a})^2 - (\gamma + 1 - \bar{a})^2 = \gamma$$

imply $\underline{a} > \gamma$. If $\bar{a} < \gamma + 1$, it is possible to perturb $\underline{a}$ and $\bar{a}$ and increase the principal's payoff

Discretion and level of effort

Lemma 2 *Suppose that the optimal effort level* γ *is interior. Let* $\tilde{A}$ denote optimal delegation set. *If* $\gamma < 1$, *then*

$$\tilde{A} = [\gamma + \sqrt{\gamma}, \gamma + 1]$$

If $\gamma \geqslant 1$, *then*

$$\tilde{A} = \left\{ \frac{3\gamma + 1}{2} \right\}$$

Suppose that the optimal delegation set is $\tilde{A} = [\underline{a}, \gamma + 1]$ for some $\gamma < \underline{a} < \gamma + 1$

The effort level γ satisfies the first order conditions:

$$(\gamma - \underline{a})^2 = \gamma$$

which imply

$$\underline{a} = \gamma + \sqrt{\gamma} < \gamma + 1$$

and, thus,

$$\gamma < 1$$

On the other hand, if $\tilde{A} = \{\underline{a}\}$ for some $\underline{a} \geqslant \gamma + 1$ then

$$(\underline{a} - \gamma)^2 - (\underline{a} - \gamma - 1)^2 = \gamma$$

which yields

$$\underline{a} = \frac{3\gamma + 1}{2} \geqslant \gamma + 1$$

and, thus,

$$\gamma \geqslant 1$$

The optimal level of effort

For every $\gamma > 0$ let $V_P(\gamma)$ denote the principal's payoff if he offers the optimal delegation set that induces the effort level γ

We have

$$V_P(\gamma) = -\int_{\gamma}^{\gamma+\sqrt{\gamma}} \left(\gamma + \sqrt{\gamma} - (\omega + \beta)\right)^2 d\omega - \int_{\gamma+\sqrt{\gamma}}^{\gamma+1} \beta^2 d\omega + v(\gamma)$$

for $\gamma < 1$, and

$$V_P(\gamma) = -\int_{\gamma}^{\gamma+1} \left(\frac{3\gamma+1}{2} - (\omega+\beta)\right)^2 d\omega + v(\gamma)$$

for $\gamma \geqslant 1$

We compute the derivative of V_P :

$$V_P'(\gamma) = \beta - \frac{1}{2}\sqrt{\gamma} + v'(\gamma)$$

for $\gamma < 1$, and

$$V_P'(\gamma) = \beta - \frac{1}{2}\gamma + v'(\gamma)$$

for $\gamma \geqslant 1$

V_P is concave (recall v is concave) and V_P' is continuous everywhere

We set $V_P'(\gamma) = 0$ and obtain a unique solution

Proposition 3 *Assume that the optimal level of effort is strictly positive.*

If $\beta - \frac{1}{2} + v'(1) < 0$, *then the optimal delegation set is* $[\gamma^* + \sqrt{\gamma^*}, \gamma^* + 1]$ *where the optimal level of effort* $\gamma^* < 1$ *satisfies*

$$\beta - \frac{1}{2}\sqrt{\gamma^*} + v'(\gamma^*) = 0$$

If $\beta - \frac{1}{2} + v'(1) \geqslant 0$, *then the optimal delegation set is* $\left\{\frac{3\gamma^*+1}{2}\right\}$ *where* $\gamma^* \geqslant 1$ *satisfies*

$$\beta - \frac{1}{2}\gamma^* + v'(\gamma^*) = 0$$

Corner solution

If $\beta \geqslant 0$, it is not optimal for the principal to induce an effort level equal to zero

If $\beta < 0$, the optimal delegation set that induces zero effort coincides with the optimal delegation set $(-\infty, \bar{a}]$, $\bar{a} < 1$, when the state is uniformly distributed over the unit interval (no moral hazard)

