Article

Engineering and Applying Quantum Contextuality

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Abstract: Endeavour of refuting hidden variable theories underlying quantum theory yielded the discipline of contextual sets. A plethora of various kinds of sets of arbitrary structure in any dimension have been developed, alongside extensive experimental validation. So, in this paper we investigate to which extent we might move beyond hidden variables and engineer contextual sets and their applications within quantum theory itself without any reference to hidden variable models. In particular, we consider possible applications of contextual sets in quantum computation, cryptography, pseudo-telepathy, and nonlocality as well as generating them from error-correction protocols, complex Hadamard gates, and simple quantum gates.

Keywords: quantum contextuality; hypergraph contextuality; MMP hypergraphs; MMP language; Kochen-Specker sets; non-Kochen-Specker contextual sets; quantum contextuality applications

1. Introduction

Quantum contextuality is a property of a set (let us call it a *contextual set*) of quantum states that precludes assignments of predetermined 0-1 values to their elements.

This feature of contextual sets has historically stemmed from procedures aimed to refute the existence of hidden classical variables underlying the quantum description of quantum systems and it has so far mostly served us as a tool for distinguishing between quantum and classical sets. Along this line experiments carried out so far [1–21], just confirm that implemented contextual sets really are contextual.

In contrast, we shall focus on engineering contextual sets and their applications without any reference to possible hidden variable models. In particular, we shall explore how quantum contextuality emerges from requirements we might expect quantum systems to possess and how we can find their application in quantum computation and networking.

In doing so, we shall not focus on how to generate new contextual sets by different methods but on how much quantum theory and quantum measurements themselves generate contextual sets and whether there are instances of say quantum computation and quantum measurements that generate none but a contextual set or are generated by none but contextual sets. For instance, a recent analysis of stabilizer operations characterizing quantum error correction and therefore quantum computation [22] received the following review: “What gives quantum computers that extra oomph over their classical digital counterparts? An intrinsic, measurable aspect of quantum mechanics called contextuality, it now emerges” [23]. But the question “which contextualities” has not been properly addressed so far.

In order to be able to speak about contextual sets we first have to introduce the language that might describe and handle them. Its syntax will enable us to carry out our elaboration of cases as well as to identify failures which might happen when we fail to adhere to it.

The language we shall make use of is the language of MMP hypergraphs (MMPHs). There are other formulations, but they can all be translated to MMPHs [24].

1.1. General hypergraphs

A general hypergraph $\mathcal{H} = (V, E)$ is a set of points/vertices $V = \{v_1, v_2, \ldots, v_k\}$ and a set of subsets/hyperedges $E = \{e_1, e_2, \ldots, e_l\}$ of them [25–28]. The cardinality of V ($|V|$) (number of vertices) is called the *order* of a hypergraph and the cardinality of E ($|E|$) (number of hyperedges) is called the *size* of a hypergraph.

1.2. MMP hypergraphs and their language

A special kind of general hypergraph on which the formalism of contextual sets is founded is the McKay-Megill-Pavičić hypergraph (MMPH).

Definition 1. **MMPH-dimension** *n is a predefined (for an assumed task or purpose) maximal possible number (n) of vertices within a hyperedge of an MMPH even when none of the processed hyperedges include n vertices.*

Definition 2. *An* **MMPH** *is a connected* hypergraph $\mathcal{H} = (V, E)$ *(where $V = \{V_1, V_2, \ldots, V_k\}$ is a set of* vertices *and $E = \{E_1, E_2, \ldots, E_l\}$ sets of* hyperedges*) of* MMPH-dim $n \geq 3$ *in which*

1. *Every vertex belongs to at least one hyperedge;*
2. *Every hyperedge contains at least* 2 *and at most n vertices;*
3. *No hyperedge shares only one vertex with another hyperedge;*
4. *Hyperedges may intersect each other in at most $n - 2$ vertices;*
5. *Numerically, an MMPH is a string of ASCII characters (vertices) organized in substrings (hyperedges) separated by commas; each string ends with a period; when 90 characters* `1...9 A...Z a...z ! " # $ % & ' ( ) * - / : ; < = > ? @ [ \ ] ^ _ ` { | } ~`*; are exhausted, one reuses them prefixed by '+, then by '++', etc.—no limit.*
6. *Graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them.*

Differences between the standard hypergraph and the MMPH formalism are illustrated in [29, Fig. 1].

Definition 3. *A k-l* MMPH *of dim $n \geq 3$ with k vertices and l hyperedges, whose i-th hyperedge contains $\kappa(i)$ vertices $(2 \leq \kappa(i) \leq n, i = 1, \ldots, l)$ to which it is impossible (possible) to assign* 1*s and* 0*s in such a way that the following* rules *hold*

(i) No two vertices within any of its hyperedges may both be assigned the value 1*;*
(ii) In any of its hyperedges, not all vertices may be assigned the value 0*.*

is called a non-binary MMPH *(***NBMMPH***) (a* binary MMPH *(***BMMPH***)).*

Definition 4. *A* **Critical NBMMPH** *is an* NBMMPH *which is minimal in the sense that removing any of its hyperedges turns it into a BMMPH.*

Definition 5. **Vertex multiplicity** *is the number of hyperedges vertex 'i' belongs to; we denote it by $m(i)$.*

Definition 6. *A* **filled** *n-dim* MMPH *is the one that is derived from an n-dim* MMPH *with at least one hyperedge containing fewer than n vertices by adding vertices to ensure all hyperedges have precisely n vertices.*

1.3. Coordinatization

Vector or state or operator representation, i.e., a *coordinatization* of vertices is operationally required for any implementation of an MMPH since a full coordinatization of vertices turns MMPH-dimension n into a dimension of a Hilbert space determined by vectors each vertex is assigned to.

Our algorithms and programs can detect the contextuality of an MMPH no matter whether its coordinatization is given (or even existent) or not. Handling of MMPHs using our algorithms embedded in programs SHORTD, MMPSTRIP, MMPSUBGRAPH, VECFIND, STATES01, and others [30–35] without taking their coordinatization into account gives us a computational advantage over handling them with a coordinatization because processing bare hypergraphs is faster than processing them with vectors assigned to their vertices.

Lemma 1. *An* NBMMPH *from Def. 3 with all $\kappa(i) = n$ (with at least one $\kappa(i)$ such that $2\le\kappa(i)<n$) which possesses a coordinatization is a* **Kochen-Specker (KS)** *(***non-Kochen-Specker (non-KS)***) set.*

Proof. Obvious. [29,36–38]. □

The paper is organized as follows.

In Sec. 2.1 we consider the misconceptions of measuring states that define structures of quantum contextuality. States are organized in sets of orthogonalities which form hyperedges of MMPHs and ultimately the MMPHs themselves. Of those states we might decide to measure some and not the others (non-KS MMPHs) but only all geometrically possible states define the structure of an MMPH (KS or noncontextual). We show that this is overlooked by many researchers in the field what results in untenable approaches to systems. In particular, we show that well-known 3D non-KS MMPHs cannot be minimal since none of them is critical and they all contain many smaller MMPHs. The same applies to 4D stabilizer + magic state non-KS MMPH which finds its application in the quantum computation.

In Sec. 2.2 we consider a fusion of entanglement and nonlocality called the pseudo-telepathy game. Such games were originally conceived as another way of disproving hidden variable theories. We analyse some recent approaches to see whether they offer us a richer method of such a verification. In particular whether the complete bases set up limits on it. We find that their role is richer than anticipated.

In Secs. 2.3 and 2.4 we point out that nonlocalities and inequalities are just other forms of quantum contextualities rather than their "applications."

In Sec. 2.5 we consider cryptography supported by MMPHs. They offer a vast support and protection to large alphabet protocols. We show that this MMPH data base is what protects the protocols and not the contextuality of MMPHs.

In Sec. 2.6 a generation of star like non-KS MMPHs from generalized complex Hadamard matrices is presented and a way of obtaining coordination for low number of complete bases is elaborated.

In Sec. 2.7 a recent attempt to construct new large 3D sets and use it to obtain a lower bound on size of non-KS sets is analysed and explained why it failed.

In Sec. 3 we discuss the obtained results.

2. Results

In our previous papers we have shown how one can generate NBMMPHs of arbitrary structure and complexity in any dimension [24,29,39–41].

Here, we shall investigate whether one can obtain NBMMPHs from quantum artefacts (models, structures, and/or measurements) or make use of them to arrive at possible quantum applications without a reference to any hidden variable theory. While doing so, we shall highlight some misconceptions and misinterpretations that might stand in the way of a successful elaboration toward such a goal.

2.1. Minimal KS sets and quantum computation

Recently, a role of quantum contextuality in quantum error correction algorithms which made use of stabilizer states has been revealed [22] and its contribution to speeding-up quantum computation has been considered [42].

In doing so, graphs and hypergraphs were employed as prevalent tools. But before we dwell on the stabilizer sets let us first discuss some relevant features of simple 3D hypergraphs/MMPHs.

There is a widespread misconception that the historic contextual sets 33-36 Bob-Schütte, 33-40 Peres, 31-37 Conway-Kochen, 117-118 Kochen-Specker, and 33-50 Pavičić sets are the smallest possible 3D contextual sets, two of which are shown in Fig. 1(b,d). However, being the smallest set means being a critical set in the sense that dropping any of the edges of such a set renders it non-contextual. Hence, since they are claimed to be the smallest, they should not contain smaller contextual sets than they themselves are. Yet, none of the aforementioned five 3D sets is critical and they all contain a large number of smaller critical KS sets [43,60].

