All Possible Equilibria - Two Types

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1. Introduction

In this note we present a complete characterization of equilibria. Depending on the value of the primitives, several different equilibria arise in our simple two type model. In principle, there could be nine different types of equilibria, each associated to a different vector d^*. Table 1 shows all of them. We present necessary and sufficient conditions for the existence of all types of equilibrium. Thus, we obtain necessary and sufficient conditions for hyperphily to be the only possible equilibrium.

Positive assortative matching can arise not only with hyperphily but also with strict or weak homophily.[1] Therefore, a full characterization of the model allows us to present necessary and sufficient conditions for the existence and uniqueness of an equilibrium with positive assortative matching.

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[1]We present closed-form solutions for densities $e(x, y)$ in each possible equilibrium. We show the sorting pattern that arises. In Table 1 we indicate if sorting is positive, negative, or random. Equilibria with weak heterophily and weak homophily (B) feature no first order stochastic dominance.

2. All possible equilibria

Characterizing each equilibrium involves going through the same process as already performed for hyperphily. First, we select agreements that satisfy condition 3 in our equilibrium definition. Then, we verify that transitions are consistent. Lastly, for each possible match, we verify that the equilibrium agreement solves the bargaining problem. Table 1 presents all possible equilibrium in a two-type model.

Table 1: All Possible Equilibria in the Two Type Model

ℓ's decision	h's decision		
	$d^*(h,\ell,h)=1$ $d^*(h,h,\ell)=0$	$d^*(h,\ell,h)=0$ $d^*(h,h,\ell)=0$	$d^*(h,\ell,h)=0$ $d^*(h,h,\ell)=1$
$d^*(\ell,\ell,h)=1$ $d^*(\ell,h,\ell)=0$	Hyperphily (positive sorting)	Weak Heterophily	Strict Heterophily (negative sorting)
$d^*(\ell,\ell,h)=0$ $d^*(\ell,h,\ell)=0$	Weak Homophily (A) (positive sorting)	Indifference (random sorting)	*Impossible*
$d^*(\ell,\ell,h)=0$ $d^*(\ell,h,\ell)=1$	Strict Homophily (positive sorting)	Weak Homophily (B)	*Impossible*

We discuss now the main results regarding equilibria other than hyperphily. First, with a supermodular production function the equilibrium cannot feature neither weak nor strict heterophily. To see this, note that $\pi^*(h,\ell) < h$ makes a h-type agent strictly prefer another agent of type h. Similarly, $\pi^*(\ell,h) < \ell$ makes an ℓ-type agent strictly prefer another agent of type ℓ. Then, $F \geq h + \ell$ is a necessary condition for both weak and strict heterophily. It is also straightforward to show that neither weak nor strict homophily can be equilibria with a submodular production function. Finally, only strict heterophily exists when h strictly prefers ℓ to h (see Section 2.1 for details).

Strict heterophily is the only equilibrium featuring negative assortative matching. Thus, negative sorting only occurs with a submodular production function. Positive assortative matching occurs both with homophily and hyperphily. Random sorting only happens if both h and ℓ are indifferent, which requires $\pi^*(\ell,h) = \ell$ and $\pi^*(h,\ell) = h$. Hence indifference, and therefore random sorting, can only happen if the production function is modular.

Figure 1 illustrates the set of primitives which lead to each possible equilibrium.[2] Equilibria with strict heterophily or strict homophily are rare, as shown in panel c of Figure 1. In strict heterophily h prefers a match with ℓ over a more productive match with another h. This can happen when ℓ's outside option is lower than h's. Therefore, although the production of the match (ℓ, h) is smaller than the production of the match (h, h), the total surplus of the match (ℓ, h) is larger than the total surplus of the match (h, h). On the other hand, strict homophily requires ℓ to strictly prefer another ℓ, which is demanding given that the match (ℓ, h) is more productive. As in the case of strict heterophily, the agent prefers a less productive match because its total surplus is larger. When r or δ increase, or when ρ decreases, the outside option becomes less relevant and therefore strict homophily and strict heterophily require stronger complementarity in production.

If agents can search while matched, the match duration depends on the bargaining outcome. Symmetric surplus splitting might not solve the bargaining problem, as it occurs in the cases of weak heterophily and weak homophily. In these equilibria, one agent is indifferent between partner types and takes a larger fraction of the total surplus in the match (ℓ, h). Uneven surplus splitting produces a larger product of surpluses because it implies a longer duration of the match and a larger total surplus. These equilibria are more likely to exist when agents care more about endogenous destruction (when r or δ are low); or when it is easier to find partners (when ρ is large). This is shown in panel b and d of Figure 1.

As frictions vanish, the outcome does not necessarily approach that of the frictionless market in Becker [1973]. Consider the index of labor market frictions $\kappa \equiv \frac{\rho}{\delta}$.[3] A larger κ implies weaker frictions. With submodularity, one would expect perfect negative sorting in a frictionless market. In contrast, we show that hyperphily, and thus positive sorting, can arise with submodularity when $\kappa \to \infty$.

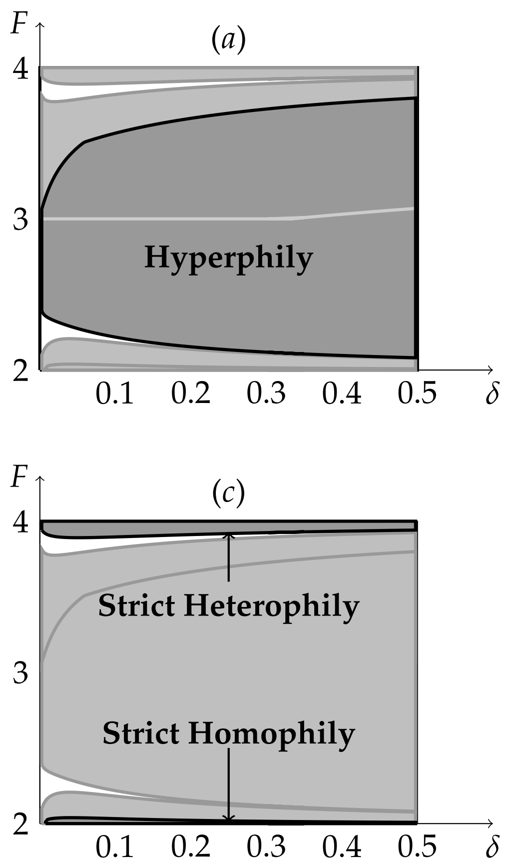
[2] The shaded areas in Figure 1 represent the set of values of F consistent with each equilibrium as a function of the destruction rate δ. We present later the corresponding figures for ρ, r, and $h - \ell$.

[3] κ is used as an index of frictions in several papers. See Ridder and van den Berg [2003] for an example.

The equilibrium outcome in the limit depends on whether it is ρ or δ what drives $\kappa \to \infty$. On one side, if κ is large because ρ is large, hyperphily does not occur. On the other side, if κ is large because δ is small, there are equilibria with hyperphily, even with a submodular production function. Patient enough ℓ-type agents are happy to trade a shorter duration of the match for a higher allocation. Interestingly, as $\delta \to 0$ sorting becomes *perfectly* positive, instead of perfectly negative as in Becker [1973].

As frictions grow, positive assortative matching becomes pervasive. When δ and r increase, or when ρ decreases, the region where hyperphily is the unique equilibrium grows.

Figure 1: The Impact of Destruction Rate δ



Note: $\ell = 1, h = 2, r = 0.1$ and $\rho = 0.1$.

2.1 Only Strict Heterophily with $d^*(h,h,\ell)=1$

Lemma 1. *$S^*(h,\ell) > S^*(h,h) \Rightarrow S^*(\ell,h) > S^*(\ell,\ell)$.*

Proof. First, since $S^*(h,\ell) > S^*(h,h)$, the third condition in the equilibrium definition guarantees $S^*(\ell,h) \geq S^*(h,\ell)$. Next, consider the following alternative agreement for (h,h): they never leave each other and they split production. Let $\widehat{S}$ denote the surplus resulting from that agreement. Then,

$$S^*(h,\ell) \geq \widehat{S} = (r+\delta)^{-1}\left[h - q^*(h,\ell)\, S^*(h,\ell) - q^*(h,h)S^*(h,h)\right]$$

We show our result by contradiction. Assume $S^*(\ell,\ell) \geq S^*(\ell,h)$. Note that $q^*(\ell,h) \geq q^*(h,h)$ and $q^*(\ell,\ell) \geq q^*(h,\ell)$, since both agents prefer low types (at least weakly). Then,

$$S^*\left(\ell,\ell\right) = (r+\delta)^{-1}\left[\ell - q^*(\ell,\ell)\, S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)\right] < \widehat{S}$$

To sum up, $\widehat{S} > S^*(\ell,\ell) \geq S^*(\ell,h) \geq S^*(h,\ell) \geq \widehat{S}$. That is our contradiction. ■

3. Details on All Possible Equilibria

3.1 Weak Heterophily

Under weak heterophily $d(\ell,\ell,h) = 1$ and $d(\ell,h,\ell) = d(h,\ell,h) = d(h,h,\ell) = 0$. Then, the steady state conditions become:

$$\begin{aligned}
e(\ell,\ell)[\delta + q(\ell,h)] + \delta e(\ell,h) &= e(\ell,\varnothing)[q(\ell,\ell) + q(\ell,h)] \\
e(\ell,\varnothing)q(\ell,\ell) &= e(\ell,\ell)[\delta + 2q(\ell,h)] \\
[e(\ell,\varnothing) + e(\ell,\ell)]q(\ell,h) &= \delta e(\ell,h) \\
\delta[e(h,\ell) + e(h,h)] &= e(h,\varnothing)[q(h,\ell) + q(h,h)] \\
e(h,\varnothing)q(h,h) &= \delta e(h,h)
\end{aligned}$$

The successful meeting rates become:

$$q(\ell,\ell) = \rho e(\ell,\varnothing) \qquad\qquad q(\ell,h) = \rho e(h,\varnothing)$$
$$q(h,\ell) = \rho[e(\ell,\varnothing) + e(\ell,\ell)] \qquad\qquad q(h,h) = \rho e(h,\varnothing)$$

Substituting these into the steady state conditions, dividing by ρ, and setting $\kappa^{-1} = \frac{\delta}{\rho}$, we get

$$\begin{aligned}
e(\ell,\ell)[\kappa^{-1} + e(h,\varnothing)] + \kappa^{-1} e(\ell,h) &= e(\ell,\varnothing)[e(\ell,\varnothing) + e(h,\varnothing)] \\
e(\ell,\varnothing)^2 &= e(\ell,\ell)[\kappa^{-1} + 2e(h,\varnothing)] \\
[e(\ell,\varnothing) + e(\ell,\ell)]e(h,\varnothing) &= \kappa^{-1} e(\ell,h) \\
\kappa^{-1}[e(h,\ell) + e(h,h)] &= e(h,\varnothing)[e(\ell,\varnothing) + e(\ell,\ell) + e(h,\varnothing)] \\
e(h,\varnothing)^2 &= \kappa^{-1} e(h,h)
\end{aligned}$$

Rewrite the third and the fourth to get the following system of two equations in the two unknowns $e(\ell,h), e(h,\varnothing)$:

$$\begin{aligned}
e(h,\varnothing)\left(\frac{1}{2} - e(\ell,h)\right) &= \kappa^{-1} e(\ell,h) \\
\kappa^{-1}\left(\frac{1}{2} - e(h,\varnothing)\right) &= \left(\frac{1}{2} - e(\ell,h) + e(h,\varnothing)\right) e(h,\varnothing)
\end{aligned}$$

Use the first of these two to write $e(\ell,h) = \frac{e(h,\varnothing)}{2[\kappa^{-1} + e(h,\varnothing)]}$, and then substitute this expression into the second. After rearranging, we get the following third-degree equation in $e(h,\varnothing)$:

$$2e(h,\varnothing)^3 + 4\kappa^{-1} e(h,\varnothing)^2 + 2\kappa^{-2} e(h,\varnothing) - \kappa^{-2} = 0$$

By computer algebra we get its real solution:

$$\begin{aligned} e(h,\varnothing) = \frac{1}{6}\sqrt[3]{8\kappa^{-3}+54\kappa^{-2}+6\sqrt{24\kappa^{-5}+81\kappa^{-4}}} \\ +\frac{2}{3}\frac{\kappa^{-2}}{\sqrt[3]{8\kappa^{-3}+54\kappa^{-2}+6\sqrt{24\kappa^{-5}+81\kappa^{-4}}}} - \frac{2}{3\kappa} \end{aligned}$$

We can now immediately write $e(\ell,h)$ and $e(h,h)$ in terms of $e(h,\varnothing)$:

$$\begin{aligned} e(\ell,h) = e(h,\ell) &= \frac{e(h,\varnothing)}{2[\kappa^{-1}+e(h,\varnothing)]} \\ e(h,h) &= \kappa e(h,\varnothing)^2 \end{aligned}$$

Next, use the normalization condition on $e(\ell,\ell)$ in the second equation to get the following second-degree equation in $e(\ell,\varnothing)$:

$$e(\ell,\varnothing)^2 + [\kappa^{-1}+2e(h,\varnothing)]e(\ell,\varnothing) + [\kappa^{-1}+2e(h,\varnothing)]\left(e(\ell,h)-\frac{1}{2}\right) = 0$$

Its positive root is

$$\begin{aligned} e(\ell,\varnothing) &= -\frac{1}{2\kappa} - e(h,\varnothing) + \sqrt{\frac{1}{4}\left(\kappa^{-1}+2e(h,\varnothing)\right)^2 - \left(\kappa^{-1}+2e(h,\varnothing)\right)\left(e(\ell,h)-\frac{1}{2}\right)} \\ &= -\frac{1}{2\kappa} - e(h,\varnothing) \\ &\quad + \sqrt{\left(\kappa^{-1}+2e(h,\varnothing)\right)\left[\frac{1}{4}\left(\kappa^{-1}+2e(h,\varnothing)\right) - \left(e(\ell,h)-\frac{1}{2}\right)\right]} \\ &= -\frac{1}{2\kappa} - e(h,\varnothing) + \frac{1}{2}\sqrt{\left(\kappa^{-1}+2e(h,\varnothing)\right)\left(\kappa^{-1}+2e(h,\varnothing)-4e(\ell,h)+2\right)} \\ &= -\frac{1}{2\kappa} - e(h,\varnothing) + \frac{1}{2}\sqrt{\left(\kappa^{-1}+2e(h,\varnothing)\right)\left(\kappa^{-1}+2e(h,\varnothing)+\frac{2\kappa^{-1}}{\kappa^{-1}+e(h,\varnothing)}\right)} \end{aligned}$$

Finally, using the fifth equation,

$$
\begin{aligned}
e(\ell,\ell) = \frac{e(\ell,h)}{\kappa e(h,\varnothing)} - e(\ell,\varnothing) = \\
&= \frac{\kappa^{-1}}{2(\kappa^{-1}+e(h,\varnothing))} + \frac{1}{2\kappa} + e(h,\varnothing) \\
&- \frac{1}{2}\sqrt{(\kappa^{-1}+2e(h,\varnothing))\left(\kappa^{-1}+2e(h,\varnothing)+\frac{2\kappa^{-1}}{\kappa^{-1}+e(h,\varnothing)}\right)}
\end{aligned}
$$

LEMMA 2. *In an equilibrium with weak heterophily there is no positive assortative matching nor negative assortative matching.*

Proof. First, we show that $e(\ell,\varnothing) > e(h,\varnothing)$. Assume that $e(\ell,\varnothing) \leq e(h,\varnothing)$. The inflow to $e(\ell,\varnothing)$ is then larger than the inflow to $e(h,\varnothing)$. But the outflow of $e(\ell,\varnothing)$ is smaller than the outflow of $e(h,\varnothing)$. Therefore if $e(\ell,\varnothing) \leq e(h,\varnothing)$, $e(\ell,\varnothing)$ and $e(h,\varnothing)$ cannot be jointly in steady state.

