Lecture Notes on

Voting Theory

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Foreword

These lecture notes have been developed for the course *Computational Social Choice* of the Artificial Intelligence MSc programme at the University of Groningen (academic years 2019/20, 2020/21, 2021/22, 2022/23, 2023/24). They cover mathematical and algorithmic aspects of voting theory: the theory of how to take collective decisions by voting.

The course has been inspired by similar courses run by colleagues at the University of Amsterdam (Ulle Endriss) and Technion (Reshef Meir).

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Chapter 1

Choosing One Out of Two

This chapter is concerned with situations in which a group of individuals is faced with the problem of choosing, collectively, one out of two alternatives. It introduces the basic ideas and definitions of standard voting theory and applies them to this binary choice setting. In such context we prove two fundamental theorems of social choice over two alternatives.

1.1 Preliminaries

We start by defining the basic framework of social choice: what a social choice context is; what a social choice function is; and some of the basic properties that one might want such functions to satisfy.

1.1.1 Social choice problems

A social choice problem arises whenever a group needs to take a collective choice, among two or more alternatives, based on the different preferences/opinions/views that the members of the group may have about those alternatives.

Choice contexts The context of such a collective choice, to which we refer as *social choice context*, consists first of all of a set of individuals and a set of alternatives:

- $N = \{1, 2, \ldots, n\}$ is a finite set of n *individuals*, also called agents or voters. So, $|N| = n$. Specific individuals are denoted by natural numbers (1, 2, 3, etc.) and by $i, j, k, \ldots$ when treated as variables.
- $A = \left\{a, b, \ldots, m^{th} letter\right\}$ is a finite set of m *alternatives*, or options. So, $|A| = m$. Specific alternatives are denoted by the initial letters of the alphabet (a, b, c, etc.) and by $x, y, z, \ldots$ when treated as variables.

We will be assuming throughout these notes that $m \geq 2$ and, later in this chapter, we will be focusing on the case $m = 2$.

Preferences and preference profiles We assume that each individual holds a *preference* $\succeq_i$ over the alternatives in A. In these lecture notes we typically assume such preferences to be linear orders[1] over A: $x \succeq_i y$ stands for "x is strictly preferred to y by i, or $x = y$". Intuitively, this means that we assume individuals to be able to rank, without ties, all the alternatives available in the choice context. Occasionally we will also deal with relaxations of this very specific type of individual preference. The set of all linear orders over A is denoted $\mathcal{L}(A)$. Formally, our assumption about individual preferences amounts therefore to require that, for all $i \in N$, $\succeq_i \in \mathcal{L}(A)$. The asymmetric (and thus irreflexive) part of each $\succeq_i$ is denoted $\succ_i$.[2] We will typically denote linear orders simply as sequences, or vectors, of alternatives from the most preferred (left) to the least preferred (right), e.g., xyz.

[1] That is, a binary relations that are transitive, total (or complete), and antisymmetric. It is therefore also reflexive.

[2] Note that $\succeq$ and $\succ$ are identical except for the fact that the first is reflexive and the second is irreflexive.

Collecting all these preferences together defines a so-called *preference profile* (or simply profile) $\mathbf{P} = \langle \succeq_1, \succeq_2, \ldots, \succeq_n \rangle$. That is, a tuple collecting all preferences of the voters in N about the alternatives in A. The set of all possible profiles for the voters in N is denoted $\mathcal{L}(A)^n$, that is, the n^{th} Cartesian power of the set of all possible linear orders over A.

Remark 1.1. We can also conveniently think of a profile $\mathbf{P}$ as a matrix

$$\mathbf{P} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \ldots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \ldots & x_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \ldots & x_{nm} \end{bmatrix},$$

where each entry x_{ij} denotes the alternative in the j^{th} position in the linear order of voter i. Using matrix notation we can write $\mathbf{P}_{ij}$ for such an entry. Similarly $\mathbf{P}_i$ denotes the vector representing the linear order $\succeq_i$ of voter i.

Example 1.1 (A preference profile)**.** Let $N = \{1, 2, 3\}$ and $A = \{a, b\}$. A possible preference profile is the triple $\langle \succeq_1, \succeq_2, \succeq_3 \rangle$ where: $\succeq_1 = ab$, $\succeq_2 = ab$ and $\succeq_3 = ba$. This profile can be represented in tabular form as follows:

1	a	b
2	a	b
3	b	a

Here each row represents a voter, and the linear order representing to the voter's ballot is rendered from the most preferred (left) to the least preferred (right).

A social choice problem, loosely defined, is then the problem of finding a 'suitable' or 'appropriate' subset of alternatives, given the preferences of the individuals about the alternatives in the choice context. Voting theory concerns the many ways in which these subsets of alternatives could be selected according to different criteria. We turn now to the two main concepts underpinning the mathematical theory of voting: social choice functions and axioms.

1.1.2 Social choice functions and voting rules

Social choice functions Given a social choice context, a social choice function describes the way in which a subset of the alternatives (that is, a *social choice*) is selected based on the preferences of individuals in N about the alternatives in A.

Definition 1.1. Let $\langle N, A \rangle$ be given. A *social choice function* or SCF for $\langle N, A \rangle$ is a function

$$f : \mathcal{L}(A)^n \to 2^A \setminus \{\emptyset\}. \tag{1.1}$$

That is, for any profile $\mathbf{P}$, an SCF outputs a non-empty set of alternatives. These are the alternatives constituting the 'social choice' made by the group N via function f, given the expressed preferences in $\mathbf{P}$. If the size of the output of an SCF is larger than 1 then the alternatives it contains can be thought of as 'tied' choices.

Voting rules and ballots We will refer to concrete social choice functions as *voting rules*. In the context of voting rules we will often refer to preferences (linear orders) as *ballots*, which are the preferences that voters reveal to the voting rule. We will therefore be talking also about *ballot profiles*. Sometimes, however, the distinction between the voters' preferences and what the voters reveal to the rule (their ballots) becomes relevant.[3] Also, sometimes the information that the rule requires is not in the format of a linear order,[4] but for now we can treat the terms preference and ballot interchangeably.

Among the best-known, and most-used, voting rules there are: the plurality rule, according to which the social choice consists of the alternatives that occur most often in first position among the preferences of the voters; the majority rule, according to which the social choice consists of the single alternative that occurs in the first position of the preferences of a majority of voters.

Rule 1 (Plurality)**.** The plurality rule is the SCF defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{Plurality}(\mathbf{P}) = \left\{x \in A \ : \ \nexists y \in A, \left|\left\{i \in N \ : \ \max_{\succeq_i}(A) = y\right\}\right| > \left|\left\{i \in N \ : \ \max_{\succeq_i}(A) = x\right\}\right|\right\}.$$

We will also refer to $|\{i \in N \ : \ \max_{\succeq_i}(A) = x\}|$ as the *plurality score* of alternative x in profile $\mathbf{P}$ and we will sometimes denote it by $\texttt{Plurality}(\mathbf{P})(x)$.

So, according to plurality, the social choice for a profile contains all alternatives for which there is no other alternative that occurs as top preference more often in the profile. Notice that this allows for ties in the social choice.

Rule 2. The majority rule is the SCF defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{Majority}(\mathbf{P}) = \begin{cases} \{x\} & \text{if } |\{i \in N \ : \ \max_{\succeq_i}(A) = x\}| \geq \lceil \frac{n+1}{2} \rceil, \\ A & \text{otherwise.} \end{cases}$$

So, an alternative is selected by the majority rule if its plurality score is larger than $\frac{n}{2}$. If no such majority-supported alternative exists, majority returns a tie among all alternatives.

Example 1.2. Applying the above rules to the profile $\langle \succeq_1, \succeq_2, \succeq_3 \rangle$ in Example 1.1 gives us

$$\texttt{Plurality}(\langle \succeq_1, \succeq_2, \succeq_3 \rangle) = \texttt{Majority}(\langle \succeq_1, \succeq_2, \succeq_3 \rangle) = \{a\}.$$

The single winner is therefore alternative a according to both rules.

A useful benchmark for the further exploration of the space of voting rules are so-called dictatorships.

Rule 3 (Dictatorship)**.** The dictatorship of a given $i \in N$ is the SCF defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{Dictatorship}_i(\mathbf{P}) = \left\{\max_{\succeq_i}(A)\right\}.$$

Dictatorships are therefore those functions whose choice, in every profile, equals the top alternative of one same voter: the dictator.

1.1.3 Basic axioms

Asking what is the best way to aggregate the preferences of a group in order to reach a social choice amounts to asking whether we can identify a 'best' social choice function f. For each $\langle N, A \rangle$ there are many social choice functions in the framework we just defined: $(2^m - 1)^{(m!)^n}$. So even just for the social choice context of Example 1.1 there are $(2^3 - 1)^{2^3} = 2562890625$ such functions, among which plurality, majority and dictatorships are just some of them.

A standard way to get a grip on this space is by imposing properties on the SCF f that we would like it to satisfy: so-called axioms. Axioms are meant to capture properties of SCFs that are considered

[3] This will be the case in Chapter 3.
[4] We will encounter such a rule in Chapter 2.

$$
\begin{tikzcd}
\mathbf{P} \arrow[r, "f"] \arrow[d, "\rho"] & f(\mathbf{P}) \\
\pi(\mathbf{P}) \arrow[ur, "f"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{P} \arrow[r, "f"] \arrow[d, "\rho"] & f(\mathbf{P}) \arrow[d, "\rho"] \\
\rho(\mathbf{P}) \arrow[r, "f"] & f(\rho(\mathbf{P}))
\end{tikzcd}
$$

Figure 1.1: Illustration of anonymity (left) and neutrality (right).

desirable. Ultimately, one would like to identify a set of such desirable properties that, all together, uniquely *characterize* f. Each of the axioms reported below captures a property of SCFs that may be considered desirable, in at least some contexts of collective decision-making. Some may be considered more intuitive than others, but no axiom can really be thought of as unquestionable.

The first group of axioms concerns some basic requirements roughly inspired by an idea of 'democratic voting' in which all voters and all alternatives are to be treated equally.

Definition 1.2 (Equal treatment axioms)**.** Let $\langle N, A\rangle$ be given. An SCF f is:

Anonymous iff for every permutation $\pi : N \to N$ and $\mathbf{P} \in \mathcal{L}(A)^n$,

$$f(\underbrace{\langle \succeq_1, \succeq_2, \ldots, \succeq_n\rangle}_{\mathbf{P}}) = f\left(\left\langle \succeq_{\pi(1)}, \succeq_{\pi(2)}, \ldots, \succeq_{\pi(n)}\right\rangle\right).$$

Intuitively, f treats all voters equally, as the only thing that matters for f is who holds which preference.

Non-dictatorial iff there exists no $i \in N$ s.t. $\forall \mathbf{P} \in \mathcal{L}(A)^n$, $f(\mathbf{P}) = \{\max_{\succeq_i}(A)\}$. It is said to be dictatorial otherwise, with i being the *dictator*.

Intuitively, there exists no voter that can unilaterally determine the social choice.

Neutral if for every permutation $\rho : A \to A$ and $\mathbf{P} \in \mathcal{L}(A)^n$,

$$\rho(f(\underbrace{\langle \succeq_1, \succeq_2, \ldots, \succeq_n\rangle}_{\mathbf{P}})) = f(\langle \rho(\succeq_1), \rho(\succeq_2), \ldots, \rho_i(\succeq_n)\rangle)$$

where for $\succeq_i = x_1 x_2 \ldots x_n$, $\rho(\succeq_i) = \rho(x_1)\rho(x_2)\ldots\rho(x_m)$.

Intuitively, f treats all alternatives equally, as the only thing that matters for f is the position that an alternative gets in each individual preference, and not its name.

Figure 1.1 provides a graphical illustration of the properties of anonymity and neutrality.

The next group of axioms concerns the ability of an SCF to respond appropriately to growing support for alternatives. They rule out questionable SCFs such as 'choose x whenever the number of voters supporting x is odd', or SCFs that allow for too many ties, such as, 'choose all alternatives supported by at least one third of the voters'. Before introducing the axioms we define one extra piece of notation. Given a profile $\mathbf{P}$ and two distinct alternatives x and y, $N^{xy}_{\mathbf{P}} = \{i \in N \ : \ x \succ_i y\}$ denotes the set of voters that strictly prefer x to y in $\mathbf{P}$. Throughout these notes we will also write $\#^{xy}_{\mathbf{P}}$ instead of $|N^{xy}_{\mathbf{P}}|$ to denote the number of voters that strictly prefer x to y in $\mathbf{P}$.

Definition 1.3 (Responsiveness axioms)**.** Let $\langle N, A\rangle$ be given. An SCF f is:

Pareto iff for all $\mathbf{P} \in \mathcal{L}(A)^n$ and for all $x \in A$, if there exists $y \in A$ s.t. for all $i \in N$, $y \succ_i x$, then

$x \notin f(\mathbf{P})$.

Intuitively, the social choice cannot contain dominated alternatives, that is, alternatives for which the unanimity of voters prefers a different alternative.

Unanimous iff for all $\mathbf{P} \in \mathcal{L}(A)^n$ and $x \in A$, if for all $i \in N$, $\max_{\succeq_i}(A) = x$, then $f(\mathbf{P}) = \{x\}$.

Intuitively, whenever an alternative is the top choice of all voters, that alternative is the unique social choice.

Monotonic iff for all $\mathbf{P} \in \mathcal{L}(A)^n$ and $x \in A$, if $x \in f(\mathbf{P})$ then $x \in f(\mathbf{P'})$, where:

- $\mathbf{P'} \in \mathcal{L}(A)^n$
- $N^{xy}_{\mathbf{P}} \subseteq N^{xy}_{\mathbf{P'}}$, for all $y \in A \setminus \{x\}$ (i.e., x improves for some voters in $\mathbf{P'}$)
- $N^{yz}_{\mathbf{P}} = N^{yz}_{\mathbf{P'}}$, for all $y, z \in A \setminus \{x\}$ (i.e., the order of other alternatives is preserved by $\mathbf{P'}$)

Intuitively, if an alternative is a social choice it remains so whenever some voters decide to rank it higher in their ballot, while leaving the relative ranking of the other alternatives unchanged.

Strongly monotonic iff for all $\mathbf{P} \in \mathcal{L}(A)^n$ and $x \in A$, if $x \in f(\mathbf{P})$ then $x \in f(\mathbf{P'})$, where:

- $\mathbf{P'} \in \mathcal{L}(A)^n$
- $N^{xy}_{\mathbf{P}} \subseteq N^{xy}_{\mathbf{P'}}$, for all $y \in A \setminus \{x\}$ (i.e., x improves for some voters in $\mathbf{P'}$)

Intuitively, if an alternative is a social choice it remains so whenever some voters decide to rank it higher in their ballot, while possibly also changing the relative ranking of the other alternatives.

Positively responsive iff for all $x \in A$ and $\mathbf{P} \in \mathcal{L}(A)^n$, if $x \in f(\mathbf{P})$ then $f(\mathbf{P'}) = \{x\}$ where:

- $\mathbf{P} \neq \mathbf{P'} \in \mathcal{L}(A)^n$
- $N^{xy}_{\mathbf{P}} \subseteq N^{xy}_{\mathbf{P'}}$, for all $y \in A \setminus \{x\}$
- $N^{yz}_{\mathbf{P}} = N^{yz}_{\mathbf{P'}}$, for all $y, z \in A \setminus \{x\}$

Intuitively, if an alternative is selected as a social choice in a profile, it becomes the only social choice in those profiles in which some voter ranks x higher.

It is worth to pause and observe how monotonicity is strengthened in two different directions by the properties of strong monotonicity and positive responsiveness.

The axioms in the last group concern the ability of an SCF to output a single social choice, therefore ruling out ties, and its ability to effectively consider the whole space of possible preferences in the voters' population.

Definition 1.4 (Imposition and resoluteness axioms)**.** Let $\langle N, A\rangle$ be given. An SCF f is:

Non-imposed iff for all $x \in A$ there exists $\mathbf{P} \in \mathcal{L}(A)^n$ s.t. $f(\mathbf{P}) = \{x\}$. It is said to be *imposed* otherwise.

Intuitively, every alternative is selected as unique social choice for at least one profile.

Resolute iff for all $\mathbf{P} \in \mathcal{L}(A)^n$, $f(\mathbf{P})$ is a singleton. It is said to be *irresolute* otherwise.

Intuitively, the social choice consists of only one alternative.

An SCF is imposed whenever there exists an alternative that can never be the social choice. It is irresolute when it admits ties.

Some immediate consequences can be drawn from the above definitions. If f is anonymous, then whenever $\mathbf{P}$ and $\mathbf{P'}$ are such that, for all distinct alternatives x and y, $\#^{xy}_{\mathbf{P}} = \#^{xy}_{\mathbf{P'}}$, it follows that

$f(\mathbf{P}) = f(\mathbf{P}')$. That is, for anonymous SCFs the only information that matters in a profile is the size of support for each alternative against each other alternative. As a direct consequence dictatorships are not anonymous. We state below some more simple observations concerning the axioms we just stated:

Fact 1.1. Let $\langle N, A\rangle$ be given, and let f be an SCF.

a) If f is positively responsive or strongly monotonic, then it is monotonic.

b) If f is non-imposed and strongly monotonic, then it is unanimous.

c) If f is Pareto, then it is unanimous.

d) If f is dictatorial, then it is unanimous.

e) If f is unanimous, then it is non-imposed.

Proof. Exercise 1.1 □

1.2 Plurality is the best ... when $m = 2$

We discuss now the plurality rule in the context of social choice problems where $m = 2$. Observe that in such cases the plurality rule (Rule 1) simplifies as follows. For all $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\begin{aligned}
\texttt{Plurality}(\mathbf{P}) &= \left\{x \in A \;\middle|\; \nexists y \in A, \left|\left\{i \in N \;:\; \max_{\succeq_i}(A) = y\right\}\right| > \left|\left\{i \in N \;:\; \max_{\succeq_i}(A) = x\right\}\right|\right\} \\
&= \{x \in A \mid \nexists y \in A, \#^{yx}_{\mathbf{P}} > \#^{xy}_{\mathbf{P}}\} \\
&= \begin{cases} \{x\} & \text{if } \#^{xy}_{\mathbf{P}} \geq \lceil \frac{n+1}{2} \rceil \\ \{y\} & \text{if } \#^{yx}_{\mathbf{P}} \geq \lceil \frac{n+1}{2} \rceil \\ A & \text{otherwise} \end{cases} \qquad (1.2) \\
&= \texttt{Majority}(\mathbf{P})
\end{aligned}$$

where y is the second element of A. So, in the case of two alternatives the plurality rule (Rule 1) and the majority rule (Rule 2) define the same social choice function. We refer to such function, as defined by Equation (1.3) as *simple majority rule.*

In this section we will also be working with the following generalization of the simple majority rule, under the assumption that $m = 2$.

> **Rule 4** (Quota)**.** Let A be such that $m = 2$. The quota rule $\texttt{Quota}_q$, where q (the quota) is an integer such that $1 \leq q \leq n+1$, is an SCF defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:
>
> $$\texttt{Quota}_q(\mathbf{P}) = \begin{cases} \{x \in A \mid |\{i \in N \;:\; \max_{\succeq_i}(A) = x\}| \geq q\} & \text{if such set is non-empty,} \\ A & \text{otherwise.} \end{cases}$$

Observe that simple majority (1.3) is, therefore, the quota rule where $q = \lceil \frac{n+1}{2} \rceil$. More generally, quota rules select an alternative as social choice whenever the support for that alternative reaches a given quota. Ties can arise when the quota are below a majority quota, or when the quota is impossible to be met (*e.g.*, $q = n+1$).

Fact 1.2. Let A be such that $m = 2$. If $\frac{n}{2} < q \leq n+1$ then:

$$\texttt{Quota}_q(\mathbf{P}) = \begin{cases} \{x\} & \text{if } \#^{xy}_{\mathbf{P}} \geq q \\ \{y\} & \text{if } \#^{yx}_{\mathbf{P}} \geq q \\ A & \text{otherwise} \end{cases} \qquad (1.3)$$

Proof. It suffices to observe that by the definition of Rule 4, for every profile $\mathbf{P}$ and alternatives $x, y \in A$, if $\frac{n}{2} < q \leq n$ then $x \in \texttt{Quota}_q(\mathbf{P})$ implies $y \notin \texttt{Quota}_q(\mathbf{P})$. □

$$\mathbf{P} = \begin{array}{c|cc} 1 & x & y \\ 2 & x & y \\ 3 & y & x \end{array} \qquad \mathbf{P}' = \begin{array}{c|cc} 1 & y & x \\ 2 & y & x \\ 3 & x & y \end{array} \qquad \mathbf{P}'' = \begin{array}{c|cc} 1 & x & y \\ 2 & y & x \\ 3 & y & x \end{array}$$

Figure 1.2: Illustration of the application of permutations to the sets of alternatives and agents in the proof of Theorem 1.1.

1.2.1 Axiomatic characterizations of simple majority

The following three theorems provide an axiomatic justification for the use of simple majority voting (Rules 1 and 2) in social choice contexts involving only two options.

Theorem 1.1 (May's theorem [May, 1952])**.** Let $\langle N, A\rangle$ be given such that n is odd and $m = 2$. The plurality rule is the only SCF that is resolute, anonymous, neutral and monotonic.

Proof. $\boxed{\text{Left-to-right}}$ Assuming that f is plurality, we need to show that f is resolute, anonymous, neutral and monotonic. See Exercise 1.2 $\boxed{\text{Right-to-left}}$ Let f be resolute, anonymous, neutral and monotonic. We need to show that $f = \texttt{Plurality}$. Observe first of all that since $m = 2$ there are only two possible ballots: xy or yx. Furthermore, since f is resolute there are only two possible outcomes: $\{x\}$ or $\{y\}$. Let us proceed towards a contradiction, and assume that $f \neq \texttt{Plurality}$. Then there exists a profile $\mathbf{P}$ where f returns a choice different from what `Plurality` returns. There are two possible cases: either $\#^{xy}_{\mathbf{P}} > \#^{yx}_{\mathbf{P}}$, or $\#^{yx}_{\mathbf{P}} > \#^{xy}_{\mathbf{P}}$.

$\boxed{\#^{xy}_{\mathbf{P}} > \#^{yx}_{\mathbf{P}}}$ By assumption, $f(\mathbf{P}) = \{y\}$ and $\texttt{Plurality}(\mathbf{P}) = \{x\}$. Now permute the alternatives to obtain a profile $\mathbf{P}'$ where $\#^{xy}_{\mathbf{P}} = \#^{yx}_{\mathbf{P}'}$ (and therefore $\#^{yx}_{\mathbf{P}} = \#^{xy}_{\mathbf{P}'}$). Then, by the neutrality of f, it follows that $f(\mathbf{P}') = \{x\}$. It is useful to notice that $\mathbf{P}$ differs from $\mathbf{P}'$ in having more voters preferring x to y. By permuting the agents in N we can then construct out of $\mathbf{P}'$ a new profile $\mathbf{P}''$ such that $N^{xy}_{\mathbf{P}''} \subseteq N^{xy}_{\mathbf{P}}$. That is, we make sure that the same agents ranking x above y in $\mathbf{P}''$ also rank x above y in $\mathbf{P}$.[5] By anonymity $f(\mathbf{P}'') = \{x\}$ and by monotonicity $f(\mathbf{P}) = \{x\}$. This is a contradiction and we can therefore conclude that $f(\mathbf{P}) = \{x\}$. Figure 1.2 gives an illustration of the type of profiles involved in the argument.

