Optimal choice of quantization axis

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There are two target parameters that one wants to optimize when using a magic frequency to implement the birefringence effect for QND measurements:

① the effective coupling strength $\chi_{eff} = (\chi_{H\uparrow} - \chi_{H\downarrow}) - (\chi_{V\uparrow} - \chi_{V\downarrow}) = 2(\chi_{H\uparrow} - \chi_{H\downarrow})|_{\omega=\omega_{magic}}$, which determines the measurement backaction strength and the rate of spin-squeezing due to the measurement backaction.

② the peak value of squeezing parameter $\xi = 2j\langle \hat{N}_A\rangle \frac{\Delta J_\perp^2}{\langle J_\parallel\rangle^2}$, which is the ultimate goal of the QND measurement purpose and is a modification based on the optimization result of χ_{eff} considering the decoherence process.

Since the final optimal choice of quantization axis towards parameter ② is based on optimizing ①, we consider the optimization target of parameter ① first.

Based on Ivan's simplification for the clock-state subspace we are considering, the tensor angular momentum element

$$\langle F,0|[(\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\hat{\vec{F}}^2\delta_{ij}]|F,0\rangle = \langle F,0|\hat{U}_z\hat{U}_z^+[\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\hat{\vec{F}}^2\delta_{ij}]\hat{U}_z\hat{U}_z^+|F,0\rangle$$

$$=\langle F,0|\hat{U}_z^+[\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\hat{\vec{F}}^2\delta_{ij}]\hat{U}_z|F,0\rangle$$

where $\hat{U}_z = e^{i\theta\hat{F}_z/\hbar}$ is an arbitrary rotation along z-axis. Therefore $[\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\delta_{ij}\hat{\vec{F}}^2]$ is invariant under z rotations, and all non-diagonal elements with x,y subscripts in the x-y-z basis or with ± subscripts in the $\sigma_\pm$-π basis are zero.

Further using $\langle F,0|\hat{F}_z|F,0\rangle=0$, $\langle F,0|\hat{F}_x|F,0\rangle=\langle F,0|\hat{F}_y|F,0\rangle$ and $\langle F,0|\hat{\vec{F}}^2|F,0\rangle=F(F+1)$, one can get

$$\langle F,0|[\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\hat{\vec{F}}^2\delta_{ij}]|F,0\rangle = F(F+1)[\frac{\delta_{ij}}{6}-\frac{2}{3}\delta_{iz}\delta_{jz}]$$

⟹ In terms of irreducible tensor components, the coupling strength can be written as

$$\chi_{\mu,F} = -\frac{2\pi\omega_0}{v_g}\vec{u}_\mu^*(\vec{r}_\perp')\cdot\langle F,0|\overleftrightarrow{\alpha}|F,0\rangle\cdot\vec{u}_\mu(\vec{r}_\perp')$$

$$= n_g\sigma_0\sum_{F'}\{C^{(0)}_{J'FF'}|u_\mu(\vec{r}_\perp')|^2 + C^{(2)}_{J'FF'}\sum_{i,j}\langle F,0|[\frac{\hat{F}_i\hat{F}_j+\hat{F}_j\hat{F}_i}{2}-3\hat{\vec{F}}^2\delta_{ij}]|F,0\rangle\cdot u_i^*(\vec{r}_\perp')u_j(\vec{r}_\perp')\}\frac{\Gamma}{4\Delta_{F'F}}$$

$$= n_g\sigma_0\sum_{F'}\{C^{(0)}_{J'FF'}|u_\mu(\vec{r}_\perp')|^2 + C^{(2)}_{J'FF'}F(F+1)[\frac{|u_\mu(\vec{r}_\perp')|^2}{6}-\frac{1}{2}|\hat{e}_\pi\cdot\vec{u}_\mu(\vec{r}_\perp')|^2\}\frac{\Gamma}{4\Delta_{F'F}}$$

$$= n_g\sigma_0\sum_{F'}\{[C^{(0)}_{J'FF'}+C^{(2)}_{J'FF'}\frac{F(F+1)}{6}]|\vec{u}_\mu(\vec{r}_\perp')|^2 - C^{(2)}_{J'FF'}\frac{F(F+1)}{2}|\hat{e}_\pi\cdot\vec{u}_\mu(\vec{r}_\perp')|^2\}\frac{\Gamma}{4\Delta_{F'F}}.$$

