Derivation of Heisenberg-Langevin Equ

Monday, November 18, 2013 9:30 PM

Ref: Kien 2005 Spontaneous emission of a Cesium atom near a nanofiber

Starting from

$$\hat{H}_{int} = -i\hbar \sum_{fpeg} \int_0^\infty d\omega \, G_{\mu eg} \hat{\sigma}_{ge}^+ \hat{a}_\mu e^{-i(\omega-\omega_0)t}$$

$$-i\hbar \sum_{mpeg} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \, G_{\nu eg} \hat{\sigma}_{ge}^+ \hat{a}_\nu e^{-i(\omega-\omega_0)t} + H.C.$$

$$[\hat{a}_\mu, \hat{a}_{\mu'}^+] = \delta(\omega-\omega')\delta_{ff'}\delta_{pp'}$$

$$[\hat{a}_\nu, \hat{a}_{\nu'}^+] = \delta(\omega-\omega')\delta(\beta-\beta')\delta_{mm'}\delta_{pp'}$$

$$\Rightarrow \frac{d}{dt}\hat{a}_\mu = -\frac{i}{\hbar}[\hat{a}_\mu, \hat{H}_{int}]$$

$$= \sum_{fpeg} \int_0^\infty d\omega' \, G_{\mu' eg}^* \hat{\sigma}_{ge} [\hat{a}_\mu, \hat{a}_{\mu'}^+] e^{i(\omega'-\omega_0)t}$$

$$= \sum_{fpeg} \int_0^\infty d\omega' \, G_{\mu' eg}^* \hat{\sigma}_{ge} \delta(\omega-\omega')\delta_{ff'}\delta_{pp'} e^{i(\omega'-\omega_0)t}$$

$$= \sum_{eg} G_{\mu eg}^* \hat{\sigma}_{ge} e^{i(\omega-\omega_0)t}$$

Similarly, we have

$$\frac{d}{dt}\hat{a}_\nu = \sum_{eg} G_{\nu eg}^* \hat{\sigma}_{ge} e^{i(\omega-\omega_0)t}.$$

The formal solution can be given by

$$\begin{cases} \hat{a}_\mu(t) = \hat{a}_\mu(t_0) + \sum_{eg} G_{\mu eg}^* \int_{t_0}^t dt' \, \hat{\sigma}_{ge}(t') e^{i(\omega-\omega_0)t'} \\ \hat{a}_\nu(t) = \hat{a}_\nu(t_0) + \sum_{eg} G_{\nu eg}^* \int_{t_0}^t dt' \, \hat{\sigma}_{ge}(t') e^{i(\omega-\omega_0)t'} \end{cases}$$

For the $\hat{\sigma}_{ge}$ operator, we can write the time-evolution Equ as

$$\frac{d}{dt}\hat{\sigma}_{ge} = -\frac{i}{\hbar}[\hat{\sigma}_{ge}, \hat{H}_{int}]$$

$$= -\sum_{fpeg} \int_0^\infty d\omega \, G_{\mu eg} [|g\rangle\langle e|, |e\rangle\langle g|] a_\mu e^{-i(\omega-\omega_0)t}$$

$$-\sum_{mpeg} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \, G_{\nu eg} [|g\rangle\langle e|, |e\rangle\langle g|] a_\nu e^{-i(\omega-\omega_0)t}$$

$$= -\sum_{fpeg} \int_0^\infty d\omega \, G_{\mu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) a_\mu e^{-i(\omega-\omega_0)t}$$

$$-\sum_{mpeg} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \, G_{\nu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) a_\nu e^{-i(\omega-\omega_0)t}$$

$$= \left. \begin{array}{l} -\sum_{fpeg} \int_0^\infty d\omega \, G_{\mu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) \hat{a}_\mu(t_0) e^{-i(\omega-\omega_0)t} \\ -\sum_{mpeg} \int_0^\infty d\omega \int_{kn_2}^{kn_1} d\beta \, G_{\nu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) \hat{a}_\nu(t_0) e^{-i(\omega-\omega_0)t} \end{array} \right\} \hat{\sigma}_{ge}^{free}$$

