Controlling the phase shift and polarizability of the light going through a nanofiber is a key technique to implementing a useful quantum interface based on the nanofiber-trapped atomic system for applications including quantum data bus/router and quantum memory. Since we have shown in another work that the phase shift for a guided mode $\delta\phi = -\frac{\Gamma_{1D}}{2\Delta} = -\frac{\Gamma_{gyd}}{2\Delta}$, the phase/polarizability controlling problem is equivalent to the decay rate control problem.

In this work, we will show that the decay rates of an atom trapped by a nanofiber can be controlled via preparing the atom in a designed state and using a properly polarized probe light. Below, we will start from the semiclassical and quantum theory of atomic decay rates, and build up the relationship between atomic transitions and the dipole orientations. Then, we will revisit the decay rate calculation result in the perspective of wave interference using the dipole image method, and lead to the discussion on the symmetry of fields and eigen-decomposition of the dyadic Green function. Finally, the landscape of control freedom will be reached based on the conclusions drawn above. This is a framework of the theory, detailed numerical calculations should be conducted after the Green function method is fully verified for this research project.

§.1. Decay rates of an atom in presence of a photonic nanostructure.

§1.1 The classical & quantum models:

There are 2 usual ways to view the decay rates. One is to treat it as a classical radiation decay problem. so that the power evolves as
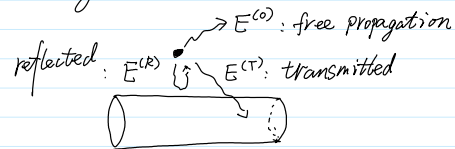
$$\frac{dP}{dt} \doteq -\Gamma P.$$

After some manipulation, the total decay rate normalized to the vacuum case can be given by

$$\frac{\Gamma}{\Gamma_{vac}} = 1 + \frac{3}{2} \, \text{Im}\left[\frac{\vec{d}^* \cdot \vec{E}^{(R)}(\vec{r}')}{d_0^2 \, k^3}\right]. \qquad (1)$$

Details of derivation can be found in, for example, R.R. Chance, et al, Adv Chem Phys. 37, 1 (1978) and Eqn (8.137) in Principles of Nano-Optics (2006). Specifically, the guided and radiative contributions are

$$\frac{\Gamma_{1D}}{\Gamma_{vac}} = \frac{\Gamma_{gyd}}{\Gamma_{vac}} = \frac{3}{2} \, \text{Im}\left[\frac{\vec{d}^* \cdot \vec{E}_{gyd}^{(R)}(\vec{r}')}{d_0^2 \, k^3}\right] \qquad (2)$$

$$\frac{\Gamma_{rad}}{\Gamma_{vac}} = 1 + \frac{3}{2} \, \text{Im}\left[\frac{\vec{d}^* \cdot \vec{E}_{rad}^{(R)}(\vec{r}')}{d_0^2 \, k^3}\right] \qquad (3)$$

E$^{(0)}$: free propagation

reflected: E$^{(R)}$ is   E$^{(T)}$: transmitted

where $E_{gyd}^{(R)}$ and $E_{rad}^{(R)}$ are the guided and radiative contributions to the reflected field components. These two quantities can be calculated through the β integration along corresponding paths. Here, the optical dipole moment of the atom is a classical treatment to a general 2-level system, which has a (real) orientation direction vector $\hat{e}_d$ and an amplitude d_0.

