Dispersive field-atom interaction Hamiltonian

Friday, December 12, 2014 10:29 AM

We consider the Hamiltonian in the dispersive regime that

$$\hat{H}_{eff} = -\hat{\vec{E}}^{(-)}(\vec{r}\,')\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}(\vec{r}\,').$$

where $\hat{\vec{E}}^{(+)}(\vec{r}) = \hat{\vec{E}}^{(+)}(r_\perp,\phi,z) = \sqrt{\frac{2\pi\hbar\omega_0}{v_g\tau}}\,\left(\vec{u}_H(r_\perp,\phi)\hat{a}_H + \vec{u}_V(r_\perp,\phi)\hat{a}_V\right)e^{i\beta z}$

↳ forwarding modes only.

By using the irreducible decomposition of the atomic polarizability, one can write

$$\hat{H}_{eff} = -\sum_{f,f'}\alpha_0(\Delta_{ff'})\Big\{C^{(0)}_{j'ff}\,\hat{\vec{E}}^{(-)}(\vec{r}\,')\cdot\hat{\vec{E}}^{(+)}(\vec{r}\,')\,\hat{\mathbb{1}}_f + iC^{(1)}_{j'ff'}\left(\hat{\vec{E}}^{(-)}(\vec{r}\,')\times\hat{\vec{E}}^{(+)}(\vec{r}\,')\right)\cdot\hat{\vec{f}}$$
$$+\,C^{(2)}_{j'ff}\,\hat{E}^{(-)}_i\hat{E}^{(+)}_j\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{1}{3}\delta_{ij}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)\Big\}.$$

The goal for this note is to rewrite the effective Hamiltonian in terms of Stokes vectors and spin operators.

$$\hat{\vec{E}}^{(-)}\cdot\hat{\vec{E}}^{(+)} = \frac{2\pi\hbar\omega_0}{v_g\tau}\left(|\vec{u}_H|^2\hat{a}^\dagger_H\hat{a}_H + |\vec{u}_V|^2\hat{a}^\dagger_V\hat{a}_V + \underbrace{\vec{u}^*_H\cdot\vec{u}_V}_{0}\,\hat{a}^\dagger_H\hat{a}_V + \underbrace{\vec{u}^*_V\cdot\vec{u}_H}_{0}\,\hat{a}^\dagger_V\hat{a}_H\right)$$

for $\phi'=0$ or π when the atoms are on the x-axis.

$$(\hat{\vec{E}}^{(-)}\times\hat{\vec{E}}^{(+)})\cdot\hat{\vec{f}} = \frac{2\pi\hbar\omega_0}{v_g\tau}\left(\vec{u}^*_H\hat{a}^\dagger_H+\vec{u}^*_V\hat{a}^\dagger_V\right)\times\left(\vec{u}_H\hat{a}_H+\vec{u}_V\hat{a}_V\right)\cdot\hat{\vec{f}}$$
$$=\frac{2\pi\hbar\omega_0}{v_g\tau}\left(\vec{u}^*_H\times\vec{u}_V\,\hat{a}^\dagger_H\hat{a}_V + \vec{u}^*_V\times\vec{u}_H\,\hat{a}^\dagger_V\hat{a}_H + \vec{u}^*_H\times\vec{u}_H\,\hat{a}^\dagger_H\hat{a}_H + \vec{u}^*_V\times\vec{u}_V\,\hat{a}^\dagger_V\hat{a}_V\right)\cdot\hat{\vec{f}}$$
$$=\frac{2\pi\hbar\omega_0}{v_g\tau}\sum_{i,j,k}\epsilon_{ijk}\left(u^{i*}_H u^j_V\hat{f}_k\hat{a}^\dagger_H\hat{a}_V + u^{i*}_V u^j_H\hat{f}_k\hat{a}^\dagger_V\hat{a}_H + u^{i*}_H u^j_H\hat{f}_k\hat{a}^\dagger_H\hat{a}_H + u^{i*}_V u^j_V\hat{f}_k\hat{a}^\dagger_V\hat{a}_V\right)$$

$$\hat{E}^{(-)}_i\hat{E}^{(+)}_j = \frac{2\pi\hbar\omega_0}{v_g\tau}\left(u^{i*}_H\hat{a}^\dagger_H + u^{i*}_V\hat{a}^\dagger_V\right)\left(u^j_H\hat{a}_H + u^j_V\hat{a}_V\right)$$
$$=\frac{2\pi\hbar\omega_0}{v_g\tau}\left(u^{i*}_H u^j_H\hat{a}^\dagger_H\hat{a}_H + u^{i*}_H u^j_V\hat{a}^\dagger_H\hat{a}_V + u^{i*}_V u^j_H\hat{a}^\dagger_V\hat{a}_H + u^{i*}_V u^j_V\hat{a}^\dagger_V\hat{a}_V\right)$$

