Derive Ephi term

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$$E_\phi^{(0)} = d_\phi k^2 G^{(0)}(\vec{r},\vec{r}') + (\vec{d}_0\cdot\nabla)\frac{1}{\rho}\frac{\partial G^{(0)}(\vec{r},\vec{r}')}{\partial\phi}$$
$$= d_\phi\left(\frac{1}{\rho^2}\frac{\partial^2 G^{(0)}(\vec{r},\vec{r}')}{\partial\phi^2} + k^2 G^{(0)}(\vec{r},\vec{r}')\right) + d_\rho\left(\frac{1}{\rho}\frac{\partial^2 G^{(0)}(\vec{r},\vec{r}')}{\partial\rho\,\partial\phi} - \frac{1}{\rho^2}\frac{\partial G^{(0)}(\vec{r},\vec{r}')}{\partial\phi}\right) + \frac{d_z}{\rho}\frac{\partial^2 G^{(0)}(\vec{r},\vec{r}')}{\partial z\,\partial\phi}$$

Using the definition that

$$d_\phi = \frac{i}{\sqrt{2}}\left(e^{-i\phi}d_+ + e^{i\phi}d_-\right),\quad d_\rho = \frac{1}{\sqrt{2}}\left(-e^{-i\phi}d_+ + e^{+i\phi}d_-\right),\quad d_z = d_0 = d_z^{(0)}$$

$d_\pm = \mp\frac{1}{\sqrt{2}}\left(d_\rho^{(0)} \pm i d_\phi^{(0)}\right)$, $d_0 = d_z^{(0)}$ ⟵ superscript "(0)" means at the atom position. $\vec{r}' = (\rho', \phi', z'=0)$.

$$G^{(0)}(\vec{r},\vec{r}') = \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = \frac{i}{2}\sum_{m=-\infty}^{\infty}\int d\beta\, e^{im\phi+i\beta z} J_m(p\rho) H^{(1)}(p\rho') \qquad \text{for } (\rho<\rho').$$

we have

$$E_\phi^{(0)} = \frac{i}{2}\int d\beta\left\{\sum_{m=-\infty}^{\infty}\left(-\frac{m^2 d_\phi}{\rho^2} + d_\phi k^2 - \frac{imd_\rho}{\rho^2} - \frac{\beta m d_z}{\rho}\right)e^{im\phi+i\beta z}J_m(p\rho)H_m^{(1)}(p\rho') + \sum_{m=-\infty}^{\infty}\frac{imd_\rho}{\rho}e^{im\phi+i\beta z}\frac{\partial J_m(p\rho)}{\partial\rho}H_m^{(1)}(p\rho')\right\}$$
$$= \frac{i}{2}\int d\beta\left\{\sum_{m=-\infty}^{\infty}\left[\left(-\frac{im^2 d_+}{\sqrt{2}\rho^2} + \frac{ik^2 d_+}{\sqrt{2}} + \frac{imd_+}{\sqrt{2}\rho^2}\right)e^{i(m-1)\phi} - \frac{\beta m d_z^{(0)}}{\rho}e^{im\phi} + \left(-\frac{im^2 d_-}{\sqrt{2}\rho^2} + \frac{ik^2 d_-}{\sqrt{2}} - \frac{imd_-}{\sqrt{2}\rho^2}\right)e^{i(m+1)\phi}\right]e^{i\beta z}J_m(p\rho)H_m^{(1)}(p\rho') + \sum_{m=-\infty}^{\infty}\frac{im}{\sqrt{2}\rho}\left[-d_+e^{i(m-1)\phi} + d_-e^{i(m+1)\phi}\right]e^{i\beta z}\frac{\partial J_m(p\rho)}{\partial\rho}H_m^{(1)}(p\rho')\right\}.$$

We can obtain the $E_\phi^{(0)}$ component associated with $e^{im\phi+i\beta z}$ term as.

$$E_{\phi,m\beta}^{(0)} = \frac{i}{2}\left[-\frac{i(m+1)^2 d_+}{\sqrt{2}\rho^2} + \frac{ik^2 d_+}{\sqrt{2}} + \frac{i(m+1)d_+}{\sqrt{2}\rho^2}\right]J_{m+1}(p\rho)H_{m+1}^{(1)}(p\rho') - \frac{i\beta m d_z^{(0)}}{2\rho}J_m(p\rho)H_m^{(1)}(p\rho')$$
$$+\frac{i}{2}\left[-\frac{i(m-1)^2 d_-}{\sqrt{2}\rho^2} + \frac{ik^2 d_-}{\sqrt{2}} - \frac{i(m-1)d_-}{\sqrt{2}\rho^2}\right]J_{m-1}(p\rho)H_{m-1}^{(1)}(p\rho')$$
$$-\frac{i}{2}\left[\frac{i(m+1)d_+}{\sqrt{2}\rho}\frac{\partial J_{m+1}(p\rho)}{\partial\rho}H_{m+1}^{(1)}(p\rho') - \frac{i(m-1)d_-}{\sqrt{2}\rho}\frac{\partial J_{m-1}(p\rho)}{\partial\rho}H_{m-1}^{(1)}(p\rho')\right]$$
$$= \frac{d_+}{2\sqrt{2}}\left[\frac{m(m+1)}{\rho^2} - k^2\right]J_{m+1}(p\rho)H_{m+1}^{(1)}(p\rho') + \frac{d_+}{2\sqrt{2}\rho}(m+1)\frac{\partial J_{m+1}(p\rho)}{\partial\rho}H_{m+1}^{(1)}(p\rho')$$
$$-\frac{i\beta m d_z^{(0)}}{2\rho}J_m(p\rho)H_m^{(1)}(p\rho')$$
$$+\frac{d_-}{2\sqrt{2}}\left[\frac{m(m-1)}{\rho^2} - k^2\right]J_{m-1}(p\rho)H_{m-1}^{(1)}(p\rho') - \frac{d_-}{2\sqrt{2}\rho}(m-1)\frac{\partial J_{m-1}(p\rho)}{\partial\rho}H_{m-1}^{(1)}(p\rho').$$

