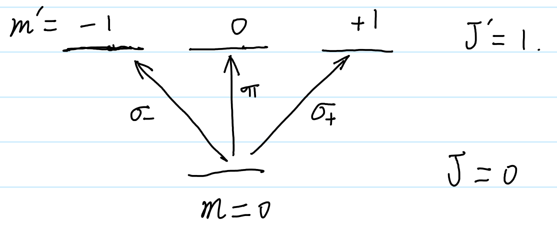
Decay rates for a simple case

Tuesday, December 03, 2013 10:20 AM

131203-1 We consider a relative simple case as below



We want to show the equivalence between the theoretical model presented in Ref [Kien2005spontaneous] and Ref [Klimov2004spontaneous].

First, we use Kien's model to calculate the total decay rates of Γ_{ee}, where $e \to \begin{cases} m' = \pm 1, 0, \\ J' = 1. \end{cases}$ yet $g \to \begin{cases} m = 0 \\ J = 0. \end{cases}$ is unique.

From Kien's paper, we have

$$\Gamma_{ee} = \gamma_{ee}^{(g)} + \gamma_{ee}^{(r)},$$

$$G_{\mu eg} = \sqrt{\frac{\omega\beta'}{4\pi\epsilon_0\hbar}}\,(\vec{d}_{eg}\cdot\vec{\mathcal{E}}^{(\mu)})\,e^{i(f\beta z + p\varphi)}$$

$$\mu_0 = (\omega_0, f, p).$$

with

$$\gamma_{ee}^{(g)} = 2\pi\sum_{fpg} G_{\mu_0 eg} G^*_{\mu_0 eg}$$

$$= \sum_{fp}\frac{\omega\beta'}{2\epsilon_0\hbar}\left(\vec{d}_{eg}\cdot\vec{\mathcal{E}}^{(\omega_0,f,p)}\right)\left(\vec{d}_{eg}\cdot\underbrace{\vec{\mathcal{E}}^{(\omega_0,f,p)}}_{\text{mode vector}}\right)^* \qquad (1)$$

Now that

$$\vec{d}_{eg} = \langle e|\hat{\vec{D}}|g\rangle e^{-i\omega_0 t}$$

$$= \langle e|-e\hat{\vec{r}}\,|g\rangle$$

$$= -e\langle J'=1, m'|\hat{\vec{r}}\,|J=0, m=0\rangle \qquad (2)$$

Using the fact that $\hat{r}$ is a first-order tensor operator and the Wigner–Eckart theorem that

$$\langle J'm'|T_q^{(k)}|Jm\rangle = (-1)^{2k}\langle J'\|T^{(k)}\|J\rangle\langle J'm'|Jm, kq\rangle$$

we have

$$\langle J'=1, m'|\hat{r}_q|J=0, m=0\rangle$$

$$= \langle J'=1\|\hat{r}\|J=0\rangle\langle J'=1, m'|J=0, m=0, 1, q\rangle$$

$$= \langle J'=1\|\hat{r}\|J=0\rangle\,\delta_{m'q} \qquad (3)$$

where $q = 0, \pm 1$ is the spherical tensor component index with

$$\begin{cases} \hat{r}_0 = \hat{r}_z \\ \hat{r}_1 = -\frac{1}{\sqrt{2}}(\hat{r}_x + i\hat{r}_y) \\ \hat{r}_{-1} = \frac{1}{\sqrt{2}}(\hat{r}_x - i\hat{r}_y). \end{cases} \qquad (4)$$

Therefore, we can rewrite

— every polarization is identical.

$(r_{-1} = \frac{1}{\sqrt{2}}(r_x - i r_y))$.

Therefore, we can rewrite

$$\vec{d}_{eg} = \vec{d}_{(J'=1,m';\,J=0,m=0)}$$

$$= d\,(\delta_{m',-1}\,\vec{e}_{-}^{\,*} + \delta_{m',0}\,\vec{e}_{0}^{\,*} + \delta_{m',1}\,\vec{e}_{+}^{\,*}) \qquad (5)$$

every polarization is identical.

with ~~the~~ reduced dipole momentum

$$d = \langle J'=1 \| -e\hat{r} \| J=0 \rangle$$

and spherical vector bases

$$\begin{cases} \vec{e}_0 = \vec{e}_z \\ \vec{e}_- = \frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y) \\ \vec{e}_+ = -\frac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y). \end{cases} \qquad (6)$$

Here, the spherical vector basis is defined in the form that for any vectors $\vec{A}$ & $\vec{B}$,

$$\begin{cases} A_q = \vec{e}_q \cdot \vec{A} \\ \vec{A} = \sum_q A_q \vec{e}_q^{\,*} = \sum_q (-1)^q A_q \vec{e}_{-q}^{\,*} = A_+\vec{e}_+^{\,*} + A_0\vec{e}_0 + A_-\vec{e}_-^{\,*} \\ \vec{A}\cdot\vec{B} = -A_1 B_{-1} + A_0 B_0 - A_{-1} B_1 = \sum_q (-1)^q A_q B_{-q}. \end{cases}$$

From the notes of PHYS 531.