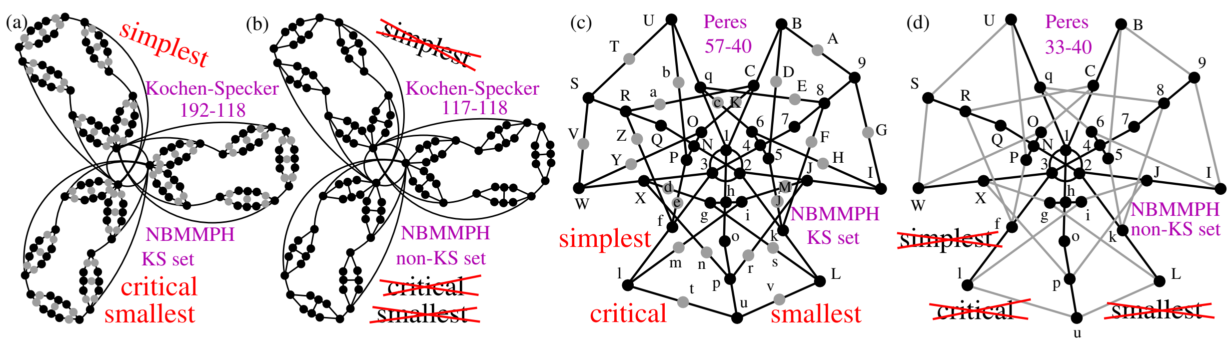

Figure 1. (a) Critical KS MBMMPH 192-118 according to [44, Fig. 6]; the simplest MMPH in its class; its coordinatization is given for the first time in [40, Supplemental Material, pp. 3,4]; (b) redrawn Γ_2 [36] following Fig. 6.8 from [45]; (c) Peres' 57-40; vertices with multiplicity 1 (grey dots) are added to [46, Fig. 9]; the simplest MMPH in its class (81-52 master); (d) Peres' 33-40 according to Fig. 9 from [46].

The sets that are genuine representatives of the smallest 3D KS sets are those with all vertices in all hyperedges (vector triples): 49-36 Bob-Schütte, 57-40 Peres, 51-37 Conway-Kochen, and 192-118 Kochen-Specker and (complex) 69-50 Pavičić's set [29, Fig. 10(e), string on p. 55] since they are all critical. Two of them are shown in Fig. 1(a,c). At the time their authors dropped vertices/vectors belonging to only one hyperedge (multiplicity 1; grey dots) just because the remaining vertices suffice for simpler KS proofs. For instance: 49-36: Bub: "Removing ... 16 rays ... that occur in only one orthogonal triple ... from the 49 rays yields ... [a] set of 33 rays" [47]. The misconception about non-KS vs. KS sets might have originated from historical statements such as Peres': "It can be shown that if a single ray is deleted from that set of 33, the contradiction disappears" [48, p. 199], which is wrong, as our program STATES01 shows [43, p. 7].

Today, we can generate arbitrarily many MMPHs in any dimension from simple vector components [43, Table 2] via an automated procedure in no CPU time. In 3D, vector components $\{0, \pm 1, \pm 2, 5\}$ yield the 97-64 class with 20 critical MMPHs which include Bub-Schütte's set, 9 non-isomorphic 51-37 MMPHs one of which is Conway-Kochen's set, a 53-38, 8 54-39, and a 55-40 [40, Table I]; $\{0, \pm 1, \pm\sqrt{2}, 3\}$ yield the 81-52 master which contains a single critical set—Peres' set; a set of 24 vector components explicitly given for the first time in [40, Supp. Material, p.3] yields the original Kochen-Specker's set; and $\{0, \pm\omega, 2\omega, \pm\omega^2, 2\omega^2\}$, where $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$, yield the 169-120 class which contains 514 critical sets the smallest of which is the 69-50 explicitly given in [29, Fig. 10(e), p. 55].

So, when we obtain (in a setup) or deal with non-KS MMPH, in order to verify whether it possesses a particular feature (in our case: being minimal) we have to first fill it up (add all missing vertices with multiplicity 1) and then check on the filled MMPH whether it is a KS one.

And vice versa, when we want to find a specified non-KS NBMMPH among subsets of a given MMP we just strip it of vertices and check it on the non-KS feature. For example, when stripping edges and vertices off Peres' KS 57-40 we might arrive at Yu and Oh's non-KS NBMMPH 13-16 [49], i.e., its filled MMPH 25-16 (non-contextual) is a subhypergraph of the 57-40.

We encounter both options when we encounter contextuality within quantum computation [22,50].

It is considered that quantum computation with qubits is most efficient when stabilizer circuits constructed via Cliford gates (Hadamard, c-NOT, and phase gate) are used for preparing qubits (in stabilizer states) which are then combined with ancillary qubits in *magic* states. Qubits in the former states cannot be used for quantum speed-up since they may be simulated on a classical computer (Gottesman-Knill theorem [50]). But their combination with ancillary qubits in the latter states can and a choice of their combined states proves to be contextual [22,51]. In particular, graph Γ (30-108) shown in [22, Fig. 2] in which the vertices correspond to two-qubit stabilizer states and their orthogonalities/hyperedges correspond to the effect of ancillary qubits in magic states on them proves to be contextual (a non-KS set).

The graph 30-108 with 30 vertices and 108 hyperedges (30-108) is presented in the MMPH language and analyzed in [29, Fig. 10(e), pp. 37,38,56]. It is shown that its filled MMPH 232-108 is contextual and therefore that it is a genuine KS set in the same way as, e.g., Peres' 57-40 above is, only it is itself not critical. It contains a single 152-71 critical subhypergraph which is shown in Fig. 2b which is the minimal critical MMPH generated from vector components $\{0, \pm 1, \pm 2, \pm 3, 5\}$ as Peres' 57-40 above is from $\{0, \pm 1, \pm\sqrt{2}, 3\}$.

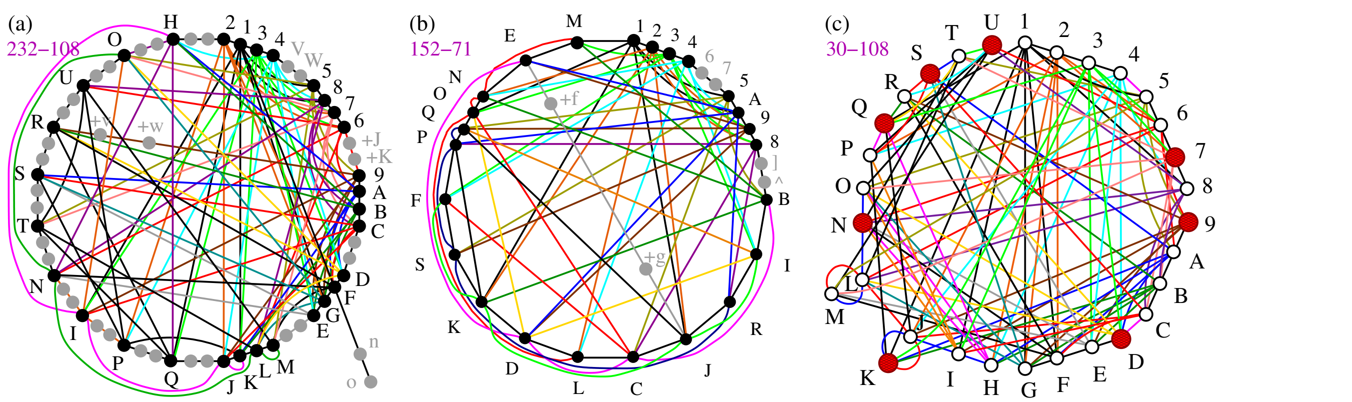

Figure 2. (a) An MMPH representation of the 232-108 which is a filled MMPH of the 30-108 non-KS MMPH from [22, Fig. 2]; not all vertices with multiplicity 1 (grey dots) are shown, but they can be easily determined from the figure and the ASCII string of the MMPH; (b) critical KS subhypergraph (152-71 MMPH) of the 232-108 MMPH; its coordinatization is given in the Appendix and its ASCII string in [29, Appendix E]; again, not all vertices with multiplicity 1 (grey dots) are shown; (c) the 30-108 obtained from 232-108 by dropping all vertices with multiplicity 1 and adapted so as to correspond to Γ from [22, Fig. 2]; red dots correspond to one of the set of vertices to which we might assign a maximal number of 1's, i.e., to the independence number α [29, Def. 4.7, Lemma 4.8].

The approach should be elaborated on contextuality of the operations supporting quantum speedup. Is there a simplified combination of stabilizer and magic states that corresponds to the 152-71 KS or 24-71 non-KS (contextual MMPH with grey vertices dropped) MMPH [51]? Are there noncontextual operations that support quantum speedup? Which other operations support the speedup? [52]

Notice that MMPHs cannot be used for universal hypergraph representation of stabilizer codes [53] because the definition of MMPH restricts a general hypergraph; e.g., a hyperedge containing only one vertex is not allowed. In order to allow this and some other features, one should rewrite all MMPH algorithms and programs.

2.2. Pseudo-telepathy games

A fusion of entanglement, nonlocality and refutation of hidden variables has recently been formulated as a pseudo-telepathy game [54] (also known as a "bipartite perfect strategy" [55]). When a state of one of two entangled particles is determined by a measurement we immediately now what a measurement of the state of the other particle would yield. We might be tempted to ascribe telepathic ability to particles, or to the measuring device, or to ourselves, or to Alice (owner of the 1st particle) over Bob (owner of the 2nd particle) but since the outcome of the measurement of the state of the first particle is completely random, the measurement of the first particle cannot serve Alice for transferring any information to Bob—so, "pseudo-telepathy."

Inasmuch as a refutation of hidden variables is tightly supported by contextual sets, pseudo-telepathy has been used to grant Alice and Bob active roles in the so-called pseudo-telepathy games. The crux of, say, a two-player (A, B) game G of dimension d is that it does not have a classical winning strategy but that it does have a quantum winning strategy (the players share a prior entangled state). Here a critic might ask why a classical wining strategy does not include a classical entangled state, but the answer might be that hidden variable theories do not assume it.

