Group velocity of the nanofiber

Thursday, August 14, 2014 3:47 PM

140814. Two ways to calculate the group velocity, v_g:

1. from the perspective of energy propagation. $\longrightarrow$ $v_g = \dfrac{P}{W} = \dfrac{\int d\vec{r}_\perp\, I(\vec{r}_\perp)}{\frac{1}{2}\int d\vec{r}_\perp\, n^2(r_\perp)\,|\vec{\mathcal{E}}^g(r_\perp)|^2}$ $\leftarrow$ Total power; $\leftarrow$ Energy stored per length

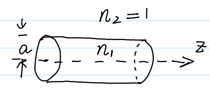
2. derive from the eigenvalue equation of the nanofiber, which determines $\beta = \beta(\omega)$ for the guided modes.

§1. Here, we consider the second approach first.

The eigenvalue equation for a single-mode nanofiber supporting HE_{11} mode is given by [see Fam Le Kien 2004 opt comm, or F. Le Kien 2005 PRA]:

$$\frac{J_0(ha)}{ha\,J_1(ha)} + \frac{n_1^2+n_2^2}{2n_1^2}\cdot\frac{K_1'(qa)}{qa\,K_1(qa)} - \frac{1}{h^2a^2} + \left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\,\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{1/2} = 0 \qquad (1)$$

where $\begin{cases} h = (n_1^2k^2-\beta^2)^{1/2} \rightarrow \text{transverse propagation constant.} \\ q = (\beta^2 - n_2^2k^2)^{1/2} \rightarrow \text{decay constant.} \end{cases}$



We let the left-hand-side part of Equ (1) be $f(h,q,k,\beta) = f(k,\beta)$, and hence Equ. (1) gives

$$f(h,q,k,\beta) = 0. \qquad (2)$$

$$\Rightarrow \frac{df}{d\beta} = \frac{\partial f}{\partial h}\cdot\left(\frac{\partial h}{\partial k}\cdot\frac{dk}{d\beta}+\frac{\partial h}{\partial\beta}\right) + \frac{\partial f}{\partial q}\cdot\left[\frac{\partial q}{\partial k}\cdot\frac{dk}{d\beta}+\frac{\partial q}{\partial\beta}\right] + \frac{\partial f}{\partial k}\cdot\frac{dk}{d\beta} + \frac{\partial f}{\partial\beta} = 0.$$

$$\Rightarrow \left[\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial k} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial k} + \frac{\partial f}{\partial k}\right]\cdot\frac{dk}{d\beta} = -\left[\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial\beta} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial\beta} + \frac{\partial f}{\partial\beta}\right]$$

$$\Rightarrow \boxed{\frac{dk}{d\beta} = -\frac{\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial\beta} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial\beta} + \frac{\partial f}{\partial\beta}}{\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial k} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial k} + \frac{\partial f}{\partial k}}}. \qquad (3)$$

Based on this, the group velocity can be calculated as

$$\boxed{v_g = \frac{d\omega}{d\beta} = c\cdot\frac{dk}{d\beta}}. \qquad (4).$$

For the detailed calculation, we have

$$\frac{\partial h}{\partial\beta} = -\frac{\beta}{h},\quad \frac{\partial q}{\partial\beta} = \frac{\beta}{q},\quad \frac{\partial h}{\partial k} = n_1^2\frac{k}{h},\quad \frac{\partial q}{\partial k} = -n_2^2\frac{k}{q}.$$

$$\frac{\partial f}{\partial h} = \frac{J_0'(ha)}{h\,J_1(ha)} - \frac{J_0(ha)J_1'(ha)}{h\,J_1^2(ha)} - \frac{J_0(ha)}{h^2a\,J_1(ha)} + \frac{2}{h^3a^2} - \frac{2\beta^2}{a^2n_1^2k^2h^3}\left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\,\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-\frac{1}{2}}\cdot\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right).$$

