Mathematical Proof of Spatial Spectrum Interpolation

In this document, we first present the mathematical proof in §1, then in §2 we explain its connection with the empirical evaluation in Appendix A of our submitted ICML manuscript. Finally, we illustrate how the mathematical proof connects to the GWRF architectural design in §3.

1 Mathematical Proof

We present a mathematical proof to establish that the spatial spectrum $\mathbf{SS}_t$ at a target transmitter position $\mathbf{P}_t$ can be approximated as a combination of spatial spectra $\{\mathbf{SS}_i\}_{i=1}^{L}$ associated with L neighboring transmitter positions $\{\mathbf{P}_i\}_{i=1}^{L}$. This interpolation is substantiated through a three-step process comprising Taylor series expansions, convex linear combinations, and error analysis.

The proof presupposes a smooth propagation environment, wherein each transmitter position $\mathbf{P}_i \in \mathbb{R}^3$ is mapped to a corresponding spatial spectrum $\mathbf{SS}_i \in \mathbb{R}^{360\times 90}$ through a function $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^{360\times 90}$. The mapping $\mathcal{T}$ is assumed to be twice continuously differentiable with respect to the position $\mathbf{P}_i = (x_i, y_i, z_i)$. This regularity condition ensures that the spatial spectrum $\mathbf{SS}_i = \mathcal{T}(\mathbf{P}_i)$ and its first- and second-order derivatives vary smoothly across the local space, a standard postulate in physics-based analyses of wave propagation [1, 2].

1.1 Step 1: Taylor Expansion of the Spatial Spectrum under the Mapping $\mathcal{T}$

Objective: Derive a mathematical relationship between the spatial spectra at two proximate transmitter positions, $\mathbf{P}_i$ and $\mathbf{P}_j$, where $\mathbf{P}_j = \mathbf{P}_i + \Delta\mathbf{P}_{ij}$, by applying a Taylor series expansion.

Statement:

$$\mathbf{SS}_j = \mathcal{T}(\mathbf{P}_i + \Delta\mathbf{P}_{ij}) = \mathbf{SS}_i + \nabla_{\mathbf{P}}\mathcal{T}(\mathbf{P}_i) \cdot \Delta\mathbf{P}_{ij} + \frac{1}{2}\Delta\mathbf{P}_{ij}^{\top}\nabla_{\mathbf{P}}^{2}\mathcal{T}(\mathbf{P}_i)\,\Delta\mathbf{P}_{ij} + \mathcal{O}\left(\|\Delta\mathbf{P}_{ij}\|^3\right)$$

Explanation:

- **Preliminaries**:
  - The spatial spectra $\mathbf{SS}_i$ and $\mathbf{SS}_j$ are defined through a mapping $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^{360\times 90}$, where $\mathbf{SS}_i = \mathcal{T}(\mathbf{P}_i)$ and $\mathbf{SS}_j = \mathcal{T}(\mathbf{P}_j)$, for transmitter positions $\mathbf{P}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ and $\mathbf{P}_j = \mathbf{P}_i + \Delta\mathbf{P}_{ij} \in \mathbb{R}^3$, with $\Delta\mathbf{P}_{ij} = (x_j - x_i, y_j - y_i, z_j - z_i)$ denoting the displacement vector between $\mathbf{P}_i$ and $\mathbf{P}_j$.
  - The matrices $\mathbf{SS}_i$ and $\mathbf{SS}_j$ represent signal strength across all directional coordinates around a receiver, with variations arising from propagation phenomena, including path loss, phase shifts, and multipath interference.
  - **Condition**: The magnitude of the displacement, $\|\Delta\mathbf{P}_{ij}\|$, is sufficiently small, ensuring that geometric perturbations remain minimal.
- **Taylor Expansion**:
  - Given a small displacement $\Delta\mathbf{P}_{ij}$, the spectrum $\mathbf{SS}_j = \mathcal{T}(\mathbf{P}_i + \Delta\mathbf{P}_{ij})$ is approximated around $\mathbf{P}_i$ by applying a Taylor series expansion, as presented in Statement.
  - **Terms**:
    * $\nabla_{\mathbf{P}}\mathcal{T}(\mathbf{P}_i) = \left(\frac{\partial\mathcal{T}}{\partial x}, \frac{\partial\mathcal{T}}{\partial y}, \frac{\partial\mathcal{T}}{\partial z}\right) \in \mathbb{R}^{360\times 90\times 3}$: The gradient tensor, quantifying the first-order sensitivity of the spatial spectrum to changes in the coordinates of $\mathbf{P}_i$.

