Master 1 Thesis

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Networks in a Dynamic Macroeconomic Model

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Abstract

In this document, three different general equilibrium models are derived in which the production part corresponds to a network structure where there are two intermediate and one final firm. The first model is static and features Cobb-Douglas technologies, while the second and third models are two-period models featuring Cobb-Douglas and CES technologies respectively. Dynamics are added via the investment decisions of households, which are embedded through a representative agent. Once the respective models are derived, different production shocks are simulated and their results analysed, analytically and numerically, if possible. The main results obtained are that the effects of a productivity shock propagate only downstream, but to the whole network and that –in general– producers adjust the origin and destination of their purchases and sales in reaction to productivity shocks. It is also verified that the existence of the network adds high complexity to the model, as opposed to the dynamics.

Acknowledgements

This dissertation is not the kind of dissertation that comes at the end of any programme. Neither undergraduate, nor master's, nor PhD and, therefore, I doubted whether this paragraph would be relevant or not and this is my conclusion: it never hurts to show gratitude. Especially in a year like this one.

First of all, I would like to dedicate this essay to the memory of Prof. Emmanuel Farhi, with the deepest of respect. His work with David Baqaee is the reason why you are reading these lines, and why I embarked on this journey that is the academic career. Although this tribute is nothing compared to others he has received, I would like to thank him -at the very least- for my enthusiasm, which was motivated by his ideas.

Of course, many thanks to Prof. Fabrice Collard, for his understanding, his great perspicacity, his unselfish help and specially for his closeness in a difficult period like this, even if we have never met in person (yet).

I would like to thank Prof. Luis Puch for being my guide along this way and for his interest and determination in his desire to improve the economics department of the Complutense through his students every day, fighting against the idea that this world is «*le meilleur des mondes possibles*» of Voltaire's Candide. Thanks also to Pablo E. García for introducing me to the networks field.

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Contents

Chapter 1

Introduction

Motivation of the research question and related literature

When discussing the micro-foundations of macroeconomics, the famous quote by R. Lucas is likely to be cited: «If these developments succeed [micro-foundations], the term 'macroeconomic' will simply disappear from use and the modifier 'micro' will become superfluous.» [15]. In the same essay, Lucas refers to macroeconomics as a surrender to the temptation to alleviate the discomfort induced by discrepancies between theory and the facts. On the other hand, K. Iwai devotes 15 pages to answer «What is macroeconomics?» [11] and points out that one of the differences usually considered to differentiate between microeconomics and macroeconomics is that they handle –respectively– microeconomic and macroeconomic variables. He even wonders whether it would be convenient to simply keep microeconomics as economic theory and relegate macroeconomics to a few notes in a statistical manual on how to aggregate microeconomic data.

Therefore, when constructing a macroeconomic model it is necessary to consider which and which level of micro-foundations are required depending on the objective of the study. In this *mémoire*, the focus is on one of the fundamental aspects of economics: interactions. Everything in economics, as in many other fields where complex systems exist -biology, physics, meteorology-, is determined by the interactions between the agents involved in a given system.

Interactions in complex systems can take many forms. One of them is that in which the elements of the system interact with each other physically, in the space. This is, for instance, the case of meteorology, where different atmospheric phenomena interact, but also the position of the moon and the sun[1]. In economics, not literally, international trade can be understood as belonging to this group. Think about Tinbergen's famous model of the gravity of international trade, following Newton's universal gravitation laws. The theoretical model that applies these laws of universal gravitation to a system is the well-known n-body problem, which has no discovered solution for $n \geq 3$, precisely because of the complexity derived from the interactions between the bodies. This aspect is central in this thesis.

Interactions between agents can also be modelled as a network. In this type of interactions, links

[1] Moreover, as D. Kelsey recalls, «Economics and weather forecasting have a lot in common. When people are not talking about the weather they talk about the economy.» [12] Largely because of their complexity, they constitute changing systems where reaching stability or their underlying solutions is impossible, and therefore they can always be debated.

between different agents have some kind of information that determines the relationship that exists between them and their impact on the network. For example, in a social network -e.g. Facebook- the same person can relate to another one by liking a post, requesting friendship or, more commonly, by insulting him/her. Logically, the consequences on the state of the system of this action depend greatly on the type of interaction undertaken.

The economy is full of networks and many of them have been previously studied by various authors. For instance, Lee, Yang et al. study the impact of the topology of the global macroeconomic network [13]; Elliot, Golub and Leduc, the supply network and its fragility [9]; and, from a more social point of view, Scherer and Cho study the contagion of risk perception [19]. In this *mémoire*, the objective is to present a general equilibrium model in which the production network is hierarchised at two levels: in the first one, a network of intermediary firms buy and sell goods from each other in order to use them as inputs. These products are also inputs for one final firm, which constitutes the second level. This approach is not unusual in economics. For example, it is used in the paper by Boscá, Díaz et al. in a study of the Spanish economy [6]. However, in their case intermediate firms do not buy and sell from each other, given the complexity arising from such interactions.

Going back to microeconomics, interaction between two or more agents is quite common. For example, in bargaining games. In these bargaining games one of the players, namely the buyer, can either accept or reject the offer made by the seller. The key component here is the time. In a static model, the buyer would always accept the offer if his valuation was higher than the price and such games would have no reason to exist. This is the second central element to be incorporated into the model developed through this document: the time dimension, i.e. dynamics.

In short, this work deals with the creation of a macroeconomic general equilibrium model whose objective is to analyse the effects of different productivity shocks on the network topology.

The cutting-edge work in this field is that of Baqaee and Farhi, which focus on shock propagation and show how non-linearities arising from interactions in the network affect propagation by varying the final impact of the shock [4, 5]. However, the authors who created this framework were Long and Plosser, in order to study business cycles [14][2]. Based on Long and Plosser's model, Acemoglu, Akcigit and Kerr create a methodology to measure the network effects of the propagation of supply and demand shocks in the economy and they apply it to several cases [1]. Finally, other leading author is Carvalho, who often applies his work to the study of the supply chain [2, 8].

However, all these studies -with the exception of Long and Plosser's seminal work- formalise a static equilibrium. Therefore, in this essay, a dynamic component is added through one of the simplest instruments: investment. This effectively provides a dynamic component, but the resultant model is technically just a concatenation of static models.

Description of the thesis

Following this introduction, in the second chapter and in order to analyse the effects produced by the incorporation of the network, a static model is studied. It features a representative agent and two intermediate companies that sell their outputs to a final company. All of them produce following Cobb-Douglas technologies.

In the third chapter the dynamic component is added through investment. The representative agent, the two intermediate firms and the final firm are kept. The capital used by the intermediate

[2] Interestingly, not once in their article the word 'network' appears.

firms in the second period is the result of the investment of the representative agent in the first period. It will be shown that the use of Cobb-Douglas functions determines that the topology of the network is invariant to possible shocks.

Therefore, in the fourth chapter, a dynamic model analogous to that from the third chapter is studied, but the Cobb-Douglas technologies are replaced by CES production functions[3]. It is shown that the network does indeed evolve by reacting to shocks in the economy.

Finally, in the fifth chapter, a brief conclusion and some thoughts on the possible direction of future research are presented.

Before moving on to the second chapter, two remarks should be made: in the models that have been developed, the productive unit can be considered both the individual firm and a sector of the economy. In this work, the two notions are assumed to be perfectly interchangeable[4]. On the other hand, it should be kept in mind that the difficulty in solving the model is due to the network structure and not to the dynamic component. These models, without the sectoral network, would be much easier to solve. However, when the production network is added, the level of complexity increases significantly. The analytical solution is derived in the specific case of the Cobb-Douglas functions, because certain interactions cancel out in the model; but it is not in the case of the CES technologies, where the solution must be numerically obtained. In an analogy inspired by the n-body problem, adding the network to a basic general equilibrium model is like going from 2 to 3 bodies, where only under certain restrictions the solution is obtainable: featuring Cobb-Douglas technologies.

[3]Constant elasticity of substitution.

[4]In general, they are not. Baqaee and Farhi have a section dedicated exclusively to industry-level aggregation [4].

Chapter 2

The static model

This first model will serve as a starting point. There is only a single period in this model: id est, it is a static problem. After the formulation and its solution, it is studied how different shocks affect the topology and the different variables of the network.

This first model is studied for two purposes: on the one hand, the following dynamic models are in fact two static problems connected through the investment decision. Therefore, in this section, it is possible to study what happens inside each period more clearly. On the other hand, the comparison between the results of the static and the dynamic models will allow to find out which effects of the shocks are caused by the dynamic component.

2.1 Model

This model features one representative agent, two intermediate firms denoted simply by 1 and 2, and one final firm f. Each intermediate firm produces one distinct intermediate good from labor, capital and both intermediate goods. In addition, the final firm produces the final good –which can be considered the GDP of the economy– exclusively from the intermediate goods. The representative agent is only able to derive utility from consumption from the final product and, therefore, only consumes that.

2.1.1 Households

The representative agent has preferences:

$$\mathcal{U}(c,h) = \ln c + \gamma \ln(1-h) \tag{2.1}$$

where c is the representative agent's consumption of the final good, h denotes its supply of labour and the parameter $\gamma \in \mathbb{R}_+$ represents the subjective level of disutility of the labour. Households are endowed with $\bar{T} = 1$ and $\bar{K}$ amounts of labor and capital, respectively. Moreover, they provide $K \leq \bar{K}$, capital, for a price q and $h \leq 1$ for a price w to the intermediate firms. Therefore, the representative agent faces the following budget constraint:

$$p_f c \leq wh + qK + \pi_f + \pi_1 + \pi_2 \tag{2.2}$$

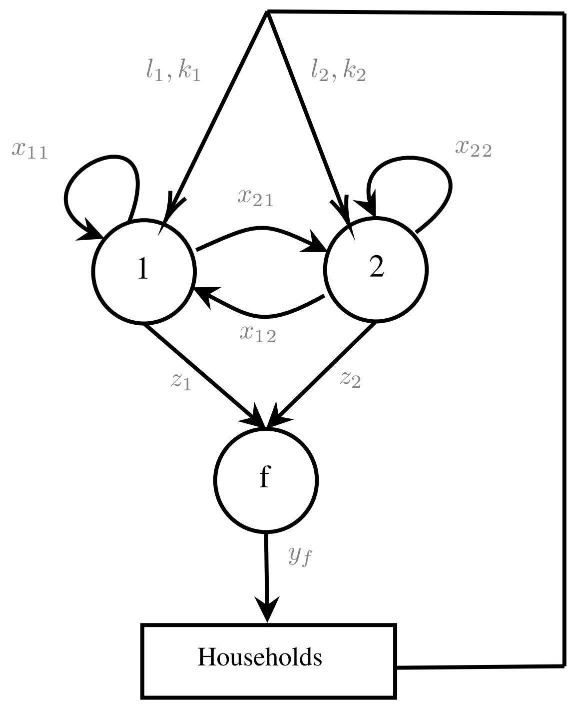


Figure 2.1: Graph of the production network from the static model.

where p_f denotes the price of the final good and it is chosen as *numéraire*, hence it equals to one. The values π_i for $i \in \{1, 2, f\}$ are the profits associated to each firm.

2.1.2 Producers

Producers form a network as shown in Figure 2.1. Intermediate firms use factors –labour and capital– and intermediate goods to produce intermediate goods[1].

Intermediate firms

Each intermediate good $i \in \{1, 2\}$ is produced by its associated firm $i \in \{1, 2\}$ and its output is denoted by y_i. Each of these firms uses some amount of labour l_i at a price w and capital k_i at a price q in production. Moreover, firm i buys output from the other firm $j = 3 - i$ in order to use it as an intermediate input. This quantity is denoted by x_{ij} and its price is p_j. In addition, it should use its own output as an input too. This quantity is denoted by x_{ii} and its price is given by p_i. Firm i produces following the Cobb-Douglas function below.

$$y_i = a_i k_i^{\alpha_1} l_i^{\alpha_2} x_{ii}^{\alpha_3} x_{ij}^{\alpha_4} \tag{2.3}$$

[1] To avoid falling into an infinite regress argument, consider that the number of intermediate goods used as input and those obtained as output can be determined simultaneously and independently. [7]

where α_k are the output elasticities of its associated factor and $\sum_{k=1}^{4} = 1$ in order to obtain constant returns to scale. The term a_i is a firm-specific Hicks-neutral productivity shifter.

Each firm i sells its output at a price p_i to be used as an intermediate input by firm j, by firm i itself, and also by the final firm f. Sold quantities are denoted by x_{ji}, x_{ii} and z_i respectively. Therefore, firm i profit function is given by:

$$\pi_i = p_i y_i - w l_i - q k_i - p_i x_{ii} - p_j x_{ij} \tag{2.4}$$

Final firm

The final firm f uses z_1 at a price p_1 and z_2 at a price p_2 –outputs of the intermediate firms– as production inputs, but not labour or capital. This firm also produces its output y_f according to a Cobb-Douglas technology, which is shown below.

$$y_f = b z_1^{\delta} z_2^{1-\delta} \tag{2.5}$$

where $\delta \in (0,1)$ is the output elasticity of z_1 and can be interpreted as the share or the weight that firm –or sector– 1 possesses in the economy. The term b is a Hicks-neutral productivity shifter and its associated shocks can be understood as those that affect the whole economy. Final firm profit is given by:

$$\pi_f = y_f - p_1 z_1 - p_2 z_2 \tag{2.6}$$

2.1.3 Equilibrium

Given parameters $\gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \delta$, shifters a_1, a_2, b and the initial capital stock $\bar{K}$, the general equilibrium of the economy is defined as a set of prices p_1, p_2, intermediate input choices $z_1, z_2, x_{11}, x_{12}, x_{21}, x_{22}$, labour and capital choices l_1, l_2, k_1, k_2, and consumption c such that:

1. the producers solve their decision problem maximizing profits by choosing the employed factors and intermediate inputs.
2. the representative agent solves its decision problem maximizing its utility subject to its budget constraint by choosing consumption and the quantities of labour and capital rented.
3. all the markets clear.

The derivation of the first-order conditions characterising the equilibrium as well as the specification of the market-emptying conditions and therefore the system of equations of the model can be found in the following section[2].

[2] The technical aspects of the solution are available in a notebook accessible through this link: `https://colab.research.google.com/drive/18bIx5uOYGbzboYSr5vJY6vtNP5GQI4wI?usp=sharing`

2.2 Solution of the model

2.2.1 System equations

Households problem

As stated above, the representative agent faces the following optimisation problem:

$$\begin{aligned} \max_{c,h} \quad & \ln c + \gamma \ln(1-h) \\ s.t.: \quad & c \leq wh + q\bar{K} + \pi_f + \pi_1 + \pi_2 \end{aligned} \tag{2.7}$$

where the budget constraint is equivalent to $c \leq y$, one of the clearing market conditions. Taking partial derivatives of the associated Lagrangian with respect c and h, a first order condition of the household problem is achieved:

$$\frac{c}{1-h} = \frac{w}{\gamma} \tag{2.8}$$

which is equivalent to $U_c = \frac{1}{w}U_{1-h}$. This is, the marginal utility of consumption equals the marginal utility of leisure weighted by their relative prices $p_f = 1$ and w.

Intermediate producers problem

Intermediate producers seek to maximise their profit given factor and intermediate goods prices, which leads to the following optimisation problem:

$$\max_{l_i, k_i, x_{ii}, x_{ij}} \quad p_i y_i - w l_i - q k_i - p_i x_{ii} - p_j x_{ij} \tag{2.9}$$

Taking the partial derivatives, the following first order conditions are achieved:

$$\begin{aligned} i. \;\; \alpha_1 \frac{p_i y_i}{k_i} &= q \\ ii. \;\; \alpha_2 \frac{p_i y_i}{l_i} &= w \\ iii. \;\; \alpha_3 \frac{p_i y_i}{x_{ii}} &= p_i \\ iv. \;\; \alpha_4 \frac{p_i y_i}{x_{ij}} &= p_j \end{aligned} \tag{2.10}$$

which is a representation of the usual solution marginal cost equals marginal revenue. Rearranging the conditions it can be obtained an expression of the form:

$$\alpha_1 = \frac{q k_i}{y_i p_i} \tag{2.11}$$

which shows a fundamental property of Cobb-Douglas functions that will be very relevant in the formation of the network: the share of total expenditure assigned to each production factor is determined by its associated technical productivities[3] α_i.

