

# Derivation of free-space E-field components.

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0/21-1 For  $r < r'$  case (I use  $r$  to replace  $r_1$ )

$$\begin{aligned} E_{z,m\phi}^{(0)} &= \frac{i}{k} \left( \frac{\partial (r B_\phi^{(0)})}{\partial r} - \frac{\partial B_r^{(0)}}{\partial \phi} \right) \\ &= \frac{i}{k} \left[ \frac{B_\phi^{(0)}}{r} + \frac{\partial B_\phi^{(0)}}{\partial r} - \frac{1}{r} \frac{\partial B_r^{(0)}}{\partial \phi} \right] \\ &= \frac{i}{2} \left[ \frac{1}{r} (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) \beta H_m^{(1)}(pr') \bar{J}_m(pr) - \frac{d_z^\circ}{r} H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \right. \\ &\quad + \frac{1}{r} (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) \beta H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) - d_z^\circ H_m^{(1)}(pr') \frac{d^2}{dr^2} \bar{J}_m(pr) \\ &\quad - \frac{1}{r} (\sin(\phi-\phi') d_\phi^\circ + \cos(\phi-\phi') d_r^\circ) \beta H_m^{(1)}(pr') \bar{J}_m(pr) \\ &\quad \left. + \frac{m}{r} \left( \frac{d_z^\circ m}{r} - (\cos(\phi-\phi') d_\phi^\circ - \sin(\phi-\phi') d_r^\circ) \beta \right) H_m^{(1)}(pr') \bar{J}_m(pr) \right] \\ &= \frac{im}{2r} \left[ \frac{d_z^\circ m}{r} + \beta \sin(\phi-\phi') d_r^\circ - \beta \cos(\phi-\phi') d_\phi^\circ \right] H_m^{(1)}(pr') \bar{J}_m(pr) \\ &\quad - \frac{1}{2} \left[ \frac{d_z^\circ}{r} + \beta (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) \right] H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \\ &\quad - \frac{1}{2} d_z^\circ H_m^{(1)}(pr') \frac{d^2}{dr^2} \bar{J}_m(pr). \end{aligned}$$

$$\begin{aligned} E_{\phi,m\beta}^{(0)} &= \frac{i}{k} \left( \frac{\partial B_r^{(0)}}{\partial z} - \frac{\partial B_z^{(0)}}{\partial r} \right) \\ &= -\frac{\beta}{2} \left[ \frac{d_z^\circ m}{r} - i (\cos(\phi-\phi') d_\phi^\circ - \sin(\phi-\phi') d_r^\circ) \beta \right] H_m^{(1)}(pr') \bar{J}_m(pr) \\ &\quad - \frac{i}{2} \left[ \frac{1}{r^2} (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) m H_m^{(1)}(pr') \bar{J}_m(pr) \right. \\ &\quad - \frac{1}{r} (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) m H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \\ &\quad \left. + (\cos(\phi-\phi') d_\phi^\circ - \sin(\phi-\phi') d_r^\circ) H_m^{(1)}(pr') \frac{d^2}{dr^2} \bar{J}_m(pr) \right] \\ &= \frac{1}{2} \left[ \frac{d_z^\circ m \beta}{r} + \left( \frac{m}{r^2} \cos(\phi-\phi') - i \beta^2 \sin(\phi-\phi') \right) d_r^\circ + \left( \frac{m}{r^2} \sin(\phi-\phi') + i \beta^2 \cos(\phi-\phi') \right) d_\phi^\circ \right] H_m^{(1)}(pr') \bar{J}_m(pr) \\ &\quad - \frac{1}{2r} (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) m H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \\ &\quad - \frac{i}{2} (\cos(\phi-\phi') d_\phi^\circ - \sin(\phi-\phi') d_r^\circ) H_m^{(1)}(pr') \frac{d^2}{dr^2} \bar{J}_m(pr) \end{aligned}$$

