This writing focuses on 2 aspects of the 2011 paper written by Dawkins and others on "dispersive optical interface based on nanofiber-trapped atoms":

1. reproducing some of ~~their~~ equations, like Equ.(4), using our quantative theory.
2. following their line of experimental setup, derive more quantative equations for atom number estimation with completely mixed state of atoms. → Also extend to clock-state case.

Part I: Reproduce the phase shift formula (Equ.(4) in Dawkins' paper) using our method.

In their experimental setup, atoms are initially prepared in a completely mixed state of $F=4$ ground state manifolds.

The phase shift for a forward-propagating guided mode, u, can be given by

$$\phi_u = 2\pi k\left(\frac{c}{v_g}\right)\sum_e \mathrm{Re}\left[\frac{|\langle e|\vec{d}|g\rangle\cdot\vec{u}_u(\vec{r}_\perp')|^2}{\hbar(\Delta_{eg}+i\Gamma/2)}\right] \leftarrow \text{ground state}: F=4,\ 6S_{1/2}.$$

$$= 2\pi k n_g \sum_{F'} \mathrm{Re}\left[\frac{|\langle J'\|\vec{d}\|J\rangle|^2}{\hbar(\Delta_{FF'}+\frac{i\Gamma}{2})}\right]\cdot C^{(0)}_{J'F'F}\ |\vec{u}_u(\vec{r}_\perp')|^2$$

$$= 2\pi k n_g |\vec{u}_u(\vec{r}_\perp')|^2 \sum_{F'} C^{(0)}_{J'F'F}\frac{|\langle J'\|d\|J\rangle|^2}{\hbar}\frac{\Delta_{FF'}}{\Delta^2_{FF'}+\Gamma^2/4}$$

$$= \frac{2\pi k}{A^u_{eff}}\frac{|\langle J'\|d\|J\rangle|^2}{\hbar\Gamma}\cdot C^{(0)}_{J'F'_{max}F}\sum_{F'}\frac{C^{(0)}_{J'F'F}}{C^{(0)}_{J'F'_{max}F}}\frac{2\tilde{\Delta}_{FF'}}{\tilde{\Delta}^2_{FF'}+1} \qquad (1)$$

where $\tilde{\Delta}_{FF'} = \Delta_{FF'}/\frac{\Gamma}{2}$.

In deriving, we have used the fact that

polarizability $\alpha(\Delta) = \alpha_0(\Delta)\sum_{F'} C^{(0)}_{J'F'F} = \frac{-|\langle J'\|d\|J\rangle|^2}{\hbar}\sum_{F'}\frac{C^{(0)}_{J'F'F}}{\Delta_{FF'}+i\Gamma/2}$

↑ only scalar polarizability is considered here.

The effective mode area is defined as $A^u_{eff} = \frac{1}{n_g|\vec{u}_u(\vec{r}_\perp')|^2}$.

On the other hand, the scattering cross section can be understood in the following way:

The energy attenuation coefficient for a collective spin ensemble can be given by

$$a = 4\pi k\,\mathrm{Im}(\alpha)\rho = \rho\,\sigma(\Delta)$$

$\sigma(\Delta)$ → scattering cross section; ρ → atomic density.

For a single atom case, we treat $\rho=1$. and hence the scattering cross section

$$\sigma(\Delta) = 4\pi k\,\mathrm{Im}[\alpha(\Delta)] = -4\pi k\frac{|\langle J'\|d\|J\rangle|^2}{\hbar}\sum_{F'}\frac{\Gamma/2\cdot C^{(0)}_{J'F'F}}{\Delta^2_{FF'}+\Gamma^2/4} = -\frac{8\pi k}{\Gamma}\frac{|\langle J'\|d\|J\rangle|^2}{\hbar}\sum_{F'}\frac{C^{(0)}_{J'F'F}}{\tilde{\Delta}^2_{FF'}+1} = \boxed{\sum_{F'}\frac{\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}}$$

As a special case, the resonant cross-section is defined at $\Delta=0$ (usually only for 2-level systems):

May not suitable for multi-level atoms.

