Introduction to Multi-step DFM-based Forecasts of Multiple Volatility Indexes

i4cast LLC

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1. Introduction

Widely watched, reported and utilized volatility indexes include VIX and others published by CBOE (https://www.cboe.com/), such as VIX (on S&P 500), VXD (on DJIA), VXN (on Nasdaq 100), RVX (on Russel 2000), VXEEM (on emerging market), VXEWZ (on Brazil ETF), GVZ (on gold), OVX (on crude oil), EVZ (on Euro), VXAZN (on Amazon), VXAPL (on Apple), VXGS (on Goldman Sachs), VXGOG (on Google), VXIBM (on IBM), etc. Volatility indexes published by global financial exchanges are regarded as among crucial indicators by many economy and market participants all over the world. Well-established holistic data-driven models able to analyze and forecast volatility indexes could serve as important tools.

Many volatility indexes, all price time-series underlying these volatility indexes, and many other relevant financial time-series are "dynamically correlated", i.e. correlated over time and cross-sectionally. A set of "dominant dynamic correlation characteristics" of these large number of time-series can be extracted by widely utilized dynamic factor models (DFMs). Those "other relevant time-series" are information-enhancing inputs of the models. Dynamic relationships represented by dominant dynamic factors are less likely contaminated by random noises and, as a result, more likely to produce more robust forecasts.

Our modeling approach is to carry out DFM volatility analysis on DFM-input time-series, which are multiple time-series including both price time-series underlying the volatility indexes and other information-enhancing time-series. Then, the volatility forecasts of the underlying time-series are transformed into multi-step forecasts of the multiple volatility indexes.

Our Dynamic Factor Based Volatility Index Forecast (DFbVIF) model will be discussed in details step by step as follows: four separate but related sets of multiple (or vector) time-series; DFM-based volatility modeling on DFM-input time-series; forecasted volatilities of DFM-input time-series; formulas and expressions of variances of underlying time-series; DFM-based dynamic forecasts of volatility indexes; dynamic volatility attributions; evaluation of forecasts of volatility indexes; numeric examples; and concluding discussions.

2. Data: Four Sets of Multiple Time-Series

Our DFM-based multi-step forecasts of multiple volatility indexes involve following four related sets of multiple time-series:

1. volatility index time-series,
2. underlying time-series,
3. info-enhancing time-series and
4. DFM-input time-series.

Underlying (CBOE's and other) volatility index time-series, VIX, VXD, etc. are market price/index time-series, S&P 500, DJIA, etc. An underlying time-series, underlying a given volatility index, is a time-series of percentage or logarithmic changes in price/value/index, with the value of volatility index representing expected volatility level of the underlying time-series, expected at a specified future time. This fact about volatility expectation indicates that forecasting volatility indexes can be made by volatility analysis on, and volatility forecast of, their underlying time-series.

Dynamic factor models (DFMs) are the most researched and useful frameworks able to model multiple dynamically correlated time-series. There is an enormous volume of widely accepted academic publications and technical whitepapers on DFMs. Doz and Fuleky (2020) is among the best overviews on DFM researches and developments. In addition to analyzing and forecasting values of multiple time-series, DFMs can also effectively analyze and forecast variance-covariance of the same set of time-series (i4cast LLC 2024 on DFVCM algorithm; Alessi, Barigozzi and Capasso 2007). This fact about multiple time-series volatility analysis indicates that forecasting volatility indexes can be made by DFM volatility modeling on the underlying time-series.

DFMs built on dynamic PCA (principal component analysis) are able to accommodate large number of time-series with greater order of vector autoregression (VAR). Some publicly available machine learning algorithms are among such DFMs of big data (e.g. LMDFM, YWpcAR and DFVCM algorithms available on AWS and offered by i4cast LLC). This crucial advantage of big data analysis indicates that we can combine the underlying time-series with many other information-enhancing time-series dynamically correlated with the underlying ones to serve as an expanded input data set of the DFM volatility analysis.

Combining underlying and info-enhancing time-series into a single set of DFM-input time-series will provide the DFM volatility analysis/forecast with more relevant information than the underlying time-series alone. The higher degree of information completeness of the DFM-input time-series indicates that they are able to increase predictive power of the DFM-based forecasts of the volatility indexes.

Forecasted variance-covariance matrix of underlying time-series is a sub-matrix of that of DFM-input time-series. Forecasts of variances of underlying time-series can be transformed into forecasts of implied variances which are squares of volatility indexes. Two different transformations are developed: one is “underlying volatility forecasts (UVF) as predictors”, and the other “quadratic autoregressive (QAR) forecasts” with QAR coefficients of underlying volatilities. Their formulars will be presented in the forms of equations of matrixes.

We denote multiple volatility index time-series as time-series vi_t of $h \times 1$ vector vi, representing h individual volatility indexes. When multiple (forecasted) volatility indexes need to be expressed in equations of matrixes, we also denote the same set of volatility indexes as time-series VI_t of $h \times h$ diagonal matrix VI. A set of h "implied variances" (squares of volatility indexes) is denoted as time-series VI_t^2 of $h \times h$ diagonal matrix VI^2,

$$VI_t^2 = VI_t \, VI_t^T = VI_t^T \, VI_t = VI_t \, VI_t \, . \tag{2.1}$$

A set of h underlying time-series is denoted as time-series $y_t^{(und)}$ of $h \times 1$ vector $y^{(und)}$. Similarly, a set of $n - h$ additional info-enhancing time-series is denoted as time-series $y_t^{(info)}$ of $(n-h) \times 1$ vector $y^{(info)}$. Therefore, a set of n DFM-input time-series is denoted as time-series y_t of $n \times 1$ vector y,

$$y_t = [\, (\, y_t^{(und)}\,)^T, \ (\, y_t^{(info)}\,)^T \,]^T \, . \tag{2.2}$$

Now, the DFM-input vector time-series y_t is ready for DFM volatility modeling.

