Mathematical Proof of Spatial Spectrum Interpolation

In the ICML manuscript, given that spatial spectrum interpolation is inspired by view interpolation theory [1] from the computer vision domain, we omitted the formal proof and instead provided empirical validation in Appendix A to demonstrate its effectiveness in the wireless domain. To provide a more complete picture of the underlying theory and its connection to the GWRF design, we present here a mathematical proof along with its connection to the empirical evaluation in Appendix A and the architectural components of GWRF.

§1 provides the mathematical proof of spatial spectrum interpolation.

§2 presents an explanation of the connection between the mathematical proof and the empirical evaluation in Appendix A of our submitted ICML manuscript.

§3 illustrates how the mathematical proof informs the GWRF architectural design.

1 Mathematical Proof

Overview: We present a mathematical proof demonstrating that the spatial spectrum $\mathbf{SS}_t$ at a target transmitter position $\mathbf{P}_t$ can be approximated as a combination of spatial spectra $\{\mathbf{SS}_i\}_{i=1}^{L}$ corresponding to L neighboring transmitter positions $\{\mathbf{P}_i\}_{i=1}^{L}$. The proof is structured in three steps: a Taylor series expansion, a convex linear combination, and an error analysis.

Assumptions: The proof presupposes that transmitters and receivers equipped with isotropic antennas are placed in free 3D space, under the following two assumptions:

ASSUMPTION 1: A smooth propagation environment, wherein each transmitter position $\mathbf{P} \in \mathbb{R}^3$ is mapped to a corresponding spatial spectrum $\mathbf{SS} \in \mathbb{R}^{360\times 90}$ through a function $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^{360\times 90}$. The mapping $\mathcal{T}$ is assumed to be twice continuously differentiable with respect to the position $\mathbf{P} = (x, y, z)$. This regularity condition ensures that the spatial spectrum $\mathbf{SS} = \mathcal{T}(\mathbf{P})$ and its first- and second-order derivatives vary smoothly across the local region of $\mathbf{P}$—a standard postulate in physics-based analyses of wave propagation [2, 3].

For example, consider the Friis Free Space Propagation Equation [4]—analogous to the mapping $\mathcal{T}$—where the received signal power S_{rx} at a receiver position $\mathbf{P}_{rx}$ is given by (note that $S_{rx} \in \mathbb{R}$ is a scalar, whereas our setting involves a full spatial spectrum $\mathbf{SS} \in \mathbb{R}^{360\times 90}$):

$$S_{rx} = S_{tx}\left(\frac{\lambda}{4\pi d}\right)^2, \quad \text{with} \quad d = \|\mathbf{P}_{tx} - \mathbf{P}_{rx}\|$$

where S_{tx} is the transmitted power, λ is the wavelength, and d is the Euclidean distance between the transmitter at $\mathbf{P}_{tx}$ and the receiver at $\mathbf{P}_{rx}$, which is a smooth function of position $\mathbf{P}$ for $d > 0$. Since $S_{rx} \propto d^{-2}$, the received power also varies smoothly with respect to positional changes in $\mathbf{P}$.

While the Friis model yields a single scalar value, the mapping $\mathcal{T}$ generalizes this concept to produce a spatial spectrum $\mathbf{SS} \in \mathbb{R}^{360\times 90}$—representing signal power across all directions around the receiver—with smoothness similarly ensured by continuous variation in free space.

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ASSUMPTION 2: We adopt the standard interpolation assumption in $\mathbb{R}^3$: it is possible to identify neighboring positions $\{\mathbf{P}_i\}_{i=1}^{L}$ such that the target position $\mathbf{P}_t$ lies within their convex hull, i.e.,

$$\sum_{i=1}^{L} w_i \mathbf{P}_i = \mathbf{P}_t$$

where the w_i are barycentric weights satisfying $w_i \geq 0$ and $\sum_{i=1}^{L} w_i = 1$.

