Introduction to Multi-step Forecast of Multivariate Volatility with Dynamic Factor Model

i4cast LLC

January 2024

Table of Content

1. Introduction

Do you need to make multi-step forecasts in multivariate volatilities of a large number time-series (e.g. those of numerous investable assets in many markets)?

Do you need to find out contributions from common factors of the time-series to the volatility forecasts (e.g., volatility components caused by common economic and market conditions)?

Do you need to find out contributions from estimated and forecasted Multivariate Auto-Covariance matrix of the common factors to the multi-step forecasts in multivariate volatilities (e.g., volatility jumps in panic, and drops in euphoria, markets)?

Do you need to find out contributions from dynamics unique to individual time-series to the volatility forecasts (e.g., volatilities due to specific trajectories of individual equity shares)?

Do you expect use of powerful dynamic factor models (DFMs) to make multi-step forecasts in multivariate values of the time-series (e.g., economic or market forecasts made by government agencies or research institutions) as well?

If all of the above questions apply to you simultaneously, a volatility model based on DFM (dynamic factor model) will offer straightforward answers to all of the questions above by a single version of an integrated and dimension-reduced multivariate analysis framework.

Multivariate GARCH models and static factor risk models are among the most popular classes of volatility models, with each class answers some of the above questions. Meanwhile, it is traditionally difficult to estimate large scale DFMs.

The advent of new machine learning algorithms able to solve for large scale DFMs (such as many long-memory dynamic factors on large number of time-series), DFM-based volatility models can now become a powerful tool at your disposal to attack real-world problems in real-time.

Academic literature on dynamic factor models (DFMs) is voluminous. Doz and Fuleky (2020) is among the latest best overviews on the history of DFM research and development. Alessi, Barigozzi and Capasso (2007) discusses benefits and performances of volatility forecasts by DFM vs. multivariate GARCH.

Following is a brief introduction to DFM-based "Dynamic Factor Variance-Covariance Model (DFVCM)" for making multi-step forecasts in multivariate volatilities.

2. Model Estimation

We denote observed multivariate time-series as vector time-series y_t , which is time-series of $n \times 1$ vector, representing n individual time-series.

To make multi-step forecasts in multivariate volatilities of the vector time-series y_t , we first estimate all coefficients and attributes of a dynamic factor model representation of y_t expressed as follows:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t \tag{2.1}$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t \tag{2.2}$$

$$u_t = g_t + r_t \tag{2.3}$$

$$g_t = D_{t,1}\, g_{t-1} + D_{t,2}\, g_{t-2} + \cdots + D_{t,q}\, g_{t-q} + e_t \tag{2.4}$$

for $j = 0,1,2, \ldots$ and $k = 1,2, \ldots$, variance and autocovariance

$$V_{(t-j),0} = Var(\, f_{t-j}\,) = E(\, f_{t-j}\, f_{t-j}^T\,) \tag{2.5}$$

$$V_{(t-j),k} = Cov(\, f_{t-j}\,,\, f_{t-j-k}\,) = E(\, f_{t-j}\, f_{t-j-k}^T\,) \tag{2.6}$$

$$W_{(t-j),0} = Var(\, g_{t-j}\,) = E(\, g_{t-j}\, g_{t-j}^T\,) \tag{2.7}$$

$$W_{(t-j),k} = Cov(\, g_{t-j}\,,\, g_{t-j-k}\,) = E(\, g_{t-j}\, g_{t-j-k}^T\,) \tag{2.8}$$