3. Volatility Model on DFM-Input Time-Series

To make multi-step forecasts of multivariate volatilities of observed DFM-input vector time-series y_t, we first estimate all coefficients and attributes of a dynamic factor model representation of y_t expressed as follows:

$$y_t = \mu_t + X_{t,0} \, f_t + u_t \, , \tag{3.1}$$

$$f_t = A_{t,1} \, f_{t-1} + A_{t,2} \, f_{t-2} + \cdots + A_{t,p} \, f_{t-p} + v_t \, , \tag{3.2}$$

$$u_t = g_t + r_t \, , \tag{3.3}$$

$$g_t = D_{t,1} \, g_{t-1} + D_{t,2} \, g_{t-2} + \cdots + D_{t,q} \, g_{t-q} + e_t \, , \tag{3.4}$$

where (random and non-random) variables

- y_t is time-series of $n \times 1$ vector of observed data,
- μ_t is time-series of $n \times 1$ vector of mean values of y_t,

- f_t is time-series of $m \times 1$ vector of common dynamic factor scores,
- $X_{t,0}$ is $n \times m$ matrix of factor loadings of dynamic factors f_t,
- $A_{t,j}$ are $m \times m$ matrixes of VAR (vector autoregressive) coefficients of factors f_t, here $n > or \gg m$,
- v_t is time-series of $m \times 1$ vector of VAR prediction errors of factors f_t,
- u_t is time-series of $n \times 1$ vector of idiosyncratic components of individual observed time-series,
- g_t is time-series of $n \times 1$ vector of unobserved dynamic components (UDCs) of idiosyncratic components u_t,
- $D_{t,k}$ are $n \times n$ diagonal matrixes of AR (autoregressive) coefficients of UDCs g_t,
- e_t is time-series of $n \times 1$ vector of AR prediction errors of UDCs g_t, and
- r_t is time-series of $n \times 1$ vector of residual random errors.

For $j, k = 0,1,2, \dots$, variances and autocovariances of common dynamic factors and idiosyncratic UDCs are estimated as:

$$V_{t-j} = Var(f_{t-j}) = E(f_{t-j} f_{t-j}^T), \tag{3.5}$$

$$V_{(t-j),(t-k)} = Cov(f_{t-j}, f_{t-k}) = E(f_{t-j} f_{t-k}^T), \tag{3.6}$$

$$W_{t-j} = Var(g_{t-j}) = E(g_{t-j} g_{t-j}^T), \tag{3.7}$$

$$W_{(t-j),(t-k)} = Cov(g_{t-j}, g_{t-k}) = E(g_{t-j} g_{t-k}^T), \tag{3.8}$$

where estimated matrixes

- V_{t-j} are $m \times m$ diagonal matrixes of variance of dynamic factors f_t,
- $V_{(t-j),(t-k)}$ are $m \times m$ matrixes of autocovariance of factors f_t,

- W_{t-j} are $n \times n$ diagonal matrixes of variance of UDCs g_t, and
- $W_{(t-j),(t-k)}$ are $n \times n$ diagonal matrixes of autocovariance of UDCs g_t.

Variances of random errors are estimated as:

$$R_t^{(v)} = Var(\, v_t \,)\,, \tag{3.9}$$

$$R_t^{(e)} = Var(\, e_t \,)\,, \tag{3.10}$$

$$R_t^{(r)} = Var(\, r_t \,)\,; \tag{3.11}$$

where estimated matrixes

- $R_t^{(v)}$ is time-series of $m \times m$ diagonal matrix of variances of VAR prediction errors v_t,
- $R_t^{(e)}$ is time-series of $n \times n$ diagonal matrix of variances of AR prediction errors e_t, and
- $R_t^{(r)}$ is time-series of $n \times n$ diagonal matrix of variances of residual errors r_t.

The DFM estimates summarized with Eqs. (3.1) through (3.11) above can be performed by algorithm DFVCM (dynamic factor variance-covariance model, https://aws.amazon.com/marketplace/pp/prodview-yvaulquatt3v2?sr=0). In DFVCM, common factors are estimated by an implementation of dynamic principal components analysis (DPCA) with 2-dimensional discrete Fourier transform (2D-DFT), and idiosyncratic UDCs are estimated by an implementation of principal components analysis (PCA) on Yule-Walker (YW) equation (whitepaper on DFVCM, https://github.com/i4cast/aws/blob/main/dynamic_factor_variance-covariance_model/publication/multi-step_forecast_of_multivariate_volatility_with_dynamic_factor_model.pdf).

The estimation of the coefficients and attributes in Eqs (3.1) to (3.11) applied following assumptions widely proposed, accepted and practiced in dynamic factor model research literature:

- mean value (of observed vector time-series) $\mu_t = 0$,
- covariance matrix (of dynamic factor scores) V_{t-j} is diagonal matrix,
- covariance matrix (of idiosyncratic components) W_{t-j} is approximated by diagonal matrix, ignoring allowed but "mild" cross-correlation,
- assuming all cross- and serial-correlations between common factors f_t and idiosyncratic UDCs g_t are $0's$.

As a result, for time-lag $j = 0,1,2,\dots$, all $n \times n$ variance-covariance matrixes of DFM-input time-series y_t are estimated as

$$\begin{aligned} C_{t-j} &= Var(\, y_{t-j}\,) \\ &= Var(\, X_{t,0}\, f_{t-j} + g_{t-j} + r_t\,) \\ &= X_{t,0}\, Var(\, f_{t-j}\,)\, X_{t,0}^T + Var(\, g_{t-j}\,) + Var(\, r_t\,) \\ &= X_{t,0}\, V_{t-j}\, X_{t,0}^T + W_{t-j} + R_t^{(r)}\,. \end{aligned} \tag{3.12}$$

