$$\frac{d}{dz}h = \frac{1}{2}\left[h_{n-1}^{(1)}(z) - \frac{h_n^{(1)}(z) + z h_{n+1}^{(1)}(z)}{z}\right]$$

$$J_q(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\tau - x\sin\tau)\, d\tau$$

Derivation of total decay rates.

① z-orientation dipole:

$$\frac{\gamma^{total}}{\gamma_0} = 1 + \frac{3}{2}\,\mathrm{Im}\,\frac{d_0 \cdot \vec{E}^{(R)}(\vec{r}', \omega_0)}{d_0^2 k^3}$$

$$E_z^{(R)} = \sum_{m=-\infty}^{\infty}\int d\beta\, \underbrace{e^{im\varphi + i\beta z}}_{1,\ \varphi=0,\ z=0}\, H_m^{(1)}(P_z\rho)\, a_{m\beta}$$

$$\Rightarrow \frac{\gamma^{total}}{\gamma_0} = 1 + \frac{3}{2}\,\mathrm{Im}\,\frac{\sum_{m=-\infty}^{\infty}\int_{C_1} d\beta\, H_m^{(1)}(P_z\rho)\, a_{m\beta}}{d_{0z}\cdot k^3}$$

$$a_{m\beta} = \frac{na}{P^2 + QR} = \frac{1}{D}\cdot\Big[\cancel{im\beta d_{0z}}\, h_z^2 P_z^2\, a J_m(h_z a) P E_{0\varphi, m\beta}$$
$$+ E_{0z m\beta}\left(P_z^2 a J_m(h_z a)\beta m P + P_z^2 k a \varepsilon h_z \tfrac{1}{2}(J_{m-1}(h_z a) - J_{m+1}(h_z a)) Q\right)$$
$$+ i h_z^2 P_z^2 a J_m(h_z a) Q B_{0\varphi, mh} - im\beta h_z P_z J_m(h_z a) S B_{0z m\beta}$$

$$= \frac{1}{D}\Big[h_z^2 P_z^2 a J_m(h_z a) P \cdot \frac{im\beta d_{0z}}{2\rho} J_m(P_z\rho) H_m^{(1)}(P_z\rho')$$
$$+ \frac{i d_{0z} P_z^2}{2} J_m(P_z\rho) H_m^{(1)}(P_z\rho')\left(P_z^2 a J_m(h_z a)\beta m P + P_z^2 \frac{k a \varepsilon h_z}{2}(J_{m-1}(h_z a) - J_{m+1}(h_z a)) Q\right)$$
$$- i h_z^2 P_z^2 a J_m(h_z a) Q \frac{d_{0z} k P_z}{4}(J_{m-1}(P_z\rho) - J_{m+1}(P_z\rho)) H_m^{(1)}(P_z\rho') - 0\Big]$$

$\rho \to \rho'$

$$= \frac{1}{D}\Big[\frac{-im\beta d_{0z} a h_z^2 P_z^2 P}{2\rho} J_m(h_z a) J_m(P_z\rho) H_m^{(1)}(P_z\rho')$$
$$+ \frac{i m\beta d_{0z} P_z^4 P}{2} J_m(h_z a) J_m(P_z\rho) H_m^{(1)}(P_z\rho')$$
$$+ \frac{i k a \varepsilon d_{0z} h_z P_z^4 Q}{4}(J_{m-1}(h_z a) - J_{m+1}(h_z a)) J_m(P_z\rho) H_m^{(1)}(P_z\rho')$$
$$- \frac{i a k d_{0z} h_z^2 P_z^3 Q}{4} J_m(h_z a)(J_{m-1}(P_z\rho) - J_{m+1}(P_z\rho)) H_m^{(1)}(P_z\rho')\Big]$$

$$= \frac{i P_z^2}{2D} \Big[-\frac{m\beta d_{0z} P(\varepsilon-1)k^2}{} J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a)$$
$$+ \frac{a k \varepsilon d_{0z} h_z P_z^2 Q}{2} (J_{m-1}(h_z a) - J_{m+1}(h_z a)) J_m(P_z a) H_m^{(1)}(P_z a)$$
$$- \frac{a k d_{0z} h_z^2 P_z Q}{2} J_m(h_z a)(J_{m-1}(P_z a) - J_{m+1}(P_z a)) H_m^{(1)}(P_z a) \Big]$$

