Regular Expressions

Regular expression

Definition

Regular expressions describe language(regular languages)

If E is a regular expression, $L(E)$ is the language it defines

RE 的定义是递归的。RE 使用三种语言上的基本操作：Union, Concatenation, Kleene Star

- Union: $L \cup M$
- Concatenation: $LM = \{wx : w \in L \text{ and } x \in M\}$
- Kleene Star: $L^* = \{\epsilon\} \cup L \cup LL \cup \dots$

Basis.

- Basis 1: If a is any symbol, then a is a RE, $L(a) = \{a\}$
- Basis 2: ϵ is a RE, $L(\epsilon) = \{\epsilon\}$
- Basis 3: $\varnothing$ is a RE, $L(\varnothing) = \varnothing$

Induction.

- Induction 1: If E_1, E_2 are RE, $E_1 + E_2$ is a RE, $L(E_1 + E_2) = L(E_1) \cup L(E_2)$
- Induction 2: If E_1, E_2 are RE, $E_1 E_2$ is a RE, $L(E_1 E_2) = L(E_1)L(E_2)$
- Induction 3: If E is a RE, then E^* is a RE, $L(E^*) = (L(E))^*$

操作优先级由高到低是 $*, \text{concatenation}, +$

Equivalence of RE and FA

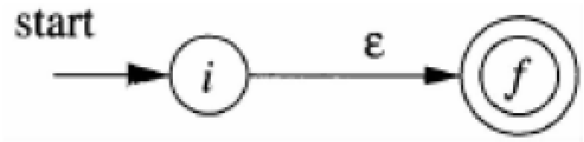
RE to FA

For every RE, there is a FA that accepts the same language

Pick ϵ-NFA

Basis.

ϵ



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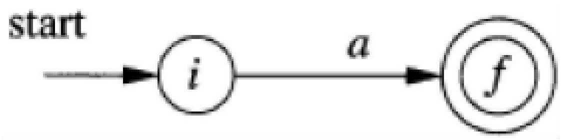
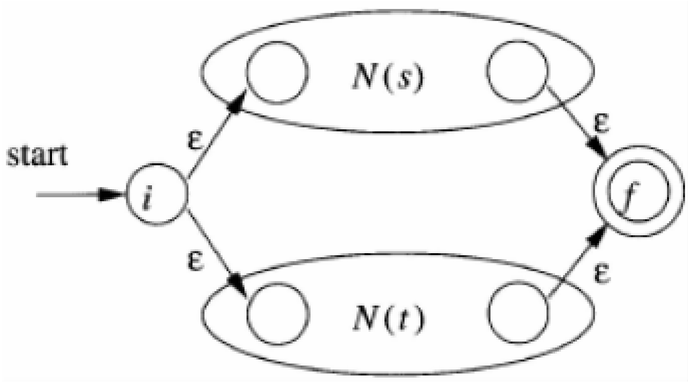
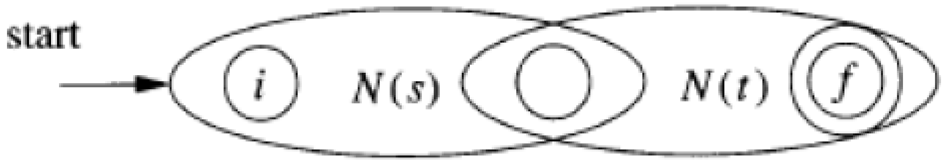


