Derivation of free-space E-field components.

Tuesday, January 21, 2014 11:02 AM

0/21-1. *For $r < r'$ case (I use r to replace r_I)*

$$E^{(0)}_{z m\beta} = \frac{i}{rk}\left(\frac{\partial (rB^{(0)}_\phi)}{\partial r} - \frac{\partial B^{(0)}_r}{\partial \phi}\right)$$

$$= \frac{i}{k}\left[\frac{B^{(0)}_\phi}{r} + \frac{\partial B^{(0)}_\phi}{\partial r} - \frac{1}{r}\frac{\partial B^{(0)}_r}{\partial \phi}\right]$$

$$= \frac{i}{2}\Big[\frac{i}{r}(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)\beta H^{(1)}_m(pr')J_m(pr) - \frac{d^0_z}{r}H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)$$

$$+ i(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)\beta H^{(1)}_m(pr')\frac{d}{dr}J_m(pr) - d^0_z H^{(1)}_m(pr')\frac{d^2}{dr^2}J_m(pr)$$

$$- \frac{i}{r}(\sin(\phi-\phi')d^0_\phi + \cos(\phi-\phi')d^0_r)\beta H^{(1)}_m(pr')J_m(pr)$$

$$+ \frac{m}{r}\left(\frac{d^0_z m}{r} - (\cos(\phi-\phi')d^0_\phi - \sin(\phi-\phi')d^0_r)\beta\right)H^{(1)}_m(pr')J_m(pr)\Big]$$

$$= \frac{im}{2r}\left[\frac{d^0_z m}{r} + \beta\sin(\phi-\phi')d^0_r - \beta\cos(\phi-\phi')d^0_\phi\right]H^{(1)}_m(pr')J_m(pr)$$

$$- \frac{1}{2}\left[\frac{id^0_z}{r} + \beta(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)\right]H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)$$

$$- \frac{i}{2}d^0_z H^{(1)}_m(pr')\frac{d^2}{dr^2}J_m(pr).$$

$$E^{(0)}_{\phi,m\beta} = \frac{i}{k}\left(\frac{\partial B^{(0)}_r}{\partial z} - \frac{\partial B^{(0)}_z}{\partial r}\right)$$

$$= -\frac{\beta}{2}\left[\frac{id^0_z m}{r} - i(\cos(\phi-\phi')d^0_\phi - \sin(\phi-\phi')d^0_r)\beta\right]H^{(1)}_m(pr')J_m(pr)$$

$$- \frac{i}{2}\Big[\frac{i}{r^2}(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)m H^{(1)}_m(pr')J_m(pr)$$

$$- \frac{i}{r}(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)m H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)$$

$$+ (\cos(\phi-\phi')d^0_\phi - \sin(\phi-\phi')d^0_r)H^{(1)}_m(pr')\frac{d^2}{dr^2}J_m(pr)\Big]$$

$$= \frac{1}{2}\left[-\frac{id^0_z m\beta}{r} + \left(\frac{m}{r^2}\cos(\phi-\phi') - i\beta^2\sin(\phi-\phi')\right)d^0_r + \left(\frac{m}{r^2}\sin(\phi-\phi') + i\beta^2\cos(\phi-\phi')\right)d^0_\phi\right]H^{(1)}_m(pr')J_m(pr)$$

$$- \frac{1}{2r}(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)m H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)$$

$$- \frac{i}{2}(\cos(\phi-\phi')d^0_\phi - \sin(\phi-\phi')d^0_r)H^{(1)}_m(pr')\frac{d^2}{dr^2}J_m(pr)$$

$$E^{(0)}_{r,m\beta} = \frac{i}{k}\left(\frac{1}{r}\frac{\partial B^{(0)}_z}{\partial \phi} - \frac{\partial B^{(0)}_\phi}{\partial z}\right)$$

$$= \frac{i}{2r}\Big[\frac{-i}{r}(-\sin(\phi-\phi')d^0_r + \cos(\phi-\phi')d^0_\phi + im\cos(\phi-\phi')d^0_r + im\sin(\phi-\phi')d^0_\phi)m H^{(1)}_m(pr')J_m(pr)$$

$$+ (-\sin(\phi-\phi')d^0_\phi - \cos(\phi-\phi')d^0_r + im\cos(\phi-\phi')d^0_\phi - im\sin(\phi-\phi')d^0_r)H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)\Big]$$

$$+ \frac{\beta}{2}\left[i(\cos(\phi-\phi')d^0_r + \sin(\phi-\phi')d^0_\phi)\beta H^{(1)}_m(pr')J_m(pr) - d^0_z H^{(1)}_m(pr')\frac{d}{dr}J_m(pr)\right]$$