$$\frac{\partial f}{\partial q} = \frac{n_1^2+n_2^2}{2n_1^2}\cdot\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{K_1'(qa)}{q^2a\,K_1(qa)}\right] + \left\{\left(\frac{n_1^2-n_2^2}{2n_1^2}\right)^2\frac{K_1'(qa)}{qa\,K_1(qa)}\cdot\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{K_1'(qa)}{q^2a\,K_1(qa)}\right] - \frac{2\beta^2}{a^2n_1^2k^2q^3}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)\right\}$$
$$\cdot\left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\cdot\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-1/2}.$$

$$\frac{\partial f}{\partial k} = -\frac{\beta^2}{n_1^2k^3}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\cdot\left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\cdot\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-1/2}.$$

$$\frac{\partial f}{\partial\beta} = \frac{\beta}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\cdot\left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\cdot\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-1/2}.$$

$$\Rightarrow D = \frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial\beta} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial\beta} + \frac{\partial f}{\partial\beta}$$
$$= -\frac{\beta J_0'(ha)}{h^2J_1(ha)} + \frac{\beta J_0(ha)J_1'(ha)}{h^2J_1^2(ha)} + \frac{\beta J_0(ha)}{h^3a\,J_1(ha)} - \frac{2\beta}{h^4a^2} + \frac{\beta(n_1^2+n_2^2)}{2q\,n_1^2}\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{\beta K_1'(qa)}{q^3a\,K_1(qa)}\right]$$
$$+ \left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\cdot\frac{K_1(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-1/2}\cdot\left\{\frac{\beta}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2 + \frac{2\beta^3}{a^2n_1^2k^2}\left(\frac{1}{h^4}-\frac{1}{q^4}\right)\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)\right.$$
$$\left. + \left(\frac{n_1^2-n_2^2}{2n_1^2}\right)^2\frac{\beta K_1'(qa)}{q^2a\,K_1(qa)}\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{K_1'(qa)}{q^2a\,K_1(qa)}\right]\right\}$$

$$N = \frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial k} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial k} + \frac{\partial f}{\partial k}$$
$$= \frac{n_1^2k\,J_0'(ha)}{h^2J_1(ha)} - \frac{n_1^2k\,J_0(ha)J_1'(ha)}{h^2J_1^2(ha)} - \frac{n_1^2k\,J_0(ha)}{h^3a\,J_1(ha)} + \frac{2n_1^2k}{h^4a^2} - \frac{n_2^2k(n_1^2+n_2^2)}{2q\,n_1^2}\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{K_1'(qa)}{q^2a\,K_1(qa)}\right]$$
$$+ \left\{\left[\frac{n_1^2-n_2^2}{2n_1^2}\cdot\frac{K_1'(qa)}{qa\,K_1(qa)}\right]^2 + \frac{\beta^2}{n_1^2k^2}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2\right\}^{-1/2}\cdot\left\{-\frac{\beta^2}{n_1^2k^3}\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)^2 + \frac{2\beta^2}{a^2n_1^2k}\left(\frac{n_2^2}{q^4}-\frac{n_1^2}{h^4}\right)\left(\frac{1}{q^2a^2}+\frac{1}{h^2a^2}\right)\right.$$
$$\left. - \left(\frac{n_1^2-n_2^2}{2n_1^2}\right)^2\frac{n_2^2k\,K_1'(qa)}{q^2a\,K_1(qa)}\left[\frac{K_1''(qa)}{q\,K_1(qa)} - \frac{(K_1'(qa))^2}{q\,K_1^2(qa)} - \frac{K_1'(qa)}{q^2a\,K_1(qa)}\right]\right\}.$$

Using the results above, one can obtain the group velocity as

$$\boxed{v_g = -c\cdot\frac{D}{N} = \frac{c}{n_g}}$$

where $\boxed{n_g = -\frac{N}{D}}$ is the group index of refraction of the waveguide for the HE_{11} mode.