[3]This name is not chosen arbitrarily. Denoting the parameter of the Cobb-Douglas function associated to the term

Final producer problem

Final producer solves an equivalent problem to that stated in (2.9). In particular:

$$\max_{z_1,z_2} \; p_f y_f - p_1 z_1 - p_2 z_2 \tag{2.12}$$

whose first order conditions are simply:

$$\begin{aligned} &i.\; \delta \frac{p_f y_f}{z_1} = p_1 \\ &ii.\; (1-\delta)\frac{p_f y_f}{z_2} = p_2 \end{aligned} \tag{2.13}$$

and the interpretation is analogous to that stated in the intermediate producers case. However, a note that may be important to make is that, if each firm i is considered as a whole economic sector, the parameter δ is a measure of the importance of the corresponding sector in the economy. However, as the model is formulated, this importance is predefined. Thus, depending on the intended use of the model this may be a major problem. For instance, it is well-known that the primary sector is replaced by the secondary and tertiary ones as the economy is developed. Therefore, this model may be not suitable for the study of the long term growth as it is impossible to capture that central effect.

Market clearing conditions

All the market-clearing conditions that take part in this economy are listed below.

1. Capital market:
$$\bar{K} = k_1 + k_2 \tag{2.14}$$

2. Labour market:
$$h = l_1 + l_2 \tag{2.15}$$

3. Intermediate good i market:
$$y_i = z_i + x_{ii} + x_{ji} \tag{2.16}$$
where $i = 1, 2$ and $j = 3 - i$.

4. Final good market[4]:
$$y_f = c \tag{2.17}$$

x_{ij} by a_{ij}, Acemoglu, Akcigit and Kerr define the input-output matrix $\mathbf{A} = (a)_{ij}$ and, from this, the Leontief inverse $\mathbf{H} = (\mathbf{I} - \mathbf{A})$, whose coefficients are called *technical coefficients* and plays a key role in the study of macroeconomic networks. [1]

[4]Note that this condition is equivalent to that obtained in (2.17), the budget constraint.

2.2.2 Solutions

Once solved the problem with the invaluable help of the latest version of SymPy, below are listed some solutions obtained. The complete solution is shown in the Appendix A1[5].

First, labour and capital variables, which are independent of the production level and are fully determined by the hyperparameters of the model and the initial endowment of capital $\bar{K}$.

$$\begin{aligned} l_1 =& \frac{\alpha_2(\alpha_3^\delta + \alpha_4^\delta - \alpha_4 - \delta)}{\alpha_2\alpha_3 - \alpha_2\alpha_4 - \alpha_2 - \alpha_3^2\gamma + 2\alpha_3\gamma + \alpha_4^2\gamma - \gamma} \\ l_2 =& \frac{\alpha_2(-\alpha_3\delta + \alpha_3 - \alpha_4\delta + \delta - 1)}{\alpha_2\alpha_3 - \alpha_2\alpha_4 - \alpha_2 - \alpha_3^2\gamma + 2\alpha_3\gamma + \alpha_4^2\gamma - \gamma} \\ k_1 =& \frac{\bar{K}(\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta)}{\alpha_3 - \alpha_4 - 1} \\ k_2 =& \frac{\bar{K}(-\alpha_3\delta + \alpha_3 - \alpha_4\delta + \delta - 1)}{\alpha_3 - \alpha_4 - 1} \end{aligned} \tag{2.18}$$

Due to the complexity of the equations, the cause of which lies in the iteration between the different firms, few conclusions can be drawn from them.

Prices of labour and capital, namely w and q depends on the production level y_f. Thus, prices react to the different shocks and are the way of adjustment of supply and demand in this model.

$$\begin{aligned} w =& y_f \frac{\alpha_2 - \alpha_3\gamma - \alpha_4\gamma + \gamma}{1 - \alpha_3 - \alpha_4} \\ q =& y_f \frac{\alpha_1}{\bar{K}(1 - \alpha_3 - \alpha_4)} \end{aligned} \tag{2.19}$$

Regarding the variables linked to intermediate producers $-z_i, x_{ii}, x_{ij}-$ as the add up to y_i, their solution in terms of y_i are a constant consisting of a jumble of parameters indicating the share of y_i that is used in that certain variable, multiplied by that certain variable. In particular, the intermediate goods used as final firm's inputs are given by:

$$\begin{aligned} z_1 =& y_1 \frac{\delta(-\alpha_3^2 + 2\alpha_3 + \alpha_4^2 - 1)}{\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta} \\ z_2 =& y_2 \frac{1 - \alpha_3^2\delta + \alpha_3^2 + 2\alpha_3\delta - 2\alpha_3 + \alpha_4^2\delta - \alpha_4^2 - \delta}{\alpha_3\delta - \alpha_3 + \alpha_4\delta - \delta + 1} \end{aligned} \tag{2.20}$$

[5] In order to understand the complexity that a network adds to a model, SymPy is able to solve the whole model except for one system of equations embedded in the general one that only includes as variables the intermediate products $x_{11}, x_{12}, x_{21}, x_{22}$.

Moreover, there is an internal system that constitutes the nucleus of the model formed by the solutions depending on y_1 and y_2 of the intermediate inputs:

$$\begin{aligned}
x_{11} =& y_1\alpha_3 \\
x_{21} =& y_1\frac{\alpha_4(-\alpha_3\delta+\alpha_3-\alpha_4\delta+\delta-1)}{\alpha_3\delta+\alpha_4\delta-\alpha_4-\delta} \\
x_{12} =& y_2\frac{\alpha_4(\delta-\alpha_3\delta-\alpha_4\delta+\alpha_4)}{(\alpha_3\delta-\alpha_3+\alpha_4\delta-\delta+1)} \\
x_{22} =& y_2\alpha_3
\end{aligned} \tag{2.21}$$

The solution of the model that depends only on productivity shocks and parameters is difficult to interpret due to its complexity and, in order to get it, it is needed to solve the system described in (2.21) with respect to $x_{11}, x_{21}, x_{12}, x_{22}$ by substituting y_1 and y_2 by their definitions from (2.3) and k_1, k_2, l_1 and l_2 by their already known values from (2.18).

However, if it is denoted by $\Theta = \Theta(\alpha_1, ..., \alpha_4, \delta)$ an undefined function of parameters –as the way it is usually done when solving differential equations– it is possible to define:

$$\begin{aligned}
\theta_1 =& a_1\Theta \\
\theta_2 =& a_2\Theta
\end{aligned} \tag{2.22}$$

and the intermediate goods elements can be expressed depending on θ_1, θ_2 as:

$$\begin{aligned}
x_{12} =& \theta_1^{\Theta}\theta_2^{\Theta} = \Theta a_1^{\Theta} a_2^{\Theta} \\
x_{22} =& \Theta x_{12} = \Theta a_1^{\Theta} a_2^{\Theta} \\
x_{11} =& \Theta\theta_1 x_{12}^{\Theta} = \Theta a_1^{\Theta} a_2^{\Theta} \\
x_{21} =& \Theta\theta_4 x_{12}^{\Theta} x_{11}^{\Theta} = \Theta a_1^{\Theta} a_2^{\Theta}
\end{aligned} \tag{2.23}$$

This yields that any firm-specific shock will affect the quantities of any intermediate good used as input for the intermediate producer. It is convenient to rewrite the right hand side expressions of (2.23) as $x_{i\cdot} = \Theta a_i^{\Theta_1} a_j^{\Theta_2}$. Thus, it is denoted by Θ_1 the exponent associated to the shock from the destiny of the input and Θ_2 the exponent associated to the shock from the other sector. Note that $i \neq j$.

From this, through definitions in (2.3) and (2.5) and the market clearing condition in (2.17), intermediate and total outputs can be computed, as well as consumption.

Finally, and going back to the solutions dependent on output, prices act as damping springs that link directly final and intermediate solutions. In particular, the price of the intermediate product i depends both on y_i and y_f.

The case for firm 1:

$$p_1 = \frac{y_f}{y_1} \frac{(\delta - \alpha_3\delta - \alpha_4\delta + \alpha_4)}{(1 + \alpha_3^2 - 2\alpha_3 - \alpha_4^2)} \tag{2.24}$$

2.3 Comparative statics

2.3.1 Propagation of shocks

From the last expression $x_{i\cdot} = \Theta a_i^{\Theta_1} a_j^{\Theta_2}$ for $i \in \{1, 2\}$ and $j = 2-i$, the propagation of the different shocks in the model can be derived directly from the definition of y_f of (2.5):

$$\mathrm{d}\ln y_f = \mathrm{d}\ln b + \delta \mathrm{d}\ln z_1 + (1-\delta)\mathrm{d}\ln z_2 \tag{2.25}$$

simplifying (2.20) as $z_i = \Theta y_i$, it is straightforward that:

$$\mathrm{d}\ln z_i = \Theta \mathrm{d}\ln y_i \tag{2.26}$$

Using the definition of y_i in the equation (2.3) and substituting the x_{ij} terms, it is possible to derive $\mathrm{d}\ln y_i$ as follows.

$$\begin{aligned}
\mathrm{d}\ln y_i =& \mathrm{d}\ln a_i + \alpha_3 \mathrm{d}\ln x_{ii} + \alpha_4 \mathrm{d}\ln x_{ij} \\
=& \mathrm{d}\ln a_i + \alpha_3(\Theta_1 \mathrm{d}\ln a_i + \Theta_2 \mathrm{d}\ln a_j) + \alpha_4(\Theta_1 \mathrm{d}\ln a_i + \Theta_2 \mathrm{d}\ln a_j) \\
=& \mathrm{d}\ln a_i + \xi_i \mathrm{d}\ln a_i + \xi_j \mathrm{d}\ln a_j
\end{aligned} \tag{2.27}$$

where the first term of the final sum describes the direct effect that a shock occurred in firm i has for firm i itself –it is a direct relation because of the use of a Cobb-Douglas function-. The second term descibres the effect coming from the network whose origin is a shock produced in the firm i itself. As shown in (2.27) there are two channels for these shocks to arrive to firm i. First, firm i acts as a supplier for itself and a positive (negative) productivity shock translates into cheaper (more expensive) inputs as more (less) quantity is produced. Second, firm i is a supplier of firm j which, in turn, is a supplier of firm i itself and the same effect occurs[6]. Finally, the last term of the sum describes the impact that a shock in firm j entails for dirm i. The same kind of interactions in the network described above applies here.

It is important to note that as shocks propagate through the network, which in the algebra is is equivalent to terms being substituted in the different equations, they lose strength – in the equations, substituted terms are multiplied by absorber constants whose value is lower than 1. This entails that $1 > \max_{k\in\{i,j\}}\{\xi_k\}$ and the comparison between ξ_i and ξ_j depends on the parameters and the relative importance that each input has for the production of good i.

A more important result, consistent with the results proved theoretically by Baqaee and Farhi [4] and empirically by Acemoglu, Akcigit and Kerr [1], is that these productivity shocks are only propagated downstream -to customer industries- as shocks produced in the final firm through variations in b do not affect the intermediate producers. However, as it has been shown, the opposite is the case: a shock in a specific firm i propagates to all its costumers, firm i itself, the other intermediate firm j and the final firm f.

[6] As with high order beliefs, these effects continue to propagate: shock in firm i impacts firm j as firm i is a supplier for j; then impacts i as j is a supplier of i, and then returns to j as it is the supplier of the supplier of the supplier, and so on. Here are only described the direct ones.

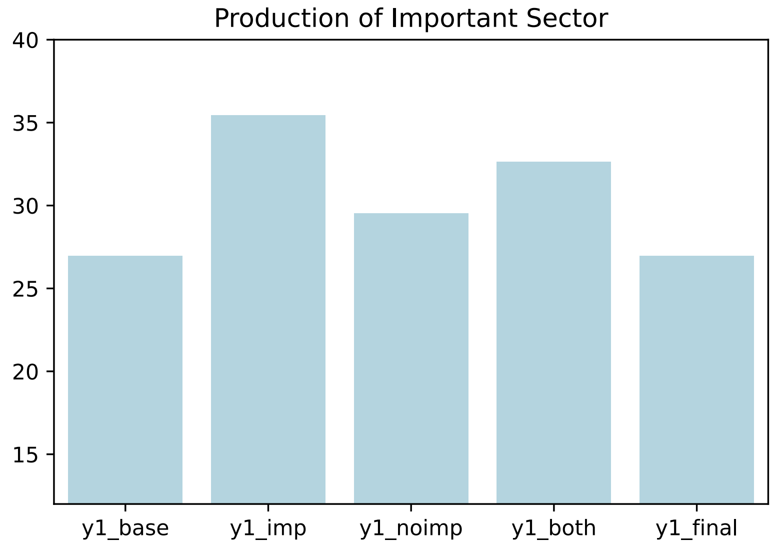


Figure 2.2: Production of the important sector with respect to different shocks.

2.3.2 Simulation of the model

A realisation of the model has been computed, as well as different shocks[7] simulated. The previous theoretical results are inevitably verified.

In the computed model there is an economy with two sectors, one of which is more important than the other as $\delta = 0.6$. All other aspects are perfectly symmetric. The shocks are equivalent quantitatively, as they produce a 20% improvement in a given productivity. However, they differ in the origin: the first shock is produced in the important sector, i.e. a_1 increases; the second is produced in the non-important one, i.e. a_2 increases; the third one is produced at the same time in both sectors, i.e. a_1 and a_2 increases, but only 10% each; and the final shock is produced in the final firm, i.e. b increases.

In Figure 2.2 the value of the production of the important sector in terms of final product for the different shocks is shown. The baseline variable is denoted as ('y1_base' variable). As expected, the highest production is reached when the shock has its origin in the same sector ('y1_imp' variable). If the shock is produced in the other intermediate sector, the production of the important one increases ('y1_noimp' variable). There is an in-between case when the increase of productivity is divided equally among the two intermediate sectors ('y1_both' variable). Finally, the shock produced in the final firm has no impact on the intermediate sectors of the economy ('y1_final' variable).

Moreover, the direct effect of the shock represents the 20% of the baseline production. Therefore, as $y_1^{base} = 26.97$, $1.20 \cdot y_1^{base} = 32.36$ and $y_1^{imp} = 35.45$, the network effect of this shock make the

[7] The detailed solution can be consulted partially in the Appendix A2 and completely and interactively in `https://colab.research.google.com/drive/1P6GDeAS_0q_mhlol-jo9zqAYJhQVHndX?usp=sharing`

output increase by more than 3 units, which corresponds to the 36.4% of the variation induced by the shock.

It is also noticed that the labor supply is constant among scenarios and fully determined by the parameters of the model.

The share that the sales of each intermediate good i to the other firm j represents as part of the total sales of good i is commonly called 'weight' in the economic networks theory. Formally this weight –following the approach by Acemoglu, Akcigit and Kerr [1]– is defined as:

$$weight_{ij} = \frac{Sales_{i \to j,t}}{TotalSales_i} = \frac{x_{ji}}{y_i} \tag{2.28}$$

and, in this model, they are not affected by the shocks. This is due to the use of the Cobb-Douglas function, as has been mentioned before and it is a direct result from the equations (2.21). It is interesting to remove this strong assumption by changing the production function and allow the network to evolve. This is done in the third chapter of this document. Before that, the dynamic component will be added to this model.

Chapter 3

The Cobb-Douglas dynamic model

The aim of the study of this second model is to test what effects are created by the addition of a dynamic component to the first static network model. Therefore, this dynamic model adopts the following approach: instead of having a one-period model, now there are two one-period models –one for each of the two periods in which the economy exists– that jointly constitutes the actual model. These period-models are linked exclusively through an usual and simply instrument: investment. A clarifying illustration of the model can be seen in Figure 3.1.