1.1 Step 1: Taylor Series Expansion of the Spatial Spectrum

Objective: Derive the relationship between the spatial spectra at two proximate transmitter positions, $\mathbf{P}_i$ and $\mathbf{P}_j$, where $\mathbf{P}_j = \mathbf{P}_i + \Delta\mathbf{P}_{ij}$ with $\Delta\mathbf{P}_{ij} = (x_j - x_i,\, y_j - y_i,\, z_j - z_i)$, by applying a Taylor series expansion under ASSUMPTION 1.

Statement:

$$\mathbf{SS}_j = \mathcal{T}\left(\mathbf{P}_i + \Delta\mathbf{P}_{ij}\right) = \mathbf{SS}_i + \nabla_{\mathbf{P}}\mathcal{T}\left(\mathbf{P}_i\right) \cdot \Delta\mathbf{P}_{ij} + \frac{1}{2}\Delta\mathbf{P}_{ij}^{\top}\nabla_{\mathbf{P}}^{2}\mathcal{T}\left(\mathbf{P}_i\right)\Delta\mathbf{P}_{ij} + \mathcal{O}\left(\left\|\Delta\mathbf{P}_{ij}\right\|^3\right)$$

Explanation:

- **Preliminaries**:
  - Spectra $\mathbf{SS}_i$ and $\mathbf{SS}_j$ are defined through a mapping $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^{360\times 90}$, where $\mathbf{SS}_i = \mathcal{T}\left(\mathbf{P}_i\right)$ and $\mathbf{SS}_j = \mathcal{T}\left(\mathbf{P}_j\right)$, for transmitter positions $\mathbf{P}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ and $\mathbf{P}_j = \mathbf{P}_i + \Delta\mathbf{P}_{ij}$, with $\Delta\mathbf{P}_{ij}$ denoting the displacement vector between $\mathbf{P}_i$ and $\mathbf{P}_j$.
  - The magnitude of the displacement $\|\Delta\mathbf{P}_{ij}\|$ is assumed to be sufficiently small, ensuring that geometric perturbations remain minimal.
- **Terms**:
  - $\nabla_{\mathbf{P}}\mathcal{T}\left(\mathbf{P}_i\right) = \left(\frac{\partial\mathcal{T}}{\partial x}, \frac{\partial\mathcal{T}}{\partial y}, \frac{\partial\mathcal{T}}{\partial z}\right) \in \mathbb{R}^{360\times 90\times 3}$: The gradient tensor, which quantifies the first-order sensitivity of the spatial spectrum to changes in the coordinates of $\mathbf{P}_i$.
  - $\nabla_{\mathbf{P}}^{2}\mathcal{T}\left(\mathbf{P}_i\right) \in \mathbb{R}^{360\times 90\times 3\times 3}$: The Hessian tensor, comprising second-order partial derivatives and capturing the curvature of the spatial spectrum with respect to $\mathbf{P}_i$.
  - $\mathcal{O}\left(\left\|\Delta\mathbf{P}_{ij}\right\|^3\right) \in \mathbb{R}^{360\times 90}$: The remainder term, which encompasses third- and higher-order derivatives and becomes negligible when $\Delta\mathbf{P}_{ij}$ is sufficiently small.

1.2 Step 2: Interpolating the Target Spectrum via Weighted Combination

Objective: Estimate the spatial spectrum $\mathbf{SS}_t$ at a target position $\mathbf{P}_t \in \mathbb{R}^3$ by blending the known spectra $\{\mathbf{SS}_i\}_{i=1}^{L}$ from nearby transmitter positions $\{\mathbf{P}_i\}_{i=1}^{L} \subset \mathbb{R}^3$.