4. Forecasts by DFM Volatility Model

According to the factor model representation discussed in the Section 3 above, s-step forecasts of time-series, f_t , g_t and y_t , based on data observed until time t can be made by dynamic equations as

$$f_{(t+s)|t} = A_{t,1}\, f_{t+s-1} + A_{t,2}\, f_{t+s-2} + \cdots + A_{t,p}\, f_{t+s-p} + v_{t+s} \tag{4.1}$$

$$g_{(t+s)|t} = D_{t,1}\, g_{t+s-1} + D_{t,2}\, g_{t+s-2} + \cdots + D_{t,q}\, g_{t+s-q} + e_{t+s} \tag{4.2}$$

$$y_{(t+s)|t} = X_{t,0}\, f_{(t+s)|t} + g_{(t+s)|t} + r_{t+s} \tag{4.3}$$

where $s = 1,2,\dots$. The random errors v_{t+s} , e_{t+s} and r_{t+s} cannot be forecasted, but can be characterized by assumed diagonal variance matrixes as

$$R_{t+s}^{(v)} = R_t^{(v)} \tag{4.4}$$

$$R_{t+s}^{(e)} = R_t^{(e)} \tag{4.5}$$

$$R_{t+s}^{(r)} = R_t^{(r)} \tag{4.6}$$

Therefore, s-step forecast of variance of common dynamic factors f_t is

$$\begin{aligned} & V_{(t+s)|t} = Var(\, f_{(t+s)|t}\,) = E(\, f_{(t+s)|t}\, f_{(t+s)|t}^T\,) \\ &= Diag(\, E(\,(\,\textstyle\sum_{j=1}^{p} A_{t,j}\, f_{t+s-j}\,)(\,\sum_{k=1}^{p} A_{t,k}\, f_{t+s-k}\,)^T\,)\,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,)\, A_{t,k}^T\,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, A_{t,j}\, V_{(t+s-j),(t+s-k)}\, A_{t,k}^T\,) + R_t^{(v)} \end{aligned} \tag{4.7}$$

where $s = 1,2, \ldots,$ and variance and autocovariance of factors, $E(\, f_{t+s-j}\, f_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (3.5) and (3.6), or by "earlier-step" forecasts Eqs. (4.7) and (4.8). Then, s-Step forecast of k-lag autocovariance matrix of dynamic factor scores f_t is

$$\begin{aligned} & V_{(t+s),(t+s-k)} = Cov(\, f_{(t+s)|t}\,,\, f_{t+s-k}\,) \\ &= E(\, f_{(t+s)|t}\, f_{t+s-k}^T\,) = E(\,\textstyle\sum_{j=1}^{p} A_{t,j}\, f_{t+s-j}\;\, f_{t+s-k}^T\,) \\ &= \textstyle\sum_{j=1}^{p} A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,) = \sum_{j=1}^{p} A_{t,j}\, V_{(t+s-j),(t+s-k)} \end{aligned} \tag{4.8}$$

where $s = 1,2, \ldots,$ $k = 1,2, \ldots,$ and, again, variance and autocovariance, $E(\, f_{t+s-j}\, f_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (3.5) and (3.6), or by earlier forecasts Eqs. (4.7) and (4.8).

Similarly, s-Step forecast of variance of idiosyncratic unobserved dynamic components (UDCs) g_t is

$$\begin{aligned} & W_{(t+s)|t} = Var(\, g_{(t+s)|t}\,) = E(\, g_{(t+s)|t}\, g_{(t+s)|t}^T\,) \\ &= Diag(\, E(\,(\,\textstyle\sum_{j=1}^{q} D_{t,j}\, g_{t+s-j}\,)(\,\sum_{k=1}^{q} D_{t,k}\, g_{t+s-k}\,)^T\,)\,) + R_t^{(e)} \\ &= \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} Diag(\, D_{t,j}\, E(\, g_{t+s-j}\, g_{t+s-k}^T\,)\, D_{t,k}\,) + R_t^{(e)} \\ &= \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j}\, W_{(t+s-j),(t+s-k)}\, D_{t,k} + R_t^{(e)} \end{aligned} \tag{4.9}$$

where $s = 1,2, \dots$, and variance and autocovariance of UDCs, $E(g_{t+s-j}\, g_{t+s-k}^T)$, are evaluated by estimates Eqs. (3.7) and (3.8), or by "earlier-step" forecasts Eqs. (4.9) and (4.10). Then, s-step forecast of k-lag autocovariance of UDCs g_t is

$$W_{(t+s),(t+s-k)} = Cov(g_{(t+s)|t}\,,\, g_{t+s-k}) = E(g_{(t+s)|t}\, g_{t+s-k}^T)$$
$$= Diag(E(\textstyle\sum_{j=1}^{q} D_{t,j}\, g_{t+s-j}\, g_{t+s-k}^T)) = \textstyle\sum_{j=1}^{q} Diag(D_{t,j}\, E(g_{t+s-j}\, g_{t+s-k}^T))$$
$$= \textstyle\sum_{j=1}^{q} D_{t,j}\, W_{(t+s-j),(t+s-k)} \qquad (4.10)$$

where $s = 1,2, \dots$, $k = 1,2, \dots$, and, again, variance and autocovariance, $E(g_{t+s-j}\, g_{t+s-k}^T)$, are evaluated by estimates Eqs. (3.7) and (3.8), or by earlier forecasts Eqs. (4.9) and (4.10).

Having

- estimated factor loadings matrix $X_{t,0}$ in Eq. (3.1),
- forecasted diagonal variance matrix $V_{[(t+s)|t],0}$ of dynamic factors f_t by Eq. (4.7),
- forecasted diagonal variance matrix $W_{[(t+s)|t],0}$ of UDCs g_t by Eq. (4.9), and
- "forecasted" diagonal variance matrix of residual errors r_t by Eq. (4.6),

s-step forecast of variance-covariance matrix of DFM-input vector time-series y_t is

$$C_{(t+s)|t} = Var(y_{(t+s)|t}) = E(y_{(t+s)|t}\, y_{(t+s)|t}^T)$$
$$= X_{t,0}\, V_{(t+s)|t}\, X_{t,0}^T + W_{(t+s)|t} + R_t^{(r)}$$
$$= X_{t,0} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(A_{t,j}\, V_{(t+s-j),(t+s-k)}\, A_{t,k}^T)\, X_{t,0}^T$$
$$+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j}\, W_{(t+s-j),(t+s-k)}\, D_{t,k}$$
$$+ X_{t,0}\, R_t^{(v)}\, X_{t,0}^T + R_t^{(e)} + R_t^{(r)}\,. \qquad (4.11)$$

where $s = 1,2, \dots$.