$$= \left\{\frac{1}{2}\left[\left(i\beta^2 + \frac{im^2}{r^2}\right)\cos(\phi-\phi') - \frac{m}{r^2}\sin(\phi-\phi')\right]d^0_r + \frac{1}{2}\left[\left(i\beta^2 + \frac{im^2}{r^2}\right)\sin(\phi-\phi') + \frac{m}{r^2}\cos(\phi-\phi')\right]d^0_\phi\right\}H^{(1)}_m(pr')J_m(pr).$$

$$+ \frac{1}{2}\left[-d^0_z\beta - \frac{1}{r}(i\cos(\phi-\phi') - m\sin(\phi-\phi'))d^0_r - \frac{d^0_\phi}{r}(i\sin(\phi-\phi') + m\cos(\phi-\phi'))\right]H^{(1)}_m(pr')\frac{d}{dr}J_m(pr).$$

Noticed that $\cos(\phi-\phi') = \frac{e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}}{2}$, $\sin(\phi-\phi') = \frac{e^{i(\phi-\phi')} - e^{-i(\phi-\phi')}}{2i}$, *each* $e^{\pm i(\phi-\phi')}$ *term will* change the mode index, so that the results we obtained above is not really the "m"-th mode components. To correct that, we redefine

$$\begin{cases} d_r = \cos(\phi-\phi')\, d_r^0 + \sin(\phi-\phi')\, d_\phi^0 = \frac{1}{\sqrt{2}}\left(-e^{-i(\phi-\phi')}d_+ + e^{+i(\phi-\phi')}d_-\right) \\ d_\phi = -\sin(\phi-\phi')\, d_r^0 + \cos(\phi-\phi')\, d_\phi^0 = \frac{i}{\sqrt{2}}\left(e^{-i(\phi-\phi')}d_+ + e^{+i(\phi-\phi')}d_-\right). \end{cases} \qquad d_\pm \equiv \mp\frac{1}{\sqrt{2}}(d_r^0 \pm i d_\phi^0)$$

Now the "$E_{\phi,m\beta}^{(0)}$" can be rewritten as

$$\text{"}E_{\phi,m\beta}^{(0)}\text{"} = \frac{1}{2}\Big[\frac{i d_z^0 m\beta}{r} + \frac{m}{\sqrt{2}r^2}\left(-e^{-i(\phi-\phi')}d_+ + e^{i(\phi-\phi')}d_-\right) - \frac{\beta^2}{\sqrt{2}}\left(e^{-i(\phi-\phi')}d_+ + e^{i(\phi-\phi')}d_-\right)\Big] H_m^{(1)}(pr')J_m(pr)$$
$$- \frac{1}{2r\sqrt{2}}\left(-e^{-i(\phi-\phi')}d_+ + e^{i(\phi-\phi')}d_-\right) m H_m^{(1)}(pr')\frac{d}{dr}J_m(pr)$$
$$+ \frac{1}{2\sqrt{2}}\left(e^{-i(\phi-\phi')}d_+ + e^{i(\phi-\phi')}d_-\right) H_m^{(1)}(pr')\frac{d^2}{dr^2}J_m(pr)$$

Considering the $E_{\phi,m\beta}^{(0)}$ only goes with $e^{im\phi+i\beta z}$ term in $E_\phi^{(0)}$ expansion, the real $E_{\phi,m\beta}^{(0)}$ should be

$$E_{\phi,m\beta}^{(0)} = -\frac{i d_z^0 m\beta}{2r} H_m^{(1)}(pr')J_m(pr)$$
$$+ \frac{d_+}{2\sqrt{2}}\left[p^2\cdot\frac{1}{p^2}\frac{d^2}{dr^2}J_{m+1}(pr) + \frac{(m+1)p}{r}\cdot\frac{1}{p}\frac{d}{dr}J_{m+1}(pr) - \left(\frac{m+1}{r^2}+\beta^2\right)J_{m+1}(pr)\right]H_{m+1}^{(1)}(pr')$$
$$+ \frac{d_-}{2\sqrt{2}}\left[p^2\cdot\frac{1}{p^2}\frac{d^2}{dr^2}J_{m-1}(pr) - \frac{(m-1)p}{r}\cdot\frac{1}{p}\frac{d}{dr}J_{m-1}(pr) + \left(\frac{m-1}{r^2}-\beta^2\right)J_{m-1}(pr)\right]H_{m+1}^{(1)}(pr').$$