Statement: Given the Taylor expansion from Step 1 and the barycentric weights (ASSUMPTION 2), the spatial spectrum at the target position $\mathbf{P}_t$ can be approximated as:

$$\mathbf{SS}_t = \sum_{i=1}^{L} w_i \mathbf{SS}_i + \sum_{i=1}^{L} w_i \nabla_{\mathbf{P}}\mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) + \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$$

where $\Delta\mathbf{P} = \max_i \|\mathbf{P}_t - \mathbf{P}_i\|$ denotes the maximum distance between the target position $\mathbf{P}_t$ and its neighboring transmitter positions $\{\mathbf{P}_i\}_{i=1}^{L}$. Given the Lipschitz continuity of $\nabla_{\mathbf{P}}\mathcal{T}$ (which follows from the mapping function $\mathcal{T}$ being twice continuously differentiable), the first-order term is bounded by $\mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$, thereby simplifying the expression to:

$$\mathbf{SS}_t = \sum_{i=1}^{L} w_i \mathbf{SS}_i + \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$$

Explanation:

- **Construction**:
  - For each neighbor $\mathbf{P}_i$, we use the Taylor expansion around $\mathbf{P}_i$ from Step 1:

$$\mathbf{SS}_t = \mathbf{SS}_i + \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) + \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$$

    This provides a local approximation of $\mathbf{SS}_t$ when $\mathbf{P}_t$ is close to $\mathbf{P}_i$.
  - We combine these approximations using barycentric weights $\{w_i\}_{i=1}^L$, where $w_i \geq 0$, $\sum_{i=1}^L w_i = 1$, and $\sum_{i=1}^L w_i \mathbf{P}_i = \mathbf{P}_t$. The weighted estimate becomes:

$$\mathbf{SS}_t = \sum_{i=1}^L w_i \mathbf{SS}_i + \sum_{i=1}^L w_i \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) + \sum_{i=1}^L w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$$

- **Simplification**:
  - *Bounding the First-Order Term*: The barycentric property ensures:

$$\sum_{i=1}^L w_i \left(\mathbf{P}_t - \mathbf{P}_i\right) = \sum_{i=1}^L w_i \mathbf{P}_t - \sum_{i=1}^L w_i \mathbf{P}_i = \mathbf{P}_t - \mathbf{P}_t = \mathbf{0}$$

    However, the gradients $\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i)$ vary across positions $\mathbf{P}_i$. Since $\mathcal{T}$ is twice differentiable, its gradient $\nabla_{\mathbf{P}} \mathcal{T}$ is Lipschitz continuous. Thus, we have:

$$\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) = \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_t) + \mathbf{E}_i, \quad \text{with} \quad \|\mathbf{E}_i\| \leq K_1 \left\|\mathbf{P}_t - \mathbf{P}_i\right\|$$

    where K_1 is the Lipschitz constant of $\nabla_{\mathbf{P}} \mathcal{T}$, quantifying the maximum rate of change of the gradient, with $K_1 = 2K$ and $K = \frac{1}{2} \sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}} \left\|\nabla_{\mathbf{P}}^2 \mathcal{T}\left(\mathbf{P}\right)\right\|$ as defined in Step 3. Substituting into the first-order term, we obtain:

$$\sum_{i=1}^L w_i \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) = \sum_{i=1}^L w_i \left[\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_t) + \mathbf{E}_i\right] \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right)$$

    This splits as:

$$\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_t) \cdot \sum_{i=1}^L w_i \left(\mathbf{P}_t - \mathbf{P}_i\right) + \sum_{i=1}^L w_i \mathbf{E}_i \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) = 0 + \sum_{i=1}^L w_i \mathbf{E}_i \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right)$$

    Bounding the error term:

$$\left\|\sum_{i=1}^L w_i \mathbf{E}_i \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right)\right\| \leq \sum_{i=1}^L w_i \left\|\mathbf{E}_i\right\| \left\|\mathbf{P}_t - \mathbf{P}_i\right\| \leq K_1 \sum_{i=1}^L w_i \left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2 \leq K_1 \left\|\Delta \mathbf{P}\right\|^2$$

    where $\sum_{i=1}^L w_i = 1$. Therefore, the first-order term is bounded by $\mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$.
  - *Residual Error Term*: The higher-order terms are:

$$\sum_{i=1}^L w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$$

    Since $\|\mathbf{P}_t - \mathbf{P}_i\| \leq \|\Delta \mathbf{P}\|$ and $\sum_{i=1}^L w_i = 1$, we have:

$$\sum_{i=1}^L w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right) \leq \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$$