5. Expressions of Underlying Volatilities

Our objective is to make forecasts of volatility indexes based on volatility expressions of underlying time-series which, in turn, can be obtained by volatility expressions of DFM-input time-series, presented in Sections 3 and 4 above.

For time-lag $j = 0,1,2,\ldots$, and forecast step $s = 1,2,\ldots$, basic volatility attributes specifically for underlying time-series $y_t^{(und)}$ include:

- common factor loadings, an $h \times m$ matrix $X_{t,0}^{(und)}$, is upper $h \times m$ block of $n \times m$ matrix $X_{t,0}$,
- autoregressive (AR) coefficients of UDCs, $h \times h$ diagonal matrixes $D_{t,j}^{(und)}$, are upper-left $h \times h$ diagonal blocks of $n \times n$ diagonal matrixes $D_{t,j}$,
- estimated/forecasted variance of UDCs, $h \times h$ diagonal matrixes $W_{t+s-j}^{(und)}$, are upper-left $h \times h$ diagonal blocks of $n \times n$ diagonal matrixes W_{t+s-j},
- estimated variances of UDC prediction errors, an $h \times h$ diagonal matrix $R_t^{(e|und)}$, is upper-left $h \times h$ diagonal block of $n \times n$ diagonal matrix $R_t^{(e)}$, and
- estimated variances of residual errors, an $h \times h$ diagonal matrix $R_t^{(r|und)}$, is upper-left $h \times h$ diagonal block of $n \times n$ diagonal matrix $R_t^{(r)}$.

Therefore, estimated variance of underlying time-series $y_t^{(und)}$, an $h \times h$ diagonal matrix σ_{t-j}^2, is a "sub-matrix" by Eq. (3.12) as

$$\begin{aligned} \sigma_{t-j}^2 &= Diag(\, C_{t-j}^{(und)}) \\ &= Diag(\, X_{t,0}^{(und)} V_{t-j} \,(\, X_{t,0}^{(und)})^T \,) + W_{t-j}^{(und)} + R_t^{(r|und)} \,, \end{aligned} \tag{5.1}$$

where time-lag $j = 0,1,2,\ldots$. Estimated volatility of $y_t^{(und)}$ is $\sigma_{t-j} = (\, \sigma_{t-j}^2 \,)^{1/2}$. Forecasted variance of underlying time-series $y_t^{(und)}$, an $h \times h$ diagonal matrix $\sigma^2_{(t+s)|t}$, is a "sub-matrix" by Eq. (4.11) as

$$\sigma^2_{(t+s)|t} = Diag(\, C^{(und)}_{(t+s)|t}\,)$$
$$= Diag(\, X^{(und)}_{t,0} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, A_{t,j}\, V_{(t+s-j),(t+s-k)}\, A^T_{t,k}\,)\,(\, X^{(und)}_{t,0}\,)^T\,)$$
$$+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} D^{(und)}_{t,j}\, W^{(und)}_{(t+s-j),(t+s-k)}\, D^{(und)}_{t,k}$$
$$+ Diag(\, X^{(und)}_{t,0}\, R^{(v)}_t\,(\, X^{(und)}_{t,0}\,)^T\,) + R^{(e|und)}_t + R^{(r|und)}_t\,, \qquad (5.2)$$

where forecast step $s = 1,2,\ldots$. Forecasted volatility of $y_t^{(und)}$ is $\sigma_{(t+s)|t} = (\,\sigma^2_{(t+s)|t}\,)^{1/2}$.

All predictors of first-step forecasts are carriers of information from current, i.e. the most recent, observations. Substituting $s = 1$ into Eq. (5.2) gives the first-step forecast of variance of underlying time-series $y_t^{(und)}$ as

$$\sigma^2_{(t+1)|t} = Diag(\, C^{(und)}_{(t+1)|t}\,)$$
$$= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, X^{(und)}_{t,0}\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A^T_{t,k}\,)\,(\, X^{(und)}_{t,0}\,)^T\,)$$
$$+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} D^{(und)}_{t,j}\, W^{(und)}_{(t+1-j),(t+1-k)}\, D^{(und)}_{t,k}$$
$$+ Diag(\, X^{(und)}_{t,0}\, R^{(v)}_t\,(\, X^{(und)}_{t,0}\,)^T\,) + R^{(e|und)}_t + R^{(r|und)}_t\,. \qquad (5.3)$$

Attributes about underlying time-series to be discussed based on Eq. (5.3) include

- basic parts of variance,
- quadratic autoregression (QAR), and
- dynamic attribution of variance.

Individual additive terms in Eq. (5.3) can be regarded as basic parts of variances of $y_t^{(und)}$. Common parts of variance of underlying time-series are based on variances and autocovariances of common dynamic factors, f_t , as

$$(\sigma^{(V)}_{t,j,k})^2$$

$$= Diag(\, X_{t,0}^{(und)}\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^{T}\,)\,(\, X_{t,0}^{(und)}\,)^{T}\,)\,, \qquad (5.4)$$

where $j, k = 1, 2, \dots, p$. Idiosyncratic parts are based on variances and autocovariances of idiosyncratic UDCs, g_t , as

$$(\sigma_{t,j,k}^{(W)})^2$$
$$= D_{t,j}^{(und)}\, W_{(t+1-j),(t+1-k)}^{(und)}\, D_{t,k}^{(und)}\,, \qquad (5.5)$$

Where $j, k = 1,2, \dots, q$. And random error part incorporating variances of all prediction/residual errors is

$$(\sigma_t^{(R)})^2$$
$$= Diag(\, X_{t,0}^{(und)}\, R_t^{(v)}\,(\, X_{t,0}^{(und)}\,)^{T}\,) + R_t^{(e|und)} + R_t^{(r|und)}\,. \qquad (5.6)$$

Then the first-step forecasts of variances of underlying time-series are sum totals of all parts of variances as

$$\sigma_{(t+1)|t}^{2}$$
$$= \sum_{j=1}^{p} \sum_{k=1}^{p} (\sigma_{t,j,k}^{(V)})^2 + \sum_{j=1}^{q} \sum_{k=1}^{q} (\sigma_{t,j,k}^{(W)})^2 + (\sigma_t^{(R)})^2\,. \qquad (5.7)$$