Comparative Statics

Proposition 4 (For $\beta < 0$ assume $\gamma^* > 0$)

i) The optimal level of effort γ^* *and the principal's payoff are increasing in* β

ii) Suppose that $c(\gamma) = k\gamma^2$ *for* $k > 0$. *Both* γ^* *and the principal's payoff are decreasing in* k

iii) Suppose that $v(\gamma) = \rho h(\gamma)$, *with* $\rho > 0$ and $h(\cdot)$ *increasing an concave. Then* $\frac{\partial \gamma^*}{\partial \rho} > 0$

The variance is increasing in the effort level

Aligned preferences: $\beta = 0$

When the effort is γ the state is uniformly distributed in the interval $[-(1-\alpha)\gamma, \alpha\gamma + 1]$

The variance is increasing in the effort level: the size of the support is $\gamma + 1$

We assume that $\alpha \in (0,1)$: the support is "increasing" in γ

The function $v(\gamma)$ is increasing

If $\alpha > \frac{1}{2}$ (the expectation is increasing in γ) and $\tilde{v}(\omega)$ is increasing and linear, then $v(\gamma)$ is increasing

The set of effort levels is bounded: $\gamma \in [0, \bar{\gamma}]$

Assumption 1
If the delegation set is $[-(1-\alpha)\bar{\gamma}, 0] \cup [1, \alpha\bar{\gamma} + 1]$ *the agent chooses the effort level* $\bar{\gamma}$

If the optimal effort level is $\gamma^* = 0$, then the optimal delegation set contains the unit interval

We focus on the case $\gamma^* > 0$ (this is the case if $v'(0) > c''(0)$)

We work with a relaxed problem:

$$\max_{a(\cdot),\gamma} -\int_{-(1-\alpha)\gamma}^{\alpha\gamma+1} \frac{(a(\omega)-\omega)^2}{\gamma+1} d\omega + v(\gamma)$$

$$\text{s.t.} \quad -(a(\omega)-\omega)^2 \geqslant -(a(\omega')-\omega)^2$$

$$\text{for every} \quad (\omega,\omega') \in [-(1-\alpha)\bar{\gamma}, \alpha\bar{\gamma}+1]^2$$

$$\frac{\partial}{\partial\gamma}\left[-\int_{-(1-\alpha)\gamma}^{\alpha\gamma+1} \frac{(a(\omega)-\omega)^2}{\gamma+1} d\omega - \frac{\gamma^2}{2}\right] \geqslant 0$$

Let $a^*(\cdot)$ and γ^* denote the solution to the relaxed problem.

We also let $D^* = \cup_{\omega\in[-(1-\alpha)\gamma^*,\alpha\gamma^*+1]} \{a^*(\omega)\}$

Lemma 3 *The solution to the relaxed problem satisfies* $a^*(-(1-\alpha)\gamma^*) = -(1-\alpha)\gamma^*$ and $a^*(\alpha\gamma^* + 1) = \alpha\gamma^* + 1$

If the equalities in Lemma 3 are not satisfied we can perturb the delegation set D^* and increase the value of the objective function

Proposition 5 *Let* γ^* *denote the optimal effort level of the* unrelaxed *problem. There exist* $0 < \underline{a} < \bar{a} < 1$ *such that the delegation set* $[-(1-\alpha)\gamma^*, \underline{a}] \cup [\bar{a}, \alpha\gamma^* + 1]$ *is optimal*

The solution to the relaxed problem must satisfy the second constraint with equality (the principal can fill the gaps in D^* and increase the objective function)

It follows from Lemma 3 that

$$\left.\frac{\partial\left(-\int_{-(1-\alpha)\gamma}^{\alpha\gamma+1}\frac{(a(\omega)-\omega)^2}{\gamma+1}d\omega-\frac{\gamma^2}{2}\right)}{\partial\gamma}\right|_{\gamma=\gamma^*} = \int_{-(1-\alpha)\gamma^*}^{\alpha\gamma^*+1}\frac{(a(\omega)-\omega)^2}{(\gamma^*+1)^2}d\omega-\gamma^*=0 \tag{1}$$

which implies that the value of the objective function is

$$-\int_{-(1-\alpha)\gamma^*}^{\alpha\gamma^*+1}\frac{(a(\omega)-\omega)^2}{(\gamma^*+1)}d\omega+v(\gamma^*)=-(\gamma^*+1)\gamma^*+v(\gamma^*)$$