Second, we show that $e(h,\ell) > e(h,h)$. Assume that $e(h,\ell) \leq e(h,h)$. The outflow of $e(h,\ell)$ is smaller than the outflow of $e(h,h)$. However the inflow to $e(h,\ell)$ is larger than the inflow to $e(h,h)$, since $e(\ell,\varnothing) > e(h,\varnothing)$. Therefore if $e(h,\ell) \leq e(h,h)$, $e(h,\ell)$ and $e(h,h)$ cannot be jointly in steady state. ■

Under weak heterophily, $d^*(\ell,\ell,h) = 1$ and $d^* = 0$ otherwise. Allocations are $\pi^*(h,h) = h$, $\pi^*(\ell,\ell) = \ell$, and $\pi^*(\ell,h) = F - \pi^*(h,\ell)$. Since h never leaves her partner, it must be the case that $S^*(h,h) = S^*(h,\ell)$, and thus $\pi^*(h,\ell) = \pi^*(h,h) = h$ (because in both cases there is no endogenous destruction). Finally, $q^*(\ell,h) = q^*(h,h)$ because an h is willing to form a new match only if she is currently unmatched, and will do so with either type. Therefore, surpluses are given by:

$$
\begin{aligned}
S^*(h,\ell) &= S^*(h,h) = [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}\, h \\
S^*(\ell,h) &= [r+\delta+q^*(h,h)]^{-1}\,[F-h-q^*(\ell,\ell)S^*(\ell,\ell)] \\
S^*(\ell,\ell) &= [r+\delta+2q^*(h,h)+q^*(\ell,\ell)]^{-1}\,\ell
\end{aligned}
$$

Consistency of the decision functions requires $G^1_{WHE} \equiv S^*(\ell, h) - S^*(\ell, \ell) > 0$.

Bargaining in match (ℓ, h)

Consider Condition 3 in our equilibrium definition. Then, it cannot be the case that $S^*(h, \ell) < S^*(\ell, h)$, since $S^*(\ell, h) \neq S^*(\ell, \ell)$. Therefore $G^2_{WHE} \equiv S^*(h, \ell) - S^*(\ell, h) \geq 0$ must hold.

There is no endogenous destruction in match (ℓ, h), thus the equilibrium agreement reaches the maximum total surplus. Under the equilibrium decision function, the equilibrium agreement maximizes the product of surpluses. Consider thus agreement c_1, where $\widehat{d}_\ell = 0$, $\widehat{d}_h = 1$. The highest product of surpluses under c_1 is reached when surplus is split equally: $\widehat{S}^{c_1}_\ell = \widehat{S}^{c_1}_h$. Condition $G^3_{WHE} \equiv S^*(h, \ell)S^*(\ell, h) - \widehat{S}^{c_1}_h \widehat{S}^{c_1}_\ell \geq 0$ must hold.

Bargaining in match (ℓ, ℓ)

In match (ℓ, ℓ) both agents leave for agents of type h. Consider an alternative agreement c_2 where agents never leave each other. Let $\widehat{S}^{c_2}_1$ be the induced surplus for both agents under c_2. If this agreement is consistent, it leads to a higher product of surpluses. Thus, $G^4_{WHE} \equiv S^*(\ell, h) - \widehat{S}^{c_2}_1 > 0$ must hold.

In agreement c_3, one agent leaves when she finds an h, while the other agent never leaves: $\widehat{d}_1(h) = 1$, $\widehat{d}_1(\ell) = 0$ and $\widehat{d}_2 = 0$. This requires $\widehat{S}^{c_3}_2 = S^*(\ell, h)$ and $\widehat{S}^{c_3}_1 \geq S^*(\ell, \ell)$. If this agreement is consistent, it leads to a higher product of surpluses. Thus, $G^5_{SHE} \equiv S^*(\ell, \ell) - \widehat{S}^{c_3}_1 > 0$ must hold.

In agreement c_4, one agent leaves the match for agents of either type while the other agent never leaves: $\widehat{d}_1 = 1$ and $\widehat{d}_2 = 0$. Note that for agent 2 to stay, she must be indifferent between this match and a match with an agent of type h. If condition G^5_{WHE} holds it cannot be the case that agent 2 obtains a value larger or equal than the value obtained in a match (ℓ, ℓ). Therefore an equilibrium with weak heterophily requires $G^6_{WHE} \equiv S^*(\ell, \ell)^2 - \widehat{S}^{c_4}_1 \widehat{S}^{c_4}_2 \geq 0$.

Bargaining in match (h, h)

In match (h, h) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

Details on equilibrium conditions

Condition $G^2_{WHE} \equiv S^*(h,\ell) - S^*(\ell,h) \geq 0$ holds if

$$F \leq \frac{2(r+\delta+q^*(h,h)) + q^*(h,\ell)}{r+\delta+q^*(h,h)+q^*(h,\ell)}\, h + \frac{q^*(\ell,\ell)\,\ell}{r+\delta+2q^*(h,h)+q^*(\ell,\ell)} \tag{WHE 1}$$

Next, in alternative agreement c_1 surpluses are

$$\widehat{S}^{c_1}_\ell = [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}\,[\widehat{\pi}_\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)]$$
$$\widehat{S}^{c_1}_h = [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}\,(F - \widehat{\pi}_\ell)$$

where $\widehat{\pi}_\ell$ is ℓ's allocation. Surplus equalization $\widehat{S}^{c_1}_\ell = \widehat{S}^{c_1}_h$ requires

$$\widehat{\pi}_\ell = \frac{r+\delta+2q^*(h,h)}{2[r+\delta+q^*(h,h)]}F + \frac{\ell q^*(\ell,\ell)(r+\delta) - hq^*(h,h)[r+\delta+2q^*(h,h)+q^*(\ell,\ell)]}{2[r+\delta+q^*(h,h)][r+\delta+2q^*(h,h)+q^*(\ell,\ell)]}$$

Condition $G^3_{WHE} \equiv S^*(h,\ell)S^*(\ell,h) - \widehat{S}^{c_1}_\ell \widehat{S}^{c_1}_h \geq 0$ holds if

$$-\frac{(r+\delta)^2}{4}\tilde{F}^2 + C_5\tilde{F} - \left(A_5 + B_5^2\right) \geq 0$$

where

$$A_5 = \frac{[r+\delta+q^*(h,\ell)+q^*(h,h)]\,h^2}{r+\delta+q^*(h,h)} + \frac{[r+\delta+q^*(h,\ell)+q^*(h,h)]\,q^*(\ell,\ell)\ell h}{[r+\delta+2q^*(h,h)+q^*(\ell,\ell)]\,[r+\delta+q^*(h,h)]}$$
$$B_5 = \frac{q^*(\ell,\ell)\,(r+\delta)\,\ell - [r+\delta+2q^*(h,h)+q^*(\ell,\ell)]\,q^*(h,h)h}{2\,[r+\delta+2q^*(h,h)+q^*(\ell,\ell)]\,[r+\delta+q^*(h,h)]}$$
$$C_5 = (r+\delta)B_5 + h[r+\delta+q^*(h,\ell)+q^*(h,h)]$$
$$\tilde{F} = [r+\delta+q^*(h,h)]^{-1}F$$

G^3_{WHE} is a concave function of F, so we must check that F lies between its two roots. Note that there exists $F = \widehat{F}$ such that $S^*(\ell, h) = S^*(h, \ell)$. At $\widehat{F}$, G^3_{WHE} is satisfied: surplus is split symmetrically with both equilibrium agreement and c_1, and total surplus is higher in the equilibrium agreement. For $F > \widehat{F}$ we have $S^*(\ell, h) > S^*(h, \ell)$, which violates G^2_{WHE}. Therefore, only the lower bound is binding:

$$F \geq \frac{2[r+\delta+q^*(h,h)]\left(C_5 - \sqrt{C_5^2 - (r+\delta)^2(A_5 + B_5^2)}\right)}{(r+\delta)^2} \tag{WHE 2}$$

Next, we consider alternative agreements in match (ℓ, ℓ). In agreement c_2, neither agent leaves the match. Surplus for both agents is

$$\widehat{S}_1^{c_2} = (r+\delta)^{-1}\left[\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)\right]$$

Condition $G^4_{WHE} \equiv S^*(\ell, h) - \widehat{S}_1^{c_2} > 0$ holds if

$$F > \ell + h \tag{WHE 3}$$

In alternative agreement c_3, agent 1 leaves when she finds an h, while agent 2 never leaves. Let $\widehat{\pi}_2$ denote agent 2's allocation. Surpluses are

$$\begin{aligned}
\widehat{S}_1^{c_3} &= [r+\delta+q^*(\ell,h)]^{-1}[2\ell - \widehat{\pi}_2 - q^*(\ell,\ell)S^*(\ell,\ell)] \\
\widehat{S}_2^{c_3} &= [r+\delta+q^*(\ell,h)]^{-1}[\widehat{\pi}_2 - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)] = S^*(\ell,h)
\end{aligned}$$

We first recover $\widehat{\pi}_2$ from $\widehat{S}_2^{c_3} = S^*(\ell, h)$:

$$\widehat{\pi}_2 = \frac{r+\delta+2q^*(\ell,h)}{r+\delta+q^*(\ell,h)}(F-h) - \frac{q^*(\ell,\ell)q^*(\ell,h)}{(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell))\,(r+\delta+q^*(\ell,h))}\ell$$

Condition $G^5_{WHE} \equiv S^*(\ell, \ell) - \widehat{S}_1^{c_3} > 0$ holds if

$$[r+\delta+3q^*(\ell,h)+q^*(\ell,\ell)]\ell < [r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)]\widehat{\pi}_2$$

Replacing $\widehat{\pi}_2$:

$$[r+\delta+q^*(\ell,h)][r+\delta+3q^*(\ell,h)+q^*(\ell,\ell)]\ell <$$
$$< [r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)]\,[r+\delta+2q^*(\ell,h)]\,(F-h) - q^*(\ell,\ell)q^*(\ell,h)\ell$$

Therefore, an equilibrium with weak heterophily requires:

$$F > h + \frac{[r+\delta+3q^*(\ell,h)+q^*(\ell,\ell)]\,[r+\delta+q^*(\ell,h)] + q^*(\ell,\ell)q^*(\ell,h)}{[r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)]\,[r+\delta+2q^*(\ell,h)]}\ell$$

Since

$$\frac{(r+\delta+3q^*(\ell,h)+q^*(\ell,\ell))\,(r+\delta+q^*(\ell,h)) + q^*(\ell,\ell)q^*(\ell,h)}{(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell))\,(r+\delta+2q^*(\ell,h))} < 1$$

condition G^5_{WHE} always holds if G^4_{WHE} holds.

In alternative agreement c_4, agent 1 leaves for any other agent, while agent 2 never leaves. Let $\widehat{\pi}_2$ denote agent 2's allocation. Surpluses are

$$\widehat{S}_1^{c_4} = [r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]^{-1}(2\ell-\widehat{\pi}_2)$$
$$\widehat{S}_2^{c_4} = [r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]^{-1}[\widehat{\pi}_2 - q^*(\ell,h)S^*(\ell,h) - q^*(\ell,\ell)S^*(\ell,\ell)]$$

Observe that

$$\widehat{S}_1^{c_4} + \widehat{S}_2^{c_4} = \frac{2\ell - q^*(\ell,h)S^*(\ell,h) - q^*(\ell,\ell)S^*(\ell,\ell)}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)}$$

and

$$2S^*(\ell,\ell) = \frac{2\ell}{r+\delta+q^*(\ell,\ell)+2q^*(\ell,h)} = \frac{2\ell - 2q^*(\ell,h)S^*(\ell,\ell)}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)}$$

Since $S^*(\ell,\ell) < S^*(\ell,h)$ by consistency and $q^*(\ell,\ell) = \rho e(\ell,\varnothing) > \rho e(h,\varnothing) = q^*(\ell,h)$, we have that $2S^*(\ell,\ell) > \widehat{S}_1^{c_4} + \widehat{S}_2^{c_4}$ and therefore $G^6_{WHE} \equiv S^*(\ell,\ell)^2 - \widehat{S}_1^{c_4}\widehat{S}_2^{c_4} \geq 0$ is always true.

To conclude we check consistency condition $G^1_{WHE} \equiv S^*(\ell,h) - S^*(\ell,\ell) > 0$. From $S^*(h,\ell) = S^*(h,h)$ we know that $F - \pi^*(\ell,h) = h$. Then, G^1_{WHE} holds if

$$F > \frac{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}\ell + h$$

Since $\frac{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)} < 1$, condition G^1_{WHE} holds if G^4_{WHE} holds.

3.2 Indifference

With indifference, $d^* = 0$ for all matches. There is complete symmetry between low and high types. Since there is no endogenous destruction, it is straightforward to calculate $e(x,y)$.

$$\frac{1}{2} - e(x,\varnothing) = 2\kappa e(x,\varnothing) \forall x \in \{h,l\}$$

Then

$$e(x,\varnothing) = \frac{-1+\sqrt{1+4\kappa}}{4\kappa}$$

for $x \in \{h,l\}$. And

$$e(x,y) = \frac{\frac{1}{2} + \frac{1-\sqrt{1+4\kappa}}{4\kappa}}{2}$$

for $x,y \in \{h,l\}$.