A common part, $(\sigma_{t,j,k}^{(V)})^2$, by Eq. (5.4) is actually factor-based portion of covariance of pair of time-lags (j, k) , while an idiosyncratic part, $(\sigma_{t,j,k}^{(W)})^2$, by Eq. (5.5) is UDC-based portion of covariance of (j, k) . Similar to definition of correlation coefficients, define "factor-based QAR (quadratic autoregressive) coefficients" as

$$a_{t,j,k}^{(V)} = \sigma_{t+1-j}^{-1}\,(\sigma_{t,j,k}^{(V)})^2\,\sigma_{t+1-k}^{-1}\,, \qquad (5.8)$$

where $j, k = 1, 2, \dots, p$, and σ_{t+1-j}^{-1} or σ_{t+1-k}^{-1} is estimated by Eq. (5.1); and define "UDC-based QAR coefficients" as

$$a_{t,j,k}^{(W)} = \sigma_{t+1-j}^{-1}\,(\sigma_{t,j,k}^{(W)})^2\,\sigma_{t+1-k}^{-1}\,, \qquad (5.9)$$

where $j, k = 1, 2, \dots, q$. With coefficients by Eqs. (5.8) and (5.9), the first-step forecasts of variances of underlying time-series $y_t^{(und)}$ can be expressed by a quadratic autoregressive (QAR) equation as

$$\sigma^2_{(t+1)|t} = \sum_{j=1}^{p} \sum_{k=1}^{p} \sigma_{t+1-j}\ a_{t,j,k}^{(V)}\ \sigma_{t+1-k}$$
$$+ \sum_{j=1}^{q} \sum_{k=1}^{q} \sigma_{t+1-j}\ a_{t,j,k}^{(W)}\ \sigma_{t+1-k} + (\sigma_t^{(R)})^2 . \qquad (5.10)$$

This is why $a_{t,j,k}^{(V)}$ and $a_{t,j,k}^{(W)}$ by Eqs. (5.8) and (5.9) are referred to as QAR coefficients. The QAR equation in Eq. (5.10) will be adapted to make QAR forecasts of implied variances (squares of volatility indexes).

By re-grouping variance parts and autocovariance parts separately, underlying variance formula Eq. (5.7) can be re-written to represent different sources of variance:

$$\sigma^2_{(t+1)|t}$$
$$= \sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, (\sigma_{t,j,k}^{(V)})^2 + \sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, (\sigma_{t,j,k}^{(V)})^2$$
$$+ \sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, (\sigma_{t,j,k}^{(W)})^2 + \sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, (\sigma_{t,j,k}^{(W)})^2$$
$$+ (\sigma_t^{(R)})^2 . \qquad (5.11)$$

where δ_{ij} is Kronecker delta ($\delta_{ij} = 1$ when $j = i$ and $\delta_{ij} = 0$ when $j \neq i$). Major dynamic sources of variances of underlying time-series $y_t^{(und)}$ shown in Eq. (5.11) are:

- "common volatility", $\sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, (\sigma_{t,j,k}^{(V)})^2$, contributions from variances of common dynamic factors,
- "common serial-correlation", $\sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, (\sigma_{t,j,k}^{(V)})^2$, contributions from vector autocovariances of common factors,
- "idiosyncratic volatility", $\sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, (\sigma_{t,j,k}^{(W)})^2$, contributions from variances of idiosyncratic UDCs,
- "idiosyncratic serial-correlation", $\sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, (\sigma_{t,j,k}^{(W)})^2$, contributions

from autocovariances of idiosyncratic UDCs, and

- "prediction error", $(\sigma_t^{(R)})^2$, contributions from dynamic factor prediction errors, UDC prediction errors, and residual errors.

A complete set of (percentage) portions of these dynamic sources (i.e. all portions adding to 1.0 or 100%) of underlying variance represent dynamic attributions of variances of underlying time-series $y_t^{(und)}$.

6. Dynamic Forecasts of Volatility Indexes

With all preparations discussed in Sections 2 through 5, we are now ready to make multi-step forecasts of multiple volatility indexes, VI_t. As outlined in Section 2, forecasts of variances of underlying time-series can be transformed into forecasts of implied variances VI_t^2. Two different transformations are developed:

- UVF forecast, and
- QAR forecast.

The UVF forecast is "volatility index forecasts with underlying volatility forecasts (UVF) as predictors", or "forecasting changes in volatility indexes with forecasted changes in underlying volatilities". It is achieved by forecasting implied variances, VI_t^2, with forecasted underlying variances, $\sigma_{(t+s)|t}^2$, as

$$VI_{(t+s)|t}^2 = \beta_t \, \sigma_{(t+s)|t}^2 , \qquad (6.1)$$

where forecast step $s = 1, 2, \ldots$ and β_t is multipliers of value-adjustment for any forecast step. Since nowcast by Eq. (6.1) should yield the known value of implied variance, i.e. $\beta_t \, \sigma_{(t+0)|t}^2 = VI_t^2$, the multipliers is

$$\beta_t = \beta_t^{(uvf)} = VI_{(t+0)|t}^2 \, \sigma_{(t+0)|t}^{-2} = VI_t^2 \, \sigma_t^{-2} \,, \tag{6.2}$$

where $\beta_t^{(uvf)}$ is an $h \times h$ diagonal matrix of value-adjustment multiplier for UVF forecasts and σ_t^2 is of estimated underlying variances. Then, the UVF forecasts of implied variances are

$$(VI_{(t+s)|t}^{(uvf)})^2 = \beta_t^{(uvf)} \, \sigma_{(t+s)|t}^2 \,, \tag{6.3}$$

and the UVF forecasts of volatility indexes are

$$VI_{(t+s)|t}^2 = (VI_{(t+s)|t}^{(uvf)})^2 \,, \tag{6.4}$$

$$VI_{(t+s)|t} = (VI_{(t+s)|t}^2)^{1/2} \,, \tag{6.5}$$

where forecast step $s = 1, 2, \ldots$.

