Transverse dyadic Green function approximation

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In our nanofiber study, we ignore losses due to fiber absorption & non-radiative decay of the atoms. We also treat the atoms as ideal localized dipoles. In principle, we should be able to ignore the longitudinal part of the dyadic Green's function of the nanofiber-trapped-single-dipole system.

As checked numerically, the guided Green's tensor $\overleftrightarrow{G}^g(\vec{r},\vec{r}')$ can be approximated as

$$\overleftrightarrow{G}^g(\vec{r},\vec{r}') \simeq \overleftrightarrow{G}^g_T(\vec{r},\vec{r}') \simeq -\frac{i}{2\beta_0\sqrt{\epsilon_{eff}}}\sum_m \vec{u}_m(\vec{r})\vec{u}^*_m(\vec{r}').$$

where
$$\epsilon_{eff} \equiv \frac{\int \epsilon(\vec{r}_\perp)\,\vec{u}^*_m(\vec{r}_\perp)\cdot\vec{u}_m(\vec{r}_\perp)\,d^2r_\perp}{\int \vec{u}^*_m(\vec{r}_\perp)\cdot\vec{u}_m(\vec{r}_\perp)\,d^2r_\perp}$$

with $\vec{u}_m(\vec{r}) = \vec{u}_m(\vec{r}_\perp)e^{i\beta_0 z} = \vec{u}_m(r_\perp)e^{i(\beta_0 z + m\phi)}$ the forward-propagating guided mode of the nanofiber.

Numerically, $\sqrt{\epsilon_{eff}} \simeq \sqrt{n_1}$ with an error $< 1\%$ in my setting ($n_1 = n_{core} = 1.4469$, $a = 225\,nm$).

With this definition of ϵ_{eff}, one can treat the nanofiber system as a homogeneous medium with effective index of refraction, $\sqrt{\epsilon_{eff}}$. Hence, an arbitrary guided field with mode index m becomes

$$\vec{E}_m(\vec{r}) = E_0\sqrt{\epsilon_{eff}}\,\vec{u}_m(\vec{r})$$

with intensity $I \sim \int d^2r_\perp |E(\vec{r})|^2 = |E_0|^2\int d^2r_\perp\,\epsilon(\vec{r}_\perp)|\vec{u}_m(\vec{r}_\perp)|^2 = |E_0|^2.$

For an active medium with losses or gain, in general, one should use

$$\vec{g}_m(\vec{r}) = E_0\sqrt{\epsilon(\vec{r})}\,\vec{u}_m(\vec{r})$$

to define the field for mode $m = \pm 1$ as being outlined in Sondergaard & Glauber's papers.

The guided modes, which we are interested in, satisfy the orthogonality & completeness conditions:

$$\begin{cases} \sum_{k_m} \vec{u}^*_m(\vec{r}_\perp)\cdot\vec{u}_m(\vec{r}_\perp') = \delta^T(\vec{r}_\perp - \vec{r}_\perp') \leftarrow \text{the transverse } \delta \text{ function with fixed } k_0 = \frac{\omega}{c}. \\ \int_S d^2r_\perp\,\epsilon(\vec{r})\,\vec{u}^*_m(\vec{r}_\perp)\cdot\vec{u}_{m'}(\vec{r}_\perp) = \epsilon_{eff}\int_S d^2r_\perp\,\vec{u}^*_m(\vec{r}_\perp)\cdot\vec{u}_{m'}(\vec{r}_\perp) = \delta_{mm'}. \end{cases}$$

The purpose of this note is to derive the analytical expression for the transverse and guided mode component of the dyadic Green function.

For simplicity, we ignore the sub/superscripts "T", "g" for all variables. All derivations here are just different trials with no guarantee of correctness.