$$\begin{aligned} E_{r,m\beta}^{(0)} &= \frac{i}{k} \left( \frac{1}{r} \frac{\partial B_z^{(0)}}{\partial \phi} - \frac{\partial B_\phi^{(0)}}{\partial z} \right) \\ &= \frac{i}{2r} \left[ \frac{1}{r} (-\sin(\phi-\phi') d_r^\circ + \cos(\phi-\phi') d_\phi^\circ + im \cos(\phi-\phi') d_r^\circ + im \sin(\phi-\phi') d_\phi^\circ) m H_m^{(1)}(pr') \bar{J}_m(pr) \right. \\ &\quad \left. + (-\sin(\phi-\phi') d_\phi^\circ - \cos(\phi-\phi') d_r^\circ + im \cos(\phi-\phi') d_\phi^\circ - im \sin(\phi-\phi') d_r^\circ) H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \right] \\ &\quad + \frac{\beta}{2} \left[ i (\cos(\phi-\phi') d_r^\circ + \sin(\phi-\phi') d_\phi^\circ) \beta H_m^{(1)}(pr') \bar{J}_m(pr) - d_z^\circ H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr) \right] \\ &= \left\{ \frac{1}{2} \left[ (i\beta^2 + \frac{im^2}{r^2}) \cos(\phi-\phi') - \frac{m}{r^2} \sin(\phi-\phi') \right] d_r^\circ + \frac{1}{2} \left[ (i\beta^2 + \frac{im^2}{r^2}) \sin(\phi-\phi') + \frac{m}{r^2} \cos(\phi-\phi') \right] d_\phi^\circ \right\} H_m^{(1)}(pr') \bar{J}_m(pr) \\ &\quad + \frac{1}{2} \left[ -d_z^\circ \beta - \frac{1}{r} (i \cos(\phi-\phi') - m \sin(\phi-\phi')) d_r^\circ - \frac{d_\phi^\circ}{r} (i \sin(\phi-\phi') + m \cos(\phi-\phi')) \right] H_m^{(1)}(pr') \frac{d}{dr} \bar{J}_m(pr). \end{aligned}$$

Noticed that  $\cos(\phi-\phi') = \frac{e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}}{2}$ ,  $\sin(\phi-\phi') = \frac{e^{i(\phi-\phi')} - e^{-i(\phi-\phi')}}{2i}$ , each  $e^{\pm i(\phi-\phi')}$  term will

change the mode index, so that the results we obtained above is not really the "m"-th mode components. To correct that, we redefine

$$\begin{cases} d_r = \cos(\phi - \phi') d_r^0 + \sin(\phi - \phi') d_\phi^0 = \frac{1}{\sqrt{2}} (-e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) \\ d_\phi = -\sin(\phi - \phi') d_r^0 + \cos(\phi - \phi') d_\phi^0 = \frac{i}{\sqrt{2}} (e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) \end{cases} \quad d_{\pm} \equiv \mp \frac{1}{\sqrt{2}} (d_r \pm i d_\phi)$$

Now the " $E_{\phi, m\beta}^{(0)}$ " can be rewritten as

$$\begin{aligned} "E_{\phi, m\beta}^{(0)}" &= \frac{1}{2} \left[ \frac{i d_z^0 m \beta}{r} + \frac{m}{\sqrt{2} r^2} (-e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) - \frac{\beta^2}{\sqrt{2}} (e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) \right] H_m^{(1)}(pr') J_m(pr) \\ &\quad - \frac{1}{2r\sqrt{2}} (-e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) m H_m^{(1)}(pr') \frac{d}{dr} J_m(pr) \\ &\quad + \frac{1}{\sqrt{2}} (e^{-i(\phi - \phi')} d_+ + e^{+i(\phi - \phi')} d_-) H_m^{(1)}(pr') \frac{d^2}{dr^2} J_m(pr) \end{aligned}$$

Considering the  $E_{\phi, m\beta}^{(0)}$  only goes with  $e^{im\phi + i\beta z}$  term in  $E_{\phi}^{(0)}$  expansion, the real  $E_{\phi, m\beta}^{(0)}$  should be

$$\begin{aligned} E_{\phi, m\beta}^{(0)} &= -\frac{i d_z^0 m \beta}{2r} H_m^{(1)}(pr') J_m(pr) \\ &\quad + \frac{d_+}{2\sqrt{2}} \left[ p^2 \frac{1}{p^2} \frac{d^2}{dr^2} J_{m+1}(pr) + \frac{(m+1)p}{r} \cdot \frac{1}{p} \frac{d}{dr} J_{m+1}(pr) - \left( \frac{m+1}{r^2} + \beta^2 \right) J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\ &\quad + \frac{d_-}{2\sqrt{2}} \left[ p^2 \frac{1}{p^2} \frac{d^2}{dr^2} J_{m-1}(pr) - \frac{(m-1)p}{r} \cdot \frac{1}{p} \frac{d}{dr} J_{m-1}(pr) + \left( \frac{m-1}{r^2} - \beta^2 \right) J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr'). \end{aligned}$$