The QAR forecast is “volatility index forecasts by quadratic autoregressive (QAR) equation with QAR coefficients of underlying volatilities”, or “making quadratic autoregressive (QAR) forecasts of volatility indexes with QAR equation of underlying volatilities”. It is achieved based on underlying volatility QAR equation in Eq. (5.10) as

$$\begin{aligned} VI_{(t+s)|t}^2 = \alpha_t + \beta_t \, (& \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} VI_{t+s-j} \; a_{t,j,k}^{(V)} \; VI_{t+s-k} \\ & + \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} VI_{t+s-j} \; a_{t,j,k}^{(W)} \; VI_{t+s-k} \,) \,, \end{aligned} \tag{6.6}$$

where forecast step $s = 1, 2, \ldots$, QAR coefficients $a_{t,j,k}^{(V)}$ and $a_{t,j,k}^{(W)}$ are estimated by Eqs. (5.8) and (5.9), respectively, and α_t and β_t are additive constants and multipliers, respectively, of value-adjustment for any forecast step, Since “1-step forecast from 1-step back” by Eq. (6.6) should yield the known value of implied variance, i.e.

$$\begin{aligned} VI_{t|(t-1)}^2 &= \beta_t \, (\textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} VI_{t-j} \; a_{t,j,k}^{(V)} \; VI_{t-k} \\ &\qquad + \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} VI_{t-j} \; a_{t,j,k}^{(W)} \; VI_{t-k} \,) + \alpha_t \\ &= VI_t^2 \,, \end{aligned} \tag{6.7}$$

and the additive constant α_t is functionally the same as, then should be proportional to, the prediction error term in Eq. (5.10), i.e.

$$\alpha_t \ VI_t^{-2} = (\sigma_t^{(R)})^2 \ \sigma_{(t+1)|t}^{-2} , \tag{6.8}$$

the additive constants of value-adjustment are

$$\alpha_t = \alpha_t^{(qar)} = VI_t^2 \ (\sigma_t^{(R)})^2 \ \sigma_{(t+1)|t}^{-2} , \tag{6.9}$$

and the multipliers of value-adjustment are

$$\begin{aligned} &\beta_t = \beta_t^{(qar)} \\ &= (VI_t^2 - \alpha_t^{(qar)}) (\textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} VI_{t-j} \ a_{t,j,k}^{(V)} \ VI_{t-k} \\ &\quad + \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} VI_{t-j} \ a_{t,j,k}^{(W)} \ VI_{t-k})^{-1} , \end{aligned} \tag{6.10}$$

where $\alpha_t^{(qar)}$ is an $h \times h$ diagonal matrix of value-adjustment additive constants for QAR forecasts, $\beta_t^{(qar)}$ is an $h \times h$ diagonal matrix of value-adjustment multipliers for QAR forecasts, and $\sigma_{(t+1)|t}^2$ is the first-step forecasts of underlying variances. Then the QAR forecasts of implied variances are

$$\begin{aligned} &(VI_{(t+s)|t}^{(qar)})^2 \\ &= \alpha_t^{(qar)} + \beta_t^{(qar)} (\textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} VI_{t+s-j} \ a_{t,j,k}^{(V)} \ VI_{t+s-k} \\ &\qquad + \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} VI_{t+s-j} \ a_{t,j,k}^{(W)} \ VI_{t+s-k}) , \end{aligned} \tag{6.11}$$

and the QAR forecasts of volatility indexes are

$$VI_{(t+s)|t}^2 = (VI_{(t+s)|t}^{(qar)})^2 , \tag{6.12}$$

$$VI_{(t+s)|t} = (VI_{(t+s)|t}^2)^{1/2} , \tag{6.13}$$

where forecast step $s = 1, 2, \ldots$. Forecasting by Eqs. (6.11) to (6.13) is an iterative process: forecasted volatility indexes at step s are inputs of forecasting implied variances at step $s + 1$.

7. Evaluating Forecasts of Volatility Indexes

Generally, potential future performances of time-series forecasts can be evaluated by comparing most recent out-of-sample forecasts against later actually observed true values being forecasted.

Specifically, the most recent out-of-sample volatility index forecasts can be denoted as

$$VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)} = VI_{(\tau(i)+s)|\tau(i)}^{(uvf)} \ \ or \ \ VI_{(\tau(i)+s)|\tau(i)}^{(qar)}, \tag{7.1}$$

where s-step forecasts of volatility indexes made at time $\tau(i)$, $VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)}$ are either UVF forecasts $VI_{(\tau(i)+s)|\tau(i)}^{(uvf)}$ or QAR forecasts $VI_{(\tau(i)+s)|\tau(i)}^{(qar)}$. A sample of (volatility index and DFM-input time-series) data points utilized to calculate out-of-sample forecasts can be labeled by either the sample ID number $i \geq 0$ or the last time-stamp $\tau(i)$ of the sample. Forecast step $s \geq 1$ and $s \leq s_{max}$. As long as the last time-stamp $\tau(i) = t - s_{max} - i$, here $i = 0, 1, 2, \ldots$, the time-stamp of the true values being forecasted $\tau(i) + s \leq t$, i.e. these true values have already been observed at time t:

$$VI_{(\tau(i)+s)|\tau(i)}^{(True)} = VI_{\tau(i)+s}^{(True)} = VI_{\tau(i)+s}. \tag{7.2}$$

The forecasts $VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)}$ are still literally "out-of-sample" when all data points observed later than $\tau(i)$, i.e. all volatility index values VI_τ and DFM-input time-series values y_τ with $\tau > \tau(i)$, are completely kept out of any calculations leading to the forecasts $VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)}$.

Then the goodness of fit of the most recent out-of-sample forecasts, $VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)}$, can be estimated by some evaluation metrics measuring certain similarities or dissimilarities between the forecasted-true value pairs, $VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)}$ and $VI_{\tau(i)+s}^{(True)}$, with sample ID $i = 0, 1, 2, \ldots$. The most popular evaluation metrics can be estimated by open-source Python library "scikit-learn" (https://scikit-learn.org/stable/, and https://scikit-learn.org/stable/modules/model_evaluation.html#regression-metrics), which include:

- "r2_score",
- "d2_absolute_error_score",
- "mean_squared_error",
- "mean_absolute_error", and
- "mean_absolute_percentage_error".