and variance of random errors

$$R_t^{(v)} = Var(\, v_t\,) \tag{2.9}$$

$$R_t^{(e)} = Var(\, e_t\,) \tag{2.10}$$

$$R_t^{(r)} = Var(\, r_t\,) \tag{2.11}$$

where (random and non-random) variables

- y_t is time-series of $n \times 1$ vector of observed data
- μ_t is time-series of $n \times 1$ vector of mean value of y_t
- $X_{t,0}$ is $n \times m$ matrix of factor loadings of dynamic factor f_t
- f_t is time-series of $m \times 1$ vector of dynamic common factor scores
- $A_{t,j}$ is $m \times m$ matrix of VAR (vector autoregressive) coefficients of factor score time-series f_t, here $m < n$, or $n \gg m$
- v_t is time-series of $m \times 1$ vector of error in VAR prediction of factor scores f_t
- u_t is time-series of $n \times 1$ vector of idiosyncratic components of individual observed time-series
- g_t is time-series of $n \times 1$ vector of unobserved dynamic components (UDCs) of idiosyncratic components u_t
- $D_{t,k}$ is $n \times n$ diagonal matrix of AR (autoregressive) coefficients of UDC time-series g_t
- e_t is time-series of $n \times 1$ vector of error in AR prediction of UDC g_t
- r_t is time-series of $n \times 1$ vector of residual random error
- $V_{(t-j),0}$ is $m \times m$ diagonal matrix of variance of dynamic factor scores f_t
- $V_{(t-j),k}$ is $m \times m$ matrix of k-lag autocovariance of dynamic factor scores f_t
- $W_{(t-j),0}$ is $n \times n$ diagonal matrix of variance of UDC g_t
- $W_{(t-j),k}$ is $n \times n$ diagonal matrix of k-lag autocovariance of UDC g_t
- $R_t^{(v)}$ is time-series of $m \times m$ diagonal matrix of variance of VAR prediction error v_t

- $R_t^{(e)}$ is time-series of $n \times n$ diagonal matrix of variance of AR prediction error e_t, and
- $R_t^{(r)}$ is time-series of $n \times n$ diagonal matrix of variance of residual error r_t

The DFM estimations summarized above can be performed jointly by two models: (1) LMDFM (long-memory dynamic factor model, https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg) estimated by an implementation of dynamic principal components analysis (DPCA) with 2-dimensional discrete Fourier transforms (2D-DFTs) (detailed in Appendix A) for estimates of common components, including $X_{t,0}$, f_t, $A_{t,j}$, $R_t^{(v)}$, $V_{(t-j),0}$, $V_{(t-j),k}$ and u_t; and (2) YWpcAR (Yule-Walker-PCA autoregression, https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6) estimated by an implementation of principal components analysis (PCA) on Yule-Walker (YW) equation (detailed in Appendix B) for estimates of idiosyncratic components, including g_t, $D_{t,k}$, $R_t^{(e)}$, $W_{(t-j),0}$, $W_{(t-j),k}$ and $R_t^{(r)}$.

In estimation of all of the above coefficients and attributes, we applied following assumptions widely proposed, accepted and practiced in dynamic factor model research literature:

- Mean value (of observed vector time-series) $\mu_t = 0$
- Covariance matrix (of dynamic factor scores) $V_{(t-j),0}$ is diagonal matrix
- Covariance matrix (of idiosyncratic components) $W_{(t-j),0}$ is approximated by diagonal matrix, ignoring allowed but "mild" cross-correlation

In order to make multi-step forecast of multivariate volatility in the next section, we need past autocovariance matrix of dynamic factor scores f_t with two separate time lags, j and k. When $0 \leq j \leq k$, the past autocovariance matrix

$$E(\, f_{t-j}\, f_{t-k}^T \,) = V_{[t-Min(j,k)],Abs(j-k)} \tag{2.12}$$

When $j \geq k \geq 0$, the past autocovariance

$$E(\, f_{t-j}\, f_{t-k}^T \,) = E(\, f_{t-k}\, f_{t-j}^T \,)^T = V_{[t-Min(j,k)],Abs(j-k)}^T \tag{2.13}$$