Anyhow, under the former strategies the maximal, classical (C) vs. quantum (Q), winning probabilities ω satisfy the following inequality:

$$\omega_C(G) < \omega_Q(G) = 1 \tag{1}$$

A two-player game G is a tuple $\langle X, Y, A, B, P, W\rangle$, where X, Y, A and B are finite sets, $P \subseteq X \times Y$ and $W \subseteq X \times Y \times A \times B$, where X and Y are the input sets, A and B are the output sets, P is a predicate on $X \times Y$ known as the *promise*, and W is the winning condition, which is a relation between inputs and outputs that has to be satisfied by Alice and Bob whenever the promise is fulfilled. Alice is asked a question $x \in X$ and she produces an answer $a \in A$. Bob, is asked a question $y \in Y$ and he produces an answer $b \in B$. They are not allowed to communicate after they have received their questions. They win if $(x, y, a, b) \in W$. Their quantum winning strategy is that they share an entangled state:

$$\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle \tag{2}$$

The KS game is defined relative to the above set of vectors. Alice receives a random n-tuple of orthogonal vectors as her input and Bob receives a single vector randomly chosen from the n-tuple as his input. Alice outputs d-tuple indicating which of her d vectors is assigned color 1 (implicitly, the other n-1 vectors are assigned color 0). Bob outputs a bit assigning a color to his vector. The requirement is that Alice and Bob assign the same color to the vector that they receive in common.

The original pseudo-telepathy KS game goes as follows [54]. Alice is given only two vectors v_1 and v_2. She chooses $d-2$ additional vectors $v_3, \ldots, v_d$, so that $v_1, \ldots, v_d$ form an orthogonal d-tuple, performs a measurement on her share of the entangled state in basis $B_a = \{|v_1\rangle, \ldots, |v_d\rangle\}$ and obtains k as the result—she assigns 1 to vector v_k. Bob is given vector v_l. He chooses $d-1$ additional vectors $w_1, \ldots, w_{d-1}$

so that $v_l, w_1, \ldots, w_{d-1}$ form an orthogonal d-tuple. The promise P states that $|v_l\rangle$ is from Alice's basis. Bob performs a measurement on his share of the entangled state in basis $B_b = \{|v\rangle, |w_1\rangle, \ldots, |w_{d-1}\rangle\}$ and obtains the output $b = 1$ if the outcome is $|v_l\rangle$ and $b = 0$ otherwise [54, Eqs. 2.2,3,4].

In other words, the original pseudo-telepathy game is just another way of proving that a KS set really is a KS set. This is what our program STATES01 does in no time for any KS in any dimension. But what we need are applications which would contribute proper quantum computation and not just refute a hidden variable theory.

Is a recent redefinition of a KS pseudo-telepathy game [55] on such a road? In the original pseudo-telepathy game the promise P states that Bob's vector is from Alice's basis. Ref. [55] changes this so that the promise P states that Bob's vector is *not orthogonal* to any of the Alice's vectors and that Alice and Bob are each given a subset of vertices, X and Y, respectively; the pseudo-telepathy game is renamed into *bipartite perfect quantum strategy*.

The maximum number of non-orthogonal vectors in an MMPH is the maximum number of pairwise non-adjacent vertices, aka the independence number and denoted by α [29, Def. 4.7]. It amounts to the number of 1s one can assign to vertices of an MMPH [29, Def. 4.4, Lemma 4.8]. An example of a set of such vertices is given in Fig. 2(c). Within an MMPH we might have different distribution of vertices contributing to α and therefore their balanced grouping in Alice and Bob's bases requires a great deal of symmetry.

Complex vectors provide such symmetries to a larger extent than real ones; see examples in 3D, 4D, and 6D [29,39,56]. In 3D, which the redefined pseudo-games consider in [55], we obtained the class 169-120 from the $\{0, \pm\omega, 2\omega, \pm\omega^2, 2\omega^2\}$ vector components the smallest among which is the highly symmetrical 69-50 KS MMPH [29,39] shown in Fig. 3(a). Its symmetry is established in the *hypergraph*-3D [29, Def. 2.2] (where the hypergraph dimensionality is defined as the maximal number of vertices in hyperedges) and is shown in Figs. 3(a) and 4(a). If we wanted to obtain its representation in a real vector space we should turn to a 6D space. It has not been done in [55, Fig. 1] where a 33-50 non-KS contextual MMPH obtainable from the 69-90 MMPH [29,39] by dropping the vertices with multiplicity 1 is considered. (By deleting the grey vertices from Fig. 3(a,b) and Fig. 4(a) we obtain Fig. 4(b).)

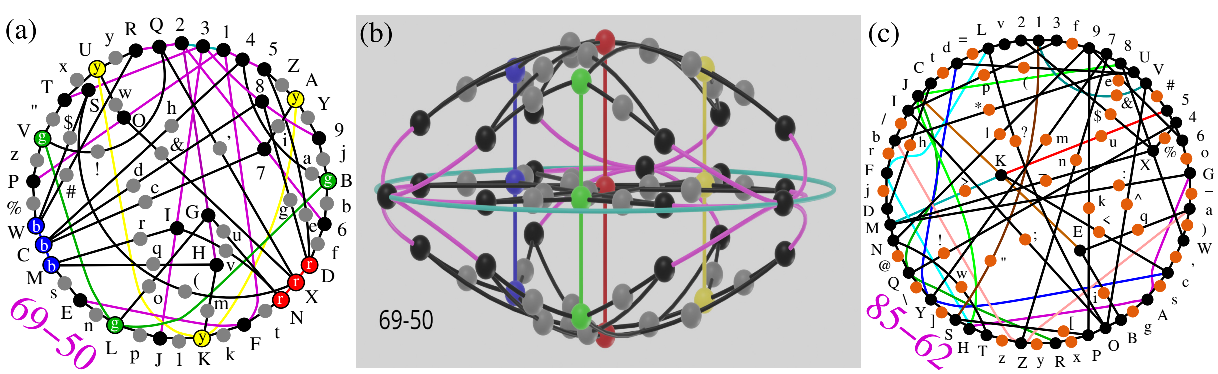

Figure 3. (a) The 69-50 MMPH from the 169-120 class [29, Fig. 10(e)] redrawn so as to flash colored vertices from [55, Fig. 1]; its variety with grey vertices of multiplicity 1 dropped—33-50—is isomorphic to [55, Fig. 1] and to Fig. 4(b) below; its coordinatization is given in the Appendix; (b) A 3D representation (hypergraph dimensionality) of the 69-50 set; side snapshot from from a Blender output obtained in [57] which the reader can interactively rotate at will; (c) the 85-62 MMPH from 169-120 class with 15 complete bases; its coordinatization is given in the Appendix.

As for the generation of masters from complex numbers apart from $\{0, \pm\omega, 2\omega, \pm\omega^2, 2\omega^2\}$ we made use of in [29,39] we can also use $\{0, \pm 1, \pm\omega, 2\omega, \pm\omega^2\}$ which give the 204-149 master which contains

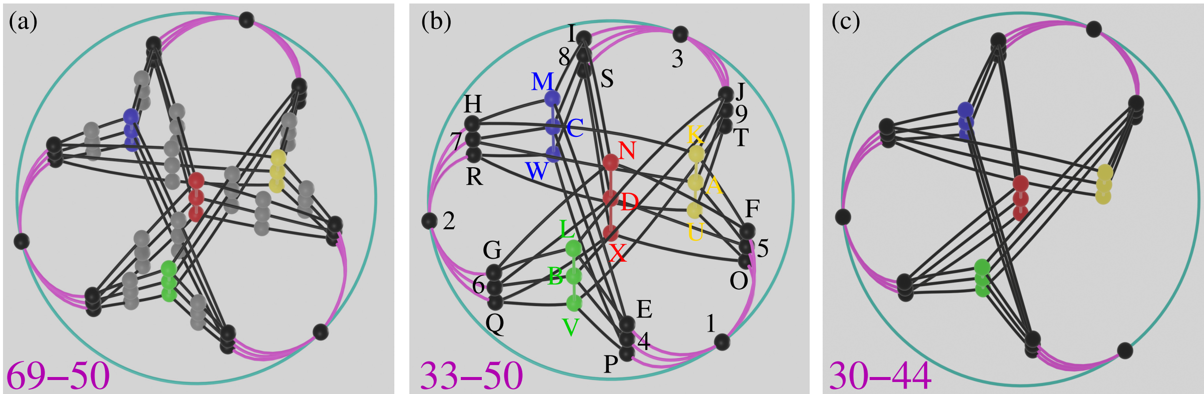

Figure 4. (a) Top snapshot of a 3D representation of the 69-50 set from a Blender output obtained in [57] which the reader can interactively rotate at will; (b) top snapshot of a 3D representation of the 33-50 set from a Blender output obtained in [58] which the reader can interactively rotate at will; (c) top snapshot of a 3D representation of the 30-44 set from a Blender output obtained in [59] which the reader can interactively rotate at will.

over 10000 critical KS MMPHs the smallest of which is the 69-50 and the largest is the 130-95, as well as $\{0, \pm1, \pm\omega, 2\omega, 2\omega^2\}$ which give the 163-102 master which contains only one critical set: the 69-50 set. The 69-50 sets obtained in any of those masters are isomorphic.

We see that we can carry out the aforementioned grouping by putting the vertices from the outer rim of the 33-50 (Fig. 4(b)), i.e., (`145,1EF,...`) in Y and the ones from the inner rim (`NDX,MCW,...`) (red, yellow, green, blue) in X. The "new record" 33-50 set of [55, Fig. 1] is isomorphic to our 33-50 and it is claimed in [55] that the set is the simplest because it contains 14 complete bases (hyperedges containing 3 vertices) while other 3D KS MMPHs contain 16 or more such hyperedges. Here we just point out that the 85-62 Fig. 3(c) has 15 such hyperedges.

As for the role of "complete bases" within a pseudo-telepathy 3D design, it does seem that the hyperedges containing vertices with multiplicity 1 might be considered as "complete bases" as well. For instance, Yu-Oh's non-KS contextual 13-16 [60, Fig. 1(a)] with nine vertices with multiplicity 1 added so as to form non-KS contextual 22-16 has 13 such complete bases as shown in Fig. 5(a). Also, the non-KS contextual 30-44 with 11 complete bases, shown in Fig. 4(c) might be considered as a legitimate candidate for a pseudo-telepathy game. From 33-50 we also obtain non-KS contextual MMPH 22-19 with seven complete bases Fig. 5(b). In 4D we can have candidates smaller than 18-9 pseudo-telepathy game for which is presented in [61], as shown in Fig. 5(c), etc.