Using the Bessel's differential equation

$$x^2 J_m''(x) + x J_m'(x) + (x^2 - m^2)J_m(x) = 0.$$
$$\Rightarrow p^2\cdot\frac{1}{p^2}\frac{d^2}{dr^2}J_{m\pm1}(pr) = -\frac{p}{r}\cdot\frac{1}{p}\frac{d}{dr}J_{m\pm1}(pr) - \left(p^2 - \frac{(m\pm1)^2}{r^2}\right)J_{m\pm1}(pr)$$
$$\Rightarrow E_{\phi,m\beta}^{(0)} = -\frac{im\beta}{2r}d_z^0 H_m^{(1)}(pr')J_m(pr)$$
$$+ \frac{d_+}{2\sqrt{2}}\left[\frac{mp}{r}\cdot\frac{d}{d(pr)}J_{m+1}(pr) + \left(\frac{m(m+1)}{r^2} - p^2 - \beta^2\right)J_{m+1}(pr)\right]H_{m+1}^{(1)}(pr')$$
$$+ \frac{d_-}{2\sqrt{2}}\left[-\frac{mp}{r}\frac{d}{d(pr)}J_{m-1}(pr) + \left(\frac{m(m-1)}{r^2} - p^2 - \beta^2\right)J_{m-1}(pr)\right]H_{m+1}^{(1)}(pr').$$

Now, we use the property that

$$J_m(pr) = \frac{m+1}{pr}J_{m+1}(pr) + \frac{d}{d(pr)}J_{m+1}(pr) = \frac{m-1}{pr}J_{m-1}(pr) - \frac{d}{d(pr)}J_{m-1}(pr).$$

hence

$$E_{\phi,m\beta}^{(0)} = -\frac{im\beta d_z^0}{2r}H_m^{(1)}(pr')J_m(pr)$$
$$+ \frac{d_+}{2\sqrt{2}}\left[\frac{mp}{r}J_m(pr) - (p^2+\beta^2)J_{m+1}(pr)\right]H_{m+1}^{(1)}(pr')$$
$$+ \frac{d_-}{2\sqrt{2}}\left[\frac{mp}{r}J_m(pr) - (p^2+\beta^2)J_{m-1}(pr)\right]H_{m-1}^{(1)}(pr')$$
$$= -\frac{im\beta d_z^0}{2r}H_m^{(1)}(pr')J_m(pr) + \frac{d_+}{2\sqrt{2}}\left[\frac{mp}{r}J_m(pr) - k^2 J_{m+1}(pr)\right]H_{m+1}^{(1)}(pr')$$
$$+ \frac{d_-}{2\sqrt{2}}\left[\frac{mp}{r}J_m(pr) - k^2 J_{m-1}(pr)\right]H_{m-1}^{(1)}(pr').$$

By exchanging J & $H^{(1)}$ functions, we reproduced Nha's result (Equ. 2.8 in their 1997 paper).

Now, we apply the same treatment to $B_{r,m\phi}^{(0)}$ & $E_{r,m\phi}^{(0)}$ terms.

$$B_{r,m\beta}^{(0)} = \frac{k}{2}\left[\frac{imd_z^0}{r}J_m(pr)H_m^{(1)}(pr') + \frac{\beta}{\sqrt{2}}d_+ J_{m+1}(pr)H_{m+1}^{(1)}(pr') + \frac{\beta}{\sqrt{2}}d_- J_{m-1}(pr)H_{m-1}(pr')\right]$$