3. Volatility Forecast

According to the factor model representation discussed in the Section 2 above, s-step forecasts of time-series, f_t, g_t and y_t, based on data observed until time t can be made by dynamic equations as

$$f_{(t+s)|t} = A_{t,1}\, f_{t+s-1} + A_{t,2}\, f_{t+s-2} + \cdots + A_{t,p}\, f_{t+s-p} + v_{t+s} \tag{3.1}$$

$$g_{(t+s)|t} = D_{t,1}\, g_{t+s-1} + D_{t,2}\, g_{t+s-2} + \cdots + D_{t,q}\, g_{t+s-q} + e_{t+s} \tag{3.2}$$

$$y_{(t+s)|t} = X_{t,0}\, f_{(t+s)|t} + g_{(t+s)|t} + r_{t+s} \tag{3.3}$$

where $s = 1,2,\ldots$. The random errors v_{t+s}, e_{t+s} and r_{t+s} cannot be forecasted, but can be characterized by assumed diagonal variance matrixes as

$$R_{t+s}^{(v)} = R_t^{(v)} \tag{3.4}$$

$$R_{t+s}^{(e)} = R_t^{(e)} \tag{3.5}$$

$$R_{t+s}^{(r)} = R_t^{(r)} \tag{3.6}$$

Therefore, s-step forecast of diagonal variance-covariance matrix of dynamic factor scores f_t is

$$\begin{aligned} &V_{[(t+s)|t],0} = Var(\, f_{(t+s)|t} \,) = E(\, f_{(t+s)|t}\, f_{(t+s)|t}^T \,) \\ &= Diag(\, E(\, (\textstyle\sum_{j=1}^{p} A_{t,j}\, f_{t+s-j} \,)(\, \sum_{k=1}^{p} A_{t,k}\, f_{t+s-k} \,)^T \,) \,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T \,)\, A_{t,k}^T \,) \,) + R_t^{(v)} \end{aligned} \tag{3.7}$$

where $s = 1,2,\ldots$, and variance and autocovariance of factors, $E(\, f_{t+s-j}\, f_{t+s-k}^T \,)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by "earlier-step" forecasts Eqs. (3.7) and (3.8). Then, s-Step forecast of k-lag autocovariance matrix of dynamic factor scores f_t is

$$V_{[(t+s)|t],k} = Cov(f_{(t+s)|t} , f_{t+s-k}) = E(f_{(t+s)|t} f_{t+s-k}^T)$$
$$= E(\sum_{j=1}^{p} A_{t,j} f_{t+s-j} f_{t+s-k}^T) = \sum_{j=1}^{p} A_{t,j} E(f_{t+s-j} f_{t+s-k}^T) \qquad (3.8)$$

where $s = 1,2,...$, $k = 1,2,...$, and, again, variance and autocovariance, $E(f_{t+s-j} f_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by earlier forecasts Eqs. (3.7) and (3.8).

Similarly, s-Step forecast of variance of unobserved dynamic components (UDCs) g_t is

$$W_{[(t+s)|t],0} = Var(g_{(t+s)|t}) = E(g_{(t+s)|t} g_{(t+s)|t}^T)$$
$$= Diag(E((\sum_{j=1}^{q} D_{t,j} g_{t+s-j})(\sum_{k=1}^{q} D_{t,k} g_{t+s-k})^T)) + R_t^{(e)}$$
$$= \sum_{j=1}^{q} \sum_{k=1}^{q} Diag(D_{t,j} E(g_{t+s-j} g_{t+s-k}^T) D_{t,k}) + R_t^{(e)} \qquad (3.9)$$

where $s = 1,2, ...$, and variance and autocovariance of UDCs, $E(g_{t+s-j} g_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by "earlier-step" forecasts Eqs. (3.9) and (3.10). Then, s-step forecast of k-lag autocovariance of UDC g_t is

$$W_{[(t+s)|t],k} = Cov(g_{(t+s)|t} , g_{t+s-k}) = E(g_{(t+s)|t} g_{t+s-k}^T)$$
$$= Diag(E(\sum_{j=1}^{q} D_{t,j} g_{t+s-j} g_{t+s-k}^T))$$
$$= \sum_{j=1}^{q} Diag(D_{t,j} E(g_{t+s-j} g_{t+s-k}^T)) \qquad (3.10)$$

where $s = 1,2,...$, $k = 1,2,...$, and, again, variance and autocovariance, $E(g_{t+s-j} g_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by earlier forecasts Eqs. (3.9) and (3.10).