Induction. 假设正则表达式 s, t 的 NFA 为 $N(s), N(t)$

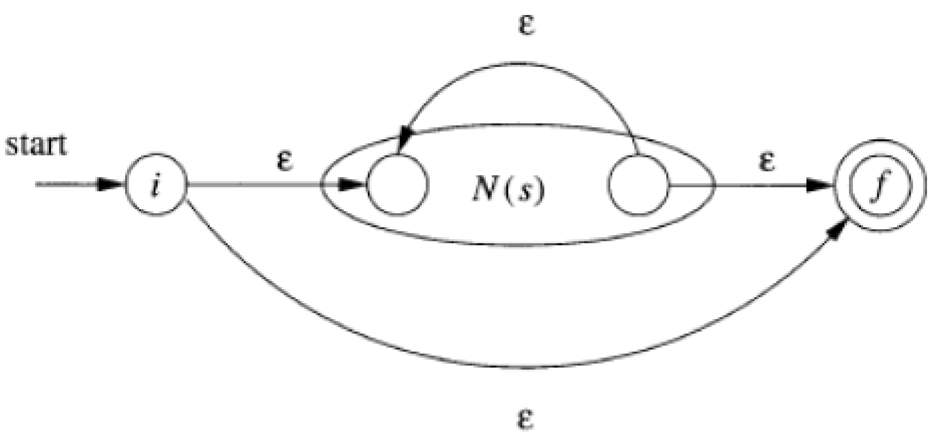
运算 $s \mid t$



运算 st



运算 s^*



FA to RE

For every FA, there is a RE defining its language

Pick DFA

使用归纳。假设 DFA 的状态为 $1, 2, \ldots, n$

定义 k-Path: A k-Path is a path through the graph of the DFA that goes through **no state numbered higher than k**

k-path 的 endpoint 没有限制，n-path 可以是任意路径

则该 DFA 对应的 RE 是所有从开始状态到接受状态的 n-path 的 RE 的集合

其正确性的证明基于对 k 的归纳，令 R_{ij}^k 为从状态 i 到状态 j 的 k-path 上的 label 表示的 RE

Basis. $k = 0$ ，此时 R_{ij}^0 是从 i 到 j 所有边上的标号的和

- 若没有边，则为 $\varnothing$
- 若 $i = j$ ，则需要加上 ϵ

Induction. 从 i 到 j 的 k-path，或者经过状态 k （一次或多次），或者不经过状态 k

- 经过状态 k ： $R_{ik}^{k-1}(R_{kk}^{k-1})^* R_{kj}^{k-1}$
- 不经过状态 k ： R_{ij}^{k-1}

故

$$R_{ij}^k = R_{ij}^{k-1} + R_{ik}^{k-1}(R_{kk}^{k-1})^* R_{kj}^{k-1}$$

The RE with the same language as the DFA is the union of R_{ij}^n where

- n is the number of states
- i is the start state
- j is one of the final states

Algebraic Laws for RE

- Union is commutative and associative, concatenation is associative
- Concatenation distributes over union
- $\varnothing$ is the identity for +, $\varnothing + R = R + \varnothing = R$
- ϵ is the identity for concatenation, $\epsilon R = R\epsilon = R$
- $\varnothing$ is the annihilator for concatenation, $\varnothing R = R\varnothing = \varnothing$

Properties of Regular Languages

Properties

A **language class** is a set of languages

Language classes have two important kinds of properties

- Decision properties （判定性）
- Closure properties （封闭性）

Closure properties: given languages in the class, an operation produces another language in the same class

Decision properties: an **algorithm** that takes a formal description of a language and tells whether or not some property holds

- Membership problem: is string w in regular language L
- Emptiness problem: does the language contain any string at all

Pumping Lemma

Let L be a regular language. Then there exists a constant n (which depends on L) such that for every string w in L such that $|w| \geqslant n$, we can break w into three strings, $w = xyz$, such that:

1. $y \neq \epsilon$
2. $|xy| \leqslant n$
3. For all $k \geqslant 0$, the string $xy^k z$ is also in L

Decision Properties of Regular Language

The membership problem

Problem: Given a string w and a regular language L , is w in L

Algorithm: Simulate the DFA, if the DFA ends in an accepting state, the answer is "yes", otherwise the answer is "no"

The emptiness problem

Problem: Given a regular language, does the language contain any string at all

Algorithm: Compute the set of states **reachable** from the start state. If at least one final state is reachable, then yes, else no.

> Basis. The start state is surely reachable from the start state.
>
> Induction. If state q is reachable from the start state, and there is an arc from q to p with any label (input or ϵ), then p is reachable

如果语言由 RE 给出，则可根据以下递归规则判断是否为空

> Basis. $\varnothing$ denotes the empty language, ϵ and a for any input symbol a do not.
>
> Induction. Suppose R is a RE. There are four cases to consider.
>
> 1. $R = R_1 + R_2$. Then $L(R)$ is empty $\iff L(R_1)$ is empty **and** $L(R_2)$ is empty
> 2. $R = R_1 R_2$. Then $L(R)$ is empty $\iff L(R_1)$ is empty **or** $L(R_2)$ is empty
> 3. $R = R_1^*$. Then $L(R)$ is not empty, it always includes at least ϵ
> 4. $R = (R_1)$. Then $L(R)$ is empty $\iff L(R_1)$ is empty

The infiniteness problem

Problem: Is a given language infinite?