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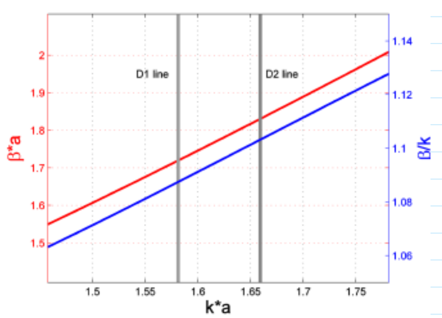
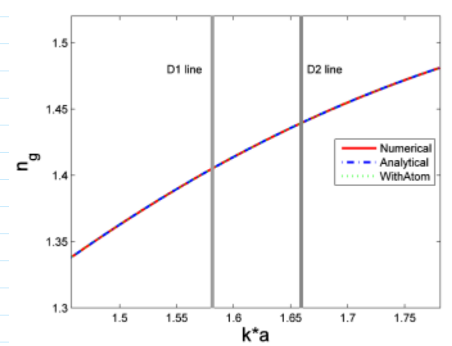
We make the bulk index of refraction of the nanofiber $n_1 = 1.4496$. The result group index of refraction for our nanofiber ($a = 225\,nm$) is $n_g = 1.4395$ @ D_2 line. The dispersion relation has been calculated both numerically & analytically to show our derivation is correct (see figures below). Meanwhile, the dispersion relation of the waveguide in presence of an atom can also be derived in a similar fashion. We show that the waveguide dispersion relation is a pure geometric effect which only depends on the boundary condition of the waveguide, and is independent of the photon emitting properties of the material & impurities. In our nanofiber case, the material dispersion is negligible ($\frac{dn}{d\lambda} \sim -0.04/um$ from Gorachand Ghosh, Opt. Commun. 163, 95-102 (1999) & the data base from refractiveindex.info website), the energy stored in the surrending atoms is also negligible. We can safely use the waveguide dispersion relationship to calculate the phase & group velocity/index of refraction of our nanofiber system.

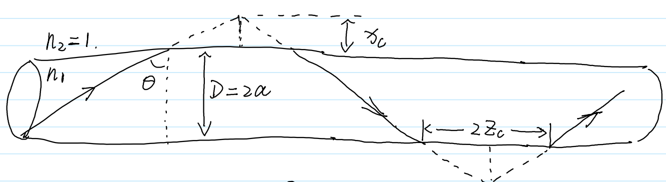


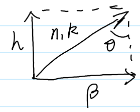


§2.1 Now, we have a straightforward way of calculating the waveguide dispersion parameters. The remaining question is what's the physics meaning of those parameters. Below, we will use the Zigzag Ray model to interpret the phase & group velocities, and relate them to the energy flow process in a waveguide.

The zigzag ray model is an effective model for understanding the wave propagation problem. It was first studied by Kogelnik, Weber and Burke in 1974. More details can be found in, for example, Kogelnik, J. Opt Soc. Am, 64. 2 (1974) for film waveguides with TE and TM modes. Here, we present some useful results for our nanofiber case.

The effective ray path propagating along a nanofiber is shown on the right. For a guided mode, the ray with an incident angle of θ penetrates into the clad with a depth of x_c due to the Goos-Haenchen effect. The effective





$$\sin\theta = \frac{\beta}{n_1 k}$$

ray propagates $2Z_c$ distance along the fiber axis direction in the effective time delay $2\tau_c$.

$$\begin{cases} Z_c = \frac{\partial\phi}{\partial\beta} \\ \tau_c = -\frac{\partial\phi}{\partial\omega}. \end{cases}$$ ← ϕ is the Goos-Haenchen phase shift, and can be calaculated for a given mode.

