Equation of motion of collective spins

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0410-1. Continued from the ScatteringNaTA. notes.

The master equation for the collective spin system can be written as

$$\frac{d\hat{\rho}}{dt} = \mathcal{D}[\hat{\rho}] = -\frac{i}{\hbar}[\hat{H}_{Loss}\hat{\rho} - \hat{\rho}\hat{H}^{+}_{Loss}] + \sum_{F_a,F_b,q,i} \hat{W}_q^{F_bF_a}(\vec{r}_i)\,\hat{\rho}\,\hat{W}_q^{F_bF_a+}(\vec{r}_i).$$

For an arbitrary atom, we can write

$$\frac{d\hat{\rho}^{(i)}}{dt} = \mathcal{D}^{(i)}[\hat{\rho}^{(i)}] = -\frac{i}{\hbar}[\hat{H}^{(i)}_{Loss}\hat{\rho}^{(i)} - \hat{\rho}^{(i)}\hat{H}^{(i)+}_{Loss}] + \sum_{F_a,F_b,q} \hat{W}_q^{F_bF_a}(\vec{r}_i)\,\hat{\rho}^{(i)}\,\hat{W}_q^{F_bF_a+}(\vec{r}_i).$$

Therefore, the expectation value of a one-body operator can be given by

$$\frac{d\langle\hat{O}^{(i)}\rangle}{dt} = \frac{d}{dt}tr(\hat{\rho}\,\hat{O}^{(i)}) = \frac{d}{dt}tr(\hat{\rho}^{(i)}\hat{O}^{(i)}) = tr(\frac{d}{dt}\hat{\rho}^{(i)}\cdot\hat{O}^{(i)})$$

$$= tr\Big(\big(-\frac{i}{\hbar}(\hat{O}^{(i)}\hat{H}^{(i)}_{Loss} - \hat{H}^{(i)+}_{Loss}\hat{O}^{(i)}) + \sum_{F_a,F_b,q} \hat{W}_q^{F_bF_a+}(\vec{r}_i)\,\hat{O}^{(i)}\,\hat{W}_q^{F_bF_a}(\vec{r}_i)\big]\cdot\hat{\rho}^{(i)}\Big)$$

$$= -\frac{i}{\hbar}\langle[\hat{O}^{(i)}\hat{H}^{(i)}_{Loss} - \hat{H}^{(i)+}_{Loss}\hat{O}^{(i)}]\rangle + \sum_{F_a,F_b,q}\langle \hat{W}_q^{F_bF_a+}(\vec{r}_i)\,\hat{O}^{(i)}\,\hat{W}_q^{F_bF_a}(\vec{r}_i)\rangle$$

$$= \langle\mathcal{D}^{(i)+}(\hat{O}^{(i)})\rangle$$

Deriving two-body operators' evolution requirs the master equation describing the evolution of any two atoms in the ensemble, i and j,

$$\frac{d\hat{\rho}^{(i,j)}}{dt} = \mathcal{D}^{(i)}[\rho^{(i,j)}] + \mathcal{D}^{(j)}[\hat{\rho}^{(i,j)}]$$

Then the expectation value of operators involving atoms i and j $(i\neq j)$ evolves as

$$\frac{d\langle\hat{a}^{(i)}\hat{b}^{(j)}\rangle}{dt} = tr\Big[\frac{d\hat{\rho}^{(i,j)}}{dt}\hat{a}^{(i)}\hat{b}^{(j)}\Big]\Big|_{i\neq j}$$

$$= \langle\mathcal{D}^{(i)+}(\hat{a}^{(i)})\,\hat{b}^{(j)}\rangle + \langle\hat{a}^{(i)}\mathcal{D}^{(j)+}(\hat{b}^{(j)})\rangle.$$

Using the relations above, one can calculate the evolution of expectation values of some collective psuedo-spin operators as follows in the clock-state subspace.