* $\nabla_{\mathbf{P}}^2 \mathcal{T}(\mathbf{P}_i) \in \mathbb{R}^{360\times 90\times 3\times 3}$: The Hessian tensor, comprising second partial derivatives and capturing the curvature of the spatial spectrum with respect to $\mathbf{P}_i$.
* $\mathcal{O}\left(\|\Delta \mathbf{P}_{ij}\|^3\right) \in \mathbb{R}^{360\times 90}$: The remainder term, encompassing third- and higher-order derivatives, which is negligible when $\Delta \mathbf{P}_{ij}$ is sufficiently small.

- **Intuition**:
  - For small positional displacements $\Delta \mathbf{P}_{ij}$, the spatial spectrum $\mathbf{SS}_j = \mathcal{T}(\mathbf{P}_i + \Delta \mathbf{P}_{ij})$ exhibits variations that are predominantly linear, as captured by the first-order term $\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot \Delta \mathbf{P}_{ij}$. This dominance arises because, in a smooth propagation environment, incremental changes in transmitter position typically induce proportional shifts in signal characteristics—such as phase or amplitude—across directional coordinates. The quadratic term $\frac{1}{2}\Delta \mathbf{P}_{ij}^\top \nabla_{\mathbf{P}}^2 \mathcal{T}(\mathbf{P}_i) \Delta \mathbf{P}_{ij}$ and higher-order corrections, while present, diminish rapidly due to the small magnitude of $\|\Delta \mathbf{P}_{ij}\|$, contributing only subtle refinements. This predominantly linear response underpins the effectiveness of the interpolation methodology, enabling accurate prediction of $\mathbf{SS}_j$ from $\mathbf{SS}_i$ when the displacement $\|\Delta \mathbf{P}_{ij}\|$ is sufficiently small.
- **Purpose**:
  - This step lays the groundwork for interpolating spatial spectra by deriving the Taylor expansion that relates spatial spectra $\mathbf{SS}_j$ to $\mathbf{SS}_i$ through the geometric transformation $\mathcal{T}_{\Delta \mathbf{P}_{ij}}$. This relationship is vital because it reveals how spectrum changes scale with small positional shifts, providing the linear approximation—via the first-order term $\nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot \Delta \mathbf{P}_{ij}$—that Step 2 leverages to combine multiple neighboring spatial spectra into an estimate at any target position.

1.2 Step 2: Interpolating the Target Spectrum via Weighted Combination

Objective: Estimate the spatial spectrum $\mathbf{SS}_t$ at a target position $\mathbf{P}_t \in \mathbb{R}^3$ by blending the known spectra $\{\mathbf{SS}_i\}_{i=1}^L$ from nearby transmitter positions $\{\mathbf{P}_i\}_{i=1}^L \subset \mathbb{R}^3$.

We use a convex combination based on barycentric weights $\{w_i\}_{i=1}^L$, where each $w_i \geq 0$, $\sum_{i=1}^L w_i = 1$, and $\sum_{i=1}^L w_i \mathbf{P}_i = \mathbf{P}_t$. These weights indicate that $\mathbf{P}_t$ lies within the convex hull of the nearby transmitters—an assumption that is reasonable in $\mathbb{R}^3$. The use of this geometric structure, along with the assumed smoothness of $\mathcal{T}$, allows for an approximate estimation of $\mathbf{SS}_t$ from its neighbors.

Statement: The approximation for the spatial spectrum at the target position $\mathbf{P}_t$ is given by:

$$\mathbf{SS}_t = \sum_{i=1}^L w_i \mathbf{SS}_i + \sum_{i=1}^L w_i \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot (\mathbf{P}_t - \mathbf{P}_i) + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$$

Using the barycentric weights $\{w_i\}_{i=1}^L$ introduced in the *Objective*, the expression simplifies to:

$$\mathbf{SS}_t = \sum_{i=1}^L w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta \mathbf{P}\|^2\right)$$

where $\Delta \mathbf{P} = \max_i \|\mathbf{P}_t - \mathbf{P}_i\|$ denotes the maximum distance between the target position and its neighboring transmitter positions $\{\mathbf{P}_i\}_{i=1}^L$.