3.1 Model

The economy from this model exists for two periods, it is created at $t = 1$, survives to $t = 2$ and then the world ends. There is a representative agent that maximises its intertemporal utility and that makes the capital formation decisions; two intermediate producers per period; and one final firm per period. All these firms maximise their intraperiod profits. As in the previous chapter, each intermediate firm produces one distinct intermediate good from labor, capital and both intermediate goods. In addition, the representative agent only consumes the final good produced by the final firm. The final firm output at $t = 1$ can be saved to form capital for $t = 2$ in a one-to-one basis.

3.1.1 Households

There is one representative agent that has preferences:

$$\mathcal{U}(c_1, c_2, h_1, h_2) = \ln c_1 + \gamma \ln(1 - h_1) + \beta \ln c_2 + \beta\gamma \ln(1 - h_2) \tag{3.1}$$

where c_t denotes the representative agent's consumption and h_t its supply of labour for $t = 1, 2$; the parameter $\gamma \in \mathbb{R}_+$ represents the subjective level of disutility of the labour and β acts as a time discounting parameter. It is rational to assume β and γ's values to be lower than one, as agents are impatients and enjoy more leisure than labour.

At the beginning of each period the representative agent is endowed with an amount of labor $\bar{T} = 1$. In addition, at the start of $t = 1$, the representative agent is endowed with a stock of capital $\bar{K}_1$. At the end of the first period, households decide which amount of final output to consume and which to invest to form capital for the second period K_2. During each period $t = 1, 2$, households

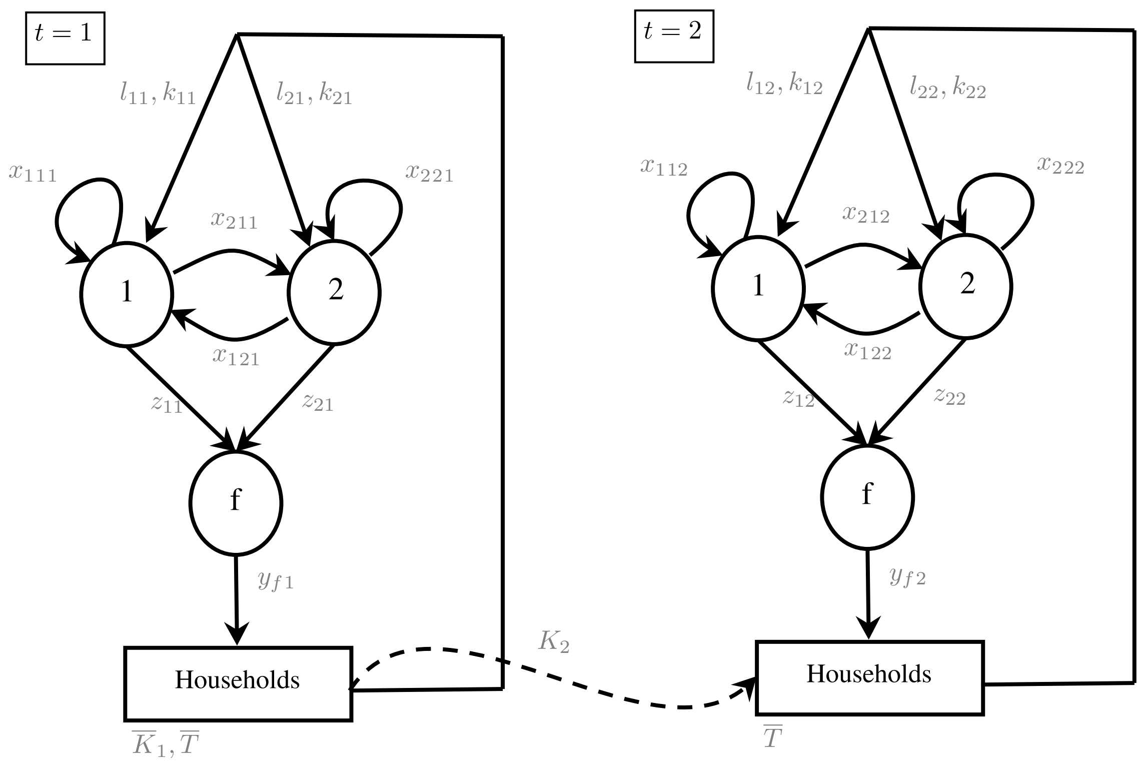


Figure 3.1: Graph of the production network from the dynamic model.

provide capital K_t and labour h_t to the intermediate firms at prices q_t and w_t respectively and consume final good c_t at a price $p_{f,t} = 1$, as they are chosen as *numéraires*.

Therefore, the representative agent faces the following budget constraint in $t = 1$:

$$p_{f,1}c_1 + p_{f,1}K_2 \leq w_1 h_1 + q_1 \bar{K}_1 + \pi_{f,1} + \pi_{1,1} + \pi_{2,1} \tag{3.2}$$

and an analogous budget constraint in $t = 2$:

$$p_{f,2}c_2 \leq w_2 h_2 + q_2 K_2 + \pi_{f,2} + \pi_{1,2} + \pi_{2,2} \tag{3.3}$$

Let denote $\Pi_t = \pi_{1,t} + \pi_{2,t} + \pi_{i,t}$ for $t = 1, 2, f$ and substitute $p_{f,1}$ and $p_{f,2}$ by their values, one. It is possible to derive an intertemporal budget constraint, as (3.2) and (3.3) are linked through K_2:

$$c_1 + \frac{c_2}{q_2} \leq w_1 h_1 + \frac{w_2 h_2}{q_2} + \Pi_1 + \frac{\Pi_2}{q_2} + q_1 \bar{K}_1 \tag{3.4}$$

where $\pi_{i,t}$ denotes the profit of firm i in period t. This equation indicates that the consumption present value at time $t = 1$ should be lower or equal to the wealth present value at that time. Moreover, the intertemporal relative price is given by $\dfrac{1}{q_2}$.

3.1.2 Producers

Producers, as shown in Figure 3.1., form a production network each period that is not affected directly by any dynamical component. Firms present identical Cobb-Douglas production functions both periods, except for shocks. Although quantities and prices may change by action of shocks and the dynamic component, firms take those as given. Hence, they exhibit an analogous behaviour both periods as they solve the optimisation problems separately. Moreover, it is also analogous to the Chapter 2 case. Therefore, the model is described for an arbitrary period $t \in \{1, 2\}$.

Intermediate firms

Each intermediate good $i \in \{1, 2\}$ is produced by its associated firm $i \in \{1, 2\}$. The output of firm i in t is denoted by $y_{i,t}$. Each of these firms uses some amount of labor $l_{i,t}$ and capital $k_{i,t}$ that hires at prices w_t and q_t respectively. Moreover, firm i buys output from the other firm $j = 3 - i$ in order to use it as an intermediate input. This quantity is denoted by $x_{ij,t}$ and its price by $p_{j,t}$. In addition, firm i should use its own output as an input too. This quantity is denoted by $x_{ii,t}$ and its price by $p_{i,t}$. Firm i exhibit a Cobb-Douglas technology:

$$y_{i,t} = a_{i,t} k_{i,t}^{\alpha_1} l_{i,t}^{\alpha_2} x_{ii,t}^{\alpha_3} x_{ij,t}^{\alpha_4} \tag{3.5}$$

where α_k are the output elasticities of its associated factor and $\sum_{k=1}^{4} \alpha_k = 1$ in order to obtain constant returns to scale. The term $a_{i,t}$ denotes a time-and-firm-specific Hicks-neutral productivity shifter.

Each firm i sells its output at a price $p_{i,t}$ to be used as an intermediate input by firm j, by firm i itself, and also by the final firm f. The sold quantities are denoted by $x_{ji,t}$, $x_{ii,t}$ and $z_{i,t}$ respectively. Therefore, firm i profit function at time t can be expressed as:

$$\pi_{i,t} = p_{i,t} y_{i,t} - w_t l_{i,t} - q_t k_{i,t} - p_{i,t} x_{ii,t} - p_{j,t} x_{ij,t} \tag{3.6}$$

Final firm

The final firm at period t uses z_{1t} at a price p_{1t} and z_{2t} at a price p_{2t} –outputs of the intermediate firms– as production inputs, but not labour or capital. This firm also produces its output y_{ft} according to a Cobb-Douglas function, which is shown below.

$$y_{f,t} = b_t z_{1,t}^{\delta} z_{2,t}^{1-\delta} \tag{3.7}$$

where $\delta \in (0, 1)$ is the output elasticity of $z_{1,t}$. The term b_t is a time-specific Hicks-neutral productivity shifter that can be interpreted as an indicator of the business cycle. Final firm profit in period t is given by:

$$\pi_{f,t} = y_{f,t} - p_{1,t} z_{1,t} - p_{2,t} z_{2,t} \tag{3.8}$$

3.1.3 Equilibrium

Given the parameters $\gamma, \beta, \alpha_1, ..., \alpha_4, \delta$, productivity shifters set $\{a_{i,t}\}$ for $(i, t) \in \{1, 2\}\text{x}\{1, 2\}$ and initial stock of capital $\bar{K}_1$, the general equilibrium of the economy is defined as a set of prices $\{p_{i,t}, w_t, q_t\}$ where $(i, t) \in \{1, 2\}\text{x}\{1, 2\}$, intermediate input choices $\{z_{i,t}, x_{ii,t}, x_{ij,t}\}$ for $(i, t) \in \{1, 2\}\text{x}\{1, 2\}$, $j = 3 - i$, labour and capital choices $l_{i,t}, k_{i,t}$ and consumptions c_1, c_2 such that:

1. each period $t \in \{1, 2\}$, the producers solve their decision problem maximizing profits by choosing the employed factors and intermediate inputs.
2. the representative agent solves its decision problem maximizing its utility subject to its intertemporal budget constraint by choosing consumption and the quantities of labour and capital rented.
3. all the markets clear.

3.2 Solution of the model

3.2.1 System equations

Households problem

The representative agent solves the following optimisation problem:

$$\begin{aligned} \max_{c,h} \quad & \ln c_1 + \beta \ln c_2 + \gamma \ln(1-h_1) + \beta\gamma \ln(1-h_2) \\ s.t.: \quad & c_1 + \frac{c_2}{q_2} \leq w_1 h_1 + \frac{w_2 h_2}{q_2} + \Pi_1 + \frac{\Pi_2}{q_2} + q_1 \bar{K}_1 \end{aligned} \tag{3.9}$$

reaching the following Euler conditions:

$$\frac{c_2}{c_1} = \frac{1}{\beta q_2} \tag{3.10}$$

where the ratio of consumptions is equal to the subjective –due to the presence of β– intertemporal price. The Euler condition regarding labour is given by:

$$\frac{w_2}{w_1} = \frac{1}{q_2} \frac{1-h_1}{1-h_2} \tag{3.11}$$

where the ratio of real wages equals the inverted ratio of leisure quantities –that is, the ratio of marginal utilities of leisure–, where the second period marginal utility is discounted through the intertemporal price $\frac{1}{q_2}$.

Moreover, inside each period $t \in \{1, 2\}$, it is obtained one condition that relates consumption and labour:

$$h_t = 1 - \gamma \frac{c_1}{w_1} \tag{3.12}$$

this is, the supply of labour h_t is determined as the total amount of labour available ($\bar{T} = 1$) minus a division of consumption –in fact, the marginal utility of consumption- and the real wage, weighted by the parameter γ that measures the subjective disutility of labor; e.g. if $\gamma = 0$, the agent does not get disutility from labour and therefore $h_t = 1$ for any set of shock shifters and parameters.

Intermediate producers

Each period $t \in \{1,2\}$ intermediate producers seek to maximise their profit given factor and intermediate goods prices, which leads to the following intraperiod optimisation problem –which is identical to the static model one–:

$$\max_{l_{i,t},k_{i,t},x_{ii,t},x_{ij,t}} \quad p_{i,t}y_{i,t} - w_t l_{i,t} - q_t k_{i,t} - p_{i,t}x_{ii,t} - p_{j,t}x_{ij,t} \tag{3.13}$$

and the same first order conditions as those obtained in (2.10) are derived here –adding in each case the appropriate temporal subindices.

Final producer

As it was the case for intermediate producers, final producer solves an identical problem to that stated in the static model for each period $t \in \{1,2\}$, id est:

$$\max_{z_{1,t},z_{2,t}} \quad p_{f,t}y_{f,t} - p_{1,t}z_{1,t} - p_{2,t}z_{2,t} \tag{3.14}$$

and the same first order conditions as those obtained in (2.13) are derived here –adding in each case the appropriate temporal subindices.

Market clearing conditions

All the market conditions that take part in the system are listed below. Although they are similar to those written in the last section, for each market there are two different equations –one for each period– and, in the final good case, they differ between $t=1$ and $t=2$ due to the fact that in $t=1$ the economy invests in capital for $t=2$, but at $t=2$ the world ends.

1. Capital markets:

$$\begin{aligned} \bar{K}_1 =& k_{1,1} + k_{2,1} \\ K_2 =& k_{1,2} + k_{2,2} \end{aligned} \tag{3.15}$$

2. Labour markets:

$$\begin{aligned} h_1 =& l_{1,1} + l_{2,1} \\ h_2 =& l_{1,2} + l_{2,2} \end{aligned} \tag{3.16}$$

3. Intermediate good i markets:

$$y_{i,t} = z_{i,t} + x_{ii,t} + x_{ji,t} \tag{3.17}$$

   where $i = 1,2$, $j = 3-i$ and $t = 1,2$.

4. Final good markets:

$$\begin{aligned} y_{f,1} =& c_1 + K_2 \\ y_{f,2} =& c_2 \end{aligned} \tag{3.18}$$

3.2.2 Solutions

As claimed earlier, this model is simply a concatenation of two static models as that described in the previous chapter. Thus, there are many similarities, but also some differences. Below are explained some of them. Solutions are available in Appendix B1. [1]

First period

Let denote by $\Theta = \Theta(\alpha_1, ..., \alpha_4, \delta, \gamma)$ any function that is exclusively combination of parameters $\alpha_1, ..., \alpha_4, \delta, \gamma$ in order to simplify the results[2] and let also express the solution in terms of outputs $y_{i,t}$ for $i \in \{1, 2, f\}$ and $t \in \{1, 2\}$. Thus, in $t = 1$, labour and capital quantities are fully determined by the parameters, as it was the case in the static model.

$$\begin{aligned} l_{1,1} &= \Theta & \qquad l_{2,1} &= \Theta \\ k_{1,1} &= \Theta \bar{K}_1 & \qquad k_{2,1} &= \Theta \bar{K}_1 \end{aligned} \tag{3.19}$$

This is, labour and capital decisions in the first period are not affected by shocks. Analogously to the static model case, prices depend on output and therefore react to the different shocks. They are the way of adjustment of supply and demand of factors in this model.

$$w_1 = y_{f,1}\Theta \qquad\qquad q_1 = \frac{y_{f,1}}{\bar{K}_1}\Theta \tag{3.20}$$

Regarding intermediate goods quantities, the solution in the first period is identical to that obtained in the static model. This fact leads to an important conclusion: shocks with origin in $t = 2$ do not affect results in the first period. Those quantities can be expressed as function of outputs as:

$$\begin{aligned} z_{1,1} &= y_{1,1}\Theta & \qquad z_{2,1} &= y_{2,1}\Theta \\ x_{11,1} &= y_{1,1}\Theta & \qquad x_{21,1} &= y_{1,1}\Theta \\ x_{12,1} &= y_{2,1}\Theta & \qquad x_{22,1} &= y_{2,1}\Theta \end{aligned} \tag{3.21}$$

and recalling that $y_{i,t} = z_{i,t} + x_{ii,t} + x_{ji,t}$, the parameter Θ that is associated to each variable indicates which share of the output $y_{i,t}$ is employed in those certain factor.