Since all agents are indifferent and there is no endogenous destruction, it must be the case that $\pi^*(h,\ell) = h$ and $\pi^*(\ell,h) = \ell$. Therefore

$$F = \ell + h \tag{IND 1}$$

is a necessary condition for indifference. Surpluses are:

$$S^*(h,\ell) = S^*(h,h) = \left[r+\delta+q^*(h,\ell)+q^*(h,h)\right]^{-1} h$$
$$S^*(\ell,h) = S^*(\ell,\ell) = \left[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)\right]^{-1} \ell$$

In matches (ℓ,ℓ) and (h,h) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

There is no endogenous destruction in match (ℓ, h), thus the equilibrium agreement has the maximum total surplus. A higher product of surpluses can only be reached if agents split surplus symmetrically. Consider thus agreement c_1, where $\widehat{d}_\ell = 0$, $\widehat{d}_h = 1$, and $\widehat{S}^{c_1}_\ell = \widehat{S}^{c_1}_h$. Condition $G^1_{IND} \equiv S^*(\ell, h)S^*(h, \ell) - \widehat{S}^{c_1}_\ell \widehat{S}^{c_1}_h \geq 0$ must hold. Surpluses are:

$$\widehat{S}^{c_1}_\ell = [r + \delta + q^*(h, \ell) + q^*(h, h)]^{-1} [\widehat{\pi}_\ell - q^*(\ell, h)S^*(\ell, h) - q^*(\ell, \ell)S^*(\ell, \ell)]$$
$$\widehat{S}^{c_1}_h = [r + \delta + q^*(h, \ell) + q^*(h, h)]^{-1} [F - \widehat{\pi}_\ell]$$

From $\widehat{S}^{c_1}_\ell = \widehat{S}^{c_1}_h$ we recover $\widehat{\pi}_\ell = \frac{1}{2}[F + q^*(\ell, h)S^*(\ell, h) + q^*(\ell, \ell)S^*(\ell, \ell)]$, and thus

$$\widehat{S}^{c_1}_\ell \widehat{S}^{c_1}_h = \frac{[F - q^*(\ell, h)S^*(\ell, h) - q^*(\ell, \ell)S^*(\ell, \ell)]^2}{4\,[r + \delta + q^*(h, \ell) + q^*(h, h)]^2} \leq S^*(\ell, h)S^*(h, \ell)$$

which requires (using $F = \ell + h$)

$$4h\ell \geq \left[h + \frac{r + \delta}{r + \delta + q^*(\ell, h) + q^*(\ell, \ell)}\ell\right]^2 \tag{IND 2}$$

Finally, an agreement c_2 where h never leaves and ℓ leaves the match in case she finds an ℓ or an h is not a candidate to solve the bargaining problem because $\widehat{S}^{c_1}_\ell + \widehat{S}^{c_1}_h > \widehat{S}^{c_2}_\ell + \widehat{S}^{c_2}_h$; this follows from $S^*(h, \ell) > S^*(\ell, \ell)$ and $S^*(h, h) > S^*(\ell, h)$.

3.3 Strict Heterophily

Under strict heterophily $d(\ell, \ell, h) = d(h, h, \ell) = 1$ and $d(\ell, h, \ell) = d(h, \ell, h) = 0$. There is complete symmetry between low and high types: since both prefer the opposite type, and everything is symmetric to begin with, we have symmetry in probability of employment: $e(\ell, \varnothing) = e(h, \varnothing)$. The symmetry condition $e(\ell, h) = e(h, \ell)$ holds in general; combining these two with the normalization conditions, we have that also $e(\ell, \ell) = e(h, h)$.

Therefore, it is enough to study the steady state conditions for the low type:

$$\begin{aligned}
[q(\ell,\ell)+q(\ell,h)]e(\ell,\varnothing) &= [\delta+q(\ell,h)]e(\ell,\ell)+\delta e(\ell,h)\\
[\delta+q(\ell,h)+q(\ell,h)]e(\ell,\ell) &= q(\ell,\ell)e(\ell,\varnothing)\\
\delta e(\ell,h) &= q(\ell,h)e(\ell,\varnothing)+q(\ell,h)e(\ell,\ell)
\end{aligned}$$

The successful meeting rates are

$$q(\ell,\ell)=\rho e(\ell,\varnothing)d(\ell,\varnothing,\ell)+\rho e(\ell,h)d(\ell,h,\ell)=\rho e(\ell,\varnothing)$$
$$q(\ell,h)=\rho e(h,\varnothing)d(h,\varnothing,\ell)+\rho e(h,h)d(h,h,\ell)=\rho[e(h,\varnothing)+e(h,h)]=\rho[e(\ell,\varnothing)+e(\ell,\ell)]$$

Substituting these into the steady state conditions we get

$$\begin{aligned}
\rho[2e(\ell,\varnothing)+e(\ell,\ell)]e(\ell,\varnothing) &= [\delta+\rho(e(\ell,\varnothing)+e(\ell,\ell))]e(\ell,\ell)+\delta e(\ell,h)\\
[\delta+2\rho(e(\ell,\varnothing)+e(\ell,\ell))]e(\ell,\ell) &= \rho e(\ell,\varnothing)e(\ell,\varnothing)\\
\delta e(\ell,h) &= \rho[e(\ell,\varnothing)+e(\ell,\ell)][e(\ell,\varnothing)+e(\ell,\ell)]
\end{aligned}$$

Dividing by ρ and cleaning up we get

$$\begin{aligned}
[2e(\ell,\varnothing)+e(\ell,\ell)]e(\ell,\varnothing) &= [\kappa^{-1}+e(\ell,\varnothing)+e(\ell,\ell)]e(\ell,\ell)+\kappa^{-1}e(\ell,h)\\
[\kappa^{-1}+2(e(\ell,\varnothing)+e(\ell,\ell))]e(\ell,\ell) &= e(\ell,\varnothing)^2\\
\kappa^{-1}e(\ell,h) &= [e(\ell,\varnothing)+e(\ell,\ell)]^2
\end{aligned} \tag{1}$$

Applying the normalization condition $e(\ell,\varnothing)+e(\ell,\ell)=\frac{1}{2}-e(\ell,h)$ to the third equation in (1) gives a second-degree equation in $e(\ell,h)$,

$$e(\ell,h)^2-(1+\kappa^{-1})e(\ell,h)+\frac{1}{4}=0$$

the acceptable solution of which is

$$e(\ell,h)=\frac{1}{2}\left(1+\kappa^{-1}-\sqrt{\kappa^{-2}+2\kappa^{-1}}\right)$$

Now rewrite the second equation in (1) using the normalization condition twice:

$$[\kappa^{-1}+1-e(\ell,h)]\left(\frac{1}{2}-e(\ell,\varnothing)-e(\ell,h)\right)=e(\ell,\varnothing)^2$$

Now note that $\kappa^{-1}+1-e(\ell,h)=\sqrt{\kappa^{-2}+2\kappa^{-1}}$ and $\frac{1}{2}-e(\ell,h)=\sqrt{\kappa^{-2}+2\kappa^{-1}}-\kappa^{-1}$, and rearrange the previous equation as

$$e(\ell,\varnothing)^2+e(\ell,\varnothing)\sqrt{\kappa^{-2}+2\kappa^{-1}}+\left(\kappa^{-1}-\sqrt{\kappa^{-2}+2\kappa^{-1}}\right)\sqrt{\kappa^{-2}+2\kappa^{-1}}=0$$

The acceptable solution to this second-degree equation in $e(\ell,\varnothing)$ is

$$e(\ell,\varnothing)=\frac{1}{2}\left(\sqrt{3(\kappa^{-2}+2\kappa^{-1})-2\kappa^{-1}\sqrt{\kappa^{-2}+2\kappa^{-1}}}-\sqrt{\kappa^{-2}+2\kappa^{-1}}\right)$$

Finally, per the normalization condition $e(\ell,\ell)=\frac{1}{2}-e(\ell,\varnothing)-e(\ell,h)$ we get

$$e(\ell,\ell)=\frac{1}{2}\left(2\sqrt{\kappa^{-2}+2\kappa^{-1}}-\kappa^{-1}-\sqrt{3(\kappa^{-2}+2\kappa^{-1})-2\kappa^{-1}\sqrt{\kappa^{-2}+2\kappa^{-1}}}\right)$$

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Proof. Since $e(\ell,\varnothing)=e(h,\varnothing)$, there is negative assortative matching if and only if $e(h,\ell)>e(h,h)$. Assume that $e(h,\ell)\leq e(h,h)$. The outflow of $e(h,\ell)$ is smaller than the outflow of $e(h,h)$. However the inflow to $e(h,\ell)$ is larger than the inflow to $e(h,h)$. Therefore if $e(h,\ell)\leq e(h,h)$, $e(h,\ell)$ and $e(h,h)$ cannot be jointly in steady state. ■

Surpluses are given by:

$$S^*(h,\ell) = [r+\delta+q^*(h,\ell)]^{-1}\left[F-\pi^*(\ell,h)-q^*(h,h)S^*(h,h)\right]$$
$$S^*(h,h) = [r+\delta+2q^*(h,\ell)+q^*(h,h)]^{-1}\, h$$
$$S^*(\ell,h) = [r+\delta+q^*(\ell,h)]^{-1}\left[\pi^*(\ell,h)-q^*(\ell,\ell)S^*(\ell,\ell)\right]$$
$$S^*(\ell,\ell) = [r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)]^{-1}\, \ell$$

Since the decision functions are symmetric, $q^*(h,h) = q^*(\ell,\ell)$ and $q^*(\ell,h) = q^*(h,\ell)$. Then, the allocation $\pi^*(\ell,h)$ is such that $S^*(\ell,h) = S^*(h,\ell)$:

$$\pi^*(\ell,h) = \frac{1}{2}\left[F - \frac{q^*(\ell,\ell)(h-\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}\right]$$

Consistency requires $G^1_{SHE} \equiv S^*(h,\ell) - S^*(h,h) > 0$ and $G^2_{SHE} \equiv S^*(\ell,h) - S^*(\ell,\ell) > 0$.

Bargaining in match (ℓ,h)

In match (ℓ,h) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

Bargaining in match (ℓ,ℓ)

In match (ℓ,ℓ) both agents leave for agents of type h. Consider an alternative agreement c_1 where agents never leave the match. Let $\widehat{S}^{c_1}_1$ be the induced surplus for both agents under c_1. If this agreement is consistent, it leads to a higher product of surpluses because both agents are better off. So $G^3_{SHE} \equiv S^*(\ell,h) - \widehat{S}^{c_1}_1 > 0$ is required for an equilibrium with strict heterophily.

In agreement c_2, one agent leaves when she finds an h, while the other agent never leaves: $\widehat{d}_1(h) = 1, \widehat{d}_1(\ell) = 0$ and $\widehat{d}_2 = 0$. This requires $\widehat{S}^{c_2}_2 = S^*(\ell,h)$ and $\widehat{S}^{c_2}_1 \geq S^*(\ell,\ell)$. If this agreement is consistent, it leads to a higher product of surpluses. So $G^4_{SHE} \equiv S^*(\ell,\ell) - \widehat{S}^{c_2}_1 > 0$ is required for an equilibrium with strict heterophily.

In agreement c_3, one agent leaves the match for agents of either type while the other agent never leaves: $\widehat{d_1} = 1$ and $\widehat{d_2} = 0$. Note that for agent 2 to stay, she must be indifferent between this match and a match with an agent of type h. If condition G^4_{SHE} holds it cannot be the case that the agent who leaves the match obtains a value larger than or equal to the value obtained in a match (ℓ, ℓ). This agreement represents a solution if the product of surpluses is larger than $[S^*(\ell, \ell)]^2$. Therefore an equilibrium with strict heterophily requires $G^5_{SHE} \equiv S^*(\ell, \ell)^2 - \widehat{S}^{c_3}_1 \widehat{S}^{c_3}_2 \geq 0$.

Bargaining in match (h, h)

In match (h, h) both agents leave for agents of type ℓ. Consider an alternative agreement c_4 where agents never leave the match. Let $\widehat{S}^{c_4}_1$ be the induced surplus for both agents under c_4. If this agreement is consistent, it leads to a higher product of surpluses. Thus, $G^6_{SHE} \equiv S^*(h, \ell) - \widehat{S}^{c_4}_1 > 0$ must hold.

In agreement c_5, one agent leaves when she finds an ℓ, while the other agent never leaves: $\widehat{d_1}(\ell) = 1$, $\widehat{d_1}(h) = 0$ and $\widehat{d_2} = 0$. This requires $\widehat{S}^{c_5}_2 = S^*(h, \ell)$ and $\widehat{S}^{c_5}_1 \geq S^*(h, h)$. If this agreement is consistent, it leads to a higher product of surpluses. Thus, $G^7_{SHE} \equiv S^*(h, h) - \widehat{S}^{c_5}_1 > 0$ must hold.

In agreement c_6, one agent leaves the match for agents of either type while the other agent never leaves: $\widehat{d_1} = 1$ and $\widehat{d_2} = 0$. Note that for agent 2 to stay, she must be indifferent between this match and a match with an agent of type ℓ. If condition G^7_{SHE} holds it cannot be the case that the agent who leaves the match obtains a value larger than or equal to the value obtained in a match (h, h). This agreement represents a solution if the product of surpluses is larger than $[S^*(h, h)]^2$. Therefore an equilibrium with strict heterophily requires $G^8_{SHE} \equiv S^*(h, h)^2 - \widehat{S}^{c_6}_1 \widehat{S}^{c_6}_2 \geq 0$.

Details on equilibrium conditions

We begin by checking alternative agreements for match (ℓ,ℓ). Surplus under agreement c_1 is

$$\widehat{S}_1^{c_1} = (r+\delta)^{-1}\left[\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)\right]$$

Condition $G^3_{SHE} \equiv S^*(\ell,h) - \widehat{S}_1^{c_1} > 0$ holds whenever

$$F > 2\ell + \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}\,(h-\ell)$$

We will later show that this condition is implied by G^6_{SHE}.

In agreement c_2, let $\widehat{\pi}_2$ denote the allocation of the agent who never leaves. Surpluses are

$$\begin{aligned}
\widehat{S}_1^{c_2} &= [r+\delta+q^*(\ell,h)]^{-1}[2\ell - \widehat{\pi}_2 - q^*(\ell,\ell)S^*(\ell,\ell)] \\
\widehat{S}_2^{c_2} &= [r+\delta+q^*(\ell,h)]^{-1}[\widehat{\pi}_2 - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)]
\end{aligned}$$

We calculate $\widehat{\pi}_2$ using $\widehat{S}_2^{c_2} = S^*(\ell,h)$, which gives $\widehat{\pi}_2 = \pi^*(\ell,h) + q^*(\ell,h)S^*(\ell,h)$ and then

$$\widehat{\pi}_2 = \frac{r+\delta+2q^*(\ell,h)}{2\,(r+\delta+q^*(\ell,h))}F - \frac{(r+\delta+2q^*(\ell,h))\,q^*(\ell,\ell)h - (r+\delta)\,q^*(\ell,\ell)\ell}{2\,(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell))\,(r+\delta+q^*(\ell,h))}$$

Using this, $G^4_{SHE} \equiv S^*(\ell,\ell) - \widehat{S}_1^{c_2} > 0$ holds whenever

$$F > \frac{2\ell[r+\delta+q^*(\ell,h)][r+\delta+3q^*(\ell,h)+q^*(\ell,\ell)] + [r+\delta+2q^*(\ell,h)]q^*(\ell,\ell)h - (r+\delta)q^*(\ell,\ell)\ell}{[r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+2q^*(\ell,h)]}$$

We will later show that this condition is implied by G^7_{SHE}.

In agreement c_3, let $\widehat{\pi}_2$ denote the allocation of the agent who never leaves.