We plug all the results above into ①, we will obtain

$$\gamma_{ee}^{(g)} = \gamma_{m'm'}^{(g)}$$

$$= \sum_{f,p} \frac{\omega\beta'}{2\epsilon_0\hbar} \left(\vec{d}_{eg}\cdot\vec{\mathcal{E}}^{(\omega_0,f,p)}\right)\cdot\left(\vec{d}_{eg}\cdot\vec{\mathcal{E}}^{(\omega_0,f,p)}\right)^*$$

$$= \sum_{f,p} \frac{\omega\beta'}{2\epsilon_0\hbar} \sum_{q,q'} (-1)^{q+q'} |d|^2 \delta_{m'q}\delta_{m'q'}^{*}\, \mathcal{E}_{-q}^{(\omega_0,f,p)} \mathcal{E}_{-q'}^{*(\omega_0,f,p)}$$

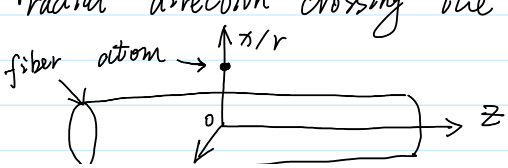
$$= \sum_{f,p} \frac{\omega\beta'}{2\epsilon_0\hbar} |d|^2 \mathcal{E}_{-m'}^{(\omega_0,f,p)} \mathcal{E}_{-m'}^{*(\omega_0,f,p)} \qquad (7)$$

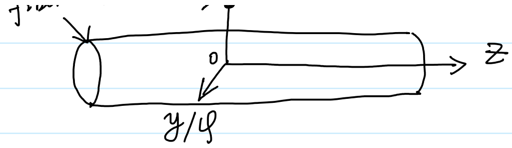
This result shows that each diagonal element of the decay rates only couples to one particular orientation of dipole momentum and field, up to a minus sign (does not matter for the final result).

For example, the decay element γ_{00} ($m'=0$) is only related to the z-orientated dipole and the z-component of the field.

The decay rates $\gamma_{11} = \gamma_{-1,-1}$ relate to the "$\pm$" spherical components of the dipole momentum and "$\mp$" components of the field.

We define the light propagation direction in the nanofiber is z-axis, the radial direction crossing the atom is x-axis as shown below,





in a classical picture, we have

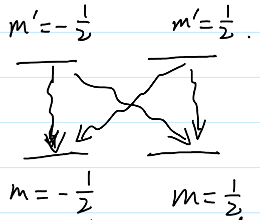
$$\begin{cases} d_{\pm1} = \mp \frac{1}{\sqrt{2}} (d_{r_\perp} \pm i d_\varphi) \\ \mathcal{E}_{\pm1} = \mp \frac{1}{\sqrt{2}} (\mathcal{E}_{r_\perp} \pm i \mathcal{E}_\varphi). \end{cases}$$

$$d_{\pm1} \cdot \mathcal{E}_{\mp} = \frac{-1}{2} (d_{r_\perp} \mathcal{E}_{r_\perp} \mp i d_{r_\perp} \mathcal{E}_\varphi \pm i d_\varphi \mathcal{E}_{r_\perp} + d_\varphi \mathcal{E}_\varphi).$$

This correlation relationship between dipole orientation & decay rate of γ_{ee} can be applied to Klimov's model for a equivalence verification against Kien's method.

Similarly, the radiation decay rates can be given in a similar form except there is an integral with respect to β from $-kn_2$ to kn_2.

140107-1. Next, we consider another atomic configuration as shown below:



$|e\rangle \to |J' = \frac{1}{2}, m' = \pm\frac{1}{2}\rangle$

$|g\rangle \to |J = \frac{1}{2}, m = \pm\frac{1}{2}\rangle.$

Now that both excited and ground states have total angular momentum $J = J' = \frac{1}{2}$.