Taking a step further and substituting labour and capital quantities by their value and solving the embedded system –as seen in the previous chapter–, these quantities can be written directly as:

$$\begin{aligned} x_{11,1} &= \Theta a_{1,1}^{\Theta_1} a_{2,1}^{\Theta_2} & \qquad x_{12,1} &= \Theta a_{1,1}^{\Theta_1} a_{2,1}^{\Theta_2} \\ x_{21,1} &= \Theta a_{1,1}^{\Theta_1} a_{2,1}^{\Theta_2} & \qquad x_{22,1} &= \Theta a_{1,1}^{\Theta_1} a_{2,1}^{\Theta_2} \end{aligned} \tag{3.22}$$

and from this, through definitions in (3.5) and (3.7), intermediate and total outputs can be computed.

[1] The Python notebook with the complete report is available here: `https://colab.research.google.com/drive/1CYvTHoVnJs2F0LixjxYvsZJVqRnsyJGR?usp=sharing`.

[2] For instance,

$$l_{1,1} = \frac{\alpha_2(\alpha_1\beta - \alpha_3 - \alpha_4 + 1)(\alpha_3\delta_1 + \alpha_4\delta_1 - \alpha_4 - \delta_1)}{(1 + \alpha_3 - \alpha_4)(\alpha_2\alpha_4 - \alpha_1\alpha_2\beta + \alpha_2\alpha_3 - \alpha_2 - \alpha_3^2\gamma - 2\alpha_3\alpha_4\gamma + 2\alpha_3\gamma - \alpha_4^2\gamma + 2\alpha_4\gamma - \gamma)}$$

The remarkable thing that is really different from the static model in this first period is that here the final product is not used exclusively for consumption, but also for capital investment. Thus, as a direct consequence of (3.18):

$$c_1 = \Theta y_{f,1} \qquad K_2 = \Theta y_{f,1} \tag{3.23}$$

and that Θ parameter should be interpreted as the share of final output that is employed in consumption or investment, depending on the case.

Finally, goods' prices link directly final and intermediate firm solutions. The price of the intermediate product i in $t = 1$ is given by:

$$p_{i,1} = \frac{y_{f,t}}{y_{i,t}}\Theta \tag{3.24}$$

Second period

In the second period, most solutions are analogous to those obtained in the first period. For instance, as far as labour is concerned, quantities are still entirely determined by the parameters:

$$l_{1,2} = \Theta \qquad l_{2,2} = \Theta \tag{3.25}$$

Wages, final and intermediate outputs and prices take the same identical form that in the initial period, this is:

$$\begin{aligned} z_{1,2} &= y_{1,2}\Theta & z_{2,2} &= y_{2,2}\Theta \\ x_{11,2} &= y_{1,2}\Theta & x_{21,2} &= y_{1,2}\Theta \\ x_{12,2} &= y_{2,2}\Theta & x_{22,2} &= y_{2,2}\Theta \end{aligned} \tag{3.26}$$

However, there is a great difference that is added by the dynamic component of the model. As it has been stated above, investment K_2 depends on the total output of the final firm $y_{f,t}$. Therefore, K_2 is affected by shocks produced in the first period. In particular, $k_{i,2}$ for any $i \in \{1, 2\}$ can be written as:

$$k_{i,2} = y_{f,1}\Theta = b_1 z_1^{\delta} z_2^{1-\delta}\Theta = b_1 a_{1,1}^{\Theta_{k_1}} a_{2,1}^{\Theta_{k_2}}\Theta \tag{3.27}$$

where $\Theta_{k_i} : \mathbb{R}^7 \to \mathbb{R}$ for $i = 1, 2$ is some non-linear function that depends on $\{\alpha_1, ..., \alpha_4, \delta, \gamma, \bar{K}\}$, the set of parameters that take part in period 1 solutions and serves as the exponent of its associated element $a_{i,1}$. Thus, substituting this result in the definitions from (3.5) and rearranging, output of intermediate firm i can be expressed as:

$$y_{i,2} = \Theta a_{i,2}(b_1 a_{1,1}^{\Theta_{k_1}} a_{2,1}^{\Theta_{k_2}})^{\alpha_1} x_{ii,2}^{\alpha_3} x_{ij,2}^{\alpha_4} \tag{3.28}$$

Although the internal system of intermediate inputs already described in (2.21) keeps the same for known values of capital, in this second period $k_{1,2}$ and $k_{2,2}$ are not fully determined by parameters of the model, but also include the productivity shifters. Thus, the solutions should be rewritten as:

$$x_{ij,2} = \Theta \left[b_1^{\Theta_{b_1}} a_{1,1}^{\Theta_{a_{1,1}}} a_{2,1}^{\Theta_{a_{2,1}}} \right] \left[a_{1,2}^{\Theta_{a_{1,2}}} a_{2,2}^{\Theta_{a_{2,2}}} \right] \tag{3.29}$$

for $(i, j) \in \{1, 2\}\times\{1, 2\}$ and where the first term in brackets comes from the $k_{i,2}$ expression that inherits shocks from $t = 1$. The second bracket is derived determined by the network iterations in this period $t = 2$.

Once these quantities are known, both final and intermediate output definitions are fully determined and can be computed substituting the already known values in (3.21), (3.5) and (3.7).

The last remaining elements of the system to be obtained are the intermediate product prices $p_{1,2}$ and $p_{2,2}$ and the price of capital q_2. The former are given by:

$$p_{1,2} = \frac{y_{f,2}}{y_{1,2}}\Theta \qquad\qquad p_{2,2} = \frac{y_{f,2}}{y_{2,2}}\Theta \tag{3.30}$$

and its behaviour is equivalent to that explained for the first period. However, q_2 acts as the intertemporal price –as it has sometimes been called through this essay– and enables the valuation of goods from one period in terms of the other, since it is the price associated to the only form of wealth transfer in the model. In particular, it is given by:

$$q_2 = \frac{y_{f,2}}{y_{f,1}}\Theta \tag{3.31}$$

3.3 Comparative statics

3.3.1 Propagation of shocks

In the case of the first period, the variation of final and intermediate outputs are identical to those in the static model. In particular –just adding the temporal index–:

$$\mathrm{d}\ln y_{f,1} = \mathrm{d}\ln b_1 + \delta\mathrm{d}\ln z_{1,1} + (1-\delta)\mathrm{d}\ln z_{2,1} \tag{3.32}$$

where $\mathrm{d}\ln z_{i,t} = \Theta\mathrm{d}\ln y_{i,1}$ and:

$$\mathrm{d}\ln y_{i,1} = \mathrm{d}\ln a_{i,1} + \xi_{i,1}\mathrm{d}\ln a_{i,1} + \xi_{j,1}\mathrm{d}\ln a_{j,1} \tag{3.33}$$

where $\xi_{j,1}$ is a constant, combination of the parameters of the model, that represent the strength of the impact that shocks originated in firm j have in the output of firm i.

This leads to an immediate and important result: shocks produced in the second periods do not affect quantities or prices in the first period.

In the second period case, the analogous result is more interesting. From the definition of $y_{f,t}$ in (3.7), it is obtained that:

$$\mathrm{d}\ln y_{f,2} = \mathrm{d}\ln b_2 + \delta\mathrm{d}\ln z_{1,2} + (1-\delta)\mathrm{d}\ln z_{2,2} \tag{3.34}$$

and directly from the solution of $z_{i,2}$ depending on $y_{i,2}$ for $i \in \{1, 2\}$ in (3.21):

$$\mathrm{d}\ln z_{i,2} = \Theta\mathrm{d}\ln y_{i,2} \tag{3.35}$$

So far the propagation mechanism is identical to the previous one. However, the solutions from (3.29) and (3.27) –which are dependent on the first period shocks through capital– should be embedded in definition (3.5). Thus:

$$\begin{aligned}
\mathrm{d}\ln y_{i,2} = & \ \mathrm{d}\ln a_{i,2} + \alpha_1 \mathrm{d}\ln k_{i,2} + \alpha_3 \mathrm{d}\ln x_{ii,2} + \alpha_4 \mathrm{d}\ln x_{ij,2} \\
= & \ \mathrm{d}\ln a_{i,2} + \alpha_1 \left(\mathrm{d}\ln b_1 + \Theta \mathrm{d}\ln a_{1,1} + \Theta \mathrm{d}\ln a_{2,2}\right) \\
& + \alpha_3 \left(\Theta \mathrm{d}\ln b_1 + \Theta \mathrm{d}\ln a_{1,1} + \Theta \mathrm{d}\ln a_{2,1} + \Theta \mathrm{d}\ln a_{1,2} + \Theta \mathrm{d}\ln a_{2,2}\right) \\
& + \alpha_4 \left(\Theta \mathrm{d}\ln b_1 + \Theta \mathrm{d}\ln a_{1,1} + \Theta \mathrm{d}\ln a_{2,1} + \Theta \mathrm{d}\ln a_{1,2} + \Theta \mathrm{d}\ln a_{2,2}\right) \\
= & \ \mathrm{d}\ln a_{i,2} + \left[\xi_{i,2} \mathrm{d}\ln a_{i,2} + \xi_{j,2} \mathrm{d}\ln a_{j,2}\right] + \left[\xi_{b_1} \mathrm{d}\ln b_1 + \xi_{i,1} \mathrm{d}\ln a_{i,1} + \xi_{j,1} \mathrm{d}\ln a_{j,1}\right]
\end{aligned} \tag{3.36}$$

this is, output from firm i is being affected by its own shock –the first term of the final sum in (3.36)–; shocks from intermediate firms in period $t = 2$ –first term in brackets–; and shocks from the past, $t = 1$, that are propagated through K_2 –last term in brackets–. The constants $\xi_{k,t}$ depend on the parameters of the model and all their values are lower than one. This yields that a shock originated in firm i and period t affects the most the output of firm i in period t, as might be expected.

The most important results of this section are straightforward from (3.34), (3.35) and (3.36): any shock originated anywhere in the economy –including period $t = 1$– has an impact in the final firm output in period $t = 2$. However, shocks originated in the final firm in $t = 2$ has no impact anywhere except in the final output in $t = 2$ itself.

In a nutshell, the result that claims that productivity shocks only propagates downstream continues to be applied. Schematically:

$$y_{1,1}, y_{2,1} \xrightarrow[\text{affect}]{} y_{f,1} \xrightarrow[\text{affects}]{} y_{1,2}, y_{2,2} \xrightarrow[\text{affect}]{} y_{f,2}$$

3.3.2 Simulation of the model

Different realisations of the model have been computed by adding different shocks to a baseline case in order to verify and clarify the previous theoretical results.

The computed economy features two firms or sectors. However, firm 1 is contributes more than the other to the final output as $\delta = 0.6$. Thus, firm $i = 1$ is simply called 'important' and firm $i = 2$ 'non important'. All other aspects are perfectly symmetric.

Shocks were intended to be equivalent quantitatively, producing a 20% improvement in a given productivity. However, they differ in the origin and the duration. There are four different types of shocks according to their origin: first, the improvement takes place in the important sector, i.e. $a_{1,t}$ increases 20%; second, the shock is originated in the non-important sector, then $a_{2,t}$ increases 20%; third, productivity improves in both sectors such that both $a_{1,t}$ and $a_{2,t}$ increases 10%[3]; and, finally, a shock produced in the final firm such that b_t increases. These shocks can be transitory –the improvement only takes place in the first period– or permanent –productivity increases both periods–. Therefore, there are eight possible shocks and every one are computed[4].

Some results are analogous to those obtained in the simulated static model. Among others: first, labor supply $l_{i,t}$ do not depend on the shocks or firm in any period $t = 1, 2$ –as it is fully determined by the parameters of the model–; second, due to the use of the Cobb-Douglas technologies, weights of the model as defined in (2.28) remain not affected by shocks.

[3] In order to keep the comparability.

[4] The detailed solution can be consulted partially in the Appendix B2 and completely and interactively in `https://colab.research.google.com/drive/1gucZvRNp4vQNgxmEiB4uiY0I3_amYyz8?usp=sharing`.

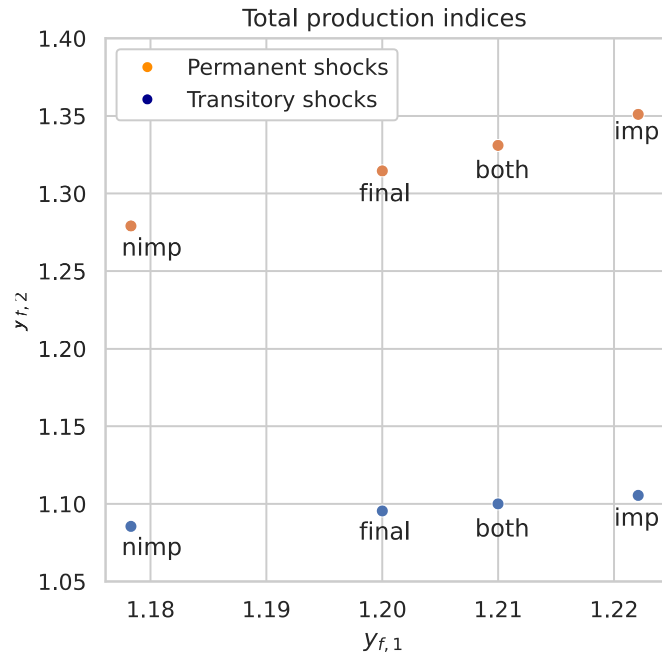


Figure 3.2: Total output of the economy when affected by the different shocks described in section 3.3.2. Each point corresponds to one unique shock and is defined by the coordinates $(r_{1,k}, r_{2,k})$ where $r_{t,k}$ is an index defined by the ratio between the total output of the economy in t when affected by the shock k and the total output of the economy without shocks. Dots in orange/blue represent permanent/transitory shocks. In addition, 'nimp' stands for the non important intermediate firm shock; 'final' for the final firm shock; 'both' for the shock that affects both intermediate firms, and 'imp' for the shock affecting the important intermediate firm. E.g.: the orange 'nimp' dot corresponds to the permanent shock in the non important intermediate firm.

However, the most relevant results on this section are those where the dynamical component takes part. Quantities $k_{1,2}$ and $k_{2,2}$ are affected by shocks in the first period, although $k_{1,1}$ and $k_{2,1}$ are fully determined by parameters and initial stock of capital $\bar{K}_1$. Focusing on the outputs, total output production improvements are measured through an index whose base is the economy without shocks as presented in Figure 3.2. Transitory shocks affect both the first and second period outputs, as with permanent shocks. It is remarkable that, although the improvement in the second period output is significantly greater for permanent shocks, the improvement in the first period output is exactly the same for permanent and transitory shocks. This situation is compatible with the fact that productivity increases in the second period do not affect results in the first period, as shown theoretically above.

Regarding the importance of the sector/firm where the shock is originated, conclusions are identical to those of the static model section. The fact that a shock in the final firm has a lower impact than an equivalent one originated simultaneously in both intermediate firms has to do with the iterations between firms in the network where the shock feeds back, as explained in section 3.3.1. A deeper work on this issue by Baqaee and Farhi, although through a different approach, shows that these interactions magnify negative shocks and attenuate positive shocks [5].

Chapter 4

The CES dynamic model

In the benchmark introduced during past sections, let denote the graph that represents the network of the intermediate producers as $G = (V, E, W)$, where V designs the set of nodes containing firms $V = \{1, 2, f\}$; E the set of directional links that represent sells –here the economy is understood as a multigraph where every node is connected to the others and sometimes also to itself– and W the weight matrix that assigns a specific weight to each edge. This weight, as introduced in (2.28) and adding the temporal index, is defined as:

$$weight_{ij,t} = \frac{Sales_{i \to j,t}}{TotalSales_{i,t}} = \frac{x_{ji,t}}{y_{i,t}} \tag{4.1}$$

Thus, in the second and third chapters of this essay, it has been shown that a supply shock originated in any firm in the economy has no effect on the topology of the network. However, this property that derives from the employment of Cobb-Douglas technologies may seem counter-intuitive as it seems unrealistic for the market not to readjust the network by reallocating resources, exploiting the –imperfect– substitutability of the intermediate goods.

Consequently, in order to study this aspect further, Cobb-Douglas technologies are abandoned in favour of CES production functions, which incorporate the substitutability notion. However, there is an additional problem: the network becomes more complex. Therefore, it is not feasible to obtain the analytical solutions of the system and the model is only numerically solved.