$\boxed{\#^{yx}_{\mathbf{P}} > \#^{xy}_{\mathbf{P}}}$ By assumption, $f(\mathbf{P}) = \{x\}$ and $\texttt{Plurality}(\mathbf{P}) = \{y\}$. An argument identical to the one for the previous case applies to conclude $f(\mathbf{P}) = \{y\}$. We thus obtain a contradiction in this case too.

In both cases we obtain a contradiction. Function f is therefore the plurality rule (1.3). □

At a high level the proof exploits this argument. Since $m = 2$ and n is odd, f can choose either the minority or the majority alternative. If it chooses the minority alternative, then by anonymity and neutrality the minority alternative should be chosen in all profiles that split N in the same way size-wise (i.e., the cell with ballot xy vs. the cell with ballot yx, or the cell with ballot yx vs. the cell with ballot xy). However, consistently selecting the minority option goes against monotonicity. Hence f must select the majority alternative.

We now look at generalizations of Theorem 1.1. First of all, the theorem assumes an odd number of voters. When that is not the case, it is reasonable for the SCF to output both alternatives (a tie). The theorem can be generalized to this setting.

Theorem 1.2 (May's theorem with ties [May, 1952])**.** Let $\langle N, A\rangle$ be given such that $m = 2$. The plurality rule is the only SCF that is anonymous, neutral and positively responsive.

Proof. See Exercise 1.3. □

[5] Note that this is precisely what is required by the second condition in the definition of monotonicity. Note furthermore that as we are handling only two alternatives, the third condition of the monotonicity axiom is trivially satisfied.

The theorem can then be further generalized to the class of quota rules.

Theorem 1.3 (May's theorem for quota rules)**.** Let $\langle N, A\rangle$ be given such that $m = 2$. Let f be an SCF that is anonymous, neutral and monotonic. Then there exists $q \in (\frac{n}{2}, n+1]$ such that $f = \texttt{Quota}_q$.

Proof. See Exercise 1.4. □

Observe that this latter theorem allows us to obtain Theorem 1.1 as a corollary.

1.2.2 Simple majority as maximum likelihood estimator

Theorem 1.1 above provides a justification of the use of simple majority in contexts involving two alternatives based on the fact that such a rule is the only one satisfying a set of arguably desirable axioms. There is also a second very prominent route to the justification of simple majority in the binary choice context of this chapter, based on its ability to correctly identify the 'right' alternative between the two, when such right alternative exists. We turn to it in this section.

Maximum likelihood estimation

Assume that there exists an objective *true* ranking $\succeq$ of the two alternatives: either $\succeq = xy$ or $\succeq = yx$.[6] Voters are uncertain about which one of the two is the true ranking, but each voter can recognize the correct option with some given probability $0.5 < p \leq 1$, which is the same for all voters. Such p represents the voters' individual accuracy and corresponds to the conditional probability $\mathbb{P}(x \succ_i y | \succeq = xy)$. That is, the probability that i ranks x above y given that x is (actually) better than y.

So, voters observe the state of the world imperfectly and vote accordingly to their imperfect observation. We can then think of each voter's ballot $\succeq_i$ as a realization from distribution p, once the true state of the world $\succeq$ has been fixed. Assuming each such vote is independent, a profile is thus a series of identically distributed independent random variables that can take two values (xy or yx).[7] The most likely state of the world is therefore the one that has the highest likelihood of generating the observed profile $\mathbf{P}$, that is: $\text{argmax}_{\succeq \in \{xy, yx\}} \mathbb{P}(\mathbf{P} | \succeq)$.

Example 1.3. Assume $N = \{1, 2, 3\}$ and $A = \{a, b\}$. Assume furthermore that $p = \frac{2}{3}$. Voters can express one of two ballots (ab or ba), and assume that the profile we observe is $\mathbf{P} = \langle ab, ab, ba\rangle$.

$$\mathbb{P}(\mathbf{P}|ab) = p^2 \cdot (1 - p),$$

while

$$\mathbb{P}(\mathbf{P}|ba) = p \cdot (1 - p)^2.$$

It follows that ranking ab is $\frac{\mathbb{P}(\mathbf{P}|ab)}{\mathbb{P}(\mathbf{P}|ba)} = \frac{p}{1-p} = 2$ times as likely as ba to have generated profile $\mathbf{P}$. The most likely state is therefore ranking ab.

Remark 1.2. If we assume that the alternatives are equally likely (i.e., that there is an equal prior for xy and yx), then the above approach is essentially equal to establishing the probability of each

[6] You can think for instance of x and y as two policies, of which only one is the truly better one for the group. Or you can thing of x and y as two candidates for a job, for which only one is the objectively better fit.

[7] You can think of a profile as a sequence of n coin tosses, where each coin corresponds to a voter, and the coin used is a biased coin with bias p towards the correct option.

state of the world given the observed profile. Continuing on the previous example, by Bayes rule:

$$\begin{aligned}\mathbb{P}(ab|\mathbf{P}) &= \frac{\mathbb{P}(\mathbf{P}|ab)\cdot\mathbb{P}(ab)}{\mathbb{P}(\mathbf{P}|ab)\cdot\mathbb{P}(ab)+\mathbb{P}(\mathbf{P}|ba)\cdot\mathbb{P}(ba)}\\ &= \frac{p^2\cdot(1-p)\cdot 0.5}{p^2\cdot(1-p)\cdot 0.5+((1-p)^2\cdot p)\cdot 0.5}\\ &= \frac{p^2\cdot(1-p)}{p^2\cdot(1-p)+(1-p)^2\cdot p} = \frac{p\cdot(1-p)}{p(1-p)+(1-p)^2} = p.\end{aligned}$$

This is the probability that ab is the correct state of the world, given the observation of profile $\mathbf{P}$.

The jury theorem

Can the maximum likelihood estimation approach described above be implemented through by SCF, the idea being that voters, as 'jurors', vote in order to track the true state of the world? We want, in other words, that the SCF selects the most likely ranking. It turns out that this is possible, and again such SCF is the plurality/simple majority rule.

Theorem 1.4 (Condorcet's jury theorem [Condorcet, 1785])**.** Fix A such that $m = 2$. Assume furthermore that $0.5 < p \leq 1$ and that, for any N such that n is odd, each profile for $\langle N, A\rangle$ is an i.i.d. sequence of random variables $\succeq_i$ generated by p. Then:

$$p_{\texttt{Majority}}(n) \leq p_{\texttt{Majority}}(n+2) \quad \text{for every odd } n \tag{1.4}$$

$$p \leq p_{\texttt{Majority}}(n) \quad \text{for every odd } n \tag{1.5}$$

$$\lim_{n\to\infty} p_{\texttt{Majority}}(n) = 1 \tag{1.6}$$

where $p_{\texttt{Majority}}(n)$ denotes the accuracy of the social choice for $\langle N, A\rangle$ determined via plurality.

Proof. First of all observe that a decision taken by plurality is correct if and only if a majority of voters has voted correctly (1.3), that is, whenever a number $h \geq \frac{n+1}{2}$ of voters votes correctly. There are $\sum_{h=\frac{n+1}{2}}^{n}\binom{n}{h}$ such majorities. For each of these, the probability that precisely that majority of h voters votes correctly is $p^h \cdot (1-p)^{n-h}$. We thus obtain that:

$$p_{\texttt{Majority}}(n) = \sum_{\frac{n+1}{2}=h}^{n}\binom{n}{h}\cdot p^h\cdot(1-p)^{n-h}. \tag{1.7}$$

We proceed to prove the three claims of the theorem.

$\boxed{(1.4)}$ The claim is a consequence of the following recursive formula:

$$p_{\texttt{Majority}}(n+2) = p_{\texttt{Majority}}(n) + \underbrace{(2p-1)\cdot\binom{n}{\frac{n+1}{2}}\cdot(p\cdot(1-p))^{\frac{n+1}{2}}}_{\phi}. \tag{1.8}$$

Given the assumption $0.5 < p \leq 1$, it is easy to see that $\phi \geq 0$ (and that $\phi > 0$ when in addition $p \neq 1$), from which we can conclude that $p_{\texttt{Majority}}(n+2) > p_{\texttt{Majority}}(n)$ as desired. We now proceed to derive (1.8) by a combinatorial argument. We are focusing on profiles of length $n+2$. It is now helpful to think of $p_{\texttt{Majority}}(n+2)$ as the probability that more than half of these $n+2$ votes in the profile are correct, and of $p_{\texttt{Majority}}(n)$ as the probability that more than half of the first n votes are correct. Call the corresponding events H_{n+2} and H_n. So: $p_{\texttt{Majority}}(n+2) = \mathbb{P}(H_{n+2})$ and $p_{\texttt{Majority}}(n) = \mathbb{P}(H_n)$ Consider now the two following events:

- $H_{n+2} \setminus H_n$ denotes the event that more than half of the first $n+2$ votes, but less than half of the first n votes, are correct;

- $H_{n+2} \cap H_n$ denotes the event that more than half of the first $n+2$ votes and more than half of the first n votes are correct.

Using these events and the standard laws of probability, we can rewrite $p_{\texttt{Majority}}(n+2)$ as follows:

$$\begin{aligned}
p_{\texttt{Majority}}(n+2) &= \mathbb{P}(H_{n+2}) \\
&= \mathbb{P}(H_{n+2} \setminus H_n) + \mathbb{P}(H_{n+2} \cap H_n) \\
&= \mathbb{P}(H_{n+2} \setminus H_n) + \mathbb{P}(H_n) - \mathbb{P}(H_n \setminus H_{n+2}) \\
&= p_{\texttt{Majority}}(n) + \underbrace{\mathbb{P}(H_{n+2} \setminus H_n)}_{\alpha} - \underbrace{\mathbb{P}(H_n \setminus H_{n+2})}_{\beta}. \qquad (1.9)
\end{aligned}$$

We proceed to establish α and β. $\boxed{\alpha}$ Event $H_{n+2} \setminus H_n$ occurs if the first n votes contain a narrow incorrect majority (the majority is decided by one voter) and the two subsequent votes are both correct. The first event happens with probability $\binom{n}{\frac{n+1}{2}} \cdot p^{\frac{n-1}{2}} \cdot (1-p)^{\frac{n+1}{2}}$ and the second with probability p^2. We thus obtain:

$$\begin{aligned}
\alpha &= \binom{n}{\frac{n+1}{2}} \cdot p^{\frac{n-1}{2}} \cdot (1-p)^{\frac{n+1}{2}} \cdot p^2 \\
&= p \cdot \binom{n}{\frac{n+1}{2}} \cdot (p \cdot (1-p))^{\frac{n+1}{2}}. \qquad (1.10)
\end{aligned}$$

$\boxed{\beta}$ In the same fashion, event $H_n \setminus H_{n+2}$ occurs if the first n votes contain a correct majority but the first $n+2$ do not, implying the last two votes are incorrect. The first event happens with probability $\binom{n}{\frac{n+1}{2}} \cdot p^{\frac{n+1}{2}} \cdot (1-p)^{\frac{n-1}{2}}$ and the second with probability $(1-p)^2$. We thus obtain:

$$\begin{aligned}
\beta &= \binom{n}{\frac{n+1}{2}} \cdot p^{\frac{n+1}{2}} \cdot (1-p)^{\frac{n-1}{2}} \cdot (1-p)^2 \\
&= (1-p) \cdot \binom{n}{\frac{n+1}{2}} \cdot (p \cdot (1-p))^{\frac{n+1}{2}}. \qquad (1.11)
\end{aligned}$$

We can now replace (1.10) and (1.11) in (1.9) and obtain (1.8) as desired:

$$\begin{aligned}
p_{\texttt{Majority}}(n+2) &= p_{\texttt{Majority}}(n) + p \cdot \binom{n}{\frac{n+1}{2}} \cdot (p \cdot (1-p))^{\frac{n+1}{2}} - (1-p) \cdot \binom{n}{\frac{n+1}{2}} \cdot (p \cdot (1-p))^{\frac{n+1}{2}} \\
&= p_{\texttt{Majority}}(n) + (2p-1) \cdot \binom{n}{\frac{n+1}{2}} \cdot (p \cdot (1-p))^{\frac{n+1}{2}}.
\end{aligned}$$

$\boxed{(1.5)}$ The claim follows directly from (1.4) and the observation that $p = p_{\texttt{Majority}}(1)$.

$\boxed{(1.6)}$ By the strong law of large numbers as n grows to infinity the average number of correct votes converges almost surely to p. As $p > 0.5$ it follows that the probability of a correct majority as n grows to infinity is 1. $\square$

The theorem is probably the first mathematical formalization of a of 'wisdom of the crowd' effect where decision making by the group outperforms that of each individual taken in isolation. It states that groups become more accurate as they grow larger (1.4), that they are more accurate that single individuals (1.5), and that infinite groups achieve perfect accuracy (1.6).

1.3 Chapter notes

The bulk of this chapter is based on introductions to voting theory provided in [Brandt et al., 2016, Ch. 2], [Taylor, 2005, Ch. 1] and [Endriss, 2011]. The Pareto principle was first formulated in [Pareto, 1919].

Theorem 1.1 was first presented in [May, 1952]. It is a beautiful example of the application of the axiomatic method to voting. Theorem 1.4 appeared first in [Condorcet, 1785] and was rediscovered several times during the last century (e.g., [Moore and Shannon, 1956]). Even though the theorem is a well-known result, it is somewhat hard to find detailed proofs of its non-asymptotic part. The proof presented here is based on the one reported in [Dietrich and Spiekermann, 2020]. The theorem has later been generalized in various directions since at least the 80s, trying to lift some of its most restrictive assumptions, such as: the homogeneity of voters' competence; voters' independence. See, for instance, [Grofman et al., 1983, Häggström et al., 2006].

1.4 Exercises

Exercise 1.1. Provide a proof of Fact 1.1

Exercise 1.2. Provide a proof of the $\boxed{\text{Left-to-right}}$ direction of Theorem 1.1.

Exercise 1.3. Provide a proof of Theorem 1.2.

Exercise 1.4. Provide a proof of Theorem 1.3. $\boxed{\text{Hint}}$ You want to show there exists a number q that has two properties: x is the only social choice if at least q voters rank x above y; and $q \in (\frac{n}{2}, n+1]$. Now collect in a set Q all numbers k that have the property that if k voters rank x over y then $\{x\}$ is the social choice. Q can be empty, or not. Reason from there.

Exercise 1.5 (Odd rule)**.** Consider the following rule.

Rule 5. Let $A = \{a, b\}$. The odd rule is the SCF defined as follows. For every $\mathbf{P}$:

$$\mathtt{Odd}(\mathbf{P}) = \begin{cases} \{a\} & \text{if } \#^{ab}_{\mathbf{P}} \text{ is odd} \\ \{b\} & \text{otherwise} \end{cases}$$

Intuitively, the rule chooses a whenever the size of the set of voters supporting it is odd, otherwise it chooses b. Determine whether the odd rule is: anonymous, neutral, unanimous, positively responsive, non-imposed, resolute. Explain your answers.

Exercise 1.6 (Iterated voting when $m = 2$)**.** Consider a series of choice contexts

$$\langle N_1, A_1 \rangle, \langle N_2, A_2 \rangle, \ldots, \langle N_k, A_k \rangle,$$

with k large (e.g., $k \geq 2^n$) and such that for every $1 \leq i < k$, $N_i = N_{i+1}$, $A_i = A_{i+1}$ and such that $m = 2$. For any series $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$ of profiles, a social choice function f would thus determine the series of social choices $f(\mathbf{P}_1), f(\mathbf{P}_2), \ldots, f(\mathbf{P}_k)$. According to your intuition, would plurality be a 'fair' way of determining such series of social choices? Justify your answer and if you think the answer is negative devise a voting rule that would behave more 'fairly' according to your insights. Again, explain your answer.

Exercise 1.7. Consider Theorem 1.4 and determine how its statement should be modified, for all its parts (1.4), (1.5) and (1.6), once the assumption $0.5 < p \leq 1$ is changed to $p = 0.5$ (first variant), and to $0 \leq p < 0.5$ (second variant). State these two variants of the theorem.

Chapter 2

Choosing One Out of Many

This chapter extends the binary social choice context of the previous chapter to situations in which a group needs to select one out of three or more alternatives, that is, when $m > 2$. We will see how this generalization introduces a wealth of challenges to our understanding of what a 'best' rule would be in such a context.

2.1 Beyond plurality

The previous chapter has singled out plurality/majority (Rules 1 and 2) as the go-to voting rule when $m = 2$, at least if we care about equality of voters and alternatives (May's theorem), or if we care about correctly estimating the correct alternative between the two available (Condorcet's jury theorem).

When more than two alternatives are available, majority may easily result in the inability to reach a decision, because no single alternative may reach the majority threshold. Plurality does away with this issue, but we will see how its choices may appear objectionable. In fact, the classic contributions to social choice theory sparked precisely from the dissatisfaction of how the plurality rule behaves when more than two alternatives are up for choice.

2.1.1 Plurality selects unpopular options

The two following examples illustrate well the most apparent shortcoming of the plurality rule: the complete disregard of all the preference information beyond the individuals' top choices.

Example 2.1 (Roman Senate, 1 century b.C.)**.** " ...the consul Africanus Dexter had been found slain, and it was uncertain whether it had died at his own hand or at those of his freedmen. When the matter came before the Roman Senate, Pliny wished to acquit them [alternative a], another senator moved that they should be banished to an island [alternative b]; and a third that they should be put to death [alternative c]" [Ordeshoek, 2003, p. 54].

$$\begin{array}{r|ccc} \#102 & a & b & c \\ \#101 & b & a & c \\ \#100 & c & b & a \end{array}$$

The standard voting practice in the Roman Senate would have consisted of a staged procedure: first a vote on a (are the freedmen innocent, and therefore to-be-aquitted, or are they guilty?), which would have been lost; second a vote on b vs. c, which would have been won by b. Pliny, who was chairing the session, favoured a, and so demanded a vote by plurality. Plurality would select a as unique social choice which, however, is hugely unpopular: it corresponds to about $\frac{1}{3}$ of the votes of all the senators; and there is one alternative, namely b, who is preferred to both a and both c by a strong majority of senators.

Example 2.2 (2002 US Presidential elections, stylized [Dasgupta and Maskin, 2004])**.** The example assumes only four kind of voters and abstracts away from the electoral college procedure in use for the election of the US president. It assumes there are 100000 voters (rows indicate multiple of 1000).

#2	Nader	Gore	Bush	Buchanan
#49	Gore	Bush	Nader	Buchanan
#48	Bush	Buchanan	Gore	Nader
#1	Buchanan	Bush	Gore	Nader

Again plurality would elect Gore, who is receiving a minority of votes. At the same time Bush appears to occupy higher positions than Gore in the voters' rankings.

2.1.2 Looking for better voting rules

What seems to be the root of plurality's problems in convincingly aggregating preferences over more than two alternatives is that it disregards all information provided by voters, except for their top choices. In this section we introduce a number of rules that, using different strategies, try to put that information at work to identify a suitable social choice.

Pairwise comparisons

One piece of information that appears to be relevant in the above examples is how often an alternative x 'beats' another alternative y in a pairwise contest, that is, whether there exists a larger support for x over y than there is for y over x. We can easily tabulate such information as a *pairwise comparison* $m \times m$ matrix $[\#^{xy}_{\mathbf{P}}]$ with $x, y \in A$. Clearly, for $x \neq y$, $\#^{xy}_{\mathbf{P}} = n - \#^{yx}_{\mathbf{P}}$ and, for any $x \in A$, $\#^{xx} = 0$.[1]

Example 2.3. The pairwise comparisons matrix for the election described in Example 2.2 is:

	Buchanan	Bush	Gore	Nader
Buchanan	0	1	49	49
Bush	99	0	49	98
Gore	51	51	0	98
Nader	51	2	2	0

Based on such information we can then identify which alternatives are considered better than which other alternatives in pairwise comparisons and thereby possibly identifying the 'best' alternatives.

Definition 2.1 (Net preference)**.** Let $\mathbf{P}$ be a profile for context $\langle N, A \rangle$. For any pair of distinct alternatives xy, the *net preference* for x over y in $\mathbf{P}$ is

$$Net_{\mathbf{P}}(xy) = \#^{xy}_{\mathbf{P}} - \#^{yx}_{\mathbf{P}}.$$

Note that $Net_{\mathbf{P}}(xy) = \#^{xy}_{\mathbf{P}} - \#^{yx}_{\mathbf{P}} = 2 \cdot \#^{xy}_{\mathbf{P}} - n$.

Definition 2.2 (Pairwise majority tournament)**.** The pairwise majority tournament (or *majority graph*) of $\mathbf{P}$ is the graph $\left\langle A, \succ^{Net}_{\mathbf{P}} \right\rangle$ where $\succ^{Net}_{\mathbf{P}} \subset A^2$ is a binary relation such that $x \succ^{Net}_{\mathbf{P}} y$ iff $Net_{\mathbf{P}}(xy) > 0$. The weighted pairwise majority tournament (or *weighted majority graph*) is the graph $\left\langle A, \succ^{Net}_{\mathbf{P}}, Net_{\mathbf{P}} \right\rangle$ where each edge $xy \in \succ^{Net}_{\mathbf{P}}$ is labeled by $Net_{\mathbf{P}}(xy)$. The *weak majority tournament*, allowing for ties, can be naturally defined as: $x \succeq^{Net}_{\mathbf{P}} y$ iff $Net_{\mathbf{P}}(xy) \geq Net_{\mathbf{P}}(yx)$.

When the profile is clear from the context we will omit reference to it and write simply $Net(xy)$, $\succ^{Net}$ and $\succeq^{Net}$.

[1] This method of representing relevant election information was known already to R. Llull [Llull, 1275] in the 13th century.

Given a majority tournament, it appears therefore natural to single out as social choice the alternative that beats any other alternative in the tournament.

Definition 2.3 (Condorcet winner)**.** Let $\left\langle A, \succ^{Net} \right\rangle$ be the majority graph of $\mathbf{P}$. An alternative $x \in A$ is the *Condorcet winner* of $\mathbf{P}$ if for all $y \in A \setminus \{x\}$, $x \succ^{Net}_{\mathbf{P}} y$. It is a *weak Condorcet winner* of $\mathbf{P}$ if for all $y \in A \setminus \{x\}$, $x \succeq^{Net}_{\mathbf{P}} y$.

So, a Condorcet winner is an alternative that beats every other alternative in a pairwise comparison—in graph-theoretical terms it is a 'source' in the majority graph. As such, it can be considered the natural social choice, at least if we care about majorities. Notice that relation $\succ^{Net}_{\mathbf{P}}$ is irreflexive and complete when n is odd, for any profile $\mathbf{P}$. However, it is not transitive in general. It follows that a Condorcet winner may not exist.

Example 2.4. The majority graph of the profile in Example 2.1 is:

$$b \succ^{Net} a \succ^{Net} c.$$

Alternative b is therefore the Condorcet winner of the profile. Then, the majority graph of the profile in Example 2.2 is:

$$\text{Gore} \succ^{Net} \text{Bush} \succ^{Net} \text{Nader} \succ^{Net} \text{Buchanan}$$

And Gore is therefore the Condorcet winner of that profile.