Using the Bessel's differential equation

$$\begin{aligned} x^2 J_m''(x) + x J_m'(x) + (x^2 - m^2) J_m(x) &= 0 \\ \Rightarrow p^2 \cdot \frac{1}{p^2} \frac{d^2}{dr^2} J_{m\pm 1}(pr) &= -\frac{p}{r} \cdot \frac{1}{p} \frac{d}{dr} J_{m\pm 1}(pr) - \left( p^2 - \frac{(m\pm 1)^2}{r^2} \right) J_{m\pm 1}(pr) \end{aligned}$$

$$\begin{aligned} \Rightarrow E_{\phi, m\beta}^{(0)} &= -\frac{i m \beta}{2r} d_z^0 H_m^{(1)}(pr') J_m(pr) \\ &\quad + \frac{d_+}{2\sqrt{2}} \left[ \frac{m p}{r} \cdot \frac{d}{d(pr)} J_{m+1}(pr) + \left( \frac{m(m+1)}{r^2} - p^2 - \beta^2 \right) J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\ &\quad + \frac{d_-}{2\sqrt{2}} \left[ -\frac{m p}{r} \frac{d}{d(pr)} J_{m-1}(pr) + \left( \frac{m(m-1)}{r^2} - p^2 - \beta^2 \right) J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr'). \end{aligned}$$

Now, we use the property that

$$J_m(pr) = \frac{m+1}{pr} J_{m+1}(pr) + \frac{d}{d(pr)} J_{m+1}(pr) = \frac{m-1}{pr} J_{m-1}(pr) - \frac{d}{d(pr)} J_{m-1}(pr).$$

hence

$$\begin{aligned} E_{\phi, m\beta}^{(0)} &= -\frac{i m \beta d_z^0}{2r} H_m^{(1)}(pr') J_m(pr) \\ &\quad + \frac{d_+}{2\sqrt{2}} \left[ \frac{m p}{r} J_m(pr) - (p^2 + \beta^2) J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\ &\quad + \frac{d_-}{2\sqrt{2}} \left[ \frac{m p}{r} J_m(pr) - (p^2 + \beta^2) J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr') \\ &= -\frac{i m \beta d_z^0}{2r} H_m^{(1)}(pr') J_m(pr) + \frac{d_+}{2\sqrt{2}} \left[ \frac{m p}{r} J_m(pr) - k^2 J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\ &\quad + \frac{d_-}{2\sqrt{2}} \left[ \frac{m p}{r} J_m(pr) - k^2 J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr'). \end{aligned}$$

By exchanging  $J$  &  $H^{(1)}$  functions, we reproduced Nha's result (Equ. 2.8 in their 1997 paper).

Now, we apply the same treatment to  $B_{r, m\beta}^{(0)}$  &  $E_{r, m\beta}^{(0)}$  terms.

$$B_{r, m\beta}^{(0)} = \frac{k}{2} \left[ \frac{i m d_z^0}{r} J_m(pr) H_m^{(1)}(pr') + \frac{\beta}{\sqrt{2}} d_+ J_{m+1}(pr) H_{m+1}^{(1)}(pr') + \frac{\beta}{\sqrt{2}} d_- J_{m-1}(pr) H_{m-1}^{(1)}(pr') \right]$$