We also need
$$\begin{cases}\hat{a}^\dagger_H\hat{a}_H = \hat{S}_0+\hat{S}_1, & \hat{a}^\dagger_V\hat{a}_V = \hat{S}_0-\hat{S}_1,\\ \hat{a}^\dagger_H\hat{a}_V = \hat{S}_2+i\hat{S}_3, & \hat{a}^\dagger_V\hat{a}_H = \hat{S}_2-i\hat{S}_3\end{cases}$$

$$\begin{cases}\vec{u}^f_H(r_\perp',\phi',z'=0) = \sqrt{2}\left[u_{r_\perp}(r_\perp')\cos\phi'\,\hat{r}_\perp + iu_\phi(r_\perp')\sin\phi'\,\hat{\phi} + f u_z(r_\perp')\cos\phi'\,\hat{z}\right]\\ \vec{u}^f_V(r_\perp',\phi',z'=0) = \sqrt{2}\left[u_{r_\perp}(r_\perp')\sin\phi'\,\hat{r}_\perp - iu_\phi(r_\perp')\cos\phi'\,\hat{\phi} + f u_z(r_\perp')\sin\phi'\,\hat{z}\right].\end{cases}$$

With all of the relationships derived above, we have

$$\hat{H}_{eff} = -\frac{2\pi\hbar\omega_0}{v_g\tau}\sum_{f,f'}\alpha_0(\Delta_{ff'})\Big\{C^{(0)}_{j'ff}\left[|u_H|^2(\hat{S}_0+\hat{S}_1)+|u_V|^2(\hat{S}_0-\hat{S}_1)\right]\hat{\mathbb{1}}_f$$
$$+iC^{(1)}_{j'ff'}\sum_{i,j,k}\epsilon_{ijk}\left[u^{i*}_H u^j_V\hat{f}_k(\hat{S}_2+i\hat{S}_3) + u^{i*}_V u^j_H\hat{f}_k(\hat{S}_2-i\hat{S}_3) + u^{i*}_H u^j_H\hat{f}_k(\hat{S}_0+\hat{S}_1) + u^{i*}_V u^j_V\hat{f}_k(\hat{S}_0-\hat{S}_1)\right]$$
$$+C^{(2)}_{j'ff}\sum_{i,j}\left[u^{i*}_H u^j_H(\hat{S}_0+\hat{S}_1) + u^{i*}_H u^j_V(\hat{S}_2+i\hat{S}_3) + u^{i*}_V u^j_H(\hat{S}_2-i\hat{S}_3) + u^{i*}_V u^j_V(\hat{S}_0-\hat{S}_1)\right]\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{1}{3}\delta_{ij}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)\Big\}.$$
$$= -\frac{2\pi\hbar\omega_0}{v_g\tau}\sum_{f,f'}\alpha_0(\Delta_{ff'})\left[\hat{A}_0\hat{S}_0+\hat{A}_1\hat{S}_1+\hat{A}_2\hat{S}_2+\hat{A}_3\hat{S}_3\right]$$

where
$$\hat{A}_0 = C^{(0)}_{j'ff}\left(|u_H|^2+|u_V|^2\right)\hat{\mathbb{1}}_f + \underbrace{iC^{(1)}_{j'ff'}\sum_{i,j,k}\epsilon_{ijk}\left(u^{i*}_H u^j_H\hat{f}_k + u^{i*}_V u^j_V\hat{f}_k\right)}_{\text{may not cancel out when }\phi'=0\text{ or }\pi} + C^{(2)}_{j'ff}\sum_{i,j}\left(u^{i*}_H u^j_H + u^{i*}_V u^j_V\right)\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{\delta_{ij}}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)$$