Surpluses are:

$$\widehat{S}_1^{c_3} = \frac{2\ell - \widehat{\pi}_2}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)}$$
$$\widehat{S}_2^{c_3} = \frac{\widehat{\pi}_2 - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)} = S^*(\ell,h)$$

We first recover $\widehat{\pi}_2$ from $\widehat{S}_2^{c_3} = S^*(\ell,h)$:

$$\widehat{\pi}_2 = \frac{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}{2\left(r+\delta+q^*(\ell,h)\right)}F - \left[\frac{q^*(\ell,\ell)\left(h-\ell\right)}{2\left(r+\delta+q^*(\ell,h)\right)} + \frac{\left(q^*(\ell,\ell)+q^*(\ell,h)\right)q^*(\ell,\ell)\ell}{\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)\left(r+\delta+q^*(\ell,h)\right)}\right]$$

$G^5_{SHE} \equiv S^*(\ell,\ell)^2 - \widehat{S}_1^{c_3}\widehat{S}_2^{c_3} \geq 0$ holds whenever

$$\left[\frac{2\ell-\widehat{\pi}_2}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}\right]\left[\frac{\pi^*(\ell,h)-q^*(\ell,\ell)S^*(\ell,\ell)}{r+\delta+q^*(\ell,h)}\right] \leq \left[S^*(\ell,\ell)\right]^2$$

Therefore an equilibrium with strict heterophily requires $A_3F^2 + B_3F + C_3 \geq 0$, where

$$\begin{aligned}
A_3 &= \frac{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}{4\left(r+\delta+q^*(\ell,h)\right)} \\
B_3 &= -\left[\frac{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}{2\left(r+\delta+q^*(\ell,h)\right)}\right]\left[\frac{q^*(\ell,\ell)\left(h+\ell\right)}{2\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)}\right] \\
&\quad -\ell - \frac{q^*(\ell,\ell)\left(h-\ell\right)}{4\left(r+\delta+q^*(\ell,h)\right)} - \frac{\left(q^*(\ell,\ell)+q^*(\ell,h)\right)q^*(\ell,\ell)}{2\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)\left(r+\delta+q^*(\ell,h)\right)}\ell \\
C_3 &= \frac{\left(r+\delta+q^*(\ell,h)+q^*(\ell,\ell)\right)\left(r+\delta+q^*(\ell,h)\right)}{\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)^2}\ell^2 + \left[\frac{q^*(\ell,\ell)\left(h+\ell\right)}{2\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)}\right] \\
&\quad \times \left[2\ell + \frac{q^*(\ell,\ell)\left(h-\ell\right)}{2\left(r+\delta+q^*(\ell,h)\right)} + \frac{\left(q^*(\ell,\ell)+q^*(\ell,h)\right)q^*(\ell,\ell)\ell}{\left(r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)\right)\left(r+\delta+q^*(\ell,h)\right)}\right]
\end{aligned}$$

Since A_3 is positive, G^5_{SHE} is a convex function of F. Therefore an equilibrium with strict heterophily requires:

$$F \notin \left(\frac{-B_3 - \sqrt{B_3^2 - 4A_3C_3}}{2A_3}, \frac{-B_3 + \sqrt{B_3^2 - 4A_3C_3}}{2A_3}\right) \tag{SHE 1}$$

Next, we check alternative agreements for match (h,h). Surplus under agreement c_4 is

$$\widehat{S}_1^{c_4} = (r+\delta)^{-1}\left[h - q^*(h,\ell)S^*(h,\ell) - q^*(h,h)S^*(h,h)\right]$$

Condition $G^6_{SHE} \equiv S^*(h,\ell) - \widehat{S}_1^{c_4} > 0$ holds whenever

$$F > 2h - \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}(h-\ell) \tag{SHE 2}$$

Note that since $h > \ell$,

$$2\ell + \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}(h-\ell) < 2h - \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}(h-\ell)$$

Therefore if $F > 2h - \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}(h-\ell) \Rightarrow F > 2\ell + \frac{q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}(h-\ell)$. If G^6_{SHE} holds, G^3_{SHE} also holds.

In agreement c_5, let $\widehat{\pi}_2$ denote the allocation of the agent who never leaves. Surpluses are

$$\widehat{S}_1^{c_5} = [r+\delta+q^*(h,\ell)]^{-1}[2h - \widehat{\pi}_2 - q^*(h,h)S^*(h,h)]$$
$$\widehat{S}_2^{c_5} = [r+\delta+q^*(h,\ell)]^{-1}[\widehat{\pi}_2 - q^*(h,h)S^*(h,h) - q^*(h,\ell)S^*(h,\ell)]$$

We calculate $\widehat{\pi}_2$ using $\widehat{S}_2^{c_5} = S^*(h,\ell)$:

$$\widehat{\pi}_2 = \frac{r+\delta+2q^*(h,\ell)}{2\left(r+\delta+q^*(h,\ell)\right)}F - \frac{\left(r+\delta+2q^*(h,\ell)\right)q^*(h,h)\ell - (r+\delta)\,q^*(h,h)h}{2\left(r+\delta+2q^*(h,\ell)+q^*(h,h)\right)\left(r+\delta+q^*(h,\ell)\right)}$$

Using this, $G^7_{SHE} \equiv S^*(h,h) - \widehat{S}_1^{c_5} > 0$ holds whenever

$$F \geq \frac{2h[r+\delta+q^*(h,\ell)][r+\delta+3q^*(h,\ell)+q^*(h,h)] + [r+\delta+2q^*(h,\ell)]q^*(h,h)\ell - (r+\delta)q^*(h,h)h}{[r+\delta+2q^*(h,\ell)+q^*(h,h)][r+\delta+2q^*(h,\ell)]} \tag{SHE 3}$$

Since $h > \ell$, if condition G^7_{SHE} holds then condition G^4_{SHE} also holds.

In agreement c_6, let $\widehat{\pi}_2$ denote the allocation of the agent who never leaves.

Surpluses are:

$$\widehat{S}_1^{c_6} = [r+\delta+q^*(h,h)+q^*(h,\ell)]^{-1}(2h-\widehat{\pi}_2)$$
$$\widehat{S}_2^{c_6} = [r+\delta+q^*(h,h)+q^*(h,\ell)]^{-1}[\widehat{\pi}_2 - q^*(h,h)S^*(h,h) - q^*(h,\ell)S^*(h,\ell)] = S^*(h,\ell)$$

We first recover $\widehat{\pi}$ from $\widehat{S}_2^{c_6} = S^*(h,\ell)$:

$$\widehat{\pi}_2 = \frac{r+\delta+2q^*(h,\ell)+q^*(h,h)}{2\,(r+\delta+q^*(h,\ell))}F - \left[\frac{q^*(h,h)\,(\ell-h)}{2\,(r+\delta+q^*(h,\ell))} + \frac{(q^*(h,h)+q^*(h,\ell))\,q^*(h,h)h}{(r+\delta+2q^*(h,\ell)+q^*(h,h))\,(r+\delta+q^*(h,\ell))}\right]$$

$G_{SHE}^8 \equiv S^*(h,h)^2 - \widehat{S}_1^{c_6}\widehat{S}_2^{c_6} \geq 0$ holds whenever

$$\left[\frac{2h-\widehat{\pi}_2}{r+\delta+q^*(h,\ell)+q^*(h,h)}\right]\left[\frac{\pi^*(\ell,h)-q^*(h,h)S^*(h,h)}{r+\delta+q^*(h,\ell)}\right] \leq [S^*(h,h)]^2$$

Therefore an equilibrium with strict heterophily requires $A_4F^2 + B_4F + C_4 \geq 0$, where

$$\begin{aligned}
A_4 &= \frac{r+\delta+2q^*(h,\ell)+q^*(h,h)}{4\,(r+\delta+q^*(h,\ell))}\\
B_4 &= -\left[\frac{r+\delta+2q^*(h,\ell)+q^*(h,h)}{2\,(r+\delta+q^*(h,\ell))}\right]\left[\frac{q^*(h,h)\,(h+\ell)}{2\,(r+\delta+2q^*(h,\ell)+q^*(h,h))}\right]\\
&\quad -h - \frac{q^*(h,h)\,(\ell-h)}{4\,(r+\delta+q^*(h,\ell))} - \frac{(q^*(h,h)+q^*(h,\ell))\,q^*(h,h)}{2\,(r+\delta+2q^*(h,\ell)+q^*(h,h))\,(r+\delta+q^*(h,\ell))}h\\
C_4 &= \frac{(r+\delta+q^*(h,\ell)+q^*(h,h))\,(r+\delta+q^*(h,\ell))}{(r+\delta+2q^*(h,\ell)+q^*(h,h))^2}h^2 + \left[\frac{q^*(h,h)\,(h+\ell)}{2\,(r+\delta+2q^*(h,\ell)+q^*(h,h))}\right]\\
&\quad \times\left[2h + \frac{q^*(h,h)\,(\ell-h)}{2\,(r+\delta+q^*(h,\ell))} + \frac{(q^*(h,h)+q^*(h,\ell))\,q^*(h,h)h}{(r+\delta+2q^*(h,\ell)+q^*(h,h))\,(r+\delta+q^*(h,\ell))}\right]
\end{aligned}$$

Since A_4 is positive, G_{SHE}^8 is a convex function of F. Therefore an equilibrium with strict heterophily requires:

$$F \notin \left(\frac{-B_4-\sqrt{B_4^2-4A_4C_4}}{2A_4}, \frac{-B_4+\sqrt{B_4^2-4A_4C_4}}{2A_4}\right) \qquad \text{(SHE 4)}$$

Finally, we check consistency of the equilibrium with strict heterophily. Condition $G^1_{SHE} \equiv S^*(h,\ell) - S^*(h,h) > 0$ becomes

$$F > \pi^*(\ell,h) + \frac{r+\delta+q^*(h,\ell)+q^*(h,h)}{r+\delta+2q^*(h,\ell)+q^*(h,h)}h$$

Replacing $\pi^*(\ell,h)$ and using $q^*(h,\ell) = q^*(\ell,h)$ and $q^*(h,h) = q^*(\ell,\ell)$ this becomes

$$F > 2h\frac{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)} - \frac{q^*(\ell,\ell)(h-\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)}$$

Since $\frac{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)} < 1$, if G^6_{SHE} holds, G^1_{SHE} also holds.

Since $S^*(\ell,h) = S^*(h,\ell)$ and $S^*(h,h) - S^*(\ell,\ell) = \frac{h-\ell}{r+\delta+2q^*(\ell,h)+q^*(\ell,\ell)} > 0$, if G^1_{SHE} holds so does condition $G^2_{SHE} \equiv S^*(\ell,h) - S^*(\ell,\ell) > 0$.

3.4 Strict Homophily

Under strict homophily $d(\ell,h,\ell) = d(h,\ell,h) = 1$ and $d(\ell,\ell,h) = d(h,h,\ell) = 0$. As was the case for Strict Heterophily, symmetry of all decision rules leads to symmetry in steady state densities: $e(\ell,\varnothing) = e(h,\varnothing)$ and $e(\ell,\ell) = e(h,h)$. The steady state conditions for the low type are (where we already use $e(h,h) = e(\ell,\ell)$):

$$\begin{aligned}
[q(\ell,\ell)+q(\ell,h)]e(\ell,\varnothing) &= \delta e(\ell,\ell) + [\delta + q(\ell,\ell)]e(\ell,h) \\
\delta e(\ell,\ell) &= q(\ell,\ell)e(\ell,\varnothing) + q(\ell,\ell)e(\ell,h) \\
[\delta + q(\ell,\ell) + q(\ell,\ell)]e(\ell,h) &= q(\ell,h)e(\ell,\varnothing)
\end{aligned}$$

The successful meeting rates are

$$\begin{aligned}
q(\ell,\ell) &= \rho e(\ell,\varnothing)d(\ell,\varnothing,\ell) + \rho e(\ell,h)d(\ell,h,\ell) = \rho[e(\ell,\varnothing)+e(\ell,h)] \\
q(\ell,h) &= \rho e(h,\varnothing)d(h,\varnothing,\ell) + \rho e(h,h)d(h,h,\ell) = \rho e(h,\varnothing) = \rho e(\ell,\varnothing)
\end{aligned}$$

Substitute these into the steady state conditions and divide by ρ to get

$$\begin{aligned}
[2e(\ell,\varnothing)+e(\ell,h)]e(\ell,\varnothing) &= \kappa^{-1}e(\ell,\ell)+[\kappa^{-1}+e(\ell,\varnothing)+e(\ell,h)]e(\ell,h) \\
\kappa^{-1}e(\ell,\ell) &= [e(\ell,\varnothing)+e(\ell,h)]^2 \\
[\kappa^{-1}+2(e(\ell,\varnothing)+e(\ell,h))]e(\ell,h) &= e(\ell,\varnothing)^2
\end{aligned}$$

Note that these are exactly the same as (1), the steady state conditions for Strict Heterophily, with $e(\ell,\ell)$ and $e(\ell,h)$ swapped. The densities are thus, up to the substitution $e(\ell,\ell) \leftrightarrow e(\ell,h)$, exactly the same as those for Strict Heterophily.

LEMMA 4. *In an equilibrium with strict homophily there is positive assortative matching.*

Proof. Since $e(\ell,\varnothing) = e(h,\varnothing)$, there is positive assortative matching if and only if $e(h,\ell) < e(h,h)$. Assume that $e(h,\ell) \geq e(h,h)$. The outflow of $e(h,\ell)$ is larger than the outflow of $e(h,h)$. However the inflow to $e(h,\ell)$ is smaller than the inflow to $e(h,h)$. Therefore if $e(h,\ell) \leq e(h,h)$, $e(h,\ell)$ and $e(h,h)$ cannot be jointly in steady state. ■

Surpluses are given by:

$$\begin{aligned}
S^*(h,\ell) &= [r+\delta+q^*(\ell,\ell)+q^*(h,\ell)+q^*(h,h)]^{-1}\,\pi^*(h,\ell) \\
S^*(h,h) &= [r+\delta+q^*(h,h)]^{-1}\,[h-q^*(h,\ell)S^*(h,\ell)] \\
S^*(\ell,h) &= [r+\delta+q^*(\ell,h)+q^*(\ell,\ell)+q^*(h,h)]^{-1}\,[F-\pi^*(h,\ell)] \\
S^*(\ell,\ell) &= [r+\delta+q^*(\ell,\ell)]^{-1}\,[\ell-q^*(\ell,h)S^*(\ell,h)]
\end{aligned}$$

Since the decision functions are symmetric, $q^*(h,h) = q^*(\ell,\ell)$ and $q^*(\ell,h) = q^*(h,\ell)$. Surplus equalization $S^*(h,\ell) = S^*(\ell,h)$ (which must hold per third point of the definition of equilibrium) gives $\pi^*(\ell,h) = \frac{1}{2}F = \pi^*(h,\ell)$. Consistency requires $G^1_{SHO} \equiv S^*(h,h) - S^*(h,\ell) > 0$ and $G^2_{SHO} \equiv S^*(\ell,\ell) - S^*(\ell,h) > 0$.