The other way to computing the decay rate is to consider the population decay process on the given

energy level. Using the Weisskopf-Wigner approximation in the weak-coupling regime, one can obtain

$$\frac{d\langle \hat{N}_e \rangle}{dt} \simeq -\Gamma \langle \hat{N}_e \rangle, \qquad \hat{N}_e = |e\rangle\langle e|.$$

← More details can be found in Appendix B of *Principles of Nano-optics* by Novotny & Hecht.

$$\Gamma = \Gamma_{gyd} + \Gamma_{rad} = \frac{8\pi\omega^2}{\hbar c^2} \sum_g \vec{d}_{eg} \cdot Im[\overleftrightarrow{G}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^* \qquad (4)$$

$$\Gamma_{vac} = \frac{4\omega^3}{3\hbar c^3} \sum_g |\vec{d}_{eg}|^2 \qquad (5)$$

$$\Gamma_{1D} = \Gamma_{gyd} = \frac{8\pi\omega^2}{\hbar c^2} \sum_g \vec{d}_{eg} \cdot Im[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^* \qquad (6)$$

$$\Gamma_{rad} = \frac{8\pi\omega^2}{\hbar c^2} \sum_g \vec{d}_{eg} \cdot Im[\overleftrightarrow{G}_{rad}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^* \qquad (7)$$

Therefore, the normalized decay rates

$$\frac{\Gamma}{\Gamma_{vac}} = \frac{6\pi c}{\omega} \sum_g \vec{d}_{eg} \cdot Im[\overleftrightarrow{G}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^* \Big/ |\vec{d}_{eg}|^2 \qquad (8)$$

$$\frac{\Gamma_{1D}}{\Gamma_{vac}} = \frac{\Gamma_{gyd}}{\Gamma_{vac}} = \frac{6\pi c}{\omega} \frac{\sum_g \vec{d}_{ge} \cdot Im[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^*}{\sum_g |\vec{d}_{eg}|^2}, \qquad (9)$$

where the optical dipole moment is defined as

$$\vec{d}_{ge} = \langle g|\hat{d}_{ge}|e\rangle = \langle g\|\hat{D}\|e\rangle \langle J, m | J', m', 1, -q\rangle \hat{e}_q^* \qquad (10)$$

with

$$\begin{cases} \hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{e}_{r_\perp} + i\hat{e}_\phi) \\ \hat{e}_0 = \hat{e}_z \\ \hat{e}_- = \frac{1}{\sqrt{2}}(\hat{e}_{r_\perp} - i\hat{e}_\phi) \end{cases} \qquad (11)$$

for $q = +1, 0, -1$ and choosing z-axis as the quantization axis.

Notice that the guided mode induced decay rate can also be calculated via Fermi's Golden which yields

$$\boxed{\frac{\Gamma_{1D}}{\Gamma_{vac}} = \frac{3\pi}{2}\left(\frac{c}{\omega}\right)^2 \frac{c}{v_g} \frac{\sum_{g,m=\pm1,f=\pm1} |\vec{d}_{eg} \cdot \vec{U}_m^f(\vec{r}_\perp')|^2}{\sum_g |\vec{d}_{eg}|^2}} \qquad (12)$$

$(\omega = \omega_{eg}).$

where $m = \pm1$ correspond to the two degenerate polarizations of the guided HE_{11} mode $\vec{U}_m(\vec{r})$, and $f = \pm1$ correspond to the forward/backward propagation directions.

Through comparing Eqns. (9) & (12), one can conclude that, to make the two results consistent, the projection of the dyadic Green function must satisfy

$$\vec{d}_{eg} \cdot Im[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^* = \frac{n_g c}{4\omega} \sum_{m,f} |\vec{d}_{eg} \cdot \vec{U}_m^f(\vec{r}_\perp')|^2 \qquad (13)$$

This is not consistent with the transverse mode decomposition of the dyadic Green function, however,

$$\overleftrightarrow{G}_{gyd}(\vec{r}, \vec{r}') = -i\frac{n_g c}{2\omega} \sum_{m,f} \vec{U}_m^f(\vec{r}_\perp) \vec{U}_m^{f*}(\vec{r}_\perp') . \qquad (14)$$

There is a factor of $-\frac{1}{2}$ difference. This may comes from the factor that the conventional dyadic Green function in Eqn (9) does not consider the forward/backward propagation directions. We should ignore signs here.