Markoff approx $\hat{\sigma}_{ge}(t') \simeq \hat{\sigma}_{ge}(t)$

$$-\sum_{fpeg} \int_0^\infty d\omega \, G_{\mu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) \sum_{e'g'} G_{\mu e'g'}^* \int_{t_0}^t dt' e^{-i(\omega-\omega_0)(t-t')} \hat{\sigma}_{g'e'}(t)$$

$$-\sum_{mpeg} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \, G_{\nu eg} (\hat{\sigma}_{gg} - \hat{\sigma}_{ee}) \sum_{e'g'} G_{\nu e'g'}^* \int_{t_0}^t dt' e^{-i(\omega-\omega_0)(t-t)} \hat{\sigma}_{g'e'}(t)$$

Using the relationships that

$$(\hat{\sigma}_{gg} - \hat{\sigma}_{ee})\hat{\sigma}_{ge'} = \delta_{gg'}\hat{\sigma}_{ge'} - \delta_{eg'}\hat{\sigma}_{ee'}$$

$$\int_{t_0}^{t} dt'\, e^{-i(\omega-\omega_0)(t-t')} G_{\mu eg} G^*_{\mu e'g'}$$

$$= \int_{-\infty}^{\infty} dt'\, e^{-i(\omega-\omega_0)(t-t')} G_{\mu eg} G^*_{\mu e'g'}$$ very sharply peaked at $t'=t$, with a width approx the inverse of the freq bandwidth.

$$= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} d\tau\, e^{-i(\omega-\omega_0)\tau} e^{-\epsilon\tau} G_{\mu eg} G^*_{\mu e'g'}$$ → Fourier Trans

$$= G_{\mu eg} G^*_{\mu e'g'} \left(\pi\delta(\omega-\omega_0) - i\,\mathbb{P}\left[\frac{1}{\omega-\omega_0}\right]\right)$$ → Cauchy's Principle part → ignore for now?

We have

$$\frac{d}{dt}\hat{\sigma}_{ge} = \hat{\sigma}_{ge}^{free} - \sum_{fpege'} \pi \left(G_{\mu eg} G^*_{\mu e'g} \hat{\sigma}_{ge'} - \cancelto{0}{G_{\mu eg} G^*_{\mu e'e} \hat{\sigma}_{ee'}} \right)$$

$$- \sum_{mpeg} \pi \int_{-kn_2}^{kn_2} d\beta \left(G_{\nu eg} G^*_{\nu eg'} \hat{\sigma}_{ge'} - \cancelto{0}{G_{\nu eg} G^*_{\nu e'e} \hat{\sigma}_{ee'}} \right) + \text{Cauchy's Principle part}$$

$$= \hat{\sigma}_{ge}^{free} - \sum_{fpege'} \pi G_{\mu eg} G^*_{\mu e'g} \hat{\sigma}_{ge'}$$

$$- \sum_{mpege'} \pi \int_{-kn_2}^{kn_2} d\beta\, G_{\nu eg} G^*_{\nu e'g} \hat{\sigma}_{ge'} + \text{Cauchy's principle part}$$

$$= \hat{\xi}_{ge} - \frac{1}{2}\sum_{e'} \Gamma_{ee'} \hat{\sigma}_{ge'}$$

where $\hat{\xi}_{ge} = \hat{\sigma}_{ge}^{free}$ + Cauchy's principle part ...

$$\Gamma_{ee'} = \gamma_{ee'}^{(g)} + \gamma_{ee'}^{(r)}$$

$$\gamma_{ee'}^{(g)} = 2\pi \sum_{fpg} G_{\mu_0 eg} G^*_{\mu_0 e'g} .$$ ← $\begin{cases} \mu_0 = (\omega_0, f, p) \\ \nu_0 = (\omega_0, \beta, m, p). \end{cases}$

$$\gamma_{ee'}^{(r)} = 2\pi \sum_{mpg} \int_{-kn_2}^{kn_2} d\beta\, G_{\nu_0 eg} G^*_{\nu_0 e'g} .$$

Similarly,

$$\frac{d}{dt}\hat{\sigma}_{ee'} = -\frac{i}{\hbar}[\hat{\sigma}_{ee'}, \hat{H}_{int}]$$

$$= -\sum_{fpe''g} \int_0^{\infty} d\omega\, G_{\mu e''g} [\hat{\sigma}_{ee'}, \hat{\sigma}^{+}_{ge''}] \hat{a}_\mu e^{-i(\omega-\omega_0)t}$$

$$- \sum_{mpe''g} \int_0^{\infty} d\omega \int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g} [\hat{\sigma}_{ee'}, \hat{\sigma}^{+}_{ge''}] \hat{a}_\nu e^{-i(\omega-\omega_0)t}$$