Example 2.5 (Condorcet paradox [Condorcet, 1785])**.** Three instantiations of the paradox for $\langle \{1, 2, 3\}, \{a, b, c\} \rangle$ (left), $\langle \{1, \dots, 303\}, \{a, b, c\} \rangle$ (center), $\langle \{1, 2, 3\}, \{a, b, c, d\} \rangle$ (right):

$$\mathbf{P}_1 = \begin{array}{r|ccc} 1 & a & b & c \\ 2 & b & c & a \\ 3 & c & a & b \end{array} \qquad \mathbf{P}_2 = \begin{array}{r|ccc} 102\times & a & b & c \\ 101\times & b & c & a \\ 100\times & c & a & b \end{array} \qquad \mathbf{P}_3 = \begin{array}{r|cccc} 1 & a & b & c & d \\ 2 & b & c & a & d \\ 3 & c & a & b & d \end{array}$$

The pairwise majority tournament in all the above profiles contains a 3-cycle $a \succ^{Net} b \succ^{Net} c \succ^{Net} a$. These are also called *majority cycles* and can arise whenever $m > 2$.

Finally, it may be worth knowing that any tournament (i.e., irreflexive, complete binary relation) can be generated via Definition 2.2 by some profile, under the assumption that n is odd (McGarvey's theorem [McGarvey, 1953]).

Rules based on pairwise comparisons

We introduce now two rules based on (unweighted) majority graphs.

Rule 6 (Condorcet)**.** The Condorcet rule is the SCF defined as follows. For every profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{Condorcet}(\mathbf{P}) = \begin{cases} \{x\} & \text{if } \forall y \in A \setminus \{x\}, x \succ^{Net}_{\mathbf{P}} y \\ A & \text{otherwise} \end{cases}$$

So, the Condorcet rule selects a Condorcet winner when it exists and lets otherwise all alternatives tie.

Rule 7 (Copeland [Llull, 1275, Copeland, 1951])**.** The Copeland rule is the SCF defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{Copeland}(\mathbf{P}) = \operatorname*{argmax}_{x \in A} C_{\mathbf{P}}(x)$$

where

$$C_{\mathbf{P}}(x) = \left|\left\{y \in A \ : \ x \succ^{Net}_{\mathbf{P}} y\right\}\right| - \left|\left\{y \in A \ : \ y \succ^{Net}_{\mathbf{P}} x\right\}\right|.$$

This value is called the *Copeland score* of x (in $\mathbf{P}$).

So the Copeland score consists of the number of times an alternative beats other alternatives, minus the number of times it is beaten by other alternatives (in the majority graph of the corresponding profile). The social choice is then the set of alternatives with the highest Copeland score. Notice that the rule disregards the margins whereby an alternative beats, or is beaten by, other alternatives: the Copeland score can be computed by just looking at the majority graph of a profile, disregarding the net preference information.

Example 2.6. Consider again Example 2.5. We have that: $\texttt{Condorcet}(\mathbf{P}_1) = \texttt{Copeland}(\mathbf{P}_2) = \{a, b, c\}$ but $\texttt{Condorcet}(\mathbf{P}_3) = \{a, b, c, d\}$ while $\texttt{Copeland}(\mathbf{P}_3) = \{a, b, c\}$.

All the above rules work on the basis of the majority graph of the profile and therefore disregard the relative positions of alternatives in the voters' rankings. This type of information is processed by the next class of rules we are going to discuss: positional scoring rules.

Remark 2.1. Another voting rule based on (unweighted) majority graphs is the Slater rule [Slater, 1961]: given the majority graph $\left\langle A, \succ^{Net}\right\rangle$ find the strict linear order $\succ$ that minimizes the number of edges one needs to switch in $\succ^{Net}$ to get $\succ$. The social choice is the singleton consisting of the top alternative in $\succ$.

Remark 2.2. There are however also voting rules based on pairwise comparisons that use the weighted majority graph. An elegant example is the Ranked-pairs rule: given the weighted majority graph $\left\langle A, \succ^{Net}, Net\right\rangle$, order the edges of relation $\succ^{Net}$ by the size of their weight (largest to smallest), then construct a linear order over A by starting from the edge with the largest weight and proceeding to add edges following their weights unless a cycle is formed. The social choice is the top alternative in the constructed order.

Positional scoring rules

Positional scoring rules are a class of SCFs that determine the social choice by assigning points to alternatives depending on how often the alternative occupies a given position in the voters' ballots. The social choice is the alternative collecting the most points.

Definition 2.4 (Positional scoring rules)**.** A positional scoring rule f is an SCF such that there exists a vector of weights (*score vector*) $\mathbf{w} = \langle w_1, w_2, \ldots, w_m\rangle$ and, for all $\mathbf{P} \in \mathcal{L}(A)^n$:

$$f(\mathbf{P}) = \operatorname*{argmax}_{x \in A} \sum_{i \in N} w_{i(x)},$$

where $i(x)$ denotes x's position in the linear order $\succeq_i$.

Observe that plurality (Rule 1) can also be considered a scoring rule, with score vector $\left\langle 1, \underbrace{0, \ldots, 0}_{m \text{ times}}\right\rangle$.

The most famous, and most widely deployed, positional scoring rule is the Borda rule.[2]

[2]The method carries the name of Jean-Charles de Borda, French mathematician and engineer [J.-C., 1781]. The method already appeared, however, in the writings of Nicholas of Cusa in the 15th century.

Rule 8 (Borda)**.** The Borda rule `Borda` is the positional scoring rule with score vector

$$\mathbf{b} = \langle m-1, m-2, \ldots, 0\rangle .$$

For every alternative x and profile $\mathbf{P}$, $\sum_{i\in N} b_{i(x)}$ is called the *Borda score* of x (in $\mathbf{P}$).

In the Borda method an alternative x collects, for each voter, as many points as the number of alternatives the voter places under x in her ballot. Such points are then summed up across voters to obtain the Borda score of the alternative in the given profile.

Example 2.7 (Borda vs. Condorcet)**.** Consider again the profile in Example 2.2. We have seen that the Condorcet winner in that profile is Gore (Example 2.4). However, the Borda rule selects Bush:

$$\sum_{i\in N} b_{i(\text{Bush})} = 3\cdot 48 + 2\cdot 50 + 1\cdot 2 \qquad \sum_{i\in N} b_{i(\text{Gore})} = 3\cdot 49 + 2\cdot 2 + 49$$

$$\sum_{i\in N} b_{i(\text{Nader})} = 3\cdot 2 + 1\cdot 49 \qquad \sum_{i\in N} b_{i(\text{Buchanan})} = 2\cdot 48$$

with the Borda score vector is $\mathbf{b} = \langle 3, 2, 1, 0\rangle$.

Example 2.8 (Borda vs. Condorcet (simpler))**.** Consider this other profile:

$$\mathbf{P} = \begin{array}{r|ccc} \#15 & a & b & c \\ \#10 & b & c & a \end{array}$$

We have $\texttt{Plurality}(\mathbf{P}) = \texttt{Condorcet}(\mathbf{P}) = \{a\}$ but $\texttt{Borda}(\mathbf{P}) = \{b\}$ with the following scores: $15 \times 2 + 0 = 30$ for a and $10 \times 2 + 15 \times 1 = 35$ for b.

The examples show that the Borda rule may fail to select a Condorcet winner when it exists. This feature (or bug?) can be shown to generalize to all positional scoring rules.

Other examples of positional scoring rules follow:

Rule 9 (Veto)**.** The veto rule `Veto` is the positional scoring rule with score vector $\Big\langle \underbrace{1, \ldots, 1}_{m-1 \text{ times}}, 0\Big\rangle$.

Rule 10 (k-Approval)**.** The k-Approval rule k`Approval`, with $1 \leq k \leq m$, is the positional scoring rule with score vector $\Big\langle \underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{m-k \text{ times}} \Big\rangle$.

Observe that, by the above definition, plurality is therefore the 1-Approval rule, and veto is the $m-1$-Approval rule.

Remark 2.3. k-Approval determines the social choice only based on the top-k alternatives of voters, and assigns identical scores to all such alternatives. This amounts to asking voters for only the set of such top k alternatives, rather than a linear order. Such non-ranked ballots are called *approval ballots* and k-Approval insists that such ballots should always contain exactly k alternatives. The voting rule known as *approval voting* [Brams, 2004] generalizes k-Approval by giving voters the freedom to express as approval ballot any subset $X \subseteq A$ of the alternatives. When working with such ballots, profiles are not tuples of linear orders any more, but tuples of subsets of alternatives: $\mathbf{X} = \langle X_1, \ldots, X_n\rangle \in (2^A)^n$.

With the previous observation in mind, we can provide the definition of approval voting in its full generality:

Rule 11 (Approval)**.** The Approval rule `Approval`, is defined as, for any approval ballot profile $\mathbf{X}$:

$$\texttt{Approval}(\mathbf{X}) = \underset{x \in A}{\text{argmax}} \,|\, \{i \in N \ : \ x \in X_i\} \,|.$$

So, `Approval` selects the alternatives that are approved by most voters and k`Approval` does exactly the same but restricting approval ballots to having size k.

An area where positional scoring rules are widely applied are sport or music competitions, such as the F1 championship or the Eurovision Song Contest competition. For example, the winner of the F1 championship is determined via a positional scoring rule based on the ranking of each pilot in a fixed number of races. Notice that each race here acts as a voter: it determines a linear order over the pilots. Over the years various score vectors have been used for the computation of winners, for example: in the 1961-1990 seasons the vector $\langle 9, 6, 4, 3, 2, 1, 0, \ldots, 0\rangle$ was used; in the 1991-2002 seasons, the vector $\langle 10, 6, 4, 3, 2, 1, 0, \ldots, 0\rangle$; and in the 2003-2008 seasons, the vector $\langle 10, 8, 6, 5, 4, 3, 2, 1, 0, \ldots, 0\rangle$. In song competitions, like the Eurovision Song Contest, juries (and the public) act as voters. The score vector of the Eurovision Song Context is currently $\langle 12, 10, 8, 7, 6, 5, 4, 3, 2, 1, 0, \ldots, 0\rangle$. For more information about the design of scoring rules for competitions see [Baumeister and Hogrebe, 2019].

Multiround rules

This third class of rules is based on the idea that ballots can be processed in an iterated manner by removing, at each round in the process, the 'less popular' alternatives from the ballots, until one alternative achieves majority. Before introducing the rules in this class we need some auxiliary notation. Let $\mathbf{P}$ be a profile and $X \subseteq A$ a set of alternatives. Then the profile restricted to X is $\mathbf{P}|_X = \langle \succeq_1 |_X, \ldots, \succeq_n |_X\rangle$, where each $\succeq_i |_X$ is the restriction of $\succeq_i$ to the alternatives in X.

A very simple implementation of the above multiround idea is the following rule.

Rule 12 (Plurality with run-off)**.** The plurality with run-off rule is the SCF defined as follows, for any profile $\mathbf{P}$:

$$\texttt{Plurality}^{RO}(\mathbf{P}) = \begin{cases} \{x\} & \text{if } |\{i \in N \ : \ \max_{\succeq_i}(A) = x\}| \geq \lceil \frac{n+1}{2} \rceil, \\ \texttt{Plurality}(\mathbf{P}|_{\mathit{Top2}}) & \text{otherwise.} \end{cases}$$

Here, $\mathit{Top2} \subseteq A$ denotes the set of the 2 alternatives in A with the two highest plurality scores in $\mathbf{P}$.

Example 2.9 (Non-monotonicity)**.** Consider the two profiles

$$\mathbf{P} = \begin{array}{c|ccc} \#8 & a & b & c \\ \#10 & c & a & b \\ \#7 & b & c & a \end{array} \qquad\qquad \mathbf{P}' = \begin{array}{c|ccc} \#6 & a & b & c \\ \#12 & c & a & b \\ \#7 & b & c & a \end{array}$$

We have that $\texttt{Plurality}^{RO}(\mathbf{P}) = \{c\}$ (b is eliminated in the first round, and c wins in the second round against a by a net preference of $17 - 8$) but $\texttt{Plurality}^{RO}(\mathbf{P}') = \{b\}$ (a is eliminated in the first round, and b wins in the second round against c by a net preference of $13 - 12$) even though in $\mathbf{P}'$ 2 voters have ranked c higher than in $\mathbf{P}$.

Consider now the function $\sigma_{\mathbf{P}} : 2^A \to 2^A$ defined as follows:

$$\sigma_{\mathbf{P}}(X) = X \setminus \{x \in A \ : \ x \text{ has lowest plurality score in } \mathbf{P}|_X\}\,. \tag{2.1}$$

That is, given a set of alternatives X, $\sigma_{\mathbf{P}}(X)$ removes from X the alternatives that are ranked highest by the smallest number of voters. For $k \in \mathbb{N}$ we denote by $\sigma^k_{\mathbf{P}}$ the k^{th}-fold iteration of $\sigma_{\mathbf{P}}$.

Rule 13 (Single transferable vote)**.** The single transferable vote rule (STV) is the SCF defined as follows, for any profile $\mathbf{P}$:

$$\mathtt{STV}(\mathbf{P}) = \sigma_{\mathbf{P}}^{k^\star - 1}(A),$$

where k^* is the step at which all alternatives are removed: $\sigma_{\mathbf{P}}^{k^\star}(A) = \emptyset$.

Intuitively, STV iteratively removes the alternatives with lowest plurality score until all alternatives are removed. So STV is still based on the plurality score, but it extracts as much information as possible from the ballot profile, thereby reducing the amount of votes 'wasted'. By means of comparison, in a standard plurality election normally more than half the votes would be 'wasted' in the sense of supporting losers of the election. STV obviates to this problem by iteratively removing plurality losers and recomputing the plurality score based on the preferential ballots. So the alternatives that are removed last constitute the social choice, which therefore consists of either one alternative with a majority support, or by alternatives with equal plurality score.[3] The STV rule goes also under other names, e.g., Hare rule [Hare, 1859],[4] instant run-off voting, ranked choice voting.[5]

2.1.3 More axioms for SCFs

We have seen in the previous section that a variety of rules, driven by different logics, can be devised for the setting when $m > 3$. It is now time to introduce a few more axioms, besides the ones we encountered in the previous chapter, that will help assess such rules.

Definition 2.5. Let a social choice context $\langle N, A\rangle$ be given. An SCF f is:

Condorcet-consistent iff for all $\mathbf{P} \in \mathcal{L}(A)^n$, if x is the Condorcet winner of $\mathbf{P}$, then $f(\mathbf{P}) = \{x\}$.

Intuitively, the rule agrees with the Condorcet rule on all profiles that give rise to transitive majority graphs.

Independent iff for any two profiles $\mathbf{P}, \mathbf{P}' \in \mathcal{L}(A)^n$ and two alternatives $x, y \in A$, if $N^{xy}_{\mathbf{P}} = N^{xy}_{\mathbf{P}'}$ and $x \in f(\mathbf{P})$ but $y \notin f(\mathbf{P})$ then $y \notin f(\mathbf{P}')$.

Intuitively, whether one alternative belongs to the social choice while the other does not depends only on how voters rank the two alternatives, and not on how they rank other alternatives.

Liberal iff for all $i \in N$ there exists $x \neq y \in A$ s.t. for every profile $\mathbf{P} \in \mathcal{L}(A)$, $i \in N^{xy}_{\mathbf{P}}$ implies $y \notin f(\mathbf{P})$ and $i \in N^{yx}_{\mathbf{P}}$ implies $x \notin f(\mathbf{P})$. Each such agent i is said to be *two-way decisive* on x and y.

Intuitively, each agent should be able to unilaterally veto, for at least two alternatives, whether the alternative is part of the social choice.

These axioms help us understand the different behaviors of various voting rules and allow us to systematize the space of such rules. See Exercise 2.3 for an attempt in this direction.

2.1.4 Impossibility results for SCFs: examples

As Exercise 2.3 (and Table 2.2) make evident, the rules we have considered so far satisfy some of the desirable properties—the axioms—we came up with, but none satisfies them all. The question then arises

[3] Several variants to this definition exist, mostly focusing on what to do when ties exist among the alternatives with lowest scores.

[4] In 1862 John Stuart Mill referred to it as "among the greates improvements yet made in the theory and practice of government". It is popular among electoral reform groups and it is widely deployed to elect representatives (among others in Ireland, Northern Ireland, Australia).

[5] It should however be noted that we dealt here with STV as a single-winner voting rule. It was in fact designed originally as a multi-winner election method (see Chapter 4).

of whether such 'ideal' social choice functions exist at all. The key message of the next section is that this is not the case, and the means to show that is through so-called *impossibility* results: theorems stating the inexistence of functions with a given set of properties. Before moving on to the next section and discuss the most influential among the impossibility results of social choice theory, we illustrate here the 'style' of such results via two simpler examples.

Fact 2.1. Let $\langle N, A \rangle$ be given such that $n = m = 2$. There exists no SCF that is anonymous, neutral and resolute.

Theorem 2.1 (Sen's impossibility of a Paretian liberal [Sen, 1970])**.** Let $\langle N, A \rangle$ be given such that $m > 2$ and $n \geq 2$. There exists no SCF that is Pareto (recall Definition 1.3) and liberal.

Proof. See Exercise 2.4.[6] □

2.2 There is no obvious social choice function when $m > 2$

In this section we present and prove what is arguably the most famous mathematical results in social choice theory.[7]

2.2.1 Social preference functions

The rules discussed in the previous section not only identify a social choice for each profile, but actually determine a weak 'social preference' over the set of alternatives. Such a social preference should be interpreted as a form of 'contingency plan': society should choose the top-ranked alternative in the social preference, but if this turns out to be unfeasible, the second alternative should be selected and so on. So, voting rules can be viewed as *social preference functions*[8] mapping profiles of individual linear orders to total preorders, which are linear orders admitting ties:[9]

$$F : \mathcal{L}(A)^n \to \mathcal{TP}(A). \tag{2.2}$$

where $\mathcal{TP}(A)$ denotes the set of all total preorders on the set of alternatives. So, for any profile $\mathbf{P}$, we refer to $F(\mathbf{P}) = \succeq^F_{\mathbf{P}}$ as the social preference determined by F in profile $\mathbf{P}$, and to $\succ^F_{\mathbf{P}}$ as its asymmetric (strict) part.

It is worth observing that SCFs and SPFs are closely related:

Remark 2.4 (Correspondence between SCFs and SPFs)**.** Let $\langle N, A \rangle$ be given.

a) For every SPF F, $\max_{F(\cdot)}(A)$ is the SCF that selects, for any profile $\mathbf{P} \in \mathcal{L}(A)^n$, the top alternatives in $\succeq^F_{\mathbf{P}}$.

b) For every SCF f, the linear order

$$f(\mathbf{P}) \succ f(\mathbf{P}|_{A \setminus \{f(A)\}}) \succ f(\mathbf{P}_{f(\mathbf{P}|_{A \setminus \{f(A)\}})}) \succ \ldots.$$

defines a social preference function $\succeq^F_{\mathbf{P}}$ where the (possibly tied) top-ranked alternatives are the

[6] This is also known as Sen's paradox and is considered an important insight in economics and political theory. For this and several other contributions, Sen obtained the Nobel prize for Economic Sciences in 1998.

[7] The Nobel prize for economics winner Paul Samuelson once wrote: "The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for a chimera, for a logical self-contradiction. [...] Now scholars all over the world-in mathematics, politics, philosophy and economics are trying to salvage what can be salvaged from Arrow's devastating discovery that is to mathematical politics what Kurt Gödel's 1931 impossibility-of-proving-consistency theorem is to mathematical logic" (cited from [Gardner, 1970, p. 120]).

[8] A more common terminology is 'social welfare function'.

[9] Technically, a total preorder is a reflexive, transitive and total binary relation.

alternatives in $f(\mathbf{P})$, the tied second-ranked alternatives are the alternatives in $f(\mathbf{P}|_{A\setminus\{f(A)\}})$ and so on. Observe that the length of $\succeq^F_{\mathbf{P}}$ is at most $m-1$ when f is resolute.

Intuitively, the social choice corresponds to the best alternatives in such a social preference. Vice versa, the social preference can be induced by iterated social choices on sets of alternatives from which the winners of the previous social choice have been removed.

It is worth observing right away that May's and Condorcet's theorems (Theorems 1.1 and 1.4) for the setting $m = 2$ can be straigthorwardly formulated for the setting of SPFs. In general, the axioms we studied so far can all be adapted to the SPF setting. Here we present three, that will be used for the impossibility result discussed later in this section.

Definition 2.6. Let $\langle N, A\rangle$ be given. A SPF F is:

Non-dictatorial iff there exists no $i \in N$ s.t. $\forall \mathbf{P} \in \mathcal{L}(A)^n$, $F(\mathbf{P}) = \succeq_i$. Otherwise f is said to be dictatorial with i being *the dictator*.

Intuitively, there exists no voter whose ballot is always identical to the social preference.

Pareto iff for all profile $\mathbf{P} \in \mathcal{L}(A)^n$ whenever $N^{xy}_{\mathbf{P}} = N$, $x \succ^F_{\mathbf{P}} y$.

Intuitively, if everybody agrees on the relative ranking of two alternatives that ranking should be part of the social preference.

Independent of irrelevant alternatives (IIA) iff for any two profiles $\mathbf{P}, \mathbf{P}' \in \mathcal{L}(A)$ and two alternatives $x, y \in A$, if $N^{xy}_{\mathbf{P}} = N^{xy}_{\mathbf{P}'}$ then $x \succeq^F_{\mathbf{P}} y$ iff $x \succeq^F_{\mathbf{P}'} y$

Intuitively, if two profiles are identical with respect to the relative ranking of two alternatives, then the relative ranking of the two alternatives in the the social preferences for those two profiles is the same.

The following proposition shows how properties such as the above ones may interact in ways that may be unexpected at first. The proposition will also be of importance in the discussion of Arrow's theorem in the next section.

Proposition 2.1. Let F be a SPF for a $\langle N, A\rangle$ s.t. $m > 2$. If F satisfies Pareto and IIA, then F always outputs a linear order.

Proof. Assume towards a contradiction that there exists a profile $\mathbf{P}$ in which x and y are tied in $F(\mathbf{P})$—$x \sim^F_{\mathbf{P}} y$. Now consider a new profile $\mathbf{P}'$ such that $N^{xy}_{\mathbf{P}} = N^{xz}_{\mathbf{P}'} \cap N^{zy}_{\mathbf{P}'}$ and $N^{yx}_{\mathbf{P}} = N^{zy}_{\mathbf{P}'} \cap N^{zx}_{\mathbf{P}'}$. That is, in $\mathbf{P}'$ the voters that were ranking x over y now place an alternative z between x and y, while the voters that were ranking y over x now place an alternative z above both y and x. By IIA we get x and y tied in $F(\mathbf{P}')$—$x \sim^F_{\mathbf{P}'} y$—and by Pareto we get $z \succ^F_{\mathbf{P}'} y$ and therefore $z \succ^F_{\mathbf{P}'} x$.