propagation direction $f=1$

$$= C^{(0)}_{j'ff}\left(|\vec{u}_H|^2+|\vec{u}_V|^2\right)\hat{\mathbb{1}}_f + C^{(2)}_{j'ff}\sum_i\left(|u^i_H|^2+|u^i_V|^2\right)\left(\hat{f}_i^2-\frac{1}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right) - 4C^{(1)}_{j'ff'}\,\mathrm{Im}\left[f u^*_z(r_\perp')u_{r_\perp}(r_\perp')\right]\hat{f}_\phi + 2C^{(2)}_{j'ff}\mathrm{Re}\left[f u^*_z(r_\perp')u_{r_\perp}(r_\perp')\right]\left(\hat{f}_{r_\perp}\hat{f}_z+\hat{f}_z\hat{f}_{r_\perp}\right)$$

$$\hat{A}_1 = C^{(0)}_{j'ff}\left(|\vec{u}_H|^2-|\vec{u}_V|^2\right)\hat{\mathbb{1}}_f + iC^{(1)}_{j'ff'}\sum_{i,j,k}\epsilon_{ijk}\left(u^{i*}_H u^j_H\hat{f}_k - u^{i*}_V u^j_V\hat{f}_k\right) + C^{(2)}_{j'ff}\sum_{i,j}\left(u^{i*}_H u^j_H - u^{i*}_V u^j_V\right)\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{\delta_{ij}}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)$$
$$= C^{(0)}_{j'ff}\left(|\vec{u}_H|^2-|\vec{u}_V|^2\right)\hat{\mathbb{1}}_f - 4C^{(1)}_{j'ff'}\,\mathrm{Im}\left[f u^*_z(r_\perp')u_{r_\perp}(r_\perp')\right]\hat{f}_\phi + 2C^{(2)}_{j'ff}\mathrm{Re}\left[f u^*_z(r_\perp')u_{r_\perp}(r_\perp')\right]\left(\hat{f}_z\hat{f}_{r_\perp}+\hat{f}_{r_\perp}\hat{f}_z\right) + C^{(2)}_{j'ff}\sum_i\left(|u^i_H|^2-|u^i_V|^2\right)\left(\hat{f}_i^2-\frac{1}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right).$$

↳ propagation direction $f=1$

$$\hat{A}_2 = iC^{(1)}_{j'ff'}\sum_{i,j,k}\epsilon_{ijk}\left(u^{i*}_H u^j_V + u^{i*}_V u^j_H\right)\hat{f}_k + C^{(2)}_{j'ff}\sum_{i,j}\left(u^{i*}_H u^j_V + u^{i*}_V u^j_H\right)\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{\delta_{ij}}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)$$
$$= 2iC^{(1)}_{j'ff'}\left[\left(-iu^*_{r_\perp}(r_\perp')u_\phi(r_\perp') - iu_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\right)\hat{f}_z + \left(f i u^*_\phi(r_\perp')u_z(r_\perp') + f i u_\phi(r_\perp')u^*_z(r_\perp')\right)\hat{f}_{r_\perp}\right]$$
$$+ C^{(2)}_{j'ff}\left[\left(-iu^*_{r_\perp}(r_\perp')u_\phi(r_\perp') + iu_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\right)\left(\hat{f}_{r_\perp}\hat{f}_\phi+\hat{f}_\phi\hat{f}_{r_\perp}\right) + \left(f i u^*_\phi(r_\perp')u_z(r_\perp') - f i u_\phi(r_\perp')u^*_z(r_\perp')\right)\left(\hat{f}_\phi\hat{f}_z+\hat{f}_z\hat{f}_\phi\right)\right]$$
$$= 4C^{(1)}_{j'ff'}\left[\mathrm{Re}\left(u^*_{r_\perp}(r_\perp')u_\phi(r_\perp')\right)\hat{f}_z - f\,\mathrm{Re}\left(u^*_\phi(r_\perp')u_z(r_\perp')\right)\hat{f}_{r_\perp}\right] + 2C^{(2)}_{j'ff}\left[\mathrm{Im}\left(u^*_{r_\perp}(r_\perp')u_\phi(r_\perp')\right)\left(\hat{f}_{r_\perp}\hat{f}_\phi+\hat{f}_\phi\hat{f}_{r_\perp}\right) + f\,\mathrm{Im}\left(u_\phi(r_\perp')u^*_z(r_\perp')\right)\left(\hat{f}_\phi\hat{f}_z+\hat{f}_z\hat{f}_\phi\right)\right]$$