$$+ \sum_{fpe''g} \int_0^{\infty} d\omega\, G^*_{\mu e''g} [\hat{\sigma}_{ee'}, \hat{\sigma}_{ge''}] \hat{a}^{+}_\mu e^{i(\omega-\omega_0)t}$$

$$+ \sum \int_0^{\infty} d\omega \int_{-kn_2}^{kn_2} d\beta\, G^*_{\nu e''g} [\hat{\sigma}_{ee'}, \hat{\sigma}_{ge''}] \hat{a}^{+}_\nu e^{i(\omega-\omega_0)t}$$

$$= -\sum_{fpe''g} \int_0^{\infty} d\omega\, G_{\mu e''g} (\delta_{e'e''}\hat{\sigma}_{eg} - \delta_{ge}\hat{\sigma}_{e''e'}) \hat{a}_\mu e^{-i(\omega-\omega_0)t}$$

$$- \sum_{mpe''g} \int_0^{\infty} d\omega \int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g} (\delta_{e'e''}\hat{\sigma}_{eg} - \delta_{ge}\hat{\sigma}_{e''e'}) \hat{a}_\nu e^{-i(\omega-\omega_0)t}$$

$$-\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\hat a_\nu\, e^{-i(\omega-\omega_0)t}$$

$$+\sum_{fpe''g}\int_0^\infty d\omega\, G^*_{ue''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\hat a_u^+\, e^{i(\omega-\omega_0)t}$$

$$+\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G^*_{\nu e''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\hat a_\nu^+\, e^{i(\omega-\omega_0)t}$$

$$= \left.\begin{aligned}
&-\sum_{fpe''g}\int_0^\infty d\omega\, G_{ue''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\hat a_u(t_0)\, e^{-i(\omega-\omega_0)t}\\
&-\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\hat a_\nu(t_0)\, e^{-i(\omega-\omega_0)t}\\
&+\sum_{fpe''g}\int_0^\infty d\omega\, G^*_{ue''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\hat a_u^+(t_0)\, e^{i(\omega-\omega_0)t}\\
&+\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G^*_{\nu e''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\hat a_\nu^+(t_0)\, e^{i(\omega-\omega_0)t}
\end{aligned}\right\}\ \hat\sigma_{ee'}^{free}.$$

$$-\sum_{fpe''g}\int_0^\infty d\omega\, G_{ue''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\sum_{e'''g'}G^*_{ue'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{g'e'''}(t')\, e^{-i(\omega-\omega_0)(t-t')}$$

$$-\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\sum_{e'''g'}G^*_{\nu e'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{g'e'''}(t')\, e^{-i(\omega-\omega_0)(t-t')}$$

$$+\sum_{fpe''g}\int_0^\infty d\omega\, G^*_{ue''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\sum_{e'''g'}G_{ue'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{e'''g'}(t')\, e^{i(\omega-\omega_0)(t-t')}$$

$$+\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G^*_{\nu e''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\sum_{e'''g'}G_{\nu e'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{e'''g'}(t')\, e^{i(\omega-\omega_0)(t-t')}$$

$$\doteq \hat\sigma_{ee'}^{free}$$

$$-\sum_{fpe''g}\int_0^\infty d\omega\, G_{ue''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\sum_{e'''g'}G^*_{ue'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{g'e'''}(t')\, e^{-i(\omega-\omega_0)(t-t')}$$

$$-\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G_{\nu e''g}\left(\delta_{e''e'}\hat\sigma_{eg}-\delta_{ge}\hat\sigma_{e''e'}\right)\sum_{e'''g'}G^*_{\nu e'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{g'e'''}(t')\, e^{-i(\omega-\omega_0)(t-t')}$$

$$+\sum_{fpe''g}\int_0^\infty d\omega\, G^*_{ue''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\sum_{e'''g'}G_{ue'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{e'''g'}(t')\, e^{i(\omega-\omega_0)(t-t')}$$

$$+\sum_{mpe''g}\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\, G^*_{\nu e''g}\left(\delta_{ge'}\hat\sigma_{ee''}-\delta_{e''e}\hat\sigma_{ge'}\right)\sum_{e'''g'}G_{\nu e'''g'}\cdot\int_{t_0}^t dt'\,\hat\sigma_{e'''g'}(t')\, e^{i(\omega-\omega_0)(t-t')}$$