Now we can use the property that

$$\begin{cases} J_m(p\rho) = \frac{m+1}{p\rho}J_{m+1}(p\rho) + \frac{d}{d(p\rho)}J_{m+1}(p\rho) \\ J_m(p\rho) = \frac{m-1}{p\rho}J_{m-1}(p\rho) - \frac{d}{d(p\rho)}J_{m-1}(p\rho) \end{cases}$$

$$\Rightarrow E_{\phi,m\beta}^{(0)} = \frac{d_+}{2\sqrt{2}}\frac{p(m+1)}{\rho}\left[\frac{(m+1)-1}{p\rho}J_{m+1}(p\rho) + \frac{\partial J_{m+1}(p\rho)}{\partial(p\rho)}\right]H_{m+1}^{(1)}(p\rho')$$
$$-\frac{i\beta m d_z^{(0)}}{2\rho}J_m(p\rho)H_m^{(1)}(p\rho')$$
$$+\frac{d_-}{2\sqrt{2}}\frac{p(m-1)}{\rho}\left[\frac{(m-1)+1}{p\rho}J_{m-1}(p\rho) - \frac{\partial J_{m-1}(p\rho)}{\partial(p\rho)}\right]H_{m-1}^{(1)}(p\rho')$$
$$-\frac{d_+k^2}{2\sqrt{2}}J_{m+1}(p\rho)H_{m+1}^{(1)}(p\rho') - \frac{d_-k^2}{2\sqrt{2}}J_{m-1}(p\rho)H_{m-1}^{(1)}(p\rho')$$
$$= \frac{d_+(m+1)}{2\sqrt{2}\rho}\left[pJ_m(p\rho) - \frac{1}{\rho}J_{m+1}(p\rho)\right]H_{m+1}^{(1)}(p\rho') - \frac{i\beta m d_z^{(0)}}{2\rho}J_m(p\rho)H_m^{(1)}(p\rho')$$
$$+\frac{d_-(m-1)}{2\sqrt{2}\rho}\left[pJ_m(p\rho) + \frac{1}{\rho}J_{m-1}(p\rho)\right]H_{m-1}^{(1)}(p\rho')$$
$$-\frac{d_+k^2}{2\sqrt{2}}J_{m+1}(p\rho)H_{m+1}^{(1)}(p\rho') - \frac{d_-k^2}{2\sqrt{2}}J_{m-1}(p\rho)H_{m-1}^{(1)}(p\rho')$$
$$= \frac{i}{2}\left[-\frac{\beta m d_z^{(0)}}{\rho}J_m(p\rho)H_m^{(1)}(p\rho') + \frac{id_+}{\sqrt{2}}\left(\left(k^2 + \frac{m+1}{\rho^2}\right)J_{m+1}(p\rho) - \frac{(m+1)p}{\rho}J_m(p\rho)\right)H_{m+1}^{(1)}(p\rho')\right.$$
$$\left.+\frac{id_-}{\sqrt{2}}\left(\left(k^2 - \frac{m-1}{\rho^2}\right)J_{m-1}(p\rho) - \frac{(m-1)p}{\rho}J_m(p\rho)\right)H_{m-1}^{(1)}(p\rho')\right]$$

Compared to Nha's expression (Eqn. 28), our expression is equivalent if

$$\begin{cases} \frac{m+1}{\rho^2}J_{m+1}(p\rho) - \frac{p}{\rho}J_m(p\rho) = 0 & ① \\ \frac{m-1}{\rho^2}J_{m-1}(p\rho) - \frac{p}{\rho}J_m(p\rho) = 0. & ② \end{cases}$$

$$\Leftrightarrow \begin{cases} (m+1)J_{m+1}(p\rho) - p\rho J_m(p\rho) = 0 \\ (m-1)J_{m-1}(p\rho) - p\rho J_m(p\rho) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (m+1)J_{m+1}(x) - xJ_m(x) = 0 \\ (m-1)J_{m-1}(x) - xJ_m(x) = 0. \end{cases}$$ but this is not true in general!