Polarizability control, input-output formalism and Master equations

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The goal of this note is to derive the input-output formalism in both frequency & time-domain and Master equations for controlling the polarization of two-mode light signal using controllable atomic polarizability of an ensemble of atoms surrounding a waveguide.

Frequency domain: First, we start with a two-level atom interacting with a dispersive traveling light, and see how the output light is changed.

$|e\rangle$, $|g\rangle$, $\Gamma_u + \Gamma_\nu$, Ω. $\leftarrow \hat{\vec{E}}^{(+)} = \hat{\vec{E}}_g^{(+)} + \hat{\vec{E}}_r^{(+)}$ We include both guided & radiation modes

$$\hat H = \hat H_F + \hat H_A + \hat H_{int},$$

with
$$\begin{cases} \hat H_F = \sum_{m,f}\int d\beta\, \hbar\omega_\beta\, \hat a_\mu^+(\beta)\hat a_\mu(\beta) + \sum_{m,p}\int d\beta\, \hbar\omega_\beta\, \hat a_\nu^+(\beta)\hat a_\nu(\beta) = \sum_{m,f} f\int d\omega\cdot\hbar\omega\, \hat a_\mu^+(\omega)\hat a_\mu(\omega) + \sum_{m,p}\int d\omega\cdot\hbar\omega\, \hat a_\nu^+(\omega)\hat a_\nu(\omega) \\ \hat H_A = \sum_g E_g\hat\sigma_{gg} + \sum_e E_e\hat\sigma_{ee} \end{cases}$$
$\longrightarrow$ chose the quantum transition between $|e\rangle$ & $|g\rangle$ states, and sum over all excited & ground states.

We have defined $\hat a_\mu(\omega) = \sqrt{\left|\frac{d\beta}{d\omega}\right|}\,\hat a_\mu(\beta) = \frac{1}{\sqrt{v_g}}\hat a_\mu(\beta)$ and $\hat a_\nu(\omega) = \frac{1}{\sqrt{c}}\hat a_\nu(\beta)$. $[\hat a_\mu(\beta), \hat a_{\mu'}^+(\beta')] = \delta_{\mu\mu'}\delta(\beta-\beta')$, $[\hat a_\mu(\omega), \hat a_{\mu'}^+(\omega')] = \delta_{\mu\mu'}\delta(\omega-\omega')$.

$[\hat a_\nu(\omega), \hat a_{\nu'}^+(\omega')] = \delta_{\nu\nu'}\delta(\beta-\beta')\delta(\omega-\omega')$

The interaction Hamiltonian is

$$\hat H_{int} = -\hat{\vec d}\cdot\hat{\vec E} = -\vec d_{eg}\cdot\hat{\vec E}^{(+)}(\vec r') - \vec d_{ge}\cdot\hat{\vec E}^{(-)}(\vec r') \leftarrow \hat{\vec d}_{eg} = \hat P_e\hat{\vec d}\hat P_g,\ \vec d_{eg} = \langle e|\hat{\vec d}|g\rangle.$$

$$\begin{cases} \text{guided field } \hat{\vec E}_g^{(+)}(\vec r) = \sum_{m,f}\int d\omega\sqrt{\frac{\hbar\omega}{v_g}}\,\hat a_\mu(\omega)\,\vec u^{(\mu)}(\vec r_\perp)e^{if\beta z} \\ \text{radiation field } \hat{\vec E}_r^{(+)}(\vec r) = \sum_{m,p}\int d\omega\int_{-kn_2}^{kn_2} d\beta\sqrt{\hbar\omega}\,\hat a_\nu(\omega)\,\vec u^{(\nu)}(\vec r_\perp)e^{i\beta z}. \end{cases}$$

If we define the light-atom coupling strengths by

$$\hbar g_\mu^{eg}(\omega) = \sqrt{\frac{\hbar\omega}{v_g}}\,\vec d_{eg}\cdot\vec u^{(\mu)}(\vec r_\perp')$$
$$\hbar g_\nu^{eg}(\omega) = \sqrt{\hbar\omega}\,\vec d_{eg}\cdot\vec u^{(\nu)}(\vec r_\perp')$$

where the atomic dipole moment vector can be obtained through $\hat{\vec d}_{eg} = \vec d_{eg}\hat\sigma_+ = \langle e|\vec D|g\rangle\hat\sigma_+$.