Now consider a different profile $\mathbf{P}''$ where we choose a different placement for z: $N^{xy}_{\mathbf{P}} = N^{xz}_{\mathbf{P}''} \cap N^{yz}_{\mathbf{P}''}$ and $N^{yx}_{\mathbf{P}} = N^{yz}_{\mathbf{P}''} \cap N^{zx}_{\mathbf{P}''}$. That is, in $\mathbf{P}''$ the voters that were ranking x over y now place the alternative z below both x and y, while the voters that were ranking y over x now place an alternative z between the two. By IIA we get again $x \sim^F_{\mathbf{P}''} y$ and by Pareto we get $y \succ^F_{\mathbf{P}''} z$ and therefore $x \succ^F_{\mathbf{P}''} z$. However $N^{xz}_{\mathbf{P}'} = N^{xz}_{\mathbf{P}''}$, but $z \succ^F_{\mathbf{P}'} x$ and $x \succ^F_{\mathbf{P}''} z$, which violates IIA. Contradiction. $\square$

The lemma tells us that SPFs that are Pareto and IIA, when there are at least three alternatives, map profiles of linear orders to linear orders: they do not allow for ties in the social preference.

2.2.2 Arrow's theorem

Arrow's theorem establishes that it is impossible to aggregate the preferences of a finite set of voters while respecting Pareto, independence of irrelevant alternatives and non-dictatorship. The three axioms are inconsistent.[10]

Theorem 2.2 (Arrow's theorem [Arrow, 1950, Arrow, 1963]). Let F be a SPF for a $\langle N, A \rangle$ s.t. $m > 2$. F satisfies Pareto and IIA if and only if it is a dictatorship.

Another way to look at the theorem is that of a characterization of the dictatorship rule, exactly like May's theorem is a characterization of the plurality rule in the $m = 2$ context: dictatorships are the only SPFs that are Pareto and IIA. In a way the theorem shows that there is no 'obvious' method—in the sense of being independent of irrelevant alternatives and Pareto—to aggregate preferences on more than 2 alternatives.

Proof strategy and decisive coalitions

The proof of the theorem presented here relies on one main definition and three lemmas. The central definition in the proof of Arrow's theorem is that of a decisive coalition of voters.

Definition 2.7 (Decisive coalition). Let F be a SPF (for a given $\langle N, A \rangle$), and $x, y \in A$. A coalition $C \subseteq N$ is *decisive for x over y* (or xy-decisive), under F, if

$$\forall \mathbf{P} \in \mathcal{L}(A)^n : \text{ if } C \subseteq N^{xy}_{\mathbf{P}} \text{ then } x \succ^F_{\mathbf{P}} y.$$

A coalition $C \subseteq N$ is *decisive* if it is decisive for every pair of alternatives. The set of xy-decisive coalitions (under F) is denoted $\mathcal{D}^{xy}_F$ (or simply $\mathcal{D}^{xy}$ when F is clear from the context). The set of decisive coalitions (under F) is denoted $\mathcal{D}$.

Intuitively, a set of voters is decisive for x over y whenever, if they all agree with ranking x over y, then x is ranked over y in the social preference.

The proof then relies on three main lemmas. The first lemma shows that if a coalition is decisive for a pair of alternatives, then it is decisive for all alternatives (a.k.a. contagion lemma).

The second one shows that if SPFs are Pareto and IIA, then their set of decisive coalitions $\mathcal{D}$ takes the form of a so-called ultrafilter. Ultrafilters are collections of sets and were originally introduced in [Cartan, 1937] to capture a handful of properties characterizing the notion of 'large set', like: i) the largest set is a large set; ii) a set is large iff its complement is not large; iii) the intersection of two large sets is large.

The third lemma is a fact about ultrafilters constructed on finite sets (like the set of voters N), and it states that every finite ultrafilter contains a singleton. It follows that if $\mathcal{D}$ is an ultrafilter, it must be finite since N is finite, and it must therefore contain a singleton decisive coalition, that is, a dictator. The intuition behind the use of ultrafilters in voting theory is that a notion of 'large set' can naturally be used to define SFPs responding to the rough intuition: issue x is ranked above y in the social preference if and only if there is a 'large set' of individuals—a 'large coalition'—supporting it.

The three lemmas

Lemma 2.1 (Contagion lemma). Let F be a SPF (for a given $\langle N, A \rangle$ with $m > 2$) which is Pareto and IIA. If $C \in \mathcal{D}^{xy}$ for some $x, y \in A$, then $C \in \mathcal{D}$.

Proof. We want to show that if $C \in \mathcal{D}^{xy}$ for some $x, y \in A$ then $C \in \mathcal{D}^{wz}$ for all $w, z \in A$. To do that we identify a profile where the fact that C decides on the relative ranking of x and y determines, by transitivity, also the relative ranking of w and z. Notice that we are assuming $m > 2$ so w or z may be equal to x or y. We provide the case in which the four alternative are distinct, the proof for the other cases with only three alternatives can be obtained in a similar fashion. So, consider any profile $\mathbf{P}$ where

[10] The enormous impact that this theorem had in social choice, economics and political theory in general was well summarized by Paul Samuelson (like Arrow, a winner of the Nobel Memorial Prize in Economic Science):"Arrow's devastating discovery is to mathematical politics what Kurt Gödel's 1931 impossibility-of-proving-consistency theorem is to mathematical logic".

each voter i in C holds preferences satisfying $w \succ_i x \succ_i y \succ_i z$ and any other voters j holds preferences satisfying $w \succ_j x$ and $y \succ_j z$:

$$\mathbf{P} = \begin{array}{c|ccccccccc} C & \dots & w & \dots & x & \dots & y & \dots & z & \dots \\ \overline{C} & & \dots & w & \dots & x & \dots & & & \\ & & \dots & y & \dots & z & \dots & & & \end{array}$$

Notice that this specification of $\mathbf{P}$ leaves it open how voters in $\overline{C}$ rank w w.r.t. z and x w.r.t y. Now $C \subseteq N^{wz}_{\mathbf{P}}$ and, since C is xy decisive, we therefore have that $x \succ^F_{\mathbf{P}} y$ and, by Pareto, we have that $w \succ^F_{\mathbf{P}} x$ and $y \succ^F_{\mathbf{P}} z$. By transitivity of the social preference function we thus have that $w \succ^F_{\mathbf{P}} z$. By the specification we gave for $\mathbf{P}$, there exists a profile $\mathbf{P}'$ such that $N^{wz}_{\mathbf{P}} = N^{wz}_{\mathbf{P}'}$. Consider any such profile $\mathbf{P}'$ and let $C \subseteq N^{wz}_{\mathbf{P}'}$. By IIA, $w \succ^F_{\mathbf{P}'} z$. We can then conclude that $C \in \mathcal{D}^{wz}$, as desired. $\square$

Lemma 2.2 (Ultrafilter lemma)**.** Let F be a SPF (for a given $\langle N, A\rangle$), that satisfies Pareto and IIA. The set $\mathcal{D}$ of decisive coalitions (for F) is an *ultrafilter* over N, that is:

i) $N \in \mathcal{D}$, i.e., the set of all individuals is a decisive coalition;

ii) $C \in \mathcal{D}$ iff $\overline{C} \notin \mathcal{D}$, i.e., a coalition is decisive if and only if its complement is not;

iii) $\mathcal{D}$ is closed under finite intersections: if $C, D \in \mathcal{D}$ then $C \cap D \in \mathcal{D}$, i.e., if two coalitions are winning then the individuals they have in common form a winning coalition.

Proof. i) The claim is a direct consequence of the assumption that F satisfies Pareto.

ii) $\boxed{\text{Left-to-right}}$ Suppose, towards a contradiction, that $C, \overline{C} \in \mathcal{D}$. Consider now a profile $\mathbf{P}$ where $C = N^{xy}_{\mathbf{P}}$ and $\overline{C} = N^{yx}_{\mathbf{P}}$. This profile must exist as SPFs admit any profile in $\mathcal{L}(A)^n$ as input, and be such that $xy, yx \in F(\mathbf{P})$. But $F(\mathbf{P})$ must be a linear order by Proposition 2.1. Contradiction.

$\boxed{\text{Right-to-left}}$ Assume $\overline{C} \notin \mathcal{D}$. Then there exists a pair of (distinct) alternatives yx such that $\overline{C} \notin \mathcal{D}^{yx}$. So assume, for $x \neq y \in A$, that $\overline{C} \notin \mathcal{D}^{yx}$. Then, by Definition 2.7, there exists a profile $\mathbf{P}$ such that $\overline{C} \subseteq N^{yx}_{\mathbf{P}}$ and $x \succ^F_{\mathbf{P}} y$. Such profile can be depicted as

$$\mathbf{P} = \begin{array}{c|ccccc} C' & \dots & x & \dots & y & \dots \\ C'' & \dots & y & \dots & x & \dots \\ \overline{C} & \dots & y & \dots & x & \dots \end{array}$$

where $C = C' \cup C''$ and C'' may be empty. Now consider the following variant of $\mathbf{P}$ constructed by placing a third alternative z (which must be available as $m > 2$) in different positions in the individual ballots:

$$\mathbf{P}' = \begin{array}{c|ccccccc} C' & \dots & x & \dots & y & \dots & z & \dots \\ C'' & \dots & y & \dots & z & \dots & x & \dots \\ \overline{C} & \dots & y & \dots & z & \dots & x & \dots \end{array}$$

The social preference $F(\mathbf{P}')$ for this profile is such that: $x \succ^F_{\mathbf{P}'} y$ by IIA; $y \succ^F_{\mathbf{P}'} z$ by Pareto; $x \succ^F_{\mathbf{P}'} z$ by transitivity of the social preference. By IIA, in any profile $\mathbf{P}''$ such that $N^{xz}_{\mathbf{P}'} = N^{xz}_{\mathbf{P}''}$ we have that $x \succ^F_{\mathbf{P}''} z$. It follows that $C' \in \mathcal{D}^{xz}$. Since, by construction, $C' \subseteq C$ we also obtain $C \in \mathcal{D}^{xz}$. Finally, by Lemma 2.1, we conclude $C \in \mathcal{D}$ as desired.

iii) Assume towards a contradiction that $C, D \in \mathcal{D}$ and $C \cap D \notin \mathcal{D}$. By the previous item, $\overline{C \cap D} \in \mathcal{D}$. Construct a profile $\mathbf{P}$ with the following features:[11]

$$\begin{array}{c|ccccccc} C \cap D & \dots & x & \dots & y & \dots & z & \dots \\ D \setminus C & \dots & y & \dots & z & \dots & x & \dots \\ C \setminus D & \dots & z & \dots & x & \dots & y & \dots \\ \overline{C \cup D} & \dots & z & \dots & y & \dots & x & \dots \end{array}$$

[11]Note the resemblance with the Condorcet paradox (Example 2.5).

We have that:

- $(C \cap D) \cup (C \setminus D) = C$, which is decisive by assumption. So, as for all $i \in C$ $x \succ_i y$, it follows that $x \succ^F_{\mathbf{P}} y$;
- $(C \cap D) \cup (D \setminus C) = D$, which is decisive by assumption. So, as for all $i \in D$ $y \succ_i z$, it follows that $y \succ^F_{\mathbf{P}} z$;
- $\overline{C \cup D} \cup (C \setminus D) \cup (D \setminus C) = \overline{C \cap D}$, which is also decisive by claim ii). So, as for all $i \in \overline{C \cap D}$ $z \succ_i x$, it follows that $z \succ^F_{\mathbf{P}} x$.

The above forces the social preference $\succeq^F_{\mathbf{P}}$ to be cyclical, which is impossible by the definition of SPFs (2.2). Contradiction.

All claims have been proven and the proof is therefore complete. $\square$

The second lemma concerns a general well-known fact about finite ultrafilters. Worth noticing is that the fact and its proof do not involve any reference to SPFs or preferences. The lemma concerns a simple set-theoretic property.

Lemma 2.3 (Existence of a dictator)**.** Let $\mathcal{D}$ be an ultrafilter on N. Then $\mathcal{D}$ is *principal*, i.e.: $\exists i \in N$ s.t. $\{i\} \in \mathcal{D}$.

Proof. Recall the definition of ultrafilter given in Lemma 2.2. Since the set of voters N is finite by assumption, the closure $\bigcap \mathcal{D}$ of $\mathcal{D}$ under finite intersections belongs to $\mathcal{D}$ by property *iii)*. We therefore have that $\bigcap \mathcal{D} \neq \emptyset$. For suppose not, then $N \notin \mathcal{D}$ by property *ii)*, against property *i)*. So, w.l.o.g., assume $i \in \bigcap \mathcal{D}$ for $i \in N$. We want to show that $\{i\} \in \mathcal{D}$. Suppose towards a contradiction that $\{i\} \notin \mathcal{D}$. By property *ii)* we have that $\overline{\{i\}} \in \mathcal{D}$, from which follows that $i \notin \bigcap \mathcal{D}$. Contradiction. Hence $\{i\} \in \mathcal{D}$. $\square$

Completing the proof

We can now pull the above lemmas together and prove the result we are after:

Proof of Theorem 2.2. $\boxed{\text{Right-to-left}}$ Assume F is a dictatorship. For each profile where $N^{xy} = N$, the dictator ranks x over y. Hence $xy \in F(\mathbf{P})$, proving Pareto. For any two profiles agreeing on the set of voters ranking x over y either the dictator belongs to that set, and therefore xy belongs to both social preferences, or it does not and therefore yx belongs to both social preferences. This proves IIA.

$\boxed{\text{Left-to-right}}$ By Lemma 2.2 the set of decisive coalitions under F is an ultrafilter. By Lemma 2.3 such ultrafilter is principal and therefore it contains a singleton. Such singleton is a decisive coalition and therefore, by Definition 2.7, the voter in the singleton is a dictator. $\square$

2.3 Social choice by maximum likelihood & closest consensus

In this section we look at two other perspectives on social choice, and some of the voting rules they inspire: the maximum likelihood estimation approach, which we already saw at work in the two alternatives case with the Condorcet jury theorem (Theorem 1.4); and the consensus-based approach.

According to the maximum likelihood estimation perspective a ‘true’ (correct, objective) ranking of the alternatives in A exists (i.e., a true social preference), and individual ballots are noisy estimates of such ranking. The voting rule should then identify the most likely ranking that has generated the observed profile, that is, it should function as a *maximum likelihood estimator* (MLE).

According to the consensus-based perspective, the social preference is a ranking of the alternatives upon which the individuals in the group would be willing to agree. It is therefore a form of ‘consensus’ intended as a preference that is ‘as close as possible’ to the individual preferences in the observed profile. According to this perspective a voting rule should therefore aim at identifying the *closest consensus* to the elicited profile.

2.3.1 The Condorcet model when $m > 2$

The model Condorcet developed in [Condorcet, 1785] for the case in which $m > 2$ is based on the following assumptions (cf. Section 1.2.2):

C0 All rankings are equally likely a priory.

C1 In each pairwise comparison each voter chooses the objectively better alternative with probability $p \in (0.5, 1]$.

C2 Each voter's opinion on a pairwise comparison is independent of her opinion on any other pairwise comparison.

C3 Each voter's opinions are independent of any other voter's opinions.

C4 Each voter's judgment defines a linear order.

The four assumptions are clearly incompatible (specifically, **C2** and **C4**). Here, following [Young, 1988], we drop **C4** and explore to what kind of voting rules assumptions **C0**-**C3** lead. It is worth stressing that even though there are intuitive reasons against dropping condition **C4** (after all it feels odd to allow for intransitive opinions), there are also natural reasons in favor of such a choice: if we take individual votes as the voter's best approximation at trying to identify the correct ranking, such votes may well be intransitive since best approximations of linear orders may fail to be transitive (cf. [Truchnon, 2008]).[12]

2.3.2 Voting rules as MLE when $m > 2$

First, under the above assumptions **C0**-**C3** we are interested in finding a voting rule that identifies the most likely social preference. Two adjustments to the notion of SPF are needed. First, individual ballots may now not be linear orders as their strict part may contain cycles. They are, instead, tournaments (cf. Definition 2.2), that is, asymmetric, complete but not necessarily transitive binary relations over A. We denote the class of tournaments over A by $\mathcal{T}(A)$. Second, there might be more than one most likely social preference. We therefore generalize the notion of SPF as follows. A generalized social preference function (GSPF) is a function:

$$G : \mathcal{T}(A)^n \to 2^{\mathcal{L}(A)} \setminus \{\emptyset\} \tag{2.3}$$

That is, it takes a vector of tournaments as input and outputs a non-empty set of linear orders. Notice that a SPF (2.2), when its output is a linear order (cf. Lemma 2.1), can therefore be viewed as a GSPF that always outputs a singleton set of linear orders.

A MLE of the correct ranking is therefore a GSPF G such that, for every profile $\mathbf{P} \in \mathcal{T}(A)^n$, $G(\mathbf{P})$ is the set of linear orders $\succeq$ that maximize the probability of the observed profile $\mathbf{P}$ given $\succeq$. That is:

$$G(\mathbf{P}) = \underset{\succeq \in \mathcal{L}(A)}{\operatorname{argmax}}(\mathbb{P}(\mathbf{P} | \succeq)). \tag{2.4}$$

We discuss a concrete application of this approach, and a slight variant of it.

The Kemeny rule

We first introduce one more piece of notation. The distance between two linear orders $\succeq$ and $\succeq'$, called *swap distance*, is defined as:[13]

$$d_{swap}(\succeq, \succeq') = \left| \left\{ xy \in A^2 \ : \ x \succ y \text{ and } y \succ' x \right\} \right| \tag{2.5}$$

[12] The alternative choice consisting of retaining **C4** while dropping **C2** requires different noise models for the random generation of linear orders. One such widely deployed model is the so-called *Mallows noise model* [Mallows, 1957].

[13] The distance is also known as, among others, Kendall tau or bubble-sort distance.

That is, the swap distance between two linear orders is given by the number of pairs of alternatives with respect to which the two orders disagree. Equivalently $d_{swap}(\succeq, \succeq')$ is the minimum number of swaps (i.e., inversions) of adjacent alternatives whereby one can obtain $\succeq'$ from $\succeq$. Yet another way to interpret this distance is to say that, if the cost of inverting any two adjacent alternatives in a linear order is 1, then in order to turn $\succeq$ into $\succeq'$ one incurs a cost of $d_{swap}(\succeq, \succeq')$.

Remark 2.5. Many alternative notions of distance can be defined. The simplest one is possibly the so-called *discrete distance*:

$$d_{discr}(\succeq, \succeq') = \begin{cases} 1 & \text{if } \succeq \neq \succeq' \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

That is, the distance is 1 for all and only the ballots that differ from $\succeq$.

In general, a distance is a function d satisfying the following properties, for any $\succeq, \succeq', \succeq''$:

Non-negativity, $d(\succeq, \succeq') \geq 0$;

Identity of indiscernibles, $d(\succeq, \succeq') = 0$ iff $\succeq = \succeq'$;

Symmetry, $d(\succeq, \succeq') = d(\succeq', \succeq)$;

Triangle inequality, $d(\succeq, \succeq') \leq d(\succeq, \succeq'') + d(\succeq'', \succeq')$.

Theorem 2.3. Let $\mathbf{P} = \langle \succeq_1, \ldots, \succeq_n \rangle \in \mathcal{T}(A)^n$ be randomly generated according to assumptions **C0-C3** above. Then

$$\underset{\succeq \in \mathcal{L}(A)}{\operatorname{argmin}} \sum_{i \in N} d_{swap}(\succeq_i, \succeq)$$

is a MLE of the correct ranking.

Proof. For ease of presentation we work with the asymmetric (strict) part of linear orders. Let $\succ \in \mathcal{L}(A)$ be the correct ranking. Let now $\succ' \in \mathcal{T}(A)$ be a tournament that agrees with $\succ$ on k pairs of alternatives. Observe that by (2.5) we have $d_{swap}(\succ', \succ) = \binom{m}{2} - k$. So, by assumptions **C1** and **C2** the probability that a voter expresses ballot $\succ'$ is

$$p^k \cdot (1-p)^{\binom{m}{2} - k} \;=\; p^{\binom{m}{2} - d_{swap}(\succ', \succ)} \cdot (1-p)^{d_{swap}(\succ', \succ)} \;=\; p^{\binom{m}{2}} \cdot \left(\frac{p}{1-p}\right)^{-d_{swap}(\succ'_i, \succ)}.$$

Then by assumption **C3** we have that the probability of a profile $\langle \succ'_1, \ldots, \succ'_n \rangle$, given that $\succ$ is the correct ranking, is proportional to

$$\prod_{i \in N} \left(\frac{p}{1-p}\right)^{-d_{swap}(\succ'_i, \succ)} = \left(\frac{p}{1-p}\right)^{-\sum_{i \in N} d_{swap}(\succ'_i, \succ)}.$$

By **C0** the rankings that are most likely to be the correct one are the ones that maximize the probability of the observed profile. In our case, by **C2** (since $p \in (0.5, 1]$, $\frac{p}{1-p} > 0.5$), those are the rankings that minimize $\sum_{i \in N} d_{swap}(\succ'_i, \succ)$, thus proving the claim. $\square$

The theorem provides an MLE justification of the following voting rule, which is known as the Kemeny rule [Kemeny, 1959].

Rule 14 (Kemeny)**.** The Kemeny rule is the SCF defined as follows. For all profiles $\mathbf{P}$:

$$\texttt{Kemeny}(\mathbf{P}) = \left\{ \max_{\succeq}(A) \;:\; \succeq \in \underset{\succeq' \in \mathcal{L}(A)}{\operatorname{argmin}} \left(\sum_{i \in N} d_{swap}(\succeq_i, \succeq') \right) \right\}$$

Fixing a linear order $\succeq$, $\sum_{i \in N} d_{swap}(\succeq_i, \succeq)$ is called the *Kemeny distance* of $\succeq$ to $\mathbf{P}$.

Intuitively, the Kemeny rule first identifies the linear orders which minimize the Kemeny distance of the order from the profile, then selects the top ranked alternatives in such linear orders.

Example 2.10 ([Young, 1995]). Let $n = 60$ and $m = 3$. Consider the following profile:

$$\mathbf{P} = \begin{array}{c|ccc} \#23 & a & b & c \\ \#17 & b & c & a \\ \#2 & b & a & c \\ \#10 & c & a & b \\ \#8 & c & b & a \end{array}$$

Notice that the profile has no Condorcet winner. There are 3! possible linear orders. For each of them we can calculate their Kemeny distance from the above profile. For example, let us consider the order bca. We have:

$$\begin{aligned} & 23 \cdot d_{swap}(abc, bca) + 17 \cdot d_{swap}(bca, bca) + 2 \cdot d_{swap}(bac, bca) + 10 \cdot d_{swap}(cab, bca) + 8 \cdot d_{swap}(cba, bca) \\ = \; & 46 + 0 + 2 + 20 + 8 \\ = \; & 76 \end{aligned}$$

The Kemeny distance for the other rankings is computed in like fashion (left to the reader). The ranking bca is in fact the one that minimizes the Kemeny distance. So $\texttt{Kemeny}(\mathbf{P}) = \{b\}$.

Observe that Theorem 2.3 does not make any specific assumption on m. In fact, when $m = 2$, the theorem allows us to draw the following conclusion:

Fact 2.2. Assume $m = 2$. Then, for any profile $\mathbf{P} \in \mathcal{L}(A)^n$: $\texttt{Plurality}(\mathbf{P}) = \texttt{Kemeny}(\mathbf{P})$.

Proof. See Exercise 2.8. $\square$

As a corollary of the above fact we have that, on two alternatives, the plurality rule (Rule 1) is a MLE (cf. Theorem 1.4) under the Condorcet's assumptions.[14] We can therefore see the Kemeny rule as a natural generalization of plurality from a maximum likelihood estimation perspective.