Bargaining in match (ℓ, h)

In match (ℓ, h) both agents leave for agents of their own type. Consider an alternative agreement c_1 where neither agent leaves and h is made indifferent. If c_1 is consistent, both agents would be better off and thus it would lead to a higher product of surpluses. Strict homophily thus requires $G^3_{SHO} \equiv S^*(\ell, \ell) - \widehat{S}^{c_1}_\ell > 0$.

In agreement c_2, ℓ never leaves the match, and h only leaves for another h. Since $S^*(\ell, \ell) > S^*(\ell, h) = S^*(h, \ell)$ by G^2_{SHO}, the highest product of surpluses occurs when $\widehat{S}^{c_2}_\ell = S^*(\ell, \ell)$. If the agreement is consistent, ℓ is strictly better off and h is weakly better off, so the product of surpluses is higher. Therefore strict homophily requires $G^4_{SHO} \equiv S^*(h, \ell) - \widehat{S}^{c_2}_h > 0$.

In agreement c_3, ℓ never leaves the match, and h always leaves. Strict homophily requires $G^5_{SHO} \equiv S^*(\ell, h)S^*(h, \ell) - \widehat{S}^{c_3}_\ell \widehat{S}^{c_3}_h \geq 0$. Note that no consistency checks are required: $\widehat{S}^{c_3}_\ell = S^*(\ell, \ell) > S^*(\ell, h)$ by G^2_{SHO}; if instead h would rather not leave the match, we are back in one of the two previous agreements.

In agreement c_4, h never leaves the match, and ℓ only leaves for another ℓ. Since $S^*(h, h) > S^*(h, \ell) = S^*(\ell, h)$ by G^1_{SHO}, the highest product of surpluses occurs when $\widehat{S}^{c_2}_h = S^*(h, h)$. If the agreement is consistent, h is strictly better off and ℓ is weakly better off, so the product of surpluses is higher. Therefore strict homophily requires $G^6_{SHO} \equiv S^*(\ell, h) - \widehat{S}^{c_4}_\ell > 0$.

In agreement c_5, h never leaves the match, and ℓ always leaves. Strict homophily requires $G^7_{SHO} \equiv S^*(\ell, h)S^*(h, \ell) - \widehat{S}^{c_5}_\ell \widehat{S}^{c_5}_h \geq 0$.

Bargaining in matches (ℓ, ℓ) **and** (h, h)

In matches (ℓ, ℓ) and (h, h) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

Details on equilibrium conditions

We first check consistency of the equilibrium with strict homophily. Condition $G^2_{SHO} \equiv S^*(\ell,\ell) - S^*(\ell,h) > 0$ becomes

$$F < \frac{r+\delta+q^*(h,\ell)+2q^*(\ell,\ell)}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)} 2\ell \tag{SHO 1}$$

$S^*(h,h) > S^*(\ell,\ell)$ holds because $h > \ell$, and thus $G^1_{SHO} \equiv S^*(h,h) - S^*(h,\ell) > 0$ holds whenever $G^2_{SHO} \equiv S^*(\ell,\ell) - S^*(\ell,h) = S^*(\ell,\ell) - S^*(h,\ell) > 0$ holds.

In agreement c_1, neither agent leaves the match. To be consistent, both agents must receive allocations at least as large as what they would receive with agents of their own type. For strict homophily to be an equilibrium, it cannot be the case that total output F is large enough to allocate h to agent h and ℓ to agent ℓ:

$$F < \ell + h \tag{SHO 2}$$

In agreement c_2, ℓ never leaves the match and h leaves when she finds an h. Surpluses are:

$$\begin{aligned}
\widehat{S}^{c_2}_\ell &= [r+\delta+q^*(h,h)]^{-1}[\widehat{\pi}_\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)] = S^*(\ell,\ell) \\
\widehat{S}^{c_2}_h &= [r+\delta+q^*(h,h)]^{-1}[F - \widehat{\pi}_\ell - q^*(h,\ell)S^*(h,\ell)]
\end{aligned}$$

From $\widehat{S}^{c_2}_\ell = S^*(\ell,\ell)$ we can recover $\widehat{\pi}_\ell$:

$$\widehat{\pi}_\ell = \frac{r+\delta+2q^*(h,h)}{r+\delta+q^*(h,h)}\ell - \frac{q^*(h,h)q^*(\ell,h)}{[r+\delta+q^*(h,h)][r+\delta+2q^*(h,h)+q^*(\ell,h)]}\frac{F}{2}$$

Condition $G^4_{SHO} \equiv S^*(h,\ell) - \widehat{S}^{c_2}_h > 0$ holds if $F < \widehat{\pi}_\ell + [r+\delta+q^*(h,h)+q^*(h,\ell)]S^*(h,\ell)$. Replacing $\widehat{\pi}_\ell$:

$$F < \frac{[r+\delta+2q^*(h,h)][r+\delta+2q^*(h,h)+q^*(h,\ell)]}{[r+\delta+q^*(h,h)+q^*(h,\ell)]q^*(h,h)} 2\ell$$

Since $\frac{r+\delta+2q^*(h,h)}{q^*(h,h)} > 1$, G^4_{SHO} holds whenever G^2_{SHO} does.

In agreement c_3, ℓ never leaves the match and h always leaves. Surpluses are:

$$\widehat{S}^{c_3}_\ell = [r+\delta+q^*(h,h)+q^*(h,\ell)]^{-1}\left[\widehat{\pi}_\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)\right] = S^*(\ell,\ell)$$
$$\widehat{S}^{c_3}_h = [r+\delta+q^*(h,h)+q^*(h,\ell)]^{-1}\left(F - \widehat{\pi}_\ell\right)$$

From $\widehat{S}^{c_3}_\ell = S^*(\ell,\ell)$ we can recover $\widehat{\pi}_\ell$:

$$\widehat{\pi}_\ell = \frac{r+\delta+2q^*(h,h)+q^*(h,\ell)}{r+\delta+q^*(h,h)}\ell - \frac{[q^*(h,h)+q^*(h,\ell)]q^*(h,\ell)}{[r+\delta+q^*(h,h)][r+\delta+2q^*(h,h)+q^*(h,\ell)]}\frac{F}{2}$$

Condition $G^5_{SHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_3}_\ell\widehat{S}^{c_3}_h \geq 0$ holds when $A_6F^2 + B_6F + C_6 \geq 0$, where:

$$\begin{aligned}
A_6 &= \frac{2[r+\delta+q^*(h,h)][r+\delta+2q^*(h,h)+q^*(h,\ell)] + [q^*(h,h)+q^*(h,\ell)]q^*(h,\ell)}{4[r+\delta+q^*(h,h)+q^*(h,\ell)][r+\delta+q^*(h,h)]^2[r+\delta+2q^*(h,h)+q^*(h,\ell)]^2}q^*(h,\ell) \\
&\quad + \frac{1}{4[r+\delta+2q^*(h,h)+q^*(h,\ell)]^2} \\
B_6 &= -\frac{2[r+\delta+q^*(h,h)][r+\delta+2q^*(h,h)+q^*(h,\ell)] + [q^*(h,h)+q^*(h,\ell)]q^*(h,\ell)}{2[r+\delta+q^*(h,h)+q^*(h,\ell)][r+\delta+q^*(h,h)]^2[r+\delta+2q^*(h,h)+q^*(h,\ell)]}\ell \\
&\quad + \frac{q^*(h,\ell)\ell}{2[r+\delta+q^*(h,h)+q^*(h,\ell)][r+\delta+q^*(h,h)]^2} \\
C_6 &= \frac{r+\delta+2q^*(h,h)+q^*(h,\ell)}{[r+\delta+q^*(h,h)+q^*(h,\ell)][r+\delta+q^*(h,h)]^2}\ell^2
\end{aligned}$$

Since $A_6 > 0$, G^5_{SHO} is a convex function of F. In order to have an equilibrium with strict homophily, F has to be smaller than the lower root or larger than the higher one. Only the first of these two conditions is relevant. To see this, note that there exists an $\widehat{F}$ such that $S^*(\ell,h) = S^*(\ell,\ell)$. For $F = \widehat{F}$, $\widehat{S}^{c_3}_\ell\widehat{S}^{c_3}_h > S^*(\ell,h)S^*(h,\ell)$ holds.[4] Therefore, F larger than the large root of $G^5_{SHO} = 0$ requires that $F > \widehat{F}$. However,

[4] If $F = \widehat{F}$, this is equivalent to $\widehat{S}^{c_3}_h > S^*(h,\ell)$ because $\widehat{S}^{c_3}_\ell = S^*(\ell,\ell) = S^*(\ell,h)$. Add $\widehat{S}^{c_3}_\ell = S^*(\ell,h)$ on both sides of the inequality and rearrange terms to get:

$$\frac{\widehat{F} - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)}{r+\delta+q^*(h,h)+q^*(h,\ell)} > \frac{\widehat{F} - 2q^*(h,h)S^*(h,\ell)}{r+\delta+q^*(h,h)+q^*(h,\ell)}$$

Comparing numerators gives $q^*(h,h) > q^*(h,\ell)$, which indeed holds.

by G^1_{SHO}, an equilibrium with strict homophily requires $F < \widehat{F}$. Therefore an equilibrium with strict homophily requires

$$F \leq \frac{-B_6 - \sqrt{B_6^2 - 4A_6C_6}}{2A_6} \tag{SHO 3}$$

In agreement c_4, h never leaves the match and ℓ leaves when she finds an ℓ. Surpluses are:

$$\widehat{S}^{c_4}_{\ell} = [r + \delta + q^*(h,h)]^{-1}[\widehat{\pi}_{\ell} - q^*(h,\ell)S^*(h,\ell)]$$
$$\widehat{S}^{c_4}_{h} = [r + \delta + q^*(h,h)]^{-1}[F - \widehat{\pi}_{\ell} - q^*(h,\ell)S^*(h,\ell) - q^*(h,h)S^*(h,h)] = S^*(h,h)$$

Note that $\widehat{S}^{c_4}_{\ell} + \widehat{S}^{c_4}_{h} \leq \widehat{S}^{c_2}_{\ell} + \widehat{S}^{c_2}_{h}$, since $-q(h,h)S^*(h,h) \leq -q^*(\ell,\ell)S^*(\ell,\ell)$. Rearrange this as $\widehat{S}^{c_4}_{\ell} \leq S^*(\ell,\ell) - S^*(h,h) + \widehat{S}^{c_2}_{h}$. Since $S^*(h,h) > S^*(\ell,\ell)$, this implies $\widehat{S}^{c_4}_{\ell} < \widehat{S}^{c_2}_{h}$. Therefore $G^6_{SHO} \equiv S^*(\ell,h) - \widehat{S}^{c_4}_{\ell} > 0$ always holds if $G^4_{SHO} \equiv S^*(h,\ell) - \widehat{S}^{c_2}_{h} > 0$ holds.

In agreement c_5, h never leaves the match and ℓ always leaves. Surpluses are:

$$\widehat{S}^{c_5}_{\ell} = [r + \delta + q^*(h,h) + q^*(h,\ell)]^{-1}\widehat{\pi}_{\ell}$$
$$\widehat{S}^{c_5}_{h} = [r + \delta + q^*(h,h) + q^*(h,\ell)]^{-1}\left[F - \widehat{\pi}_{\ell} - q^*(h,\ell)S^*(h,\ell) - q^*(h,h)S^*(h,h)\right] = S^*(h,h)$$

Note that $\widehat{S}^{c_5}_{\ell} + \widehat{S}^{c_5}_{h} \leq \widehat{S}^{c_3}_{\ell} + \widehat{S}^{c_3}_{h}$, since $-q(h,h)S^*(h,h) \leq -q^*(\ell,\ell)S^*(\ell,\ell)$. Next, the higher surplus under c_3, $\widehat{S}^{c_3}_{\ell} = S^*(\ell,\ell)$, is lower than the higher surplus under c_5, $\widehat{S}^{c_5}_{h} = S^*(h,h)$. Therefore $\widehat{S}^{c_5}_{\ell}\widehat{S}^{c_5}_{h} < \widehat{S}^{c_3}_{\ell}\widehat{S}^{c_3}_{h}$. But then $G^7_{SHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_5}_{\ell}\widehat{S}^{c_5}_{h} \geq 0$ always holds if $G^5_{SHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_3}_{\ell}\widehat{S}^{c_3}_{h} \geq 0$ holds.

3.5 Weak Homophily (A)

Under weak homophily (A) $d(h,\ell,h) = 1$ and $d(\ell,h,\ell) = d(\ell,\ell,h) = d(h,h,\ell) = 0$. The steady state conditions are

$$
\begin{aligned}
[q(\ell,\ell) + q(\ell,h)]e(\ell,\varnothing) &= \delta e(\ell,\ell) + [\delta + q(h,h)]e(\ell,h) \\
\delta e(\ell,\ell) &= q(\ell,\ell)e(\ell,\varnothing) \\
[\delta + q(h,h)]e(\ell,h) &= q(\ell,h)e(\ell,\varnothing) \\
[q(h,\ell) + q(h,h)]e(h,\varnothing) &= \delta e(h,\ell) + \delta e(h,h) \\
\delta e(h,h) &= q(h,h)e(h,\varnothing) + q(h,h)e(h,\ell)
\end{aligned}
$$

The successful meeting rates are

$$
\begin{aligned}
q(\ell,\ell) &= \rho e(\ell,\varnothing)d(\ell,\varnothing,\ell) + \rho e(\ell,h)d(\ell,h,\ell) = \rho e(\ell,\varnothing) \\
q(\ell,h) &= \rho e(h,\varnothing)d(h,\varnothing,\ell) + \rho e(h,h)d(h,h,\ell) = \rho e(h,\varnothing) \\
q(h,\ell) &= \rho e(\ell,\varnothing)d(\ell,\varnothing,h) + \rho e(\ell,\ell)d(\ell,\ell,h) = \rho e(\ell,\varnothing) \\
q(h,h) &= \rho e(h,\varnothing)d(h,\varnothing,\ell) + \rho e(h,\ell)d(h,\ell,h) = \rho[e(h,\varnothing) + e(h,\ell)]
\end{aligned}
$$

Substituting these in the steady state conditions and dividing by ρ we get

$$
\begin{aligned}
[e(\ell,\varnothing) + e(h,\varnothing)]e(\ell,\varnothing) &= \kappa^{-1}e(\ell,\ell) + [\kappa^{-1} + e(h,\varnothing) + e(h,\ell)]e(\ell,h) \\
\kappa^{-1}e(\ell,\ell) &= e(\ell,\varnothing)^2 \\
[\kappa^{-1} + e(h,\varnothing) + e(h,\ell)]e(\ell,h) &= e(h,\varnothing)e(\ell,\varnothing) \\
[e(\ell,\varnothing) + e(h,\varnothing) + e(h,\ell)]e(h,\varnothing) &= \kappa^{-1}[e(h,\ell) + e(h,h)] \\
\kappa^{-1}e(h,h) &= [e(h,\varnothing) + e(h,\ell)]^2
\end{aligned}
$$