Explanation:

- **Construction**:
  - For each neighbor $\mathbf{P}_i$, apply the Taylor expansion from Step 1 to approximate $\mathbf{SS}_t$:

    $$\mathbf{SS}_t = \mathbf{SS}_i + \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot (\mathbf{P}_t - \mathbf{P}_i) + \mathcal{O}\left(\|\mathbf{P}_t - \mathbf{P}_i\|^2\right)$$

    This expansion provides a reasonable approximation of the spectrum's variation around $\mathbf{P}_i$ when $\|\mathbf{P}_t - \mathbf{P}_i\|$ is sufficiently small.

- Combine the approximations from all L neighbors using barycentric weights w_i:

$$\mathbf{SS}_t \approx \sum_{i=1}^{L} w_i \left[\mathbf{SS}_i + \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right)\right] + \sum_{i=1}^{L} w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$$

where $\mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$ denotes the residual error from each Taylor expansion, capturing the influence of higher-order terms. This weighted combination leverages the neighbors' spectra and gradients to construct an approximate estimate of $\mathbf{SS}_t$.

- **Simplification**:
  - *Expanding the Approximation*: We begin with the constructed approximation for the spatial spectrum $\mathbf{SS}_t$ at the transmitter position $\mathbf{P}_t$:

$$\mathbf{SS}_t \approx \sum_{i=1}^{L} w_i \mathbf{SS}_i + \sum_{i=1}^{L} w_i \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) + \sum_{i=1}^{L} w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$$

where $\mathbf{SS}_i = \mathcal{T}\left(\mathbf{P}_i\right)$ represents the known spectra at neighboring points $\mathbf{P}_i$, w_i are the barycentric weights, $\nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right)$ denotes the gradient of the spectrum function $\mathcal{T}$ at $\mathbf{P}_i$, and $\mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right)$ captures higher-order residual errors. This expression balances zeroth-order contributions, first-order corrections, and residual terms.

  - *Applying the Barycentric Condition*: Use the property of barycentric weights:

$$\sum_{i=1}^{L} w_i \left(\mathbf{P}_t - \mathbf{P}_i\right) = \sum_{i=1}^{L} w_i \mathbf{P}_t - \sum_{i=1}^{L} w_i \mathbf{P}_i = \mathbf{P}_t \sum_{i=1}^{L} w_i - \mathbf{P}_t = \mathbf{P}_t \cdot 1 - \mathbf{P}_t = \mathbf{0}$$

Since $\sum_{i=1}^{L} w_i = 1$ and $\mathbf{P}_t = \sum_{i=1}^{L} w_i \mathbf{P}_i$, the weighted displacements cancel out:

$$\sum_{i=1}^{L} w_i \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right) \cdot \left(\mathbf{P}_t - \mathbf{P}_i\right) = \left(\sum_{i=1}^{L} w_i \nabla_{\mathbf{P}} \mathcal{T}\left(\mathbf{P}_i\right)\right) \cdot \mathbf{0} = 0$$

  - *Bounding the Residual Error*: Define $\Delta\mathbf{P} = \max_i \left\|\mathbf{P}_t - \mathbf{P}_i\right\|$ as the maximum distance from $\mathbf{P}_t$ to any $\mathbf{P}_i$. Each residual term satisfies $\mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right) \leq \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$. Since the weights sum to one ($\sum_{i=1}^{L} w_i = 1$), the total error is:

$$\sum_{i=1}^{L} w_i \mathcal{O}\left(\left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2\right) \leq \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right) \sum_{i=1}^{L} w_i = \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$$

This shows that the approximation error is second-order, decreasing quadratically as the neighborhood around $\mathbf{P}_t$ becomes tighter.