$$\sigma_0 = \sigma(\Delta=0) = -8\pi k\frac{d^2_{eg}}{\hbar\Gamma} = -\frac{8\pi k}{\hbar\Gamma}\sum_{F'}|\langle J'\|d\|J\rangle|^2 C^{(0)}_{J'F'F}. \quad \text{with } \Gamma = \frac{4}{3\hbar}d^2_{eg}k^3 \Rightarrow \sigma_0 = \frac{6\pi}{k^2} = \frac{3\lambda^2}{2\pi}.$$

We substitute the expression of σ_0 into Eq(1) of ϕ_u, and obtain

resonant cross-section for $F\leftrightarrow F'$ transition: $\boxed{\sigma_{F'} = \frac{-8\pi k}{\hbar\Gamma}|\langle J'\|d\|J\rangle|^2 C^{(0)}_{J'F'F}}$

$$\boxed{\phi_u = \frac{-\sigma_5}{4A^u_{eff}}\cdot\sum_{F'}\frac{\sigma_{F'}}{\sigma_5}\frac{2\tilde{\Delta}_{FF'}}{\tilde{\Delta}^2_{FF'}+1} = -\phi^u_{max}\sum_{F'}2\frac{\sigma_{F'}}{\sigma_5}\frac{\tilde{\Delta}_{F'}}{\tilde{\Delta}^2_{F'}+1}}$$

with $\phi^u_{max} = \frac{\sigma_5}{4A^u_{eff}}$, $\frac{\sigma_{F_1'}}{\sigma_{F_2'}} = \frac{C^{(0)}_{J'F_1'F}}{C^{(0)}_{J'F_2'F}} \rightarrow \frac{\sigma_4}{\sigma_5}(F=4) = \frac{21}{44}$, $\frac{\sigma_3}{\sigma_5}(F=4) = \frac{7}{44}$.

Therefore, with H & V modes, the rotation angle on the Poincaré sphere is

$\phi_H - \phi_V$ ...

Therefore, with H & V modes, the rotation angle on the Poincaré sphere is

$$\boxed{\begin{aligned}\phi &\doteq \phi_H - \phi_V \\ &= -\frac{\sigma_S}{4A_{eff}} \cdot \sum_{F'} \frac{\sigma_{F'}}{\sigma_S} \frac{2\tilde{\Delta}_{FF'}}{\tilde{\Delta}^2_{FF'}+1} = -\phi_{max} \sum_{F'} 2\frac{\sigma_{F'}}{\sigma_S} \frac{\tilde{\Delta}_{F'}}{\tilde{\Delta}^2_{F'}+1}\end{aligned}}$$

$\leftarrow$ Eq. (4) of Dawkin's paper! for D_2 line transitions.

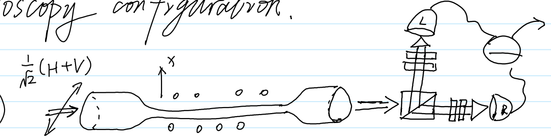
with $A_{eff} = \dfrac{1}{n_g\left(|\vec{u}_H(\vec{r}_\perp')|^2 - |\vec{u}_V(\vec{r}_\perp')|^2\right)}$

$\phi_{max} = \dfrac{\sigma_S}{4A_{eff}} = \dfrac{OD}{4}$ with optical depth $OD = \dfrac{\sigma_S}{A_{eff}}$.

ϕ reaches maximum when $\tilde{\Delta}_{FF'} = -1$ or $\Delta_{FF'} = -\frac{\Gamma}{2}$.

Part II: Minimum detectable atom number using a mixed state of atoms on the ground states.

We continue to use our usual H-V birefringence spectroscopy configuration.



With N atoms, the signal power measured is

$\Delta P_S = P_0 \sin\phi_N \simeq N P_0 \phi$ (P_0 is the input power, ϕ is the rotation/atom.)

The shot noise is

$\Delta P_{SN} = \sqrt{\dfrac{P_0 \hbar\omega_0}{2\eta\tau_{pd}}}$ (η is the quantum efficiency, τ_{pd} is the response time of the photon detector.