For now, we will use Eqn. (12) instead of Eqn. (9), and apply Eqn. (14) without using the negative sign to have

$$\boxed{\begin{aligned} \frac{\Gamma_{1D}}{\Gamma_{vac}} &= 3\pi\left(\frac{c}{\omega}\right) \frac{\sum_g \vec{d}_{eg} \cdot Im[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^*}{\sum_g |\vec{d}_{eg}|^2} \\ &= 3\pi\frac{c}{\omega} \cdot \frac{\sum_g |\vec{d}_{eg}|^2 \hat{e}_d \cdot Im[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \hat{e}_d^*}{\sum_g |\vec{d}_{eg}|^2} \end{aligned}} \qquad (15)$$

where we have used the property that

$$\begin{cases} \vec{d}_{eg} = d_{eg} \cdot \hat{e}_d & \text{with } \hat{e}_d \text{ a unit vector indicating the direction of } \vec{d}_{eg}. \\ \vec{d}_{eg} \cdot \text{Im}[\overleftrightarrow{G}] \cdot \vec{d}_{eg}^* = d_{eg} d_{eg}^* \left(\hat{e}_d \cdot \text{Im}[\overleftrightarrow{G}] \cdot \hat{e}_d^* \right) = |d_{eg}|^2 \, \hat{e}_d \cdot \text{Im}[G] \cdot \hat{e}_d^* \end{cases}$$

For the simple case that $J' = 1$, $J = 0$, there is only one ground state, and hence

$$\frac{\Gamma_{1D}}{\Gamma_{vac}} = 3\pi \frac{c}{\omega} \, \hat{e}_d \cdot \text{Im}[\overleftrightarrow{G}_{gyd}(\vec{r}', \vec{r}')] \cdot \hat{e}_d^*. \qquad (16)$$

In this case, if there is only σ_+ decay, then $\hat{g} = 1$,

$$\begin{cases} \hat{e}_{-\hat{g}}^* = \hat{e}_-^* = \frac{1}{\sqrt{2}}(\hat{e}_{r_\perp} + i\hat{e}_\phi) = -\hat{e}_+ \neq 0. \\ \hat{e}_0 = \hat{e}_z = 0 \\ \hat{e}_- = \frac{1}{\sqrt{2}}(\hat{e}_{r_\perp} - i\hat{e}_\phi) = 0 \end{cases} \Rightarrow \begin{cases} \hat{e}_{r_\perp} = i\hat{e}_\phi \\ \hat{e}_z = 0. \end{cases}$$

Therefore, a σ_+ decay corresponds to a $\vec{d} = \frac{1}{\sqrt{2}}(i\hat{e}_{r_\perp} + \hat{e}_\phi)$ dipole referenced in Eqns ① ~ ③. Similarly, one can find the other quantum decay transition and classical dipole orientation correlations. In sum, we have the following correspondence.

$$\begin{cases} \sigma_+ \text{ decay} \longleftrightarrow \vec{d} \propto \frac{1}{\sqrt{2}}(i\hat{e}_{r_\perp} + \hat{e}_\phi) \\ \sigma_\pi \text{ decay} \longleftrightarrow \vec{d} \propto \hat{e}_z. \\ \sigma_- \text{ decay} \longleftrightarrow \vec{d} \propto \frac{1}{\sqrt{2}}(-i\hat{e}_{r_\perp} + \hat{e}_\phi) \end{cases} \qquad (17).$$

The numerical check has been performed against the classical & quantum approaches to calculate the guided & radiative decay rates using Eqns ②, ③ & ⑫. They match up pretty well. The Green function representation will be checked later.