$$\hat{A}_3 = -C^{(1)}_{j'ff'}\sum_{i,j,k}\epsilon_{ijk}\left(u^{i*}_H u^j_V - u^{i*}_V u^j_H\right)\hat{f}_k + iC^{(2)}_{j'ff}\sum_{i,j}\left(u^{i*}_H u^j_V - u^{i*}_V u^j_H\right)\left(\frac{\hat{f}_i\hat{f}_j+\hat{f}_j\hat{f}_i}{2}-\frac{\delta_{ij}}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)$$

$$= -2C^{(1)}_{j'ff'}\Big[\big(-i\,u^*_{r_\perp}(r_\perp')u_\phi(r_\perp') + i\,u_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\big)\hat{f}_z + \big(-f i\,u^*_\phi(r_\perp')u_z(r_\perp') + f i\,u_\phi(r_\perp')u^*_z(r_\perp')\big)\hat{f}_{r_\perp}\Big]$$

$$+ iC^{(2)}_{j'ff}\Big[\big(-i\,u^*_{r_\perp}(r_\perp')u_\phi(r_\perp') - i\,u_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\big)(\hat{f}_{r_\perp}\hat{f}_\phi + \hat{f}_\phi\hat{f}_{r_\perp}) + \big(-f i\,u^*_\phi(r_\perp')u_z(r_\perp') - f i\,u_\phi(r_\perp')u^*_z(r_\perp')\big)(\hat{f}_\phi\hat{f}_z + \hat{f}_z\hat{f}_\phi)\Big]$$

$$= 4C^{(1)}_{j'ff'}\Big[\mathrm{Im}\big(u_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\big)\hat{f}_z + f\,\mathrm{Im}\big(u_\phi(r_\perp')u^*_z(r_\perp')\big)\hat{f}_{r_\perp}\Big] + 2C^{(2)}_{j'ff}\Big[\mathrm{Re}\big(u^*_{r_\perp}(r_\perp')u_\phi(r_\perp')\big)(\hat{f}_{r_\perp}\hat{f}_\phi + \hat{f}_\phi\hat{f}_{r_\perp}) + f\,\mathrm{Re}\big(u^*_\phi(r_\perp')u_z(r_\perp')\big)(\hat{f}_\phi\hat{f}_z + \hat{f}_z\hat{f}_\phi)\Big]$$

We can further define $\chi_{of} = -\frac{2\pi\omega_0}{v_g\tau}\alpha(\Delta_f) = \frac{2\pi\omega_0}{v_g\tau}\frac{|d_{F'_{max}F}|^2}{\hbar\Delta_F}$

therefore, the effective Hamiltonian can be given by

$$H_{eff} = \sum_f \hbar\chi_{of}\sum_{f'}\frac{1}{1+\delta_{ff'}/\Delta_f}\left(\hat{A}_0\hat{S}_0 + \hat{A}_1\hat{S}_1 + \hat{A}_2\hat{S}_2 + \hat{A}_3\hat{S}_3\right).$$

Now, Let us consider including both forward-propagating and backward-propagating modes:

$$\hat{\vec{E}}^{(+)}(\vec{r}') = \sum_{F=\pm1}\hat{\vec{E}}^{(+)}_F(\vec{r}') = \sqrt{\frac{2\pi\hbar\omega_0}{v_g\tau}}\sum_{F=\pm1}\left(\vec{u}_{HF}(r_\perp')\hat{a}_H + \vec{u}_{VF}(r_\perp')\hat{a}_V\right)e^{iF\beta z}$$

The Hamiltonian becomes

$$\hat{H}_{eff} = -\sum_{F,F'}\hat{\vec{E}}^{(-)}_F(\vec{r}')\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_{F'}(\vec{r}') = \underbrace{-\hat{\vec{E}}^{(-)}_+\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_+}_{\text{forward in forward out.}}\ \underbrace{-\hat{\vec{E}}^{(-)}_+\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_-}_{\text{backward in forward out.}}\ \underbrace{-\hat{\vec{E}}^{(-)}_-\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_+}_{\text{forward in backward out}}\ \underbrace{-\hat{\vec{E}}^{(-)}_-\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_-}_{\text{backward in backward out}}$$

Each term of the Hamiltonian above corresponds to various scenarios. For the case that we only care about the forward-propagating modes, the first two terms yield the output and the last two terms are loss. If we further ignore the scattering among atoms through the nanofiber, then only the first term is important for us, which is what I have derived in the last part above.