The Borda rule as MLE

We have shown that the Kemeny rule is a MLE under the assumptions **C0-C3**. By varying such assumptions it is possible to obtain similar MLE characterizations for different rules (cf. [Conitzer and Sandholm, 2005]). Here we show how this can be done for the Borda rule (Rule 8) by modifying in a specific way the error model (condition **C2**) in order to require a specific probability for a voter to rank an alternative at position k under the condition that such alternative is the top alternative in the correct ranking. In the case of Borda we are not interested in estimating the most likely true ranking, but the most likely true winner (i.e., the most likely top alternative in the correct ranking).[15]

Theorem 2.4 (Borda as MLE [Conitzer and Sandholm, 2005]). Let $\mathbf{P} = \langle \succeq_1, \ldots, \succeq_n \rangle \in \mathcal{L}(A)^n$ be randomly generated according to assumptions **C0**, **C1**, **C3**, **C4** and assuming that, for all $k \in \mathbb{N}$ such that $0 \leq k \leq m - 1$:

$$\mathbb{P}(\ i(x) = k \quad : \quad \max_{\succeq}(A) = x\ ) \propto 2^{m-k}.$$

where $i(x)$ denotes the rank of x in i's ballot in $\mathbf{P}$ and $\succeq$ is the correct order. Then the Borda rule is a MLE of the top alternative in $\succeq$.

[14] Notice also that condition **C4** of the Condorcet model is trivially satisfied when $m = 2$.

[15] Technically this involves a MLE that is a SCF rather than a GSPF.

Proof. By the constraint on $\mathbb{P}(i(x) = k | \max_{\succeq}(A) = x)$, the probability of observing profile $\mathbf{P}$, given the true preference ranks x as first, is proportional to

$$\prod_{i=1}^{n} 2^{m-i(x)} = 2^{\left(\sum_{i=1}^{n} m-i(x)\right)}.$$

Notice that $\sum_{i=1}^{n} m - i(x) = \sum_{i \in N} m - i(x)$ is precisely the Borda score (Rule 8). It follows that the alternative with the highest Borda score is also the alternative that is most likely the top alternative given the observed profile $\mathbf{P}$. $\square$

2.3.3 Social choice as the closest consensus

As briefly mentioned above, the consensus-based approach to social choice is based on the following idea: we should first identify a compromise profile—a *consensus*—which is as close as possible to the observed profile, and than use that compromise profile to determine the social choice. Depending on how the notions of 'consensus' and 'closeness' are made precise, different voting rules may be characterized. In this section we will discuss four main examples of characterizations of voting rules in terms of types of consensus and distance.

As a first example we mention again the Kemeny rule (Rule 14). We can think of the Kemeny rule as follows. First we identify a preference that is as close as possible—by swap distance—to the observed profile. That preference is then taken up by every agent creating a strongly unanimous profile where everybody agrees on the relative position of all alternatives. We can then take the top-ranked alternative in that unanimous preference to determine the social choice. So the rule can be reformulated as follows:

$$\texttt{Kemeny}(\mathbf{P}) = \left\{ \max_{\mathbf{P}'_i}(A) \ : \ i \in N \text{ and } \mathbf{P}' \in \underset{\mathbf{P}'' \in \mathcal{S}}{\text{argmin}} \left(\sum_{j \in N} d_{swap}(\mathbf{P}_j, \mathbf{P}''_j) \right) \right\} \tag{2.7}$$

where $\mathcal{S}$ is the class of profiles $\mathbf{P} \in \mathcal{L}(A)^n$ such that $\forall i, j \in N \succeq_i = \succeq_j$ (strong unanimity profiles). So we can justify the Kemeny rule both from an MLE and a consensus-based perspective. Recall that $\mathbf{P}_i$ denotes the i'th projection of $\mathbf{P}$, i.e., the linear order $\succeq_i$ of i in $\mathbf{P}$.

As a second simple example, we show that the plurality rule (Rule 1) is the SCF that minimizes the discrete distance (Equation (2.6)) from a consensual profile where every voter ranks the same alternative first.

Theorem 2.5. For all $\mathbf{P} \in \mathcal{L}(A)^n$

$$\texttt{Plurality}(\mathbf{P}) = \left\{ \max_{\mathbf{P}'_i}(A) \ : \ i \in N \text{ and } \mathbf{P}' \in \underset{\mathbf{P}'' \in \mathcal{U}}{\text{argmin}} \left(\sum_{j \in N} d_{discr}(\mathbf{P}_j, \mathbf{P}''_j) \right) \right\}$$

where $\mathcal{U}$ is the class of profiles $\mathbf{P} \in \mathcal{L}(A)^n$ such that $\forall i, j \in N$ $\max_{\succeq_i}(A) = \max_{\succeq_j}(A)$ (unanimity profiles).

Proof. See Exercise 2.9. $\square$

Notice furthermore that since $\mathbf{P}' \in \mathcal{U}$, the top-ranked alternative of each voter is the same. That is why the social choice is determined by the top-ranked alternative of any agent i. So the plurality rule is the rule that selects the alternatives that are unanimously top-ranked in profiles that minimize the total distance—measured by d_{discr}—from the input profile.

A similar characterization exists also for the Borda rule (Rule 8).

Rules	Consensus class	Distance
Plurality	Unanimous ($\mathcal{U}$)	Discrete (d_{discr})
Borda	Unanimous ($\mathcal{U}$)	Swap (d_{swap})
Kemeny	Strongly unanimous ($\mathcal{S}$)	Swap (d_{swap})
Dodgson	Condorcet ($\mathcal{C}$)	Swap (d_{swap})

Table 2.1: Types of consensus classes and distances for the characterizations of Plurality, Borda, Kemeny and Dodgson.

Theorem 2.6. For all $\mathbf{P} \in \mathcal{L}(A)^n$

$$\texttt{Borda}(\mathbf{P}) = \left\{ \max_{\mathbf{P}'_i}(A) \ : \ i \in N \text{ and } \mathbf{P}' \in \underset{\mathbf{P}'' \in \mathcal{U}}{\text{argmin}} \left(\sum_{j \in N} d_{swap}(\mathbf{P}_j, \mathbf{P}''_j) \right) \right\}$$

where $\mathcal{U}$ is the class of unanimity profiles.

Proof. See Exercise 2.10. □

Intuitively, under this characterization the Borda rule is the rule that selects the alternatives that are unanimously top-ranked in profiles that minimize the total distance—measured this time by d_{swap}—from the input profile.

Finally, we introduce the last rule of this chapter, which was explicitly defined in terms of closest consensus, the Dodgson rule [Dodgson, 1876].

Rule 15 (Dodgson)**.** The Dodgson rule `Dodgson` is the SCF defined as follows. For all profiles $\mathbf{P}$:

$$\texttt{Dodgson}(\mathbf{P}) = \left\{ \texttt{Condorcet}(\mathbf{P}') \ : \ \mathbf{P}' \in \underset{\mathbf{P}'' \in \mathcal{C}}{\text{argmin}} \left(\sum_{j \in N} d_{swap}(\mathbf{P}''_j, \mathbf{P}_j) \right) \right\}$$

where $\mathcal{C}$ is the class of profiles $\mathbf{P} \in \mathcal{L}(A)^n$ for which there exists a Condorcet winner (Condorcet profiles) and `Condorcet` is the Condorcet rule (Rule 6).

Intuitively the rule outputs the alternatives that are Condorcet winners in the Condorcet profiles that are closest—according to d_{swap}—to the input profile. Clearly such rule is Condorcet-consistent by definition.

Table 2.3.3 recapitulates the characterizations of the rules dealt with in this section in terms of type of consensus (strongly unanimous, unanimous or Condorcet) and type of distance (swap or discrete). It is worth noticing that the consensus classes considered are ordered by increasing generality: $\mathcal{S} \subset \mathcal{U} \subset \mathcal{C}$.

2.4 Chapter notes

The bulk of this chapter is again based on the introductions to voting theory provided in [Brandt et al., 2016, Ch. 2], [Taylor, 2005, Ch. 1], [Endriss, 2011] and, in addition, [Taylor and Pacelli, 2008].

The proof of Arrow's theorem presented above is based on [Hansson, 1976]. The first proof of Arrow's theorem based on the ultrafilter argument is due to [Fishburn, 1970] and [Kirman and Sondermann, 1972]. The theorem has been proved with a wealth of different arguments. See for instance [Barberá, 1980, Reny, 2001, Geanakoplos, 2005].

Various generalizations and variants of Theorem 2.4 are studied in [Conitzer and Sandholm, 2005]. For an overview of results about the MLE and consensus-based approaches to voting rules we refer to the overview in [Elkind and Slinko, 2016].

2.5 Exercises

Exercise 2.1. Consider the following rule:

> **Rule 16** (Symmetric Borda)**.** The symmetric Borda voting rule is the SCF defined as follows. For all $\mathbf{P} \in \mathcal{L}(A)^n$:
> $$\texttt{Borda}^s(\mathbf{P}) = \underset{x \in A}{\text{argmax}} \sum_{y \in A} Net^{xy}_{\mathbf{P}}$$

That is, the rule selects the alternatives that beat all other alternatives (in the corresponding majority graph) by the largest margin. Prove that for any profile $\mathbf{P}$: $\texttt{Borda}(\mathbf{P}) = \texttt{Borda}^s(\mathbf{P})$.

Exercise 2.2. Come up with a ballot profile that has the property that all rules mentioned in Table 2.2 output a different social choice.

Exercise 2.3. Consider Table 2.2. Rows correspond to voting rules, and columns to axioms that we introduced in this and the previous chapter. For the four rules of Plurality, Copeland, Borda, STV, determine whether they satisfy (✓) or do not satisfy (no mark) the axioms given in the columns of the table. Provide a proof for each such statement. By doing so you are completing Table 2.2.

Exercise 2.4. Provide a proof of Theorem 2.1. $\boxed{\text{Hint}}$ argue towards a contradiction and construct a profile in which liberalism and Pareto together would force an empty social choice, which we know is impossible by the definition of SCF.

Exercise 2.5. Prove that every ultrafilter $\mathcal{U}$ on a an arbitrary set X (see the definition of ultrafilter provided in Lemma 2.2) is closed under supersets. That is: if $Y \in \mathcal{U}$ and $Y \subseteq Z \subseteq X$ then $Z \in \mathcal{U}$.

Exercise 2.6. Let $A = \{a, b\}$, n odd and let $\texttt{Plurality}^\star$ denote the SPF corresponding to the plurality rule we introduced as SCF (Rule 1). Consider the set of decisive coalitions $\mathcal{D}_{\texttt{Plurality}^\star}$ under $\texttt{Plurality}^\star$. For each of the three properties of ultrafilters (cf. Lemma 2.2) prove that $\mathcal{D}_{\texttt{Plurality}^\star}$ satisfies that property or provide a counter-example if it does not.

Exercise 2.7 (Properties of the Kemeny rule)**.** Determine whether the Kemeny rule satisfies the properties in Table 2.2. Justify your answers.

Exercise 2.8. Prove Fact 2.2.

Exercise 2.9. Prove Theorem 2.5.

Exercise 2.10. Prove Theorem 2.6.

Exercise 2.11. Determine whether the Kemeny and Dodgson rules satisfy neutrality (Definition 1.2). Justify your answer.

	Resoluteness	Unanimity	Condorcet-cons.	Independence	Monotonicity	Liberalism
Dictatorship	✓	✓		✓	✓	
Plurality						
Condorcet		✓	✓	✓	✓	
Copeland						
Borda						
STV						

Table 2.2: Table stating which rules satisfy which axioms (only cases for Dictatorships and the Condorcet rule are provided).

Chapter 3

Eliciting Truthful Ballots

The previous chapters tacitly assumed that the ballots provided by the voters report the voters' genuine individual preferences: the voting rule can access truthful information and establish an appropriate social choice, depending on the specific logic driving the rule. But what if the ballots provided by the voters do not correspond to their true preferences? Voting rules may create incentives for voters to misrepresent their true preferences by submitting a 'manipulated' or 'tactical' ballot to the rule, and thereby steering the rule's outcome towards better—for those voters—alternatives.

3.1 Preliminaries: strategic voting

We start by introducing the notion of strategic manipulation that we are going to focus on, and use it to define a new property of SCFs, called non-manipulability or strategy-proofness.

3.1.1 Voting strategically

We start with three examples illustrating how voters could manipulate the social choice in their favour by misrepresenting their true preferences to the voting rule.

Example 3.1 (Roman senate, continued)**.** Consider again the profile discussed in Example 2.1

$$\begin{array}{r|ccc} \#102 & a & b & c \\ \#101 & b & a & c \\ \#100 & c & b & a \end{array}$$

Pliny demanded a vote by plurality that would lead to the choice of $\{a\}$. For 100 voters (bottom row) this amounts to the worst possible alternative being selected as social choice. They have an incentive to misrepresent their ballots by submitting b as their top-ranked alternative rather than c. The plurality rule would then be applied to the profile

$$\begin{array}{r|ccc} \#102 & a & b & c \\ \#101 & b & a & c \\ \#100 & b & c & a \end{array}$$

leading to the choice of $\{b\}$. This is in fact what happened in the Roman senate after Pliny's decision to vote by plurality (see for instance [Szpiro, 2010]).

Example 3.2 ([Zwicker, 2016])**.** Consider the following profile for $N = \{1, \dots 7\}$ and $A = \{a, b, c, d, e\}$:

$$\mathbf{P} = \begin{array}{c|ccccc} \#2 & a & b & c & d & e \\ \#3 & d & e & b & c & a \\ \#2 & e & c & a & d & b \end{array}$$

The Borda rule selects $\{e\}$ with a score of 17. For 2 voters—those with ballot *abcde*—this is the worst possible alternative. It suffices for one of them, let it be voter 1, to modify his ballot to *dabce* in order to have d instead selected as Borda winner. In this new profile

$$\mathbf{P}^\bullet = \begin{array}{c|ccccc} \#1 & a & b & c & d & e \\ \#1 & d & a & b & c & e \\ \#3 & d & e & b & c & a \\ \#2 & e & c & a & d & b \end{array}$$

the outcome is preferred by voter 1 with respect to the one in $\mathbf{P}$: $\texttt{Borda}(\mathbf{P}^\bullet) \succ_1 \texttt{Borda}(\mathbf{P})$.

Example 3.3 ([Conitzer and Walsh, 2016])**.** Let now $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$, and suppose the social choice is carried out using plurality with an alphabetic tie-breaking rule ($a > b > c$). Consider now the following ballot profile:

$$\mathbf{P} = \begin{array}{c|ccc} 1 & a & b & c \\ 2 & b & a & c \\ 3 & c & b & a \end{array}$$

In this case the social choice would be $\{a\}$, making agent 3 the least happy with the choice since a is ranked at the bottom of her preference. However, were 3 to misrepresent her ballot by reporting $b \succ_3 c \succ_3 a$ instead, b would become the social choice, granting her a better outcome.

3.1.2 Manipulability and strategy-proofness

The examples above show: that the Borda rule (Rule 8), in profiles where a single winner is selected, is manipulable by a single voter;[1] that the plurality rule (Rule 1) can be manipulated by a coalition of voters, or by a single voter when it is supplemented by a deterministic tie-breaker (that enforces resoluteness).

We can make all the above forms of manipulation of a social choice process mathematically precise. In this chapter, however, we will focus on single-voter manipulability of resolute rules as illustrated in Example 3.3. This is the definition we will be working with in this chapter:

Definition 3.1. A resolute SCF f (for a given $\langle N, A \rangle$) is (single-voter) *manipulable* if there exist two profiles

$$\mathbf{P} = \langle \succeq_1, \dots, \succeq_i, \dots \succeq_n \rangle \text{ and } \mathbf{P}^\bullet = \langle \succeq_1, \dots, \succeq_{i-1}, \succeq_i^\bullet, \succeq_{i+1}, \dots, \succeq_n \rangle$$

such that $f(\mathbf{P}^\bullet) \succ_i f(\mathbf{P})$. We say then that i is a *manipulator*, that $\succeq_i$ is i's *truthful* ballot and $\succeq_i^\bullet$ is i's *untruthful* (or manipulated, or strategic) ballot.[a]

[a]Notice that by writing $f(\mathbf{P}^\bullet) \succ_i f(\mathbf{P})$ we slightly abuse notation as the output of f is, technically, a singleton.

Intuitively, a SCF is said to be manipulable whenever there exist situations (profiles) in which some voter has an incentive to submit an untruthful ballot to the SCF, that is, by manipulating his balllot the social

[1]The French Academy of Sciences, of which Borda was a member, did experiment with the Borda rule but did not pursue its use further exactly because of this susceptibility to manipulation. Borda famously responded to this criticism by stating "My scheme is intended for only honest men" [Black, 1958].

choice is an alternative the manipulator prefers over the alternative that would be selected, were he to vote truthfully.

It is worth noticing that the definition makes a full information assumption on the manipulator: the manipulator needs to know the ballots of all other agents in order to be able to select the right manipulated ballot. This is in general unrealistic, but it is a reasonable assumption when n is small (e.g., deliberative committees) or when accurate enough information is available in the form of opinion pools (e.g., political elections). The assumption can also be interpreted as a worst-case assumption: as we cannot anticipate the level of information of a potential manipulator, we should conservatively assume that the manipulator has access to full information.

Remark 3.1 (Manipulability for irresolute SCFs)**.** We defined manipulability only for resolute rules. This is a simplifying—and arguably unrealistic—assumption, although we saw that resoluteness is a by-product of natural requirements such as IIA and Pareto (Proposition 2.1). It is however possible to define manipulability also in the context of irresolute SCFs. In such a case we need to specify how voters may rank sets of alternatives. Two natural definitions have been explored (cf. [Taylor, 2005]):

- For an *optimistic* manipulator i, $f(\mathbf{P}) \succ_i f(\mathbf{P'})$ whenever $\max_{\succeq_i}(f(\mathbf{P})) \succ_i \max_{\succeq_i}(f(\mathbf{P'}))$;
- For a *pessimistic* manipulator i, $f(\mathbf{P}) \succ_i f(\mathbf{P'})$ whenever $\min_{\succeq_i}(f(\mathbf{P})) \succ_i \min_{\succeq_i}(f(\mathbf{P'}))$.

The problem with manipulability is that it creates the possibility for the process of social choice to be mislead as profiles may no longer represent the true preferences of the voters. In addition, manipulability creates indirect incentives for voters to invest energy and time into anticipating each others' manipulative behavior, rather than investing them on the substance of the decision at issue: if i knows that j has an incentive to manipulate, she will try to adapt her ballot accordingly, at which point j may need to reconsider her ballot again, and so on. This motivates the following axiom.

Definition 3.2. Let $\langle N, A\rangle$ be given. A resolute SCF F is **strategy-proof** iff it is not manipulable.

That is, no voter ever has an incentive to misrepresent her ballot.[2]

3.1.3 Strategy-proofness and responsiveness axioms

Strategy-proofness in resolute SCFs is directly linked to the responsiveness properties introduced in Definition 1.3. In particular:

Lemma 3.1. Let $\langle N, A\rangle$ be given. Let f be a resolute and strategy-proof SCF. Then:

(a) f is monotonic;

(b) f is independent;

(c) if f is non-imposed, it is Pareto.

Proof. (a) We prove the claim by contraposition. So assume f is not monotonic (Definition 1.3). Then there exist two profiles $\mathbf{P}, \mathbf{P'} \in \mathcal{L}(A)^n$ such that:

- $N^{xy}_{\mathbf{P}} \subseteq N^{xy}_{\mathbf{P'}}$, for all $y \in A \setminus \{x\}$;
- $N^{yz}_{\mathbf{P}} = N^{yz}_{\mathbf{P'}}$, for all $y, z \in A \setminus \{x\}$;
- $f(\mathbf{P}) = \{x\}$ and $x \notin f(\mathbf{P'})$.

[2] In game-theoretic terms [Osborne and Rubinstein, 1994] we say that reporting a truthful ballot is a dominant strategy in the voting game $\langle N, \mathcal{L}(A)_i, f, \succeq_i\rangle$ where players' are endowed with a linear order over the decision alternatives as preference, their actions are all the possible linear orders i (the ballots) and the game outcome is determined by the SCF f.

That is, in $\mathbf{P}'$ some more voters may rank x over y while keeping the rankings of all other alternatives as in $\mathbf{P}$, and shifting the winner to a different alternative from x. Let then $f(\mathbf{P}') = \{z\}$ with $z \neq x$. Notice there may be many such voters needed to shift the winner. So there exists a sequence of intermediate profiles $\mathbf{P} = \hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_k = \mathbf{P}'$ differing only in the ballot of one voter and such that at some point $1 < \ell \leq k$ we have that $f(\hat{\mathbf{P}}_{\ell-1}) = \{x\}$ and $f(\hat{\mathbf{P}}_\ell) = \{z\}$. So w.l.o.g. let us assume that $\mathbf{P}$ and $\mathbf{P}'$ are such profiles differing only in the ballot of one voter i who has raised the ranking of x w.r.t. one other alternative y in her ballot. Observe that $i \in N^{yx}_{\mathbf{P}} \cap N^{xy}_{\mathbf{P}'}$. There are two cases. $\boxed{x \succ_i z}$ Then in $\mathbf{P}'$ (where still $x \succ'_i z$ by how $\mathbf{P}'$ is constructed) i can manipulate by submitting ballot $\succeq_i$. $\boxed{z \succ_i x}$ Then in $\mathbf{P}$ i can manipulate by submitting ballot $\succeq'_i$. In both cases f is therefore manipulable, which proves the claim.

(b) We prove the claim by contraposition. So assume f is not independent (Definition 2.5). Then there exist two profiles $\mathbf{P}, \mathbf{P}' \in \mathcal{L}(A)^n$ and two alternatives $x, y \in A$, such that $N^{xy}_{\mathbf{P}} = N^{xy}_{\mathbf{P}'}$, $x \in f(\mathbf{P})$ (therefore $y \notin f(\mathbf{P})$ by resoluteness) and $y \in f(\mathbf{P}')$ (therefore $f(\mathbf{P}') = \{y\}$ by resoluteness). It follows that there exists $i \in N$ such that $\succeq_i \neq \succeq'_i$. By an argument analogous to the one provided in (a) we can focus w.l.o.g. on profiles $\mathbf{P}$ and $\mathbf{P}'$ that differ only in the ballot of i. There are two cases. $\boxed{x \succ_i y}$ Then in $\mathbf{P}'$ (where still $x \succ'_i y$ by how the profile is constructed) i can manipulate by submitting ballot $\succeq_i$. $\boxed{y \succ_i x}$ Then in $\mathbf{P}$ i can manipulate by submitting ballot $\succeq'_i$. In both cases f is therefore manipulable, which proves the claim.

(c) See Exercise 3.1.

All items have been proven and the proof is complete. □

The gist of the above lemma is that any failure of the axioms of monotonicity or independence in resolute SCFs generates opportunities for tactical voting. This provides therefore an extra argument for insisting those properties are satisfied by social choice functions.

3.2 There is no obvious strategy-proof social choice when $m > 2$

Two immediate strategy-proof SCFs come to mind. First, dictatorships are trivially strategy proof as all ballots, except the dicator's, are irrelevant to determine the social choice. Second, simple majority—that is, plurality when $m = 2$—is also strategy-proof: reporting a manipulated ballot inevitably lowers the plurality score of the top alternative (see Theorem 3.1 below). The question we are after in this section then is: are there 'nice' (non-dictatorial) SCFs that are strategy-proof when there are more than two alternatives?