For the radiative decay rate in the quantum approach, the following relationships have been used (deducted from Le Kien's 2005 paper):

$$\frac{\Gamma_{rad}}{\Gamma_{vac}} = \frac{3\pi}{2}\left(\frac{c}{\omega}\right)^2 c \int_{-kn_2}^{kn_2} d\beta \frac{\sum_{g,m} |\vec{d}_{eg} \cdot \vec{U}_m^{(r)}(\vec{r}')|^2}{\sum_g |\vec{d}_{eg}|^2} = \frac{3\pi c}{\omega_0} \frac{\sum_g \vec{d}_{eg} \cdot \text{Im}[\overleftrightarrow{G}_{rad}(\vec{r}', \vec{r}')] \cdot \vec{d}_{eg}^*}{\sum_g |d_{eg}|^2} \qquad (18)$$

$$\overleftrightarrow{G}_{rad}(\vec{r}, \vec{r}') = \frac{ic^2}{2\omega} \int_{-kn_2}^{kn_2} d\beta \cdot \vec{U}_m^{(r)}(\vec{r}) \vec{U}_m^{(r)}(\vec{r}') \qquad (19)$$
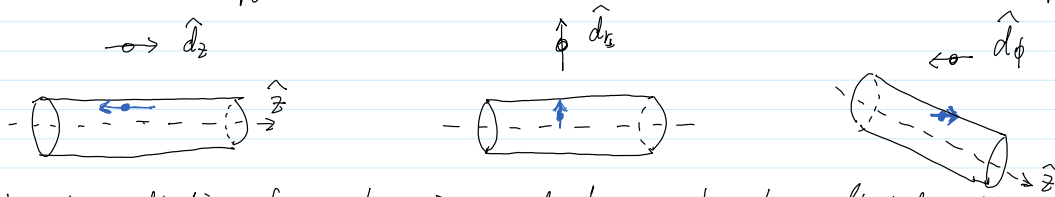
↳ The radiation modes.

To be confirmed. ←

§ 1.2. Image method to understand the radiation decay problem.

From the classical approach, we find that the decay rates are correlated (only) with the reflected fields. This can be understand straightforwardly using the image method.

For electric dipoles orientated in $\hat{z}$, $\hat{r}_\perp$ and $\hat{\phi}$ directions while itself is located in x-axis, the configuration of the mirror dipoles are shown as below for the nanofiber:

z-axis, the configuration of the mirror dipoles are shown as below for the nanofiber:

As the radiation from the image dipole equals the reflected field, we can see that the image dipole of the z-orientated dipole source has an opposite oscillation direction and an equal amount of oscillation magnitude. Therefore, the radiation from the image dipole at the real dipole position will yield a destructive interference with the radiation from the real dipole. This interference quasi-periodically alternates from very strong to very weak as a function of $r_\perp$.

Similarly, the $\hat{r}_\perp$-orientated dipole will have a constructive interference with the image dipole, since the image dipole has the same orientation yet a smaller amplitude of oscillation compared to the real dipole. The oscillating pattern of the interference as a function of $r_\perp$ is weak compared to the z-orientation case.

The $\hat{\phi}$ orientation case has a weak destructive interference. The strength should be between the two other cases.

Using this simple direct method, one can explain the decay rate as a function of dipole orientations. It also inspire us to consider the symmetry determined by the nanofiber and the control landscape of "dipole" orientations which will be introduced next.

§2. Emission rate surface and control landscape.

As discussed above, the decay rates is proportional to $\vec{d}_{eg} \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')] \cdot \vec{d}_{eg}^* = |\vec{d}_{eg}|^2 \, \hat{e}_d \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')] \cdot \hat{e}_d^*$.
The symmetry of dipole orientations and the position of the atom will strongly determine the emission property.

§2.1 Theory of emission rate surface.

Physically, for an arbitrary orientation of a dipole, the decay rates should always be positive. This works for guided, radiative and the total rates. It implies that (ignore subscripts for different contributions)

$$\forall \hat{e}_d, \quad \hat{e}_d \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')] \cdot \hat{e}_d^* > 0. \qquad (20)$$

$\Rightarrow \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]$ is a positive definite tensor.