In general, each term of the Hamiltonian can be written as

$$\hat{H}^{FF'}_{eff} = -\hat{\vec{E}}^{(-)}_F\cdot\overleftrightarrow{\alpha}\cdot\hat{\vec{E}}^{(+)}_{F'} = \sum_f\hbar\chi_{of}\sum_{f'}\frac{1}{1+\delta_{ff'}/\Delta_f}\left[\hat{A}^{FF'}_0\hat{S}_0 + \hat{A}^{FF'}_1\hat{S}_1 + \hat{A}^{FF'}_2\hat{S}_2 + \hat{A}^{FF'}_3\hat{S}_3\right],$$

where, following the same derivation procedure, the $\hat{A}^{FF'}$ operators can be given as follows.

$$\hat{A}^{FF'}_0 = C^{(0)}_{j'ff}\left(\vec{u}^*_{HF}\cdot\vec{u}_{HF'} + \vec{u}^*_{VF}\cdot\vec{u}_{VF'}\right)\hat{\mathbb{1}}_f + C^{(2)}_{j'ff}\sum_i\left(u^{i*}_{HF}u^i_{HF'} + u^{i*}_{VF}u^i_{VF'}\right)\left(\hat{f}_i^2 - \tfrac{1}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right)$$

$$+ 2iC^{(1)}_{j'ff'}\left[F\,u^*_z(r_\perp')u_{r_\perp}(r_\perp') - F'\,u^*_{r_\perp}(r_\perp')u_z(r_\perp')\right]\hat{f}_\phi + C^{(2)}_{j'ff}\left[F\,u^*_z(r_\perp')u_{r_\perp}(r_\perp') + F'\,u^*_{r_\perp}(r_\perp')u_z(r_\perp')\right](\hat{f}_{r_\perp}\hat{f}_z + \hat{f}_z\hat{f}_{r_\perp}).$$

$$\hat{A}^{FF'}_1 = C^{(0)}_{j'ff}\left[\vec{u}^*_{HF}\cdot\vec{u}_{HF'} - \vec{u}^*_{VF}\cdot\vec{u}_{VF'}\right]\hat{\mathbb{1}}_f + 2iC^{(1)}_{j'ff'}\left[F\,u^*_z(r_\perp')u_{r_\perp}(r_\perp') - F'\,u^*_{r_\perp}(r_\perp')u_z(r_\perp')\right]\hat{f}_\phi$$

$$+ C^{(2)}_{j'ff}\left[F\,u^*_z(r_\perp')u_{r_\perp}(r_\perp') + F'\,u^*_{r_\perp}(r_\perp')u_z(r_\perp')\right](\hat{f}_{r_\perp}\hat{f}_z + \hat{f}_z\hat{f}_{r_\perp}) + C^{(2)}_{j'ff}\sum_i\left[u^{i*}_{HF}u^i_{HF'} - u^{i*}_{VF}u^i_{VF'}\right]\left(\hat{f}_i^2 - \tfrac{1}{3}\hat{\vec{f}}\cdot\hat{\vec{f}}\right).$$

$$\hat{A}^{FF'}_2 = 2C^{(1)}_{j'ff'}\left[2\mathrm{Re}\big(u^*_{r_\perp}(r_\perp')u_\phi(r_\perp')\big)\hat{f}_z - \big(F'\,u^*_\phi(r_\perp')u_z(r_\perp') + F\,u_\phi(r_\perp')u^*_z(r_\perp')\big)\hat{f}_{r_\perp}\right]$$

$$- iC^{(2)}_{j'ff}\left[i2\,\mathrm{Im}\big(u^*_{r_\perp}(r_\perp')u_\phi(r_\perp')\big)(\hat{f}_{r_\perp}\hat{f}_\phi + \hat{f}_\phi\hat{f}_{r_\perp}) + \big(F\,u_\phi(r_\perp')u^*_z(r_\perp') - F'\,u^*_\phi(r_\perp')u_z(r_\perp')\big)(\hat{f}_\phi\hat{f}_z + \hat{f}_z\hat{f}_\phi)\right]$$