  - *Final Result*: Combining all terms:

$$\mathbf{SS}_t = \sum_{i=1}^L w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right) + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right) = \sum_{i=1}^L w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$$

    The error is quadratic in $\|\Delta \mathbf{P}\|$, meaning it decreases proportionally to the square of the maximum distance between the target position and its neighboring transmitters.

1.3 Step 3: Quantifying the Interpolation Error

Objective: Derive an upper bound on the interpolation error to assess the approximation accuracy.

Statement: The interpolation error $\epsilon = \left\|\mathbf{SS}_t - \sum_{i=1}^{L} w_i \mathbf{SS}_i\right\|$ satisfies the bound:

$$\epsilon \leq K \cdot \max_i \|\mathbf{P}_t - \mathbf{P}_i\|^2, \quad \text{with} \quad K = \frac{1}{2} \sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}} \left\|\nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P})\right\|$$

where K is a constant that depends on the second-order derivatives of the function $\mathcal{T}$, and encapsulates the maximum curvature of $\mathcal{T}$ over the convex hull of the neighboring positions $\{\mathbf{P}_i\}_{i=1}^{L}$.

Explanation:

- **Derivation**:
  - From Step 2, the interpolation error ϵ is expressed as:

$$\epsilon = \left\|\mathbf{SS}_t - \sum_{i=1}^{L} w_i \mathbf{SS}_i\right\| = \left\|\mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right)\right\|$$

  - $\mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right)$ is bounded by the second derivatives of $\mathcal{T}$. By Taylor's theorem, for a twice continuously differentiable function $\mathcal{T}$, there exists a constant $K > 0$ such that:

$$\|\mathcal{T}(\mathbf{P}_t) - \mathcal{T}(\mathbf{P}_i) - \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot (\mathbf{P}_t - \mathbf{P}_i)\| \leq K \|\mathbf{P}_t - \mathbf{P}_i\|^2$$

    where $K = \frac{1}{2} \sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}} \left\|\nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P})\right\|$ bounds the acceleration of the spatial spectrum's variation across the interpolation region. Smoother environments—those with gentler second-order changes—yield smaller K, tightening the error bound $\epsilon \leq K \cdot \|\Delta\mathbf{P}\|^2$. For example, in free-space propagation, K is negligible, whereas near obstacles—where signal strength fluctuates rapidly—K grows significantly.
    * $\nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P})$ is the *Hessian matrix* of $\mathcal{T}$ at $\mathbf{P}$, quantifying how sensitively the spectrum changes in response to second-order variations in the transmitter coordinates.
    * $\left\|\nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P})\right\|$ denotes the *operator norm* of the Hessian—a scalar measure of its "magnitude," representing the maximum rate of curvature of $\mathcal{T}$ in any direction.
    * $\sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}}$ takes the *supremum* (least upper bound) of the Hessian norm over all points in the convex hull of the neighboring positions $\{\mathbf{P}_i\}$. This represents the worst-case curvature of $\mathcal{T}$ within the local interpolation region.
  - Summing over all neighbors' spatial spectra with barycentric weights w_i, and noting that $\sum_{i=1}^{L} w_i = 1$, the total error satisfies:

$$\epsilon \leq \sum_{i=1}^{L} w_i \cdot K \|\mathbf{P}_t - \mathbf{P}_i\|^2 \leq K \cdot \max_i \|\mathbf{P}_t - \mathbf{P}_i\|^2$$