Approach I: Starting from wave equations:

$$\begin{cases} -\nabla\times(\nabla\times\overleftrightarrow{G}(\vec{r},\vec{r}')) + k_0^2\,\epsilon(\vec{r})\,\overleftrightarrow{G}(\vec{r},\vec{r}') = \overleftrightarrow{\mathbb{I}}\,\delta(\vec{r}-\vec{r}') & \text{①} \\ -\nabla\times(\nabla\times\vec{u}_m(\vec{r})) + k^2\,\epsilon(\vec{r})\,\vec{u}_m(\vec{r}) = 0. & \text{②} \end{cases}$$

Notice that I have used $k_0 = \frac{\omega}{c}$ for the dyadic Green function, which is determined by the photon emitter; I use k as the wave vector for the nanofiber mode equation, which is free of choice in general. However, in our case, we ignore the loss of energy in the absorption & re-radiation process, and hence can assume $k_0 = k$ and $\beta \approx \beta_{11}$, where

β is the longitudinal or z-component of the radiating wave vector, and β_{11} is the corresponding z-component of the wave vector of the bare nanofiber HE_{11} mode.

There is a virtual "mode-matching" process when the radiated field from the atom to the nanofiber system. Although the total k is the same in this process, the components of the wave vector are disturbed and re-distributed. Since the Green's tensor is the sum of all possible reaction paths, we should have some integrals over those redistributed wave vector components.

To understand this physics picture may be a key to solve this problem. Below, I can only provide a few trials along this thread, but there are other points that I may have missed. I will relax some of my assumptions later.

Now, we use the properties that

$$\begin{cases} \nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2 = \nabla\nabla\cdot - \nabla_\perp^2 - \frac{\partial^2}{\partial z^2}. \\ \nabla\cdot(\epsilon(\vec{r})\vec{u}_m(\vec{r})) = \nabla\cdot(\epsilon_{eff}\vec{u}_m(\vec{r})) = 0 = \nabla\cdot(\epsilon_{eff}\overleftrightarrow{G}(\vec{r},\vec{r}\,')). \end{cases}$$

$\leftarrow$ For transverse modes. Boundary condition should work? There may be a problem here.

and equations ① & ② become

$$\begin{cases} \nabla_\perp^2 \overleftrightarrow{G}(\vec{r},\vec{r}\,') - \beta^2 \overleftrightarrow{G}(\vec{r},\vec{r}\,') + k_0^2\epsilon\overleftrightarrow{G}(\vec{r},\vec{r}\,') = \overleftrightarrow{\mathbb{I}}\,\delta(\vec{r}-\vec{r}\,'). & ③ \\ \nabla_\perp^2\vec{u}_m(\vec{r}) + k^2\epsilon\,\vec{u}_m(\vec{r}) = \beta_0^2\,\vec{u}_m(\vec{r}). & ④ \end{cases}$$

We assume $\overleftrightarrow{G}(\vec{r},\vec{r}\,') = \sum_m \vec{u}_m(\vec{r})\vec{A}_m(\vec{r}\,')$,

and equation ③ gives

$$\sum_m\left[\nabla_\perp^2\vec{u}_m(\vec{r}) - \beta^2\vec{u}_m(\vec{r}) + k_0^2\epsilon\,\vec{u}_m(\vec{r})\right]\vec{A}_m(\vec{r}\,') = \overleftrightarrow{\mathbb{I}}\,\delta(\vec{r}-\vec{r}\,'). \qquad ⑤$$

Substituting ④ into the equation above gives

$$-\sum_m\left[(\beta^2-\beta_0^2) + (k^2-k_0^2)\epsilon(\vec{r})\right]\vec{u}_m(\vec{r})\vec{A}_m(\vec{r}\,') = \overleftrightarrow{\mathbb{I}}\,\delta(\vec{r}-\vec{r}\,'). \qquad ⑥$$

As discussed before, $k^2 - k_0^2 = 0$, so

$$\vec{u}_m(\vec{r})\vec{A}_m(\vec{r}\,') = -\frac{\overleftrightarrow{\mathbb{I}}\,\delta(\vec{r}-\vec{r}\,')}{\beta^2-\beta_0^2}.$$

Now, we apply $\int d^2r_\perp\, \vec{u}_m^*(\vec{r})\cdot$ to both sides and use the orthogonality relation of modes, we obtain

$$\vec{A}_m(\vec{r}\,') = -\frac{\vec{u}_m^*(\vec{r}\,')}{(\beta^2-\beta_0^2)\epsilon_{eff}}$$

$$\Rightarrow \overleftrightarrow{G}(\vec{r},\vec{r}\,') = -\sum_m\frac{\vec{u}_m(\vec{r})\vec{u}_m^*(\vec{r}\,')}{(\beta^2-\beta_0^2)\epsilon_{eff}} = -\sum_m\frac{\vec{u}_m(\vec{r}_\perp)\vec{u}_m^*(\vec{r}_\perp\,')}{(\beta^2-\beta_0^2)\epsilon_{eff}}e^{i\beta(z-z')}.$$

