Green function of free-space radiation under paraxial approximation

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We consider the Maxwell-Helmholtz wave equation for a monochromatic wave:

$$(\nabla^2 + k^2) E(\vec{r_\perp}, z) = 0,$$

where we have assumed the wave is propagating along z-direction, and we can focus on only one component of the field vector due to the symmetry of separable variables. Using the fact that

$$k^2 = \frac{\omega^2}{c^2} n^2 = k_0^2 \frac{\varepsilon}{\varepsilon_0} \text{ in SI units or } k^2 = k_0 \varepsilon \text{ in Gauss units.}$$

with $\begin{cases} k_0 = \frac{\omega}{c} \text{ is the vacuum wave number,} \\ \varepsilon = \varepsilon_0 (1+\chi) \text{ in SI unit or } \varepsilon = 1 + 4\pi\chi \text{ in Gauss units.} \end{cases}$ electric susceptibility

$\Rightarrow$ The wave equation in CGS: $(\nabla^2 + k_0^2) E(\vec{r_\perp}, z) = -4\pi k_0^2 \chi E(\vec{r_\perp}, z)$ ①

Now, the solution of the wave equation above can be formulated as

$$E(\vec{r_\perp}, z) = \mathcal{E}(\vec{r_\perp}, z) e^{ik_0 z},$$

and hence the equation about the field envelope $\mathcal{E}(\vec{r_\perp}, z)$ obeys

$$\left[\nabla_T^2 + 2ik_0 \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}\right] \mathcal{E}(\vec{r_\perp}, z) = -4\pi k_0^2 \chi \mathcal{E}(\vec{r_\perp}, z). \quad ②$$

Now, we assume the field envelope is concentrated in a small area in the x-y plane along the propagation direction, which is named as "paraxial" approximation. Or, for a Gaussian wave we will focus on later, the paraxial approximation can be safely used under the condition that

$$z \gg Z_R = \frac{\pi W_0^2}{\lambda}, \quad \text{with } W_0 \text{ the waist of the beam.}$$

We also assume that the field component varies slowly on z direction (or maybe smoothly vary with a period $\sim \lambda \ll 1m$). Therefore, we can ignore the $\frac{\partial^2}{\partial z^2}$ term in Eqn. ② ($\left|k_0 \frac{\partial}{\partial z} \mathcal{E}\right| \gg \left|\frac{\partial^2}{\partial z^2} \mathcal{E}\right|$). So, the wave equation becomes

$$\left[\frac{\partial}{\partial z} - \frac{i}{2k_0} \nabla_T^2\right] \mathcal{E}(\vec{r_\perp}, z) = i 2\pi k_0 \chi \mathcal{E}(\vec{r_\perp}, z). \quad ③$$

On the right-hand-side is the source term, with $\chi = \alpha \delta(\vec{r} - \vec{r}')$, and α is the polarizability. Due to the presence of the δ-function, the source term is equivalent to $f(\vec{r}) = i2\pi k_0 \alpha \delta(\vec{r} - \vec{r}') \mathcal{E}(\vec{r_\perp}', z')$ which only depends on the field at $\vec{r}'$. We define the Green's function for solving this problem as

$$\mathcal{E}(\vec{r_\perp}, z) = \int d^2 r_\perp K(\vec{r_\perp}, z; \vec{r_\perp}', z') \mathcal{E}(\vec{r_\perp}', z') = \int d^2 r_\perp \cdot K(\vec{r_\perp} - \vec{r}', z - z') \mathcal{E}(\vec{r_\perp}', z') \quad ④$$

The corresponding Green's function $K(\vec{r_\perp}, z; \vec{r_\perp}', z')$ satisfies the following equation by substituting ④ into ③ yet with a unitary strength source ($i 2\pi k_0 \alpha \mathcal{E}(\vec{r}') \to 1$):

$$\left[\frac{\partial}{\partial z} - \frac{i}{2k_0} \nabla_T^2\right] K(\vec{r_\perp}, z; \vec{r_\perp}', z') = \delta^{(2)}(\vec{r_\perp} - \vec{r_\perp}') \delta(z - z'). \quad ⑤$$