$$\frac{d\langle\hat{J}_z\rangle}{dt} = \sum_i^{N_A}\frac{1}{2}\frac{d\langle\hat{\sigma}_z^{(i)}\rangle}{dt} = \sum_i^{N_A}\frac{1}{2}\langle\mathcal{D}^{(i)+}(\hat{\sigma}_z^{(i)})\rangle$$

$$= \frac{1}{2}\sum_i^{N_A}\Big\{-\frac{i}{\hbar}\langle[\hat{\sigma}_z^{(i)}\hat{H}^{(i)}_{Loss} - \hat{H}^{(i)+}_{Loss}\hat{\sigma}_z^{(i)}]\rangle + \sum_{F_a,F_b,q}\langle\hat{W}_q^{F_bF_a+}(\vec{r}_i)\,\hat{\sigma}_z^{(i)}\,\hat{W}_q^{F_bF_a}(\vec{r}_i)\rangle\Big\}$$

$$= \frac{1}{2}\sum_i^{N_A}\Big\{-\gamma_s^+\langle\hat{\sigma}_z^{(i)}\rangle - \gamma_s^-\langle\hat{\mathbb{1}}^{(i)}\rangle$$
$$+\frac{1}{4}\gamma_{44}\langle(\hat{\mathbb{1}}^{(i)}+\hat{\sigma}_z^{(i)})\,\hat{\sigma}_z^{(i)}\,(\hat{\mathbb{1}}^{(i)}+\hat{\sigma}_z^{(i)})\rangle$$
$$+\frac{1}{4}\gamma_{33}\langle(\hat{\mathbb{1}}^{(i)}-\hat{\sigma}_z^{(i)})\,\hat{\sigma}_z^{(i)}\,(\hat{\mathbb{1}}^{(i)}-\hat{\sigma}_z^{(i)})\rangle$$
$$+\frac{1}{4}\gamma_{43}\langle(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})\,\hat{\sigma}_z^{(i)}\,(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)})\rangle$$
$$+\frac{1}{4}\gamma_{34}\langle(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)})\,\hat{\sigma}_z^{(i)}\,(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})\rangle\Big\}$$

$$= -\gamma_s^+\langle\hat{J}_z\rangle - \frac{\gamma_s^-}{2}\langle\hat{N}_A\rangle + \frac{1}{4}\sum_i^{N_A}\Big\{\gamma_{44}\langle(\hat{\mathbb{1}}^{(i)}+\hat{\sigma}_z^{(i)})\rangle - \gamma_{33}\langle(\hat{\mathbb{1}}^{(i)}-\hat{\sigma}_z^{(i)})\rangle + \gamma_{43}\langle(\hat{\mathbb{1}}^{(i)}-\hat{\sigma}_z^{(i)})\rangle - \gamma_{34}\langle(\hat{\mathbb{1}}^{(i)}+\hat{\sigma}_z^{(i)})\rangle\Big\}$$

$$= -\gamma_s^+\langle\hat{J}_z\rangle - \frac{\gamma_s^-}{2}\langle\hat{N}_A\rangle + \frac{1}{4}\sum_i^{N_A}\big[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})\langle\hat{\mathbb{1}}^{(i)}\rangle + (\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34})\langle\hat{\sigma}_z^{(i)}\rangle\big]$$

$$= \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - \gamma_s^+\Big]\langle\hat{J}_z\rangle + \frac{1}{2}\Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\langle\hat{N}_A\rangle.$$

Since $\hat{J}_z^2 = \frac{1}{4}\sum_{i,j}^{N_A}\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)} = \frac{1}{4}\sum_i^{N_A}\hat{\mathbb{1}}^{(i)} + \frac{1}{4}\sum_{i\neq j}^{N_A}\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)} = \frac{1}{4}\hat{N}_A + \frac{1}{4}\sum_{i\neq j}^{N_A}\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}$,

one can write the evolution of $\langle\hat{J}_z^2\rangle$ into the one-body operator dynamics of $\hat{N}_A$ and the two-body one can write the evolution of $\langle \hat{J}_z^2 \rangle$ into the one-body operator dynamics of $\hat{N}_A$ and the two-body operator dynamics part of $\langle \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)} \rangle$. In the nanofiber system case, all atoms experience the same field, and hence one may be able to replace $\sum_{i\neq j}^{N_A} \to N_A(N_A-1)$.