Key Idea: if the DFA has n states, and the language contains ant string of length n or more, then the language is infinite.

> Suppose that L is a regular language accepted by a DFA with n states. Then L is infinite $\iff L$ contains a word whose length is at least n.
>
> $\Leftarrow$: 如果一个有 n 个状态的 DFA 接受一个长度至少为 n 的字符串 w，则在其路径上至少有一个状态出现两次（PHP）。设其路径为
>
> $$s_1 s_2 \ldots s_i \ldots s_i \ldots s_n$$
>
> 设从 s_1 到第一个 s_i 的标号组成串 x ，从第一个 s_i 到第二个 s_i 的标号组成串 y ，从第二个 s_i 到 s_n 组成字符串 z ，则该 DFA 可接受所有形如 $xy^i z (i \geqslant 0)$ 的字符串 (i. e. pumping lemma)
>
> $\Rightarrow$: trivial

However, there are an infinite number of strings of length $> n$, and we can't test them all

Key Idea: if there is a string of length $\geqslant n$ in L , then there is a string of length between n and $2n-1$

> Suppose that L is a regular language accepted by a DFA with n states. Then L is infinite $\iff L$ contains a word whose length is between n and $2n-1$.
>
> $\Rightarrow$: 因为 L 是 infinite，其包含一个 string 长度至少为 n ，设 w 为 L 包含的长度至少为 n 的 string 中最短的

> $$w = w_1 w_2 \ldots w_N$$
>
> 设 DFA 接受其的状态顺序为 $a_0 a_1 \ldots a_N$ ，根据 PHP ，其中必有一个状态至少出现两次，设其为 a_i ，且 $a_i = a_j (i < j)$ ，故 $z = w_1 w_2 \ldots w_i w_{j+1} \ldots w_N$ 也属于 L 。根据 w 的定义，有 $|z| < n$ ，又 $|w| = z + j - i$ 且 $j - i \leqslant n$ ，故 $|w| < 2n$ 。根据其定义 $|w| \geqslant n$ ，故可得
>
> $$n \leqslant |w| \leqslant 2n - 1$$
>
> $\Leftarrow$: 同上（pumping lemma）

根据以上两个 key idea，检查所有长度在 n 到 $2n-1$ 之间的串即可

然而在实际中这样的算法代价也是不可接受的。所以一般是通过检查 DFA 对应的图中是否有环来判断语言是否 infinite。具体而言分为以下几步

1. Eliminate states not reachable from the start state
2. Eliminate states that do not reach a final state
3. Test if the remaining transition graph has any cycles

The equivalence problem

Problem: Given regular languages L and M, is $L = M$

Algorithm: Constructing the **product DFA** from DFA for L, M

设 L 与 M 的 DFA 有状态集合 Q 与 R ，则 product DFA 有状态集合 $Q \times R$

对于 product DFA 而言

- start state: $[q_0, r_0]$
- transition: $\delta([q, r], a) = [\delta_L(q, a), \delta_M(r, a)]$
- final states: states $[q, r]$ that exactly one of q and r is a final state of its own DFA

由于 product DFA 的 accept state 是两个状态中有且仅有一个是原本的 accept state，则如果一个 string w 被 product DFA 接受，说明其被一个 DFA 接受而不被另一个 DFA 接受

$L = M \iff$ the product DFA's language is empty

The containment problem

Problem: Given regular languages L and M , is $L \subseteq M$

Algorithm: use the product DFA

final state: states $[q, r]$ that q is the final state, r is not

$L \subseteq M \iff$ the product DFA's language is empty

The Minimum-State DFA for a Regular Language

Equivalence of states

Given a DFA A, find the DFA with the fewest states accepting $L(A)$ (equivalence)

> Equivalence of states: states p and q are *equivalent* if
>
> For all input strings w , $\delta(p, w)$ is an accepting state $\iff$ $\delta(q, w)$ is an accepting state
>
> If two states are not equivalent, then we say they are *distinguishable*. That is, state p is distinguishable from state q if there is at least one string w such that one of $\delta(p, w)$ and $\delta(q, w)$ is accepting, and the other is not accepting.