Therefore, the dipole matrix element

$$\begin{aligned} \vec{d}_{eg} &= \langle e | \hat{\vec{D}} | g \rangle e^{-i\omega_0 t} \\ &= \langle J' = \tfrac{1}{2}, m' | -e\hat{\vec{r}} | J = \tfrac{1}{2}, m \rangle \\ &= \sum_{q=\pm1,0} \langle J' = \tfrac{1}{2} \| -e\hat{r} \| J = \tfrac{1}{2} \rangle \langle J' = \tfrac{1}{2}, m' | J = \tfrac{1}{2}, m, 1, q \rangle \vec{e}_q^{\,*} \\ &= \sum_{q=\pm1,0} (-1)^{-\frac{1}{2}+m'} \langle J' = \tfrac{1}{2} \| -e\hat{r} \| J = \tfrac{1}{2} \rangle \sqrt{2} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m & q & -m' \end{pmatrix} \vec{e}_q^{\,*} \\ &= \sum_{q=\pm1,0} (-1)^{m'-\frac{1}{2}} \sqrt{2} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m & q & -m' \end{pmatrix} d . \vec{e}_q^{\,*} \end{aligned}$$

where I have used

3j-symbol.

$$\langle J_3, m_3 | J_1, m_1, J_2, m_2 \rangle = (-1)^{J_1 - J_2 + m_3} \sqrt{2J_3+1} \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & -m_3 \end{pmatrix},$$

$$d = \langle J' = \tfrac{1}{2} \| -e\hat{r} \| J = \tfrac{1}{2} \rangle.$$

Equation ① can now be written as

$$\gamma_{ee}^{(g)} = \sum_{f,p,q} \frac{\omega\beta'}{2\epsilon_0\hbar} (\vec{d}_{eg} \cdot \vec{\mathcal{E}}^{(\omega_0,f,p)}) \cdot (\vec{d}_{eg} \cdot \vec{\mathcal{E}}^{(\omega_0,f,p)})^*$$

Equation ④ can now be written as

$$\gamma_{ee}^{(g)} = \sum_{f,p,g} \frac{\omega\beta'}{2\epsilon_0\hbar} \left(\overrightarrow{d_{eg}}\cdot\vec{\mathcal{E}}^{(\omega_0,f,p)}\right)\cdot\left(\overrightarrow{d_{eg}}\cdot\vec{\mathcal{E}}^{(\omega_0,f,p)}\right)^*$$

$$= \sum_{f,p,m} \frac{\omega\beta'}{2\epsilon_0\hbar} \left| \sum_{q=\pm1,0} (-1)^{q+m'-\frac{1}{2}}\sqrt{2} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m & q & -m' \end{pmatrix} d\cdot \mathcal{E}_{-q}^{(\omega_0,f,p)} \right|^2$$

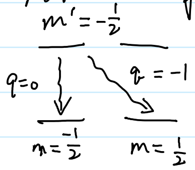
The 3-j symbol part is zero unless the following selection rules satisfy

$$\begin{cases} m_1+m_2+m_3 = m+q-m'=0 \\ J_1+J_2+J_3 = \frac{1}{2}+1+\frac{1}{2} = 2 \quad \text{is an integer (or an even integer if } m_1=m_2=m_3=0). \checkmark \\ |m_i| \le J_i \quad \checkmark \\ |J_1-J_2| \le J_3 \le J_1+J_2 \Leftrightarrow |1-\frac{1}{2}| \le \frac{1}{2} \le 1+\frac{1}{2} \quad \checkmark \end{cases}$$

Once we have $m=\pm\frac{1}{2}$, $q=\pm1, 0$, and $m'=\pm\frac{1}{2}$, all selection rules are satisfy, except for there is a constrain on $m+q-m'=0$.

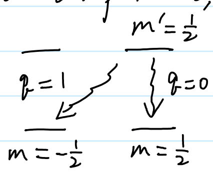
This result shows that, for a given component index q, there is a fixed m & m' correlation.