Now, we consider the dynamics of the expectation value of the atomic operator, $\hat{N}_A = \sum_i^{N_A} \hat{\mathbb{I}}^{(i)}$.

$$
\begin{aligned}
\frac{d\langle \hat{N}_A\rangle}{dt} &= \sum_i^{N_A} \frac{d}{dt}\langle \hat{\mathbb{I}}^{(i)}\rangle = \sum_i^{N_A} \langle \mathcal{D}^{(i)\dagger}(\hat{\mathbb{I}}^{(i)})\rangle \\
&= \sum_i^{N_A} \Big\{ -\gamma_s^+ \langle \hat{\mathbb{I}}^{(i)}\rangle - \gamma_s^- \langle \hat{\sigma}_z^{(i)}\rangle + \sum_{F_a,F_b,q} \langle \hat{W}_q^{F_bF_a\dagger}(\vec{r}_i)\hat{W}_q^{F_bF_a}(\vec{r}_i)\rangle \Big\} \\
&= -\gamma_s^+\langle \hat{N}_A\rangle - 2\gamma_s^-\langle \hat{J}_z\rangle + \sum_i^{N_A} \frac{1}{4}\Big[\gamma_{44}\langle (\hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)})^2\rangle + \gamma_{33}\langle (\hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)})^2\rangle \\
&\qquad + \gamma_{43}\langle (\hat{\sigma}_x^{(i)} - i\hat{\sigma}_y^{(i)})(\hat{\sigma}_x^{(i)} + i\hat{\sigma}_y^{(i)})\rangle + \gamma_{34}\langle (\hat{\sigma}_x^{(i)} + i\hat{\sigma}_y^{(i)})(\hat{\sigma}_x^{(i)} - i\hat{\sigma}_y^{(i)})\rangle \Big] \\
&= -\gamma_s^+\langle \hat{N}_A\rangle - 2\gamma_s^-\langle \hat{J}_z\rangle + \frac{1}{2}\sum_i^{N_A}\Big[\gamma_{44}\langle \hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)}\rangle + \gamma_{33}\langle \hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)}\rangle + \gamma_{43}\langle \hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)}\rangle + \gamma_{34}\langle \hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)}\rangle \Big] \\
&= -\gamma_s^+\langle \hat{N}_A\rangle - 2\gamma_s^-\langle \hat{J}_z\rangle + \frac{1}{2}\sum_i^{N_A}\Big[(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34})\langle \hat{\mathbb{I}}^{(i)}\rangle + (\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34})\langle \hat{\sigma}_z^{(i)}\rangle \Big] \\
&= \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - \gamma_s^+\Big]\langle \hat{N}_A\rangle + \Big[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - 2\gamma_s^-\Big]\langle \hat{J}_z\rangle .
\end{aligned}
$$

For the evolution of the expectation value of the two-body operator $\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\big|_{i\neq j}$, we have

$$
\begin{aligned}
\frac{d\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle}{dt}\Big|_{i\neq j} &= \langle \mathcal{D}^{(i)\dagger}(\hat{\sigma}_z^{(i)})\hat{\sigma}_z^{(j)}\rangle + \langle \hat{\sigma}_z^{(i)}\mathcal{D}^{(j)\dagger}(\hat{\sigma}_z^{(j)})\rangle \\
&= \Big\langle \Big\{ \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - \gamma_s^+\Big]\hat{\sigma}_z^{(i)} + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\hat{\mathbb{I}}^{(i)} \Big\}\cdot \hat{\sigma}_z^{(j)} \Big\rangle \\
&\quad + \Big\langle \hat{\sigma}_z^{(i)} \cdot \Big\{ \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - \gamma_s^+\Big]\hat{\sigma}_z^{(j)} + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\hat{\mathbb{I}}^{(j)} \Big\} \Big\rangle \\
&= \Big[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\Big]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\cdot\Big[\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle + \langle \hat{\sigma}_z^{(i)}\hat{\mathbb{I}}^{(j)}\rangle\Big]
\end{aligned}
$$