Algorithm: table-filling algorithm

- Construct a table with all pairs of states
- If you find a string that **distinguishes** two states (takes exactly one to an accepting state), mark that pair
- Algorithm is a **recursion** on the length of the shortest distinguishing string

> Basis. If p is an accepting state and q is non-accepting, then the pair $[p, q]$ is distinguishable
>
> Induction. Let p and q be states such that for some input symbol a, $r = \delta(p, a)$ and $s = \delta(q, a)$ are a pair of states known to be distinguishable. Then $[p, q]$ is a pair of distinguishable states.

在算法结束后，所有未被标记的 pair 是等价的状态

> If two states are not distinguished by the table-filling algorithm, then the states are equivalent.
>
> Proof. 假设命题错误，故存在至少一对状态 p, q 满足
>
> - p, q is distinguishable.
> - the algorithm does not find p and q to be distinguishable
>
> 令这样的状态对为 bad pair。则对于所有 bad pairs，都存在一个 string 来区分两个状态。令其中 string 最短的 pair 为 $[p, q]$ ，区分其的最短 string 为 $w = a_1 a_2 \dots a_n$

> 显然 $w \neq \epsilon$，否则在 Basis 阶段就被区分，即 $n \geqslant 1$。考虑 $r=\delta(p,a_1), s=\delta(q,a_1)$，显然，$a_2a_3\dots a_n$ 区分 $[r,s]$，但是 $a_2a_3\dots a_n$ 比任何能区分 bad pair 的 string 都短，故 $[r,s]$ 不是 bad pair，算法将其标记为 distinguishable。于是根据 Induction 的部分，算法发现 $[p,q]$ 是 distinguishable，因为 $r=\delta(p,a_1), s=\delta(q,a_1)$ 是 distinguishable。矛盾。故原命题正确

> 对于 Equivalence problem，也可以通过这个算法解决。若判断两个 regular languages L,M 是否等价，可以将其状态合并，即 $Q\cup R$，然后对其应用 table-filling algorithm
>
> $L=M \iff$ 两个 DFA 的开始状态 q_0,r_0 等价

Minimize DFA

要最小化一个 DFA，首先要合并所有不可区分的状态

状态的合并可按照如下步骤

- 假设 $q_1,q_2,\dots q_k$ 是不可区分的
- 将其用一个代表状态 q 代替
- 显然 $\delta(q_1,a),\delta(q_2,a),\dots\delta(q_k,a)$ 也是不可区分的（否则其中必有至少一对被算法标记为 distinguishable）
- 令 $\delta(q,a)$ 为 $\{p:p=\delta(q_i,a), i=1,2,\dots k\}$ 的代表元素

合并之后，还要去掉所有从开始状态不可达的状态

事实上，状态的等价是一种等价关系，所以所有等价的状态将状态集划分为多个不相交的等价类

> The equivalence of states is transitive
>
> Proof. 假设 $[p,q]$ 与 $[q.r]$ 是等价的，但 $[p,r]$ 不等价。根据定义，存在一个输入 string w 满足 $\delta(p,w)$ 与 $\delta(r,w)$ 中有且仅有一个是接收状态
>
> w. l. o. g. $\delta(p,w)$ 是接收状态。考虑 $\delta(q,w)$，若其为接收状态，$[q,r]$ 为 distinguishable，若其为非接受状态，$[p,q]$ 为 distinguishable。均与假设矛盾

故最小化 DFA 即为

- 消去所有从开始不可达的状态
- 将状态根据等价关系（由 table-filling algorithm 得到）划分为等价类
- 将含有不止一个状态的等价类合并为一个代表状态
- 合并状态转移

包含开始状态的等价类为新 DFA 的开始状态，包含接受状态的等价类为新 DFA 的接收状态

Minimized DFA can't be beaten

> If A is a DFA, and M the DFA constructed from A by the algorithm of minimizing DFA, then M has as few states as any DFA equivalent to A