It is significant that even the most symmetric MMPHs require orthogonal connections of their groups (X, Y) of "complete bases" via "incomplete" bases, i.e., those whose hyperedges contain less than d vertices, to establish a pseudo-telepathy game (e.g., "Alice and Bob win except when they output the two orthogonal vectors." [55, p. 3, left column, 3rd par.]). It is because, although, e.g., for 69-50 $\alpha = 36$ and for 33-50 $\alpha = 9$, X and Y do not contain just non-orthogonal vertices (by definition of α). The others, also contained in them, might be mutually orthogonal. In [62] this was handled in a misleading way of presentation which is for all intents and purposes simply wrong. Take the following sentence from that reference: "Conway and Kochen's CK-33 set is a critical KS set ... It has 20 [complete] bases. Table VI illustrates CK-33." This is doubly wrong: first, CK-33 is *not* critical as we pointed above; second, in a pseudo-telepathy game we cannot consider only X and Y hyperedges, as in their Table VI, because only all hyperedges considered together build the contextuality. The remaining 13 hyperedges are missing in

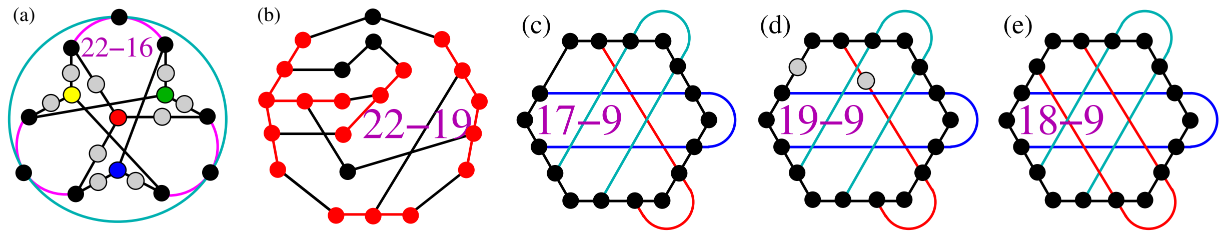

Figure 5. (a) Partially "extended" Yu-Oh non-KS set which is still contextual and has thirteen complete bases; (b) contextual non-KS sub-hypergraph of the 33-50 non-KS contextual set with seven complete bases; (c) a contextual critical non-KS subset of the 18-9 KS MMPH obtained by means of a weak deletion of a vertex [28, Sec. 7.4]; (d) non-contextual *weakly extended* 17-9; (e) contextual *strongly extended* 17-9.

their Table VI. In their caption, text, or comments we were not able to find a reference to them. So, it seems that the protocol of the game at least does not clearly include that references in contrast to [55] which does include them (notwithstanding the non-orthogonality in the latter vs. orthogonality in the former scheme). Complete bases alone (X, Y) just do not build a contextual set [63].

To sum up, while the pseudo-telepathy games have arbitrarily many suitable small and large, real and complex MMPH candidates for implementation, they remain, at least for the time being, just another way of proving that chosen MMPHs really are KS or non-KS MMPHs. But whether their applications beyond disproving hidden variable theories exist remains to be addressed.

2.3. Contextuality vs. nonlocality

Since contextuality can be converted into nonlocality [64] and vice versa [65], nonlocality is not an "application" of contextuality, but just another of its forms.

2.4. Inequalities

Various inequalities which are and those which are not noncontextuality inequalities are analyzed and discussed in detail in [29]. In particular, it is shown that the so-called GLS inequality is violated by arbitrarily many MMPHs and that it is therefore not a noncontextuality inequality. The noncontextuality inequalities serve for theoretical and/or experimental verification of contextuality of an MMPH and are therefore not "applications" of quantum contextual sets.

2.5. Quantum cryptography

In [66] a quantum cryptography protocol based on a KS scheme from [67] has been proposed and claimed to be "protected" by KS contextuality. It is assumed that the KS architecture would give the protocol a quantum advantage. But does it?

Let us consider the following KS BB84-like protocol however not limited to four states (a pair of orthogonal states). Since over 100×100 entangled angular momentum dimensionality is experimentally achievable [68], the number of orthogonal pairs is practically unlimited. The protocol classifies as a large alphabet protocol [69].

- Alice sends outputs from gates/hyperedges of a chosen MMPH to Bob in blocks; she can repeat sending from the same MMPHs or pick up new ones;
- Bob stores Alice's sending in a quantum memory (e.g., photons in fiber loops);
- Alice informs Bob about which sending belonged to which hyperedge over a classical channel (with a delay);

- Bob reads them off, scrambles them, and sends them back to her over the quantum channel; scrambling code is Bob's message (still unknown to Alice);
- Alice stores Bob's sending into a quantum memory;
- Bob informs Alice of the scrambling code over a classical channel (with a delay);
- After an agreed number of exchanged blocks they can announce some messages over a classical channel to check whether Eve is in the quantum channel;
- After Alice correlated the reflected sending with the original ones via Bob's code, she learns how to measure each of them from the quantum memory and read off Bob's message.

A 4D example of the protocol is provided in [24, pp. 21,21].

However, although MMPHs are used in the protocol and implementation, their KS contextual vs. classical noncontextual properties are not. Alice sends quantum state messages without juxtaposing them to any predetermined classical scheme. The only contra-position which might be encountered is with a classical predetermination-led eavesdropper (Eve). But Eve knows that Alice and Bob's protocol is quantum and will therefore not base her interception on some predetermined hidden-variable model. Instead she will adopt the quantum gate balanced output procedure.

Hence, the KS large alphabet does offer higher security not because of contextuality but simply because it can draw inputs from a vastly abundant source of MMPHs. Actually, the MMPHs can be non-contextual as well as KS or non-KS contextual. So, the KS QKD protocols do not offer a quantum advantage over any other possible large alphabet protocols.

2.6. Generalized complex Hadamard matrices and star-like contextual MMPHs

Most known quantum computation algorithms are based on the the quantum Fourier transform and the real Hadamard (H) transform is a special case of it [70, Sec. 3.3] or [71, Sec. 3.3]. When implemented it takes the role of an H-gate within a quantum network. As we stressed above, it belongs to a collection of Cliford gates which themselves alone do not offer a quantum speed-up of the network. Whether a generalized complex H-transform can offer the speed-up seems to be an open question.

The real H matrix $H = [H_{ij}]$ is a matrix which satisfies the conditions: (1) $HH^T = nI_n$; (2) $|h_{ij}| = 1$ $\forall 1 \le i, j \le n$. That means that

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}, \quad \text{and} \quad \text{recursively constructed matrices of order } 2^n \tag{3}$$

are all Hadamard matrices. Their order is even, but this recursive generalization does not yield matrices of all even orders, e.g., 6, 10, etc. are missing. (Note that the next H in Eq. (3) is H_8 containing four H_4's.)

A complex generalization, on the other hand, seems to fill the gaps—it is conjectured ("Complex Hadamard conjecture" [72, p. 68]) that it yields complex H matrices for any even order of n. So far there are constructive proofs for even n's up to 70. The considered generalization is a *quaternary complex Hadamard matrix* (a kind of the *Butson matrices* [72, Sec. 4.1]) of even order n which is an $n \times n$ matrix with entries from $\{\pm 1, \pm i\}$ such that $HH^* = nI_n$, where * stands for conjugate transpose. This condition is a substitute for the condition (2) above. In particular we have:

$$H_{c2} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \quad H_{c4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix}, \quad H_{c6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i & -i & i \\ 1 & i & -1 & i & -i & -i \\ 1 & -i & i & -1 & i & -i \\ 1 & -i & -i & 1 & i & i \\ 1 & i & -i & -i & i & -1 \end{bmatrix}, \quad \text{etc.} \tag{4}$$

Another kind of Butson Hadamard matrices which exists for other orders of n than the real Hadamard matrices are those with 1, ω, and ω^2 ($\omega = e^{2\pi i/3}$ being the cube root of 1) as their elements [73, p. 3]. Two smallest of them are

$$\Omega_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \quad \Omega_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega^2 & \omega & \omega & 1 \end{bmatrix}. \tag{5}$$

An analogous coordinatization of the smallest 6D star-like 21-7 KS MMPH [74] might have prompted Lisoněk to add a third H condition to the aforementioned two:

$$(3)\forall\, k,l\ (1 \le k,l \le n,\ k \ne l)\ \sum_{j=1}^{n} h_{kj}^2 h_{lj}^2 = 0. \tag{6}$$

thus arriving at what he calls S-H matrix [75] with elements including $\zeta_g = e^{2\pi i/g}$ (g-th root of 1), which he then uses to prove the following theorem

Theorem 1. *(Lisoněk 2019) Suppose that there exists an S-H matrix of order n (n even); then there exists a KS hypergraph k-l in* $\mathbb{C}^n$ *such that* $k \le \binom{n+1}{2}$ *and* $l = n+1$.

From [74], it is not clear whether an S-H matrix for $n = 8$ exists, but the star-like 8D 36-9 MMPH does exist and one of its coordinatization is real ($\{0, \pm 1\}$) [56] while finding a coordinatization already for the star-like 10D 55-11 MMPH [41], [24, Fig. 13] is computationally too demanding. So, implementations of KS MMPHs with $k = \binom{n+1}{2}$ is presently highly unlikely.