- **Intuition**:
  - **Interpretability of** K: The constant K directly quantifies the worst-case curvature of $\mathcal{T}$ over the interpolation region, as governed by the Hessian $\nabla^2_{\mathbf{P}} \mathcal{T}$. In free-space propagation (low curvature, small K), the bound tightens significantly, whereas near obstacles or multipath-rich environments (high curvature, large K), the error grows predictably. This ties the theoretical guarantee to physically observable phenomena, ensuring the result is not merely abstract but grounded in wave physics.
  - **Scope of the Error Bound**: The derived bound applies specifically to the linear convex combination of neighboring spectra, as defined by barycentric weights. While this provides a foundational guarantee for basic linear interpolation, the GWRF architecture extends beyond this linear regime. By integrating a neural network, GWRF learns non-linear corrections to the weighted spectral average, capturing higher-order correlations and environmental discontinuities (e.g., diffraction, reflection, scattering) that the linear model cannot represent. Thus, the proven bound serves as a baseline for the worst-case error in linear neighbor spatial spectrum interpolation.

2 Connecting to Empirical Spectrum Interpolation in Appendix A

Following the theoretical framework established in Steps 1, 2, and 3, we now connect these steps to the empirical spectrum interpolation method presented in Appendix A, titled "Why Incorporate Neighbors." This appendix demonstrates that the spatial spectrum of a target transmitter can be effectively approximated by interpolating the spectra of its geographically proximate neighbors, leveraging their similarity in propagation behavior. Here, we illustrate how our proof provides a rigorous mathematical foundation for these empirical insights.

2.1 Connection to Step 1: Taylor Series Expansion

In Step 1, we model the spatial spectrum as a smooth function under a diffeomorphic mapping $\mathcal{T}$, applying a Taylor expansion to approximate the spectrum at a target position based on a nearby neighbor. This step establishes that small positional changes lead to predictable, smooth variations in the spectrum. This step is formalized as:

$$\mathbf{SS}_j = \mathbf{SS}_i + \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \Delta \mathbf{P}_{ij} + \frac{1}{2} \Delta \mathbf{P}_{ij}^{\top} \nabla_{\mathbf{P}}^2 \mathcal{T}\left(\mathbf{P}_i\right) \Delta \mathbf{P}_{ij} + \mathcal{O}\left(\left\|\Delta \mathbf{P}_{ij}\right\|^3\right)$$

This aligns directly with Appendix A's Observation #1, which shows that neighboring transmitters share similar propagation paths due to their proximity, as evidenced by 80% of reflection point distances being less than 0.1 meters apart. The smoothness of $\mathcal{T}$ justifies this empirical similarity, providing a theoretical basis for why spectra from nearby transmitters can be used to approximate a target spectrum. By proving that the spectrum varies continuously and predictably with position, Step 1 supports the feasibility of the empirical interpolation approach.

2.2 Connection to Step 2: Interpolating the Target Spectrum via Weighted Combination

Step 2 builds on this foundation by developing a weighted combination method to interpolate the target spectrum $\mathbf{SS}_t$ from the spectra of L neighboring transmitters $\{\mathbf{SS}_i\}_{i=1}^{L}$. Using barycentric weights w_i, where $\sum_{i=1}^{L} w_i = 1$ and $\mathbf{P}_t = \sum_{i=1}^{L} w_i \mathbf{P}_i$, the approximation simplifies to:

$$\mathbf{SS}_t = \sum_{i=1}^{L} w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$$

This step connects to Appendix A's Observation #2, where an empirical interpolation using six neighboring spectra achieves an average PSNR of 20.5 dB, suggesting a notable correlation with the target spectrum. While Appendix A uses a heuristic weighted summation, Step 2 offers a more formal mathematical framework, introducing weights informed by Taylor expansion and geometric relationships. This formulation aims to improve the interpretability and potential accuracy of the interpolation, providing theoretical support for the observed empirical behavior.