Using

$$\left(z\frac{d}{dz} - n^2\right) J_n(z) = -z^2 J_n(z)$$

$$\Rightarrow z\frac{d}{dz} J_n(z) = (n^2 - z^2) J_n(z)$$

$$\Rightarrow a_{m\beta} = \frac{i P_z^2}{2D} \Big[-m\beta d_{0z}(\varepsilon-1)k^2 P \cdot J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a)$$
$$+ k\varepsilon d_{0z} P_z^2 Q (m^2 - h_z^2 a^2) J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a)$$
$$- k d_{0z} h_z^2 Q (m^2 - P_z^2 a^2) J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a) \Big]$$

$$= \frac{i P_z^2}{2D} \cdot J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a) \cdot d_{0z} \cdot k \Big[-m\beta(\varepsilon-1)k P + \varepsilon P_z^2 m^2 Q$$
$$- h_z^2 Q m^2 + P_z^2 h_z^2 a^2 Q(1-\varepsilon) \Big]$$

$$= \frac{i d_{0z} k(\varepsilon-1) P_z^2}{2D} J_m(h_z a) J_m(P_z a) H_m^{(1)}(P_z a)$$
$$\cdot \left[m\beta k P + \beta^2 m^2 Q + a^2 P_z^2 h_z^2 Q \right]$$

$$= -\frac{i d_{0z} P_z^2 P J_m(P_z a)}{2 m k \beta D} \left(m\beta k P + \beta^2 m^2 Q + a^2(\varepsilon k^2 - \beta^2)(k^2 - \beta^2) Q \right)$$

$$= -\frac{i d_{0z} P_z^2 J_m(P_z a)}{2D} \left(P^2 + \frac{\beta m}{k} Q P + \frac{a^2 h_z^2 k}{\beta m} Q P - \frac{a^2 h_z^2 \beta}{k m} Q P \right)$$

Might be useful:

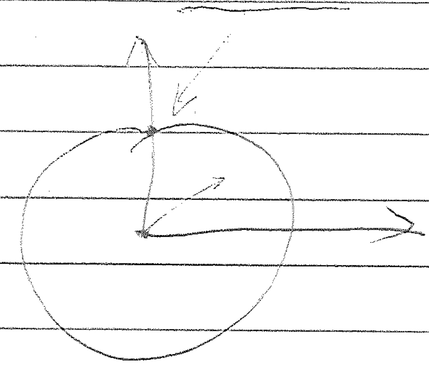
$$H_m^{(1)}(z) = J_m(z) + i Y_m(z), \quad \left(\frac{d}{z\,dz}\right)^m \{ z^\nu H_\nu^{(1)}(z) \} = z^{\nu-m} H_{\nu-m}^{(1)}(z)$$

$$z\frac{d}{dz} H_\nu^{(1)}(z) + \nu H_\nu^{(1)}(z) = z H_{\nu-1}^{(1)}(z), \quad z\frac{d}{dz} H_\nu^{(1)}(z) - \nu H_\nu^{(1)}(z) = -z H_{\nu+1}^{(1)}(z),$$

$$\frac{d}{dz}\{ z^\nu Y_\nu(z) \} = z^\nu Y_{\nu-1}(z), \quad Y_\nu'(z) = \frac{1}{2}(Y_{\nu-1}(z) - Y_{\nu+1}(z))$$

$$z Y_\nu'(z) + \nu Y_\nu(z) = z Y_{\nu-1}(z), \quad z Y_\nu'(z) - \nu Y_\nu(z) = -z Y_{\nu+1}(z).$$

$$\hat{P} = \frac{1}{2}\left[\left(\frac{e^{-T/\tau_c}+1}{2}\right)\hat{\mathbb{1}} + \left(\frac{e^{-T/\tau_c}-1}{2}\vec{e_z} + e^{-T/\tau_c}\vec{e_y}\right)\hat{\vec{\sigma}} \right]$$



Derivation Continue:

$$QP = -\cancel{\ } \beta m k^3 h_z P_z a J_m(h_z a) H_m^{(1)}(P_z a)(\varepsilon - 1)$$
$$\left(h_z J_m(h_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a) - P_z H_m^{(1)}(P_z a)\frac{d}{d(P_z a)} J_m(h_z a)\right)$$

$$QR = -h_z^2 P_z^2 a^2 k^2 \left(h_z J_m(h_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a) - P_z H_m^{(1)}(P_z a)\frac{d}{d(h_z a)} J_m(h_z a)\right)$$
$$\left(h_z J_m(h_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a) - P_z \varepsilon H_m^{(1)}(P_z a)\frac{d}{d(h_z a)} J_m(h_z a)\right]$$
$$= -h_z^2 P_z^2 a^2 k^2 \Big(h_z^2 J_m^2(h_z a)\left(\frac{d}{d(P_z a)} H_m^{(1)}(P_z a)\right)^2 + \varepsilon P_z^2 H_m^2(P_z a)\left(\frac{d}{d(h_z a)} J_m(h_z a)\right)^2$$
$$- h_z P_z \varepsilon J_m(h_z a) H_m^{(1)}(P_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a)\frac{d}{d(h_z a)} J_m(h_z a)$$
$$- h_z P_z J_m(h_z a) H_m^{(1)}(P_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a)\frac{d}{d(h_z a)} J_m(h_z a) \Big]$$

$$R = ak\left[h_z^2 P_z J_m(h_z a)\frac{d}{d(P_z a)} H_m^{(1)}(P_z a) - h_z P_z^2 \varepsilon H_m^{(1)}(P_z a)\frac{d}{d(h_z a)} J_m(h_z a)\right]$$