whereas

$$
\begin{aligned}
\frac{d\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle}{dt}\Big|_{i\neq j} &= \langle \mathcal{D}^{(i)\dagger}(\hat{\mathbb{I}}^{(i)})\hat{\sigma}_z^{(j)}\rangle + \langle \hat{\mathbb{I}}^{(i)}\mathcal{D}^{(j)\dagger}(\hat{\sigma}_z^{(j)})\rangle \\
&= \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - \gamma_s^+\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle \\
&\quad + \Big[\frac{1}{2}(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - \gamma_s^+\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle \\
&= \Big[(\gamma_{44}+\gamma_{33}) - 2\gamma_s^+\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle \\
&\quad + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle .
\end{aligned}
$$

Similarly, $\frac{d\langle \hat{\sigma}_z^{(i)}\hat{\mathbb{I}}^{(j)}\rangle}{dt}\Big|_{i\neq j} = \frac{d\langle \hat{\mathbb{I}}^{(j)}\hat{\sigma}_z^{(i)}\rangle}{dt}\Big|_{i\neq j}$ can be retrieved by simply exchanging $i \leftrightarrow j$ from the expression above for $\frac{d\langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle}{dt}\Big|_{i\neq j}$. In our result, we can let $\mathrm{Sym}(\hat{\mathbb{I}}^{(i)}\sigma_z^{(j)}) = \frac{1}{2}(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)} + \hat{\sigma}_z^{(i)}\hat{\mathbb{I}}^{(j)})$,

$$
\begin{aligned}
\Rightarrow \frac{d\langle \mathrm{Sym}(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)})\rangle}{dt} &= \Big[(\gamma_{44}+\gamma_{33}) - 2\gamma_s^+\Big]\langle \mathrm{Sym}(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)})\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle \\
&\quad + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle .
\end{aligned}
$$

We also have

$$
\begin{aligned}
\frac{d\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle}{dt}\Big|_{i\neq j} &= \langle \mathcal{D}^{(i)\dagger}(\hat{\mathbb{I}}^{(i)})\hat{\mathbb{I}}^{(j)}\rangle + \langle \hat{\mathbb{I}}^{(i)}\mathcal{D}^{(j)\dagger}(\hat{\mathbb{I}}^{(j)})\rangle \\
&= \Big[(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - 2\gamma_s^+\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle + \Big[\frac{1}{2}(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - \gamma_s^-\Big]\cdot\Big(\langle \hat{\sigma}_z^{(i)}\hat{\mathbb{I}}^{(j)}\rangle + \langle \hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)}\rangle\Big). \\
&= \Big[(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - 2\gamma_s^+\Big]\langle \hat{\mathbb{I}}^{(i)}\hat{\mathbb{I}}^{(j)}\rangle + \Big[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - 2\gamma_s^-\Big]\langle \mathrm{Sym}(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)})\rangle
\end{aligned}
$$

Therefore, we have

$$
\frac{d\langle \hat{J}_z^2\rangle}{dt} = \frac{1}{4}\frac{d}{dt}\langle \hat{N}_A\rangle + \frac{1}{4}\sum_{i\neq j}^{N_A}\frac{d\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle}{dt}
$$

$$= \frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - \gamma_s^+\right]\langle \hat{N}_A\rangle + \frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - 2\gamma_s^-\right]\langle \hat{J}_z\rangle$$
$$+\frac{1}{4}\sum_{i\neq j}^{N_A}\left\{\left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34})-2\gamma_s^+\right]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle + \left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\cdot\langle Sym(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)})\rangle\right\}$$

This equation couples to other equations as a complete set.