Notice that, as discussed earlier, the presence of the atom disturbs the wave vector components. Meanwhile, there is a degeneracy over β_0 for $m=\pm1$. In the end, we should integrate over β around β_0 to give the correct $\overleftrightarrow{G}$.

$$\Rightarrow \overleftrightarrow{G}(\vec{r},\vec{r}\,') = -\sum_m\oint_{\beta_0}\frac{d\beta}{2\pi}\frac{\vec{u}_m(\vec{r}_\perp)\vec{u}_m^*(\vec{r}_\perp\,')e^{i\beta(z-z')}}{\epsilon_{eff}(\beta^2-\beta_0^2)}$$

$$= -i\sum_m \mathrm{Res}\left[\frac{\vec{u}_m(\vec{r}_\perp)\vec{u}_m^*(\vec{r}_\perp\,')}{\epsilon_{eff}(\beta^2-\beta_0^2)}e^{i\beta(z-z')}\right]_{\text{around } \beta_0}$$

$$= -\frac{i}{\cdots}\cdots e^{i\beta_0(z-z')}$$

$$= -i\sum_m \mathrm{Res}\left[\frac{U_m(\vec{r}_\perp)\, U_m^*(\vec{r}_\perp')}{\epsilon_{eff}(\beta^2-\beta_0^2)} e^{i\beta(z-z')}\right]_{\text{around } \beta_0}$$

$$\boxed{= -\frac{i}{2\epsilon_{eff}\beta_0} \vec{U}_m(\vec{r}_\perp)\vec{U}_m^*(\vec{r}_\perp')\, e^{i\beta_0(z-z')}}.$$

Compare to our numerical calculation, there is a $\sqrt{\epsilon_{eff}}$ difference, which I haven't figure out where it might come from. Maybe it doesn't matter.

Below, we relax our assumptions on $k=k_0$, and will do the integal over k. We can process from equation ⑥, and will reach a similar result except for the pre-factor.

The pre-factor is determined by

$$C = -\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\int_0^\infty \frac{dk}{2\pi}\frac{1}{(\beta^2-\beta_0^2)+(k^2-k_0^2)\epsilon}$$ ← integals should only cover the guided modes regime for our case.

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_0^\infty dk \lim_{\varepsilon\to 0}\frac{1}{(k+i\varepsilon)^2\epsilon - [k_0^2\epsilon - (\beta^2-\beta_0^2)]}$$

$$= -\frac{1}{4\pi^2}\int d\beta\int dk \lim_{\varepsilon\to 0}\frac{1}{\left[(k+i\varepsilon)\epsilon - \sqrt{k_0^2\epsilon-(\beta^2-\beta_0^2)}\right]\left[(k+i\varepsilon)\epsilon + \sqrt{k_0^2\epsilon-(\beta^2-\beta_0^2)}\right]}$$

$$= -\frac{i}{4\pi}\int d\beta \frac{1}{\sqrt{k_0^2\epsilon-(\beta^2-\beta_0^2)}}$$

$$\simeq -\frac{i}{4\pi k_0\sqrt{\epsilon}}\int d\beta \frac{1}{1-\frac{\beta^2-\beta_0^2}{2k_0^2\epsilon}}$$

$$= +\frac{ik_0\sqrt{\epsilon}}{2\pi}\int d\beta \frac{1}{\beta^2-\beta_0^2-2k_0^2\epsilon}$$

$$= \frac{ik_0\sqrt{\epsilon}}{2\pi}\int d\beta\frac{1}{(\beta+\sqrt{\beta_0^2+2k_0^2\epsilon})(\beta-\sqrt{\beta_0^2+2k_0^2\epsilon})}$$

$$= \frac{-k_0\sqrt{\epsilon}}{2\sqrt{\beta_0^2+2k_0^2\epsilon}}.$$

This is different from the numerical result by a factor on the order of k_0, which comes from $\int dk \cdots$.

The Link between Ivan's note and the Sondergaard's paper:

In the referenced derivations, a 3D complete & orthogonal set of modes is employed. The results may not be able to adapt to our case directly, yet we can show that the two sets of results are equivalent. I will outline the proof briefly below.