  - *Final Result*: Combining the cancellation of the first-order term and the error bound, the approximation simplifies to:

$$\mathbf{SS}_t = \sum_{i=1}^{L} w_i \mathbf{SS}_i + \mathcal{O}\left(\left\|\Delta\mathbf{P}\right\|^2\right)$$

This result shows that $\mathbf{SS}_t$ can be estimated as a convex combination of the known spectra $\{\mathbf{SS}_i\}$, with an approximation error that becomes negligible as $\Delta\mathbf{P} \to 0$.

1.3 Step 3: Quantifying the Interpolation Error

Objective: Derive an upper bound on the interpolation error to assess the approximation accuracy.

Statement: The interpolation error $\epsilon = \left\|\mathbf{SS}_t - \sum_{i=1}^{L} w_i \mathbf{SS}_i\right\|$ satisfies:

$$\epsilon \leq C \cdot \max_i \left\|\mathbf{P}_t - \mathbf{P}_i\right\|^2$$

where the term C is a constant that depends on the second-order derivatives of the mapping function $\mathcal{T}$, and $\mathbf{SS}_t = \mathcal{T}(\mathbf{P}_t)$, $\mathbf{SS}_i = \mathcal{T}(\mathbf{P}_i)$.

Explanation:

- **Context**: The error ϵ arises from truncating the Taylor expansion in Step 2, where $\mathbf{SS}_t$ was approximated using a weighted sum of neighboring signal strengths $\sum_{i=1}^{L} w_i \mathbf{SS}_i$. This step quantifies the loss due to ignoring higher-order terms.
- **Derivation**:
  - From Step 2, the interpolated estimate is $\mathbf{SS}_t = \sum_{i=1}^{L} w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right)$, where $\Delta\mathbf{P} = \max_i \|\mathbf{P}_t - \mathbf{P}_i\|$. The error is expressed as:

$$\epsilon = \left\| \mathbf{SS}_t - \sum_{i=1}^{L} w_i \mathbf{SS}_i \right\| = \left\| \mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right) \right\|$$

  - Residual $\mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right)$ is bounded by the second derivatives of $\mathcal{T}$. By Taylor's theorem, for a twice continuously differentiable function $\mathcal{T}$, there exists a constant $C > 0$:

$$\|\mathcal{T}(\mathbf{P}_t) - \mathcal{T}(\mathbf{P}_i) - \nabla_{\mathbf{P}}\mathcal{T}(\mathbf{P}_i) \cdot (\mathbf{P}_t - \mathbf{P}_i)\| \leq C \|\mathbf{P}_t - \mathbf{P}_i\|^2$$

    where $C = \frac{1}{2} \sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}} \left\| \nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P}) \right\|$ encapsulates the maximum curvature of the mapping $\mathcal{T}$ over the region spanned by neighboring transmitters. In essence, C bounds the acceleration of the spatial spectrum's variation across the interpolation region. Smoother environments (gentler second-order changes) yield smaller C, tightening the error bound $\epsilon \leq C \cdot (\Delta\mathbf{P})^2$. For example, in free-space propagation, C is negligible, while near obstacles (where signal strength fluctuates rapidly), C grows significantly:
    * $\nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P})$ is the Hessian matrix of $\mathcal{T}$ at position $\mathbf{P}$, quantifying how sensitively the spectrum changes with second-order variations in the transmitter coordinates.
    * $\left\| \nabla^2_{\mathbf{P}} \mathcal{T}(\mathbf{P}) \right\|$ denotes the operator norm of the Hessian, a scalar measure of its "magnitude" and thus the maximum rate of curvature in any direction.
    * $\sup_{\mathbf{P} \in \text{conv}\{\mathbf{P}_i\}}$ takes the *supremum* (least upper bound) of the Hessian norm over all points in the convex hull of the neighboring positions $\{\mathbf{P}_i\}$. This represents the worst-case curvature within the local region.
  - Summing over all neighbors' spatial spectra with barycentric weights w_i and noting $\sum_{i=1}^{L} w_i = 1$, the total error satisfies:

$$\epsilon \leq \sum_{i=1}^{L} w_i \cdot C \|\mathbf{P}_t - \mathbf{P}_i\|^2 \leq C \cdot \max_i \|\mathbf{P}_t - \mathbf{P}_i\|^2 .$$

