Testing equality between incidence rates

0. Definitions:

X_{ij} : # cases region i, subregion j.

E_{ij} : # elements exposed (e.g. population of i,j) region i, subregion j.

I. Models :

(i) Poisson: X_{ij} i.i.d, $X_{ij} \sim \mathcal{P}(E_{ij}\theta_i)$, θ_i dependent only on i (county).

(ii) Binomial: X_{ij} i.i.d, $X_{ij} \sim B(E_{ij}, \theta_i)$, " " " .

QUESTIONS

1. Assume 2 counties, $i \in \{1,2\}$ s.t. $\theta_1 = \theta_2$ under the Poisson or the binomial assumption. MLE? Case of k counties?

FIRST. Poisson MLE?

$$X_{1j} \sim P(E_{1j}\theta) \qquad j \in \{1, \dots, J_1\}$$
$$X_{2j} \sim P(E_{2j}\theta) \qquad j \in \{1, \dots, J_2\}$$

Thus, the likelihood function is given by:

$$\mathcal{L}(\theta; x_{11}, \dots, x_{1J_1}, x_{21}, \dots, x_{2J_2}) = \prod_{j=1}^{J_1} f_\theta(x_{1j}) \prod_{j=1}^{J_2} f_\theta(x_{2j}) =$$

$$= \prod_{j=1}^{J_1} e^{-E_{1j}\theta} \frac{(E_{1j}\theta)^{x_{1j}}}{x_{1j}!} \prod_{j=1}^{J_2} e^{-E_{2j}\theta} \frac{E_{2j}\theta^{x_{2j}}}{x_{2j}!} =$$

$$= \frac{e^{-\sum_{j}^{J_1} E_{1j}\theta} \prod_{j=1}^{J_1} E_{1j}^{x_{1j}} \theta^{\sum_{j=1}^{J_1} x_{1j}}}{\prod_{j}^{J_1} x_{1j}!} \frac{e^{-\sum_{j}^{J_2} E_{2j}\theta} \prod_{j=1}^{J_2} E_{1j}^{x_{2j}} \theta^{\sum_{j=1}^{J_2} x_{2j}}}{\prod_{j}^{J_2} x_{2j}!}$$

$$\ln\left(\prod_{i} E_{1j}^{x_{1j}}\right) = \sum \ln (E_{ij})^{x_{1j}} = $$
$$= \sum x_{ij} \ln(E_{ij})$$

$$\Rightarrow \ln L(\theta; x_{11}, \ldots, x_{2J_2}) =$$

$$= -\theta \sum_{j=1}^{J_1} E_{1j} + \sum_{j=1}^{J_1} x_{1j} \ln(E_{1j}) - \ln\left(\prod_{j=1}^{J_1} x_{1j}!\right) + \sum_{j=1}^{J_1} x_{1j} \ln(\theta)$$
$$-\theta \sum_{j=1}^{J_2} E_{2j} + \sum_{j=1}^{J_2} x_{2j} \ln(E_{2j}) - \ln\left(\prod_{j=1}^{J_2} x_{2j}!\right) + \sum_{i=1}^{u} x_{2j} \ln(\theta)$$

$$\Rightarrow \frac{\partial \ln L(\theta; x \ldots)}{\partial \theta} = -\sum_{j=1}^{J_1} E_{1j} + \frac{\sum_{j=1}^{J_1} x_{1j}}{\theta} - \sum_{j=1}^{J_2} E_{2j} + \frac{\sum_{j=1}^{J_2} x_{2j}}{\theta} = 0$$

$$\Longleftrightarrow \frac{\sum_{j=1}^{J_1} x_{1j} + \sum_{j=1}^{J_2} x_{2j}}{\theta} = \sum_{j=1}^{J_1} E_{1j} + \sum_{j=2}^{J_2} E_{2j}$$

$$\longmapsto \quad \hat{\theta}_{ML} = \frac{\sum_{j=1}^{J_1} X_{1j} + \sum_{j=1}^{J_2} X_{2j}}{\sum_{j=1}^{J_1} E_{1j} + \sum_{j=2}^{J_2} E_{2j}} = \frac{\#\text{ cases (total)}}{\text{population (total)}}$$

TOTAL := Country 1 ∪ Country 2

CSO: $$\frac{\partial^2 \ln L(\theta; \vec{x})}{\partial \theta^2} = -\frac{\sum_{j=1}^{J_1} X_{1j}}{\theta^2} - \frac{\sum_{j=2}^{J_2} X_{2j}}{\theta^2} < 0$$

$\forall \theta \in \Theta$ ✓

generalize for K countries!

$$\boxed{\hat{\theta}^{K}_{ML} = \frac{\sum_{i=1}^{K} X_{i,\cdot}}{\sum_{i=1}^{K} E_{i,\cdot}}}$$

to DO:

- BINOMIAL CASE