The main conclusion using this model is that

$\boxed{v_p = \frac{c}{n_p} = \frac{Z_c}{\tau_c}}$ → The phase velocity equals the effective velocity along fiber axis in the clad due to GH effect. (5)

$\boxed{v_g = \frac{c}{n_g} = \frac{Z_1 + 4Z_c}{\tau_1 + 4\tau_c}}$ → The group velocity equals the effective velocity along z through the whole system. (6)

Above, we have let Z_1 the traveling distance across the nanofiber diameter projected to the fiber axis, and τ_1 the corresponding travel time of the light in the nanofiber region. They satisfy that

$$\begin{cases} Z_1 = D\cdot\tan\theta = 2a\cdot\frac{\beta}{h} \\ \tau_1 = m_1 D/(c\cdot\cos\theta) = \frac{2a\beta m_1}{c n_1 k}. \end{cases}$$

where m_1 is the group index of refraction of the bulk material of the nanofiber,

$$m_1 = n_1 + \omega\frac{dn_1}{d\omega} = n_1 - \lambda\frac{dn_1}{d\lambda}.$$

Due to the small material dispersion of the nanofiber, we could use $m_1 \simeq n_1$ for quick estimation. $Z_1 + 4Z_c$ & $\tau_1 + 4\tau_c$ are the total light travel distance & time in one period of the total internal refrection of the guided mode. n_1 is the usual bulk material index of refraction of the nanofiber, which equals to the phase index of the material.

Proof of Eqns. (5) & (6):

Eqn (5) is trivial to show by noticing that for a given configuration of waveguide n_p & V_p are constant, so that

$$Z_c = \frac{\partial\phi}{\partial\beta} = \frac{\partial\phi}{\partial(n_p k)} = \frac{\partial\phi}{\partial(\frac{n_p}{c}\omega)} = \frac{c}{n_p}\frac{\partial\phi}{\partial\omega} = V_p\frac{\partial\phi}{\partial\omega} = -V_p\cdot\tau_c.$$ (My method. The sign doesn't give any difference)

Eqn (6) can be shown through using the total internal reflection condition and geometry relationships that

$$\begin{cases} kDn_1\cos\theta - 2\phi = \nu\pi & (\nu \text{ is the mode order}). \\ \beta = kn_1\sin\theta. \end{cases}$$

Differentiate the equations above with respect to β, and obtain

$$\begin{cases} \frac{D\cos\theta}{c}(n_1 + \omega\frac{dn_1}{d\omega})\frac{d\omega}{d\beta} - kDn_1\sin\theta\frac{d\theta}{d\beta} - \frac{d}{d\beta}(2\phi) = 0 \\ 1 = \frac{\sin\theta}{c}(n_1 + \omega\frac{dn_1}{d\omega})\frac{d\omega}{d\beta} + kn_1\cos\theta\frac{d\theta}{d\beta} \end{cases}$$

Using the relation that $\phi = \phi(\omega, \beta)$

$$\frac{d\phi}{d\beta} = \frac{\partial\phi}{\partial\beta} + \frac{\partial\phi}{\partial\omega}\cdot\frac{d\omega}{d\beta} \quad \& \quad m_1 = n_1 + \omega\frac{dn_1}{d\omega}$$

one can solve for $\frac{d\omega}{d\beta}$ and obtain

$$\frac{d\omega}{d\beta} = \frac{D\tan\theta + \frac{\partial(2\phi)}{\partial\beta}}{\frac{m_1 D}{c\cdot\cos\theta} - \frac{\partial(2\phi)}{\partial\omega}}.$$

Notice that from geometry $\begin{cases} D\cdot\tan\theta = Z_1, & \frac{m_1 D}{c\cdot\cos\theta} = \tau_1, \\ \frac{\partial(2\phi)}{\partial\beta} = 2Z_c, & -\frac{\partial(2\phi)}{\partial\omega} = 2\tau_c. \end{cases}$

$$\Rightarrow v_g = \frac{d\omega}{d\beta} = \frac{Z_1 + 2Z_c}{\tau_1 + 2\tau_c}. \quad \boxed{\text{QED}}$$

§2.2. We have shown that the phase velocity/index is related to the light flow in the clad region. Any more staight interpertation for the group velocity/index?

In fact, after some algebra, one can show that interpertation for the group velocity / index?

In fact, after some algebra, one can show that

$$\boxed{n_g = \frac{c}{v_g} = \frac{\tilde{n}_g Z_1 + 2 n Z_c}{Z_1 + 2Z_c}}$$

where $\tilde{n}_g = \frac{m_1}{\sin\theta} = \frac{m_1 n_1}{n_p}$ is the pseudo group index of refraction calculated from the over-simplied zigzag-ray model without considering the Goos-Hanchen effect.