$$= \hat\sigma_{ee'}^{free}$$

$$-\int_0^\infty d\omega\left[\sum_{fpe'''g} G_{ue'g}\, G^*_{ue'''g}\ \hat\sigma_{ee'''}-\sum_{fpe''e'''} G_{ue''e}\, G^*_{ue'''e'}\ \hat\sigma_{e''e'''}\right]\cdot\left(\pi\delta(\omega-\omega_0)-i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)$$

$$-\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\left[\sum_{mpe'''g} G_{\nu e'g}\, G^*_{\nu e'''g}\ \hat\sigma_{ee'''}-\sum_{mpe''e'''} G_{\nu e''e}\, G^*_{\nu e'''e'}\ \hat\sigma_{e''e'''}\right]\cdot\left(\pi\delta(\omega-\omega_0)-i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)$$

$$+\int_0^\infty d\omega\left[\sum_{fpe''g'} G^*_{ue''e'}\, G_{ue''g'}\ \hat\sigma_{eg'}-\sum_{fp\,gg'} G^*_{ueg}\, G_{ue'g'}\ \hat\sigma_{gg'}\right]\cdot\left(\pi\delta(\omega-\omega_0)+i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)$$

$$+\int_0^\infty d\omega\int_{-kn_2}^{kn_2} d\beta\left[\sum G^*_{\nu e''e'}\, G_{\nu e''g'}\ \hat\sigma_{eg'}-\sum G^*_{\nu eg}\, G_{\nu e'g'}\ \hat\sigma_{gg'}\right]\cdot\left(\pi\delta(\omega-\omega_0)+i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)$$

$$
+\int_0^\infty d\omega \left[\sum_{fpe''g'} G^*_{\mu e''e'} G_{\mu e''g'} \hat{\sigma}_{eg'} - \sum_{fp}\sum_{gg'} G^*_{\mu eg} G_{\mu e'g'} \hat{\sigma}_{gg'}\right]\cdot\left(\pi\delta(\omega-\omega_0) + i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)
$$

$$
+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot \left[\sum_{mpe''g'} G^*_{\nu e''e'} G_{\nu e''g'} \hat{\sigma}_{eg'} - \sum_{mp}\sum_{gg'} G^*_{\nu eg} G_{\nu e'g'} \hat{\sigma}_{gg'}\right]\cdot\left(\pi\delta(\omega-\omega_0) + i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)
$$

$$
\left\{
\begin{array}{l}
G_{\mu e''e'} \propto d_{e''e'} = 0, \quad G_{\nu e''e'} = G_{\mu e''e} = G_{\nu e''e} = G_{\mu e'''e'} = G_{\nu e'''e'} = 0. \\
\sum_{g,g'} \hat{\sigma}_{gg'} = 1 - \sum_{e''} \hat{\sigma}_{e''e'} \quad ?
\end{array}
\right.
$$