In Ivan's recent notes, the dyadic Green's function is

$$\overleftrightarrow{G}(\vec{r},\vec{r}') = \int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\int_0^\infty\frac{dk}{2\pi}\frac{\vec{u}_{\beta k}(\vec{r}_\perp)\vec{u}_{\beta k}^*(\vec{r}_\perp')}{\epsilon(\vec{r})(k^2-k_0^2)}e^{i\beta(z-z')},$$ ← integals ensure all degenerate modes are included.

The Sondergaard's paper essentially gives

$$\overleftrightarrow{G}(\vec{r},\vec{r}') = \sum_n \frac{\vec{E}_n(\vec{r})[\vec{E}_n(\vec{r}')]^*}{N_n\lambda_n},$$

where λ_n is defined by equation (21) in the paper. That is

$$-\nabla\times\nabla\times\vec{E}_n + k_0^2\epsilon(\vec{r})\vec{E}_n = \lambda_n\epsilon(\vec{r})\vec{E}_n.$$

Compared with Equ. ②, the equation above implies that

$$\lambda_n = (k^2 - k_0^2)\,\epsilon(\vec{r}).$$

This implies Ivan's result is equivalent to Sondergaard's result.

Notice that, the 3D completeness & orthogonality conditions are compatible with our 2D transverse modes $\vec{u}_m(\vec{r}_\perp)$. In principle we should be able to derive from the general case.

Approach II: Starting from

$$\overleftrightarrow{G}(\vec{r},\vec{r}') = \sum_m \int \frac{d\beta}{2\pi} \int d^2 k_\perp \frac{\vec{u}_m(\vec{r}_\perp)\,\vec{u}_m^*(\vec{r}_\perp')}{\epsilon(\vec{r})\,(k^2 - k_0^2)}\, e^{i\beta(z-z')}$$ ← I have adapted the integals to include the consideration on mode degeneracy.

$$\epsilon(k^2 - k_0^2) = \Delta\beta^2 + \Delta k_\perp^2.$$

Turns out, this is similar to the $\int\frac{d\beta}{2\pi}\int\frac{dk}{2\pi}$ integal case.

Unless $\Delta k_\perp^2 = 0$, the result may not have a correct scale.

Even if $\Delta k_\perp^2 = 0$, the factor of $\epsilon(\vec{r})$ is still position dependent, and gives 1 for $r_\perp > a$.

Or, $\Delta k_\perp^2 = -\Delta\beta^2$ for $\Delta k^2 = 0$?

Maybe I used a wrong integal form? To obtain a correct form of integal, we may need to prove

$$\int \boxed{}\ \frac{\vec{u}_m(\vec{r})\,\vec{u}_m^*(\vec{r}')}{\epsilon(k^2 - k_0^2)} = -\frac{i}{2\beta_0\sqrt{\epsilon_{eff}}}\,\vec{u}_m(\vec{r})\,\vec{u}_m^*(\vec{r}').$$

I have reached the possible solution.

Approach III: Starting from

$$\begin{cases} \overleftrightarrow{G}(\vec{r},\vec{r}') = \left(\overleftrightarrow{\mathbb{I}} + \dfrac{\nabla\nabla}{k^2}\right) G_0(\vec{r},\vec{r}') \\ G_0(\vec{r},\vec{r}') = \displaystyle\int \frac{d^3 k}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k_0^2 - k^2}. \end{cases}$$

By applying the boundary conditions, one should be able to obtain the dyadic Green function, too.

But how to get the factor $\frac{1}{\beta_0}$ out from mode integals?

One may be able to use the properties that

$$u_\phi \ \&\ u_{r_\perp} = \frac{\boxed{}}{\omega^2\epsilon - \beta^2}\cdot\begin{pmatrix} E_z \\ H_z \end{pmatrix}$$

The integal over β can give the pole at $\omega^2\epsilon$, but not β_0.

The secret may still relies on the integal $\int\frac{d^3k}{(2\pi)^3}\frac{1}{k_0^2 - k^2}$.

So, how to handle it to give $\frac{1}{\beta_0\sqrt{\epsilon_{eff}}}$ may be the key to find an efficient way to reach the correct solution...