For example, if $m'=-\frac{1}{2}$,



$$\begin{cases} q=-1 \Rightarrow m=\frac{1}{2}, \text{ the 3-j symbol } \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & +\frac{1}{2} \end{pmatrix} = -\frac{\sqrt{3}}{3}, \text{ associated with } \mathcal{E}_1^{(\omega_0,f,p)} \\ q=0 \Rightarrow m=-\frac{1}{2}, \text{ the 3-j symbol } \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{6}}, \text{ associated with } \mathcal{E}_0^{(\omega_0,f,p)} \\ q=1 \Rightarrow m=-\frac{3}{2} \rightarrow \text{not exists.} \end{cases}$$

For comparison, if $m'=\frac{1}{2}$,



$$\begin{cases} q=-1 \Rightarrow m=\frac{3}{2} \Rightarrow \text{not exists} \\ q=0 \Rightarrow m=\frac{1}{2}, \text{ the 3-j symbol } \begin{pmatrix} \frac{1}{2}, & 1 & \frac{1}{2} \\ \frac{1}{2}, & 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{6}}, \text{ associated with } \mathcal{E}_0^{(\omega_0,f,p)} \\ q=1 \Rightarrow m=-\frac{1}{2}, \text{ the 3-j symbol } \begin{pmatrix} \frac{1}{2}, & 1, & \frac{1}{2} \\ -\frac{1}{2}, & 1, & -\frac{1}{2} \end{pmatrix} = -\frac{\sqrt{3}}{3}, \text{ associated with } \mathcal{E}_{-1}^{(\omega_0,f,p)} \end{cases}$$

For the two cases of $|e\rangle = |J'=\frac{1}{2}, m'=\pm\frac{1}{2}\rangle$, we have

$$\gamma_{m'=-\frac{1}{2},m'=-\frac{1}{2}}^{(g)} = \sum_{f,p} \frac{\omega\beta'}{\epsilon_0\hbar} \left| \frac{-1}{\sqrt{3}} d\cdot \mathcal{E}_1^{(\omega_0,f,p)} + \frac{1}{\sqrt{6}} d\cdot \mathcal{E}_0^{(\omega_0,f,p)} \right|^2$$

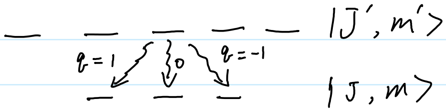
$$= \sum_{f,p} \frac{\omega\beta'}{\epsilon_0\hbar} |d|^2 \cdot \frac{1}{3} \left| -\mathcal{E}_1^{(\omega_0,f,p)} + \frac{1}{\sqrt{2}} \mathcal{E}_0^{(\omega_0,f,p)} \right|^2.$$

$$\gamma_{m'=\frac{1}{2},m'=\frac{1}{2}}^{(g)} = \sum_{f,p} \frac{\omega\beta'}{\epsilon_0\hbar} \cdot \frac{|d|^2}{3} \cdot \left| \frac{1}{\sqrt{2}} \mathcal{E}_0^{(\omega_0,f,p)} - \mathcal{E}_{-1}^{(\omega_0,f,p)} \right|^2.$$

The radiation decay rates can be written in the similar form yet with an integle w.r.t β.

140107-2. Now, we consider a general case that

$$\begin{cases} |e\rangle = |J', m'\rangle \\ |g\rangle = |J, m\rangle \end{cases}$$



We go through the same process as above, and obtain:

$$\overrightarrow{d_{eg}} = \sum (-1)^{J-1+m'} \sqrt{2J'+1} \begin{pmatrix} J & 1 & J' \\ m & q & m' \end{pmatrix} \vec{e}_q^{\,*}$$

process as above, and obtain:

$$\vec{d}_{eg} = \sum_{q=\pm1,0} (-1)^{J-1+m'} \sqrt{2J'+1} \begin{pmatrix} J & 1 & J' \\ m & q & -m' \end{pmatrix} \vec{e}_q^{\,*}$$

$$\gamma_{ee}^{(g)} = \sum_{f,p,g} \frac{\omega\beta'|d|^2}{2\epsilon_0\hbar} \Big| \underbrace{\sum_{q=\pm1,0} (-1)^{J-1+m'} \sqrt{2J'+1} \begin{pmatrix} J & 1 & J' \\ m & q & -m' \end{pmatrix}}_{d_q^{eg}} \varepsilon_{-q}^{(\omega_0,f,p)} \Big|^2$$

$d = \langle J' \| \hat{D} \| J \rangle$.

The selection rule gives

$$m - m' + q = 0$$

with $q = \pm 1, 0$.

If we consider the hyperfine structure of an atom, we only need to replace

$$d_{eg}^{(q)} = (-1)^{I+J'-M'} \langle J' \| \hat{D} \| J \rangle \sqrt{(2F+1)(2F'+1)} \cdot \underbrace{\begin{Bmatrix} J' & F' & I \\ F & J & 1 \end{Bmatrix}}_{\text{6-j symbol}} \cdot \begin{pmatrix} F & 1 & F' \\ M & q & -M' \end{pmatrix}$$

The radiation decay rates and other non-diagonal elements of the decay rate matrix can be obtained in a similar way.