Now the interaction Hamiltonian can be rewritten as

$$\hat H_{int} = -\sum_{m,f}\int d\omega\cdot\hbar\left[g_\mu^{eg}(\omega)\hat a_\mu\hat\sigma_{eg} + g_\mu^{eg*}(\omega)\hat a_\mu^+\hat\sigma_{ge}\right]$$
$$-\sum_{m,p}\int d\omega\int_{-kn_2}^{kn_2} d\beta\,\hbar\left[g_\nu^{eg}(\omega)\hat a_\nu\hat\sigma_{eg} + g_\nu^{eg*}(\omega)\hat a_\nu^+\hat\sigma_{ge}\right].$$

To sum up, the total Hamiltonian in the frequency domain is

$$\hat H = \sum_{m,f} f\int d\omega\cdot\hbar\omega\,\hat a_\mu^+\hat a_\mu(\omega) + \sum_{m,p}\int d\omega\int_{-kn_2}^{kn_2} d\beta\,\hbar\omega\,\hat a_\nu^+\hat a_\nu(\omega) + \sum_g E_g\hat\sigma_{gg} + \sum_e E_e\hat\sigma_{ee}$$
$$-\sum_{e,g}\left\{\sum_{m,f}\int d\omega\cdot\hbar\left[g_\mu^{eg}(\omega)\hat a_\mu\hat\sigma_{eg} + g_\mu^{eg*}(\omega)\hat a_\mu^+\hat\sigma_{ge}\right] + \sum_{m,p}\int d\omega\cdot\int_{-kn_2}^{kn_2} d\beta\,\hbar\left[g_\nu^{eg}(\omega)\hat a_\nu\hat\sigma_{eg} + g_\nu^{eg*}(\omega)\hat a_\nu^+\hat\sigma_{ge}\right]\right\} \quad (1)$$

If all quantum operators are not explicitly time-dependent, the Heisenberg equations of motion can be given by

$$\begin{cases} \frac{d\hat a_\mu}{dt} = \frac{i}{\hbar}[\hat H, \hat a_\mu] = -if\omega\,\hat a_\mu(\omega) + i\sum_{e,g} g_\mu^{eg*}(\omega)\,\hat\sigma_{ge} & (1) \\ \frac{d\hat a_\nu}{dt} = \frac{i}{\hbar}[\hat H, \hat a_\nu] = -i\omega\,\hat a_\nu(\omega) + i\sum_{eg} g_\nu^{eg*}(\omega)\,\hat\sigma_{ge} & (2) \\ \frac{d\hat\sigma_{ge}}{dt} = \frac{i}{\hbar}[\hat H, \hat\sigma_{ge}] = -i\omega_{eg}\hat\sigma_{ge} - i\int_0^\infty d\omega\sum_{i,m,f}\left[(g_\mu^{ig}\hat\sigma_{ie} - g_\mu^{ei}\hat\sigma_{gi})\hat a_\mu + (g_\mu^{gi*}\hat\sigma_{ie} - g_\mu^{ie*}\hat\sigma_{gi})\hat a_\mu^+\right] & \\ \qquad - i\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\cdot\sum_{i,m,p}(g_\nu^{ig}\hat\sigma_{ie} - g_\nu^{ei}\hat\sigma_{gi})\hat a_\nu + (g_\nu^{gi*}\hat\sigma_{ie} - g_\nu^{ie*}\hat\sigma_{gi})\hat a_\nu^+] & (3) \end{cases}$$

Used: $[\hat\sigma_{ij}, \hat\sigma_{i'j'}] = \delta_{ji'}\hat\sigma_{ij'} - \delta_{ij'}\hat\sigma_{i'j}$, all indeces are summed over all excited & ground states.

Notice that, for the scenario of real-time control on the atomic state with a fast changing light signal, each equation above will have a $\boxed{\frac{\partial\hat O}{\partial t}}$ term added at the end. One can use the new set of the equations to correct the error due to the spontaneous emission, and evolve the atomic or photonic state to a target state. Some conclusions drawn in Q ~~tomography~~ as a branch of optimization theory also works in control theory as well.