Combined with the fact that $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]$ has all elements being real, we can conclude that

① $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]$ is Hermitian symmetric (as it is when we look at the guided & radiative dyadic Green functions in terms of fiber modes).

② $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]$ must have 3 positive eigenvalues g_1, g_2, g_3 in the descenting order corresponding to 3 orthonormal eigenvectors $\hat{v}_1, \hat{v}_2$ and $\hat{v}_2$.

② $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})]$ must have 3 positive eigenvalues g_1, g_2, g_3 in the descenting order corresponding to 3 orthonormal eigenvectors $\hat{v}_1, \hat{v}_2$ and $\hat{v}_3$.

As we will see later, those eigenvectors correspond to the dipole orientations that yield maximum and minimum emission rate and another one in-between.

The second property of $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})]$ says that

$$\boxed{\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] = \sum_{i=1,2,3} g_i \, \hat{v}_i \hat{v}_i}$$ (21)

$\hookrightarrow$ outer/tensor product of the real vector v_i.

We call v_i are the principal axes of $\text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})]$ at the atom position $\vec{r}$.

Now that, we can project the dipole orientation vector onto the principal axes to give

$$\hat{e}_d = d_1 \hat{v}_1 + d_2 \hat{v}_2 + d_3 \hat{v}_3$$ (22)

$$\begin{cases} d_1 = \hat{e}_d \cdot \hat{v}_1 = \hat{e}_d^T \cdot \hat{v}_1 \\ d_2 = \hat{e}_d \cdot \hat{v}_2 = \hat{e}_d^T \cdot \hat{v}_2 \\ d_3 = \hat{e}_d \cdot \hat{v}_3 = \hat{e}_d^T \cdot \hat{v}_3 \end{cases}$$

with $|d_1|^2 + |d_2|^2 + |d_3|^2 = 1$ to make sure $\hat{e}_d$ is on the unit sphere of directions.

Therefore,

$$\hat{e}_d \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot \hat{e}_d^* = \sum_i \hat{e}_d \cdot (\hat{v}_i \cdot \hat{v}_i) \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot (\hat{v}_i \cdot \hat{v}_i) \cdot \hat{e}_d^* = \sum_{i=1}^{3} |d_i|^2 \, \hat{v}_i \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot \hat{v}_i$$

$$= \sum_{i=1,2,3} |d_i|^2 \, g_i$$ (23)

For instance, one can find

$$\boxed{\Gamma_{1D} = \frac{4\pi w^2}{\hbar c^2} |\vec{d}_{eg}|^2 \cdot \sum_{i=1}^{3} |d_i|^2 \, g_i = |d_1|^2 \Gamma_1 + |d_2|^2 \Gamma_2 + |d_3|^2 \Gamma_3}$$ (24)

where $\Gamma_i = \frac{4\pi w^2}{\hbar c^2} |\vec{d}_{eg}|^2 \cdot \hat{v}_i \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot \hat{v}_i = \frac{4\pi w^2}{\hbar c^2} |\vec{d}_{eg}|^2 \, g_i$. (25)

As can be seen, Γ_{1D} is a quadratic (convex) combination of Γ_i, with constrain $\sum_i |d_i|^2 = 1$. And hence

$$\Gamma_{min} \le \Gamma_{1D} \le \Gamma_{max}$$

$$\Gamma_{max} = \Gamma_1, \quad \Gamma_{med} = \Gamma_2, \quad \Gamma_{min} = \Gamma_3$$

In other word, Γ_{1D} lies on the surface defined via the quadratic equation of ㉔ with the dipole orientation projected onto the coordinate system spanned by the 3 principal axes. We call this quadratic surface as emission rate suface, which only depends on the symmetry of the boundary condition and the position of the atom.
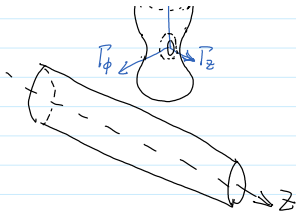
More examples of this emission rate surface method can be found in, for example, W.L. Vos, et al. PRA 80, 053802 (2009).