- **Intuition**:
  - **Interpretability of C**: The constant C directly quantifies the worst-case curvature of $\mathcal{T}$ over the interpolation region, as governed by the Hessian $\nabla^2_{\mathbf{P}}\mathcal{T}$. In free-space propagation (low curvature, small C), the bound tightens significantly, whereas near obstacles or multipath-rich environments (high curvature, large C), the error grows predictably. This ties the theoretical guarantee to physically observable phenomena, ensuring the result is not merely abstract but grounded in wave physics.
  - **Scope of the Error Bound**: The derived bound applies specifically to the linear convex combination of neighboring spectra, as defined by barycentric weights. While this provides a foundational guarantee for basic linear interpolation, the GWRF architecture extends beyond this linear regime. By integrating a neural network, GWRF learns non-linear corrections to the weighted spectral average, capturing higher-order correlations and environmental discontinuities (e.g., diffraction, reflection, scattering) that the linear model cannot represent. Thus, the proven bound serves as a baseline for the worst-case error in linear neighbor spatial spectrum interpolation.

2 Connecting to Empirical Spectrum Interpolation in Appendix A

Following the theoretical framework established in Steps 1, 2, and 3, we now connect these steps to the empirical spectrum interpolation method presented in Appendix A, titled "Why Incorporate Neighbors." This appendix demonstrates that the spatial spectrum of a target transmitter can be effectively approximated by interpolating the spectra of its geographically proximate neighbors, leveraging their similarity in propagation behavior. Here, we illustrate how our proof provides a rigorous mathematical foundation for these empirical insights.

2.1 Connection to Step 1: Taylor Expansion under the Mapping $\mathcal{T}$

In Step 1, we model the spatial spectrum as a smooth function under a diffeomorphic mapping $\mathcal{T}$, applying a Taylor expansion to approximate the spectrum at a target position based on a nearby neighbor. This step establishes that small positional changes lead to predictable, smooth variations in the spectrum. This step is formalized as:

$$\mathbf{SS}_j = \mathbf{SS}_i + \nabla_{\mathbf{P}} \mathcal{T}(\mathbf{P}_i) \cdot \Delta\mathbf{P}_{ij} + \frac{1}{2} \Delta\mathbf{P}_{ij}^\top \nabla_{\mathbf{P}}^2 \mathcal{T}(\mathbf{P}_i) \Delta\mathbf{P}_{ij} + \mathcal{O}\left(\|\Delta\mathbf{P}_{ij}\|^3\right)$$

This aligns directly with Appendix A's Observation #1, which shows that neighboring transmitters share similar propagation paths due to their proximity, as evidenced by 80% of reflection point distances being less than 0.1 meters apart. The smoothness of $\mathcal{T}$ justifies this empirical similarity, providing a theoretical basis for why spectra from nearby transmitters can be used to approximate a target spectrum. By proving that the spectrum varies continuously and predictably with position, Step 1 supports the feasibility of the empirical interpolation approach.

2.2 Connection to Step 2: Interpolating the Target Spectrum via Weighted Combination

Step 2 builds on this foundation by developing a weighted combination method to interpolate the target spectrum $\mathbf{SS}_t$ from the spectra of L neighboring transmitters $\{\mathbf{SS}_i\}_{i=1}^L$. Using barycentric weights w_i, where $\sum_{i=1}^L w_i = 1$ and $\mathbf{P}_t = \sum_{i=1}^L w_i \mathbf{P}_i$, the approximation simplifies to:

$$\mathbf{SS}_t = \sum_{i=1}^L w_i \mathbf{SS}_i + \mathcal{O}\left(\|\Delta\mathbf{P}\|^2\right)$$

This step connects to Appendix A's Observation #2, where an empirical interpolation using six neighboring spectra achieves an average PSNR of 20.5 dB, suggesting a notable correlation with the target spectrum. While Appendix A uses a heuristic weighted summation, Step 2 offers a more formal mathematical framework, introducing weights informed by Taylor expansion and geometric relationships. This formulation aims to improve the interpretability and potential accuracy of the interpolation, providing theoretical support for the observed empirical behavior.