To satisfy the magic wavelength condition that

$$\chi_{H\uparrow}-\chi_{H\downarrow} = \chi_{V\downarrow}-\chi_{V\uparrow},$$

we will have

$$\sum_{F'}\{[C^{(0)}_{J'4F'}+\frac{5}{3}C^{(2)}_{J'4F'}]\frac{1}{\Delta_{4F'}} - [C^{(0)}_{J'3F'}+2C^{(2)}_{J'3F'}]\frac{1}{\Delta_{3F'}}\}\cdot[|\vec{u}_H(\vec{r}_\perp')|^2+|u_V(\vec{r}_\perp')|^2]$$

$$=\sum_{F'}[\frac{10C^{(2)}_{J'4F'}}{\Delta_{4F'}} - \frac{6C^{(2)}_{J'3F'}}{\Delta_{3F'}}]\cdot[|\hat{e}_\pi\cdot\vec{u}_H(\vec{r}_\perp')|^2+|\hat{e}_\pi\cdot\vec{u}_V(\vec{r}_\perp')|^2]$$

$$\Longleftrightarrow \sum_{F'}(\frac{A_{4F'}}{\Delta_{4F'}}-\frac{A_{3F'}}{\Delta_{3F'}})[|\vec{u}_H(\vec{r}_\perp')|^2+|\vec{u}_V(\vec{r}_\perp')|^2] = \sum_{F'}(\frac{B_{4F'}}{\Delta_{4F'}}-\frac{B_{3F'}}{\Delta_{3F'}})\cdot[|\hat{e}_\pi\cdot\vec{u}_H(\vec{r}_\perp')|^2+|\hat{e}_\pi\cdot\vec{u}_V(\vec{r}_\perp')|^2]$$

with

$$\begin{cases} A_{FF'} = C^{(0)}_{J'FF'} - \frac{F(F+1)}{6}C^{(2)}_{J'FF'} \\ B_{FF'} = \frac{F(F+1)}{\ } C^{(2)} \end{cases}$$

with $\begin{cases} A_{FF'} = C^{(0)}_{J'FF'} - \frac{F(F+1)}{6} C^{(2)}_{J'FF'} \\ B_{FF'} = \frac{F(F+1)}{2} C^{(2)}_{J'FF'} . \end{cases}$

$$\Rightarrow \sum_{F'} \left(\frac{A_{4F'}}{\Delta_{4F'}} - \frac{A_{3F'}}{\Delta_{3F'}} \right) = \frac{|\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 + |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2}{|\vec{u}_H(\vec{r}_I')|^2 + |\vec{u}_V(\vec{r}_I')|^2} \sum_{F'} \left(\frac{B_{4F'}}{\Delta_{4F'}} - \frac{B_{3F'}}{\Delta_{3F'}} \right).$$

The effective coupling strength at a magic frequency is then

$$\chi_{eff} = 2(\chi_{H\uparrow} - \chi_{H\downarrow})$$

$$= 2\left\{ \sum_{F'} \left[\frac{A_{4F'}}{\Delta_{4F'}} - \frac{A_{3F'}}{\Delta_{3F'}} \right] \cdot |\vec{u}_H(\vec{r}_I')|^2 - \sum_{F'} \left[\frac{B_{4F'}}{\Delta_{4F'}} - \frac{B_{3F'}}{\Delta_{3F'}} \right] \cdot |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 \right\} \frac{\Gamma}{4} n_g \sigma_0$$

$$= 2\left[\frac{|\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 + |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2}{|\vec{u}_H(\vec{r}_I')|^2 + |\vec{u}_V(\vec{r}_I')|^2} \cdot |\vec{u}_H(\vec{r}_I')|^2 - |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 \right] \sum_{F'} \left(\frac{B_{4F'}}{\Delta_{4F'}} - \frac{B_{3F'}}{\Delta_{3F'}} \right) \frac{\Gamma}{4} n_g \sigma_0$$

$$= -2n_g \underbrace{\left[\frac{|\vec{u}_V(\vec{r}_I')|^2}{|\vec{u}_H(\vec{r}_I')|^2 + |\vec{u}_V(\vec{r}_I')|^2} |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 - \frac{|\vec{u}_H(\vec{r}_I')|^2}{|\vec{u}_H(\vec{r}_I')|^2 + |\vec{u}_V(\vec{r}_I')|^2} |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2 \right]}_{\text{quantization-axis dependent factor } Q_\pi.} \cdot \underbrace{\sum_{F'} \left(\frac{B_{4F'}}{\Delta_{4F'}} - \frac{B_{3F'}}{\Delta_{3F'}} \right) \frac{\Gamma}{4}}_{\text{atomic structure-dependent factor } B. \sim \frac{\Gamma}{4\Delta_{eff}}} \sigma_0$$

Discussions:

<1> Regarding factor B:

B is detuning-dependent relative to the hyperfine splitting.