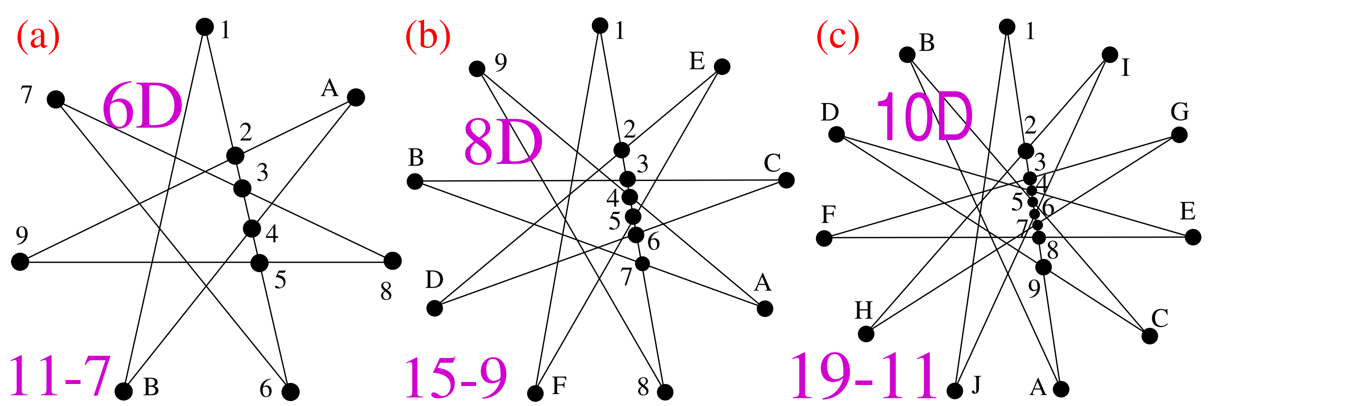

Figure 6. Star-like non-KS contextual MMPHs $(2n-1)$-$(n+1)$ with one complete hyperedge generated from KS contextual MMPHs $\frac{n}{2}(n+1)$-$(n+1)$ obtained in [41]; (a) the smallest critical 6D non-KS 11-7 which, unlike its generator KS 21-7, allows a coordinatization from $\{0, \pm 1\}$ vector components; (b,c) also allow such coordinatizations; strings and coordinatizations are given in the Appendix.

However, a prospect of connecting generalized Hadamard gates with smaller non-KS contextual MMPHs we might derive from the $(2n-1)$-$(n+1)$ star-like KS MMPHs is intriguing since their coordinatizations are easily computable for higher dimensions. Notice that a deletion of all inner vertices of a star-like MMPH (leaving only its tip vertices) yields a non-KS contextual MMPH. This explains why an n-gon (another form of a star [29, Fig. 5(a,c)]) is contextual for any even n and noncontextual for any odd n—the Schläfli symbol of a regular star is $[n+1/\frac{n}{2}]$ ($\frac{n}{2}$ has to be an integer). Hence, any contextual star-like MMPH between a non-KS $(n+1)$-$(n+1)$ and a KS $\frac{n}{2}(n+1)$-$(n+1)$ is critical.

Recalling that the stabilizer MMPHs as well as the quantum game ones are all non-KS contextual, this option of obtaining non-KS critical contextual MMPHs with arbitrarily many complete hyperedges offers us a possibility of having applications in quantum computation. We present three such non-KS MMPHs with single complete hyperedges in lower dimensions in Fig. 6 and their strings and coordinatizations in the Appendix.

2.7. *Contextuality—noncontextuality dichotomy*

Both KS and non-KS sets are contextual. To handle them we make use of the MMP hypergraph language [24,29,31,37] since it greatly helps in resolving impasses.

Let us trace one of them. The Clifton hypergraph $\mathcal{G}_1$ [76]—aka *bug* [77]—shown in Fig. 7(a), is a contextual non-KS MMPH. The filled $\mathcal{G}_1$ shown in Fig. 7(b) is neither a KS nor a non-KS contextual MMPH, i.e., it is a noncontextual MMPH.

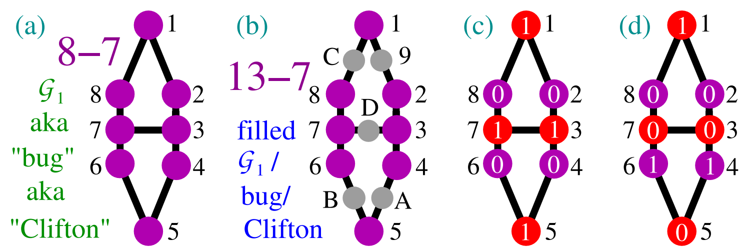

Figure 7. (a) 8-7 contextual non-KS bug/Clifton; (b) noncontextual 13-7 filled bug/Clifton; (c,d) two mutually contradictory Clifton-like "proofs".

Among others Clifton [76], Ramanathan, Rosicka, Horodecki, Pironio, Horodecki and Horodecki [78], and most recently Williams and Constantin [79, p. 2] claim that the following proposition holds.

Proposition 1. *Vertices* 1 *and* 5 *of* $\mathcal{G}_1$ *cannot both be assigned value* '1'.

Their "proof" runs as follows. Suppose $v(1) = v(5) = 1$. Then the condition (i) of Def. 3 implies $v(2) = v(8) = v(4) = v(6) = 0$ and condition (ii) implies $v(3) = v(7) = 1$. However, condition (i) does not allow that, hence a contradiction. See Fig. 7(c).

Ergo, they seem not to be aware of the fact that the 8-7 $\mathcal{G}_1$ is contextual and that the contradiction stems from that same contextuality, i.e., that the "contradiction" does not prove Proposition 1—it is merely a consequence of the contextuality. To see this, let us consider the opposite proposition.

Proposition 2. *In* $\mathcal{G}_1$*, vertex* 5 *cannot be assigned value* '0' *once vertex* 1 *is assigned value* '1'. *(And vice versa.)*

The "proof" would run as follows. Suppose $v(1) = 1$ and $v(5) = 0$. Then the condition (i) of Def. 3 implies $v(2) = v(8) = 0$ and $v(4) = v(6) = 1$ and condition (i) implies $v(3) = v(7) = 0$. However, condition (ii) does not allow that, hence a contradiction. See Fig. 7(d).

We see that both propositions are wrong and that the "contradictions" are just impossibilities to assign 1-0 values to all vertices of a contextual set so as to satisfy the conditions (i) and (ii) of Def. 3.

An MMPH to which Proposition 1 *can* be validly applied is the *filled* $\mathcal{G}_1$ shown in Fig. 7(b) as the reader can easily convince her/himself by assigning 1s and 0s to its vertices. However, the *filled* $\mathcal{G}_1$ is neither a non-KS nor a KS contextual MMPH—it is a noncontextual MMPH.

That invalid Proposition 1 ensued a number of untenable results and/or claims in the literature. Let us consider a recent ambitious endeavour undertaken by Williams and Constantin [79].

In Sec.II.C of their paper they construct $\mathcal{G}_4$—four nested $\mathcal{G}_1$s—sketched in [79, Fig. 3]. We give a detailed hypergraph of $\mathcal{G}_4$ in Fig. 8(a).

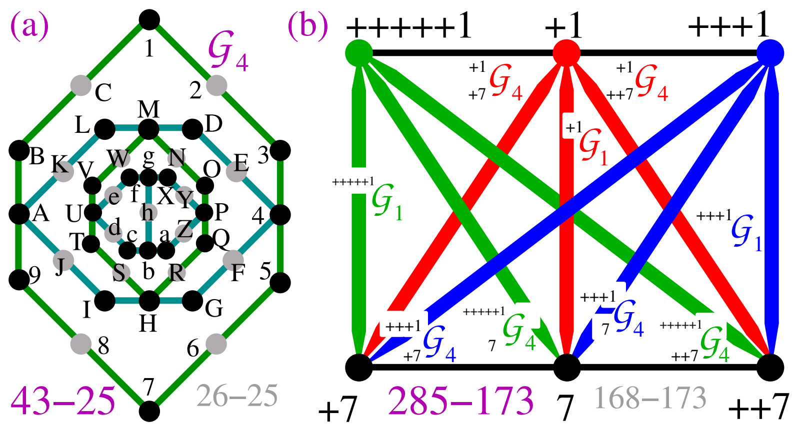

Figure 8. (a) $\mathcal{G}_4$—nested $\mathcal{G}_1$s [79, Fig. 3]; (b) "168-direction" set [79, Fig. 4]; the top line is the inner triangle and the bottom one is the outer triangle of [79, Fig. 4]; note that a clique/triangle in the graph notation is represented by a line in the hypergraph notation; '+'s, '++'s, . . . refer to repeated notations of successive $\mathcal{G}$s for automated processing.

In their *Proposition 2* [79, p. 2, bottom] Williams and Constantin claim that (following their Proposition 2—the Proposition 1 above) `U` and `P` of $\mathcal{G}_4$ cannot both be assigned value 1. But since the Proposition 1 does not hold, that claim does not hold either.

Not being aware of that untenability, by means of $\mathcal{G}_4$s and $\mathcal{G}_1$s they construct a 168-direction set [79, p. 2, bottom], whose MMPH version is shown in Fig. 8(b), and claim that, following their assumption that opposite vertices of $\mathcal{G}_1$s and of nested $\mathcal{G}_1$-like shells within $\mathcal{G}_4$s cannot both be assigned value 1, the 168-direction set is a non-KS contextual MMPH. However, since the assumption is false their deduction is not valid although the 168 MMPH *is* a non-KS MMPH simply because it contains $\mathcal{G}_1$ which is a non-KS contextual set.

The facts are as follows.

(i) $\mathcal{G}_1$ 8-7 is a non-KS MMPH, $\mathcal{G}_4$ 26-25 is a non-KS MMPH because it contains $\mathcal{G}_1$s which are non-KS MMPHs, and the 168-173 MMPH is a non-KS contextual MMPH because it contains $\mathcal{G}_1$s which are non-KS MMPHs;

(ii) neither filled $\mathcal{G}_1$ 13-7, nor filled $\mathcal{G}_4$ 43-25, nor filled 168-direction MMPH 285-173 are either KS or non-KS sets as follows from our programs which are freely available from our repository [80]:
COMMAND: states01 -1 < **285-173**
OUTPUT: **285-173** admits {0,1} state

(iii) there is a fundamental difference between the 168-direction MMPH and, say, the original Kochen-Specker's MMPH shown in Fig. 1. While both of their stripped versions, 168-173 and 117-118, respectively, are non-KS and therefore contextual, their filled versions, 285-173 and 192-118, respectively, differ fundamentally—the 285-173 is a noncontextual/classical set whilst the 192-118 is a critical KS contextual/quantum set;

(iv) next, the coordinatization of 285-173 indicated in [79, Fig. 4] is not as simple as presented. A coordinatization of 192-168 (obtained more than 50 years after [36]) requires at least 24 vector components and a computation over several months on a supercomputer [40, Supp. Mat.]. Note that the 192-118 contains 15 filled $\mathcal{G}_1$s and that all vertices in them have to be different;

(v) the 285-173 contains 3 filled $\mathcal{G}_1$s and 6 $\mathcal{G}_4$s and finding vector components and computing all the vertices/vectors are practically unfeasible even on supercomputers. Note that a coordinatization of 168-173 must follow from a coordinatization of 285-173.