事实上，对于一个 DFA A ，其所有最小的 DFA 都是同构的

> Proof. 假设 A 是一个 DFA，最小化其得到 DFA M，假设存在一个 DFA N 与 A, M 接受一样的 language，但有更少的状态。考虑 M, N 组合的 DFA，首先
>
> - M, N 的开始状态是等价的，因为 $L(M) = L(N)$
> - 若状态 $[p, q]$ 等价，则 $[\delta(p, a), \delta(q, a)]$ 等价
>
> > 对于每个 M 中的状态 q ， 其与 N 中至少一个状态等价：
> >
> > Proof: by induction
> >
> > Basis. 如果 q 是 M 的开始状态，其与 N 的开始状态等价
> >
> > 假设从 M 的开始状态到达 q 的最短路径长度为 k
> >
> > I. H. 若从 M 的开始状态到达 q 的最短路径长度短于 k ，则存在一个 N 中的状态与其等价
> >
> > Induction. 假设从 M 的开始状态到 q 的最短路径的 string 为 $w = xa$ ，长为 k。假设从 M 开始状态出发沿 x 到达 r ，从 N 开始状态出发沿 x 到达 p ，根据 I. H. ，$[p, r]$ 等价。故 $[\delta_M(r, a), \delta_N(p, a)]$ 等价
>
> 由于根据前提，N 状态数少于 M，故至少有两个 M 中的状态与 N 中的同一个状态等价。因此这两个状态等价，这与 M 中所有状态都不等价矛盾，故前提错误，不存在这样的 N

Closure Properties of Regular Language

Closure Under Union

If L and M are regular languages, then so is $L \cup M$

Proof. 如果 L, M 为正则语言，则其有对应的 RE R, S ，即 $L = L(R), M = L(S)$ ，根据 RE 的定义，有 $L \cup M = L(R) \cup L(S) = L(R + S)$，显然 $L \cup M$ 为正则语言

Concatenation 与 Kleene Closure 的证明同理，根据 RE 的定义即可证明其封闭性：RS 是 LM 的 RE，R^* 是 L^* 的 RE

Closure Under Complementation

If L is a regular language over alphabet Σ , then $\overline{L} = \Sigma^* - L$ is also a regular language

Proof. Let $L = L(A)$ for some DFA $A = (Q, \Sigma, \delta, q_0, F)$, then $\overline{L} = L(B)$, where B is the DFA $(Q, \Sigma, \delta, q_0, Q - F)$. 即构造一个 DFA B 使得其所有接收状态都为 A 中的非接受状态，这样对于任意字符串 w 都有

$$w \in L(B) \iff \delta(q_0, w) \in Q - F \iff \delta(q_0, w) \notin F \iff w \notin L(A)$$
$$L(B) = \Sigma^* - L = \overline{L}$$

即 $\overline{L}$ 是由 DFA B 定义的正则语言

Closure Under Insertion

If L and M are regular languages, then so is $L \cap M$

Proof. 可以通过 De Morgan‘s Law 以及 Complementation 和 Union 的封闭性证明

$$L \cap M = \overline{\overline{L} \cup \overline{M}}$$

也可通过直接构造 DFA 证明。设 L, M 对应的 DFA 为 A, B ，则构造 product DFA C ，其中

$$[p, q] \in F_C \iff p \in F_A \text{ and } q \in F_B$$

这样对于任意字符串 w 有

$$\begin{aligned} w \in L(C) &\iff \delta([p_0, q_0], w) = [p, q] \\ &\iff \delta(p_0, w) = p \text{ and } \delta(q_0, w) = q \\ &\iff w \in L(A) \text{ and } w \in L(B) \\ &\iff w \in L(A) \cap L(B) \end{aligned}$$

则有 $L \cap M = L(C)$

> 已知 $L_1 = \{0^n 1^n : n \geqslant 0\}$ 不是 RE
>
> L_2 为有相等个 0 和 1 的串的集合，则 L_2 不是 RE
>
> Proof. 若 L_2 是 RE，则 $L_2 \cap L(0^* 1^*) = L_1$ 为 RE，矛盾，故 L_2 不是 RE

Closure Under Difference

If L and M are regular languages, then so is $L - M$

Proof. $L - M = L \cap \overline{M}$ ，根据 complementation 与 insertion 的封闭性，$L - M$ 也是 RE

Closure Under Reversal

对于一个字符串 $w = a_1 a_2 \ldots a_n$ ，其 reversal $w^R = a_n a_{n-1} \ldots a_1$

对于一个语言 L ，$L^R = \{w^R : w \in L\}$

If L is a regular language, so is L^R

> Automaton-based proof
>
> 对于 L 的 DFA A
>
> 1. 将 A 的转换图中的边反转
> 2. 令 A 的开始状态为唯一的接收状态
> 3. 添加一个开始状态 q_0 ，以及从 q_0 到其他接收状态的 ϵ-transition
>
> 则 $w^R \in L(A^R) \iff w \in L(A)$

RE-based Proof. 假设 L 的 RE 为 E，存在一个 RE E^R 使得 $L(E^R) = (L(E))^R$。对 E 的长度归纳

Basis. $\{\epsilon\}^R = \{\epsilon\}, \{a\}^R = \{a\}$

Induction.