Since n_g is a convex function of Z_1 & $2Z_c$ bounded by $\tilde{n}_g$ & n, n_g is always within the "light-cone" defined by the bulk phase index of refraction and the pseudo-group index of the fiber. Those two dependence seems only rely on the properties of the fiber and its geometry.

$\tilde{v}_g = \frac{c}{m_1}\sin\theta$

§2.3. Beyond the geometric approach, we can also rederive the phase & group index/velocity in terms of power flow & energy stored in the nanofiber. Specifically, we will pinpoint some of the roles of transverse and longitudinal components of the EM fields.

We define the total power carried by a guided mode in a waveguide as

$$P = \int d^2r_\perp \cdot \langle S_z \rangle = \frac{1}{2}\int d^2 r_\perp \, Re(\vec{E}\times\vec{H}^*)\cdot\hat{z} = \frac{1}{2}\int d^2 r_\perp \cdot Re(\vec{E}\times\vec{H}^*)_z .$$

The P for our nanofiber system refers to the power flow along the fiber axis (z) in any cross section of the fiber.

Notice that, for a plant wave in a homogeneous medium, $\vec{E}, \vec{B} \sim e^{i\vec{k}\cdot\vec{x}}$, one can use $\vec{E} = \frac{\omega}{k^2}\vec{B}\times\vec{k}$ to simplify the expression of P to let it solely depend on $\vec{E}$ or $\vec{B}$. But for our waveguide case, this trick does not work in general, and

$$P = \frac{1}{2}\int d^2 r_\perp \, Re(\vec{E}_\perp \times \vec{H}_\perp^*)_z$$

only depend on the transverse components of E & B fields.

Similarly, we can define the energy stored per unit length along z-axis of the fiber as

$$\begin{aligned}
W &= \int d^2 r_\perp \cdot w \\
&= \frac{1}{2}\int d^2 r_\perp \cdot \vec{E}\cdot\vec{D} + \frac{1}{2}\int d^2 r_\perp \cdot \vec{B}\cdot\vec{H} \\
&= \frac{1}{4}\int d^2 r_\perp \cdot Re(\vec{E}\cdot\vec{D}^*) + \frac{1}{4}\int d^2 r_\perp \cdot Re(\vec{B}\cdot\vec{H}^*) \\
&= \frac{1}{4}\int d^2 r_\perp \cdot \varepsilon(\vec{r}_\perp)|\vec{E}|^2 + \frac{1}{4}\int d^2 r_\perp \cdot \mu(\vec{r}_\perp)|H|^2 . \\
&= W^\varepsilon + W^\mu
\end{aligned}$$

← Using the complex expression of fields
We ignore the material dispersion and loss for our case.

W^μ ↳ magnetic energy per unit length.

W^ε ↳ electronic energy per unit length.

As we will show later, $W^\varepsilon = W^\mu$, and hence

$$W = 2W^\varepsilon = \frac{1}{2}\int d^2 r_\perp \cdot \varepsilon(\vec{r}_\perp)|\vec{E}|^2 .$$

We split the guided mode fields into transverse and longitudinal (z) components, and have

$$\begin{cases} \vec{E}(\vec{r}_\perp, z; t) = (\vec{E}_\perp(\vec{r}_\perp) + \vec{E}_z(\vec{r}_\perp))\, e^{-i\omega t + i\beta z} \\ \vec{H}(\vec{r}_\perp, z; t) = (\vec{H}_\perp(\vec{r}_\perp) + \vec{H}_z(\vec{r}_\perp))\, e^{-i\omega t + i\beta z} \end{cases}$$

Based on the Maxwell equations that

$$\begin{cases} \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{H} = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t} \end{cases}$$