(vi) of critical non-KS sets contained in 117-118 and 168-173 sets, some of which are given in Table 1, only 1% mutually overlap, meaning that the only benefit of the 168-173 set is to be a source of new non-KS sets.

Table 1. Critical non-KS hypergraphs contained in 117-118 and 168-173, obtained via our program STATES01 on a supercomputer. Among 100,000 hypergraphs obtained from each of them there are 229 and 663 non-isomorphic ones, respectively, the smallest and the largest of which are shown in the table below.

117-118			168-173		
vertices	hyperedges	No.	vertices	hyperedges	No.
8	7	1	8	7	1
14	10	1	-	-	-
-	-	-	14	11	2
15	11	1	15	11	5
17	12	1	-	-	-
-	-	-	16	13	2
-	-	-	17	13	4
18	13	1	18	13	6
...	...	...	...	...	...
44	32	5	44	32	2
...	...	...	...	...	...
-	-	-	43	34	4
47 Max	34 Max	1	-	-	-
...	...	...	...	...	...
-	-	-	64	47	3
-	-	-	67 Max	49 Max	2

Williams and Constantin introduced noncontextual sets [79, p.3, Sec. III.A, Def. 3] and called them "*non-KS*" sets in contradistinction with the term non-KS introduced two years earlier [81], to mean sets/MMPHs which are quantum contextual MMPHs but not the KS MMPHs. Therefore we denote their "*non-KS*" MMPHs as *non-Q* MMPHs. They define their *non-Q* set as follows.

Definition 7. *In $\mathcal{H}^n$, $n \geq 3$, there is a set A of vectors $\boldsymbol{w}$'s pointing to the points on an $(n-1)$-dim sphere S^{n-1}—called a* non-Q*—that admits a valuation map: $A \to \{0,1\}$ such that*
(i) $n(-\boldsymbol{w}) = n(\boldsymbol{w})$ for all vectors in A;
(ii) $\sum_{i\in I} v(\boldsymbol{w}_i) \leq 1$ for all sets of mutually orthogonal vectors $\{\boldsymbol{w}_i\}_{i\in I}$ in A;
(iii) $\sum_{i\in I} v(\boldsymbol{w}_i) = 1$ for all sets of n mutually orthogonal vectors $\{\boldsymbol{w}_i\}_{i\in I}$ in A

Further, Williams and Constantin [79, p.6, Sec. III.B] put forward:

Proposition 3. *The union of any non-Q set with its antipodal is also a* non-Q *set.*

In effect, this boils down to a claim that a union of any of "non-Q" sets with a coordinatization with its antipodal set with a coordinatization is "non-Q" set with a coordinatization. This is incorrect. Let us consider details.

When dealing with sets Williams and Constantin actually deal with vectors themselves, not with hyperedges the vectors determine. So, if we took the upper pentagon of vectors from Fig. 9 to be A, then the lower pentagon of vectors would be $-A$. But the pentagons themselves, are formed by the hyperedges

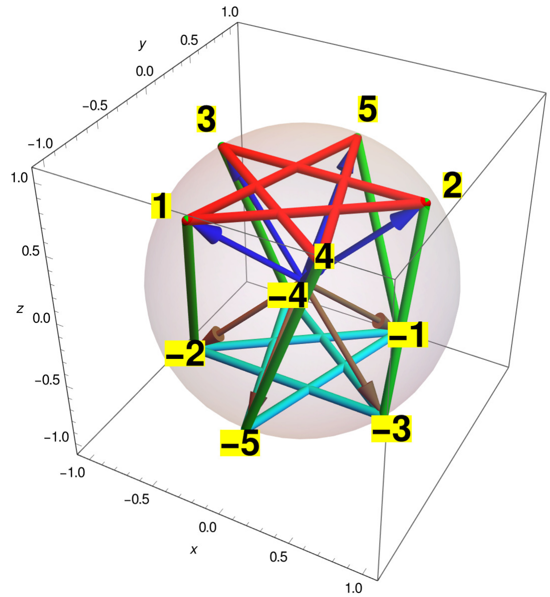

Figure 9. Double-pentagon 10-11.

which not only connect the heads of just two mutually orthogonal end vectors, but also contain a third vector in between them since they live in the 3D space—Cf. Fig. 5 in [29]. The fact that we carry out our calculations with only two end vectors eventually does not change the fact that every hyperedge in 3D should be a triple of vectors—the geometry of orthogonality requires three vectors orthogonal to each other to define the hyperedges. Whether we discard data of measurements carried out in one of the three directions does not change the geometry of vectors in 3D. The problem with the proposed construction is that such third vectors cannot have coordinatizations and that therefore the whole construction is inconsistent in spite of the fact that filled A's, i.e., pentagons, do have a coordinatization.

The vectors, by their orthogonality, should define the vertical (green) hyperedges shown in Fig. 9. All these hyperedges together should define a hypergraph/set by the orthogonalities of vectors. But the vectors cannot transfer a coordinatization to the hypergraph in 3D since it contains square loops, e.g., `1287` (1,2,-3,-2) [31, Sec. 5(x)].

Antipodal vectors `-3`, `-2`, … do have a coordinatization: $-3 = ((\sqrt{3}-\sqrt{5}/2, \sqrt{1+1/\sqrt{5}}/2, -5^{-1/4})$, $-2 = (-1/\sqrt{2}, -\sqrt{1/2-1/\sqrt{5}}, -5^{-1/4})$, …, but they cannot build a hypergraph together with `3`, `2`, … Third vectors that should be there in the middle of the hyperedges/triples (orthogonal to two end vectors)—Cf. Fig. 5 in [29]—simply do not exist.

When we take the negative vectors `-1`,…,`-5` from Fig. 9 to mean just hypergraph vertices and rename them so as to read `6`,…,`A`, we obtain the string which represents the double-pentagon together with the vertical green edges (with `1`,…,`5` also taken to be just vertices and not any more vectors):

10-15 `12,23,34,45,51,17,28,39,4A,A1,67,78,89,9A,A6.`

It is obviously a non-KS set, i.e., a contextual set:

COMMAND: states01 -1 < **10-15**

OUTPUT: **10-15** fails (admits no 0,1 state)

because it contains pentagons which are contextual. Notice that, e.g., `1B2,23,3B4,45,51.` is also a critical non-KS MMPH.

If we filled up both pentagons and vertical green hyperedges, we would obtain the following 25-15 MMPH: `1C2,2D3,3E4,4F5,5G1,1B7,2H8,3I9,4JA,AK1,6L7,7M8,8N9,9OA,AP6.` which is noncontextual (non-Q):

COMMAND: states01 -1 < **10-15**

OUTPUT: **(10-15)** passes (admits 0,1 state)

but it does not have a coordinatization.

Taken together, A might be a set of vectors with a coordinatization, that form a contextual hypergraph pentagon with a coordinatization, but the $A \cup -A$ is an oxymoron—it should be a hypergraph with a coordinatization since both A and $-A$ do have a coordinatization, but it cannot be a hypergraph with coordinatization in 3D since the hypergraph with a square loop cannot have a coordinatization in 3D.

This brings the validity of the lower bound for the maximal non-Q [79, p. 6, Eq. (24)] into question since its derivation [79, p. 5] assumes that $A \cup -A$ is a noncontextual set with a coordinatization, but that is incorrect since we proved above that their pentagon-hyperegraph cannot have a coordinatization.

Then they proceed as follows: "Hence a lower bound for the size of a [non-KS] set can be established by finding the smallest set of directions that any given non-Q set avoids" and arrive at Eq. (29) of [79].

But since one cannot say whether a non-Q set "avoids" non-KS sets since it might not have a coordinatization, this result might be incorrect, either.

3. Discussion

In this paper we search for possible emergence of contextuality from quantum theory or quantum measurements unrelated to any hidden variable theory quantum experiments might beat. To be more precise, we explore the implications of rejecting the premise that a set of predetermined measurement outcomes constitutes a valid starting point for finding applications of quantum contextuality. On the road to this goal we encounter a number of obstacles such as incorrect results, misconceptions, misfocusing, and some terminological and methodological clumsiness. Some of them we bring forward in detail.

We start with an emergence of contextuality which can definitely satisfy our restriction, i.e., which emerges directly from protocols of quantum computation and its error correction procedures and which we elaborate on in Sec. 2.1. Here we deal with a procedure developed in Ref. [22] (see also [23]) where Hadamard gate applied on qubit states combined with ancillary qubits in magic states yield contextual non-KS MMPHs shown in Fig. 2(a). The obtained MMPHs is not critical, hence not minimal, and therefore we generate a KS MMPH which contains it and from the latter MMPH we obtain a smaller critical MMPH shown in Fig. 2(b). This procedure of dealing with non-KS vs. KS sets has predecessors from more then half century ago about which there are still widespread misconceptions today. So, we actually start the section with a discussion of misconceptions which boil down to the fact that, e.g., the famous 3D non-KS MMPHs like original Kochen-Specker's 117-118 or Peres' 33-40 sets are taken to be the smallest (minimal, simplest) of their class in spite of the evidence that they are not critical sets as pointed out in Fig. 1.