$$
\begin{aligned}
E^{(0)}_{r,m\beta} &= \frac{d_+}{2\sqrt{2}}\left[-i\beta^2 J_{m+1}(pr) + \frac{ip}{r}\frac{d}{d(pr)}J_{m+1}(pr) - \frac{i(m+1)^2}{r^2}J_{m+1}(pr) + \frac{i(m+1)}{r^2}J_{m+1}(pr) - \frac{i(m+1)p}{r}\frac{d}{d(pr)}J_{m+1}(pr)\right]H^{(1)}_{m+1}(pr') \\
&+ \frac{d_-}{2\sqrt{2}}\left[i\beta^2 J_{m-1}(pr) - \frac{ip}{r}\frac{d}{d(pr)}J_{m-1}(pr) + \frac{i(m-1)^2}{r^2}J_{m-1}(pr) + \frac{i(m-1)}{r^2}J_{m-1}(pr) - \frac{i(m-1)p}{r}\frac{d}{d(pr)}J_{m-1}(pr)\right]H_{m-1}(pr') \\
&- \frac{\beta}{2}d^0_z\frac{d}{dr}J_m(pr)\,H^{(1)}_m(pr') \\
&= \frac{\beta p d^0_z}{4}\left[J_{m+1}(pr) - J_{m-1}(pr)\right]H^{(1)}_m(pr') \\
&+ \frac{d_+}{2\sqrt{2}}\left[-i\beta^2 J_{m+1}(pr) - \frac{imp}{r}\frac{d}{d(pr)}J_{m+1}(pr) - \frac{im(m+1)}{r^2}J_{m+1}(pr)\right]H^{(1)}_{m+1}(pr') \\
&+ \frac{d_-}{2\sqrt{2}}\left[i\beta^2 J_{m-1}(pr) - \frac{imp}{r}\frac{d}{d(pr)}J_{m-1}(pr) + \frac{im(m-1)}{r^2}J_{m-1}(pr)\right]H^{(1)}_{m-1}(pr') \\
&= \frac{\beta p d^0_z}{4}\left[J_{m+1}(pr) - J_{m-1}(pr)\right]H^{(1)}_m(pr') \\
&+ \frac{d_+}{2\sqrt{2}}\left[-i\beta^2 J_{m+1}(pr) - \frac{imp}{r}J_m(pr)\right]H^{(1)}_{m+1}(pr') \\
&+ \frac{d_-}{2\sqrt{2}}\left[i\beta^2 J_{m-1}(pr) + \frac{imp}{r}J_m(pr)\right]H^{(1)}_{m-1}(pr').
\end{aligned}
$$

For a further check, we see that

$$
\begin{aligned}
B^{(0)}_{z,m\beta} &= \frac{ikd_+}{2\sqrt{2}}\left[\frac{p(m+1)}{pr}J_{m+1}(pr) + p\frac{d}{d(pr)}J_{m+1}(pr)\right]H^{(1)}_{m+1}(pr') \\
&+ \frac{ikd_-}{2\sqrt{2}}\left[-\frac{p(m-1)}{pr}J_{m-1}(pr) + p\frac{d}{d(pr)}J_{m-1}(pr)\right]H^{(1)}_{m-1}(pr') \\
&= \frac{ikp}{2\sqrt{2}}J_m(pr)\left[d_+H^{(1)}_{m+1}(pr') - d_-H^{(1)}_{m-1}(pr')\right]
\end{aligned}
$$

$$
B^{(0)}_{\phi,m\beta} = \frac{ik}{2}\left[\frac{\beta d_-}{\sqrt{2}}J_{m-1}(pr)H^{(1)}_{m-1}(pr') - \frac{\beta d_+}{\sqrt{2}}J_{m+1}(pr)H^{(1)}_{m+1}(pr') + \frac{ip d^0_z}{2}\left(J_{m-1}(pr) - J_{m+1}(pr)\right)H^{(1)}_m(pr')\right]
$$

$$
\begin{aligned}
E^{(0)}_{z,m\beta} &= \frac{id^0_z}{2}\left[-p^2\frac{d^2}{d(pr)^2}J_m(pr) - \frac{p^2}{pr}\cdot\frac{d}{d(pr)}J_m(pr) + p^2\frac{m^2}{(pr)^2}J_m(pr)\right]H^{(1)}_m(pr') \\
&+ \frac{d_+}{2\sqrt{2}}\left[\frac{\beta(m+1)}{r}J_{m+1}(pr) + \beta p\frac{d}{d(pr)}J_{m+1}(pr)\right]H^{(1)}_{m+1}(pr') \\
&+ \frac{d_-}{2\sqrt{2}}\left[\frac{\beta(m-1)}{r}J_{m-1}(pr) - \beta p\frac{d}{d(pr)}J_{m-1}(pr)\right]H^{(1)}_{m-1}(pr') \\
&= \frac{id^0_z}{2}p^2 J_m(pr)H^{(1)}_m(pr') \\
&+ \frac{d_+\beta p}{2\sqrt{2}}J_m(pr)H^{(1)}_{m+1}(pr') + \frac{d_-\beta p}{2\sqrt{2}}J_m(pr)H^{(1)}_{m-1}(pr') \\
&= \frac{p}{2}J_m(pr)\left[id^0_z p H^{(1)}_m(pr') + \frac{d_+\beta}{\sqrt{2}}H^{(1)}_{m+1}(pr') + \frac{d_-\beta}{\sqrt{2}}H^{(1)}_{m-1}(pr')\right]
\end{aligned}
$$

These results are consistent with Nha's result.