① For $\Delta >> \Delta^{min}_{F_1'F_2'}$ (detuning is far greater than hyperfine splitting),

$$B \doteq \frac{\Gamma}{4\Delta}\left[\sum_{F'}(B_{4F'} - B_{3F'})\right] = \frac{\Gamma}{4\Delta}\left[\sum_{F'}\left(10 C^{(2)}_{J'4F'} - 6 C^{(2)}_{J'3F'}\right)\right] = 0.$$

where we have used the fact that $\sum_{F'} C^{(2)}_{J'FF'} = 0$.

In fact, to obtain the magic frequency, we must couple $C^{(2)}_{J'FF'}$ with $\frac{1}{\Delta_{FF'}}$, which requires $\Delta \sim \Delta^{min}_{F_1'F_2'}$.

② For $\Delta \sim \Delta^{min}_{F_1'F_2'}$, we have two magic wavelengths which are close to $\omega_{33'}$ or $\omega_{44'}$.

It also ensures $B \neq 0$, yet B could be negative for the case of using some quantization-axis ($\Delta_{44'} < 0$, $\Delta_{33'} > 0$).

<2> Regarding the factor of Q_π:

$$Q_\pi = R_1 |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 - R_2 |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2 = R_1 x - R_2 y$$

with $R_1 > 0$, $R_2 > 0$, $R_1 - R_2 = 1$. In our case, $R_1 > R_2$.

We also defined $\begin{cases} x \equiv |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 = \hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I') \vec{u}_H^*(\vec{r}_I') \cdot \hat{e}_\pi^* = \hat{e}_\pi \cdot \left(\sum_{i=1,2} g_H^i \hat{e}_H^i \hat{e}_H^{i*}\right) \cdot \hat{e}_\pi^* > 0 \\ y \equiv |\hat{e}_\pi \cdot |\vec{u}_V(\vec{r}_I')|^2 = \hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I') \vec{u}_V(\vec{r}_I') \cdot \hat{e}_\pi^* = \hat{e}_\pi \cdot \left(\sum_{i=1} g_V \hat{e}_\phi \hat{e}_\phi^*\right) \cdot \hat{e}_\pi^* > 0 \end{cases}$

where the positive semidefinit tensor $\vec{u}_H(\vec{r}_I')\vec{u}_H^*(\vec{r}_I')$ is diagonalized as $\sum_{i=1,2} g_H^i \hat{e}_H^i \hat{e}_H^{i*}$ with eigenvectors on the x-z plane ($\hat{e}_i$) and nonegtive eigenvalues g_H^i. Similarly, $\vec{u}_V(\vec{r}_I)\vec{u}_V^*(\vec{r}_I) = g_V^i \hat{e}_\phi \hat{e}_\phi^*$ with eigenvector along $\hat{\phi}$.

So, $Q_\pi = Q_\pi(x, y)$ is guaranteed to reach an extrem value when either x or y is zero and the other is using its eigenvalue, since eigenvectors of $\vec{u}_H(\vec{r}_I)\vec{u}_H^*(\vec{r}_I')$ and $\vec{u}_V(\vec{r}_I)\vec{u}_V^*(\vec{r}_I')$ are guaranteed to be orthogonal.

That is when we choose the eigenvector directions as the quantization axis, we can find the maximal value of the absolute value of $Q_{\sigma\pi}$. The absolute value of Q_{π} is determined by the eigenvalues of g_H^i & g_V^i.

Maybe wrong, as we are solving for λ_{magic}.