The next possible application we consider (In Sec. 2.2) are pseudo-telepathy games. When a state of one of two entangled particles is determined by a measurement we immediately know what a measurement of the state of the other particle would yield. Someone might be tempted to ascribe telepathic ability to the participants but since they cannot send information to each other via entanglement the term pseudo-telepathy was introduced to name the games. Originally the pseudo-telepathy games dealt with KS MMPHs, i.e., they carried out verification of orthogonalities between entangled states belonging to complete bases (hyperedges with n vertices, where n is the dimension of the MMPH) and they served for disproving hidden variable theories. Recently, games with non-KS MMPHs were considered with verification of orthogonalities or non-orthogonalities and we checked whether they can offered applications beyond invalidating hidden variable theories. Surprisingly we find that the games were limited by misconceived lower bound of non-KS MMPHs containing a certain number of complete bases. We show that various sets of vectors components generate corresponding master sets which generate their class of smaller KS MMPHs. In 3D every such KS MMPH contains a lot of vertices contained in just one hyperedge. When we remove them we obtain non-KS MMPHs which are predominantly non-critical. The smallest critical non-KS is always smaller than the smallest critical KS and the lowest number of complete bases of non-KS MMPHs is always smaller than the number of complete bases in the smallest KS MMPHs, stopping simplifying the game with the latter number of complete bases is unfounded. When analysing approaches we focused on MMPHs generated by complex vectors and found that some of them, due to their symmetry, can be given a 3D representation which offers the readers interactive views from any angle through links of Fig. 4. Whether the considered versions of the pseudo-telepathy games can have applications beyond disproving hidden variable theories remains to be addressed. We do not find any.

In Secs. 2.3 and 2.4 we point out that nonlocality and inequalities, respectively, being just other forms of contextuality cannot be considered to be its "application." In addition we stress the GLS inequality is violated by arbitrarily many MMPHs and that it is therefore not a noncontextuality inequality.

Recently proposed quantum cryptography based on a KS protocol is claimed to be "protected" by contextuality. In Sec. 2.5 we consider its protocol and find that it would be protected by contextuality only if an eavesdropper assumed that Alice and Bob exchange messages organized under predetermined assignments of bits to vertices of a chosen MMPH. But since Eve knows that Alice and Bob's assignments are quantum she is not going to base her interception on some predetermined hidden-variable mode and therefore Alice and Bob cannot achieve any contextual "protection." Yet, the MMPH large alphabet protocol does offer higher security simply because it can draw inputs from a vastly abundant source of MMPHs.

An interesting relationship between generalized complex Hadamard matrices and star-like MMPHs is considered in Sec. 2.6. With the proviso that complex Hadamard gates will find their applications in quantum computation, Lisoněk's theorem 1 yields existence of MMPHs for every S-Hadamard matrix of even order n which establishes a relation between S-Hadamard matrices and MMPHs. The non-KS versions of them are manageable but details of correlations await elaboration.

In Sec. 2.7 we consider an attempt to construct a large noncontextual MMPH in 3D and to use it to arrive at a lower bound on size of a contextual MMPH which starts with an incorrect theorem what invalidate further steps of the procedure. The theorem is a rather well-known Clifton theorem which claims that the opposite vertices of the so-called "bug" cannot both be assigned value '1'. Since we are able to prove an opposite claim Clifton theorem turns out to be false. The whole construction of a large MMPH which follows this false assumption is instructive and we present it in some detail with the help of *Mathematica* calculations and 3D representation shown in Fig. 9 since the procedure show how one can generate MMPHs via method which differs from those ones supported by our previous programs.

Taken together we show that there is only a handful of possible genuinely quantum applications of contextual MMPHs. We find that they require further elaboration and that they have not been properly addressed as of yet because researches have been focused on disproving hidden variable theories and finding smallest and simplest sets, games, and protocols instead of just using available sets of any kind, size, and dimension to investigate their further possible usage. This is so far as physics of contextuality is concerned. Finding new methods of generation of new contextual sets would remain a valuable contribution to the hypergraph theory as a mathematical discipline.

4. Methods

The methods we employ to manage quantum contextual sets are based on algorithms and programs developed within the MMPH language, including VECFIND, STATES01, MMPSTRIP, MMPSHUFFLE, SUBGRAPH, LOOP, SHORTD, and ONE as referenced in [29,30,32,33,35,82–84]. These resources are freely accessible at http://puh.srce.hr/s/Qegixzz2BdjYwFL. MMPHs can be visualized as hypergraph figures, consisting of dots and lines, and can also be represented as strings of ASCII characters. This representation enables the simultaneous processing of billions of MMPHs using supercomputers and clusters. To facilitate this processing, we have developed additional dynamical programs to manage and parallelize tasks involving any number of MMPH vertices and edges. For 3D representations we made use of Blender and Mathematica.

5. Conclusion

The field of contextual sets has emerged from an endeavour to invalidate the possibility of hidden variables underlying quantum mechanics. After more than half a century of diligent research and development, automated methods of generation of plethora of various kinds of sets of arbitrary structure in any dimension have been developed, alongside extensive experimental validation. Therefore, in this paper we investigate whether one can obtain contextual sets from quantum artefacts (models, structures, and/or measurements) or make use of them to arrive at possible quantum applications without a reference to any hidden variable theory. In doing so, we highlighted and resolved some misconceptions that might hinder reaching the goal. A more detailed summary of the achieved results is given in Sec. 3

Author Contributions: Conceptualization; Data curation; Formal analysis; Funding acquisition; Investigation; Methodology; Project administration; Resources; Supervision; Validation; Visualization; Writing – original draft; Writing – review & editing.

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Abbreviations

The following abbreviations are used in the manuscript:

MMPH	McKay-Megill-Pavičić hypergraph (Definition 2)
NBMMPH	Non-binary McKay-Megill-Pavičić hypergraph (Definition 3)
BMMPH	Binary McKay-Megill-Pavičić hypergraph (Definition 3)
KS	Kochen-Specker (Lemma 1)
non-KS	Non-Kochen-Specker (Lemma 1)

Appendix

69-50 `123,145,267,389,9YA,5ZA,4aB,6bB,7cC,8dC,5eD,6fD,8gD,4hC,7iA,9jB,1EF,2GH,3IJ,KkF,KlJ,KmH,LnE,LoG,LpJ,MqH,MrI,MsE,NtF,NuG,NvI,1OP,2QR,3ST,UwO,UxT,UyR,VzP,V!Q,V"T,W#R,W$S,W%P,X&O,X'Q,X(S,BLV,CMW,AKU,DNX.` 1=(0,0,1), 2=(0,1,0), 3=(1,0,0), 4=(1,-1,0), 5=(1,1,0), 6=(1,0,-1), 7=(1,0,1), 8=(0,1,1), 9=(0,1,-1), A=(-1,1,1), B=(1,1,1), C=(1,1,-1), D=(1,-1,1), E=$(\omega,-1,0)$, F=$(\omega,1,0)$, G=$(1,0,-\omega)$, H=$(1,0,\omega)$, I=$(0,\omega,1)$, J=$(0,\omega,-1)$, K=$(-1,\omega^2,\omega)$, L=$(1,\omega^2,\omega)$, M=$(1,\omega^2,-\omega)$, N=$(1,-\omega^2,\omega)$, O=$(1,\omega,0)$, P=$(1,-\omega,0)$, Q=$(\omega,0,-1)$, R=$(\omega,0,1)$, S=$(0,1,\omega)$, T=$(0,1,-\omega)$, U=$(-1,\omega,\omega^2)$, V=$(1,\omega,\omega^2)$, W=$(1,\omega,-\omega^2)$, X=$(1,-\omega,\omega^2)$, Y=$(2\omega,\omega,\omega)$, Z=$(\omega,-\omega,2\omega)$, a=$(-\omega,-\omega,2\omega)$, b=$(-\omega,2\omega,-\omega)$, c=$(-\omega,2\omega,\omega)$, d=$(2\omega,-\omega,\omega)$, e=$(-\omega,\omega,2\omega)$, f=$(\omega,2\omega,\omega)$, g=$(2\omega,\omega,-\omega)$, h=$(\omega,\omega,2\omega)$, i=$(\omega,2\omega,-\omega)$, j=$(2\omega,-\omega,-\omega)$, k=$(1,-\omega^2,2\omega)$, l=$(2\omega,1,\omega^2)$, m=$(\omega^2,2\omega,-1)$, n=$(-1,-\omega^2,2\omega)$, o=$(-\omega^2,2\omega,-1)$, p=$(2\omega,-1,-\omega^2)$, q=$(-\omega^2,2\omega,1)$, r=$(2\omega,-1,\omega^2)$, s=$(1,\omega^2,2\omega)$, t=$(-1,\omega^2,2\omega)$, u=$(\omega^2,2\omega,1)$, v=$(2\omega,1,-\omega^2)$, w=$(\omega^2,-1,2\omega)$, x=$(2\omega,\omega^2,1)$, y=$(1,2\omega,-\omega^2)$, z=$(-\omega^2,-1,2\omega)$, !=$(-1,2\omega,-\omega^2)$, "=$(2\omega,-\omega^2,-1)$, #=$(-1,2\omega,\omega^2)$, $=$(2\omega,-\omega^2,1)$, %=$(\omega^2,1,2\omega)$, &=$(-\omega^2,1,2\omega)$, ’=$(1,2\omega,\omega^2)$, (=$(2\omega,\omega^2,-1)$