When we only conside one fine structure excited level as all the other excited fine structure levels are relatively far-detuned, then the $\hat a_\mu^+$ & $\hat a_\nu^+$ vanish in the $\frac{d\hat\sigma_{ge}}{dt}$ equation. That is

$$\frac{d\hat\sigma_{ge}}{dt} = -i\omega_{eg}\hat\sigma_{ge} - i\int_0^\infty d\omega\sum_i\left[\sum_{m,f}(g_\mu^{ig}\hat\sigma_{ie} - g_\mu^{ei}\hat\sigma_{gi})\hat a_\mu + \int_{-kn_2}^{kn_2} d\beta\sum_{m,p}(g_\nu^{ig}\hat\sigma_{ie} - g_\nu^{ei}\hat\sigma_{gi})\hat a_\nu\right] \quad (4)$$

Integrating the equations above will give the following solutions:

$$\{\hat a_\mu(t) = \hat a_\mu(t_0)\,e^{-if\omega(t-t_0)} + i\sum g^{eg*}\int^t dt'\,e^{-if\omega(t-t')}\,\hat\sigma \ldots \quad (5)$$

Integrating the equations above will give the following solutions:

$$\begin{cases} \hat{a}_u(t) = \hat{a}_u(t_0)\, e^{-if\omega(t-t_0)} + i\sum_{e,g} g_u^{eg*} \int_{t_0}^{t} dt'\, e^{-if\omega(t-t')}\, \hat{\sigma}_{ge}(t') & (5) \\ \hat{a}_\nu(t) = \hat{a}_\nu(t_0)\, e^{-i\omega(t-t_0)} + i\sum_{e,g} g_\nu^{eg*} \int_{t_0}^{t} dt'\, e^{-i\omega(t-t')}\, \hat{\sigma}_{ge}(t') & (6) \end{cases}$$

One can substitute the solution above into Equ. (4), and use the standard Markov approximation that

$$\int_0^\infty d\omega\, e^{-i\omega f(t-t')} |g_i^{eg}(\omega)|^2 \hat{\sigma}_{ge}(t') \simeq \pi\delta(t-t') |g_i^{eg}(\omega_{eg})|^2 \hat{\sigma}_{ge}(t) \quad \text{with } i = u, \nu,$$

to obtain

$$\begin{aligned} \frac{d\hat{\sigma}_{ge}}{dt} = & -i\omega_{eg}\hat{\sigma}_{ge} - i\int_0^\infty d\omega \sum_i \Big\{ \sum_{m,f} (g_u^{ig}\hat{\sigma}_{ie} - g_u^{ei}\hat{\sigma}_{gi})\hat{a}_u(t_0)\, e^{-if\omega(t-t_0)} + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} (g_\nu^{ig}\hat{\sigma}_{ie} - g_\nu^{ei}\hat{\sigma}_{gi})\hat{a}_\nu(t_0)\, e^{-i\omega(t-t_0)} \Big\} \\ & + \pi \sum_{i,e',g'} \Big\{ \sum_{m,f} (g_u^{ig} g_u^{e'g'*}\hat{\sigma}_{ie}\hat{\sigma}_{g'e'} - g_u^{ei} g_u^{e'g'*}\hat{\sigma}_{gi}\hat{\sigma}_{g'e'}) + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} (g_\nu^{ig} g_\nu^{e'g'*}\hat{\sigma}_{ie}\hat{\sigma}_{g'e'} - g_\nu^{ei} g_\nu^{e'g'*}\hat{\sigma}_{gi}\hat{\sigma}_{g'e'}) \Big\} \\ = & -i\omega_{eg}\hat{\sigma}_{ge} - i\int_0^\infty d\omega \sum_i \Big\{ \sum_{m,f} (g_u^{ig}\hat{\sigma}_{ie} - g_u^{ei}\hat{\sigma}_{gi})\hat{a}_u(t_0)\, e^{-if\omega(t-t_0)} + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} (g_\nu^{ig}\hat{\sigma}_{ie} - g_\nu^{ei}\hat{\sigma}_{gi})\hat{a}_\nu(t_0)\, e^{-i\omega(t-t_0)} \Big\} \\ & - \pi \sum_{g',e'} \Big[\sum_{m,f} g_u^{eg'} g_u^{e'g'*} + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} g_\nu^{eg'} g_\nu^{e'g'*} \Big] \hat{\sigma}_{ge'} \\ = & -i\omega_{eg}\hat{\sigma}_{ge} - \sum_{e'} \frac{\Gamma_{ee'}}{2}\hat{\sigma}_{ge'} \\ & - i\int_0^\infty d\omega \sum_i \Big\{ \sum_{m,f} (g_u^{ig}\hat{\sigma}_{ie} - g_u^{ei}\hat{\sigma}_{gi})\hat{a}_u(t_0)\, e^{-if\omega(t-t_0)} + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} (g_\nu^{ig}\hat{\sigma}_{ie} - g_\nu^{ei}\hat{\sigma}_{gi})\hat{a}_\nu(t_0)\, e^{-i\omega(t-t_0)} \Big\} \end{aligned} \quad (7)$$