We obtain

$$\begin{cases} \nabla_\perp \times \vec{E}_\perp = i\omega\mu \vec{H}_z & (7) \\ \nabla_\perp \times \vec{H}_\perp = -i\omega\varepsilon \vec{E}_z & (8) \\ \nabla_\perp \times \vec{E}_z + i\beta \hat{e}_z \times \vec{E}_\perp = i\omega\mu \vec{H}_\perp & (9) \\ \nabla_\perp \times \vec{H}_z + i\beta \hat{e}_z \times \vec{H}_\perp = -i\omega\varepsilon \vec{E}_\perp & (10) \end{cases}$$

Now that (9) $\cdot \vec{H}_\perp^* - \vec{E}_z \cdot$ (8)* gives

$$\underline{(\nabla_\perp \times \vec{E}_z)\cdot \vec{H}_\perp^*} + i\beta \hat{e}_z \cdot (\vec{E}_\perp \times \vec{H}_\perp^*) - \underline{\vec{E}_z \cdot (\nabla_\perp \times \vec{H}_\perp^*)} = i\omega\mu |\vec{H}_\perp|^2 - i\omega\varepsilon |\vec{E}_z|^2$$

$$\Longleftrightarrow \nabla_\perp \cdot (\vec{E}_z \times \vec{H}_\perp^*) + i\beta (\vec{E}_\perp \times \vec{H}_\perp^*)\cdot \hat{e}_z = i\omega\mu |\vec{H}_\perp|^2 - i\omega\varepsilon |\vec{E}_z|^2. \quad (11)$$

Similarly, (7) $\cdot \vec{H}_z^* - \vec{E}_\perp \cdot$ (10)* gives

$$(\nabla_\perp \times \vec{E}_\perp)\cdot \vec{H}_z^* - \vec{E}_\perp \cdot (\nabla_\perp \times \vec{H}_z^*) + i\beta\, \vec{E}_\perp \cdot (\hat{e}_z \times \vec{H}_\perp^*) = i\omega\mu |\vec{H}_z|^2 - i\omega\varepsilon |\vec{E}_\perp|^2$$

$$\Longleftrightarrow \nabla_\perp \cdot (\vec{E}_\perp \times \vec{H}_z^*) - i\beta (\vec{E}_\perp \times \vec{H}_\perp^*)\cdot \hat{e}_z = i\omega\mu |\vec{H}_z|^2 - i\omega\varepsilon |\vec{E}_\perp|^2. \quad (12)$$

Next, we integrate Equs. (11) & (12) over the cross section of the nanofiber, and use the fact that

$$\int d^2 r_\perp\, \nabla_\perp \cdot (\vec{E}_z \times \vec{H}_\perp^*) = \oint_\infty d\vec{l} \cdot (\vec{E}_z \times \vec{H}_\perp^*) \xrightarrow{\text{field components decay exponentially when } r_\perp \to \infty \text{ for the fiber mode.}} 0 = \int d^2 r_\perp \cdot \nabla_\perp \cdot (\vec{E}_\perp \times \vec{H}_z^*)$$

We obtain two important relationships independent of propagation direction & polarization direction

$$\begin{cases} \beta P = 2\omega (W_\perp^\mu - W_z^\varepsilon) & (13) \\ \beta P = 2\omega (W_\perp^\varepsilon - W_z^\mu) & (14) \end{cases}$$

where the portions of the time-averaged stored energy per unit length are defined as

$$\begin{cases} W_\perp^\mu = \frac{1}{4}\int d^2 r_\perp \cdot \mu |\vec{H}_\perp|^2 \\ W_z^\mu = \frac{1}{4}\int d^2 r_\perp \cdot \mu |\vec{H}_z|^2 \\ W_\perp^\varepsilon = \frac{1}{4}\int d^2 r_\perp \cdot \varepsilon |\vec{E}_\perp|^2 \\ W_z^\varepsilon = \frac{1}{4}\int d^2 r_\perp \cdot \varepsilon |\vec{E}_z|^2 \end{cases}$$

and hence $W^\mu = W_\perp^\mu + W_z^\mu$, $W^\varepsilon = W_\perp^\varepsilon + W_z^\varepsilon$, $W_\perp = W_\perp^\mu + W_\perp^\varepsilon$, $W_z = W_z^\mu + W_z^\varepsilon$.