Furthermore, in general, to make evaluation of time-series forecasts genuinely meaningful, we must compare the goodness of fit of the model-based forecasts with an objective benchmark, e.g. "goodness of fit of the most relevant random forecasts".

Specifically, a set of most relevant random forecasts of volatility index is of a normal distribution:

$$VI_{(\tau(i)+s)|\tau(i)}^{(Random)} \sim N(\, VI_{\tau(i)} \,,\, \sigma_{\tau(i)}^{2}(s) \,) \,, \tag{7.3}$$

where $\sigma_{\tau(i)}^{2}(s)$, $i \geq 0$, is variance of s-step difference of the volatility indexes. When using an evaluation metric based on mean values, as those listed above, to calculate the benchmark of goodness, we can replace a Monte Carlo simulation with Eq. (7.3) by simple "flat forecasts" as

$$VI_{(\tau(i)+s)|\tau(i)}^{(Flat)} = VI_{\tau(i)} \,. \tag{7.4}$$

Therefore, an objective evaluation score, $Score(s)$, of s-step volatility index forecasts can be estimated as follows:

$$\begin{aligned} &Goodness(s) \\ &= Metric(\, VI_{(\tau(i)+s)|\tau(i)}^{(Forecast)} \,,\, VI_{(\tau(i)+s)|\tau(i)}^{(True)} \,;\, i = 0,1,\dots \,) \,, \end{aligned} \tag{7.5}$$

$$\begin{aligned} &Benchmark(s) \\ &= Metric(\, VI_{(\tau(i)+s)|\tau(i)}^{(Flat)} \,,\, VI_{(\tau(i)+s)|\tau(i)}^{(True)} \,;\, i = 0,1,\dots \,) \,, \end{aligned} \tag{7.6}$$

$$\begin{aligned} &Score(s) \\ &= Goodness(s) - Benchmark(s) \,. \end{aligned} \tag{7.7}$$

When denoting:

$$y_{pred} = \{ VI^{(Forecast)}_{(\tau(i)+s)|\tau(i)} \,;\, i = 0,1,2, \dots \} ,$$

$$or \;\; \{ VI^{(Flat)}_{(\tau(i)+s)|\tau(i)} \,;\, i = 0,1,2, \dots \} , \qquad (7.8)$$

$$y_{true} = \{ VI^{(True)}_{(\tau(i)+s)|\tau(i)} \,;\, i = 0,1,2, \dots \} , \qquad (7.9)$$

and using Python scikit-learn library to calculate evaluation scores, the evaluation formulas Eqs. (7.5) and (7.6) can be calculated as:

$$\begin{aligned} & Metric(VI^{(Forecast\ or\ Flat)}_{(\tau(i)+s)|\tau(i)} , VI^{(True)}_{(\tau(i)+s)|\tau(i)} \,;\, i = 0,1, \dots) \\ &= Metric(y\_pred, y\_true) \\ &= \;\; r2\_score(y\_true, y\_pred, \dots) \\ & or \;\; d2\_absolute\_error\_score(y\_true, y\_pred, \dots) \\ & or \;\; mean\_squared\_error(y\_true, y\_pred, \dots) \\ & or \;\; mean\_absolute\_error(y\_true, y\_pred, \dots) \\ & or \;\; mean\_absolute\_percentage\_error(y\_true, y\_pred, \dots) \, . \qquad (7.10) \end{aligned}$$

For a “metric of similarity”, e.g. r2_score or d2_absolute_error_score, a positive evaluation score, $Score(s) > 0$, indicates effective forecasts, and the higher score the better. On the other hand, for a “metric of dissimilarity”, e.g. mean_squared_error, mean_absolute_error, or mean_absolute_percentage_error, a negative evaluation score, $Score(s) < 0$, indicates effective forecasts, and the lower score the better.

8. Dynamic Volatility Attributions

One of the primary objectives of volatility analysis is to make “volatility attribution”: evaluating contributions from various sources to estimated, nowcasted or forecasted volatility measures. The simplest source of volatility is a pure random noise – assuming all multivariate/individual serial-correlations are zeros. In real data sets, however, serial-correlations are quite often not zeros. In general, multivariate volatility values are sums of contributions from two sources: (1) variance-covariances and (2) vector and/or individual autocovariances. Positive serial-correlations increase volatility levels, while negative ones decrease them. Non-zero autocovariances also make term-structures of volatilities different from that of pure random noise: “volatility level is proportional to square root of time horizon”.

Dynamic volatility forecasts, with Eqs. (3.7) to (3.11) in whitepaper by i4cast LLC (2025) and with Eqs. (6.6) to (6.11) here in Section 6 of this paper, demonstrate that equations for immediate (as opposed to “forward”) one-step volatility forecasts incorporate all volatility sources expressed in the latest available data values and model estimates. Therefore, any specific formula of dynamic volatility forecasts has a corresponding set of volatility attributions associated with it.

Specifically, associated with the UVF forecasts Eq. (6.3) of volatility indexes are dynamic volatility attributions the same as volatility attributions of the underlying time-series discussed in i4cast LLC (2025). It can be summarized with contributions from 5 sources of volatility:

- “common volatility”, from variances of dynamic common factors;
- “common serial-correlation”, from vector autocovariances of dynamic common factors;
- “idiosyncratic volatility”, from variance of idiosyncratic UDCs;
- “idiosyncratic serial-correlation”, from autocovariance of idiosyncratic UDCs;
- “prediction error”, from (a) dynamic factor prediction errors, (b) UDC prediction errors, and (c) residual errors.

