Eigenfunction expansion of Green functions

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The Green function for our nanofiber problem satisfies

$$\left[-\nabla\times\nabla\times + n^2 k_0^2\right] G(\vec{r},\vec{r}\,') = \mathbb{I}\,\delta^{(3)}(\vec{r}-\vec{r}\,'), \qquad (1)$$

where $n = n(\vec{r}_\perp) = \sqrt{\epsilon(\vec{r}_\perp)}$, and G is dyad. We let $k_0 = \frac{\omega}{c}$.

correspondingly the bare-fiber modes (eigenmodes) satisfy

$$\left[-\nabla\times\nabla\times + n^2 k^2\right] \vec{u}_{\vec{k}}(\vec{r}) = 0 \qquad \text{with } k^2 = \frac{\omega_k^2}{c^2}. \qquad (2)$$

Now, we can define a vector function

$$\vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}) = n(\vec{r})\,\vec{u}_{\vec{k}}(\vec{r}), \qquad (3)$$

and hence

$$\underbrace{\left[-\frac{1}{n}\nabla\times\nabla\times\frac{1}{n} + k^2\right]}_{\mathcal{H}_k \to \text{ a Hermitian operator introduced by Glauber and Lewenstein.}} \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}) = 0. \iff \overbrace{\left[-\frac{1}{n}\nabla\times\nabla\times\frac{1}{n} + k_0^2\right]}^{\mathcal{H}_{k_0}} \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}) = \overbrace{(k_0^2 - k^2)}^{\lambda_{\vec{k}}}\,\vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}) \qquad (4)$$

Therefore, $(\vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}), \lambda_{\vec{k}})$ is a set of eigenfunction and eigenvalue for $\mathcal{H}_{k_0}\vec{u}^{\,\epsilon}_{\vec{k}} = \lambda_{\vec{k}}\vec{u}^{\,\epsilon}_{\vec{k}}$.

The adjoint eigenvalue equation $\mathcal{H}^+_{k_0}\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}}(\vec{r}) = \lambda^*_{\vec{k}}\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}}(\vec{r})$ will have a set of eigensolutions $(\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}}(\vec{r}), \lambda^*_{\vec{k}})$. In our case, we let $\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}}(\vec{r}) = \vec{u}^{\,\epsilon *}_{\tilde{\vec{k}}}(\vec{r})$ with $\tilde{\vec{k}} = (\omega, -m, -f)$ & $\vec{u}^{\,\epsilon}_{\tilde{\vec{k}}}(\vec{r}) = \vec{u}^{\,\epsilon *}_{\vec{k}}(\vec{r})$.

$\hookrightarrow$ propagation direction

$\hookrightarrow$ polarizability.

Such, the corresponding Green function G_ϵ for $\vec{u}^{\,\epsilon}(\vec{r})$ can be defined as

$$G_\epsilon(\vec{r},\vec{r}\,') = n(\vec{r})\,G(\vec{r},\vec{r}\,')\,n(\vec{r}\,'). \qquad (5)$$

Satisfying

$$\mathcal{H}_{k_0} G_\epsilon(\vec{r},\vec{r}\,') = \mathbb{I}\,\delta^{(3)}(\vec{r}-\vec{r}\,'). \quad \text{with } \mathcal{H}_{k_0} = -\frac{1}{n}\nabla\times\nabla\times\frac{1}{n} + k_0^2, \qquad (6)$$

Notice that, here, we only consider the transverse modes and components, since the longitudinal components are negligible for neutral and ideal dipole sources.

The orthogonality relationship reads

$$\int \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r})\cdot\left[\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}'}(\vec{r})\right]^* d\vec{r} = \int n^2\,\vec{u}_{\vec{k}}(\vec{r})\cdot\vec{u}^{\,*}_{\vec{k}'}(\vec{r})\,d\vec{r} = \delta_{mm'}\delta_{ff'}\,\delta(\beta-\beta'). \qquad (7)$$

For ~~the~~ 2-D transverse plane integral, we can write

$$\int \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r}_\perp)\cdot\vec{u}^{\,\epsilon *}_{\vec{k}'}(\vec{r}_\perp)\,d\vec{r}_\perp = \int n^2\,\vec{u}_{\vec{k}}(\vec{r}_\perp)\cdot\vec{u}^{\,*}_{\vec{k}'}(\vec{r}_\perp)\,d\vec{r}_\perp = \delta_{mm'}\delta_{ff'} \qquad (8)$$

with $\beta = \beta'$ for our single-mode fiber.