We know the outgoing free-space Green's function solution $\frac{e^{ik(\vec{r} - \vec{r}')}}{|\vec{r} - \vec{r}'|} \approx \frac{C}{z - z'} e^{\frac{ik|\vec{r_\perp} - \vec{r_\perp}'|^2}{2(z - z')}}$

under the paraxial approximation. So, we can assume

$$K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \frac{C}{z-z'} \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right],$$

and need to solve C using Equ ⑤.

For $\vec{r}_\perp \neq \vec{r}_\perp{}'$ and $z = z'$, the right-hand side of Equ ⑤ gives 0. On the left-hand side,

$$\frac{\partial}{\partial z} K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \left[\frac{-C}{(z-z')^2} - \frac{iCk_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')^3}\right] \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right]$$

$$\vec{\nabla}_T K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \frac{iCk_0|\vec{r}_\perp - \vec{r}_\perp{}'|}{(z-z')^2} \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right] \hat{R} \longrightarrow \frac{\vec{r}_\perp - \vec{r}_\perp{}'}{|\vec{r}_\perp - \vec{r}_\perp{}'|}$$

$$\nabla_T^2 K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \vec{\nabla}_T \cdot (\vec{\nabla}_T K) = \left[\frac{i2Ck_0}{(z-z')^2} - \frac{Ck_0^2|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{(z-z')^3}\right] \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right]$$

We can see that $\left[\frac{\partial}{\partial z} - \frac{i}{2k_0}\nabla_T^2\right] K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = 0$ works.

When $\vec{r}_\perp = \vec{r}_\perp{}'$, $z = z'$, we integrate over space, then the right-hand-side of Equ ⑤ is 1.

Using the fact that $\frac{\partial}{\partial z}$ & ∇_T^2 are independent of $\int d^2 r_\perp \int dz$, and

$$\int d^2 r_\perp \, K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \int d^2 r_\perp \cdot \frac{C}{z-z'} \cdot \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right]$$

$$= \frac{i2\pi}{k} C.$$

$\Rightarrow$ LHS $= \frac{i2\pi C}{k_0} = 1 =$ RHS.

$\Rightarrow C = \frac{-ik_0}{2\pi}$.

$\Rightarrow$ The Green's function can be written as

$$K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') = \frac{-ik_0}{2\pi(z-z')} \exp\left[\frac{ik_0|\vec{r}_\perp - \vec{r}_\perp{}'|^2}{2(z-z')}\right].$$

Therefore, when there is an atom interacting with an input field $E_{in} = \mathcal{E}_0 e^{ik_0 z}$, the output field can be written in the following form under Born approximation:

$$E_{out}(\vec{r}_\perp, z) = E_{in}(\vec{r}_\perp, z) + i2\pi k_0 \int \chi \, K(\vec{r}_\perp, z; \vec{r}_\perp{}'; z') \, E_{in}(\vec{r}_\perp{}'; z') \, d^3 r'$$

$$= E_{in}(\vec{r}_\perp, z) + i2\pi k_0 \alpha \, K(\vec{r}_\perp, z; \vec{r}_\perp{}', z') \, E_{in}(\vec{r}_\perp{}', z').$$

Such far, the results derived here match up with Ivan's notes on "Wigner-Weisskopf and decoherence - 20130529".

Here are some notes on deriving the results above:

<1> In the process of deriving Equ ③ from ②, I have divided $i2k_0$ on both sides Equ ②, which yield a proper "$i2\pi k_0$" factor compared with Ivan's notes, and fits

the corresponding equation into the convenient form of parabolic equations (see time-dependent Schrödinger equation for a free particle or the diffussion equations). Although this divided factor "$i2k_0$" normalizes the coefficient in front of $\frac{\partial}{\partial z}$, and seems giving a proper dimension on the source term, the necessarity of this "normalization" process is not clear to me. As long as we reach some convention on the "normalization" process, I will rewrite the Green's function formalism for nanofiber project under the same convention.