In essence, this model is analogous to the one in the previous chapter, but using CES production functions that allow for progress. Thereby, in this third chapter is presented a dynamical model with two periods and two intermediate firms and whose outline corresponds also to that shown in Figure 3.1.

4.1 Model

The economy from this model exists for two periods, it is created at $t = 1$, survives to $t = 2$ and then the world ends. There is a representative agent that maximises its intertemporal utility and that makes the capital formation decisions; two intermediate producers per period; and one final firm per period. All these firms maximise their intraperiod profits. Each intermediate firm produces one distinct intermediate good from labor, capital and both intermediate goods. In addition, the representative agent only consumes the final good produced by the final firm. The final firm output at $t = 1$ can be saved to form capital for $t = 2$ in a one-to-one basis.

4.1.1 Households

The households description is identical to that on section 3.1.1. The utility function of the representative agent is given by:

$$\mathcal{U}(c_1, c_2, h_1, h_2) = \ln c_1 + \gamma(1 - h_1) + \beta \ln c_2 + \beta\gamma(1 - h_2) \tag{4.2}$$

where c_t denotes the representative agent's consumption and ht its supply of labour for $t = 1, 2$; the parameter $\gamma \in \mathbb{R}$ represents the subjective level of disutility of the labour and β acts as a time discounting parameter.

The intertemporal budget constraint that the agent faces is also identical:

$$c_1 + \frac{c_2}{q_2} \leq w_1 h_1 + \frac{w_2 h_2}{q_2} + \Pi_1 + \frac{\Pi_2}{q_2} + q_1 \bar{K}_1 \tag{4.3}$$

where Π_t denotes the joint profit of firms, q_t the price of capital and w_t the price of labor factor in $t = 1, 2$, $\bar{K}_1$ is the initial stock of capital.

4.1.2 Producers

Producers form exactly the same network that the one shown in Figure 3.1. and described in section 3.1.2. However, although the final firm keeps its Cobb-Douglas function, intermediate firms feature CES technologies.

Intermediate firms

Each intermediate good $i \in \{1, 2\}$ is produced by its associated firm $i \in \{1, 2\}$. The output of fitm i in t is denoted by $y_{i,t}$. Each of these firms employs some amount of labor $l_{i,t}$ and capital $k_{i,t}$ that hires at prices w_t and q_t respectively. They also take as inputs the intermediate goods $x_{ii,t}$ and $x_{ij,t}$ at prices $p_{i,t}$ and $p_{j,t}$ respectively. The production function of firm i is defined by a CES function:

$$y_{i,t} = a_{i,t} k_{i,t}^{\alpha_1} l_{i,t}^{\alpha_2} \left(\omega x_{ii,t}^{\rho} + (1 - \omega) x_{ij,t}^{\rho} \right)^{\frac{1-\alpha_1-\alpha_2}{\rho}} \tag{4.4}$$

where $\omega \in (0, 1)$ is a share parameter that determines the importance of each input in the production of intermediate good i –previously this role was played by α_3 and α_4–. Constant ρ is the substitution parameter, the crux of the matter in this chapter. It can also be written as $\rho = \frac{\sigma-1}{\sigma}$, where $\sigma \in (0, \infty)$ stands for the elasticity of substitution between the two inputs. If $\sigma > 1$ ($\sigma < 1$), intermediate goods are known as substitute (complementary) inputs and then, a positive shock affecting $x_{ii,t}$ will translate into an increase (reduction) of the marginal product of $x_{ii,t}$ with respect to the marginal product of $x_{ij,t}$ [3]. Note that Leontief's minimum production functions and Cobb-Douglas are special cases of this general CES technology. The former is the case when inputs cannot be substituted at all; whereas in the latter the effect of a shock to the marginal production is the same for both inputs. The asymmetry caused by different impacts on marginal productions –which ultimately determine a change in optimal quantities– leads to the desired changes in the weights of the network defined in (4.1).

Final firm

The final firm at period t continues using a Cobb-Douglas functional form and $z_{1,t}$ and $z_{2,t}$ as inputs. Therefore, its output $y_{f,t}$ equals to:

$$y_{t,f} = b_t z_{1,t}^{\delta} z_{2,t}^{1-\delta} \tag{4.5}$$

where $\delta \in (0,1)$ is the output elasticity of $z_{1,t}$. Thus, final firm profit in period t is given by:

$$\pi_{f,t} = y_{f,t} - p_{1,t} z_{1,t} - p_{2,t} z_{2,t} \tag{4.6}$$

4.1.3 Equilibrium

The equilibrium in this model is defined analogously to that in the previous chapter:

Given the parameters γ, β, α_1, α_2, δ, ρ and ω, productivity shifters set $\{a_{i,t}\}$ for $(i,t) \in \{1,2\}\times\{1,2\}$ and initial stock of capital $\bar{K}_1$, the general equilibrium of the economy is defined as a set of prices $\{p_{i,t}, w_t, q_t\}$ where $(i,t) \in \{1,2\}\times\{1,2\}$, intermediate input choices $\{z_{i,t}, x_{ii,t}, x_{ij,t}\}$ for $(i,t) \in \{1,2\}\times\{1,2\}$, $j = 3-i$, labour and capital choices $l_{i,t}, k_{i,t}$ and consumption quantities c_1, c_2 such that:

1. each period $t \in \{1,2\}$, the producers solve their decision problem maximizing profits by choosing the employed factors and intermediate inputs.
2. the representative agent solves its decision problem maximizing its utility subject to its intertemporal budget constraint by choosing consumption and the quantities of labour and capital rented.
3. all the markets clear.

4.2 Solution of the model

Households problem

The problem that solves the representative agent is identical to that described in section 3.2.1. This is, households optimisation problem can be written as:

$$\begin{aligned} \max_{c,h} \quad & \ln c_1 + \beta \ln c_2 + \gamma \ln(1-h_1) + \beta\gamma \ln(1-h_2) \\ s.t.: \quad & c_1 + \frac{c_2}{q_2} \leq w_1 h_1 + \frac{w_2 h_2}{q_2} + \Pi_1 + \frac{\Pi_2}{q_2} + q_1 \bar{K}_1 \end{aligned} \tag{4.7}$$

where the following Euler conditions were obtained:

$$\begin{aligned} \frac{c_2}{c_1} =& \frac{1}{\beta q_2} \\ \frac{w_2}{w_1} =& \frac{1}{q_2} \frac{1-h_1}{1-h_2} \end{aligned} \tag{4.8}$$

Moreover, inside each period $t \in \{1, 2\}$, it was obtained one condition that related consumption and labour:

$$h_t = 1 - \gamma \frac{c_1}{w_1} \tag{4.9}$$

Intermediate producers

This is where the major difference lies between this and the last chapter. Each period $t \in \{1, 2\}$ intermediate producers seek to maximise their profits given factor and intermediate good prices, which leads to the following intraperiod optimisation problem:

$$\max_{l_{i,t}, k_{i,t}, x_{ii,t}, x_{ij,t}} \quad p_{i,t} y_{i,t} - w_t l_{i,t} - q_t k_{i,t} - p_{i,t} x_{ii,t} - p_{j,t} x_{ij,t} \tag{4.10}$$

and the following first order conditions are derived:

$$\begin{aligned}
&i.\ \ \alpha_1 \frac{p_{i,t} y_{i,t}}{k_{i,t}} = q_t \\
&ii.\ \ \alpha_2 \frac{p_{i,t} y_{i,t}}{l_{i,t}} = w_t \\
&iii.\ \ \omega p_{i,t} \frac{(1-\alpha_1-\alpha_2) y_{i,t}}{x_{ii,t}^{\rho}} \frac{1}{\omega x_{ii,t}^{\rho} + (1-\omega) x_{ij,t}^{\rho}} = p_{i,t} \\
&iv.\ \ (1-\omega) p_{i,t} \frac{(1-\alpha_1-\alpha_2) y_{i,t}}{x_{ij,t}^{\rho}} \frac{1}{\omega x_{ii,t}^{\rho} + (1-\omega) x_{ij,t}^{\rho}} = p_{j,t}
\end{aligned} \tag{4.11}$$

which is still the marginal revenue –in the left hand side– equaling the marginal cost of each input. Note that $\rho = 0$ case is equivalent to the equations derived from the Cobb-Douglas in the second and third chapters. The reason why the term associated to the other input appears –e.g. $x_{ii,t}$ in the fourth condition– is that CES functions allow or penalizes one input to be substituted by the other according to the parameter of substitution ρ as described in 4.1.2., and the solution takes this into account.

Final producer problem

Final producer solves exactly the same problem that in chapters two and three. In particular, for $t \in \{1, 2\}$:

$$\max_{z_{1,t}, z_{2,t}} \quad p_{f,t} y_{f,t} - p_{1,t} z_{1,t} - p_{2,t} z_{2,t} \tag{4.12}$$

and the same first order conditions as those obtained in (2.13) are derived here –adding in each case the appropriate temporal subindices.

Market clearing conditions

The set of market clearing conditions of this model is identical to that described in section 3.2.1. for the previous model:

1. Capital markets:

$$\begin{aligned}\bar{K}_1 =& k_{1,1} + k_{2,1} \\ K_2 =& k_{1,2} + k_{2,2}\end{aligned} \tag{4.13}$$

2. Labour markets:

$$\begin{aligned}h_1 =& l_{1,1} + l_{2,1} \\ h_2 =& l_{1,2} + l_{2,2}\end{aligned} \tag{4.14}$$

3. Intermediate good i markets:

$$y_{i,t} = z_{i,t} + x_{ii,t} + x_{ji,t} \tag{4.15}$$

   where $i = 1, 2$, $j = 3 - i$ and $t = 1, 2$.

4. Final good markets:

$$\begin{aligned}y_{f,1} =& c_1 + K_2 \\ y_{f,2} =& c_2\end{aligned} \tag{4.16}$$

Numerical solution

The consequence of conditions from (4.11) being more complex than those of the Cobb-Douglas case is that it is not feasible to solve the model analytically and therefore only a numerical solution can be implemented.

Thus, the numerical solutions of the model corresponding to different parametrisations are shown in Appendix C. Each of these is related to a specific shock as it is described in the following section 4.3. The computation has been carried out through the 'fcsolve' function in MATLAB, taking as initial guesses –which are updated– the computed solutions from the baseline of third chapter.

4.3 Comparative statics

Sixteen shocks are simulated through variations in the productivity shifters with respect to two baseline solutions of the model –one with substitute inputs, one with complementary inputs–, following the approach employed in chapters 2 and 3[1]. The implemented shocks are designed to describe all existing possibilities which are qualitatively different. In particular:

1. Depending on the parameter ρ of the CES function, intermediate products act as substitute or complementary inputs. Thus, shocks should cover both cases $\rho < 1$ and $\rho > 1$.

2. Shocks can be transitory –changes in production exclusively during the first period– or permanent –productivity variation– and both cases are studied.

3. Finally, as in the first and second chapters, shocks differ in their origin. Thus, shocks occurred in the important and no important intermediate firm –considering firm 1 important if $\delta > 0.5$ and firm 2 otherwise–, in both intermediate firms simultaneously or exclusively in the final firm. The four cases are covered.

[1] The Python notebook with the complete report is available the following link: `https://colab.research.google.com/drive/1fDQciuHxB8IdV0I262sYDUe3HDU_iG7N?usp=sharing`

Therefore, the resulting sixteen shocks are all the possible combinations of these characteristics. Each of these shocks are named 'i_j_k' where $i \in \{c, s\}$ designs if inputs are complementary or substitute; $j \in \{t, p\}$ if the shock is transitory or permanent; and $k \in \{imp, nimp, both, fin\}$ whether the shock is originated in the 'important' firm, in the 'non-important' one, in both simultaneously or in the final firm, respectively.

Two fundamental aspects are studied: first, the impact of shocks to total output –to check that it is consistent with the past results and also to evaluate the effect of ρ–. Second, the evolution in the topology –weights– of the production network is analyzed.

To this end, a 20% productivity increase is implemented.

Impacts on output

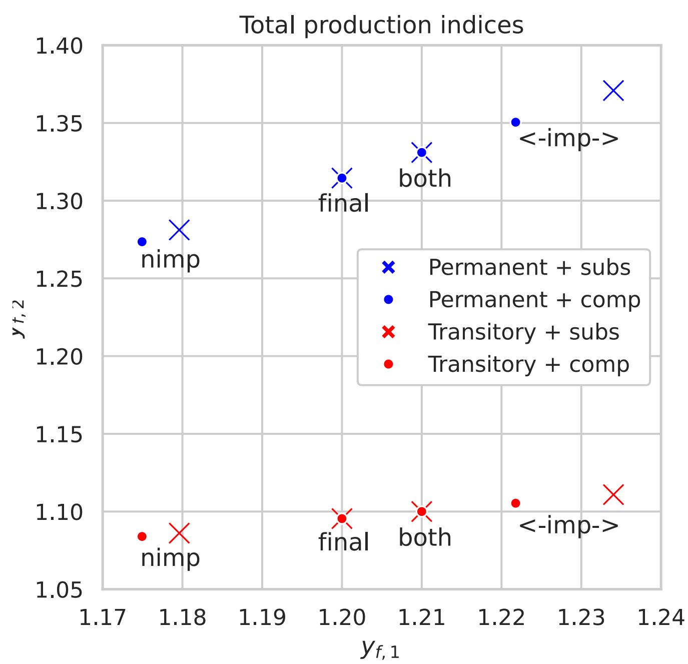


Figure 4.1: Total output of the economy when affected by the different shocks described in section 4.3. Each point corresponds to one unique shock and is defined by the coordinates $(r_{1,k}, r_{2,k})$ where $r_{t,k}$ is an index defined by the ratio between the total output of the economy in t when affected by the shock k and the total output of the economy without shocks. Points in red/blue represent transitory/permanent shocks. Crosses stand for solutions where $\rho = 0.5$ and therefore intermediate inputs are substitute, whereas dots stand for complementary inputs cases, $\rho = -0.5$. In addition, 'nimp' stands for the non important intermediate firm shock; 'final' for the final firm shock; 'both' for the shock that affects both intermediate firms, and 'imp' for the shock affecting the important intermediate firm. E.g.: the red 'nimp' cross corresponds to the transitory shock in the non important intermediate firm where the intermediate inputs are substitute, $\rho = 0.5$.

Figure 4.1. summarises the impact of shocks on the final output. Much can be gleaned from this. For instance, all the results obtained in the Cobb Douglas case still apply: shocks in the important firm have the greatest effect on output, shocks from the first period impact the result of the second period, but the opposite its not true, etc. This is because all these aspects have nothing to do with the intermediate production function chosen.

The main point of this analysis involves the elasticity of substitution and is as follows: shocks have greater impacts as the elasticity of substitution is higher. This result is very intuitive: as the elasticity of substitution increases, the use of the good of the firm in which the shock occurs is more appropriate to be employed as an input on behalf of the other good that has not been (as) affected by the shock.

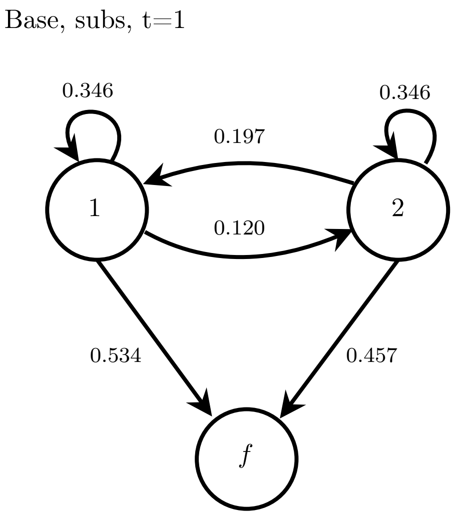
A brief example: let suppose that a method to duplicate donated blood is discovered, but it only works for a certain type of blood (input). If this occurs for the –universal donor– 0 negative type, the impact of the shock in the public health (final good) would be much greater that if this were to happen to the –universal recipient– AB positive type, precisely because of the ability to substitute other inputs.