We add Equs. (13) & (14) to give

$$\beta P = \omega (W_\perp - W_z)$$

$$\Rightarrow \boxed{v_p = \frac{c}{n_p} = \frac{\omega}{\beta} = \frac{P}{W_\perp - W_z}}. \quad (15)$$

We substract these two equations to have

$$W^\mu = W^\varepsilon. \quad (16)$$

As has been shown in various ways, the group velocity can be given by (for example, see Appendix B of Kogelnik's 1974 paper).

$$\boxed{v_g = \frac{c}{n_g} = \frac{P}{W} = \frac{P}{W_\perp + W_z}}. \quad (17)$$

The ratio of Equs. (15) & (17) gives

$$\boxed{\frac{v_p}{v_g} = \frac{W_\perp + W_z}{W_\perp - W_z} \geq 1} \text{ for positive energy portions.}$$

$v_p = v_g$ valid i.f.f. $W_z = 0$.

The non-transverse nature of the nanofiber determines that $v_p > v_g$, or $n_g > n_p$.

§2.4. Another question for the nanofiber modes is how the evanescent field portion contributes to v_g & v_p. On the other hand, can we define a simple expression as ε_{eff} to reprensent the energy flow property when we quantize the traveling wave in a waveguide?

From the Kogelnik 1974 paper, for the TE & TM modes of a planary waveguide, the phase velocity can be simply defined as the energy flow in the clad

$$\boxed{v_p = \frac{P_c}{W_c} = \frac{c}{n_p}},$$

We would expect the same expression works for the HE mode as well — which is a mixture of TE & TM modes. However, the expression for v_g is a little complicated even for TE & TM modes only.

Below, let us start from the $v_g = \frac{P}{W}$ relationship to see if we can define something like

$$\varepsilon_{eff} = n_{eff}^2 = \frac{n_1^2 \int_{core} d^2 r_\perp |E|^2 + n_2^2 \int_{clad} d^2 r_\perp |E|^2}{\int d^2 r_\perp |E|^2}. \quad ?$$

For our fiber's HE modes, we have (from Qi's Scattering notes)

$$\begin{cases} \vec{E}_\perp = -\frac{1}{kn^2} \hat{e}_z \times (\beta \vec{H}_\perp + i \nabla_\perp H_z) \\ \vec{H}_\perp = \frac{1}{k} \hat{e}_z \times (\beta \vec{E}_z + i \nabla_\perp E_z) \\ E_z = i \frac{1}{kn^2} \hat{e}_z \cdot (\nabla_\perp \times \vec{H}_\perp) \\ H_z = -i \frac{1}{k} \hat{e}_z \cdot (\nabla_\perp \times \vec{E}_\perp) \end{cases}$$

$$\Rightarrow P = \frac{1}{2} \int d^2 r_\perp \, \mathrm{Re} (\vec{E}_\perp \times \vec{H}_\perp^*)_z$$

with

$$\vec{E}_\perp \times \vec{H}_\perp^*$$

$$= \frac{1}{k} \vec{E}_\perp \times [\hat{e}_z \times (\beta \vec{E}_z^* - i \nabla_\perp E_z^*)]$$

$$= \frac{1}{k} \hat{e}_z [\vec{E}_\perp \cdot (\beta \vec{E}_z^* - i \nabla_\perp E_z^*)] - \frac{1}{k} (\beta \vec{E}_z^* - i \nabla_\perp E_z^*) \underbrace{(\vec{E}_\perp \cdot \hat{e}_z)}_{0}$$

$$= -\frac{i}{k} \hat{e}_z (\vec{E}_\perp \cdot \nabla_\perp E_z^*)$$

$$\Rightarrow \boxed{P = \frac{1}{2k} \int d^2 r_\perp \cdot \mathrm{Im} [\vec{E}_\perp \cdot \nabla_\perp E_z^*]}.$$

Not so simple? To be continued ...