The completeness relationship reads

$$\sum_{\vec{k}} \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r})\left[\tilde{\vec{u}}^{\,\epsilon}_{\vec{k}}(\vec{r}\,')\right]^* = \sum_{\vec{k}} n(\vec{r}_\perp)\,n(\vec{r}_\perp')\,\vec{u}_{\vec{k}}(\vec{r})\,\vec{u}^{\,*}_{\vec{k}}(\vec{r}\,') = \mathbb{I}\,\delta^{T}(\vec{r}-\vec{r}\,') \qquad (9)$$

We can assume

$$G_\epsilon(\vec{r},\vec{r}\,') = \sum_{\vec{k}} \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r})\,\vec{A}_{\vec{k}}(\vec{r}\,')$$

Equ. (6) gives

$$\mathcal{H}_{k_0}\sum_{\vec{k}} \vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r})\,\vec{A}^{\,\epsilon}_{\vec{k}}(\vec{r}\,') = \mathbb{I}\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\iff \sum_{\vec{k}}\left[\mathcal{H}_{k_0}\vec{u}^{\,\epsilon}_{\vec{k}}(\vec{r})\right]\vec{A}^{\,\epsilon}_{\vec{k}}(\vec{r}\,') = \mathbb{I}\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Leftrightarrow \quad \sum_{\vec{k}} \left[\mathcal{H}_{k_0} \vec{\mathcal{U}}_{\vec{k}}^{\epsilon}(\vec{r}) \right] \vec{A}_{\vec{k}}^{\epsilon}(\vec{r}') = \mathbb{1}\, \delta^{(3)}(\vec{r}-\vec{r}')$$

$$\Leftrightarrow \quad \sum_{\vec{k}} (k_0^2 - k^2)\, \vec{\mathcal{U}}_{\vec{k}}^{\epsilon}(\vec{r})\, \vec{A}_{\vec{k}}^{\epsilon}(\vec{r}') = \mathbb{1}\, \delta^{(3)}(\vec{r}-\vec{r}') = \sum_{\vec{k}} \vec{\mathcal{U}}_{\vec{k}}^{\epsilon}(\vec{r}) \left[\tilde{\vec{\mathcal{U}}}_{\vec{k}}^{\epsilon}(\vec{r}') \right]^*$$

$$\Rightarrow \tilde{\vec{A}}_{k}(\vec{r}') = \frac{\left[\tilde{\vec{\mathcal{U}}}_{\vec{k}}^{\epsilon}(\vec{r}') \right]^*}{k_0^2 - k^2}$$

$$\Rightarrow \tilde{G}(\vec{r}, \vec{r}') = \sum_{\vec{k}} \frac{\vec{\mathcal{U}}_{\vec{k}}^{\epsilon}(\vec{r}) \left[\tilde{\vec{\mathcal{U}}}_{\vec{k}}^{\epsilon}(\vec{r}') \right]^*}{k_0^2 - k^2} = n(\vec{r}) \left[\sum_{\vec{k}} \frac{c^2\, \vec{\mathcal{U}}_{\vec{k}}(\vec{r})\, \vec{\mathcal{U}}_{\vec{k}}^*(\vec{r}')}{\omega_0^2 - \omega^2} \right] n(\vec{r}')$$

$$\Rightarrow G(\vec{r}, \vec{r}') = \sum_{\vec{k}} \frac{c^2\, \vec{\mathcal{U}}_{\vec{k}}(\vec{r})\, \vec{\mathcal{U}}_{\vec{k}}^*(\vec{r}')}{\omega_0^2 - \omega^2} = \sum_{\beta, m, f=\pm 1} \frac{c^2\, \vec{\mathcal{U}}_{\mu}(r_\perp)\, \vec{\mathcal{U}}_{\mu}^*(r_\perp')}{\omega_0^2 - \omega^2}\, e^{i f \beta (z - z') + i m (\phi - \phi')}$$