Having

- estimated factor loadings matrix $X_{t,0}$ in Eq. (2.1),
- forecasted diagonal variance matrix $V_{[(t+s)|t],0}$ of dynamic factor scores f_t by Eq. (3.7),
- forecasted diagonal variance matrix $W_{[(t+s)|t],0}$ of unobserved dynamic

components (UDCs) g_t by Eq. (3.9), and

- "forecasted" diagonal variance matrix of residual errors r_t by Eq. (3.6),

s-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$C_{(t+s)|t} = Var(\, y_{(t+s)|t} \,) = E(\, y_{(t+s)|t} \, y^T_{(t+s)|t} \,)$$
$$= X_{t,0} \, V_{[(t+s)|t],0} \, X^T_{t,0} + W_{[(t+s)|t],0} + R^{(r)}_t \tag{3.11}$$

where $s = 1,2, \ldots$.

4. Simple Forecast as Benchmark

There are two classes of simple forecasts widely practiced for multivariate variance-covariance matrix of observed vector time-series y_t .

The simplest calculated forecast is to use a sample-based variance-covariance matrix as forecasted matrix:

$$C^{(Sample)}_t = K^{-1} \sum_{k=0}^{K} (\, y_{t-k} - \mu_t \,)\,(\, y_{t-k} - \mu_t \,)^T \tag{4.1}$$

$$C^{(Sample)}_{(t+s)|t} = C^{(Sample)}_t \tag{4.2}$$

where forecast step $s = 1,2, \ldots$.

A factor-based forecast is to use a variance-covariance matrix estimated by a static factor model as forecasted matrix:

$$C^{(Static)}_t = X_t \, V_t \, X^T_t + W_t + R_t \tag{4.3}$$

$$C^{(Static)}_{(t+s)|t} = C^{(Static)}_t \tag{4.4}$$

where $s = 1,2, \ldots$; and factor loadings matrix X_t , diagonal factor variance matrix V_t , diagonal UDC variance matrix W_t and diagonal residual error variance matrix R_t are estimated coefficients and attributes of a static factor model expressed as follows:

$$y_t = \mu_t + X_t f_t + u_t$$

$$u_t = g_t + r_t$$

$$V_t = Var(f_t) = E(f_t f_t^T)$$

$$W_t = Var(g_t) = E(g_t g_t^T)$$

$$R_t = Var(r_t) = E(r_t r_t^T)$$

Another widely practiced class of static factor models is "fundamental risk factor analysis", in which factor loadings matrix is pre-determined based on certain fundamental analysis theory or framework ahead of factor model estimation. Therefore, fundamental factor analysis is able to make meaningful explanations about multivariate volatility structure, but variance-covariance matrix of fundamental factor score time-series is not diagonal.

Since we have already estimated a dynamic factor model, we can replace estimates of a static factor model by estimates, not forecasts, of our dynamic factor model as follows:

$$C_t^{(Estimate)} = X_{t,0} V_{t,0} X_{t,0}^T + W_{t,0} + R_t^{(r)} \tag{4.5}$$

$$C_{(t+s)|t}^{(Estimate)} = C_t^{(Estimate)} \tag{4.6}$$

where $s = 1,2, \dots$.

The above simple forecasts, Eq. (4.2), (4.4) or (4.6), can serve as benchmarks in evaluation of our multi-step forecast in multivariate volatility by dynamic factor model.

<u>5. Evaluation of Volatility Forecast</u>

Directly examining quality or accuracy of the forecasted variance-covariance matrix $C_{(t+s)|t}$ itself is a complicated undertaking in theory and difficult (for a large number of time-series) task in practice.