Note that the first equation is simply the sum of the second and the third, so we can drop it. From the last equation together with the normalization $e(h,\varnothing) + e(h,\ell) =$

$\frac{1}{2} - e(h,h)$ we get $\kappa^{-1}e(h,h) = \frac{1}{4} - e(h,h) + e(h,h)^2$ and can thus recover $e(h,h)$:

$$e(h,h) = \frac{1}{2}\left(1 + \kappa^{-1} - \sqrt{\kappa^{-2} + 2\kappa^{-1}}\right)$$

Next, use the second equation in the normalization condition for the low-types to get

$$e(\ell,h) = \frac{1}{2} - e(\ell,\ell) - e(\ell,\varnothing) = \frac{1}{2} - e(\ell,\varnothing)\left[1 + \kappa e(\ell,\varnothing)\right]$$

Use this expression together with the normalization for high-types in the third equation to get

$$\left[\kappa^{-1} + \frac{1}{2} - e(h,h)\right]\left[\frac{1}{2} - e(\ell,\varnothing)\left(1 + \kappa e(\ell,\varnothing)\right)\right] = e(h,\varnothing)e(\ell,\varnothing)$$

Using the (known) expression for $e(h,h)$ and rearranging we can express $e(h,\varnothing)$ in terms of $e(\ell,\varnothing)$ and κ^{-1}:

$$e(h,\varnothing) = \frac{1}{2}e(\ell,\varnothing)^{-1}\left[\kappa^{-1} + \sqrt{\kappa^{-2} + 2\kappa^{-1}}\right]\left[\frac{1}{2} - e(\ell,\varnothing)\left(1 + \kappa e(\ell,\varnothing)\right)\right]$$

Finally, in the fourth steady state condition we can use the expressions for $e(h,h)$, $e(\ell,h)$ and $e(h,\varnothing)$, together with the normalization $e(h,\varnothing) + e(h,\ell) = \frac{1}{2} - e(h,h) = \sqrt{\kappa^{-2} + 2\kappa^{-1}} - \kappa^{-1}$ to get

$$\begin{aligned}\left[e(\ell,\varnothing) + \sqrt{\kappa^{-2} + 2\kappa^{-1}} - \kappa^{-1}\right]\frac{1}{2}\left[\kappa^{-1} + \sqrt{\kappa^{-2} + 2\kappa^{-1}}\right]\left[\frac{1}{2} - e(\ell,\varnothing)\left(1 + \kappa e(\ell,\varnothing)\right)\right] = \\ = \kappa^{-1}e(\ell,\varnothing)\left[\frac{1}{2} - e(\ell,\varnothing)\left(1 + \kappa e(\ell,\varnothing)\right) + \frac{1}{2}\left(1 + \kappa^{-1} - \sqrt{\kappa^{-2} + 2\kappa^{-1}}\right)\right]\end{aligned}$$

This is a cubic equation in the sole variable $e(\ell,\varnothing)$, which can be solved by computer algebra (it has one real solution for all values of κ^{-1}). We can then express the other densities in terms of $e(\ell,\varnothing)$: the second steady state condition gives $e(\ell,\ell)$, while $e(\ell,h) = e(h,\ell)$ and $e(h,\varnothing)$ are given above.

LEMMA 5. *In an equilibrium with weak homophily (A) there is positive assortative matching.*

Proof. First, we show that $e(\ell, \varnothing) > e(h, \varnothing)$. Assume that $e(\ell, \varnothing) \leq e(h, \varnothing)$. The inflow to $e(\ell, \varnothing)$ is then larger than the inflow to $e(h, \varnothing)$. But the outflow of $e(\ell, \varnothing)$ is smaller than the outflow of $e(h, \varnothing)$. Therefore if $e(\ell, \varnothing) \leq e(h, \varnothing)$, $e(\ell, \varnothing)$ and $e(h, \varnothing)$ cannot be jointly in steady state.

Second, we show that $e(h, \ell) < e(h, h)$. Assume that $e(h, \ell) \geq e(h, h)$. The outflow of $e(h, \ell)$ is larger than the outflow of $e(h, h)$. However the inflow to $e(h, \ell)$ is smaller than the inflow to $e(h, h)$, since $e(\ell, \varnothing) > e(h, \varnothing)$. Therefore if $e(h, \ell) \geq e(h, h)$, $e(h, \ell)$ and $e(h, h)$ cannot be jointly in steady state. ■

Surpluses are given by:

$$\begin{aligned}
S^*(h,\ell) &= [r+\delta+q^*(h,\ell)+q^*(\ell,\ell)]^{-1}\,[\pi^*(h,\ell)-q^*(h,h)S^*(h,h)] \\
S^*(h,h) &= [r+\delta+q^*(h,h)]^{-1}\,[h-q^*(h,\ell)S^*(h,\ell)] \\
S^*(\ell,h) &= [r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]^{-1}\,[F-\pi^*(h,\ell)] \\
S^*(\ell,\ell) &= [r+\delta+q^*(\ell,\ell)]^{-1}\,[\ell-q^*(\ell,h)S^*(\ell,h)]
\end{aligned}$$

Note that $q^*(\ell, h) = q^*(h, h)$, as an h is available only if she is unemployed, and will match with either type. Since h is indifferent between types, $S^*(h, \ell) = S^*(h, h) = [r + \delta + q^*(h, \ell) + q^*(h, h)]^{-1}h$. This allows us to recover $\pi^*(h, \ell)$:

$$\pi^*(h,\ell) = \frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}h$$

Consistency requires $G^1_{WHO} \equiv S^*(\ell, \ell) - S^*(\ell, h) > 0$.

Bargaining in match (ℓ, h)

In match (ℓ, h) agent h never leaves, while ℓ leaves if she finds an ℓ. If $S^*(h, \ell) < S^*(\ell, h)$, the third point of the definition of equilibrium could not hold, as ℓ is not indifferent between partner types. Thus $G^2_{WHO} \equiv S^*(h, \ell) - S^*(\ell, h) \geq 0$ must hold.

Consider an alternative agreement c_1 where neither agent leaves and h is made indifferent. If c_1 is consistent, ℓ would be made better off, so it would lead to a higher product of surpluses. Weak homophily thus requires $G^3_{WHO} \equiv S^*(\ell,\ell) - \widehat{S}^{c_1}_{\ell} > 0$.

In agreement c_2, ℓ never leaves, and h always leaves (note that h can only leave for both types or for none), and surplus is split equally. This agreement invalidates the equilibrium with weak homophily if it is consistent and generates a higher product of surpluses. Therefore $G^4_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_2}_{\ell}\widehat{S}^{c_2}_{h} \geq 0$ must hold whenever c_2 is consistent.

If agreement c_2 is not consistent, an alternative agreement c_3 where h always leaves, ℓ never leaves, and $\widehat{S}^{c_3}_{\ell} = S^*(\ell,\ell)$, invalidates the equilibrium if it generates a higher product of surpluses. Therefore $G^5_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_3}_{\ell}\widehat{S}^{c_3}_{h} \geq 0$ must hold.

In agreement c_4, ℓ leaves when she meets another ℓ, h always leaves, and surplus is split equally. $G^6_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_4}_{\ell}\widehat{S}^{c_4}_{h} \geq 0$ must hold.

Finally, all surpluses must be non-negative. By consistency, it is only necessary to verify $G^7_{WHO} \equiv S^*(\ell,h) \geq 0$. Note that in all previous equilibria, it was straightforward to see that all surpluses were non-negative.

Bargaining in matches (ℓ,ℓ) and (h,h)

In matches (ℓ,ℓ) and (h,h) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

Details on equilibrium conditions

Condition $G^2_{WHO} \equiv S^*(h,\ell) - S^*(\ell,h) \geq 0$ holds if

$$F \leq h + \frac{r+\delta+q^*(\ell,h)+2q^*(\ell,\ell)}{r+\delta+q^*(h,\ell)+q^*(h,h)}h$$

Since $q^*(\ell,h) = q^*(h,h)$ and $q^*(\ell,\ell) > q^*(h,\ell)$ (an ℓ currently matched with an h will agree to switch to an ℓ), we have $F < 2h$, which always holds.

In agreement c_1, neither agent leaves the match. To be consistent, both agents must receive allocations at least as large as what they would receive with agents of their own type. For weak homophily to be an equilibrium, it cannot be the case that total output F is large enough to allocate h to agent h and ℓ to agent ℓ:

$$F < \ell + h \tag{WHO 1}$$

In agreement c_2, ℓ never leaves, h always leaves, and $\widehat{S}_\ell^{c_2} = \widehat{S}_h^{c_2}$. Surpluses are:

$$\begin{aligned}
\widehat{S}_\ell^{c_2} &= [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}[\widehat{\pi}_\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)] \\
\widehat{S}_h^{c_2} &= [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}[F-\widehat{\pi}_\ell]
\end{aligned}$$

From $\widehat{S}_\ell^{c_2} = \widehat{S}_h^{c_2}$ we can recover $\widehat{\pi}_\ell$:

$$\widehat{\pi}_\ell = \frac{1}{2}F + \frac{(r+\delta)q^*(\ell,h)[F-\pi^*(h,\ell)] + \ell q^*(\ell,\ell)[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]}{2[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]}$$

Consistency of c_2 requires $\widehat{S}_\ell^{c_2} \geq S^*(\ell,\ell)$ and $\widehat{S}_h^{c_2} < S^*(h,h)$. The latter condition is implied by G^3_{WHO}. Since $\widehat{S}_\ell^{c_2} = \widehat{S}_h^{c_2}$, the former is equivalent to $\widehat{S}_\ell^{c_2} + \widehat{S}_h^{c_2} \geq 2S^*(\ell,\ell)$, which holds if

$$\begin{aligned}
F \geq F^4_{WHO} \equiv{} & \frac{[2r+2\delta+2q^*(h,\ell)+2q^*(h,h)+q^*(\ell,\ell)][r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]+[r+\delta+2q^*(h,\ell)+2q^*(h,h)]q^*(\ell,h)}\ell \\
&+ \frac{[r+\delta+2q^*(h,\ell)+2q^*(h,h)][r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)]q^*(\ell,h)}{[r+\delta+q^*(h,\ell)+q^*(h,h)]\left\{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]+[r+\delta+2q^*(h,\ell)+2q^*(h,h)]q^*(\ell,h)\right\}}h
\end{aligned}$$

Condition $G^4_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}_\ell^{c_2}\widehat{S}_h^{c_2} \geq 0$ becomes

$$\frac{h[F-\pi^*(h,\ell)]}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)} \geq \frac{(F-\widehat{\pi}_\ell)^2}{r+\delta+q^*(h,\ell)+q^*(h,h)}$$

Replacing $\widehat{\pi}_\ell$, this can be rewritten as $A_7F^2 + B_7F + C_7 \geq 0$, with:

$$\begin{aligned}
A_7 &= -\left[\frac{[r+\delta+q^*(\ell,\ell)]^2 + q^*(\ell,h)q^*(\ell,\ell)}{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)]}\right]^2 \\
B_7 &= 4\,\frac{r+\delta+q^*(h,\ell)+q^*(h,h)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}h + \left\{[r+\delta+q^*(\ell,\ell)]^2 + q^*(\ell,h)q^*(\ell,\ell)\right\} \times \\
&\quad \times 2\,\frac{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]q^*(\ell,\ell)\ell - q^*(\ell,h)\,(r+\delta)\,\pi^*(h,\ell)}{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]^2[r+\delta+q^*(\ell,\ell)]^2} \\
C_7 &= -4\,\frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}h^2 \\
&\quad - \left[\frac{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]q^*(\ell,\ell)\ell - q^*(\ell,h)\,(r+\delta)\,\pi^*(h,\ell)}{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)]}\right]^2
\end{aligned}$$

Since $A_7 < 0$, G^4_{WHO} is a concave function of F. In order to have an equilibrium with weak homophily of type 1, F has to be between the two roots of $G^4_{WHO} = 0$, but only if agreement c_2 is consistent. Therefore, equilibrium with weak homophily of type 1 requires:

$$F \in \begin{cases} \left[2\ell, F^4_{WHO}\right) \cup \left[\frac{-B_7+\sqrt{B_7^2-4A_7C_7}}{2A_7}, \frac{-B_7-\sqrt{B_7^2-4A_7C_7}}{2A_7}\right] \text{ if } F^4_{WHO} \leq \frac{-B_7+\sqrt{B_7^2-4A_7C_7}}{2A_7} \\ \left[2\ell, \frac{-B_7-\sqrt{B_7^2-4A_7C_7}}{2A_7}\right] \text{ if } \frac{-B_7+\sqrt{B_7^2-4A_7C_7}}{2A_7} < F^4_{WHO} \leq \frac{-B_7-\sqrt{B_7^2-4A_7C_7}}{2A_7} \\ \left[2\ell, F^4_{WHO}\right) \text{ if } F^4_{WHO} > \frac{-B_7-\sqrt{B_7^2-4A_7C_7}}{2A_7} \end{cases} \tag{WHO 2}$$

In agreement c_3, ℓ never leaves, h always leaves, and $\widehat{S}^{c_3}_\ell = S^*(\ell,\ell)$. Surpluses are:

$$\begin{aligned}
\widehat{S}^{c_3}_\ell &= [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}[\widehat{\pi}_\ell - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h)] = S^*(\ell,\ell) \\
\widehat{S}^{c_3}_h &= [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}(F - \widehat{\pi}_\ell)
\end{aligned}$$

From $\widehat{S}^{c_3}_\ell = S^*(\ell,\ell)$ we can recover $\widehat{\pi}_\ell$:

$$\widehat{\pi}_\ell = \frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(\ell,\ell)}\ell - \frac{[q^*(h,\ell)+q^*(h,h)]q^*(\ell,h)[F-\pi^*(h,\ell)]}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]}$$

Condition $G^5_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}^{c_3}_\ell \widehat{S}^{c_3}_h \geq 0$ becomes

$$\frac{h[r+\delta+q^*(\ell,\ell)] + q^*(\ell,h)[F-\widehat{\pi}_\ell]}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}[F-\pi^*(h,\ell)] \geq \ell(F-\widehat{\pi}_\ell)$$

Replacing $\widehat{\pi}_\ell$, this can be rewritten as $A_8F^2 + B_8F + C_8 \geq 0$, with:

$$\begin{aligned}
A_8 &= q^*(\ell,h)\frac{[q^*(h,\ell)+q^*(h,h)]q^*(\ell,h) + [r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]^2} \\
B_8 &= -\ell q^*(\ell,h)\frac{r+\delta+2q^*(h,\ell)+2q^*(h,h)+q^*(\ell,\ell)}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]} - \ell + \frac{r+\delta+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}h \\
&\quad - 2\,\frac{q^*(\ell,h)^2\,\pi^*(h,\ell)[q^*(h,\ell)+q^*(h,h)]}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]^2} - \frac{q^*(\ell,h)\pi^*(h,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)} \\
C_8 &= -\pi^*(h,\ell)\frac{r+\delta+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}h + \left[\ell + \frac{q^*(\ell,h)\pi^*(h,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}\right] \\
&\quad \times \left[\frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(\ell,\ell)}\ell + \frac{[q^*(h,\ell)+q^*(h,h)]q^*(\ell,h)\pi^*(h,\ell)}{[r+\delta+q^*(\ell,\ell)][r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]}\right]
\end{aligned}$$