<2>. Beyond the "ansatz" method I used above to solve for the Green's function, we can also use the "Fourier-Laplace Transformation" method to solve it. To make it clear, Let us replace $z \to t$, so for non-source position $\vec{r}$, the Green's function satisfies

$$\begin{cases} \frac{\partial K}{\partial t} - \frac{i}{2k_0}\nabla_T^2 K = 0. \\ K_0 = K(\vec{r_\perp}, t=0; \vec{r_\perp}', 0) = \delta(\vec{r_\perp} - \vec{r_\perp}') \end{cases} \quad (\delta \text{ is the Dirac delta function}).$$

The Fourier transform $\tilde{K}(\vec{\xi}, t)$ of $K(\vec{r_\perp}, t) = K(\vec{r_\perp}, t; \vec{r_\perp}', t'=0)$ is given by

$$\tilde{K}(\vec{\xi}, t) = \exp[-\tfrac{i}{2k_0}|\vec{\xi}|^2] \quad (t \geq 0),$$

by substituting $K(\vec{r_\perp}, t) = \int d^2\xi \cdot \tilde{K}(\vec{\xi}, t) e^{-i\vec{\xi}\cdot\vec{r_\perp}}$ back into the first equation.

The Fourier-Laplace transformation $\hat{K}(\vec{\xi}, \eta)$ of $K(\vec{r_\perp}, t)$ is hence

$$\hat{K}(\vec{\xi}, \eta) = \int_0^\infty dt\, \tilde{K}(\vec{\xi}, t) e^{-\eta\cdot t} = \frac{1}{\eta + \frac{i}{2k_0}|\vec{\xi}|^2}.$$

If we invert these transformations, we should be able to obtain the solution for $K(\vec{r_\perp}, z)$ (replaced $t \to z$) is in the form of

$$K(\vec{r_\perp}, z) = K(\vec{r_\perp} - \vec{r_\perp}', z - z') = \frac{c}{z - z'} \exp\left[\frac{ik_0|\vec{r_\perp} - \vec{r_\perp}'|^2}{2(z - z')}\right].$$

<3> The final result of K shows that, under our convention, the Green's function is equivalent to $\underbrace{\text{the free-space Green's function}}_{\text{paraxial approximation of 3D}}$ times a factor of $\frac{k_0}{i2\pi}$. That is

$$\frac{k_0}{i2\pi}\frac{e^{ik_0\cdot(\vec{r}-\vec{r}')}}{|\vec{r}-\vec{r}'|} \xrightarrow{\text{paraxial approximation}} K(\vec{r_\perp}, z; \vec{r_\perp}', z')\cdot \underbrace{e^{ik_0(z-z')}}_{\text{propagation phase carrier}}.$$

This extra factor beyond the free-space solution is a result of the choice of convention on manipulating the wave equation.

<4>. The Green's function we derived here is a propagator in z-evolution of the field's envelope. It satisfies that

$$\mathcal{E}(\vec{r}_\perp, z) = \int d^2 r_\perp' \cdot K(\vec{r}_\perp, z; \vec{r}_\perp', z') \cdot \mathcal{E}(\vec{r}_\perp', z'),$$

$$\mathcal{E}^*(\vec{r}_\perp', z') = \int d^2 r_\perp \cdot \mathcal{E}^*(\vec{r}_\perp, z) \; K(\vec{r}_\perp, z; \vec{r}_\perp', z')$$

$$K^*(\vec{r}_\perp, z; \vec{r}_\perp', z') = - K(\vec{r}_\perp', z'; \vec{r}_\perp, z).$$