TSPL: Polymorphic Lambda Calculus and The Calculus of Constructions

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1 Polymorphic lambda calculus

The polymorphic lambda calculus, also called System F, was discovered independently by Girard (1972) and Reynolds (1974).

Let A, B, C range over types, and L, M, N range over terms. We write $\Gamma \vdash A : \mathsf{type}$ if A is a well-formed type, and we write $\Gamma \vdash M : A$ if M is a term of type A, where Γ is an environment of pairs of the form $X : \mathsf{type}$ and $x : A$.

$\boxed{\Gamma \vdash_{\mathrm{F}} A : \mathsf{type}}$

$$\text{typ id}\;\frac{(X : \mathsf{type}) \in \Gamma}{\Gamma \vdash X : \mathsf{type}}$$

$$\text{fun}\;\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma \vdash B : \mathsf{type}}{\Gamma \vdash (A \to B) : \mathsf{type}} \qquad \text{all}\;\frac{\Gamma,\, X : \mathsf{type} \vdash B : \mathsf{type}}{\Gamma \vdash (\forall X.\, B) : \mathsf{type}}$$

$\boxed{\Gamma \vdash_{\mathrm{F}} M : A}$

$$\text{id}\;\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}$$

$$\text{fun abs}\;\frac{\Gamma,\, x : A \vdash N : B}{\Gamma \vdash (\lambda x{:}A.\, N) : A \to B} \qquad \text{fun app}\;\frac{\Gamma \vdash L : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash (L\ M) : B}$$

$$\text{typ abs}\;\frac{\Gamma,\, X : \mathsf{type} \vdash N : B}{\Gamma \vdash (\Lambda X.\, N) : \forall X.\, B} \qquad \text{typ app}\;\frac{\Gamma \vdash L : \forall X.\, B \qquad \Gamma \vdash A : \mathsf{type}}{\Gamma \vdash (L\ A) : [X \mapsto A]B}$$

The reduction rules, not including congruences, are:

$$\begin{aligned}(\lambda x{:}A.\, N)\ M &\longrightarrow [x \mapsto M]N \\ (\Lambda X.\, N)\ A &\longrightarrow [X \mapsto A]N\end{aligned}$$

Product, unit, sum, and empty types can be defined in terms of these, as can natural numbers.

$$
\begin{aligned}
A \times B &\stackrel{\text{def}}{=} \forall Z.\, (A \to B \to Z) \to Z \\
(V, W) &\stackrel{\text{def}}{=} \Lambda Z.\, \lambda k{:}A{\to}B{\to}Z.\, k\; V\; W \\
\mathsf{fst}\ L &\stackrel{\text{def}}{=} L\; A\; (\lambda x{:}A.\, \lambda y{:}B.\, x) \\
\mathsf{snd}\ L &\stackrel{\text{def}}{=} L\; B\; (\lambda x{:}A.\, \lambda y{:}B.\, y) \\[1ex]
1 &\stackrel{\text{def}}{=} \forall Z.\, Z \to Z \\
() &\stackrel{\text{def}}{=} \Lambda Z.\, \lambda z : Z.\, z \\[1ex]
A + B &\stackrel{\text{def}}{=} \forall Z.\, (A \to Z) \to (B \to Z) \to Z \\
\mathsf{inl}\ V &\stackrel{\text{def}}{=} \Lambda Z.\, \lambda h{:}A{\to}Z.\, \lambda k{:}B{\to}Z.\, h\, V \\
\mathsf{inr}\ W &\stackrel{\text{def}}{=} \Lambda Z.\, \lambda h{:}A{\to}Z.\, \lambda k{:}B{\to}Z.\, k\, W \\
\mathsf{case}\ L\ \mathsf{of}\ \{\, \mathsf{inl}\ x \Rightarrow M;\ \mathsf{inr}\ y \Rightarrow N \,\} : C &\stackrel{\text{def}}{=} L\; C\; (\lambda x{:}A.\, M)\; (\lambda y{:}B.\, N) \\[1ex]
0 &\stackrel{\text{def}}{=} \forall Z.\, Z \\
\mathsf{case}\ L\ \mathsf{of}\ \{\,\} : C &\stackrel{\text{def}}{=} L\, C \\[1ex]
\mathsf{Nat} &\stackrel{\text{def}}{=} \forall Z.\, (Z \to Z) \to Z \to Z \\
\mathsf{Z} &\stackrel{\text{def}}{=} \Lambda Z.\, \lambda s{:}Z{\to}Z.\, \lambda z{:}Z.\, z \\
\mathsf{S} &\stackrel{\text{def}}{=} \lambda n{:}\mathsf{Nat}.\, \Lambda Z.\, \lambda s{:}Z{\to}Z.\, \lambda z{:}Z.\, s\; (n\; Z\; s\; z) \\
m + n &\stackrel{\text{def}}{=} m\; \mathsf{Nat}\; \mathsf{S}\; n \\
m \times n &\stackrel{\text{def}}{=} m\; \mathsf{Nat}\; (\lambda x{:}\mathsf{Nat}.\, n + x)\; \mathsf{Z} \\
m^n &\stackrel{\text{def}}{=} m\; \mathsf{Nat}\; (\lambda x{:}\mathsf{Nat}.\, n \times x)\; (\mathsf{S}\; \mathsf{Z})
\end{aligned}
$$

2 Calculus of Constructions

The calculus of constructions was proposed by Coquand and Huet (1988). It is the basis of the system used in Coq.

Let A, B, C, L, M, N range over constructions (which encompass both terms and types), and let s range over either type or prop, which are called *sorts*. If $\Gamma \vdash A : \mathsf{type}$ then we say A is a type, while if $\Gamma \vdash M : A$ we say term M has type A, where Γ is an environment of pairs of the form $x : A$ (which includes $x : \mathsf{type}$).

Where we previously wrote $\forall X.\, B[X]$ we now write $\forall x{:}\mathsf{type}.\, B[x]$, and where we previously wrote $\Lambda X.\, B[X]$ we now write $\lambda x{:}\mathsf{type}.\, B[x]$.

$\boxed{\Gamma \vdash_{\mathrm{F}} M : A}$

$$\text{id}\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \text{type}\frac{}{\Gamma \vdash \mathsf{prop} : \mathsf{type}} \qquad \text{all}\frac{\Gamma,\, x : A \vdash B : s}{\Gamma \vdash (\forall x : A.\, B) : s}$$

$$\text{abs}\frac{\Gamma,\, x : A \vdash N : B}{\Gamma \vdash (\lambda x{:}A.\, N) : \forall x{:}A.\, B} \qquad \text{app}\frac{\Gamma \vdash L : \forall x{:}A.\, B \qquad \Gamma \vdash M : A}{\Gamma \vdash (L\ M) : [x \mapsto M]B}$$

$$\text{conv}\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s \qquad A =_\beta B}{\Gamma \vdash M : B}$$

We have the following abbreviation.

$$A \to B \stackrel{\text{def}}{=} \forall x{:}A.\, B \qquad \text{if } x \notin B$$

Not including congruences, there is only one reduction rule.

$$(\lambda x{:}A.\, N)\ M \longrightarrow [x \mapsto M]N$$

System F is included in the Calculus of Constructions. We can also define many other things, such as equality of terms of type A.

$$(x =_A y) \stackrel{\text{def}}{=} \forall P : A{\to}\mathsf{prop}.\, P\ x \to P\ y$$

References

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