Since A_8 is positive, G^5_{WHO} is a convex function of F. Thus, an equilibrium with weak homophily of type 1 requires

$$F \notin \left(\frac{-B_8 - \sqrt{B_8^2 - 4A_8C_8}}{2A_8}, \frac{-B_8 + \sqrt{B_8^2 - 4A_8C_8}}{2A_8}\right) \tag{WHO 3}$$

In agreement c_4, ℓ leaves when she meets an ℓ, h always leaves, and $\widehat{S}^{c_4}_\ell = \widehat{S}^{c_4}_h$. Surpluses are:

$$\begin{aligned}
\widehat{S}^{c_4}_\ell &= [r+\delta+q^*(\ell,\ell)+q^*(h,\ell)+q^*(h,h)]^{-1}[\widehat{\pi}_\ell - q^*(\ell,h)S^*(\ell,h)] \\
\widehat{S}^{c_4}_h &= [r+\delta+q^*(\ell,\ell)+q^*(h,\ell)+q^*(h,h)]^{-1}[F-\widehat{\pi}_\ell]
\end{aligned}$$

From $\widehat{S}^{c_4}_\ell = \widehat{S}^{c_4}_h$ we can recover $\widehat{\pi}_\ell$:

$$\widehat{\pi}_\ell = \frac{1}{2}F + \frac{1}{2}q^*(\ell,h)\frac{F-\pi^*(h,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}$$

Consistency of c_4 requires $\widehat{S}^{c_4}_\ell < S^*(\ell,\ell)$. Note that c_4 generates a lower total surplus than c_2; since surplus is split equally in both cases, this gives $\widehat{S}_\ell^{c_4} < \widehat{S}_\ell^{c_2}$. Therefore $\widehat{S}_\ell^{c_4}\widehat{S}_h^{c_4} < \widehat{S}_\ell^{c_2}\widehat{S}_h^{c_2}$ and *in*consistency of c_2 implies consistency of c_4. But then, the only way in which G^4_{WHO} can hold and G^6_{WHO} not hold is if c_4 is consistent. Therefore, it is enough to check the product of surpluses. Condition $G^6_{WHO} \equiv S^*(\ell,h)S^*(h,\ell) - \widehat{S}_\ell^{c_4}\widehat{S}_h^{c_4} \geq 0$ becomes

$$\frac{h[F-\pi^*(h,\ell)]}{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+q^*(h,\ell)+q^*(h,h)]} \geq \frac{(F-\widehat{\pi}_\ell)^2}{[r+\delta+q^*(\ell,\ell)+q^*(h,\ell)+q^*(h,h)]^2}$$

Replacing $\widehat{\pi}_\ell$, this can be rewritten as $A_9F^2 + B_9F + C_9 \geq 0$, with:

$$\begin{aligned}
A_9 &= -[r+\delta+q^*(\ell,\ell)]^2 \\
B_9 &= 4\frac{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+q^*(h,\ell)+q^*(h,h)+q^*(\ell,\ell)]^2}{r+\delta+q^*(h,h)+q^*(h,\ell)}h \\
&\quad - 2q^*(\ell,h)[r+\delta+q^*(\ell,\ell)]\frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}h \\
C_9 &= -4\frac{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)][r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)]^3}{[r+\delta+q^*(h,h)+q^*(h,\ell)]^2}h^2 \\
&\quad - \left[\frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}q^*(\ell,h)h\right]^2
\end{aligned}$$

Since A_9 is negative, G^6_{WHO} is a concave function of F. Thus, an equilibrium with weak homophily of type 1 requires

$$F \in \left[\frac{-B_9+\sqrt{B_9^2-4A_9C_9}}{2A_9}, \frac{-B_9-\sqrt{B_9^2-4A_9C_9}}{2A_9}\right] \tag{WHO 4}$$

Condition $G^7_{WHO} \equiv S^*(\ell,h) \geq 0$ holds if $F \geq \pi^*(h,\ell)$. If instead $F < \pi^*(h,\ell)$, note that $G^6_{WHO} < 0$. Therefore, if G^6_{WHO} holds, so does G^7_{WHO}.

Finally, we check consistency of the equilibrium with weak homophily of type 1. Condition $G^1_{WHO} \equiv S^*(\ell,\ell) - S^*(\ell,h) > 0$ holds if

$$F < \frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}h + \ell$$

Since $\frac{r+\delta+q^*(h,\ell)+q^*(\ell,\ell)+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)} > 1$, if G^3_{WHO} holds, so does G^1_{WHO}.

3.6 Weak Homophily (B)

Under weak homophily (B) $d(\ell,h,\ell) = 1$ and $d(h,\ell,\ell) = d(\ell,\ell,h) = d(h,h,\ell) = 0$. The steady state conditions are

$$\begin{aligned}
[q(\ell,\ell)+q(\ell,h)]e(\ell,\varnothing) &= \delta e(\ell,\ell)+\delta e(\ell,h) \\
\delta e(\ell,\ell) &= q(\ell,\ell)e(\ell,\varnothing)+q(\ell,\ell)e(\ell,h) \\
[\delta+q(\ell,\ell)]e(\ell,h) &= q(\ell,h)e(\ell,\varnothing) \\
[q(h,\ell)+q(h,h)]e(h,\varnothing) &= [\delta+q(\ell,\ell)]e(h,\ell)+\delta e(h,h) \\
\delta e(h,h) &= q(h,h)e(h,\varnothing)
\end{aligned}$$

The successful meeting rates are

$$\begin{aligned}
q(\ell,\ell) &= \rho e(\ell,\varnothing)d(\ell,\varnothing,\ell)+\rho e(\ell,h)d(\ell,h,\ell) = \rho[e(\ell,\varnothing)+e(\ell,h)] \\
q(\ell,h) &= \rho e(h,\varnothing)d(h,\varnothing,\ell)+\rho e(h,h)d(h,h,\ell) = \rho e(h,\varnothing) \\
q(h,\ell) &= \rho e(\ell,\varnothing)d(\ell,\varnothing,h)+\rho e(\ell,\ell)d(\ell,\ell,h) = \rho e(\ell,\varnothing) \\
q(h,h) &= \rho e(h,\varnothing)d(h,\varnothing,\ell)+\rho e(h,\ell)d(h,\ell,h) = \rho e(h,\varnothing)
\end{aligned}$$

Substituting these in the steady state conditions and dividing by ρ we get

$$\begin{aligned}
[e(\ell,\varnothing)+e(\ell,h)+e(h,\varnothing)]e(\ell,\varnothing) &= \kappa^{-1}e(\ell,\ell)+\kappa^{-1}e(\ell,h) \\
\kappa^{-1}e(\ell,\ell) &= [e(\ell,\varnothing)+e(\ell,h)]^2 \\
[\kappa^{-1}+e(\ell,\varnothing)+e(\ell,h)]e(\ell,h) &= e(h,\varnothing)e(\ell,\varnothing) \\
[e(\ell,\varnothing)+e(h,\varnothing)]e(h,\varnothing) &= [\kappa^{-1}+e(\ell,\varnothing)+e(\ell,h)]e(h,\ell)+\kappa^{-1}e(h,h) \\
\kappa^{-1}e(h,h) &= e(h,\varnothing)^2
\end{aligned}$$

Note that these are the same as the steady state conditions for Weak Homophily (A) once ℓ and h are switched. Therefore the densities are the same as those for Weak Homophily (A) with the substitutions $e(\ell,\varnothing) \leftrightarrow e(h,\varnothing)$ and $e(\ell,\ell) \leftrightarrow e(h,h)$.

LEMMA 6. *In an equilibrium with weak homophily (B) there is no positive assortative matching nor negative assortative matching.*

Proof. Since The steady state conditions are the same as the steady state conditions for Weak Homophily (A) once ℓ and h are switched, $e(\ell,\varnothing) < e(h,\varnothing)$. We show that $e(h,h) > e(h,\ell)$. Assume that $e(h,\ell) \geq e(h,h)$. The outflow of $e(h,\ell)$ is larger than the outflow of $e(h,h)$. However since $e(\ell,\varnothing) < e(h,\varnothing)$ the inflow to $e(h,\ell)$ is smaller than the inflow to $e(h,h)$. Therefore if $e(h,\ell) \geq e(h,h)$, $e(h,\ell)$ and $e(h,h)$ cannot be jointly in steady state. ■

Surpluses are given by:

$$
\begin{aligned}
S^*(\ell,h) &= [r+\delta+q^*(\ell,h)+q^*(h,h)]^{-1}\,[\pi^*(\ell,h)-q^*(\ell,\ell)S^*(\ell,\ell)] \\
S^*(\ell,\ell) &= [r+\delta+q^*(\ell,\ell)]^{-1}\,[h-q^*(\ell,h)S^*(\ell,h)] \\
S^*(h,\ell) &= [r+\delta+q^*(h,\ell)+q^*(h,h)]^{-1}\,[F-\pi^*(\ell,h)] \\
S^*(h,h) &= [r+\delta+q^*(h,h)]^{-1}\,[h-q^*(h,\ell)S^*(h,\ell)]
\end{aligned}
$$

Note that $q^*(h,\ell) = q^*(\ell,\ell)$, as an ℓ is available only if she is unemployed, and will match with either type. Since ℓ is indifferent between types, $S^*(\ell,h) = S^*(\ell,\ell) = [r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]^{-1}\ell$. This allows us to recover $\pi^*(\ell,h)$:

$$
\pi^*(\ell,h) = \frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}\ell
$$

Consistency requires $G^8_{WHO} \equiv S^*(h,h) - S^*(h,\ell) > 0$.

Bargaining in match (h,ℓ)

In match (h,ℓ) agent ℓ never leaves, while h leaves if she finds an h. If $S^*(\ell,h) < S^*(h,\ell)$, the third point of the definition of equilibrium could not hold, as h is not indifferent between partner types. Thus $G^{9}_{WHO} \equiv S^*(\ell, h) - S^*(h, \ell) \geq 0$ must hold.

Consider an alternative agreement c_5 where neither agent leaves and ℓ is made indifferent. If c_5 is consistent, h would be made better off, so it would lead to a higher product of surpluses. Weak homophily thus requires $G^{10}_{WHO} \equiv S^*(h, h) - \widehat{S}^{c_5}_h > 0$.

In agreement c_6, h never leaves, and ℓ always leaves (note that ℓ can only leave for both types or for none), and surplus is split equally. This agreement invalidates the equilibrium with weak homophily if it is consistent and generates a higher product of surpluses. Therefore $G^{11}_{WHO} \equiv S^*(h, \ell)S^*(\ell, h) - \widehat{S}^{c_6}_h \widehat{S}^{c_6}_\ell \geq 0$ must hold whenever c_6 is consistent.

If agreement c_6 is not consistent, an alternative agreement c_7 where ℓ always leaves, h never leaves, and $\widehat{S}^{c_7}_h = S^*(h, h)$, invalidates the equilibrium if it generates a higher product of surpluses. Therefore $G^{12}_{WHO} \equiv S^*(h, \ell)S^*(\ell, h) - \widehat{S}^{c_7}_h \widehat{S}^{c_7}_\ell \geq 0$ must hold.

In agreement c_8, h leaves when she meets another h, ℓ always leaves, and surplus is split equally. $G^{13}_{WHO} \equiv S^*(h, \ell)S^*(\ell, h) - \widehat{S}^{c_8}_h \widehat{S}^{c_8}_\ell \geq 0$ must hold.

Finally, all surpluses must be non-negative. By consistency, it is only necessary to verify $G^{14}_{WHO} \equiv S^*(h, \ell) \geq 0$. Note that in all previous equilibria, it was straightforward to see that all surpluses were non-negative.

Bargaining in matches (h, h) and (ℓ, ℓ)

In matches (h, h) and (ℓ, ℓ) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

Details on equilibrium conditions

Condition $G^{9}_{WHO} \equiv S^*(\ell, h) - S^*(h, \ell) \geq 0$ holds if

$$F \leq \ell + \frac{r + \delta + q^*(h, \ell) + 2q^*(h, h)}{r + \delta + q^*(\ell, h) + q^*(\ell, \ell)}\ell \tag{WHO 5}$$

In agreement c_5, neither agent leaves the match. As in weak homophily of type 1, we require

$$F < \ell + h$$

In agreement c_6, h never leaves, ℓ always leaves, and $\widehat{S}_h^{c_6} = \widehat{S}_\ell^{c_6}$. Condition G^{11}_{WHO} mirrors G^4_{WHO}. Thus, weak homophily of type 2 requires:

$$F \in \begin{cases} \left[2\ell, F^{11}_{WHO}\right) \cup \left[\frac{-B_{10}+\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}}, \frac{-B_{10}-\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}}\right] \text{ if } F^{11}_{WHO} \leq \frac{-B_{10}+\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}} \\ \left[2\ell, \frac{-B_{10}-\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}}\right] \text{ if } \frac{-B_{10}+\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}} < F^{11}_{WHO} \leq \frac{-B_{10}-\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}} \\ \left[2\ell, F^{11}_{WHO}\right) \text{ if } F^{11}_{WHO} > \frac{-B_{10}-\sqrt{B_{10}^2-4A_{10}C_{10}}}{2A_{10}} \end{cases} \tag{WHO 6}$$

with

$$F^{11}_{WHO} \equiv \frac{[2r+2\delta+2q^*(\ell,h)+2q^*(\ell,\ell)+q^*(h,h)][r+\delta+q^*(h,\ell)+q^*(h,h)]}{[r+\delta+q^*(h,h)][r+\delta+q^*(h,\ell)+q^*(h,h)]+[r+\delta+2q^*(\ell,h)+2q^*(\ell,\ell)]q^*(h,\ell)}h$$
$$+\frac{[r+\delta+2q^*(\ell,h)+2q^*(\ell,\ell)][r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)]q^*(h,\ell)}{[r+\delta+q^*(\ell,h)+q^*(\ell,\ell)]\left\{[r+\delta+q^*(h,h)][r+\delta+q^*(h,\ell)+q^*(h,h)]+[r+\delta+2q^*(\ell,h)+2q^*(\ell,\ell)]q^*(h,\ell)\right\}}\ell$$

and

$$\begin{aligned} A_{10} &= -\left[\frac{[r+\delta+q^*(h,h)]^2+q^*(h,\ell)q^*(h,h)}{[r+\delta+q^*(h,\ell)+q^*(h,h)][r+\delta+q^*(h,h)]}\right]^2 \\ B_{10} &= 4\,\frac{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}{r+\delta+q^*(h,\ell)+q^*(h,h)}\ell+\left\{[r+\delta+q^*(h,h)]^2+q^*(h,\ell)q^*(h,h)\right\}\times \\ &\quad\times 2\,\frac{[r+\delta+q^*(h,\ell)+q^*(h,h)]q^*(h,h)h-q^*(h,\ell)\,(r+\delta)\,\pi^*(\ell,h)}{[r+\delta+q^*(h,\ell)+q^*(h,h)]^2[r+\delta+q^*(h,h)]^2} \\ C_{10} &= -4\,\frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(h,\ell)+q^*(h,h)}\ell^2 \\ &\quad-\left[\frac{[r+\delta+q^*(h,\ell)+q^*(h,h)]q^*(h,h)h-q^*(h,\ell)\,(r+\delta)\,\pi^*(\ell,h)}{[r+\delta+q^*(h,\ell)+q^*(h,h)][r+\delta+q^*(h,h)]}\right]^2 \end{aligned}$$