$$
\begin{aligned}
\frac{d}{dt}\hat{\sigma}_{gg'} &= -\frac{i}{\hbar}\left[\hat{\sigma}_{gg'}, \hat{H}_{int}\right] \\
&= -\sum_{fpeg''} \int_0^\infty d\omega\, G_{\mu eg''} \left[\hat{\sigma}_{gg'}, \hat{\sigma}_{eg''}\right] \hat{a}_\mu e^{-i(\omega-\omega_0)t} \\
&\quad - \sum_{mpeg''} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta\, G_{\nu eg''} \left[\hat{\sigma}_{gg'}, \hat{\sigma}_{eg''}\right] \hat{a}_\nu e^{-i(\omega-\omega_0)t} \\
&\quad + \sum_{fpeg''} \int_0^\infty d\omega\, G^*_{\mu eg''} \left[\hat{\sigma}_{gg'}, \hat{\sigma}_{g''e}\right] \hat{a}^+_\mu e^{i(\omega-\omega_0)t} \\
&\quad + \sum_{mpeg''} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot G^*_{\nu eg''} \left[\hat{\sigma}_{gg'}, \hat{\sigma}_{g''e}\right] \hat{a}^+_\nu e^{i(\omega-\omega_0)t} \\
&= -\sum_{fpeg''} \int_0^\infty d\omega\, G_{\mu eg''} \left(\delta_{g'e}\hat{\sigma}_{gg''} - \delta_{g''g}\hat{\sigma}_{eg'}\right) \hat{a}_\mu e^{-i(\omega-\omega_0)t} \\
&\quad - \sum_{mpeg''} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta\, G_{\nu eg''} \left(\delta_{g'e}\hat{\sigma}_{gg''} - \delta_{g''g}\hat{\sigma}_{eg'}\right) \hat{a}_\nu e^{-i(\omega-\omega_0)t} \\
&\quad + \sum_{fpeg''} \int_0^\infty d\omega\, G^*_{\mu eg''} \left(\delta_{g'g''}\hat{\sigma}_{ge} - \delta_{eg}\hat{\sigma}_{g''g'}\right) \hat{a}^+_\mu e^{i(\omega-\omega_0)t} \\
&\quad + \sum_{mpeg''} \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot G^*_{\nu eg''} \left(\delta_{g'g''}\hat{\sigma}_{ge} - \delta_{eg}\hat{\sigma}_{g''g'}\right) \hat{a}^+_\nu e^{i(\omega-\omega_0)t} \\
&= -\int_0^\infty d\omega \left[\sum_{fpg''} G_{\mu g'g''}\hat{\sigma}_{gg''} - \sum_{fpe} G_{\mu eg}\hat{\sigma}_{eg'}\right] \cdot \hat{a}_\mu e^{-i(\omega-\omega_0)t} \\
&\quad - \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \left[\sum_{mpg''} G_{\nu g'g''}\hat{\sigma}_{gg''} - \sum_{mpe} G_{\nu eg}\hat{\sigma}_{eg'}\right] \cdot \hat{a}_\nu e^{-i(\omega-\omega_0)t} \\
&\quad + \int_0^\infty d\omega \left[\sum_{fpe} G^*_{\mu eg'}\hat{\sigma}_{ge} - \sum_{fpg''} G^*_{\mu gg''}\hat{\sigma}_{g''g'}\right] \hat{a}^+_\mu e^{i(\omega-\omega_0)t} \\
&\quad + \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \left[\sum_{mpe} G^*_{\nu eg'}\hat{\sigma}_{ge} - \sum_{mpg''} G^*_{\nu gg''}\hat{\sigma}_{g''g'}\right] \hat{a}^+_\nu e^{i(\omega-\omega_0)t} \\
&= +\int_0^\infty d\omega \sum_{fpe} G_{\mu eg}\hat{\sigma}_{eg'} \cdot \hat{a}_\mu e^{-i(\omega-\omega_0)t} \\
&\quad + \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \sum_{mpe} G_{\nu eg}\hat{\sigma}_{eg'} \cdot \hat{a}_\nu e^{-i(\omega-\omega_0)t} \\
&\quad + \int_0^\infty d\omega \sum_{fpe} G^*_{\mu eg'}\hat{\sigma}_{ge}\, \hat{a}^+_\mu e^{i(\omega-\omega_0)t} \\
&\quad + \int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot \sum_{mpe} G^*_{\nu eg'}\hat{\sigma}_{ge}\, \hat{a}^+_\nu e^{i(\omega-\omega_0)t} \\
&= \left.
\begin{aligned}
&+\int_0^\infty d\omega \sum_{fpe} G_{\mu eg}\hat{\sigma}_{eg'}\, \hat{a}_\mu(t_0) e^{-i(\omega-\omega_0)t} \\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \sum_{mpe} G_{\nu eg}\hat{\sigma}_{eg'} \cdot \hat{a}_\nu(t_0) e^{-i(\omega-\omega_0)t} \\
&+\int_0^\infty d\omega \sum_{fpe} G^*_{\mu eg'}\hat{\sigma}_{ge}\, \hat{a}^+_\mu(t_0) e^{i(\omega-\omega_0)t}
\end{aligned}
\right\} \hat{\sigma}^{free}_{gg'}
\end{aligned}
$$

$$
\left.\begin{aligned}
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \sum_{mpe} G_{\nu eg} \hat{\sigma}_{eg'} \hat{a}_{\nu}(t_0) e^{-i(\omega-\omega_0)t}\\
&+\int_0^\infty d\omega \sum_{fpe} G^*_{\mu eg'} \hat{\sigma}_{ge} \hat{a}^\dagger_{\mu}(t_0) e^{i(\omega-\omega_0)t}\\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot \sum_{mpe} G^*_{\nu eg'} \hat{\sigma}_{ge} \hat{a}^\dagger_{\nu}(t_0) e^{i(\omega-\omega_0)t}
\end{aligned}\right\} \hat{\sigma}^{free}_{gg'}
$$