For our nanofiber case, we would guess the $\hat{r}$, $\hat{\phi}$ and $\hat{z}$ defines the principal axes of the emission rate surface. One possible total Γ_{1D} surface is sketched as below

$$\Gamma_{\hat{r}} > \Gamma_{\hat{\phi}} > \Gamma_{\hat{z}}$$
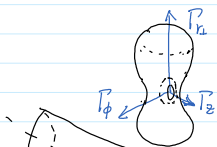
The radial length from the principal coordinate's origin to

$$\Gamma_{r_1} > \Gamma_\phi > \Gamma_z$$

The radial length from the principal coordinate's origin to the surface points indicates the decay/emission rate for a dipole orientated along corresponding directions.

The exact surface configuration can be calculated using the dyadic Green function data under the constrain that $\sum |d_i|^2 = 1$. Contributions for a given mode (forward/polarisation) can also be decomposited easily using the analytical expression of $\overleftrightarrow{G}$.

Since there are only 2 degrees of freedom over the control space of the dipole orientations, there is at least one symmetric mirror on the emission rate surface. As long as the dipole can be orientated on a half of the emission rate surface, all possible emission rates can be reached.

§2·2 Control landscape of atom-photonic interfaces.

For a quantum state of a real atom, the dipole moment is defined in Eqn. ⑩ with reduced sperical unit vector basis $\hat{e}_q$. Via Eqn. ⑰, all those possible transition and hence the projected dipole moment can be mapped to a classical dipole with dipole orientation $\hat{e}_d$. In this process, the vector $\hat{e}_d$ is, in general, complex.

However, from Eqn. ㉒ for the principal axis decomposition of $\hat{e}_d$ and Eqn. ㉔ for the principal decomposition of Γ, we notice that we actually only care about the magnitude of d_i which are the magnitude of the principle coefficients of $\hat{e}_d$. Therefore, one can safely use $\hat{e}_d = \sum_i |d_i| \hat{v}_i$ to represent the equivalent classical dipole moment orientations. Then, all the analysis of emission rate surface can be applied to the state-dependent control landscape of decay rates / phase shift / transformation of polarizability. The 3 principal decay rate axes Γ_i determines the rang of controlability. We only need a quarter of the surface mapping with all positive principal direction parameters to get full freedom of control.

Obviously, this landscape of control geometry works for pure ground & excited states case. It can be generalized to mixed states of an ensemble of atoms, by noticing that the equivalent dipole orientation vector can be a statistical mixture of pure dipole transitions.

In general, we have

→ some constant determined by ω.

$$\Gamma = C(\omega) \; \vec{d} \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot \vec{d}^*$$

$$= C(\omega) \cdot \text{tr}[\vec{d} \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})] \cdot \vec{d}^*]$$

$$= C(\omega) \cdot \text{tr}[(\vec{d}^*\vec{d}) \cdot \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})]]$$

$$\Gamma = C(\omega) \, \text{tr}[\overleftrightarrow{\rho} \, \text{Im}[\overleftrightarrow{G}(\vec{r},\vec{r})]] \qquad (2.6)$$

$$= C(\omega) \cdot \mathrm{tr}\left[(\vec{d}^* \vec{d}\,) \cdot \mathrm{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')] \right]$$

$$\boxed{\Gamma = C(\omega) \cdot \mathrm{tr}\left[\overleftrightarrow{P}\, \mathrm{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')] \right]} \qquad (26)$$

where $\overleftrightarrow{P}$ is the dipole moment product matrix defined as

$$\boxed{\overleftrightarrow{P} = \vec{d}^* \vec{d} = \sum_j P_j\, \vec{d}_j^* \vec{d}_j = \sum_j P_j\, \overleftrightarrow{P}_j}, \qquad \text{with } \sum_j P_j = 1. \qquad (27)$$

where $\vec{d}_j$ are the dipole moment determined by a quantum transition between pure state of excited and ground states; P_j is the probability when d_j occurs. The Hermitian property of $\overleftrightarrow{P}$ & $\mathrm{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]$ ensures the non-negativity of the trace of their product.