In agreement c_7, h never leaves, ℓ always leaves, and $\widehat{S}_h^{c_7} = S^*(h,h)$. Condition

G^{12}_{WHO} mirrors G^{5}_{WHO}. Thus, weak homophily of type 2 requires

$$F \notin \left(\frac{-B_{11} - \sqrt{B_{11}^2 - 4A_{11}C_{11}}}{2A_{11}}, \frac{-B_{11} + \sqrt{B_{11}^2 - 4A_{11}C_{11}}}{2A_{11}} \right) \tag{WHO 7}$$

with

$$\begin{aligned}
A_{11} &= q^*(h,\ell)\frac{[q^*(\ell,h)+q^*(\ell,\ell)]q^*(h,\ell)+[r+\delta+q^*(h,h)][r+\delta+q^*(h,h)+q^*(h,\ell)]}{[r+\delta+q^*(h,h)][r+\delta+q^*(h,h)+q^*(h,\ell)]^2} \\
B_{11} &= -hq^*(h,\ell)\frac{r+\delta+2q^*(\ell,h)+2q^*(\ell,\ell)+q^*(h,h)}{[r+\delta+q^*(h,h)][r+\delta+q^*(h,h)+q^*(h,\ell)]} - h + \frac{r+\delta+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}\ell \\
&\quad - 2\,\frac{q^*(h,\ell)^2\,\pi^*(\ell,h)[q^*(\ell,h)+q^*(\ell,\ell)]}{[r+\delta+q^*(h,h)][r+\delta+q^*(h,h)+q^*(h,\ell)]^2} - \frac{q^*(h,\ell)\pi^*(\ell,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)} \\
C_{11} &= -\pi^*(\ell,h)\frac{r+\delta+q^*(h,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}\ell + \left[h + \frac{q^*(h,\ell)\pi^*(\ell,h)}{r+\delta+q^*(h,\ell)+q^*(h,h)}\right] \\
&\quad \times \left[\frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(h,h)}h + \frac{[q^*(\ell,h)+q^*(\ell,\ell)]q^*(h,\ell)\pi^*(\ell,h)}{[r+\delta+q^*(h,h)][r+\delta+q^*(h,h)+q^*(h,\ell)]}\right]
\end{aligned}$$

In agreement c_8, h leaves when she meets an h, ℓ always leaves, and $\widehat{S}^{c_8}_h = \widehat{S}^{c_8}_\ell$. Condition G^{13}_{WHO} mirrors G^{6}_{WHO}. Thus, weak homophily of type 2 requires

$$F \in \left[\frac{-B_{12} + \sqrt{B_{12}^2 - 4A_{12}C_{12}}}{2A_{12}}, \frac{-B_{12} - \sqrt{B_{12}^2 - 4A_{12}C_{12}}}{2A_{12}} \right] \tag{WHO 8}$$

with

$$\begin{aligned}
A_{12} &= -[r+\delta+q^*(h,h)]^2 \\
B_{12} &= 4\frac{[r+\delta+q^*(h,\ell)+q^*(h,h)][r+\delta+q^*(\ell,h)+q^*(\ell,\ell)+q^*(h,h)]^2}{r+\delta+q^*(\ell,\ell)+q^*(\ell,h)}\ell \\
&\quad - 2q^*(h,\ell)[r+\delta+q^*(h,h)]\frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}\ell \\
C_{12} &= -4\frac{[r+\delta+q^*(h,\ell)+q^*(h,h)][r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)]^3}{[r+\delta+q^*(\ell,\ell)+q^*(\ell,h)]^2}\ell^2 \\
&\quad - \left[\frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}q^*(h,\ell)h\right]^2
\end{aligned}$$

Condition $G^{14}_{WHO} \equiv S^*(h,\ell) \geq 0$ holds if $F \geq \pi^*(\ell,h)$. If instead $F < \pi^*(\ell,h)$, $G^{13}_{WHO} < 0$. Therefore, if G^{13}_{WHO} holds, so does G^{14}_{WHO}.

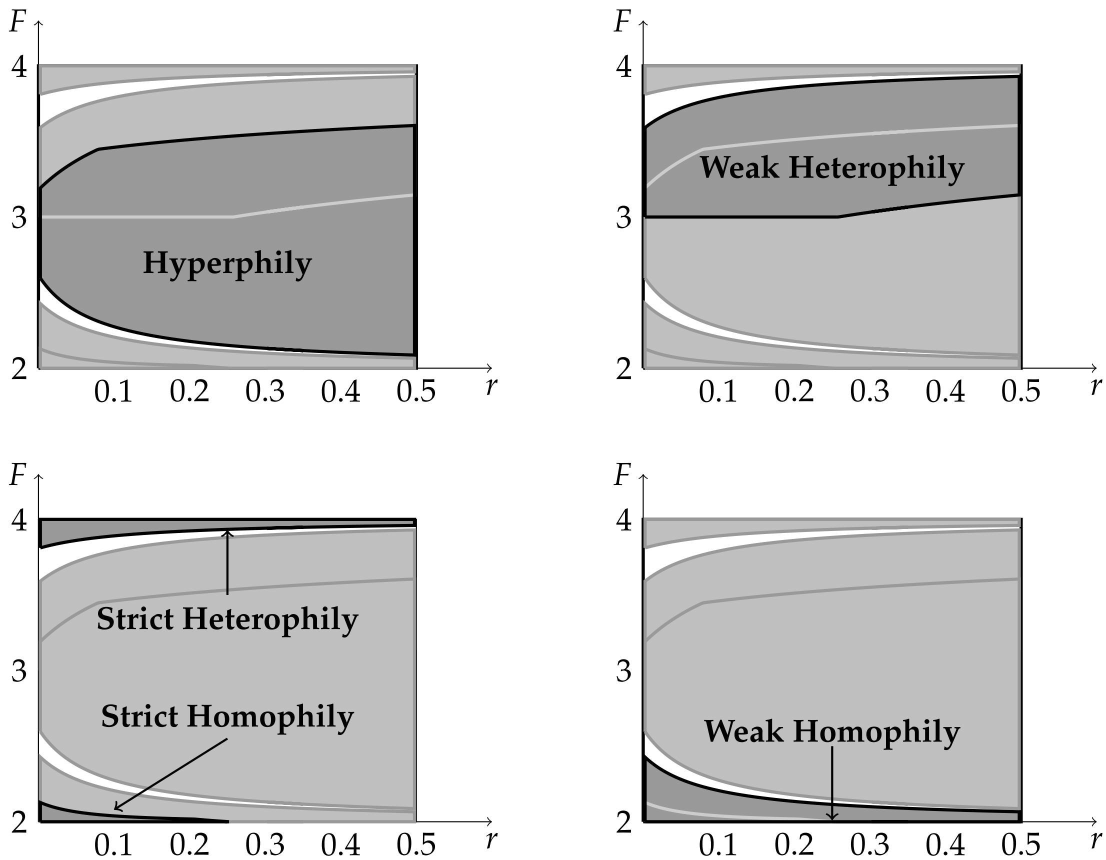
Finally, we check consistency of the equilibrium with weak homophily of type 2. Condition $G^{8}_{WHO} \equiv S^*(h,h) - S^*(h,\ell) > 0$ holds if

$$F < \frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)}\ell + h$$

Since $\frac{r+\delta+q^*(\ell,h)+q^*(h,h)+q^*(\ell,\ell)}{r+\delta+q^*(\ell,h)+q^*(\ell,\ell)} > 1$, if G^{10}_{WHO} holds, so does G^{8}_{WHO}.

4. Additional Figures

Figure 2: The Impact of Discount Rate r



Note: $\ell = 1, h = 2, \delta = 0.05$ and $\rho = 0.1$

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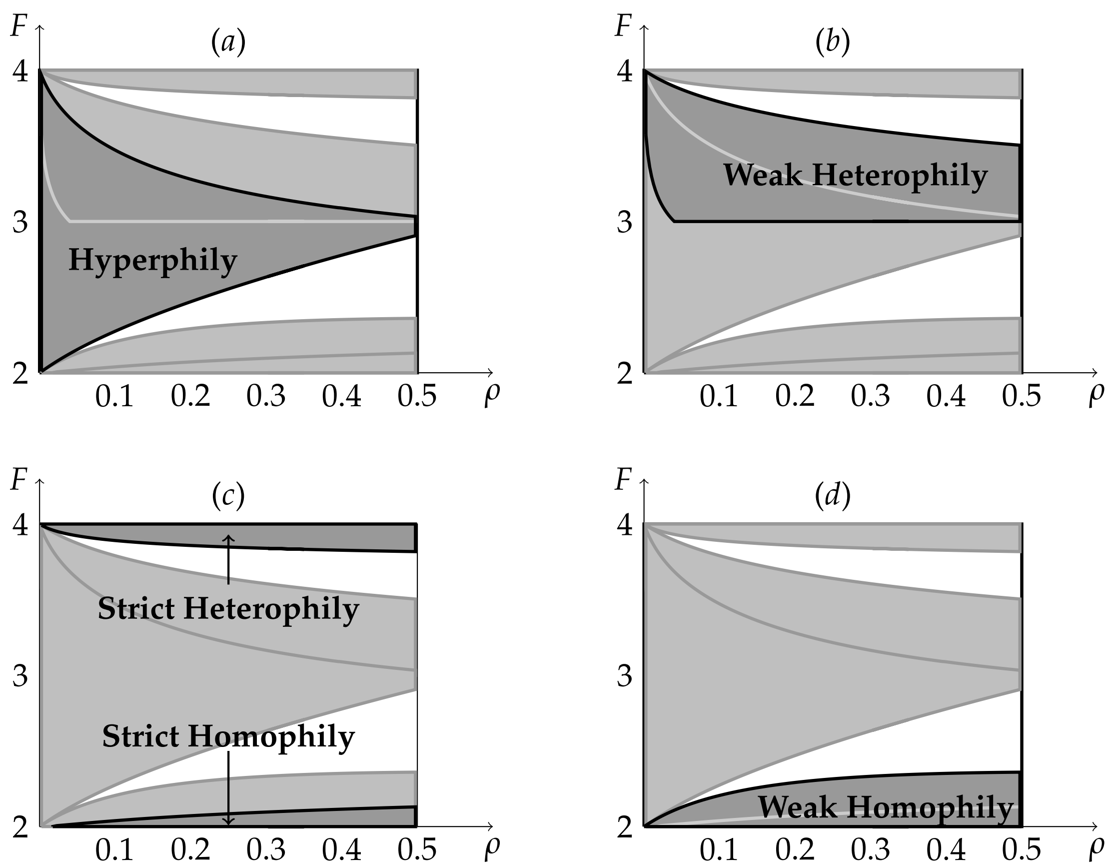
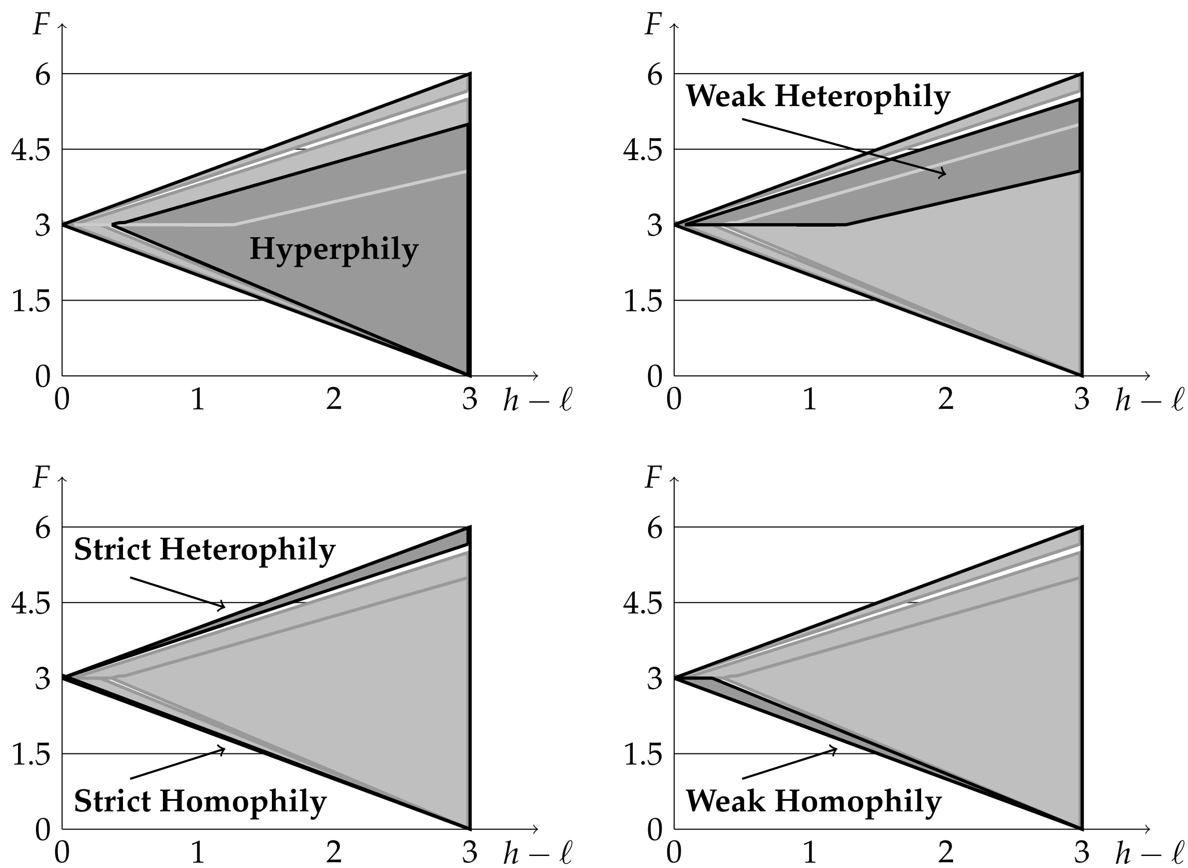
Figure 3: The Impact of Search Intensity ρ



Note: $\ell = 1, h = 2, r = 0.1$ and $\delta = 0.05$.

Figure 4: The Impact of $h - \ell$



Note: $r = 0.1, \rho = 0.1, \delta = 0.05$ and $\ell + h = 3$ with $\ell \in (0, 1.5)$ and $h \in (1.5, 3)$