It follows from Assumption 1 that there exist $0 < \underline{a} < \bar{a} < 1$ such that the delegation set $[-(1-\alpha)\gamma^*, \underline{a}] \cup [\bar{a}, \alpha\gamma^* + 1]$ satisfies equality (1)

Finally, given the delegation set $[-(1-\alpha)\gamma^*, \underline{a}] \cup [\bar{a}, \alpha\gamma^* + 1]$, the second order conditions of the agent's problem are satisfied

The variance is decreasing in the effort level

Aligned preferences: $\beta = 0$

When the effort is γ the state is uniformly distributed in the interval $[\alpha\gamma, 1-(1-\alpha)\gamma]$

The variance is decreasing in the effort level: the size of the support is $1-\gamma$

We assume that $\alpha \in (0,1)$: the support is “decreasing” in γ

The function $v(\gamma)$ is increasing

If $\alpha > \frac{1}{2}$ (the expectation is increasing in γ) and $\tilde{v}(\omega)$ is increasing and linear, then $v(\gamma)$ is increasing

The set of effort levels is bounded: $\gamma \in [0, \bar{\gamma}]$, with $\bar{\gamma} < 1$

Assumption 2
If the delegation set is $\left\{ \frac{1+(2\alpha-1)\bar{\gamma}}{2} \right\}$ *the agent chooses the effort level* $\bar{\gamma}$

The optimal effort level is strictly positive: $\gamma^* > 0$

As in the previous case, we work with a relaxed problem:

$$\max_{a(\cdot),\gamma} - \int_{\alpha\gamma}^{1-(1-\alpha)\gamma} \frac{(a(\omega)-\omega)^2}{1-\gamma} d\omega + v(\gamma)$$

$$\text{s.t.} \quad -(a(\omega)-\omega)^2 \geqslant -(a(\omega')-\omega)^2$$

$$\text{for every} \quad (\omega,\omega') \in [0,1]^2$$

$$\frac{\partial}{\partial\gamma}\left[-\int_{\alpha\gamma}^{1-(1-\alpha)\gamma} \frac{(a(\omega)-\omega)^2}{1-\gamma} d\omega - \frac{\gamma^2}{2}\right] \geqslant 0$$

Let $a^*(\cdot)$ and γ^* denote the solution to the relaxed problem.

We also let $D^* = \cup_{\omega\in[\alpha\gamma^*,1-(1-\alpha)\gamma^*]} \{a^*(\omega)\}$

Lemma 4 *The solution to the relaxed problem satisfies* $a^*(\alpha\gamma^*) > \alpha\gamma^*$ and $a^*(1-(1-\alpha)\gamma^*) < 1-(1-\alpha)\gamma^*$. *Furthermore,* D^* *is convex*

Interval delegation: by filling the gaps the principal increases the objective function and relaxes the constraint

We show that $D^* \cap [\alpha\gamma^*, 1-(1-\alpha)\gamma^*] \neq \emptyset$: if this is not true, the principal is better off by choosing the delegation set $\left\{\frac{1+(2\alpha-1)\bar{\gamma}}{2}\right\}$ which induces the largest effort level $\bar{\gamma}$

Finally, if $D^* \cap [\alpha\gamma^*, 1-(1-\alpha)\gamma^*] \neq \emptyset$, then $D^* \subset (\alpha\gamma^*, 1-(1-\alpha)\gamma^*)$ (if not, then perturbations of D^* increase the value of the objective function)

Proposition 6 *Let* γ^* *denote the optimal effort level. The optimal delegation set is* $[\underline{a}, \bar{a}]$ *for some* $\alpha\gamma^* < \underline{a} < \bar{a} < 1 - (1-\alpha)\gamma^*$

With the delegation set $[\underline{a}, \bar{a}]$ the second order conditions of the agent's problem are satisfied

Conclusions

- We introduce endogenous states in the canonical delegation model
- When effort does not affect the variance, the optimal delegation set has a floor
- When the variance is increasing in the effort level, the optimal delegation set has a gap
- When the variance is decreasing in the effort level, the optimal delegation set has both a floor and a ceiling