A widely practiced evaluation technique is to measure quality or accuracy of forecasted variance of a weighted aggregation of the time-series, with forecasted variance of the aggregate made by the forecasted variance-covariance matrix, as

$$(\sigma_{(t+s)|t}^{(w)})^2 = w^T C_{(t+s)|t} w. \tag{5.1}$$

where w is a $n \times 1$ vector of weights for aggregation and $(\sigma_{(t+s)|t}^{(w)})^2$ is forecasted variance of aggregated time-series

$$y_t^{(w)} = w^T y_t = y_t^T w, \tag{5.2}$$

The vector of weights, w, is set according to relevant application(s) of business or research. If one of the elements in the vector w is 1 and all others are $0s$, the weighted aggregate, $y_t^{(w)}$, is essentially a selected individual time-series.

To measure the accuracy of forecasted variance $(\sigma_{(t+s)|t}^{(w)})^2$ by Eq. (5.1), a "realized z-score squared of the forecasts" defined by

$$(z_{(t+s)|t}^{(w)})^2 = (y_{t+s}^{(w)} - \mu_{t+s}^{(w)})^2 / (\sigma_{(t+s)|t}^{(w)})^2, \tag{5.3}$$

is handy, where observation $y_{t+s}^{(w)}$ and estimate $\mu_{t+s}^{(w)}$ are made at time $t+s$, while forecast $(\sigma_{(t+s)|t}^{(w)})^2$ is made at earlier time t. According to Litterman and Winkelmann (1998), Patton (2011), Menchero, Morozov and Pasqua (2013) and Fan, Furger and Xiu (2015), the accuracy of portfolio volatility forecasts over a given time period $(t+1) \in [t_1, t_2]$ can be measured by bias statistic $BS_{t_1,t_2}^{(w)}$, log-likelihood $LL_{t_1,t_2}^{(w)}$, and Q-statistic $QS_{t_1,t_2}^{(w)}$ defined as

$$BS_{t_1,t_2}^{(w)} = [\frac{1}{t_2-t_1} \sum_{t=t_1-1}^{t_2-1} (z_{t+1|t}^{(w)})^2]^{1/2}, \tag{5.4}$$

$$LL_{t_1,t_2}^{(w)} = - \frac{1/2}{t_2-t_1+1} \sum_{t=t_1-1}^{t_2-1} [\ln(2\pi) + (z_{t+1|t}^{(w)})^2 + \ln(\sigma_{t+1|t}^{(w)})^2], \tag{5.5}$$

$$QS_{t_1,t_2}^{(w)} = \frac{1}{t_2-t_1+1} \sum_{t=t_1-1}^{t_2-1} [(z_{t+1|t}^{(w)})^2 - \ln(z_{t+1|t}^{(w)})^2], \tag{5.6}$$

where $sd_{t=t_1}^{t_2}(\cdot)$ denotes sample standard deviation and $w_{t+1|t}^{(QP)}$ is time-varying stock weights of the minimum variance portfolio obtained by quadratic programing (QP) using the predicted covariance matrix $C_{t+1|t}$. A bias statistic $BS_{t_1,t_2}^{(w)} > 1$ or $BS_{t_1,t_2}^{(w)} < 1$ shows an under- or over-prediction of volatility. A higher log-likelihood $LL_{t_1,t_2}^{(w)}$ or a lower Q-statistic $QS_{t_1,t_2}^{(w)}$ indicates more accurate forecasts.

6. Examples

A real-world example of a large set of time-series is a collection of many years of weekly performance time-series of more than 50 equity, fixed income, financial index and physical commodity investment funds publicly traded in the U.S. exchanges.

7. Discussion

The numeric examples demonstrate that the DFM-based "Dynamic Factor Variance-Covariance Model (DFVCM)" is capable to generate better multi-step forecasts in multivariate volatilities.