Regarding the rest of the variables, there is an interesting result. As it can be seen in the Appendix C, labour in this model is no longer fully determined by the parameters of the model and depend on shocks, since readjustments in the network change the marginal product of labour and therefore the optimal quantities.

Impacts on weights

The most interesting point, and the reason why this model is written, is that the network does not keep the weights constant –as defined in the equation (4.1)–, but readjusts in reaction to a shock, both to alleviate the needs arising from the fact that the inputs are complementary, as well as to exploit the substitutability of the factors. One example of this is represented in Figure 4.2. where a shock in the 'important' firm occurs, it is shown that the network readjusts to take advantage of the gains from the productivity increase. For instance, after the shock, firm 1 starts to sell more to firm 2 in order to increase the latter's output, as its input is also necessary to obtain the final product. For the same reason, firm 2 concentrates more of its sales towards the final firm. And, as a consequence, firm 1 uses more of its own output as input, substituting part of the input that came from firm 2. Thus, positive and negative shocks end up affecting all the nodes of the network, as we have studied above, but also the weights of the links. Of course, if the shock was produced in the final firm or in both firms at the same time, weights would have stayed the same.

It is true that the model assumes that there are no frictions when changing weights –sales–, but this seems unrealistic: inertia is a central property in economics and these changes tend to be slow. Therefore, a 'sticky' weights model could be more realistic, as is the case when it comes to firms changing prices. Moreover, as Fadinger et al. shown, input-output matrices are very similar within countries of the same level of development and technological progress is the essential cause of change –which is consistent with the results of this final model–. Therefore, most of the time, due to the stability of the input-output matrices, the use of the Cobb-Douglas functions in order to facilitate the study of the properties of networks could be justified.



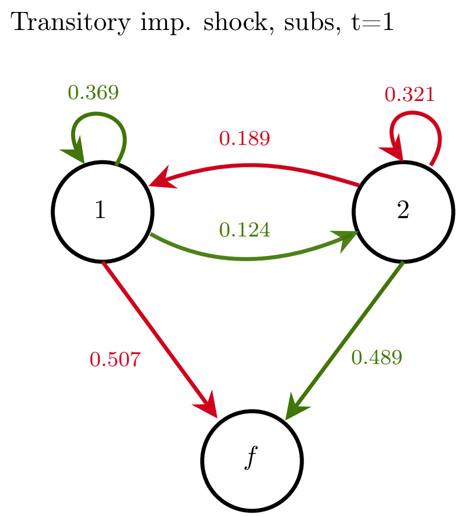


Figure 4.2: Graphs of two different economies in period 1. The left one is an economy with substitute inputs and where the firm 1 is the 'more important' of the intermediate firms, in the sense that has been explained previously. This economy acts as the baseline. The graph from the right represents exactly the same economy, but a positive productivity shock occurred in the firm 1. [ht]The arrows represent the direction of sales, and the number associated to it designs the weight of the link as defined in equation 4.1. Arrows in red (green) denote that a lower (higher) weight is achieved in the shock case than in the baseline case.

Chapter 5

Conclusion

There are two main conclusions to be drawn from this dissertation. A first one which comes directly from the results of the work and a second one, less concrete, and methodological. In addition, a note on the empirical implementation of the model is given at the end.

Regarding the first conclusion, the main insight obtained from the paper is that the network structure makes the origin of the shocks determinant in the impact they have on the macroeconomic variables they affect -in this particular model they do not affect, for example, labour- and that the dynamic component does not change this fact. In turn, the structure of the network is determined by the shocks themselves, as well as by the factor requirements that are implicit in the production functions. Finally, it is found that by adding dynamics the classical result about the way shocks propagate holds: supply shocks propagate downstream.

In general, the set of consequences of adding the production network and the dynamic component the way it is done here are disjoint. Thus, while the basic models in chapters 3 and 4 combine both approaches, the essence of both can be studied in the cases of a one period network model -as in chapter 2- or a two-period dynamic model, respectively.

Regarding the methodological insights, it is clear that both the dynamics and the network add complexity to the system, but it is the latter that really jeopardises the usefulness of the system and even the obtainment of a result. For example, only a numerical could be obtained in the fourth chapter.

Therefore, in order to make progress in this field, it is likely that other paths will have to be taken. In physics, the field in which complex systems have been present for the longest time, one of the usual solutions is to treat these systems at the macroscopic level and give them a stochastic nature that they do not possess –recall Poincaré's famous quote *«chance is only the measure of our ignorance»* and also Einstein's *«God does not play dice with the universe»*–. This approach is used, for example, by Maxwell and Boltzmann to calculate the distribution of velocities in a system of particles of an ideal gas and, from there, it is possible to compute the values of some macroscopic variables, such as pressure and temperature [16].

However, this approach is usually valid in thermodynamics because the state of particular particles does not matter, which is not always the case in economics. If complexity is ignored or the variables where complexity is present are modelled –for example, by defining a representative consumer and a representative producer– a scenario similar to that of many 'standard' macroeconomic models is obtained.

That is to say, the complexity of economic systems may not allow their analysis from the chaos or complex systems theory, at least in a global -general equilibrium- way, without a relevant change of perspective.

Thus, if it is undesirable to ignore the micro study of the particles because their individual outcome is relevant, their dynamics will have to be simplified or modelled in some way. As architects often say, it is needed to model the system keeping the complex and avoiding the complicated in order to derive valuable insights. As an example, Lambiotte and Schaub propose to write the relation between structure and (linear) dynamics as:

$$\mu \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = L\mathbf{x} \tag{5.1}$$

where the left-hand side of the equation denotes the effect of topology on spreading and the right-hand side the uncover structure from dynamics [17].

Regarding a possible empirical implementation, the approach proposed by Acemoglu, Akcigit and Kerr [1] could be followed. However, their model assumes an economy whose weights are fixed due to the use of Cobb-Douglas technologies. If it is desired to consider a sectoral network whose weights are variable as in chapter 4, a compositional model could be used to predict the evolution of the weights in the network using certain explanatory variables such as the aforementioned level of development of the economy and also incorporating productivity shocks measured through the TFP. These results could be embedded in form of an input-output matrix in the empirical model proposed by the authors. Moreover, as shown by Morais and Thomas-Agnan, impacts of the covariates can be computed –including shock variables– and therefore it could be possible to estimate which impact has any productivity shock on each network weight [18].

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Appendix A1: Complete solution of the static model

Labor and capital quantities

$$l_1 = \frac{\alpha_2(\alpha_3^\delta + \alpha_4^\delta - \alpha_4 - \delta)}{\alpha_2\alpha_3 - \alpha_2\alpha_4 - \alpha_2 - \alpha_3^2\gamma + 2\alpha_3\gamma + \alpha_4^2\gamma - \gamma}$$

$$l_2 = \frac{\alpha_2(-\alpha_3\delta + \alpha_3 - \alpha_4\delta + \delta - 1)}{\alpha_2\alpha_3 - \alpha_2\alpha_4 - \alpha_2 - \alpha_3^2\gamma + 2\alpha_3\gamma + \alpha_4^2\gamma - \gamma}$$

$$k_1 = \frac{\bar{K}(\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta)}{\alpha_3 - \alpha_4 - 1}$$

$$k_2 = \frac{\bar{K}(-\alpha_3\delta + \alpha_3 - \alpha_4\delta + \delta - 1)}{\alpha_3 - \alpha_4 - 1}$$

System of inputs and intermediate outputs

For the solution of the 'internal' system of inputs, first some parameters are defined:

$$\theta_1 = \alpha_3 a_1 k_1^{\alpha_1} l_1^{\alpha_2}$$

$$\theta_2 = \alpha_4 a_2 k_2^{\alpha_1} l_2^{\alpha_2} \frac{\alpha_4 + \delta - \alpha_3\delta - \alpha_4\delta}{1 - \alpha_4\delta - \delta - \alpha_3 + \alpha_3\delta}$$

$$\theta_3 = \alpha_3 a_2 k_2^{\alpha_1} l_2^{\alpha_2}$$

$$\theta_4 = \alpha_4 a_1 k_1^{\alpha_1} l_1^{\alpha_2} \frac{\alpha_3 - \alpha_3\delta + \delta - \alpha_4 - 1}{\alpha_4\delta - \delta - \alpha_4 + \alpha_3\delta}$$

Thus, the solution of the system is given by:

$$x_{12} = \left(\theta_1^{\frac{\alpha_3\alpha_4}{1-\alpha_3}} \theta_2^{1-\alpha_3} \theta_3^{\alpha_3} \theta_4^{\alpha_4}\right)^{\frac{(1-\alpha_3)}{\alpha_3^2 - 2\alpha_3 - \alpha_4^2 + 1}}$$

$$x_{22} = \frac{\theta_3}{\theta_2} x_{12}$$

$$x_{11} = (\theta_1 x_{12}^{\alpha_4})^{\frac{1}{1-\alpha_3}}$$

$$x_{21} = \theta_4 x_{12}^{\alpha_4} x_{11}^{\alpha_3}$$

From here, y_1 and y_2 are recovered through their definitions:

$$y_1 = a_1 k_1^{\alpha_1} l_1^{\alpha_2} x_{11}^{\alpha_3} x_{12}^{\alpha_4}$$

$$y_2 = a_2 k_2^{\alpha_1} l_2^{\alpha_2} x_{22}^{\alpha_3} x_{21}^{\alpha_4}$$

Intermediate inputs for the final good

The quantities of the intermediate good employed as input for the final firm:

$$z_1 = y_1 \frac{\delta(-\alpha_3^2 + 2\alpha_3 + \alpha_4^2 - 1)}{\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta}$$

$$z_2 = y_2 \frac{1 - \alpha_3^2\delta + \alpha_3^2 + 2\alpha_3\delta - 2\alpha_3 + \alpha_4^2\delta - \alpha_4^2 - \delta}{\alpha_3\delta - \alpha_3 + \alpha_4\delta - \delta + 1}$$

Final output and consumption

Total output and consumption:

$$y_f = b z_1^{\delta} z_2^{1-\delta}$$

$$c = y_f$$

Prices

Prices of the economy, including wages:

$$p_1 = \frac{y_f}{y_1} \frac{\delta + \alpha_4 - \delta\alpha_4 - \delta\alpha_3}{\alpha_3^2 - 2\alpha_3 - \alpha_4^2 + 1}$$

$$p_2 = \frac{y_f}{y_2} \frac{\alpha_3\delta - \alpha_3 + \alpha_4\delta + 1}{\alpha_3^2 - 2\alpha_3 - \alpha_4^2 + 1}$$

$$q = \frac{y_f}{\bar{K}_1} \frac{\alpha_1}{1 - \alpha_3 - \alpha_4}$$

$$w = y_f \frac{\alpha_2 + \gamma - \alpha_3\gamma - \alpha_4\gamma}{1 - \alpha_3 - \alpha_4}$$

Appendix A2: Report of the simulation of the static model

Arguments given

Key 'Imp' designs the shock in the important firm, 'N. Imp' in the non-important one, 'Both' in both and 'Final' in the final firm.

Parameter	Baseline	Imp.	N.Imp.	Both	Final
γ	0.8	0.8	0.8	0.8	0.8
α_1	0.25	0.25	0.25	0.25	0.25
α_2	0.25	0.25	0.25	0.25	0.25
α_3	0.25	0.25	0.25	0.25	0.25
α_4	0.25	0.25	0.25	0.25	0.25
δ	0.6	0.6	0.6	0.6	0.6
$\bar{K}$	10	10	10	10	10
a_1	10	12	10	11	10
a_2	10	10	12	11	10
b	1	1	1	1	1.2

Results

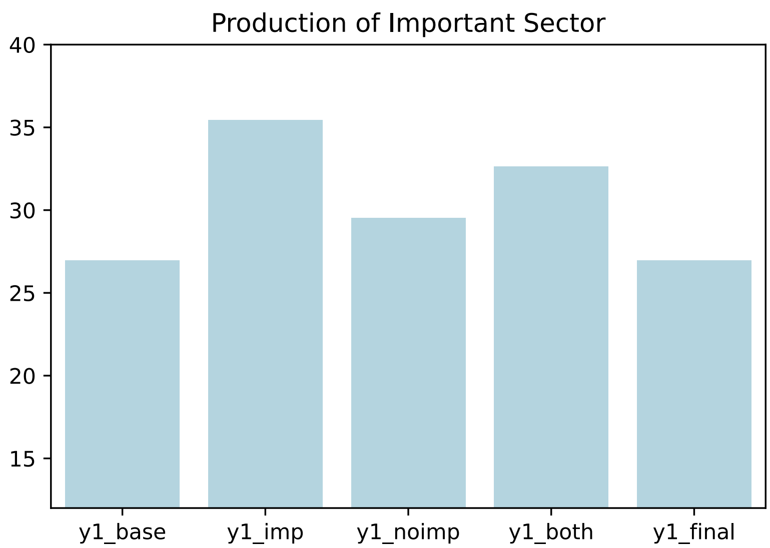
Variable	Baseline	Imp.	N.Imp.	Both	Final
h	0.3846	0.3846	0.3846	0.3846	0.3846
l_1	0.2115	0.2115	0.2115	0.2115	0.2115
l_2	0.1731	0.1731	0.1731	0.1731	0.1731
k_1	5.5000	5.5000	5.5000	5.5000	5.5000
k_2	4.5000	4.5000	4.5000	4.5000	4.5000
x_{11}	6.7415	8.8619	7.3849	8.1572	6.7415
x_{12}	6.7415	7.3849	8.8629	8.1572	6.7415
x_{21}	5.5158	7.2507	6.0422	6.6741	5.5158
x_{22}	5.5158	6.0422	7.2507	6.6741	5.5158
y_1	26.9660	35.4477	29.5397	32.6288	26.9660
y_2	22.0631	24.1689	29.0027	26.6963	22.0631
y_f	12.5066	15.2840	14.7367	15.1329	15.0079
c	12.5066	15.2840	14.7367	15.1329	15.0079
z_1	14.7087	19.3351	16.1126	17.7975	14.7087
z_2	9.8058	10.7417	12.8901	11.8650	9.8058
p_1	0.5102	0.4743	0.5488	0.5102	0.6122
p_2	0.5102	0.3881	0.4490	0.4174	0.5009
q	0.6253	0.7642	0.7368	0.7566	0.7504
w	16.2585	19.8692	19.1578	19.6728	19.5102
Utility	2.1378	2.3384	2.3019	2.3285	2.3202

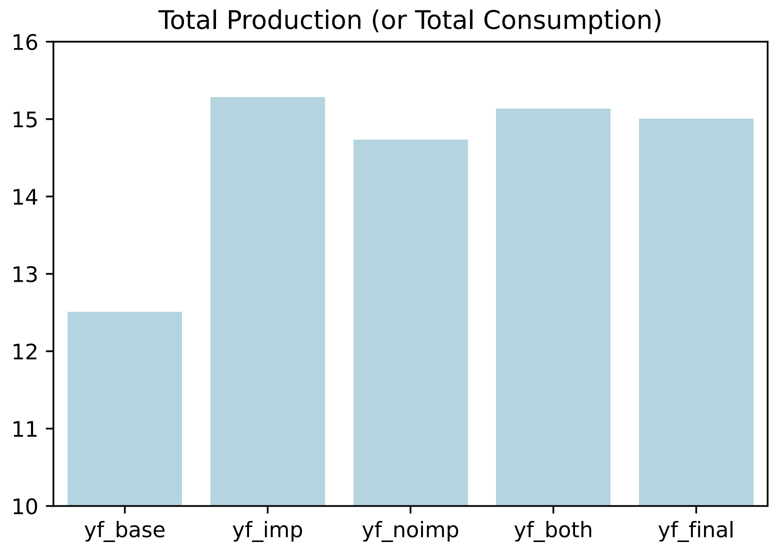
Weights of the network

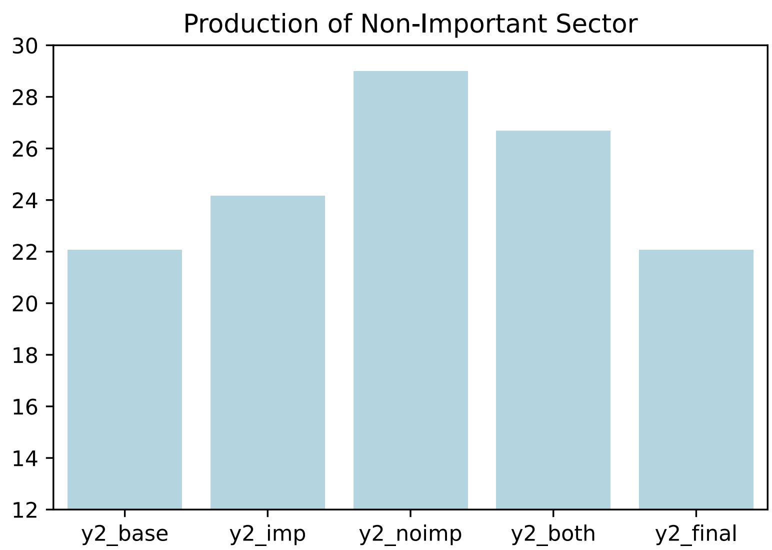
Defined as $weight_{ij} = \frac{Sales_{i \to j}}{TotalSales_i} = \frac{x_{ji}}{y_i}$