3.2.1 Strategy-proofness when $m = 2$

For social choice contexts with two alternatives we can prove this simple consequence of May's theorem:

Theorem 3.1. Let $\langle N, A \rangle$ be given such that $m = 2$ and n is odd. A resolute SCF is anonymous, neutral and strategy-proof if and only if it is plurality.

Proof. $\boxed{\text{Right-to-left}}$ Left as exercise. $\boxed{\text{Left-to-right}}$ It follows from Theorem 1.1 and Lemma 3.1 (monotonicity). □

3.2.2 Strategy-proofness when $m > 2$

We prove a fundamental theorem—akin to Arrow's (Theorem 2.2)—showing that there exists no resolute SCF that is at the same time non-imposed, strategy-proof and non-dictatorial.

Theorem 3.2 (Gibbard-Satterthwaite theorem [Gibbard, 1973, Satterthwaite, 1975]). Let f be a resolute SCF for a $\langle N, A \rangle$ s.t. $m > 2$. f is non-imposed and strategy-proof if and only if it is a dictatorship.

Proof strategy and blocking coalitions

The proof we present relies on a concept related to the concept of decisive coalitions in the proof of Arrow's theorem: blocking coalitions.

Definition 3.3 (Blocking coalition)**.** Let f be a resolute SCF (for a given $\langle N, A\rangle$), and $x, y \in A$. A coalition $C \subseteq N$ is *blocking for y by x*, under f, if

$$\forall \mathbf{P} \in \mathcal{L}(A)^n: \text{ if } C \subseteq N^{xy}_{\mathbf{P}} \text{ then } y \notin f(\mathbf{P}).$$

A coalition $C \subseteq N$ is blocking if it is blocking w.r.t. every pair of alternatives. The set of all blocking coalitions is denoted $\mathcal{B}$. The set of blocking coalitions for yx (under f) is denoted $\mathcal{B}^{yx}_f$ (or simply $\mathcal{B}^{yx}$ when f is clear from the context).

Observe that if $\mathcal{B}$ contains a singleton then there exists a voter i who is blocking for every pair of alternatives. Therefore, for any profile $\mathbf{P} \in \mathcal{L}(A)^n$, $f(\mathbf{P}) = \{\max_{\succeq_i}(A)\}$, that is, i is a dictator (Definition 1.2).

The proof then relies on Lemma 3.1 and two further lemmas. The first one is an ultrafilter lemma for the set of blocking coalitions: any resolute SCF that is strategy-proof and non-imposed induces a set of blocking coalitions that takes the form of an ultrafilter. The second lemma is Lemma 2.3, which we used also in the proof of Arrow's theorem in the previous chapter.

The ultrafilter lemma for resolute, non-imposed, strategy-proof SCFs

Lemma 3.2 (Ultrafilter lemma)**.** Let f be a resolute SCF (for a given $\langle N, A\rangle$), that satisfies non-imposition and strategy-proofness. The set $\mathcal{B}$ of blocking coalitions (for f) is an *ultrafilter* over N, that is:

i) $N \in \mathcal{B}$, i.e., the set of all individuals is a blockig coalition;

ii) $C \in \mathcal{B}$ iff $\overline{C} \notin \mathcal{B}$, i.e., a coalition is blocking if and only if its complement is not;

iii) $\mathcal{B}$ is closed under finite intersections: if $C, C' \in \mathcal{B}$ then $C \cap C' \in \mathcal{B}$, i.e., if two coalitions are blocking then the individuals they have in common form a blocking coalition.

Proof. i) By non-imposition, for any alternative x there exists a profile $\mathbf{P}$ such that $f(\mathbf{P}) = \{x\}$. Now take such profile $\mathbf{P}$ and construct a profile $\mathbf{P'}$ such that $N^{xy}_{\mathbf{P'}} = N$ for all $y \neq x$. By strategy-proofness and Proposition 3.1 (monotonicity), $f(\mathbf{P'}) = \{x\}$. Again by Proposition 3.1 (independence), it follows that for any unanimous profile $\mathbf{P''}$ on x, $y \notin f(\mathbf{P''})$ for any alternative $y \neq x$. That is, $N \in \mathcal{B}$.

ii) $\boxed{\text{Left-to-right}}$ Suppose, towards a contradiction, that $C, \overline{C} \in \mathcal{B}$. Consider now a profile $\mathbf{P}$ where: for all $i \in N$ the top two alternatives in each $\succeq_i$ are either x or y; and $C = N^{xy}_{\mathbf{P}}$ and $\overline{C} = N^{yx}_{\mathbf{P}}$. This profile must exist as SCFs admit any profile in $\mathcal{L}(A)^n$ as input, and be such that $x \notin F(\mathbf{P})$ and $y \notin F(P)$. Hence there exists $z \in A \setminus \{x, y\}$ such that $f(\mathbf{P}) = \{z\}$, against Pareto (Lemma 3.1).

$\boxed{\text{Right-to-left}}$ Assume $\overline{C} \notin \mathcal{B}$. Then there exists a profile $\mathbf{P}$ and alternatives x and y such that $\overline{C} \subseteq N^{xy}_{\mathbf{P}}$ and $f(\mathbf{P}) = \{y\}$. By Lemma 3.1 (independence) for every profile $\mathbf{P'}$ such that $N^{xy}_{\mathbf{P}} = N^{xy}_{\mathbf{P'}}$, $x \notin f(\mathbf{P'})$. That is, $N^{yx}_{\mathbf{P}}$ is a blocking coalition for x by y. Observe that $N^{yx}_{\mathbf{P}} \subseteq C$. Then by Lemma 3.1 (monotonicity) C is also blocking for x by y. By adapting the argument provided earlier for Lemma 2.1 (contagion lemma) we can conclude that $C \in \mathcal{B}$ (see Exercise 3.3).

iii) We proceed towards a contradiction and assume that $C, D \in \mathcal{B}$ and $C \cap D \notin \mathcal{B}$. By the previous item, $\overline{C \cap D} \in \mathcal{B}$. Construct a profile $\mathbf{P}$ with the following two features. One, all ballots are such that the top 3 alternatives are either x, y or z. Second, the relative rankings of the alternatives in $\{x, y, z\}$ are as follows:[3]

$$\begin{array}{c|ccc} C \cap D & x & y & z \\ D \setminus C & y & z & x \\ C \setminus D & z & x & y \\ \overline{C \cup D} & z & y & x \end{array}$$

We have that:

- $(C \cap D) \cup (C \setminus D) = C$, which is blocking by assumption. So, as for all $i \in C$ $x \succeq_i y$, it follows that $y \notin f(\mathbf{P})$;
- $(C \cap D) \cup (D \setminus C) = D$, which is decisive by assumption. So, as for all $i \in D$ $y \succeq_i z$, it follows that $z \notin f(\mathbf{P})$;
- $\overline{C \cup D} \cup (C \setminus D) \cup (D \setminus C) = \overline{C \cap D}$, which is also blocking by claim ii). So, as for all $i \in \overline{C \cap D}$ $z \succeq_i x$, it follows that $x \notin f(\mathbf{P})$.

Therefore $f(\mathbf{P}) \cap \{x, y, z\} = \emptyset$. However, for all $w \notin \{x, y, z\}$ there exists $w' \in \{x, y, z\}$ such that $w' \succeq_i w$. By Proposition 3.1 (Pareto) then $w \notin f(\mathbf{P})$. It follows that $f(\mathbf{P}) = \emptyset$, which is impossible by the definition of SCF (1.1). Contradiction.

All claims have been proven. □

The proof

Proof of Theorem 3.2. Right-to-left It is straightforward to prove that a dictatorship is non-imposed and strategy-proof. Left-to-right By Lemma 2.2 the set of blocking coalitions under f is an ultrafilter. By Lemma 2.3 such ultrafilter is principal and therefore it contains a singleton. Such singleton is a blocking coalition and therefore, by Definition 3.3 and the resoluteness of f, for any profile the social choice must consist of the top-ranked alternative of the voter in the singleton. Therefore f is a dictatorship. □

3.3 Coping with manipulability

We showed that strategy-proofness is unattainable in general. Yet there are many ways to deal with this inherent limitation of social choice functions. We review some of them in this section.

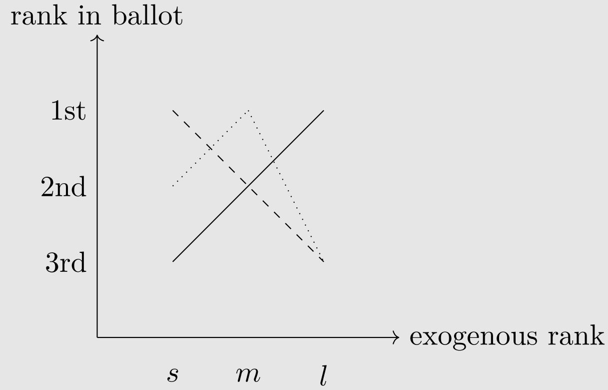
3.3.1 Strategy-proofness when individual preferences are 'coherent'

Example 3.4 (after [Zwicker, 2016])**.** Three friends want to go on a hike together. There are three types of hikes: short s, medium m and long l. So the social choice context is $\langle \{1, 2, 3\}, \{s, m, l\} \rangle$. They hold the following true preferences:

$$\mathbf{P} = \begin{array}{c|ccc} 1 & l & m & s \\ 2 & s & m & l \\ 3 & m & s & l \end{array}$$

Observe that these preferences are aligned with the natural ordering of the alternatives based on distance, from longest to shortest lms: an individual either prefers longer hikes over shorter ones; or vice versa; or she prefers medium distance hikes. Let us denote this linear order by $l \gg m \gg s$. No individual ranks the extremes (l and s) above the middle alternative (m). Diagrammatically, with the line corresponding to 1, the dashed line to 2 and the dotted line to 3:

[3] Cf. the same item in Lemma 2.2



There is a Condorcet winner in this profile: m. This guarantees the top-ranked alternative for 3 and the second-best for 1 and 2. Furthermore, neither 1 nor 2 could obtain a better result by submitting a different ballot unless such ballot is not aligned any more with the natural ordering of the alternatives $l \gg m \gg s$.

Single-peakedness as a form of preference coherence

The example illustrates that, when individual preferences are 'similar enough', then manipulability ceases to be an issue. There are many ways in which this notion of 'similarity' or 'coherence' across individual preferences can be made precise. Here we present one among the most well-known, and the one that was first studied in the literature.

Definition 3.4 (Single-peakedness [Black, 1948])**.** A ballot $\succeq_i \in \mathcal{L}(A)$ is *single-peaked* with respect to a linear order $\gg \in \mathcal{L}(A)$ if, for all $x, y \in A$:

$$\text{if } t_i \gg x \gg y \text{ or } y \gg x \gg t_i \text{ then } x \succ_i y,$$

where $t_i = \max_{\succeq_i}(A)$ is called the *peak of* i (in $\succeq_i$). We say then that $\succeq_i$ is single-peaked if there exists an order $\gg$ with respect to which $\succeq_i$ is single-peaked. The set of single-peaked linear orders over A is denoted $\mathcal{SPL}(A)$.

A profile of linear orders is *single-peaked* if there exists a linear order $\gg \in \mathcal{L}(A)$ s.t for all $i \in N$, $\succeq_i$ is single-peaked with respect to $\gg$. For a given profile $\mathbf{P} \in \mathcal{SPL}(A)^n$ we denote by $\mathbf{t_P}$ the vector of peaks of the ballots in $\mathbf{P}$ ordered by $\gg$.

Intuitively, a ballot is single-peaked whenever there exists an exogenous ranking on the alternatives such that whenever an alternative x lies between the maximal of the ballot (the peak) and another alternative y, then the ballot ranks x over y. A profile is single-peaked if all its ballots are single-peaked w.r.t. the *same* exogenous ranking. The exogenous ranking $\gg$ represents a property (e.g., being left in the political spectrum, being cheap, being far, etc.) that the alternatives enjoy to an increasing or decreasing degree. The reader should check that profiles illustrating the Condorcet's paradox in Example 2.5 are not single-peaked. Finally, observe that when $m = 2$ all profiles are trivially single-peaked.

Single-peaked profiles, or profiles that are single-peaked with high probability,[4] are natural in many settings (e.g., political elections, facility locations, committee decision-making). Empirical studies have also shown that single-peakedness is also sometimes the result of deliberative processes changing agents' preferences (cf. [List et al., 2013]).

Social choice on single-peaked preferences

We want to see whether if restrict the domain of SCFs to single-peaked profiles, one can obtain strategy-proof social choice functions. First of all observe that a SCF defined on single-peaked profiles is a

[4] Such notion is developed in [Regenwetter et al., 2006].

function

$$f : \mathcal{SPL}(A)^n \to 2^A \setminus \{\emptyset\} \,. \tag{3.1}$$

Consider then the following voting rule:

Rule 17. The median rule is the SCF defined as follows. For all $\mathbf{P} \in \mathcal{SPL}(A)^n$

$$\texttt{Median}(\mathbf{P}) = \{x \in A \ : \ x \text{ is median in } \mathbf{t_P}\}$$

Observe that with the profile $\mathbf{P}$ of Example 3.4 $\texttt{Median}(\mathbf{P}) = \texttt{Condorcet}(\mathbf{P}) = \{m\}$. The Condorcet winner is the top-ranked alternative of the median voter.

Theorem 3.3 (Black's theorem [Black, 1948])**.** Let $\mathbf{P} \in \mathcal{SPL}(A)^n$ for context $\langle N, A \rangle$ with $m > 2$. Then:

(a) If n is even, the weak majority tournament $\succeq^{Net}_{\mathbf{P}}$ (Definition 2.2) is transitive and if n is odd, the majority tournament $\succ^{Net}_{\mathbf{P}}$ is transitive.

(b) If n is odd then $\texttt{Median}(\mathbf{P})$ is the singleton containing the Condorcet winner in $\mathbf{P}$. If n is even then $\texttt{Median}(\mathbf{P})$ is the set of weak Condorcet winners in $\mathbf{P}$.

(c) For n odd, $\texttt{Median}$ is strategy-proof.

Proof. (a) We have to consider two cases: n even and n odd. $\boxed{n \text{ odd}}$ We want to prove that, for all distinct $x, y, z \in A$, $x \succ^{Net} y$ and $y \succ^{Net} z$ implies $x \succ^{Net} z$. There are three cases to consider: either y lies between x and z in $\gg$ ($B(xyz)$); or x lies between y and z ($B(yxz)$); or z lies between x and y ($B(xzy)$).

$\boxed{B(xyz)}$ By assumption $x \succ^{Net} y$ we have that $N^{xy} > \frac{n}{2}$. Since y is between x and z in $\gg$, by single-peakedness it follows that for every $i \in N^{xy}$, $y \succ_i z$. From this, by the transitivity of individual preferences we obtain $N^{xz} > \frac{n}{2}$ and therefore $x \succ^{Net} z$ as desired.

$\boxed{B(yxz)}$ By assumption $x \succ^{Net} y$ we have that $N^{xy} > \frac{n}{2}$. Now suppose towards a contradiction that $z \succeq^{Net} x$. By single-peakedness, all agents $i \in N^{zx}$ would prefer z over y by transitivity of $\gg$, against the assumption that $y \succ^{Net} z$. We thus conclude that $x \succ^{Net} z$ as desired.

$\boxed{B(xzy)}$ By assumption $x \succ^{Net} y$ and $y \succ^{Net} z$ there exists an agent $i \in N^{xy} \cap N^{yz}$ such that $x \succ_i y \succ_i z$. However, this preference is not single-peaked under the assumption that z lies between x and y. As we assumed that $\mathbf{P} \in \mathcal{SPL}(A)^n$ this case is therefore impossible.

In all possible cases we conclude $x \succ^{Net} z$ as desired, proving the claim. $\boxed{n \text{ even}}$ The argument is similar and needs to consider the possibility of split majorities in pairwise comparisons.

(b) There are two cases. $\boxed{n \text{ odd}}$ Then $\texttt{Median}(P)$ is a singleton $\{t_i\}$. We need to show that for all $x \in A$, $|N^{t_i x}| > |N^{x t_i}|$. Now consider the vector of all individual peaks ordered by $\gg$:

$$\mathbf{t_P} = \left\langle \underbrace{\ldots}_{L}, t_i, \underbrace{\ldots}_{R} \right\rangle .$$

For all voters j whose peaks appear in L we have that t_i is preferred over any alternative x to the right of t_i in $\gg$ (and therefore to any peak in R). Vice versa, for all voters j whose peaks appear in R we have that t_i is preferred over any alternative y to the left of t_i in $\gg$ (and therefore to any peak in L). It follows that t_i beats any other alternative in a pairwise comparison, i.e., it is a Condorcet winner (Definition 2.3).

$\boxed{n \text{ even}}$ Then $\texttt{Median}(\mathbf{P})$ contains two elements $\{t_i, t_j\}$. Assume w.l.o.g. that $t_i \gg t_j$. Now let:

$$\mathbf{t}_{\mathbf{P}} = \left\langle \underbrace{\ldots}_{L}, t_i, t_j \underbrace{\ldots}_{R} \right\rangle.$$

We reason in a similar fashion as in the case for n odd to conclude that t_i and t_j are weak Condorcet winners.

(c) Let $\mathbf{P}$ be given, and let i be the manipulator. Assume w.l.o.g. that i's peak lies to the right (w.r.t. $\gg$) of $\texttt{Median}(\mathbf{P})$. She has two ways to manipulate her ballot by submitting an untruthful peak t'_i. $\boxed{t_i \gg t'_i}$ Submit a ballot with peak t'_i further to the right of t_i. In such a case $\texttt{Median}(\mathbf{P})$ would not change. $\boxed{t'_i \gg t_i}$ Submit a ballot with peak t'_i to the left of t_i. In such a case, if $\texttt{Median}(\mathbf{P})$ changes, it changes to an alternative further away (w.r.t. $\gg$) to i's truthful peak and is therefore less preferable. So no manipulation for i exists.

The proof is complete. □

In this section we showed how the issue of manipulability can be sidestepped when it is possible to make certain assumptions about the coherence of the voters' true preferences. If restricting the domain of SCFs in such a way is not possible, there are a number of other avenues that have been explored in order to manage the issue of manipulability. We turn to them now, just sketching the key ideas behind each approach.

3.3.2 Making manipulation difficult

We sketch now three ways in which manipulations, even though possible, may be considered difficult or unlikely: by making them computationally intractable, by making their informational requirements matter, and by introducing randomness in the SCFs. We discuss them in turn.

Using computational complexity

The key intuition behind a computation-driven approach to cope with manipulability is the following: even if a SCF is manipulable, finding the manipulating strategy may be an intractable computational problem. If that is the case, a wanna-be manipulator would then be constrained by the difficulty of finding an appropriate manipulation.

The task of finding a suitable manipulation can be phrased exactly in terms of the following decision problem:

Given A partial profile $\mathbf{P}_{-i}$ without the vote of the manipulator i; and i's preferred alternative x

Question Is there a ballot $\succeq_i$ that i can submit so that $f(\mathbf{P}_{-i}, \succeq_i) = \{x\}$?

It has been shown (e.g., [Bartholdi et al., 1989]) that the above decision problem is solvable in time polynomial (in $n + m$) for many of the rules we discussed (e.g., Plurality, Borda) by using a simple greedy algorithm that works as follows:

Initialization: Rank x at the top;

Repeat: Check whether another alternative can be ranked immediately below the previous one without making x lose; if that is the case, do so; if that is not possible, manipulation is impossible.

However, the problem has been shown to be intractable (NP-hard) for some other common rules like, in particular, STV (Rule 13) [Bartholdi and Orlin, 1991]. A whole stream of literature has since then contributed to the study of the computational complexity of the manipulation problem, and variants thereof, for various voting rules (see [Conitzer and Walsh, 2016] for an overview).

Using uncertainty

Definition 3.1 can be modified in order to make it sensitive to the knowledge that is actually available to the manipulator.

Definition 3.5 (Dominating manipulations [Conitzer et al., 2011])**.** Fix a true profile $\mathbf{P}$ (for a given $\langle N, A\rangle$). Let $IS : N \to 2^{\mathcal{L}(A)^n}$ assign to each voter a set of profiles (information set) such that $\mathbf{P} \in IS(i)$ for all $i \in N$, and all profiles in $IS(i)$ agree on $\succeq_i$. A resolute SCF f is *vulnerable to dominating manipulations* (in $\mathbf{P}$) if there exist a voter i and ballot $\succeq_i^\bullet$ s.t

(a) $f(\mathbf{P}'_{-i}, \succeq_i^\bullet) \succeq_i f(\mathbf{P}')$ for all $\mathbf{P}' \in IS(i)$,

(b) $f(\mathbf{P}'_{-i}, \succeq_i^\bullet) \succ_i f(\mathbf{P}')$ for some $\mathbf{P}' \in IS(i)$

where $\mathbf{P}'_{-i}$ denotes the partial profile obtained from $\mathbf{P}'$ by removing i's ballot. Ballot $\succeq_i^\bullet$ is called a *dominating manipulation* by i. A SCF is said to be *immune to dominating manipulations* iff it is not vulnerable to dominating manipulations in any $\mathbf{P}$.

Observe that dominating manipulations reduce to simple manipulations (Definition 3.1) whenever a manipulator has full information, that is, his information set is $\{\mathbf{P}\}$. The definition allows then to break the negative result of Gibbard-Satterthwaite theorem by exploiting the fact that the larger the information set, the harder it becomes to find a dominating manipulation.

Fact 3.1. The Borda rule with a deterministic tie-breaking rule is immune to dominating manipulation when the manipulator's information set is $\subseteq$-maximal (i.e., the manipulator has no information).

Proof. See Exercise 3.8. □

Using randomization

One way to rule out (deterministic) manipulations altogether is by introducing randomization in the workings of SCFs. A randomized SCF (RSCF) is a function

$$r : \mathcal{L}(A)^n \to \Delta(A) \tag{3.2}$$

where $\Delta(A) = \{\mathbf{x} \in \mathbb{R}^m \ : \ \forall a \in A, \mathbf{x}_a \geq 0 \text{ and } \sum_{a\in A} \mathbf{x}_i = 1\}$ is the set of all lotteries (distributions) over A.

An elegant theorem by Gibbard [Gibbard, 1977] shows that the random draw of one agent (the so-called randomized dictatorship) from N is the only RSCF that is 'lottery' strategy-proof (in the specific sense of stochastic dominance[5]) and 'lottery' efficient (in the sense of never assigning positive probability to alternatives that are Pareto dominated).

3.3.3 Letting manipulation be

What happens if every voter is allowed to act as a manipulator? In particular, can we quantify exactly how bad the social choice on the fully manipulated profile would be when compared with the social choice supported by a truthful profile? To answer this question requires a game-theoretic perspective on voting.[6]

Any (resolute) SCF f for a given context $\langle N, A\rangle$ induces an ordinal game $G = \langle N, A, \mathcal{L}(A)^n, \mathbf{P}, f\rangle$ where:

- N is the set of players;

[5] A lottery p stochastically dominates a lottery q for i iff $\sum_{x\in A: x\succ_i y} p(x) > \sum_{x\in A:x\succ_i y} q(x)$, for any $y \in A$. A randomized rule is strategy-proof, w.r.t. stochastic dominance if for all $i \in N$, it never selects a lottery which is stochastically dominated by another lottery for i.