with

$$\Gamma_{ee'} = 2\pi \sum_{g'} \Big[\sum_{m,f} g_u^{eg'} g_u^{e'g'*} + \int_{-kn_2}^{kn_2} d\beta \sum_{m,p} g_\nu^{eg'} g_\nu^{e'g'*} \Big] \quad (8)$$

The first term is the decay going into the guided modes and the second term is the decay going into the radiative modes.

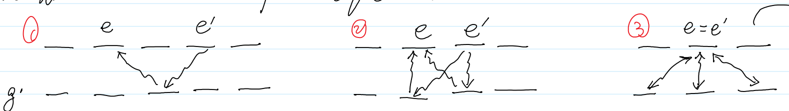
We can expand each term for the guided-mode part as below.

$$\begin{aligned} 2\pi g_u^{eg'} g_u^{e'g'*} &= \frac{2\pi\sqrt{\omega_{eg'}\omega_{e'g'}}}{\hbar v_g} \left(\langle e|\hat{d}|g'\rangle \cdot \vec{u}_u(\vec{r}_\perp')\right)\left(\vec{u}_u^*(\vec{r}_\perp') \cdot \langle g'|\hat{d}|e'\rangle\right) \\ &\simeq \frac{2\pi\omega_{eg'}}{\hbar v_g} \langle e|\hat{d}|g'\rangle \cdot \left(\vec{u}_u(\vec{r}_\perp')\vec{u}_u^*(\vec{r}_\perp')\right) \cdot \langle g'|\hat{d}|e'\rangle \\ &= \frac{1}{\hbar} \langle e|\hat{d}|g'\rangle \cdot \mathrm{Im}[\overleftrightarrow{G}^{(+)}(\vec{r}_\perp', \vec{r}_\perp'; \omega = \omega_{eg'})] \cdot \langle g'|\hat{d}|e'\rangle \end{aligned} \quad (9)$$

For the excited states $|e\rangle = |n'P_{J'}, F', m'\rangle$ and ground states $|g\rangle = |nS_J, F, m\rangle$, we have

$\langle e|\hat{d}|g\rangle = \langle n'P_{J'}\|d\|nS_J\rangle \sum_{q=\pm1,0} O_{JF}^{J'F'} C_{F'm'}^{Fm;1q}\, \hat{e}_q^*$ in the basis of $|F',m'\rangle\langle F,m|$.

The allowed transitions for Eq. (9) are sketched below:



gives $\Gamma_e = \Gamma_{ee}$ decay terms

For the guided mode contribution to the decay rates $\Gamma_{ee'}$, we have

$$\Gamma_{ee'}^{g} = \sum_F \sum_m \frac{|\langle n'P_{J'}\|d\|nS_J\rangle|^2}{\hbar} \sum_{q,q'=\pm1,0} O_{JF}^{J'F'} C_{F'm'}^{Fm;1q}\, O_{JF}^{J'F''} C_{F''m''}^{Fm;1q'}\, \hat{e}_q^* \cdot \mathrm{Im}(\overleftrightarrow{G}) \cdot \hat{e}_{q'}.$$

We may use the properties of Clebsch-Gordan coefficients below to simplify our result.

$$\sum_{F'=|F_1-F_2|}^{F_1+F_2} \sum_{m'=-F}^{F} \langle F_1, m_1; 1, q | F'm'\rangle\langle F'm' | F_1, m_2; 1, q'\rangle = \langle F_1, m_1; 1, q | F_1, m_2; 1, q'\rangle = \delta_{m_1 m_2}\delta_{qq'}.$$

$$\sum_{m_1, q} \langle F, m | F, m_1, 1q\rangle\langle F, m_1, 1q | F', m'\rangle = \langle F, m | F', m'\rangle = \delta_{FF'}\delta_{mm'}.$$

①, ② terms always cancel when $\mathrm{Im}[\overleftrightarrow{G}^{(+)}]$ is isomorphic — that is for the vacuum case. But in general, those terms may not cancel (there could be other relations I haven't thought of).
Leave this for future works. The rest of derivation is partially in the paper draft.