In the shot-noise limit, $\Delta P_S = \Delta P_{SN}$.

We set $\eta = 1$, and $\tau_{dp} = \frac{1}{\gamma_S}$ equals the photon scattering time.

$$\begin{cases} \gamma_S = \sigma(\Delta)\dfrac{I(\vec{r}_\perp')}{\hbar\omega_0} \approx \dfrac{I(\vec{r}_\perp')}{\hbar\omega_0}\displaystyle\sum_{F'}\frac{\sigma_{F'}}{1+\tilde{\Delta}^2_{FF'}} \\ I(\vec{r}_\perp') = \dfrac{c\varepsilon_0 n}{2}|E|^2\Big|_{\vec{r}_\perp=\vec{r}_\perp'} = \dfrac{P_0}{A_{in}(\vec{r}_\perp')} \end{cases} \Rightarrow \Delta P_{SN} = P_0\sqrt{\frac{1}{2A_{in}}}\sqrt{\sum_{F'}\frac{\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}}.$$

Aside: For a Gaussian beam, the peak intensity $I_p = \dfrac{P_0}{\pi w^2/2}$, $A_{in} = \dfrac{\pi w^2}{2}$.

For our nanofiber
$$\begin{cases} I(\vec{r}_\perp') = \dfrac{c\varepsilon_0}{2}|E_0|^2\left|\dfrac{(\vec{u}_H(\vec{r}_\perp')+\vec{u}_V(\vec{r}_\perp'))}{\sqrt{2}}\right|^2 = \dfrac{c\varepsilon_0}{2}\dfrac{|E_0|^2}{2}\left(|\vec{u}_H(\vec{r}_\perp')|^2+|\vec{u}_V(\vec{r}_\perp')|^2\right) = \dfrac{1}{2}\left(I_H(\vec{r}_\perp')+I_V(\vec{r}_\perp')\right) \text{ for } \phi'=0\ \&\ \pi. \\ P_0 = \dfrac{c\varepsilon_0}{2}\displaystyle\int n(\vec{r}_\perp)\frac{|E_0|^2}{2}|\vec{u}_H(\vec{r}_\perp)+\vec{u}_V(\vec{r}_\perp)|^2 d\vec{r}_\perp \\ \quad = \dfrac{c\varepsilon_0}{2}\dfrac{|E_0|^2}{2}\left[\displaystyle\int n(\vec{r}_\perp)\left(|\vec{u}_H(\vec{r}_\perp)|^2+|\vec{u}_V(\vec{r}_\perp)|^2\right)d\vec{r}_\perp + \underbrace{\int_0^\infty\!\!\int_0^{2\pi} n(\vec{r}_\perp)\cdot\left[|u_{r}|^2+|u_z|^2-|u_\phi|^2\right]\sin 2\phi \cdot d\phi\cdot r_\perp dr_\perp}_{=0 \text{ for odd function } \sin 2\phi}\right] \\ \quad = \dfrac{1}{2}(P_H+P_V). \end{cases}$$

Alternatively, using the circular polarization basis,

$$\begin{cases} \vec{u}_H = \frac{1}{\sqrt{2}}(\vec{u}_+ + \vec{u}_-) \\ \vec{u}_V = \frac{1}{i\sqrt{2}}(\vec{u}_+ - \vec{u}_-) \end{cases} \Leftrightarrow \begin{cases} u_+ = \frac{1}{\sqrt{2}}(\vec{u}_H + i\vec{u}_V) \\ u_- = \frac{1}{\sqrt{2}}(\vec{u}_H - i\vec{u}_V) \end{cases} \Rightarrow \begin{cases} I(\vec{r}_\perp') = \frac{c\varepsilon_0}{2}\frac{|E_0|^2}{2}\left(|\vec{u}_+(\vec{r}_\perp')|^2+|\vec{u}_-(\vec{r}_\perp')|^2\right) = \frac{1}{2}\left[I_+(\vec{r}_\perp')+I_-(\vec{r}_\perp')\right] \text{ for } \phi=0\ \&\ \pi. \\ P_0 = \frac{1}{2}(P_+ + P_-), \text{ with } P_\mu = \frac{c\varepsilon_0}{2}\int n(\vec{r}_\perp)|E_0|^2|\vec{u}_\mu(\vec{r}_\perp)|^2 d\vec{r}_\perp \end{cases}$$

$$\Rightarrow A_{in} = \frac{P_H + P_V}{I_H(\vec{r}_\perp') + I_V(\vec{r}_\perp')} = \frac{P_+ + P_-}{I_+(\vec{r}_\perp') + I_-(\vec{r}_\perp')}$$