[6] For an introduction to game theory see [Osborne and Rubinstein, 1994].

- A is the set of outcomes;
- $\mathcal{L}(A)$ is the strategy space of each agent (this could be coarsened to A, e.g., for plurality);
- $\mathbf{P}$ are the (true) preferences of the players;
- f is the outcome function mapping strategy profiles to outcomes.

A (pure-strategy) Nash Equilibrium of G is a profile $\mathbf{P} = \langle \succeq_1, \ldots, \succeq_n \rangle$ such that there exists no player $i \in N$ such that for some $\succeq_i'$

$$f(\mathbf{P}_{-i}, \succeq_i') \succ_i f(\mathbf{P})$$

That is, no player can profit by unilaterally misrepresenting her preference with a different ballot. No agent has a profitable deviation in $\mathbf{P}$ and is therefore playing a so-called *best response*.

Example 3.5. Recall Example 3.3 with $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$, and f be plurality with an alphabetic tie-breaking rule ($a > b > c$). Assume the following profile of true preferences

$$\mathbf{P} = \begin{array}{c|ccc} 1 & a & c & b \\ 2 & b & c & a \\ 3 & c & b & a \end{array}$$

From this profile agent 3 has a best response consisting of ballot *bca*, leading to a NE where $\{b\}$ is the social choice. Similarly, agent 2 has also a best response consisting of ballot *cba*, which leads to a NE where, instead, $\{c\}$ is the social choice.

The question that has been asked in the iterative voting literature [Brânzei et al., 2013, Meir, 2018a] is how far a NE is from the truthful profile, once such NE is obtained through a sequence of best responses starting at the truthful profile. A way to make this question precise for the class of scoring rules (Definition 2.4) is by using the so-called (dynamic) price of anarchy:

$$PoA(f) = \min_{\mathbf{P}} \min_{\mathbf{P}' \in NE_{\mathbf{P}}} \frac{s_f(f(\mathbf{P}), \mathbf{P})}{s_f(f(\mathbf{P}'), \mathbf{P}')} \tag{3.3}$$

where $s_f(x, \mathbf{P})$ denotes the score of x given $\mathbf{P}$ according to f, and $NE_{\mathbf{P}}$ is the set of NE reachable from $\mathbf{P}$ via best-response dynamics. That is, PoA denotes the worst case ratio between the true score of an alternative and the score it would receive in equilibrium.

It has been shown [Brânzei et al., 2013] that $PoA(\texttt{Plurality})$, with a deterministic tie-breaker, is close to 1. In other words, full-blown manipulation of the plurality rule does not have much impact on the quality of the outcome. A much more negative result has been proven for the Borda rule (cf. [Brânzei et al., 2013]).

3.4 Chapter notes

The bulk of this chapter is again based on the introductions to voting theory provided in [Brandt et al., 2016, Ch. 2], [Taylor, 2005, Ch. 1], [Endriss, 2011] and [Taylor and Pacelli, 2008].

The proof of Theorem 3.2 is based on [Batteau et al., 1981]. Since the original articles [Gibbard, 1973, Satterthwaite, 1975] many alternative proofs of the theorem have been presented, e.g.: [Barberá, 1983, Benoît, 2000].

Section 3.3 is based on several sources: [Conitzer and Walsh, 2016] (complexity of manipulation), [Brandt, 2018] (randomization), [Meir, 2018a] (voting games), [Conitzer et al., 2011] (information).

3.5 Exercises

Exercise 3.1. Prove item *iii)* of Lemma 3.1.

Exercise 3.2. Define a variant of manipulability for irresolute SCFs as follows: a SCF f (for a given $\langle N, A\rangle$) is *manipulable* if there exist two profiles

$$\mathbf{P} = \langle \succeq_1, \ldots, \succeq_i, \ldots \succeq_n\rangle \text{ and } \mathbf{P}^\bullet = \langle \succeq_1, \ldots, \succeq_{i-1}, \succeq_i^\bullet, \succeq_{i+1}, \ldots, \succeq_n\rangle$$

such that $f(\mathbf{P}^\bullet)$ and $f(\mathbf{P})$ are singletons and $f(\mathbf{P}^\bullet) \succ_i f(\mathbf{P})$.

Prove that the Plurality rule (Rule 1) is not manipulable according to the above definition.

Exercise 3.3. Let F be a resolute SCF (for a given $\langle N, A\rangle$) which is non-imposed and strategy-proof. Prove that if $C \in \mathcal{B}^{xy}$ for some $x, y \in A$, then $C \in \mathcal{B}$. This is the equivalent of the contagion lemma (Lemma 2.1) for the Gibbard-Sattertwhaite theorem (Theorem 3.2).

Exercise 3.4. In no more than three sentences discuss the differences between Arrow's theorem (Theorem 2.2) and the Gibbard-Sattertwhaite theorem (Theorem 3.2).

Exercise 3.5. For $|A| = m$, $|\mathcal{L}(A)^n| = (m!)^n$. Now fix an order $\gg$. How many single-peaked profiles exist for $\gg$? Justify your answer.

Exercise 3.6. Determine whether the median voter rule (Rule 17) satisfies: unanimity, independence, monotonicity. Justify your answers.

Exercise 3.7. Use the variant of May's theorem with ties (Theorem 1.2) to prove that, when $m = 2$, the median rule (Rule 17) is anonymous, neutral and positively responsive.

Exercise 3.8. Prove Fact 3.1.

Exercise 3.9. Does the Gibbard-Satterthwaite theorem (Theorem 3.2) still hold if we drop the assumption of non-imposition? Justify your answer. If the theorem still holds explain how the proof can go through without that assumption. If the theorem does not hold any more, and therefore there exist SCFs which is strategy-proof and non-dictatorial, provide one of such functions.

Exercise 3.10. A binary relation $\preceq$ over A is said to be a *dichotomous preference* if there exists $X \subseteq A$ such that, for all $x, y \in A$: $x \succeq y$ if and only if $x \in X$ or $y \notin X$. In other words, a dichotomous preference is a total preorder (an order admitting ties) that consists of only two indifference classes: the class X of top elements, and the class $\overline{X}$ of bottom elements. Dichotomous preferences correspond directly to approval ballots (see Remark 2.3), where the approval ballot is the set X of top elements in the preference of a voter. Now consider the approval voting rule `Approval` (Rule 11) and:

a) Prove that $\texttt{Approval}(\mathbf{P}) = \texttt{Condorcet}(\mathbf{P}) = \texttt{Borda}(\mathbf{P})$ where $\mathbf{P}$ is a profile of dichotomous preferences (recall Rule 6 and Rule 8);

b) Determine whether `Approval` is manipulable by an optimistic manipulator and whether it is manipulable by a pessimistic manipulator (see Remark 3.1 for the relevant definitions). For each claim provide a proof or a counterexample.

Chapter 4

Choosing Some Out of Many

The social choice problem we considered so far concerned how to select *one* 'best' alternative given a profile of individual rankings, or more generally a set of tied 'best' alternatives. In this chapter we present results on the related problem of selecting one 'best' set of alternatives—a so-called *committee*—or, more generally, one tied set of sets of 'best' alternatives. This problem is ofter referred to as *multi-winner voting* or *committee selection*. It is a much more recent, and therefore less consolidated, area of research in social choice and many directions of research in this area are still open.

Many collective decision making scenarios fall under the multi-winner voting setting. For example: selecting a representative body in the political context (a parliament), corporate context (a board) or academic context (a council); shortlisting finalists in a competition based on the opinions of a set of experts or judges; selecting locations in so-called facility location problems [Farahani and Hekmatfar, 2009]; selecting proposals or projects based on the opinions of citizens in digital democracy platforms (e.g., LiquidFeedback [Behrens et al., 2014] or Polis [Small et al., 2021]); selecting recommendations from crowdsourced data in a collaborative filtering context [Chakraborty et al., 2019]; selecting validators in distributed computing protocols [Cevallos and Stewart, 2021].

We follow the same approach used to introduce and discuss social choice functions in Chapter 1: we provide the general definition of functions for committee selection; we then provide concrete examples of such functions; we finally show how also these functions can be studied from an axiomatic point of view.

4.1 Preliminaries

4.1.1 Multi-winner social choice

We are interested in situations where a set of alternatives—the *committee*—of a given integer size $1 \leq k \leq m$, is to be selected based on the preferences over A of n individuals. Given a context $\langle N, A \rangle$ a multi-winner SCF or committee-selection function (CSF), for a given committee size $1 \leq k \leq n$, is a function

$$f_k : \mathcal{L}(A)^n \to 2^{\{X \subseteq A \ : \ |X|=k\}} \tag{4.1}$$

Given a profile of linear orders, the function outputs a set of sets of alternatives (i.e., a set of committees), all of size k. Obviously a CSF f_1 is equivalent to a SCF, as it selects a committee of size 1. When we leave k unspecified, we talk about a family of CSFs mapping every integer k in $\{1, \dots n\}$ to the CSF f_k. In this chapter, we refer to families of CSFs as committee selection or *multi-winner (voting) rules*.

Remark 4.1. Notice that there is a natural way in which the problem of committee selection can be reframed as a social choice problem where committees of size k *are* the alternatives relevant for the social choice. In other words the social choice context would be $\left\langle N, \left\{X \subseteq 2^A \ : \ |X| = k\right\}\right\rangle$, and voters would need to submit ballots ranking all elements in such a set. This would amount to asking voters to rank $\binom{m}{k} = \frac{n!}{k!(n-k)!}$ alternatives! Given a profile of such ballots a SCF could be applied to select a tied set of committees. The type in (4.1) takes a different—and for obvious reasons more practical—approach selecting committees using only information that consists of rankings of alternatives.

4.1.2 Design principles behind CSFs

At the beginning of the chapter we mentioned several application domains for CSFs. Different such domains may require CSFs to be driven by different broad principles:

Excellence-driven domains Here the issue is to select the best k alternatives from A. A typical example of this type of committee selection problems are shortlisting processes for job or funding applications.

Diversity-driven domains Here the issue is to select k alternatives that can cover all possible views on A that are present among the voters. Typical examples of this type are: facility location problems (e.g., where to place k new schools in order to serve the schooling needs of a whole city or district), movie selection for an airline entertainment system, product selection for the homepage of an internet store.

Proportionality-driven domains Here the issue is to select k alternatives that can represent the views of N and their relative sizes as faithfully as possible. The typical example are parliamentary elections where one wants to allocate parliament seats proportionally to citizens' political preferences.

All the above domains come intuitively with different requirements on how the CSF should ideally behave in order to select suitable committees. In the next sections we will look at several rules more or less loosely inspired by the above principles—first for preference ballots, and then for approval ballots—and at axiomatic results for them. We will first focus on rules taking as input preferential ballots as per Equation (4.1) (Sections 4.2 and 4.3) and then move to the more extensively studied setting of approval-based committee selection rules (Sections 4.4 and 4.4.3).

4.2 Some multi-winner voting rules

In this section we will first present examples of excellence- and diversity-driven rules. Finally we also provide examples of multi-round rules for committee selection akin to those we encountered in Chapter 2, which are commonly used in practice.

4.2.1 Excellence-driven rules

Best-k multi-winner rules are based on the notion of social preference function (SPF) we encountered in Chapter 2. Recall that these functions map profiles of linear orders (the individual preferences) to a total preorder (Equation (2.2)), that is, a ranking that may contain ties. Notice, however, that each total preorder $\preceq$ can be represented by a set of linear orders, that is, all the possible linear orders that one obtains by resolving the ties in $\preceq$.[1] So, in this section we will treat the output $G(\mathbf{P})$ of an SPF G on a profile $\mathbf{P}$ as a set of linear orders rather than as a total preorder. The intuition behind best-k multi-winner rules is to form a committee by simply selecting the top k alternatives in some of those linear orders representing possible social preferences. Before moving to define the rules in this class formally we need one more piece of notation: to denote the top k element of a linear order $\succeq \in A^2$ we write $\max^k_{\succeq}(A)$. Here is the formal definition:

Definition 4.1 (Best-k rules)**.** Given an SPF G a best-k rule $\texttt{Best}^G$ (for G) is the family of CSFs defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$ and integer $1 \leq k \leq n$

$$\texttt{Best}^G_k(\mathbf{P}) = \left\{ X \subseteq A \ : \ \text{ exists } \succeq \in G(\mathbf{P}) \text{ s.t. } X = \max^k_{\succeq}(A) \right\}.$$

[1] By means of an example, this means that we treat total preorders like $x\{y,w\}z$, where y and w are tied alternatives, as sets of linear orders $\{xywz, xwyz\}$ where the tie between y and w has been resolved.

A family of CSFs f is said to be a best-k rule whenever there exists a G such that, for every $1 \leq k \leq n$, $f_k = \texttt{Best}_k^G(\mathbf{P})$.

Depending on the choice of G in Definition 18 we obtain different multi-winner rules. In particular, if we take G to be based on specific scoring functions we obtain, for example:

Rule 18. The following are best-k ($1 \leq k \leq n$) multi-winner rules:

k-Plurality i.e., $\texttt{Best}_k^{\texttt{Plurality}}(\mathbf{P})$, also known as single non-transferable vote;

k-Approval i.e., $\texttt{Best}_k^{k\texttt{Approval}}(\mathbf{P})$, also known as bloc voting;

k-Borda i.e., $\texttt{Best}_k^{\texttt{Borda}}(\mathbf{P})$.

where `Plurality`, k`Approval` and `Borda` refer to the SPFs ranking alternatives by their plurality, approval and, respectively, Borda scores.

One can in the same fashion use the Copeland (Rule 7), Kemeny (Rule 14) or Dodgson (Rule 15) scores to obtain the corresponding rankings and then cut them to the top k alternatives to obtain the committee. These are sometimes referred to also as 'score-and-cut' and 'rank-and-cut' rules.

4.2.2 Diversity-driven rules

Several diversity-driven multi-winner rules can be formulated as follows:

Definition 4.2 (Chamberlin-Courant [Chamberlin and Courant, 1985])**.** For an integer $1 \leq k \leq n$ and scoring vector $\mathbf{w}$ (Definition 2.4) a Chamberlin-Courant rule for size k is defined as follows. For any profile $\mathbf{P} \in \mathcal{L}(A)^n$:

$$\texttt{CC}_k^{\mathbf{w}}(\mathbf{P}) = \underset{C, |C|=k}{\text{argmax}} \sum_{x \in C} \sum_{i \in N_{\mathbf{P}}^{Cx}} w_{i(x)}$$

where $N_{\mathbf{P}}^{Cx} = \{i \in N \ : \ x = \max_{\succeq_i}(C)\}$ is the set of voters i that rank x as top among the alternatives in C (x is then said to be the *representative* of i in C); and $i(x)$ as usual denotes the position of x in $\succeq_i$. Quantity $\sum_{x \in C} \sum_{i \in N_{\mathbf{P}}^{Cx}} w_{i(x)}$ is called the *representativeness value* of a given committee C.

By varying the scoring vector $\mathbf{w}$ we thus obtain different diversity-driven families of CSFs. For example:

Rule 19. The Borda-based k-Chamberlin-Courant multi-winner rule is the rule $\texttt{CC}_k^{\mathbf{w}}$ where $\mathbf{w}$ is the Borda scoring vector.

The intuition behind this rule is simple. It forms the committee that contains as highly ranked (by Borda score) alternatives as possible by as many voters as possible. So, if an alternative is to be added to the committee, alternatives with higher ranks from agents that do not yet have a representative in the committee will be preferred over alternatives with lower ranks from agents who already have representatives in the committee.

4.2.3 Sequential (choose-and-repeat) rules

These are multi-winner variants of the multi-stage rules we encountered in Chapter 2.

Rule 20 (Sequential plurality)**.** The sequential plurality rule is the CSF such that, for any profile $\mathbf{P} \in \mathcal{L}(A)^n$ and integer $k \geq 1$:

$$\texttt{SPlurality}_k(\mathbf{P}) = \left\{X \ : \ \text{exist } x^1, \ldots, x^k \text{ s.t. } X = \left\{x^1, \ldots, x^k\right\}\right\}$$

where each x^i is inductively defined as follows, with $1 \leq \ell < k$:

- $x^1 = \texttt{Plurality}(\mathbf{P})$, assuming tie-breaking;
- $x^{\ell+1} = \texttt{Plurality}(\mathbf{P}|_{A \setminus \{x^1, \ldots, x^\ell\}})$, assuming tie-breaking.

Perhaps surprisingly, notice that sequential plurality and k-plurality (Rule 18) are actually different rules as they may elect different committees (see Exercise 4.1).

Rule 21 (Single transferable vote for committees)**.** The single transferable vote rule for committee selection (CSTV) is the CSF defined as follows, for any profile $\mathbf{P}$ and $k \leq 1$:

$$\texttt{CSTV}_k(\mathbf{P}) = \left\{ X \ : \ \text{exists } S^\ell = \left\langle X^\ell, \mathbf{P}^\ell \right\rangle \text{ s.t. } \left| X^\ell \right| = k \right\}$$

where stages S^ℓ are recursively defined as follows:

$$S^0 = \left\langle X^0, \mathbf{P}^0 \right\rangle \text{ where } X^0 = \emptyset, \mathbf{P}^0 = \mathbf{P}$$

$$S^{\ell+1} = \begin{cases} \left\langle X^\ell \cup \{x\}, \left(\mathbf{P}^\ell|_{A \setminus \{x\}} \right)_{-C} \right\rangle & \text{if } \texttt{Plurality}(\mathbf{P}^\ell)(x) \geq q \text{ (assuming tie-breaking)} \\ \left\langle X^\ell, \mathbf{P}^\ell|_{A \setminus \{y^\ell\}} \right\rangle & \text{otherwise} \end{cases}$$

where: $\texttt{Plurality}(\mathbf{P}^\ell)$ denotes the plurality scoring function in the profile at stage ℓ; y^ℓ is the alternative with lowest plurality score in the profile at stage ℓ (i.e., $y \in \text{argmin}_{z \in A}(\texttt{Plurality}(\mathbf{P}^\ell)(z))$); $C \subseteq N$ is a set of size q consisting of voters who rank x as top in their preferences in profile $\mathbf{P}^\ell$; and $q = \lfloor \frac{n}{k+1} \rfloor + 1$.

Intuitively, at each stage the following happens in multi-winner STV: an alternative is added to the committee if the plurality score of such alternative meets the threshold q (there may be many such sets), and if that is the case, the alternative is discarded as well as a set of voters of size q (there may be many such sets); otherwise no alternative is added to the committee and a plurality loser (there may be many such alternatives) is discared. The above is repeated until a committee of size k is constructed.

Observe that the rule is non-deterministic, involving several tie-breaking decisions at each step. Intuitively all different committees resulting from different tie-breaking decisions are recorded in the output of the rule.[2]

Example 4.1 ([Faliszewski et al., 2017])**.** Consider the following profile for $\{1, \ldots, 6\}$ and $\{a, b, c, d, e\}$:

$$\mathbf{P} = \begin{array}{c|ccccc} 1 & a & b & c & d & e \\ 2 & e & a & b & d & c \\ 3 & d & a & b & c & e \\ 4 & c & b & d & e & a \\ 5 & c & b & e & a & d \\ 6 & b & c & d & e & a \end{array}$$

We have:

$$\begin{aligned} \texttt{CSTV}_2(\mathbf{P}) &= \{\{b, c\}, \ldots \text{left to the reader} \ldots\} \\ \texttt{Best}_2^{\texttt{Plurality}}(\mathbf{P}) &= \{\{c, a\}, \{c, b\}, \{c, d\}, \{c, e\}\} \\ \texttt{CC}_2^{\langle m-1, m-2, \ldots, 0 \rangle}(\mathbf{P}) &= \{\{a, c\}\} \\ \texttt{PAV}_2(\mathbf{P}) &= \{\{a, b\}\} \end{aligned}$$

[2] This is sometimes referred to as the parallel-universe tie-breaking model.

In the $\mathtt{CC}_2^{\langle m-1,m-2,\ldots,0\rangle}$ alternative a represents voters in $\{1,2,3\}$ and c represents voters in $\{4,5,6\}$. In the $\mathtt{PAV}_2$ the winning committee $\{a,b\}$ obtains 6.5 points.

Remark 4.2 (Droop quota)**.** The quota $q = \lfloor \frac{n}{k+1} \rfloor + 1$ in the committee STV (Rule 21) is known as the Droop quota, from Richmond Droop, English lawyer and mathematician who introduced it (see [Droop, 1881]). It is the most commonly used quota for STV committee elections, for instance in the Republic of Ireland. The quota is the smallest integer that guarantees that no candidate who would reach the quota would then have no place available in the committee. So it is a generalization of the idea of simple majority when electing committees of size $k = 1$. It can be derived as follows. The quota q should satisfy two constraints: $k \cdot q \leq n$, i.e., the number of candidates meeting the quota cannot exceed the number of voters; and $n \leq k \cdot q + (q-1)$, i.e., there cannot be a $k+1^{th}$ candidate who meets the quota. From this we obtain:

$$\left\lceil \frac{n+1}{k+1} \right\rceil = \left\lfloor \frac{n}{k+1} \right\rfloor + 1 \leq q \leq \frac{n}{k}.$$

Taking the smallest q satisfying the above inequalities thus gives us the Droop quota.

4.3 Axiomatic results

In this section we showcase the application of the axiomatic method to the case of multi-winner rules. We focus on the generalization of the notion of Condorcet-consistency (recall Definition 2.5) to the case of committee selection.

4.3.1 Condorcet consistency and monotonicity

Definition 4.3 (Condorcet committees [Gehrlein, 1985])**.** Let a profile $\mathbf{P}$ for context $\langle N, A \rangle$ be given. A set $C \subseteq A$ is a *weak Condorcet committee* if, for all $x \in C$ and $y \notin C$, $\#_{\mathbf{P}}^{xy} \geq \#_{\mathbf{P}}^{yx}$. For a given profile $\mathbf{P}$, we denote its set of Condorcet commitees of size k by $\mathtt{CC_k}(\mathbf{P})$.

Intuitively, a committee is Condorcet consistent whenever it does not elect any alternative for which a different alternative is preferred by a majority of voters. Notice that the set of Condorcet committees for a given size k may obviously be empty.

Example 4.2 ([Barberá and Coelho, 2008])**.**

$$\mathbf{P} = \begin{array}{c|ccccc} 1 & a & b & c & d & e \\ 2 & a & b & e & c & d \\ 3 & a & b & d & e & c \\ 4 & c & d & e & a & b \\ 5 & e & c & d & a & b \\ 6 & d & e & c & a & b \end{array}$$

We have that $\mathtt{CC}_1(\mathbf{P}) = \{\{\mathtt{a}\}\}$, $\mathtt{CC}_2(\mathbf{P}) = \{\{\mathtt{a},\mathtt{b}\}\}$ and $\mathtt{CC}_3(\mathbf{P}) = \{\{\mathtt{c},\mathtt{d},\mathtt{e}\}\}$.

Definition 4.4. A family of CSFs f (for a given $\langle N, A \rangle$) is:

Condorcet consistent (or) **stable** iff for all $\mathbf{P} \in \mathcal{L}(A)^n$, and $1 \leq k \leq n$, $f_k(\mathbf{P}) \subseteq \mathtt{CC_k}(\mathbf{P})$ whenever $\mathtt{CC_k}(\mathbf{P}) \neq \emptyset$.

Intuitively, the rule always selects a weak Condorcet committee when one exists.