Appendix A. LMDFM Algorithm

The LMDFM (long-memory dynamic factor model, https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg) is estimated by an implementation of dynamic principal components analysis (DPCA), reviewed by Doz and Fuleky (2020), with 2-dimensional discrete Fourier transforms (2D-DFTs) summarized as follows:

- Estimating variance-covariance matrixes (VCMs) and autocovariance matrixes (ACMs) of observed vector (i.e. multiple) time-series y_t , $C_{j,k} = Cov(y_{t-j} , y_{t-k}) = E(y_{t-j}\, y_{t-k}^T)$., $j,k = 0,1,\ldots,p$, assuming $E(y_t) = 0$.

- Combining VCMs and ACMs, $C_{j,k}$, by applying 2-dimensional discrete Fourier transforms (2D-DFTs) on VCMs and ACMs.
- Referring resulted transforms (by 2D-DFT) as spectral density matrixes (SDMs): $\{ \Omega_{q,r} \} = DFT_{2D}(\{ C_{j,k} \})$.
- Applying principal components analysis (PCA) on each of the SDMs, $\Omega_{q,r}$, $q,r = 0,1,\ldots,p$.
- Estimating principal components (PCs) of original VCMs and ACMs by applying inverse 2D-DFTs on PC-represented (dimension-reduced) SDMs.
- This way, PCs of each of original VCMs and ACMs contain dynamic information from all of VCMs and ACMs.
- If observed vector time-series can be reasonably assumed as locally stationary, the 2D-DFTs become simplified as weighted 1D-DFTs, with exactly the same "weights of the Bartlett window" shown by Doz and Fuleky (2020).

The LMDFM can be utilized to estimate common components, including $X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, $V_{(t-j),0}$, $V_{(t-j),k}$ and u_t , shown in Section "2. Model Estimation".

Appendix B. YWpcAR Algorithm

The YWpcAR (Yule-Walker-PCA autoregression, https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6) model is estimated by an implementation of principal components analysis (PCA) on Yule-Walker (YW) equation summarized as follows:

- Applying principal components analysis (PCA) on sample variance-autocovariance matrix (VACM) in Yule-Walker (YW) equation.
- Replacing elements of sample VACM by PCA-based common components.
- Constructing PCA-based YW equation with replacing elements of matrix and vector in YW equation by correspondent PCA-based common components of VACM.
- Estimating AR model coefficients by PCA-based YW equation.
- Combining individual principal component score time-series into an unobserved dynamic component (UDC) time-series.
- Forecasting expected value and variance of observed time-series with UDC time-series of YW-PCA AR model.

The YWpcAR can be utilized to estimate idiosyncratic components, including g_t, $D_{t,k}$, $R_t^{(e)}$, $W_{(t-j),0}$, $W_{(t-j),k}$ and $R_t^{(r)}$, shown in Section “2. Model Estimation”.

References

L. Alessi, M. Barigozzi and M. Capasso (2007). “Dynamic factor GARCH: Multivariate volatility forecast for a large number of series”. LEM Working Paper Series, No. 2006/25, Laboratory of Economics and Management (LEM), Pisa.

C. Doz and P. Fuleky (2020). “Chapter 2, Dynamic Factor Models” In *Macroeconomic Forecasting in the Era of Big Data: Theory and Practice*, Ed. P. Fuleky, Advanced Studies in Theoretical and Applied Econometrics, Volume 52. Springer.

J. Fan, A. Furger and D. Xiu (2015). “Incorporating Global Industrial Classification Standard into Portfolio Allocation: A Simple Factor-Based Large Covariance Matrix Estimator with High Frequency Data”. Working Paper, No. 15-01, University of Chicago Booth School of

Business, 2015.

R. Litterman and K. Winkelmann (1998). “Estimating covariance matrices”. Risk Management Series, Goldman Sachs Group, Inc., 1998.

J. Menchero, A. Morozov and A. Pasqua (2013). “Predicting risk at short horizons: a case study for the USE4D model”. MSCI Model Insight, MSCI, 2013.

A.J. Patton (2011). “Volatility forecast comparison using imperfect volatility proxies”. Journal of Econometrics, 2011, 160 (1), 246–256.

[EOF]