Therefore, in the shot-noise limit, the minimum detectable atom number

$$\boxed{\begin{aligned} N_{min} &= \sqrt{\frac{1}{2A_{in}}}\sqrt{\sum_{F'}\frac{\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}}\Big/|\phi| \\ &= 2A_{eff}\sqrt{\frac{1}{2A_{in}}}\sqrt{\sum_{F'}\frac{\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}}\Big/\left(\sum_{F'}\sigma_{F'}\frac{\tilde{\Delta}_{FF'}}{\tilde{\Delta}^2_{FF'}+1}\right) \\ &= \sqrt{\frac{2A^2_{eff}}{A_{in}}}\sqrt{\sum_{F'}\frac{\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}}\Big/\left(\sum_{F'}\frac{\tilde{\Delta}_{FF'}\sigma_{F'}}{\tilde{\Delta}^2_{FF'}+1}\right) \approx \sqrt{\frac{2A^2_{eff}}{A_{in}\sigma_0}} \end{aligned}}$$

Part III. To extend this result to the QND measurement case with atoms in clock states, we can use the solution for using magic frequencies that

$$\phi = \chi_{eff} = \frac{-\sigma_0}{A_{eff}} \frac{\Gamma}{4\Delta_{eff}}$$

where

$$\begin{cases} \dfrac{1}{A_{eff}} = 2k_g \dfrac{|\vec{u}_V(\vec{r}_i)|^2 \cdot |\hat{e}_\pi \cdot \vec{u}_H(\vec{r}_i)|^2 - |\vec{u}_H(\vec{r}_i)|^2 \cdot |\hat{e}_\pi \cdot \vec{u}_V(\vec{r}_i)|^2}{(|\vec{u}_H(\vec{r}_i)|^2 + |\vec{u}_V(\vec{r}_i)|^2)} \\ \dfrac{\Gamma}{4\Delta_{eff}} = \sum_{F'} \left(\dfrac{10 C^{(2)}_{J'F'F}}{\Delta_{4F'}} - \dfrac{6 C^{(2)}_{J'F'F}}{\Delta_{3F'}} \right) \dfrac{\Gamma}{4} \equiv \dfrac{1}{2} \dfrac{1}{\tilde{\Delta}_{eff}}. \end{cases}$$

Using the formulas developed earlier: the minimum detectable atom number becomes

$$\boxed{\begin{aligned} N_{min} &= \sqrt{\frac{1}{2A_{in}}} \sqrt{\sum_{F,F'} \frac{\sigma_{FF'}}{\tilde{\Delta}^2_{FF'}+1}} \Big/ |\phi| \\ &= \sqrt{\frac{2A^2_{eff}}{A_{in}}} \frac{\tilde{\Delta}_{eff}}{\sigma_0} \sqrt{\sum_{F,F'} \frac{\sigma_{FF'}}{\tilde{\Delta}^2_{FF'}+1}} \\ &\simeq \sqrt{\frac{2A^2_{eff}}{A_{in}\sigma_0}} \tilde{\Delta}_{eff} \sqrt{\sum_{F,F'} \frac{1}{\tilde{\Delta}^2_{FF'}+1}}. \end{aligned}}$$

For the clock-state atoms,

$$\boxed{\sigma_{FF'} = -\sum_q \frac{8\pi k}{\hbar\Gamma} |\langle J'\|d\|J\rangle|^2 \, |O^{J'F'}_{JF}|^2 \cdot |C^{F,0;1,q}_{F',q}|^2}$$

ν_0

$\tilde{\Delta}_{eff}$