3 Connecting to GWRF Framework

In this section, we bridge the theoretical basis established in § 1 with the GWRF architectural design. To clarify, the error bound presented in § 1 applies to linear interpolation. Since GWRF adopts a neural architecture that is inherently non-linear, this bound does not directly apply to GWRF.

3.1 Geometry-Aware Scene Representation (Step 1)

Step 1 establishes that the spectrum function $\mathcal{T}$ is smooth, allowing Taylor expansions to approximate spectra at nearby spatial positions based on positional differences. This smoothness is foundational to GWRF's geometry-aware scene representation, which encodes spatial relationships between a target transmitter and its neighboring points.

- **Transformer Encoder with Cross-Attention**: GWRF utilizes a Transformer encoder to process the spatial spectra of neighboring transmitters, denoted as $\{\mathbf{SS}_i, \mathbf{P}_i\}_{i=1}^{L}$, alongside positional encodings $\mathcal{E}\left(\mathbf{P}_{\text{neighbor}} - \mathbf{P}_{\text{target}}\right)$. This attention mechanism captures the relative geometric differences emphasized in Step 1, enabling the model to learn a representation that generalizes across positions. The cross-attention layer dynamically weights each neighbor's contribution based on its proximity to the target transmitter, mirroring the local approximation inherent in the Taylor expansion.

- **Latent Voxel Features**: Each spatial position is represented by a latent voxel vector $\mathbf{v}$, which encapsulates complex wireless signal propagation behaviors (e.g., reflection, diffraction). This rich representation aligns with the higher-order terms of the Taylor expansion, allowing GWRF to model intricate, position-dependent propagation effects while maintaining the smoothness assumption of the function $\mathcal{T}$.

3.2 Neural-Driven Ray Tracing (Step 2)

In Step 2, we interpolate the target spectrum $\mathbf{SS}_t$ as a weighted combination of neighboring spectra, using barycentric weights to form a convex approximation. GWRF operationalizes this concept through its neural-driven ray tracing algorithm, enhancing the interpolation with neural flexibility.

- **Feature Aggregation**: The ray tracing aggregates latent voxel features along propagation paths using a multi-head Transformer. This aggregation process parallels Step 2's weighted combination, where the Transformer's attention mechanism learns a sophisticated, position-sensitive weighting scheme. By synthesizing features across voxels, GWRF interpolates the target spectrum in a way that adapts to the wireless scene environment.
- **Multi-Head Attention for Complex Interactions**: With ten attention heads, GWRF captures diverse interactions among voxel features, such as amplitude and phase variations. This design reflects Step 2's goal of creating a precise yet flexible interpolation, adept at handling the multifaceted nature of wireless signal propagation.

3.3 Error Control (Step 3)

Step 3 provides an error bound for the linear interpolation, expressed as $\epsilon \leq K \cdot \max_i \|\mathbf{P}_t - \mathbf{P}_i\|^2$, showing that accuracy improves when neighbors are closer to the target. GWRF's design inherently enforces this principle through its neighbor selection and training strategies.

- **Proximity-Based Neighbor Selection**: The GWRF framework selects L neighboring transmitters (randomly varying between 3 and 10) based on their Euclidean distance to the target. This encourages small positional differences $\|\mathbf{P}_t - \mathbf{P}_i\|$, aligning with Step 3's insight that, under linear interpolation, the error is bounded by $\epsilon \leq K \cdot (\max_i \|\mathbf{P}_t - \mathbf{P}_i\|)^2$. While this bound formally applies to the linear setting, GWRF extends beyond it: the geometry-aware Transformer and neural-driven ray tracing enable more expressive, data-driven interpolation. These components allow the model to capture nonlinear propagation behaviors and generalize even when neighbors are not extremely close. In dense deployments, proximity helps reduce approximation error, while in sparse settings, the GWRF framework remains robust through its neurally driven learning process.
- **End-to-End Training with ℓ_2 Loss**: The model is optimized using an ℓ_2 loss on the predicted spectrum, directly minimizing the interpolation error ϵ.

References

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