For instance, in the case of complete mixture of all possible quantum transitions, the $\sigma_\pm, \pi$ transitions all have $\frac{1}{3}$ probability to occur. Using Eqn. (17), we have

$$
\begin{cases}
\sigma_+ \text{ transition} \iff \overleftrightarrow{P}_+ = \frac{|d_0|^2}{2}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}(i \ \ 1 \ \ 0) = \frac{|d_0|^2}{2}\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\[3mm]
\pi \text{ transition} \iff \overleftrightarrow{P}_0 = |d_0|^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(0 \ 0 \ 1) = |d_0|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[3mm]
\sigma_- \text{ transition} \iff \overleftrightarrow{P}_- = \frac{|d_0|^2}{2}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}(-i \ \ 1 \ \ 0) = \frac{|d_0|^2}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{cases} \qquad (28)
$$

$\Rightarrow$ The dipole moment product matrix for this completely mixed state is

$$\overleftrightarrow{P} = \frac{1}{3}\overleftrightarrow{P}_+ + \frac{1}{3}\overleftrightarrow{P}_0 + \frac{1}{3}\overleftrightarrow{P}_- = \frac{|d_0|^2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{|d_0|^2}{3}\overleftrightarrow{\mathbb{I}}. \qquad (29)$$
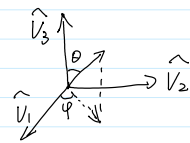
$$\Rightarrow \Gamma = \frac{C(\omega)}{3}|d_0|^2\, \mathrm{tr}\left[\mathrm{Im}[\overleftrightarrow{G}(\vec{r},\vec{r}')]\right] = \frac{1}{3}\underbrace{(\Gamma_1 + \Gamma_2 + \Gamma_3)}_{\text{principal decay rates.}} \qquad (30)$$

This result can also be understood from a classical picture that a completely mixed state has the equivalent dipole moment evenly distributed in all directions. That is in the principal coordinate, if we let

$$\Gamma(\theta,\phi) = \Gamma_1 \cos^2\phi \sin^2\theta + \Gamma_2 \sin^2\phi \sin^2\theta + \Gamma_3 \cos^2\theta$$

be the directional decay rate with a dipole orientated along (θ,ϕ). The decay rate can be written as

$$\Gamma = \int \frac{d\Omega}{4\pi}\Gamma(\theta,\phi) = \frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta \cdot \sin\theta \cdot \Gamma(\theta,\phi) = \frac{1}{3}(\Gamma_1 + \Gamma_2 + \Gamma_3).$$

In general, we have

$$\boxed{\begin{aligned} \Gamma &= C(\omega)\sum_i (\hat{v}_i \cdot \overleftrightarrow{P} \cdot \hat{v}_i)\cdot g_i \\ d_i' &= \sqrt{\hat{v}_i \cdot \overleftrightarrow{P} \cdot \hat{v}_i}, \quad \text{for } i = 1, 2, 3 \end{aligned}} \qquad (31)$$

define the equivalent dipole orientation on the emission rate surface for an arbitrary transition dipole product matrix $\overleftrightarrow{P}$.

rate surface. for an arbitrary transition dipole product matrix ρ.

As can be seen, $\vec{\rho}$ is analog to the density operator, and the emission rate surface is similar to the Bloch sphere or Poincaré sphere which have been widely used to analyze quantum states and polarization states. The emission rate surface map may be a nice tool for us to think about quantum control of the atom—nanophotonic interface more easily.