Since the equation to determine the magic wavelength & B is also determined by

$$B = \frac{|\vec{u}_H(\vec{r}_I')|^2 + |\vec{u}_V(\vec{r}_I')|^2}{|\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 + |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2} \sum_{F'} \left(\frac{A_{4F'}}{\Delta_{4F'}} - \frac{A_{3F'}}{\Delta_{3F'}}\right)\frac{\Gamma}{4}$$

Since $x + y = |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_I')|^2 + |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_I')|^2$ is convex with respect to $\hat{e}_\pi$ changing from one eigenvector to another for $\vec{u}_H(\vec{r}_I') \perp \vec{u}_V(\vec{r}_I')$ $(\phi = 0, \pi)$, one could expect B reaches extremes when $x+y$ is minimized when $\hat{e}_\pi$ pointing to the eigenvector directions.

To the end, the optimal choice of quantization axis may be determined by the condition that

$$\frac{\mathrm{Max}\,[R_1 x - R_2 y]}{\mathrm{Min}\,[x + y]}$$

not sure about this part.

which should be able to confirmed with numerics, and should be along any of the eigenvectors $\{g_H^2, g_V^1\}$.

Numerics:

For $r_I' = 1.5a$, $a = 225\,nm$, one can find that

$$\begin{cases} g_H^1 = 6.1792 \times 10^{11}, & \hat{e}_H^1 = -0.8996 i \hat{e}_x + 0.4368 \cdot \hat{e}_z \\ g_H^2 = -4.1831 \times 10^{-5} \simeq 0, & \hat{e}_H^2 = -0.4368 i \cdot \hat{e}_x - 0.8996 \cdot \hat{e}_z \\ g_V^1 = 3.3222 \times 10^{11}, & \hat{e}_V^1 = -i \hat{e}_y. \end{cases}$$

direction of the H mode $\hat{e}_H = \frac{\vec{u}_H(\vec{r}_I')}{|\vec{u}_H(\vec{r}_I')|}$.

So, if we use $\hat{e}_{\sigma\pi} = \hat{e}_H^{1*} = 0.8996 i \hat{e}_x + 0.4368 \hat{e}_z = -i\hat{e}_H$,

we can have the maximum/minimum value of $Q_{\sigma\pi}$.

However, a quantization axis cannot be complex, and we want to find the real-space counterpart of those complex vectors. Below is how to find the real-number one.

We show that the optimal quantization axis satisfies

$$|\hat{e}_\pi \cdot \hat{e}_H^1| = |\hat{e}_\pi| \cdot |\hat{e}_H^1| = 1$$

we let $\hat{e}_H^1 = [ia, b]$ in the $\{\hat{e}_x, \hat{e}_z\}$ basis, with $a^2 + b^2 = 1$.

We let a real vector $\hat{e}_{\sigma\pi} = [c, d]$ in the same basis, with $c^2 + d^2 = 1$.

$\Rightarrow$ when $\hat{e}_{\sigma\pi}$ is optimal,

$$|\hat{e}_\pi \cdot \hat{e}_H^1|^2 = |iac + bd|^2 = a^2c^2 + b^2d^2 = 1$$

$$\Leftrightarrow \quad a^2c^2 + (1-a^2)(1-c^2) = 1$$

$$\Leftrightarrow \quad 2a^2c^2 = a^2 + c^2$$

$$\Rightarrow \begin{cases} c^2 = \dfrac{a^2}{2a^2-1} = \dfrac{a^2}{a^2-b^2} \\ d^2 = \dfrac{-b^2}{a^2-b^2} \end{cases}$$

$\{1 = c^2 = a^2,\ d^2 = b^2 = 0;$ or.

Therefore, depending on the value of a^2 & b^2, either c^2 or d^2 is negative, if $b \neq 0$.
Seems there is no real vector that can give the same maximum value of $|\hat{e}_\pi \cdot \hat{e}_H|$ as the complex vector $\hat{e}_\pi = \hat{e}_H^{*}$ applies.

To find the optimal quatization axis and ensure the axis is along a real vector, we can do the following.

$$\begin{aligned} |\hat{e}_\pi \cdot \hat{e}_H|^2 &= a^2c^2 + b^2d^2 \\ &= a^2c^2 + (1-a^2)(1-c^2) = 1 + 2a^2c^2 - (a^2+c^2) \\ &= 1 - a^2 + (2a^2-1)c^2 \\ &\leq b^2 + (2a^2-1) = a^2 < 1. \end{aligned}$$

The maximum is reached when $c = \pm 1$, and hence $d = 0$. Or, $\hat{e}_\pi = [\pm 1, 0]$.
That is when the quantization is pointing along x-axis.