CIS 1600 Review

Andrew Kim

4 December 2025

Contents

1 Graphs

1.1 Introduction to Graphs

L6T Prove that the sum of the degrees of all nodes in a graph is twice the number of edges.

L6T In any graph there are an even number of vertices of odd degree.

L6T Prove that every graph with n vertices and m edges has at least $n - m$ connected components.

L6T Prove that every connected graph with n vertices has at least $n - 1$ edges.

1.2 Trees

L6T Prove that every tree with at least two vertices has at least two leaves, and deleting a leaf from an n-vertex tree produces a tree with $n - 1$ vertices.

L7T The following are equivalent:
(1) G is a tree.
(2) G is connected and has exactly $n - 1$ edges.
(3) G is minimally connected.
(4) G contains no cycle but $G + \{x, y\}$ does.
(5) Any two vertices of G are linked by a unique path.

L7T Every connected graph $G = (V, E)$ contains a spanning tree.

L9T If T is a full binary tree with k internal vertices, prove that T has $2k + 1$ vertices and $k + 1$ leaves.

L9T Prove that any binary tree of height at most h has at most 2^h leaves.

1.3 Hamiltonian & Eulerian Graphs

L9T For any integer $n \geq 3$, show that if all vertices in a simple graph G satisfy $\deg(v) \geq n/2$, then G has a Hamiltonian cycle.

L11T If $\delta(G) \geq 2$ then G contains a cycle.

L11T Prove that a connected graph G is Eulerian if and only if every vertex of G has even degree.

1.4 Graph Coloring

L11T Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

2 Probability

L6H An urn contains 5 white balls and 7 black balls. At each draw, the color is recorded and the ball is replaced along with two more of the same color. Compute the probability that the first two draws are black and the next two are white.

L6H A medical test for a certain condition has arrived in the market. According to the case studies, when the test is performed on an affected person, the test comes up positive 95% of the times and yields a "false negative" 5%. When the test is performed on a person not suffering from the medical condition the test comes up negative in 99% of the cases and yields a "false positive" in 1% of the cases. If 0.5% of the population actually have the condition, what is the probability that the person has the condition given that the test is positive?

L6H A transmitter sends binary bits, 80% 0's and 20% 1's. When a 0 is sent, the receiver will detect it correctly 80% of the time. When a 1 is sent, the receiver will detect it correctly 90% of the time. What is the probability that a 1 is sent and a 1 is received? If a 1 is received, what is the probability that a 1 was sent?

L6H Two cards are drawn without replacement. Let A be "same value," B be "first card is an ace." Are A and B independent?

L6H A fair coin is tossed twice. Let A be "first toss heads," B be "second toss heads," C be "two heads or two tails." Are A, B, C pairwise independent? Are they mutually independent?

3 Random Variables

3.1 Expectation

L8T Compute $E[X]$ when rolling a single fair die.

L8T Compute the expected sum when rolling two fair dice.

L8T Use linearity of expectation to compute the expected sum of two dice.

L8T In the hat-check problem with n hats, compute the expected number of people who get their own hat back.

L8T Throw n balls independently and uniformly into n bins. Compute the expected number of empty bins.

L9H Suppose there are k people and n days in a year. Compute the expected number of pairs of people who share a birthday.

L11H Compute the expected number of cereal boxes needed to collect all n coupons in the coupon collector's problem.

3.2 Variance, Markov's Theorem, and Chebyshev's Theorem

L9H Use Markov's inequality to bound $\Pr[X \geq 3n/4]$ when X is the number of heads in n fair coin flips.

L9H Use Markov's inequality to bound $\Pr[X \geq 7]$ when X is the result of a die roll.

L9H In the hat-check problem, compute $Var(X)$, where X is the number of people who get their own hat.

L10H Use Chebyshev's inequality to bound $\Pr[X \geq 3n/4]$ for the number of heads X in n fair coin flips.

4 Relations and Matchings

4.1 Matchings

L11H Prove that a matching M in a graph G is maximum if and only if G contains no M-augmenting path.

L12T Let $G = (X, Y, E)$ be a bipartite graph. For any set $S \subseteq X$, let $N_G(S)$ denote the set of vertices in Y that are adjacent to vertices in S. Prove that G contains a matching that saturates every vertex in X if and only if for every subset $S \subseteq X$, $|N_G(S)| \geq |S|$. The condition "for all $S \subseteq X$, $|N(S)| \geq |S|$" is called Hall's condition.

4.2 Relations

L11H How many relations are there on a set A of n elements?

L12T Let A be a set of size n. How many reflexive relations are there on A?

L13T Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Let $R_1 = \{(1, a), (1, c), (2, c), (3, a)\}$ and $R_2 = \{(1, b), (1, c), (1, d), (2, b)\}$. Compute $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \setminus R_2$, and $R_2 \setminus R_1$. [file:2]

L13T Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose R_1 consists of ordered pairs (a, b) where a is a student who has taken course b, and R_2 consists of ordered pairs (a, b) where a is a student who requires course b to graduate. Describe the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 \setminus R_2$, and $R_2 \setminus R_1$. [file:2]

L13T Let R be a relation on a set A. Prove that R is symmetric if and only if $R = R^{-1}$. [file:2]

L13T Let R be a relation on a set A. The powers of R are defined by $R^1 = R$ and $R^{n+1} = R^n \circ R$. Prove that R is transitive if and only if $R^n \subseteq R$ for all integers $n \geq 1$. [file:2]

5 Probabilistic Method

L13T Let $G = (V, E)$ be a connected graph on $n \geq 2$ vertices with minimum degree $\delta(G) = \delta$. Use the probabilistic method to prove that G contains a dominating set of size at most

$$\frac{n(1 + \ln(1 + \delta))}{1 + \delta}$$

L13T Let $R(k, k)$ be the diagonal Ramsey number. Prove that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$. In particular, show that $R(k, k) > \lfloor 2^{k/2} \rfloor$ for all integers $k \geq 3$.