	Baseline	Imp.	N.Imp.	Both	Final
w_{12}	0.204546	0.204546	0.204546	0.204546	0.204546
w_{21}	0.305555	0.305555	0.305555	0.305555	0.305555
w_{11}	0.250000	0.250000	0.250000	0.250000	0.250000
w_{22}	0.250000	0.250000	0.250000	0.250000	0.250000
w_{1f}	0.545454	0.545454	0.545454	0.545454	0.545454
w_{2f}	0.444444	0.444444	0.444444	0.444444	0.444444

Some graphics




Total Production (or Total Consumption)
16
15
14
13
12
11
10
yf_base
yf_imp
yf_noimp
yf_both
yf_final


Production of Non-Important Sector
30
28
26
24
22
20
18
16
14
12
y2_base
y2_imp
y2_noimp
y2_both
y2_final

Appendix B1: Complete solution of the Cobb-Douglas model

First period

Labor and capital quantities

$$l_{1,1} = -\frac{\alpha_2(\alpha_1\beta - \alpha_3 - \alpha_4 + 1)(\alpha_3\delta_1 + \alpha_4\delta_1 - \alpha_4 - \delta_1)}{(-\alpha_3 + \alpha_4 + 1)(\alpha_1\alpha_2\beta - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_2 + \alpha_3^2\gamma + 2\alpha_3\alpha_4\gamma - 2\alpha_3\gamma + \alpha_4^2\gamma - 2\alpha_4\gamma + \gamma)}$$

$$l_{2,1} = \frac{\alpha_2(\alpha_1\beta - \alpha_3 - \alpha_4 + 1)(\alpha_3\delta_1 - \alpha_3 + \alpha_4\delta_1 - \delta_1 + 1)}{(1 - \alpha_3 + \alpha_4)(\alpha_1\alpha_2\beta - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_2 + \alpha_3^2\gamma + 2\alpha_3\alpha_4\gamma - 2\alpha_3\gamma + \alpha_4^2\gamma - 2\alpha_4\gamma + \gamma)}$$

$$k_{1,1} = \frac{\bar{K}(\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta)}{\alpha_3 - \alpha_4 - 1}$$

$$k_{2,1} = \frac{\bar{K}(-\alpha_3\delta + \alpha_3 - \alpha_4\delta + \delta - 1)}{\alpha_3 - \alpha_4 - 1}$$

System of inputs and intermediate outputs

For the solution of the 'internal' system of inputs, first some parameters are defined:

$$\begin{aligned}
\theta_1 =& \alpha_3 a_{1,1} k_{1,1}^{\alpha_1} l_{1,1}^{\alpha_2} \\
\theta_2 =& \alpha_4 a_{2,1} k_{2,1}^{\alpha_1} l_{2,1}^{\alpha_2} \frac{\alpha_4 + \delta - \alpha_3\delta - \alpha_4\delta}{1 - \alpha_4\delta - \delta - \alpha_3 + \alpha_3\delta} \\
\theta_3 =& \alpha_3 a_{2,1} k_{2,1}^{\alpha_1} l_{2,1}^{\alpha_2} \\
\theta_4 =& \alpha_4 a_{1,1} k_{1,1}^{\alpha_1} l_{1,1}^{\alpha_2} \frac{\alpha_3 - \alpha_3\delta + \delta - \alpha_4 - 1}{\alpha_4\delta - \delta - \alpha_4 + \alpha_3\delta}
\end{aligned}$$

Thus, the solution of the system is given by:

$$\begin{aligned}
x_{12,1} =& \left(\theta_1^{\frac{\alpha_3\alpha_4}{1-\alpha_3}} \theta_2^{1-\alpha_3} \theta_3^{\alpha_3} \theta_4^{\alpha_4}\right)^{\frac{(1-\alpha_3)}{\alpha_3^2 - 2\alpha_3 - \alpha_4^2 + 1}} \\
x_{22,1} =& \frac{\theta_3}{\theta_2} x_{12,1} \\
x_{11,1} =& \left(\theta_1 x_{12,1}^{\alpha_4}\right)^{\frac{1}{1-\alpha_3}} \\
x_{21,1} =& \theta_4 x_{12,1}^{\alpha_4} x_{11,1}^{\alpha_3}
\end{aligned}$$

From here, $y_{1,1}$ and $y_{1,1}$ are recovered through their definitions:

$$\begin{aligned}
y_{1,1} =& a_{1,1} k_{1,1}^{\alpha_1} l_{1,1}^{\alpha_2} x_{11,1}^{\alpha_3} x_{12,1}^{\alpha_4} \\
y_{2,1} =& a_{1,1} k_{2,1}^{\alpha_1} l_{2,1}^{\alpha_2} x_{22,1}^{\alpha_3} x_{21,1}^{\alpha_4}
\end{aligned}$$

Intermediate inputs for the final good

The quantities of the intermediate good employed as input for the final firm:

$$\begin{aligned}
z_{1,1} =& y_{1,1} \frac{\delta(1 - \alpha_3 + \alpha_4)(\alpha_3 + \alpha_4 - 1)}{\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta} \\
z_{2,1} =& y_{2,1} \frac{(\delta - 1)(1 - \alpha_3 + \alpha_4)(\alpha_3 + \alpha_4 - 1)}{\alpha_3\delta - \alpha_3 + \alpha_4\delta - \delta + 1}
\end{aligned}$$

Final output and consumption

$$\begin{aligned}
y_{f,1} =& b_1 z_{1,1}^{\delta} z_{2,1}^{1-\delta} \\
c_1 =& y_{f,1} \frac{1 - \alpha_3 - \alpha_4}{1 - \alpha_1 - \alpha_3 - \alpha_4}
\end{aligned}$$

Prices

Prices of the economy, including wages:

$$p_{1,1} = \frac{y_{f,1}}{y_{1,1}} \frac{\alpha_3\delta - \delta - \alpha_4 + \delta\alpha_4}{(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)}$$

$$p_{2,1} = \frac{y_{f,1}}{y_{2,1}} \frac{\alpha_3 - \alpha_3\delta - \alpha_4\delta + \delta - 1}{(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)}$$

$$q_1 = \frac{y_{f,1}}{\bar{K}_1} \frac{\alpha_1}{1-\alpha_3-\alpha_4}$$

$$w_1 = y_{f,1} \frac{\alpha_1\alpha_2\beta - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_2 + \alpha_3^2\gamma + 2\alpha_3\alpha_4\gamma - 2\alpha_3\gamma + \alpha_4^2\gamma - 2\alpha_4\gamma + \gamma}{(1-\alpha_3-\alpha_4)(\alpha_1\beta - \alpha_3 - \alpha_4 + 1)}$$

Second period

Labor and capital quantities

$$l_{1,2} = \frac{\alpha_2(\alpha_3\delta + \alpha_4\delta - \alpha_4 - \delta)}{(\alpha_3 - \alpha_4 - 1)(\alpha_2 - \alpha_3\gamma - \alpha_4\gamma + \gamma)}$$

$$l_{2,2} = \frac{\alpha_2(\alpha_3\delta + \alpha_4\delta - \alpha_3 - \delta + 1)}{(1-\alpha_3+\alpha_4)(\alpha_2 - \alpha_3\gamma - \alpha_4\gamma + \gamma)}$$

$$k_{1,2} = \frac{-\alpha_1\beta y_{f1}(\alpha_3\delta_1 + \alpha_4\delta_1 - \alpha_4 - \delta_1)}{((-\alpha_3+\alpha_4+1)(\alpha_1\beta - \alpha_3 - \alpha_4 + 1))}$$

$$k_{2,2} = \frac{\alpha_1\beta y_{f1}(\alpha_3\delta_1 - \alpha_3 + \alpha_4\delta_1 - \delta_1 + 1)}{(-\alpha_3+\alpha_4+1)(\alpha_1\beta - \alpha_3 - \alpha_4 + 1)}$$

System of inputs and intermediate outputs

For the solution of the 'internal' system of inputs, first some parameters are defined:

$$\theta_1 = \alpha_3 a_{1,2} k_{1,2}^{\alpha_1} l_{1,2}^{\alpha_2}$$

$$\theta_2 = \alpha_4 a_{2,2} k_{2,2}^{\alpha_1} l_{2,2}^{\alpha_2} \frac{\alpha_4 + \delta - \alpha_3\delta - \alpha_4\delta}{1 - \alpha_4\delta - \delta - \alpha_3 + \alpha_3\delta}$$

$$\theta_3 = \alpha_3 a_{2,2} k_{2,2}^{\alpha_1} l_{2,2}^{\alpha_2}$$

$$\theta_4 = \alpha_4 a_{1,2} k_{1,2}^{\alpha_1} l_{1,2}^{\alpha_2} \frac{\alpha_3 - \alpha_3\delta + \delta - \alpha_4 - 1}{\alpha_4\delta - \delta - \alpha_4 + \alpha_3\delta}$$

Thus, the solution of the system is given by:

$$x_{12,2} = \left(\theta_1^{\frac{\alpha_3\alpha_4}{1-\alpha_3}}\theta_2^{1-\alpha_3}\theta_3^{\alpha_3}\theta_4^{\alpha_4}\right)^{\frac{(1-\alpha_3)}{\alpha_3^2-2\alpha_3-\alpha_4^2+1}}$$

$$x_{22,2} = \frac{\theta_3}{\theta_2}x_{12,2}$$

$$x_{11,2} = \left(\theta_1 x_{12,2}^{\alpha_4}\right)^{\frac{1}{1-\alpha_3}}$$

$$x_{21,2} = \theta_4 x_{12,2}^{\alpha_4}x_{11,2}^{\alpha_3}$$

From here, $y_{1,2}$ and $y_{2,2}$ are recovered through their definitions:

$$y_{1,2} = a_{1,2}k_{1,2}^{\alpha_1}l_{1,2}^{\alpha_2}x_{11,2}^{\alpha_3}x_{12,2}^{\alpha_4}$$

$$y_{2,2} = a_{1,2}k_{2,2}^{\alpha_1}l_{2,2}^{\alpha_2}x_{22,2}^{\alpha_3}x_{21,2}^{\alpha_4}$$

Intermediate inputs for the final good

The quantities of the intermediate good employed as input for the final firm:

$$z_{1,2} = y_{1,2}\frac{\delta(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)}{\alpha_3\delta+\alpha_4\delta-\alpha_4-\delta}$$

$$z_{2,2} = y_{2,2}\frac{(\delta-1)(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)}{\alpha_3\delta-\alpha_3+\alpha_4\delta-\delta+1}$$

Final output and consumption

$$y_{f,2} = b_2 z_{1,2}^{\delta}z_{2,2}^{1-\delta}$$

$$c_2 = y_{f,2}$$

Prices

Prices of the economy, including wages:

$$
\begin{aligned}
p_{1,2} =& \frac{y_{f,2}}{y_{1,2}} \frac{\alpha_3\delta - \delta - \alpha_4 + \delta\alpha_4}{(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)} \\
p_{2,2} =& \frac{y_{f,2}}{y_{2,2}} \frac{\alpha_3 - \alpha_3\delta - \alpha_4\delta + \delta - 1}{(1-\alpha_3+\alpha_4)(\alpha_3+\alpha_4-1)} \\
q_2 =& \frac{y_{f,2}}{\beta y_{f,1}} \frac{\alpha_3+\alpha_4-1-\alpha_1\beta}{\alpha_3+\alpha_4-1} \\
w_2 =& y_{f,2} \frac{\alpha_3\gamma - \alpha_2 - \gamma + \alpha_4\gamma}{\alpha_3+\alpha_4-1}
\end{aligned}
$$

Appendix B2: Report of the simulation of the Cobb-Douglas model

Arguments given

Letter 'T' stands for 'Transitory' and 'P' for 'Permanent', whereas 'Imp' designs the shock in the important firm, 'N.Imp' in the non-important one, 'Both' in both and 'Fin' in the final firm.

Param.	Base	T.Imp.	T.N.Imp.	T.Both	T.Fin	P.Imp.	P.N.Imp.	P.Both	P.Fin
β	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
γ	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
α_1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
α_2	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
α_3	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
α_4	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
δ	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$\bar{K}$	1	1	1	1	1	1	1	1	1
$a_{1,1}$	10	10	12	11	10	10	12	11	10
$a_{2,1}$	10	12	10	11	10	12	10	11	10
$a_{1,2}$	10	10	10	10	10	10	12	11	10
$a_{2,2}$	10	10	10	10	10	12	10	11	10
b_1	1	1	1	1	1.2	1	1	1	1.2
b_2	1	1	1	1	1	1	1	1	1.2

Results

Var	Base	T.Imp.	T.N.Imp	T.Both	T.Fin	T.Imp	T.N.Imp.	T.Both	T.Fin
h_1	0.467	0.467	0.467	0.467	0.467	0.467	0.467	0.467	0.467
h_2	0.385	0.385	0.385	0.385	0.385	0.467	0.467	0.467	0.467
$l_{1,1}$	0.257	0.257	0.257	0.257	0.257	0.257	0.257	0.257	0.257
$l_{1,2}$	0.212	0.212	0.212	0.212	0.212	0.212	0.212	0.212	0.212
$l_{2,1}$	0.210	0.210	0.210	0.210	0.210	0.210	0.210	0.210	0.210
$l_{2,2}$	0.173	0.173	0.173	0.173	0.173	0.173	0.173	0.173	0.173
$k_{1,1}$	0.550	0.550	0.550	0.550	0.550	0.550	0.550	0.550	0.550
$k_{1,2}$	0.685	0.837	0.807	0.828	0.821	0.837	0.807	0.828	0.821
$k_{2,1}$	0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450
$k_{2,2}$	0.560	0.684	0.660	0.678	0.672	0.684	0.660	0.678	0.672
$x_{11,1}$	2.348	3.087	2.572	2.841	2.348	3.087	2.572	2.841	2.348
$x_{11,2}$	2.378	2.629	2.582	2.616	2.605	3.456	2.828	3.166	2.605
$x_{12,1}$	2.348	2.572	3.087	2.841	2.348	2.572	3.087	2.841	2.348
$x_{12,2}$	2.378	2.629	2.582	2.616	2.605	2.880	3.394	3.166	2.605
$x_{21,1}$	1.921	2.526	2.105	2.325	1.921	2.526	2.105	2.325	1.921
$x_{21,2}$	1.946	2.151	2.112	2.141	2.132	2.828	2.314	2.590	2.132
$x_{22,1}$	1.921	2.105	2.526	2.325	1.921	2.105	2.526	2.325	1.921
$x_{22,2}$	1.946	2.151	2.112	2.141	2.132	2.357	2.777	2.590	2.132
$y_{1,1}$	9.393	12.347	10.290	11.366	9.393	12.347	10.290	11.366	9.393
$y_{1,2}$	9.514	10.517	10.327	10.465	10.422	13.825	11.313	12.663	10.422
$y_{2,1}$	7.685	8.419	10.102	9.299	7.685	8.419	10.102	9.299	7.685
$y_{2,2}$	7.784	8.605	8.449	8.562	8.527	9.426	11.107	10.360	8.527
$y_{f,1}$	4.356	5.324	5.133	5.271	5.228	5.324	5.133	5.271	5.228
$y_{f,2}$	4.412	4.878	4.790	4.854	4.833	5.961	5.644	5.873	5.800
c_1	3.112	3.803	3.667	3.765	3.734	3.803	3.667	3.765	3.734
c_2	4.412	4.878	4.790	4.854	4.833	5.961	5.644	5.873	5.800
$z_{1,1}$	5.123	6.735	5.612	6.199	5.123	6.735	5.612	6.199	5.123
$z_{1,2}$	5.189	5.737	5.633	5.708	5.685	7.541	5.612	6.907	5.685
$z_{2,1}$	3.416	3.742	4.490	4.133	3.416	3.742	4.490	4.133	3.416
$z_{2,2}$	3.460	3.824	3.755	3.805	3.790	4.189	4.936	4.605	3.790
$p_{1,1}$	0.510	0.474	0.549	0.510	0.612	0.474	0.549	0.510	0.612
$p_{1,2}$	0.510	0.510	0.510	0.510	0.510	0.474	0.549	0.510	0.612
$p_{2,1}$	0.510	0.569	0.457	0.510	0.612	0.569	0.457	0.510	0.612
$p_{2,2}$	0.510	0.510	0.510	0.510	0.510	0.569	0.457	0.510	0.612
q_1	2.178	2.662	2.567	2.636	2.614	2.662	2.567	2.636	2.614
q_2	1.772	1.603	1.633	1.611	1.618	1.959	1.924	1.950	1.942
w_1	4.668	5.704	5.500	5.648	5.601	5.704	5.500	5.648	5.601
w_2	5.736	6.341	6.226	6.310	6.284	7.749	7.337	7.635	7.540
$Util$	1.509	1.790	1.739	1.776	1.764	1.950	1.870	1.928	1.910