$$\hat{A}^{FF'}_3 = -2iC^{(1)}_{j'ff'}\left[2i\,\mathrm{Im}\big(u_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\big)\hat{f}_z + \big(F\,u_\phi(r_\perp')u^*_z(r_\perp') - F'\,u^*_\phi(r_\perp')u_z(r_\perp')\big)\hat{f}_{r_\perp}\right]$$

$$+ C^{(2)}_{j'ff'}\left[2\mathrm{Re}\big(u_{r_\perp}(r_\perp')u^*_\phi(r_\perp')\big)(\hat{f}_{r_\perp}\hat{f}_\phi + \hat{f}_\phi\hat{f}_{r_\perp}) + \big(F\,u_\phi(r_\perp')u^*_z(r_\perp') + F'\,u^*_\phi(r_\perp')u_z(r_\perp')\big)(\hat{f}_\phi\hat{f}_z + \hat{f}_z\hat{f}_\phi)\right]$$

Now, depending on how do we define the atomic operators $\hat{f}_{r_\perp}$, $\hat{f}_\phi$ and $\hat{f}_z$, we should be able to further simplify the anti-commutators like $(\hat{f}_{r_\perp}\hat{f}_\phi + \hat{f}_\phi\hat{f}_{r_\perp})$...

Now, Let us look at things in another way — we can also use Clebsch-Gordan coefficients to rederive the general light-atom interaction Hamiltonian! (See Ben's nanofiberinteraction.pdf) The detailed derivation for the clock states has been typed in the Nanofiber paper draft.

For the derivation in the first Part, we have made $\phi' = 0$ or π. If we relax this

For the derivation in the first part, we have made $\phi' = 0$ or π. If we relax this condition, the major difference is that in $\hat{\vec{E}}^{(-)} \cdot \hat{\vec{E}}^{(+)}$ calculation, we will have extra terms

$$\vec{u}_H^* \cdot \vec{u}_V \hat{a}_H^+ \hat{a}_V + \vec{u}_V^* \cdot \vec{u}_H \hat{a}_V^+ \hat{a}_H = 4 \vec{u}_H^* \cdot \vec{u}_V \hat{S}_2$$

$$= 2 \sin 2\phi' \cdot (|u_r|^2 - |u_\phi|^2 + |u_z|^2) \cdot \hat{S}_2 .$$

Therefore, $\hat{A}_2$ will have an extra term

$$4 C^{(0)}_{j'ff} \vec{u}_H^* \cdot \vec{u}_V \hat{\mathbb{1}}_f = 2 C^{(0)}_{j'ff} \sin 2\phi' \cdot (|u_r|^2 - |u_\phi|^2 + |u_z|^2) \hat{\mathbb{1}}_f .$$

To sum up, the general formulas for $\hat{A}_i$ $(i = 0, 1, 2, 3)$ can now be given by

$$\hat{A}_0 = C^{(0)}_{j'ff} (|u_H|^2 + |u_V|^2) \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} \sum_{i,j,k} \epsilon_{ijk} (u_H^{i*} u_H^j \hat{f}_k + u_V^{i*} u_V^j \hat{f}_k) + C^{(2)}_{j'ff} \sum_{i,j} (u_H^{i*} u_H^j + u_V^{i*} u_V^j) \left(\frac{\hat{f}_i \hat{f}_j + \hat{f}_j \hat{f}_i}{2} - \frac{\delta_{ij}}{3} \hat{\vec{f}} \cdot \hat{\vec{f}} \right)$$

$$= C^{(0)}_{j'ff} (\vec{u}_H^* \cdot \vec{u}_H + \vec{u}_V^* \cdot \vec{u}_V) \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} (\vec{u}_H^* \times \vec{u}_H + \vec{u}_V^* \times \vec{u}_V) \cdot \hat{\vec{f}} + C^{(2)}_{j'ff} [\vec{u}_H^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_H + \vec{u}_V^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_V]$$

$\leftarrow$ with $\hat{\overleftrightarrow{T}} = \mathrm{sym}(\hat{\vec{f}} \otimes \hat{\vec{f}}) - \frac{\hat{\mathbb{1}}_3}{3} \hat{\vec{f}} \cdot \hat{\vec{f}}$ is symmetric.