- $E = E_1 + E_2$, then $E^R = E_1^R + E_2^R$
- $E = E_1 E_2$, then $E^R = E_2^R E_1^R$
- $E = E_1^*$, then $E^R = (E_1^R)^*$

Closure Under Homomorphisms

A string homomorphism is a **function** on strings that works by substituting a particular string for each symbol

If h is a homomorphism on alphabet Σ, and $w = a_1 a_2 \dots a_n$ is a string of symbols in Σ ,then $h(w) = h(a_1)h(a_2)\dots h(a_n)$

For language L

$$h(L) = \{h(w) : w \in L\}$$

If L is a regular language over alphabet Σ , and h is a homomorphism on Σ, then $h(L)$ is also regular.

Proof. 令 R 为 L 的 RE，对 R 应用 h 得到 $h(R)$，则 $h(R)$ 定义语言 $h(L)$ ，证明为对 R 的子表达式 E 归纳证明 $h(L(E)) = L(h(E))$

Basis.

如果 E 是 ϵ 或 $\varnothing$ ，则 $h(E) = E$ ，因为 h 不影响 ϵ 或 $\varnothing$ 。显然，$L(h(E)) = L(E)$ 。又因为 $L(E)$ 为 $\{\epsilon\}$ 或 $\{\}$ ，则 $h(L(E)) = L(E)$ ，故 $L(h(E)) = h(L(E))$

如果 E 是 a， $L(E) = \{a\}$ ，根据定义，$h(L(E)) = \{h(a)\}$ ，同理，$h(E)$ 是符号 $h(a)$ 组成的 string，则 $L(h(E)) = \{h(a)\}$ ，故 $L(h(E)) = h(L(E))$

Induction.

Union: 根据定义，有 $E = F + G, h(E) = h(F + G) = h(F) + h(G)$ ，根据 + 的定义

$$
\begin{gathered}
L(E) = L(F) \cup L(G) \\
L(h(E)) = L(h(F) + h(G)) = L(h(F)) \cup L(h(G))
\end{gathered}
$$

又因为 h 独立地应用于语言中的每个字符串，有

$$
h(L(E)) = h(L(F) \cup L(G)) = h(L(F)) \cup h(L(G))
$$

根据 I. H.， $L(h(F)) = h(L(F)), L(h(G)) = h(L(G))$ ，故

$$
h(L(E)) = L(h(E))
$$

Concatenation 与 Kleene star 的证明类似

可以证明将 h 应用于一个语言 L 的 RE 可以得到一个定义了语言 $h(L)$ 的 RE

Closure Under Inverse Homomorphisms

Suppose h is a homomorphism from alphabet Σ to strings in another (possibly the same) alphabet T. Let L be a language over alphabet T. Then

$$
h^{-1}(L) = \{w : w \in \Sigma^*, h(w) \in L\}
$$

If h is a homomorphism from alphabet Σ to alphabet T, and L is a regular language of T, then $h^{-1}(L)$ is also a regular language

Proof. 令 $L = L(A)$ ，其中 $A = (Q, T, \delta, q_0, F)$ 。定义 DFA

$$
B = (Q, \Sigma, \gamma, q_0, F)
$$

其中转换函数定义为 $\gamma(q, a) = \delta(q, h(a))$

B 对于输入 a 的转换的结果是 A 对于输入串 $h(a)$ 的转换结果。根据字符串 w 的长度归纳，可以轻松地得出 $\gamma(q_0, w) = \delta(q_0, h(w))$ ，由于 A, B 的接收状态相同，故

$$
\begin{gathered}
w \in L(B) \iff h(w) \in L(A) \\
L(B) = \{w : w \in \Sigma^*, h(w) \in L\} = h^{-1}(L)
\end{gathered}
$$