$$\begin{aligned}
E_{r,m\beta}^{(c)} &= \frac{d_+}{2\sqrt{2}} \left[ -i\beta^2 J_{m+1}(pr) + \frac{iP}{r} \frac{d}{d(pr)} J_{m+1}(pr) - \frac{i(m+1)^2}{r^2} J_{m+1}(pr) + \frac{i(m+1)}{r^2} J_{m+1}(pr) - \frac{i(m+1)P}{r} \frac{d}{d(pr)} J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\
&\quad + \frac{d_-}{2\sqrt{2}} \left[ i\beta^2 J_{m-1}(pr) - \frac{iP}{r} \frac{d}{d(pr)} J_{m-1}(pr) + \frac{i(m-1)^2}{r^2} J_{m-1}(pr) + \frac{i(m-1)}{r^2} J_{m-1}(pr) - \frac{i(m-1)P}{r} \frac{d}{d(pr)} J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr') \\
&\quad - \frac{\beta d_0}{2} \frac{d}{dr} J_m(pr) H_m^{(1)}(pr') \\
&= \frac{\beta P d_0}{4} [J_{m+1}(pr) - J_{m-1}(pr)] H_m^{(1)}(pr') \\
&\quad + \frac{d_+}{2\sqrt{2}} \left[ -i\beta^2 J_{m+1}(pr) - \frac{i m P}{r} \frac{d}{d(pr)} J_{m+1}(pr) - \frac{i m(m+1)}{r^2} J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\
&\quad + \frac{d_-}{2\sqrt{2}} \left[ i\beta^2 J_{m-1}(pr) - \frac{i m P}{r} \frac{d}{d(pr)} J_{m-1}(pr) + \frac{i m(m-1)}{r^2} J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr') \\
&= \frac{\beta P d_0}{4} [J_{m+1}(pr) - J_{m-1}(pr)] H_m^{(1)}(pr') \\
&\quad + \frac{d_+}{2\sqrt{2}} \left[ -i\beta^2 J_{m+1}(pr) - \frac{i m P}{r} J_m(pr) \right] H_{m+1}^{(1)}(pr') \\
&\quad + \frac{d_-}{2\sqrt{2}} \left[ i\beta^2 J_{m-1}(pr) + \frac{i m P}{r} J_m(pr) \right] H_{m-1}^{(1)}(pr').
\end{aligned}$$

For a further check, we see that

$$\begin{aligned}
B_{z,m\beta}^{(c)} &= \frac{ikd_+}{2\sqrt{2}} \left[ \frac{P(m+1)}{pr} J_{m+1}(pr) + P \frac{d}{d(pr)} J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\
&\quad + \frac{ikd_-}{2\sqrt{2}} \left[ -\frac{P(m-1)}{pr} J_{m-1}(pr) + P \frac{d}{d(pr)} J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr') \\
&= \frac{i k P}{2\sqrt{2}} J_m(pr) [d_+ H_{m+1}^{(1)}(pr') - d_- H_{m-1}^{(1)}(pr')] \\
B_{\theta,m\beta}^{(c)} &= \frac{i k}{2} \left[ \frac{\beta d_-}{\sqrt{2}} J_{m-1}(pr) H_{m-1}^{(1)}(pr') - \frac{\beta d_+}{\sqrt{2}} J_{m+1}(pr) H_{m+1}^{(1)}(pr') + \frac{i P d_0}{2} (J_{m-1}(pr) - J_{m+1}(pr)) H_m^{(1)}(pr') \right] \\
E_{z,m\beta}^{(c)} &= \frac{id_0}{2} \left[ -P^2 \frac{d^2}{d(pr)^2} J_m(pr) - \frac{P^2}{pr} \frac{d}{d(pr)} J_m(pr) + P^2 \frac{m^2}{(pr)^2} J_m(pr) \right] H_m^{(1)}(pr') \\
&\quad + \frac{d_+}{2\sqrt{2}} \left[ \frac{\beta(m+1)}{r} J_{m+1}(pr) + \beta P \frac{d}{d(pr)} J_{m+1}(pr) \right] H_{m+1}^{(1)}(pr') \\
&\quad + \frac{d_-}{2\sqrt{2}} \left[ \frac{\beta(m-1)}{r} J_{m-1}(pr) - \beta P \frac{d}{d(pr)} J_{m-1}(pr) \right] H_{m-1}^{(1)}(pr') \\
&= \frac{id_0}{2} P^2 J_m(pr) H_m^{(1)}(pr') \\
&\quad + \frac{d_+ \beta P}{2\sqrt{2}} J_m(pr) H_{m+1}^{(1)}(pr') + \frac{d_- \beta P}{2\sqrt{2}} J_m(pr) H_{m-1}^{(1)}(pr') \\
&= \frac{P}{2} J_m(pr) \left[ id_0 P H_m^{(1)}(pr') + \frac{d_+ \beta}{\sqrt{2}} H_{m+1}^{(1)}(pr') + \frac{d_- \beta}{\sqrt{2}} H_{m-1}^{(1)}(pr') \right]
\end{aligned}$$

These results are consistent with Nha's result.