$$\hat{A}_1 = C^{(0)}_{j'ff} (|\vec{u}_H|^2 - |\vec{u}_V|^2) \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} \sum_{i,j,k} \epsilon_{ijk} (u_H^{i*} u_H^j \hat{f}_k - u_V^{i*} u_V^j \hat{f}_k) + C^{(2)}_{j'ff} \sum_{i,j} (u_H^{i*} u_H^j - u_V^{i*} u_V^j) \left(\frac{\hat{f}_i \hat{f}_j + \hat{f}_j \hat{f}_i}{2} - \frac{\delta_{ij}}{3} \hat{\vec{f}} \cdot \hat{\vec{f}} \right)$$

$$= C^{(0)}_{j'ff} (\vec{u}_H^* \cdot \vec{u}_H - \vec{u}_V^* \cdot \vec{u}_V) \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} (\vec{u}_H^* \times \vec{u}_H - \vec{u}_V^* \times \vec{u}_V) \cdot \hat{\vec{f}} + C^{(2)}_{j'ff} [\vec{u}_H^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_H - \vec{u}_V^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_V]$$

$$\hat{A}_2 = 4 C^{(0)}_{j'ff} \vec{u}_H^* \cdot \vec{u}_V \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} \sum_{i,j,k} \epsilon_{ijk} (u_H^{i*} u_V^j + u_V^{i*} u_H^j) \hat{f}_k + C^{(2)}_{j'ff} \sum_{i,j} (u_H^{i*} u_V^j + u_V^{i*} u_H^j) \left(\frac{\hat{f}_i \hat{f}_j + \hat{f}_j \hat{f}_i}{2} - \frac{\delta_{ij}}{3} \hat{\vec{f}} \cdot \hat{\vec{f}} \right)$$

$$= 4 C^{(0)}_{j'ff} \vec{u}_H^* \cdot \vec{u}_V \hat{\mathbb{1}}_f + i C^{(1)}_{j'ff'} (\vec{u}_H^* \times \vec{u}_V + \vec{u}_V^* \times \vec{u}_H) \cdot \hat{\vec{f}} + C^{(2)}_{j'ff} [\vec{u}_H^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_V + \vec{u}_V^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_H]$$

$$\hat{A}_3 = - C^{(1)}_{j'ff'} \sum_{i,j,k} \epsilon_{ijk} (u_H^{i*} u_V^j - u_V^{i*} u_H^j) \hat{f}_k + i C^{(2)}_{j'ff} \sum_{i,j} (u_H^{i*} u_V^j - u_V^{i*} u_H^j) \left(\frac{\hat{f}_i \hat{f}_j + \hat{f}_j \hat{f}_i}{2} - \frac{\delta_{ij}}{3} \hat{\vec{f}} \cdot \hat{\vec{f}} \right)$$

$$= - C^{(1)}_{j'ff'} [\vec{u}_H^* \times \vec{u}_V - \vec{u}_V^* \times \vec{u}_H] \cdot \hat{\vec{f}} + i C^{(2)}_{j'ff} [\vec{u}_H^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_V - \vec{u}_V^* \cdot \hat{\overleftrightarrow{T}} \cdot \vec{u}_H] .$$

Those coefficients in a given projective quantum state space have interesting geometry meanings which may help us better understand the state- and field-dependence of light-atom interactions.

For example:

- $\vec{u}_a \cdot \vec{u}_b \rightarrow$ projective length of vectors.
- $(\vec{u}_a^* \times \vec{u}_b) \cdot \vec{f} \rightarrow$ volume of a 3-vector-spanned hexahedron.
- $\vec{u}_a^* \cdot \overleftrightarrow{T} \cdot \vec{u}_b = \mathrm{tr}\,(\overleftrightarrow{T} \cdot (\vec{u}_b \vec{u}_a^*)) \rightarrow$ something related to a "quatric form", could be the projected axes length of a ellipsoid if $\overleftrightarrow{T}$ is positive-semidefinite.

$\uparrow$ symmetric tensor, $\vec{v}\, \overleftrightarrow{T} \vec{v} = \sum_i \underset{\substack{\uparrow \\ \text{real}}}{\lambda_i} \vec{t}_i \vec{t}_i$