Weights of the network

Defined as $weight_{ij,t} = \frac{Sales_{i\rightarrow j,t}}{TotalSales_{i,t}} = \frac{x_{ji,t}}{y_{i,t}}$

Var	Base	T.Imp.	T.N.Imp	T.Both	T.Fin	T.Imp	T.N.Imp.	T.Both	T.Fin
$w_{12,1}$	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205
$w_{12,2}$	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205
$w_{21,1}$	0.306	0.306	0.306	0.306	0.306	0.306	0.306	0.306	0.306
$w_{21,2}$	0.306	0.306	0.306	0.306	0.306	0.306	0.306	0.306	0.306
$w_{11,1}$	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
$w_{22,1}$	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
$w_{11,2}$	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
$w_{22,2}$	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
$w_{1f,1}$	0.545	0.545	0.545	0.545	0.545	0.545	0.545	0.545	0.545
$w_{1f,2}$	0.545	0.545	0.545	0.545	0.545	0.545	0.545	0.545	0.545
$w_{2f,1}$	0.444	0.444	0.444	0.444	0.444	0.444	0.444	0.444	0.444
$w_{2f,2}$	0.444	0.444	0.444	0.444	0.444	0.444	0.444	0.444	0.444

Appendix C: Report of the simulation of the CES model

Substitute inputs $\rho = 0.5$

Arguments given

Letter 'T' stands for 'Transitory' and 'P' for 'Permanent', whereas 'Imp' designs the shock in the important firm, 'N.Imp' in the non-important one, 'Both' in both and 'Fin' in the final firm.

Param.	Base	T.Imp.	T.N.Imp.	T.Both	T.Fin	P.Imp.	P.N.Imp.	P.Both	P.Fin
β	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
γ	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
α_1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
α_2	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
δ	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$\bar{K}$	12	12	12	12	12	12	12	12	12
$a_{1,1}$	10	10	12	11	10	10	12	11	10
$a_{2,1}$	10	12	10	11	10	12	10	11	10
$a_{1,2}$	10	10	10	10	10	10	12	11	10
$a_{2,2}$	10	10	10	10	10	12	10	11	10
b_1	1	1	1	1	1.2	1	1	1	1.2
b_2	1	1	1	1	1	1	1	1	1.2
ρ	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

Results

	s_base	s_t_i	s_t_ni	s_t_bo	s_t_f	s_p_i	s_p_ni	s_p_bo	s_p_f
$z_{1,1}$	18.46	24.93	19.9	22.33	18.46	24.93	19.9	22.33	18.46
$z_{2,1}$	12.31	13.26	16.62	14.89	12.31	13.26	16.62	14.89	12.31
$z_{1,2}$	10.24	11.38	11.13	11.27	11.22	15.37	11.99	13.63	11.22
$z_{2,2}$	6.83	7.59	7.42	7.51	7.48	8.18	10.02	9.09	7.48
$l_{1,1}$	0.26	0.28	0.25	0.26	0.26	0.28	0.25	0.26	0.26
$l_{2,1}$	0.2	0.19	0.22	0.2	0.2	0.19	0.22	0.2	0.2
$l_{1,2}$	0.22	0.22	0.22	0.22	0.22	0.23	0.2	0.22	0.22
$l_{2,2}$	0.17	0.17	0.17	0.17	0.17	0.16	0.18	0.17	0.17
$k_{1,1}$	6.74	7.1	6.39	6.74	6.74	7.1	6.39	6.74	6.74
$k_{1,2}$	2.52	3.11	2.97	3.05	3.02	3.27	2.81	3.05	3.02
$k_{2,1}$	5.26	4.9	5.61	5.26	5.26	4.9	5.61	5.26	5.26
$k_{2,2}$	1.96	2.42	2.32	2.38	2.36	2.26	2.47	2.38	2.36
$x_{11,1}$	11.97	18.14	11.33	14.48	11.97	18.14	11.33	14.48	11.97
$x_{12,1}$	5.32	5.13	7.91	6.44	5.32	5.13	7.91	6.44	5.32
$x_{21,1}$	4.15	6.07	4.06	5.02	4.15	6.07	4.06	5.02	4.15
$x_{22,1}$	9.33	8.7	14.35	11.29	9.33	8.7	14.35	11.29	9.33
$x_{11,2}$	6.64	7.38	7.21	7.31	7.28	11.18	6.83	8.84	7.28
$x_{12,2}$	2.95	3.28	3.21	3.25	3.23	3.17	4.77	3.93	3.23
$x_{21,2}$	2.3	2.56	2.5	2.53	2.52	3.74	2.45	3.06	2.52
$x_{22,2}$	5.18	5.75	5.62	5.7	5.67	5.37	8.65	6.89	5.67
q_1	0.65	0.81	0.77	0.79	0.78	0.81	0.77	0.79	0.78
q_2	0.97	0.87	0.89	0.88	0.89	1.08	1.05	1.07	1.06
w_1	16.82	20.75	19.84	20.35	20.18	20.75	19.84	20.35	20.18
w_2	11.32	12.58	12.3	12.46	12.4	15.52	14.51	15.07	14.88
$p_{1,1}$	0.51	0.47	0.56	0.51	0.61	0.47	0.56	0.51	0.61
$p_{2,1}$	0.51	0.58	0.45	0.51	0.61	0.58	0.45	0.51	0.61
$p_{1,2}$	0.51	0.51	0.51	0.51	0.51	0.47	0.56	0.51	0.61
$p_{2,2}$	0.51	0.51	0.51	0.51	0.51	0.58	0.45	0.51	0.61
$y_{1,1}$	34.57	49.14	35.29	41.83	34.57	49.14	35.29	41.83	34.57
$y_{2,1}$	26.95	27.1	38.88	32.62	26.95	27.1	38.88	32.62	26.95
$y_{1,2}$	19.19	21.31	20.84	21.1	21.02	30.29	21.27	25.54	21.02
$y_{2,2}$	14.96	16.62	16.25	16.45	16.39	16.71	23.43	19.91	16.39
c_1	11.21	13.83	13.22	13.56	13.45	13.83	13.22	13.56	13.45
c_2	8.71	9.68	9.46	9.58	9.54	11.94	11.16	11.59	11.45
$y_{f,1}$	15.69	19.37	18.51	18.99	18.83	19.37	18.51	18.99	18.83
$y_{f,2}$	8.71	9.68	9.46	9.58	9.54	11.94	11.16	11.59	11.45

Complementary inputs $\rho = -0.5$

Arguments given

Letter 'T' stands for 'Transitory' and 'P' for 'Permanent', whereas 'Imp' designs the shock in the important firm, 'N.Imp' in the non-important one, 'Both' in both and 'Fin' in the final firm.

Param.	Base	T.Imp.	T.N.Imp.	T.Both	T.Fin	P.Imp.	P.N.Imp.	P.Both	P.Fin
β	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
γ	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
α_1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
α_2	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
δ	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$\bar{K}$	12	12	12	12	12	12	12	12	12
$a_{1,1}$	10	10	12	11	10	10	12	11	10
$a_{2,1}$	10	12	10	11	10	12	10	11	10
$a_{1,2}$	10	10	10	10	10	10	12	11	10
$a_{2,2}$	10	10	10	10	10	12	10	11	10
b_1	1	1	1	1	1.2	1	1	1	1.2
b_2	1	1	1	1	1	1	1	1	1.2
ρ	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5

	c_base	c_t_i	c_t_ni	c_t_bo	c_t_f	c_p_i	c_p_ni	c_p_bo	c_p_f
$z_{1,1}$	17.99	23.77	19.55	21.77	17.99	23.77	19.55	21.77	17.99
$z_{2,1}$	11.99	13.03	15.84	14.51	11.99	13.03	15.84	14.51	11.99
$z_{1,2}$	9.86	10.89	10.68	10.84	10.8	14.39	11.61	13.12	10.8
$z_{2,2}$	6.57	7.26	7.12	7.23	7.2	7.89	9.41	8.75	7.2
$l_{1,1}$	0.26	0.25	0.26	0.26	0.26	0.25	0.26	0.26	0.26
$l_{2,1}$	0.21	0.21	0.2	0.21	0.21	0.21	0.2	0.21	0.21
$l_{1,2}$	0.21	0.21	0.21	0.21	0.21	0.21	0.22	0.21	0.21
$l_{2,2}$	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17
$k_{1,1}$	6.64	6.54	6.75	6.64	6.64	6.54	6.75	6.64	6.64
$k_{1,2}$	2.42	2.96	2.84	2.93	2.9	2.91	2.89	2.93	2.9
$k_{2,1}$	5.36	5.46	5.25	5.36	5.36	5.46	5.25	5.36	5.36
$k_{2,2}$	1.95	2.38	2.29	2.36	2.34	2.43	2.25	2.36	2.34
$x_{11,1}$	9.41	11.9	10.68	11.39	9.41	11.9	10.68	11.39	9.41
$x_{12,1}$	7.18	7.97	9.28	8.69	7.18	7.97	9.28	8.69	7.18
$x_{21,1}$	5.79	7.51	6.4	7.01	5.79	7.51	6.4	7.01	5.79
$x_{22,1}$	7.59	8.64	9.56	9.19	7.59	8.64	9.56	9.19	7.59
$x_{11,2}$	5.16	5.7	5.59	5.67	5.65	7.21	6.34	6.86	5.65
$x_{12,2}$	3.94	4.35	4.27	4.33	4.31	4.83	5.51	5.24	4.31
$x_{21,2}$	3.17	3.51	3.44	3.49	3.48	4.55	3.8	4.22	3.48
$x_{22,2}$	4.16	4.6	4.51	4.57	4.56	5.23	5.68	5.54	4.56
q_1	0.64	0.78	0.75	0.77	0.76	0.78	0.75	0.77	0.76
q_2	0.96	0.87	0.88	0.87	0.88	1.06	1.04	1.05	1.05
w_1	16.39	20.02	19.26	19.83	19.67	20.02	19.26	19.83	19.67
w_2	10.89	12.04	11.81	11.98	11.93	14.71	13.87	14.5	14.32
$p_{1,1}$	0.51	0.47	0.55	0.51	0.61	0.47	0.55	0.51	0.61
$p_{2,1}$	0.51	0.57	0.45	0.51	0.61	0.57	0.45	0.51	0.61
$p_{1,2}$	0.51	0.51	0.51	0.51	0.51	0.47	0.55	0.51	0.61
$p_{2,2}$	0.51	0.51	0.51	0.51	0.51	0.57	0.45	0.51	0.61
$y_{1,1}$	33.2	43.18	36.63	40.17	33.2	43.18	36.63	40.17	33.2
$y_{2,1}$	26.77	29.64	34.69	32.39	26.77	29.64	34.69	32.39	26.77
$y_{1,2}$	18.19	20.1	19.71	20.01	19.92	26.15	21.75	24.21	19.92
$y_{2,2}$	14.67	16.21	15.9	16.13	16.07	17.95	20.6	19.52	16.07
c_1	10.93	13.35	12.84	13.22	13.11	13.35	12.84	13.22	13.11
c_2	8.38	9.26	9.08	9.22	9.18	11.32	10.67	11.15	11.02
$y_{f,1}$	15.3	18.69	17.97	18.51	18.36	18.69	17.97	18.51	18.36
$y_{f,2}$	8.38	9.26	9.08	9.22	9.18	11.32	10.67	11.15	11.02

Weights of the network

The table is given transposed for convenience.

	$w_{11,1}$	$w_{11,2}$	$w_{12,1}$	$w_{12,2}$	$w_{22,1}$	$w_{22,2}$	$w_{21,1}$	$w_{21,2}$	$w_{1f,1}$	$w_{1f,2}$	$w_{2f,1}$	$w_{2f,2}$
s_base	0.346	0.346	0.12	0.12	0.346	0.346	0.197	0.197	0.534	0.534	0.457	0.457
c_base	0.284	0.284	0.175	0.175	0.284	0.284	0.268	0.268	0.542	0.542	0.448	0.448
s_t_im	0.369	0.346	0.124	0.12	0.321	0.346	0.189	0.197	0.507	0.534	0.489	0.457
s_t_ni	0.321	0.346	0.115	0.12	0.369	0.346	0.203	0.197	0.564	0.534	0.427	0.457
s_t_bo	0.346	0.346	0.12	0.12	0.346	0.346	0.197	0.197	0.534	0.534	0.457	0.457
s_t_fi	0.346	0.346	0.12	0.12	0.346	0.346	0.197	0.197	0.534	0.534	0.457	0.457
s_p_im	0.369	0.369	0.124	0.124	0.321	0.321	0.189	0.189	0.507	0.507	0.489	0.489
s_p_ni	0.321	0.321	0.115	0.115	0.369	0.369	0.203	0.203	0.564	0.564	0.427	0.427
s_p_bo	0.346	0.346	0.12	0.12	0.346	0.346	0.197	0.197	0.534	0.534	0.457	0.457
s_p_fi	0.346	0.346	0.12	0.12	0.346	0.346	0.197	0.197	0.534	0.534	0.457	0.457
c_t_im	0.276	0.284	0.174	0.175	0.292	0.284	0.269	0.268	0.55	0.542	0.44	0.448
c_t_ni	0.292	0.284	0.175	0.175	0.276	0.284	0.268	0.268	0.534	0.542	0.457	0.448
c_t_bo	0.284	0.284	0.175	0.175	0.284	0.284	0.268	0.268	0.542	0.542	0.448	0.448
c_t_fi	0.284	0.284	0.175	0.175	0.284	0.284	0.268	0.268	0.542	0.542	0.448	0.448
c_p_im	0.276	0.276	0.174	0.174	0.292	0.292	0.269	0.269	0.55	0.55	0.44	0.44
c_p_ni	0.292	0.292	0.175	0.175	0.276	0.276	0.268	0.268	0.534	0.534	0.457	0.457
c_p_bo	0.284	0.284	0.175	0.175	0.284	0.284	0.268	0.268	0.542	0.542	0.448	0.448
c_p_fi	0.284	0.284	0.175	0.175	0.284	0.284	0.268	0.268	0.542	0.542	0.448	0.448