Committee monotonic iff for all $\mathbf{P} \in \mathcal{L}(A)^n$ and $1 \leq k < n$,

- if there exists $C \in f_k(\mathbf{P})$ then there exists $C' \in f_{k+1}(\mathbf{P})$ s.t. $C \subset C'$;
- if there exists $C' \in f_{k+1}(\mathbf{P})$ then there exists $C \in f_k(\mathbf{P})$ s.t. $C \subset C'$.

Intuitively, an alternative selected for a small committee should be selected also in larger committees.

Observe that the first condition of committee monotonicity would suffice to express the axiom if f is a family of resolute CSFs.

Theorem 4.1 ([Barberá and Coelho, 2008])**.** There exists no family of CSFs f such that f is Condorcet consistent and committee monotonic.

Proof. Assume towards a contradiction that this is not the case: there exists f_k such that, for any k, f_k is Condorcet consistent and committee monotonic. Let $\mathbf{P}$ be the profile of Example 4.2. Since f_2 is Condorcet consistent by assumption, $\{a, b\} \in f_2(\mathbf{P})$. Similarly, $\{c, d, e\} \in f_3(\mathbf{P})$. Contradiction. □

Theorem 4.2 ([Elkind et al., 2017])**.** A family of CSFs f is committee-monotonic if and only if it is a best-k rule.

Sketch of proof. Right-to-left Best-k rules are committee monotonic by construction (Definition 4.1). Left-to-right The proof is by construction. Given a committee monotonic multi-winner rule f we can define a SPF G^f which, for every profile $\mathbf{P}$, chooses a social preference (total pre-order) $\preceq$ such that the alternatives in $\preceq$ with rank k are precisely the elements in $f_k(\mathbf{P}) \backslash f_{k-1}(\mathbf{P})$ (recall Remark 2.4). Given G Definition 4.1 gives us a best-k rule as desired. □

4.4 Approval-based committee selection rules

In this chapter we are concerned with a type of social choices about which voters do not provide direct information: voters' ballots concern preferences about alternatives, but what is selected by the committee-selection function is a set of candidates. There is therefore a mismatch between the preferential information individuals provide and the social choice that is made based on those preferences. Ideally, one would like to have access to the preferences that individuals have about committees, rather than alternatives, but that is for obvious reasons unfeasible in general (recall Remark 4.1). One approach to still pursue this idea is to request voters to submit simpler types of ballots—known as approval ballots (recall Remark 2.3)—and then simply assume that voters will in general prefer committees that contain more candidates that they approve of. In this section we explore the above approach to committee selection working with approval-based CSFs or, in short, AB-CSFs.

4.4.1 Approval-based rules

Recall that i's approval ballot is a set $X_i \subseteq A$ denoting the alternatives that i approves of. An approval ballot profile is therefore a tuple $\mathbf{X} \in (2^A)^n$. We will also be using the following notation: given an approval profile and an alternative a, we denote with N^a the set of voters who approve a.

Excellence

We already encountered, in Chapter 2, one rule that worked with this type of ballots: *approval voting* (Rule 11).

Rule 22 (Multi-Winner Approval)**.** The multi-winner approval voting rule `Approval` for committee size k, is defined as follows, for any approval profile $\mathbf{X} \in (2^A)^n$:

$$\texttt{Approval}_k(\mathbf{X}) = \underset{C:|C|=k}{\operatorname{argmax}} \sum_{x \in C} | \{i \in N \ : \ x \in X_i\} |.$$

So, `MWApproval` selects the set of alternatives that carry the largest number of approvals in the submitted ballots. Recalling Rule 11, it is worth observing that $\texttt{Approval} = \texttt{Approval}_1$, that is $\texttt{Approval}_k$ reduces to (single-winner) approval voting when the committee has size 1. The rule is also a direct application of the exellence logic of Definition 4.1 to approval profiles (see Exercise 4.4).

Multi-winner approval voting 'greedily' maximizes the number of approvals. This, however, may leave some voters without even just one representative—that is, an alternative they approve of—in the selected committee, as the following example illustrates.

Example 4.3 ([Lackner and Skowron, 2023])**.** A small club of 12 members needs to select a steering committee of size $k = 4$. There are 7 candidates/alternatives: $A = \{a, b, c, d, e, f, g\}$. This is the approval profile, rendered by providing the counts of identical ballots:

$$\mathbf{X} = \begin{array}{r|l} \#3 & \{a, b\} \\ \#3 & \{a, c\} \\ \#2 & \{a, d\} \\ \#1 & \{b, c, f\} \\ \#1 & \{e\} \\ \#1 & \{f\} \\ \#1 & \{g\} \end{array}$$

The approval counts of each alternative are: 8 for a; 4 for b and c; 2 for d and f; 1 for e and g. We therefore have that $\texttt{Approval}_4(\mathbf{X}) = \{\{a, b, c, d\}, \{a, b, c, f\}\}$, selecting two tied committees. The first committee ($\{a, b, c, d\}$) leaves three voters without representation. The second committee ($\{a, b, c, f\}$) leaves two voters without representation.

Diversity

Instead of greedily maximizing approvals, the following rule attempts instead to guarantee as many voters as possible with a representative in the selected committee—that is, guarantee a committee which is as diverse as possible in terms of representation of the voters.

Rule 23 (Approval Chamberlin Courant)**.** The multi-winner approval-based Chamberlin-Courant voting rule `ACC`, is defined as follows, for any approval ballot profile $\mathbf{X} \in (2^A)^n$ and committee size k:

$$\texttt{ACC}_k(\mathbf{X}) = \underset{C:|C|=k}{\operatorname{argmax}} | \{i \in N \ : \ C \cap X_i \neq \emptyset\} |.$$

Intuitively, approval-based Chamberlin-Courant looks for the committees that maximize the number of voters with at least one representative in the committee.[3] This is the same logic driving the class of rules captured by Definition 4.2, transferred to approval ballots (see Exercise 4.5).

Example 4.4 (Example 4.3 continued.)**.** Applying $\texttt{ACC}_4$ to the profile of Example 4.3 we obtain one single committee: $\{a, f, e, g\}$. This committee guarantees one representative to each voter.

Proportionality via 'diminishing returns' from representation

Between greedily maximizing the number of approvals (multi-winner approval), and just maximizing representation while disregarding approval counts above 1 (Chamberlin-Courant), a proportionality-driven

[3] The rule was first introduced by the Danish mathematician Thorvald Thiele in 1895 and later rediscovered by the American political scientists John Chamberlin and Paul Courant in [Chamberlin and Courant, 1985].

logic would suggest the need to strike a trade-off between these two extremes. This can be achieved by incorporating a form of 'diminishing returns' in the representation provided by a committee. Intuitively, while having one representative in the selected committee yields some degree of 'satisfaction' for the voter, any further representative would still increase this 'satisfaction' but by an increasingly smaller amount. The harmonic numbers series $1, \frac{1}{2}, \frac{1}{3}, \ldots$ can be used to capture such a form of diminishing returns of representation and underpins the definition of the following rule.

Rule 24. The proportional approval voting rule $\mathtt{PAV}_k$ for committee size k is defined as follows, for any approval profile $\mathbf{X} \in (2^A)^n$:

$$\mathtt{PAV}_k(\mathbf{X}) = \underset{C, |C|=k}{\text{argmax}} \sum_{i \in N} \sum_{1 \leq \ell \leq |X_i \cap\ C|} \frac{1}{\ell}$$

That is, the rule finds the committee that maximizes a score that is computed by giving to each agent i as many points as $1 + \frac{1}{2} + \ldots + \frac{1}{\ell}$, where ℓ is the number of alternatives in the committee that are approved by i. As a consequence that the rule will always give priority to committees that contain alternatives representing more agents rather than committees that contain many alternatives representing only few agents, like in the approval-based Chamberlin-Courant rule. At the same time, alternatives that are more often approved of will be more likely to be included in the committee, reflecting a form of proportionality.

Example 4.5 (Example 4.3 continued)**.** Applying $\mathtt{PAV}_4$ to the profile of Example 4.3 we obtain one single committee: $\{a, b, c, f\}$. Here, one agent (the one voting for $\{b, c, f\}$) obtains three representatives and therefore a satisfaction of $1 + \frac{1}{2} + \frac{1}{3}$, 6 agents (those voting for $\{a, b\}$ and $\{a, c\}$) a satisfaction of $1 + \frac{1}{2}$, and 1 (that voting for $\{f\}$) a satisfaction of 1. Notice that $\{a, b, c, f\}$ happens to be one of the two committees selected by $\mathtt{Approval}_4$, namely the one containing only two voters without representatives.

Proportional approval voting was first proposed by the Danish mathematician Thorvald Thiele at the end of the 19th century in [Thiele, 1895], at the dawn of electoral suffrage in the Nordic countries. The rule was used in practice at the beginning of the 20th century in Sweden.

Remark 4.3 (Thiele methods)**.** Proportional approval voting is the main example of the class of approval-based committee selection functions known as *Thiele methods*. These can be thought of committee-selection functions parametrized by a non-decreasing weighting function $w : \mathbb{N} \to \mathbb{R}$ where $w(0) = 0$. This function models the level of 'sastisfaction' that voters have by obtaining a given number of alternatives in the selected committee. For example, in $\mathtt{PAV}_k$, such weighting function is given by the harmonic number series. Thiele methods then select the committees that maximize the total sum, over all agents, of the obtained weights. This optimization may be computationally hard, so rules such as $\mathtt{PAV}_k$ can be given a sequential formulation: one constructs the committee iteratively by selecting, at each stage, the alternative that would guarantee the highest weight to the committee. We refer to [Lackner and Skowron, 2023] for more details on this type of rules.

Proportionality via cost-balancing

Another approach to mediate between approval voting and Chamberlin-Courant uses the intuition that including an alternative in the committee incurs a 'cost', which is shared equally among all voters approving that alternative. In this view, the best committees are therefore those that 'balance' such costs in order to treat all voters as fairly as possible.

The following rule operationalizes the above intuition by assuming that the cost of adding one specific candidate to the committee is always 1, and this cost is balanced among all agents that support the alternative. So, the cost that every supporter of a incurs by adding a to the committee is 1 (the cost of the new alternative), plus the sum of the costs already incurred by a's supporters, divided by how many a's supporters there are. The intuition here is that the cost of any new alternative a, which is added to the committee, is spread in such a way that all a's supporters will have incurred the same cost after a has been added.

Rule 25. The sequential Phragmén voting rule $\texttt{SeqPhrag}_k$ for committee size k, constructs the winning committee $C = \{x_1, \ldots, x_k\}$ as follows, for any approval profile $\mathbf{X} \in (2^A)^n$, and $1 \leq \ell$:

$$x_\ell = \operatorname*{argmin}_{x \in A \setminus \{x_1, \ldots, x_{\ell-1}\}} \frac{1 + \sum_{i \in N^x} \mathsf{cost}_{\ell-1}(i)}{|N^x|}, \qquad \text{(ties broken deterministically)}$$

where $\mathsf{cost}_0(i) = 0$ for every $i \in N$ and

$$\mathsf{cost}_{\ell+1}(i) = \begin{cases} \frac{1+\sum_{i \in N^x} \mathsf{cost}_\ell(i)}{|N^x|} & \text{if } i \in N^x, \\ \mathsf{cost}_\ell(i) & \text{otherwise.} \end{cases}$$

At stage k a committee of the desired size is formed:

$$\texttt{SeqPhrag}_k(\mathbf{X}) = \{x_1, \ldots, x_k\}.$$

Intuitively, sequential Phragmén[4] constructs a winning committee in rounds by adding one candidate at a time. The candidate added is the one that imposes the smallest burden in terms of (balanced) cost on its supporters. The rule has also a natural *continuous* formulation: voters receive money at a constant rate of one unit of money per unit of time; as soon as the supporters of an alternative x have received enough money to pay together for x, x is added to the committee and their budgets are zeroed; the process continues until a committee of the desired size is formed. The intuition behind this process is that the candidates are chosen which impose the the smallest costs on their supporters and, therefore, which keep the amount of resources invested in each individual voter as equal as possible.

Example 4.6 (Example 4.3 continued)**.** Applying $\texttt{SeqPhrag}_4$ to the profile of Example 4.3 we compute the winning committee as follows:

$$\begin{array}{lll} x_1 = a & \text{balanced cost } a\text{'s supporters } \frac{1}{8} & \\ x_2 = b & \text{balanced cost } b\text{'s supporters } \frac{1+\frac{3}{8}}{4} = \frac{11}{32}, & \text{tie broken in favor of } b \text{ vs. } c \\ x_3 = c & \text{balanced cost } c\text{'s supporters } \frac{1+\frac{3}{8}+\frac{11}{32}}{4} = \frac{55}{128} & \\ x_4 = d & \text{balanced cost } c\text{'s supporters } \frac{1+\frac{2}{8}}{2} = \frac{5}{8} & \end{array}$$

So, the committee selected by sequential Phragmén is $\{a, b, c, d\}$, which is the first of the two committees tied by approval voting in Example 4.3.

4.4.2 Axiomatic results for approval-based rules

We refer to the very recent overview by [Lackner and Skowron, 2023].

4.5 Chapter notes

Multi-winner rules are a very recent, and rapidly developing, area of research within computational social choice. The chapter is based on [Barberá and Coelho, 2008, Elkind et al., 2017, Faliszewski et al., 2017] and the recent survey [Lackner and Skowron, 2023].

[4] Lars Edvard Phragmén was a Swedish mathematician, who published several works on election methods and was actively involved in the Swedish electoral reform debate at the end of the 19th century.

4.6 Exercises

Exercise 4.1. Construct a profile, and fix a k, such that $\texttt{SPlurality}_k(\mathbf{P})$ (Rule 20) and $\texttt{Best}_k^{\texttt{Plurality}}(\mathbf{P})$ (one of the rules in Rule 18) output different committees of size k for $\mathbf{P}$. Try to make such profile as small as possible. Based on your example explain the difference between the two rules.

Exercise 4.2. Construct a profile, and fix a k, such that $\texttt{Best}_k^{k\texttt{Approval}}$ (one of the rules in Rule 18) and $\texttt{PAV}_k$ (Rule 24) output different committees of size k for $\mathbf{P}$. Try to make such profile as small as possible. Based on your example explain the difference between the two rules.

Exercise 4.3. Every committee selection function f_k defines a social choice function when $k = 1$. What are the social choice functions defined by:

i) $\texttt{CC}_1^{\mathbf{w}}$, where $\mathbf{w}$ is the Borda scoring vector (Rule 19);

ii) $\texttt{CC}_1^{\mathbf{w}}$, where $\mathbf{w}$ is the ℓ-approval vector, where $1 \leq \ell \leq m$ (recall Rule 11);

iii) $\texttt{PAV}_1$ (Rule 24) where we assume the approval ballots to be determined by 1-approval (recall Rule 10).

Explain your answers.

Exercise 4.4. Recall the multi-winner approval rule (Rule 22) and the class of best-k rules (Definition 4.1). Prove that $\texttt{Approval}_k = \texttt{Best}_k^{\texttt{Approval}}$ on approval profiles.

Exercise 4.5. Under what conditions are $\texttt{ACC}_k$ (Rule 23) and $\texttt{CC}_k^{\mathbf{w}}$ (Rule 19) the same rule?

Chapter 5

Topics for Final Papers

This is a list of topics, in no specific order and with relevant blibliography, to be chosen from for the final papers.

Liquid democracy Liquid democracy is a form of voting where voting rights can be delegated transitively. It has been advocated in [Behrens et al., 2014] and used by, among others, the Pirate Party in Germany and France. It is implemented in the LiquidFeedback software.[1] A few lines of research in liquid democracy may be broadly identified. First, papers have tried to assess potential weaknesses of voting with transitive delegations. Problems the literature has focused on include: delegation cycles and the failure of individual rationality in multi-issue voting [Christoff and Grossi, 2017, Brill and Talmon, 2018]; accuracy of group decisions as compared to those achievable via direct voting in non-strategic settings [Kahng et al., 2018, Caragiannis and Micha, 2019, Halpern et al., 2021, Zhang and Grossi, 2022], as well as strategic ones [Bloembergen et al., 2019]; issues related to power [Zhang and Grossi, 2021]. Second, in response to these issues research has focused on the development of better behaved delegation schemes, e.g.: multiple delegations [Gölz et al., 2018]; complex delegations like delegations to a majority of trustees [Colley et al., 2020]; dampened delegations [Boldi et al., 2011]. Finally, implementations of liquid democracy for real-world applications have also been studied [Kling et al., 2015, Paulin, 2020]. See also [Behrens, 2017] for a history of the concept.

Participatory budgeting The participatory budgeting problem consists in selecting a set of projects within a given budget, based on the preferences of citizens in a given area (e.g., municipality).[2] It is a generalization of multi-winner voting where alternatives bear a cost and the selected committee (or 'bundle' in the participatory budgeting terminology) cannot exceed a predefined budget. A number of papers have recently proposed and studied different voting rules for participatory budgeting and studied their properties: [Benade et al., 2017, Aziz et al., 2018, Goel et al., 2019, Jain et al., 2020, Los et al., 2022]. Again the aim is to identify 'optimal' rules. See [Aziz and Shah, 2020] and especially [S. and Maly, 2023] for recent comprehensive overviews of this rapidly developing area of research.

Algorithms for online deliberation support Software for digital democracy aims at supporting online deliberation at scale among citizens. An example is Pol.is[3] [Small et al., 2021], which has been used for high-profile citizens' consultation in several countries around the world and is currently being piloted also in the Netherlands. Software like Pol.is relies on algorithms that solve social choice problems arising during mass deliberation such as which content to present to users as a 'summary' of the current state of the deliberative process. In [Halpern et al., 2023] the authors move the first steps towards the application of voting theory to this application domain.

[1] https://liquidfeedback.com/en/

[2] Maybe good to know is that a participatory budgeting pilots have been run in Groningen too (2019 in Oosterparkwijk, 2020 in Helpman).

[3] https://pol.is/home

Voting with preference intensity The type of voting covered in these lecture notes adhered to the 'one-voter-one-vote' principle. This makes it impossible for voters to signal the 'intensity' of their preferences (how much I like x over y). Voting rules have been suggested that make the signaling of preference intensities possible. Here are some examples: cumulative voting [Glasser, 1959], storable voting [Casella, 2005, Casella et al., 2006], quadratic voting [Lalley and Weyl, 2018].

Evaluative voting We are accustomed to an idea of voting in which we are asked to express our preferences, rather than judging on the quality of a candidate or a policy proposal. A number of new voting methods have been proposed which more clearly frame the social choice problem as a problem of evaluation: approval & disapproval voting [Alcantud and Laruelle, 2013], majority judgments [Balinski and Laraki, 2014].

Randomized voting In the early days of democracy randomization played an important role and was considered a quintessentially fair mechanism (e.g., in the Athenian democracy public officials were selected by lottery). Growing research is currently focusing on the use of randomization to improve on standard deterministic voting-based social choice. A good recent overview article is [Brandt, 2018].

Epistemic social choice Epistemic social choice develops the insights of jury theorems such as Condorcet's by making more realistic assumptions over voters competence and independence. A good recent overview article is [Pivato, 2019].

Judgment aggregation We have studied the social choice problem as a problem of aggregation of preferences, but more generally it can be framed as a problem of aggregation of logical formulas. This is the perspective taken in so-called judgment aggregation [Endriss, 2016, Grossi and Pigozzi, 2014].

Strategic voting In Chapter 3 we discussed the issue of strategic or tactical voting, and touched on some basic game-theoretic aspects. Extensive literature exists on strategic voting and a comprehensive overview of the topic is [Meir, 2018b].

Doodle pool voting Doodle pools work with a form of approval voting (cf. Rule 11) giving rise to interesting forms of manipulative behavior. A few recent papers have looked into this issue [James et al., 2015, Obraztsova et al., 2017, Anthony and Chung, 2018] and modeled Doodle pools as a special type of voting games.

Agenda manipulation SCFs are executed by central authorities (government, businesses, etc.). Such central authorities running the voting mechanisms may have the power to decide the agenda (the alternatives) of a social choice context. Can they, by changing such set, obtain better outcomes for themselves? This is the issue of agenda setting or agenda manipulation. An introduction to the topic, with further relevant references, is [Taylor, 2005, Section 2.4].

Incomplete preferences and elicitation In this course we assumed profiles consisted of complete descriptions of the preferences of all agents. However, in many settings, eliciting the full linear order from each agent may be unfeasible (e.g., because of the sheer size of the set of alternatives). Recent work has extended some of the notions we studied to the setting with incomplete preferences: [Terzopoulou and Endriss, 2019, Kruger and Terzopoulou, 2020]. A related line of research has looked at the problem of how to best elicit voter's preferences in order to determine winners, depending on the different voting rules: how many queries, and how complex, does a voting rule need in order to compute a social choice? A good starting point for this line of work is [Brandt et al., 2016, Ch. 10]. The problem is also directly relevant to applications in participatory democracy (see, e.g. [Lee et al., 2014]).

Participation axioms An SCF suffers of the *no show paradox* whenever there exist profiles in which an agent would do better by not casting their ballot under the rule. An SCF is then said to satisfy the participation axiom if it does not suffer of the no show paradox. A good introduction to this axiom, how it relates to monotonicity and strategy-proofness axioms, as well as relevant related literature, can be found in [Brandt et al., 2016, Ch. 2].

Voting theory and machine learning Several interfaces exist between voting theory as we studied it in this course and techniques in machine learning. I mention two important ones here.

Ensemble classification An established approach in Machine Learning to improve the performance in classification tasks is by 'combining' the decisions of various individual classifiers. One way to combine classifiers is to use voting and some papers have explored how different voting rule affect performance in classification tasks by ensembles [Mu et al., 2009, Cornelio et al., 2019].

Clustering Clustering is directly related to multi-winner voting (Chapter 4), and fairness notions developed in voting theory can inform fairer forms of clustering than standard off-the-shelf ones, such as k-means. This very recent paper elaborates on this link [Kellerhals and Peters, 2023].

Voting theory and blockchain Voting theory has several applications in blockchain (see [Grossi, 2022] for a recent overview). Here I list three main points of contact.

Validators selection Multi-winner voting (Chapter 4) has important applications in the selection of validators in some blockchain protocols. An excellent paper delving into this connection with respect to a leading consensus protocol (Polkadot[4]) is [Cevallos and Stewart, 2021].

Sybil-resilient voting A key problem of voting mechanisms over the internet is the possibility of Sybil attacks (that is, voters can generate arbitrarily many identities thereby manipulating the outcome). Considerable research has been dedicated to the development of voting mechanisms that can be, to a smaller or larger extent, resistant to this form of attack [Conitzer and Yokoo, 2010, Todo et al., 2011, Waggoner et al., 2012, Wagman and Conitzer, 2008, Wagman and Conitzer, 2014, Shahaf et al., 2019].

Mechanisms for the blockchain oracle problem Blockchains are state machines whose computation history is updated in a decentralized fashion. While the technology has been used to support electronic currencies (e.g., Bitcoin) the technology has not yet been convincingly deployed beyond that application domain. A key problem for a broader deployment of the technology resides in the possibility of making factual information available on-chain which is not directly handled by the blockchain consensus protocol. This is sometimes referred to as the Blockchain oracle problem. Relevant papers: [Goel et al., 2020, Adler et al., 2018, Cai et al., 2020, Mao et al., 2013].

[4] https://polkadot.network/

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