The evolution of the variance of $\hat{J}_z$ measurement can then be written as

$$\frac{d\Delta \hat{J}_z^2}{dt} = \frac{d}{dt}\left(\langle \hat{J}_z^2\rangle - \langle \hat{J}_z\rangle^2\right) = \frac{d}{dt}\langle \hat{J}_z^2\rangle - 2\langle \hat{J}_z\rangle\frac{d}{dt}\langle \hat{J}_z\rangle$$
$$= \frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34}) - \gamma_s^+\right]\langle \hat{N}_A\rangle + \frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34}) - 2\gamma_s^-\right]\langle \hat{J}_z\rangle$$
$$+\frac{1}{4}\sum_{i\neq j}^{N_A}\left\{\left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34})-2\gamma_s^+\right]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle + \left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\cdot\langle Sym(\hat{\mathbb{I}}^{(i)}\hat{\sigma}_z^{(j)})\rangle\right\}$$
$$-2\langle \hat{J}_z\rangle\left\{\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - \gamma_s^+\right]\langle \hat{J}_z\rangle + \frac{1}{2}\left[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34}) - \gamma_s^-\right]\langle \hat{N}_A\rangle\right\}.$$
$$= -\left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\right]\langle \hat{J}_z\rangle^2 + \frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34})-2\gamma_s^-\right]\langle \hat{J}_z\rangle$$
$$-\left[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-\gamma_s^-\right]\langle \hat{J}_z\rangle\langle \hat{N}_A\rangle + \frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}+\gamma_{43}+\gamma_{34})-\gamma_s^+\right]\langle \hat{N}_A\rangle$$
$$+\frac{1}{4}\sum_{i\neq j}^{N_A}\left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34})-2\gamma_s^+\right]\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle + \frac{N_A}{4}(N_A-1)\left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\langle Sym(\hat{\mathbb{I}}^{(1)}\hat{\sigma}_z^{(2)})\rangle$$
$$= -\left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\right]\langle \hat{J}_z\rangle^2 + \left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\right]\cdot\Big[\underbrace{\frac{1}{4}\langle \hat{N}_A\rangle + \frac{1}{4}\sum_{i\neq j}^{N_A}\langle \hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\rangle}_{\langle \hat{J}_z^2\rangle}\Big]$$
$$-\frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}-3\gamma_{43}-3\gamma_{34})-\gamma_s^+\right]\langle \hat{N}_A\rangle + \frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34})-2\gamma_s^-\right]\langle \hat{J}_z\rangle$$
$$-\left[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-\gamma_s^-\right]\langle \hat{J}_z\rangle\langle \hat{N}_A\rangle + \frac{N_A}{4}(N_A-1)\left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\langle Sym(\hat{\mathbb{I}}^{(1)}\hat{\sigma}_z^{(2)})\rangle.$$
$$= \left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\right]\Delta J_z^2 - \frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}-3\gamma_{43}-3\gamma_{34})-\gamma_s^+\right]\langle \hat{N}_A\rangle$$
$$+\frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34})-2\gamma_s^-\right]\langle \hat{J}_z\rangle - \left[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-\gamma_s^-\right]\langle \hat{J}_z\rangle\langle \hat{N}_A\rangle$$
$$+\frac{N_A}{4}(N_A-1)\left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\langle Sym(\hat{\mathbb{I}}^{(1)}\hat{\sigma}_z^{(2)})\rangle.$$

Considering the measurement backaction, we will have the $-\kappa(\Delta J_z^2)^2$ term correction for the variance dynamics.