$$
\begin{aligned}
&+\int_0^\infty d\omega \sum_{fpe} G_{\mu eg} \hat{\sigma}_{eg'} \cdot \sum_{e'g''} G^*_{\mu e'g''} \hat{\sigma}_{g''e'} \int_{t_0}^{t} dt' e^{-i(\omega-\omega_0)(t-t')}\\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \sum_{mpe} G_{\nu eg} \hat{\sigma}_{eg'} \cdot \sum_{e'g''} G^*_{\nu e'g''} \hat{\sigma}_{g''e'} \int_{t_0}^{t} dt' e^{-i(\omega-\omega_0)(t-t')}\\
&+\int_0^\infty d\omega \sum_{fpe} G^*_{\mu eg'} \hat{\sigma}_{ge} \cdot \sum_{e'g''} G_{\mu e'g''} \hat{\sigma}_{e'g''} \int_{t_0}^{t} dt' e^{i(\omega-\omega_0)(t-t')}\\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot \sum_{mpe} G^*_{\nu eg'} \hat{\sigma}_{ge} \sum_{e'g''} G_{\nu e'g''} \hat{\sigma}_{e'g''} \int_{t_0}^{t} dt' e^{i(\omega-\omega_0)(t-t')}
\end{aligned}
$$

$$
\begin{aligned}
= \hat{\sigma}^{free}_{gg'}
&+\int_0^\infty d\omega \sum_{fpe} G_{\mu eg} \hat{\sigma}_{eg'} \cdot \sum_{e'g''} G^*_{\mu e'g''} \hat{\sigma}_{g''e'} \left(\pi\delta(\omega-\omega_0) - i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)\\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \sum_{mpe} G_{\nu eg} \hat{\sigma}_{eg'} \cdot \sum_{e'g''} G^*_{\nu e'g''} \hat{\sigma}_{g''e'} \left(\pi\delta(\omega-\omega_0) - i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)\\
&+\int_0^\infty d\omega \sum_{fpe} G^*_{\mu eg'} \hat{\sigma}_{ge} \cdot \sum_{e'g''} G_{\mu e'g''} \hat{\sigma}_{e'g''} \left(\pi\delta(\omega-\omega_0) + i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)\\
&+\int_0^\infty d\omega \int_{-kn_2}^{kn_2} d\beta \cdot \sum_{mpe} G^*_{\nu eg'} \hat{\sigma}_{ge} \sum_{e'g''} G_{\nu e'g''} \hat{\sigma}_{e'g''} \left(\pi\delta(\omega-\omega_0) + i\mathbb{P}\left(\frac{1}{\omega-\omega_0}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \hat{\sigma}^{free}_{gg'}
&+\pi\Big[\sum_{fpee'g''} G_{\mu_0 eg} G^*_{\mu_0 e'g''} \hat{\sigma}_{ee'} \delta_{g'g''} + \int_{-kn_2}^{kn_2} d\beta \sum_{mpee'g''} G_{\nu_0 eg} G^*_{\nu_0 e'g''} \hat{\sigma}_{ee'} \delta_{g'g''}\\
&+ \sum_{fpee'g''} G^*_{\mu_0 eg'} G_{\mu_0 e'g''} \hat{\sigma}_{gg''} \delta_{ee'} + \int_{-kn_2}^{kn_2} d\beta \sum_{mpee'g''} G^*_{\nu_0 eg'} G_{\nu_0 e'g''} \hat{\sigma}_{gg''} \delta_{ee'}\Big]
\end{aligned}
$$

$+$ Cauchy's principle value parts.

$$
\begin{aligned}
= \pi\Big[&\sum_{fpee'} G_{\mu_0 eg} G^*_{\mu_0 e'g} \hat{\sigma}_{ee'} + \int_{-kn_2}^{kn_2} d\beta \sum_{mpee'} G_{\nu_0 eg} G^*_{\nu_0 e'g} \hat{\sigma}_{ee'}\\
&+ \sum_{fpeg''} G^*_{\mu_0 eg'} G_{\mu_0 eg''} \hat{\sigma}_{gg''} + \int_{-kn_2}^{kn_2} d\beta \sum_{mpeg''} G^*_{\nu_0 eg'} G_{\nu_0 eg''} \hat{\sigma}_{gg''}\Big]\\
&+ \hat{\xi}_{gg'} \rightarrow \hat{\sigma}^{free}_{gg'} + \text{Cauchy's principle parts}
\end{aligned}
$$