3 Connecting to GWRF Framework

In this section, we bridge the theoretical basis established in § 1 with the GWRF architectural design. To clarify, the error bound presented in § 1 applies to linear interpolation. Since GWRF adopts a neural architecture that is inherently non-linear, this bound does not directly apply to GWRF.

3.1 Geometry-Aware Scene Representation (Step 1)

Step 1 establishes that the spectrum function $\mathcal{T}$ is smooth, allowing Taylor expansions to approximate spectra at nearby spatial positions based on positional differences. This smoothness is foundational to GWRF's geometry-aware scene representation, which encodes spatial relationships between a target transmitter and its neighboring points.

- **Transformer Encoder with Cross-Attention**: GWRF utilizes a Transformer encoder to process the spatial spectra of neighboring transmitters, denoted as $\{\mathbf{SS}_i, \mathbf{P}_i\}_{i=1}^L$, alongside positional encodings $\mathcal{E}(\mathbf{P}_{\text{neighbor}} - \mathbf{P}_{\text{target}})$. This attention mechanism captures the relative geometric differences emphasized in Step 1, enabling the model to learn a representation that generalizes across positions. The cross-attention layer dynamically weights each neighbor's contribution based on its proximity to the target transmitter, mirroring the local approximation inherent in the Taylor expansion.

- **Latent Voxel Features**: Each spatial position is represented by a latent voxel vector $\mathbf{v}$, which encapsulates complex wireless signal propagation behaviors (e.g., reflection, diffraction). This rich representation aligns with the higher-order terms of the Taylor expansion, allowing GWRF to model intricate, position-dependent propagation effects while maintaining the smoothness assumption of the function $\mathcal{T}$.

3.2 Neural-Driven Ray Tracing (Step 2)

In Step 2, we interpolate the target spectrum $\mathbf{SS}_t$ as a weighted combination of neighboring spectra, using barycentric weights to form a convex approximation. GWRF operationalizes this concept through its neural-driven ray tracing algorithm, enhancing the interpolation with neural flexibility.

- **Feature Aggregation**: The ray tracing aggregates latent voxel features along propagation paths using a multi-head Transformer. This aggregation process parallels Step 2's weighted combination, where the Transformer's attention mechanism learns a sophisticated, position-sensitive weighting scheme. By synthesizing features across voxels, GWRF interpolates the target spectrum in a way that adapts to the wireless scene environment.
- **Multi-Head Attention for Complex Interactions**: With ten attention heads, GWRF captures diverse interactions among voxel features, such as amplitude and phase variations. This design reflects Step 2's goal of creating a precise yet flexible interpolation, adept at handling the multifaceted nature of wireless signal propagation.

3.3 Error Control (Step 3)

Step 3 provides an error bound for the linear interpolation, expressed as $\epsilon \leq C \cdot \max_i \|\mathbf{P}_t - \mathbf{P}_i\|^2$, showing that accuracy improves when neighbors are closer to the target. GWRF's design inherently enforces this principle through its neighbor selection and training strategies.

- **Proximity-Based Neighbor Selection**: The GWRF framework selects L neighboring transmitters (randomly varying between 3 and 10) based on their Euclidean distance to the target. This encourages small positional differences $\|\mathbf{P}_t - \mathbf{P}_i\|$, aligning with Step 3's insight that, under linear interpolation, the error is bounded by $\epsilon \leq C \cdot (\max_i \|\mathbf{P}_t - \mathbf{P}_i\|)^2$. While this bound formally applies to the linear setting, GWRF extends beyond it: the geometry-aware Transformer and neural-driven ray tracing enable more expressive, data-driven interpolation. These components allow the model to capture nonlinear propagation behaviors and generalize even when neighbors are not extremely close. In dense deployments, proximity helps reduce approximation error, while in sparse settings, the GWRF framework remains robust through its neurally driven learning process.
- **End-to-End Training with ℓ_2 Loss**: The model is optimized using an ℓ_2 loss on the predicted spectrum, directly minimizing the interpolation error ϵ.

References

[1] Joseph B Keller. Geometrical Theory of Diffraction. *Journal of the Optical Society of America*, 52(2):116–130, 1962.

[2] Weng Cho Chew. *Waves and Fields in Inhomogenous Media*. John Wiley & Sons, 1999.