$$\Rightarrow \frac{d\Delta J_z^2}{dt} = -\kappa(\Delta J_z^2)^2 + \left[(\gamma_{44}+\gamma_{33}-\gamma_{43}-\gamma_{34}) - 2\gamma_s^+\right]\Delta J_z^2 - \frac{1}{4}\left[\frac{1}{2}(\gamma_{44}+\gamma_{33}-3\gamma_{43}-3\gamma_{34})-\gamma_s^+\right]\langle \hat{N}_A\rangle$$
$$+\frac{1}{4}\left[(\gamma_{44}-\gamma_{33}-\gamma_{43}+\gamma_{34})-2\gamma_s^-\right]\langle \hat{J}_z\rangle - \left[\frac{1}{2}(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-\gamma_s^-\right]\langle \hat{J}_z\rangle\langle \hat{N}_A\rangle$$
$$+\frac{N_A}{4}(N_A-1)\left[(\gamma_{44}-\gamma_{33}+\gamma_{43}-\gamma_{34})-2\gamma_s^-\right]\langle Sym(\hat{\mathbb{I}}^{(1)}\hat{\sigma}_z^{(2)})\rangle.$$

The last two lines of the equation above are proportional to $\langle \hat{J}_z\rangle$, which may be negligible for a short measurement.

Similarly,

$$\frac{d\langle \hat{J}_x\rangle}{dt} = \frac{1}{2}\sum_i^{N_A}\langle \mathcal{D}^{(i)+}(\hat{\sigma}_x^{(i)})\rangle$$
$$= \frac{1}{2}\sum_i^{N_A}\left[-\gamma_s^+\langle \hat{\sigma}_x^{(i)}\rangle - \gamma_s^-\frac{1}{2}\left(\langle \hat{\sigma}_x^{(i)}\hat{\sigma}_z^{(i)}\rangle + \langle \hat{\sigma}_z^{(i)}\hat{\sigma}_x^{(i)}\rangle\right) + \sum_{F_a,F_b,q}\langle \hat{W}_q^{F_aF_b\dagger}(\vec{r}_i)\,\hat{\sigma}_x^{(i)}\,\hat{W}_q^{F_aF_b}(\vec{r}_i)\rangle\right]$$
$$= \frac{1}{2}\sum_i^{N_A}\Big[-\gamma_s^+\langle \hat{\sigma}_x^{(i)}\rangle + 0 + \Big\langle \frac{\gamma_{44}}{4}(\hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)})\hat{\sigma}_x^{(i)}(\hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)}) + \frac{\gamma_{33}}{4}(\hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)})\hat{\sigma}_x^{(i)}(\hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)})$$
$$+\frac{\gamma_{43}}{4}(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})\hat{\sigma}_x^{(i)}(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)}) + \frac{\gamma_{34}}{4}(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)})\hat{\sigma}_x^{(i)}(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})\Big\rangle\Big]$$
$$= \frac{1}{2}\sum_i^{N_A}\Big[-\gamma_s^+\langle \hat{\sigma}_x^{(i)}\rangle + \Big\langle \frac{\gamma_{44}}{4}(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)})(\hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)}) + \frac{\gamma_{33}}{4}(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})(\hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)})$$
$$+\frac{\gamma_{43}}{4}(\hat{\mathbb{I}}^{(i)}-\hat{\sigma}_z^{(i)})(\hat{\sigma}_x^{(i)}+i\hat{\sigma}_y^{(i)}) + \frac{\gamma_{34}}{4}(\hat{\mathbb{I}}^{(i)}+\hat{\sigma}_z^{(i)})(\hat{\sigma}_x^{(i)}-i\hat{\sigma}_y^{(i)})\Big\rangle\Big]$$
$$= \frac{1}{2}\sum_i^{N_A}\Big[-\gamma_s^+\langle \hat{\sigma}_x^{(i)}\rangle + \Big\langle \frac{\gamma_{44}}{4}\times 0 + \frac{\gamma_{33}}{4}\times 0 + \frac{\gamma_{43}}{4}\times 0 + \frac{\gamma_{34}}{4}\times 0\Big\rangle\Big]$$
$$= -\gamma_s^+\langle \hat{J}_x\rangle$$