85-62 123,456,789,9e6,9f3,AgB,ChD,EiB,FjD,GHA,IJC,KJE,LHF,MND,OPB,QRJ,STH,UV8,WX8, Pk9,Nl9,Ym7,Zn7,Go6,Ip3,aqE,brF,csA,dtC,Ku5,Lv2,TwN,RxP,ZyR,ZzT,Q!4,S"1,V#5,X$2,X%4,V&1, c’W,d(U,a)W,b*U,dcY,baZ,JH8,a-G,b/I,M:G,O;I,c<K,d=L,M>K,O?L,Q@N,S[P,Y\Q,Y]S,V^O,X_M. 1=$(1,-1,\omega)$, 2=$(1,-\omega,1)$, 3=$(\omega,-1,1)$, 4=$(1,\omega,-1)$, 5=$(1,1,-\omega)$, 6=$(\omega,1,-1)$, 7=$(0,1,-1)$, 8=$(1,0,0)$, 9=$(0,1,1)$, A=$(\omega,1,0)$, B=$(1,-\omega^2,\omega)$, C=$(\omega,0,1)$, D=$(1,\omega,-\omega^2)$, E=$(1,0,-\omega)$, F=$(1,-\omega,0)$, G=$(\omega,-1,0)$, H=$(0,0,1)$, I=$(\omega,0,-1)$, J=$(0,1,0)$, K=$(1,0,\omega)$, L=$(1,\omega,0)$, M=$(1,\omega^2,-\omega)$, N=$(1,1,-1)$, O=$(1,-\omega,\omega^2)$, P=$(1,-1,1)$, Q=$(1,0,1)$, R=$(1,0,-1)$, S=$(1,1,0)$, T=$(1,-1,0)$, U=$(0,1,-\omega)$, V=$(0,1,\omega)$, W=$(0,\omega,-1)$, X=$(0,\omega,1)$, Y=$(-1,1,1)$, Z=$(1,1,1)$, a=$(1,\omega^2,\omega)$, b=$(1,\omega,\omega^2)$, c=$(-1,\omega^2,\omega)$, d=$(-1,\omega,\omega^2)$, e=$(2\omega,-1,1)$, f=$(2\omega,1,-1)$, g=$(-1,\omega^2,2\omega)$, h=$(-1,2\omega,\omega^2)$, i=$(\omega^2,2\omega,1)$, j=$(\omega^2,1,2\omega)$, k=$(2\omega,\omega,-\omega)$, l=$(2\omega,-\omega,\omega)$, m=$(2\omega,\omega,\omega)$, n=$(2\omega,-\omega,-\omega)$, o=$(\omega^2,\omega,2\omega)$, p=$(\omega^2,2\omega,\omega)$, q=$(-\omega^2,2\omega,-1)$, r=$(-\omega^2,-1,2\omega)$, s=$(1,-\omega^2,2\omega)$, t=$(1,2\omega,-\omega^2)$, u=$(-\omega,2\omega,\omega^2)$, v=$(-\omega,\omega^2,2\omega)$, w=$(\omega,\omega,2\omega)$, x=$(\omega,2\omega,\omega)$, y=$(-\omega,2\omega,-\omega)$, z=$(-\omega,-\omega,2\omega)$, !=$(-1,2\omega,1)$, "=$(-1,1,2\omega)$, #=$(2\omega,-\omega,\omega^2)$, $=$(2\omega,\omega^2,-\omega)$, %=$(2\omega,-\omega^2,\omega)$, &=$(2\omega,\omega,-\omega^2)$, ’=$(2\omega,1,\omega^2)$, (=$(2\omega,\omega^2,1)$,)=$(2\omega,-1,-\omega^2)$, *=$(2\omega,-\omega^2,-1)$, -=$(-1,-\omega^2,2\omega)$, /=$(-1,2\omega,-\omega^2)$, :=$(1,\omega^2,2\omega)$, ;=$(1,2\omega,\omega^2)$, <=$(\omega^2,2\omega,-1)$, ==$(\omega^2,-1,2\omega)$, >=$(-\omega^2,2\omega,1)$, ?=$(-\omega^2,1,2\omega)$, @=$(-\omega,2\omega,\omega)$, [=$(-\omega,\omega,2\omega)$, \=$(\omega,2\omega,-\omega)$,]=$(\omega,-\omega,2\omega)$, ^=$(2\omega,\omega^2,-1)$, _=$(2\omega,-1,\omega^2)$

152-71 Coordinatization. 1=(0,0,0,1), 2=(0,0,1,0), 3=(0,1,0,0), 4=(1,0,0,0), 5=(0,1,1,0), 6=(0,-1,1,2), 7=(0,1,-1,1), 8=(1,0,0,1), 9=(1,0,0,-1), A=(0,1,-1,0), B=(1,1,-1,-1), C=(1,-1,1,-1), D=(1,1,1,1), E=(1,-1,-1,1), F=(0,0,1,1), G=(2,0,1,-1), H=(1,3,-1,1), I=(0,0,1,-1), J=(1,1,0,0), K=(1,-1,0,0), L=(0,1,0,-1), M=(1,0,1,0), N=(0,1,0,1), O=(1,0,-1,0), P=(1,1,1,-1), Q=(1,-1,1,1), R=(-1,1,1,1), S=(1,1,-1,1), T=(2,3,-3,0), U=(3,-1,1,0), V=(2,3,3,0), W=(-3,1,1,0), X=(1,-1,1,0), Y=(-1,1,2,0), Z=(1,1,3,0), a=(3,3,-2,0), b=(1,1,-1,0), c=(-1,2,1,0), d=(1,1,1,0), e=(1,-2,1,0), f)=(1,3,0,-1), g=(-3,2,0,3), h=(1,3,0,1), i=(3,-2,0,3), j=(1,-1,0,3), k=(-3,3,0,2), l=(1,1,0,3), m=(3,3,0,-2), n=(1,1,0,1), o=(-2,1,0,1), p=(1,1,0,-1), q=(2,-1,0,1), r=(1,0,3,-1), s=(-3,0,2,3), t=(1,0,3,1), u=(3,0,-2,3), v=(1,0,1,-1), w=(2,0,-1,1), x=(1,0,1,1), y=(-2,0,1,1), z=(1,0,-1,1), !=(-1,0,1,2), "=(1,0,1,3), #=(3,0,3,-2), $=(0,1,1,1), %=(0,1,1,-2), &=(0,1,1,-1), ’=(0,2,-1,1), (=(0,-1,1,1),)=(0,2,1,1), *=(0,1,3,1), -=(0,3,-2,3), /=(0,1,3,-1), :=(0,-3,2,3), ;=(1,-2,2,-3), <=(5,-1,1,3), ==(1,-1,1,3), >=(2,1,-1,0), ?=(0,1,-1,2), @=(3,1,-1,-1), [=(1,1,-1,-3), \=(2,-1,1,0),]=(1,1,3,-1), ^=(-1,2,0,1), _=(-1,1,3,1), ‘=(1,2,0,-1), .=(-1,0,2,1), |=(1,-3,1,-1), =(1,0,2,-1), $\sim$=(1,3,-1,-1), +1=(1,1,-3,1), +2=(1,-2,0,1), +3=(1,-1,3,1), +4=(1,2,0,1), +5=(1,0,-2,1), +6=(1,3,1,1), +7=(1,0,2,1), +8=(1,-3,-1,1), +9=(1,-2,-2,3), +A=(5,-1,-1,-3), +B=(1,2,2,3), +C=(5,1,1,-3), +D=(0,1,1,2), +E=(3,-1,-1,1), +F=(1,-1,-1,3), +G=(2,1,1,0), +H=(1,3,2,2), +I=(5,-3,1,1), +J=(1,1,-1,3), +K=(1,1,2,0), +L=(1,0,-1,2), +M=(-1,3,1,1), +N=(0,1,2,-1), +O=(3,-1,1,1), +P=(1,-3,-2,2), +Q=(5,3,-1,1), +R=(1,-1,3,5), +S=(2,-2,-3,1), +T=(0,1,2,1), +U=(3,1,-1,1), +V=(1,0,1,2), +W=(1,3,1,-1), +X=(1,3,-2,-2), +Y=(5,-3,-1,-1), +Z=(1,1,1,-3), +a=(1,1,-2,0), +b=(0,1,-2,1), +c=(-3,1,1,1), +d=(1,0,1,-2), +e=(1,-3,1,1), +f=(1,-1,5,3), +g=(2,-2,1,-3), +h=(-1,1,1,3), +i=(1,2,-1,0), +j=(0,-1,2,1), +k=(3,1,1,-1), +l=(1,3,-2,2), +m=(5,-3,-1,1), +n=(0,2,1,-1), +o=(3,-1,1,-1), +p=(-1,3,2,2), +q=(5,3,-1,-1), +r=(0,-2,1,1), +s=(3,1,1,1), +t=(-1,1,0,2), +u=(1,-1,-3,1), +v=(1,-1,0,2), +w=(1,-1,3,-1), +x=(1,1,0,2), +y=(1,1,-3,-1), +z=(1,1,0,-2), +!=(1,1,3,1)

11-7 123456,67,738,859,92A,A4B,B1. 1=(0,1,0,0,0,0), 2=(0,0,0,0,0,1), 3=(0,0,0,0,1,0), 4=(0,0,0,1,0,0), 5=(0,0,1,0,0,0), 6=(1,0,0,0,0,0), 7=(0,0,0,1,0,-1), 8=(0,0,0,1,0,1), 9=(0,1,0,0,1,0), A=(0,1,0,0,-1,0), B=(0,0,1,0,0,1)

15-9 12345678,89,94A,A7B,B3C,C6D,D2E,E5F,F1. 1=(0,1,0,0,0,0,0,0), 2=(0,0,0,0,0,0,0,1), 3=(0,0,0,0,0,0,1,0), 4=(0,0,0,0,0,1,0,0), 5=(0,0,0,0,1,0,0,0), 6=(0,0,0,1,0,0,0,0), 7=(0,0,1,0,0,0,0,0), 8=(1,0,0,0,0,0,0,0), 9=(0,0,0,0,0,0,1,-1), A=(0,0,0,0,0,0,1,1), B=(0,0,0,0,1,1,0,0), C=(0,0,0,0,1,-1,0,0), D=(0,0,0,0,1,1,1,0), E=(0,0,0,0,0,1,-1,0), F=(0,0,0,0,0,1,1,0)

19-11 123456789A,AB,B5C,C9D,D4E,E8F,F3G,G7H,H2I,I6J,J1. 1=(0,1,0,0,0,0,0,0,0,0), 2=(0,0,0,0,0,0,0,0,0,1), 3=(0,0,0,0,0,0,0,0,1,0), 4=(0,0,0,0,0,0,0,1,0,0), 5=(0,0,0,0,0,0,1,0,0,0), 6=(0,0,0,0,0,1,0,0,0,0), 7=(0,0,0,0,1,0,0,0,0,0), 8=(0,0,0,1,0,0,0,0,0,0), 9=(0,0,1,0,0,0,0,0,0,0), A=(1,0,0,0,0,0,0,0,0,0), B=(0,0,0,0,0,0,0,1,1,-1), C=(0,0,0,0,0,0,0,0,1,1), D=(0,0,0,0,0,0,0,0,1,-1), E=(0,0,0,0,0,0,1,0,1,1), F=(0,0,0,0,0,0,1,0,0,-1), G=(0,0,0,0,0,0,1,0,0,1), H=(0,0,0,0,0,0,0,1,1,0), I=(0,0,0,0,0,0,0,1,-1,0), J=(0,0,0,0,0,0,0,1,1,1)

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