Similarly, associated with the QAR forecasts Eq. (6.11) of volatility indexes are dynamic volatility attributions with the same 5 sources of volatility, but with a different set of calculations: those based on QAR equation Eq. (6.11), of course. With QAR forecasts Eqs. (6.11) to (6.13), the first 1-step forecast of implied variance is

$$\begin{aligned} & VI^2_{(t+1)|t} \\ &= \alpha_t^{(qar)} + \beta_t^{(qar)} \Big(\textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} VI_{t+1-j} \; a_{t,j,k}^{(V)} \; VI_{t+1-k} \\ &\qquad + \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} VI_{t+1-j} \; a_{t,j,k}^{(W)} \; VI_{t+1-k} \Big) . \end{aligned} \tag{8.1}$$

We can define factor-explained parts of implied variance by QAR forecast as

$$(VI_{t,j,k}^{(V)})^2 = \beta_t^{(qar)} VI_{t+1-j} \; a_{t,j,k}^{(V)} \; VI_{t+1-k} , \tag{8.2}$$

where $j, k = 1, 2, \ldots, p$, define UDC-explained parts of implied variance as

$$(VI_{t,j,k}^{(W)})^2 = \beta_t^{(qar)} VI_{t+1-j} \; a_{t,j,k}^{(W)} \; VI_{t+1-k} , \tag{8.3}$$

where $j, k = 1, 2, \ldots, q$. Therefore, the first 1-step QAR forecasts can be expressed in five volatility sources as

$$\begin{aligned} & VI^2_{(t+1)|t} \\ &= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij} \, (VI_{t,j,k}^{(V)})^2 + \sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij}) \, (VI_{t,j,k}^{(V)})^2 \\ &+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij} \, (VI_{t,j,k}^{(W)})^2 + \sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij}) \, (VI_{t,j,k}^{(W)})^2 \\ &+ \alpha_t^{(qar)} . \end{aligned} \tag{8.4}$$

Here, each of the five terms represents an aggregate dynamic source of volatility, measured by contribution to forecasted implied variance from the source:

- the first term, can be labeled as “common volatility”, is of variances of dynamic common factors;
- the second term, as “common serial-correlation”, is of vector autocovariances of dynamic common factors;
- the third term, as “idiosyncratic volatility”, is of variance of idiosyncratic UDCs;

- the fourth term, as "idiosyncratic serial-correlation", is of autocovariance of idiosyncratic UDCs;
- the fifth term, as "prediction error", is of (a) dynamic factor prediction errors, (b) UDC prediction errors, and (c) residual errors.

Volatility analysis based on dynamic factor models jointly estimate both variance-covariances and vector/individual autocovariances of large number of time-series. Traditional static factor model volatility analyses, however, do not estimate any autocovariances at all. Therefore, dynamic volatility attributions provide more and deeper insights than static volatility attributions.

9. Examples

To test Dynamic Factor Based Volatility Index Forecast (DFbVIF) model, an example of input data table is tabulated with publicly available real data sets. The table contains multiple weekly time-series for several years ending as of the last Friday of the most recent calendar year. Each row is of an individual time-series. The first 14 rows are of 14 volatility indexes published by CBOE (https://www.cboe.com/) listed here, in a format of,

- [symbol of volatility index], on [name of underlying index/price/value] ([symbol of underlying time-series]),

as follows:

- VIX, on S&P 500 (SPY);
- VXD, on DJIA (DIA);
- VXN, on Nasdaq 100 (QQQM);
- RVX, on Russel 2000 (VTWO);

- VXEEM, on emerging market (EEM);
- VXEWZ, on Brazil ETF (EWZ);
- GVZ, on gold (GLD);
- OVX, on crude oil (USO);
- EVZ, on Euro FX (FXE);
- VXAZN, on Amazon (AMZN);
- VXAPL, on Apple (AAPL);
- VXGS, on Goldman Sachs (GS);
- VXGOG, on Google (GOOGL);
- VXIBM, on IBM: (IBM).

Remaining 55 rows of the table are of "DFM-input time-series", including logarithmic changes in 14 underlying time-series (indexes/prices/values underlying the 14 volatility indexes) and in 41 information-enhancing time-series dynamically correlated with the underlying time-series. The info-enhancing ones include logarithmic changes in indexes/prices of equities, fixed incomes, commodities, and FXs.

A dynamic factor model, DFVCM algorithm (Appendix A), embedded in DFbVIF model estimated and forecasted the variance-covariances and vector autocovariances of the DFM-input time-series. Then, DFbVIF transformed the forecasted variances of the underlying time-series into forecasted implied variances of the volatility indexes, by either UVF or QAR transformation. Forecasted volatility index levels are square roots of the forecasted implied variances.

10. Discussion

The numeric example described above demonstrates that the Dynamic Factor Based Volatility Index Forecast (DFbVIF) model can generate statistically good and consistent multi-step forecasts of multiple volatility indexes. The performance of DFbVIF forecasts is consistently better than an objective benchmark: performance of simple flat forecasts, even at multiple forecasting steps, such as 10 steps.

Dynamic attributions of forecasts by DFbVIF can reveal useful information in terms of common vs. idiosyncratic factors, and cross-sectional vs. serial correlations.

Numbers of common factors and idiosyncratic UDCs for better QAR (quadratic autoregressive) forecasts may be different from those for better UVF (underlying volatility forecasts as predictors) forecasts. Or, in another words, QAR and UVF forecasts may need to be tuned separately.

Appendix A. DFVCM Algorithm

DFVCM (dynamic factor variance-covariance model, https://aws.amazon.com/marketplace/pp/prodview-yvaulquatt3v2?sr=0) algorithm

- employs dynamic factor models (DFM) on big data sets, and
- makes multi-step forecasts of multivariate volatilities (represented by large variance-covariance matrix) of DFM-input time-series.

DFVCM makes all estimates and forecasts in three separate steps:

1. Step One, estimate and forecast all values related to "common dynamic factors" (common to many or all DFM-input time-series), e.g. Eqs. (3.1), (3.2), (3.5), (3.6) and (3.9), by using LMDFM (long-memory dynamic factor model) algorithm (see Appendix B).

2. Step Two, estimate and forecast all values related to "idiosyncratic UDCs" (specific to individual DFM-input time-series), e.g. Eqs. (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11), by using YWpcAR (Yule-Walker-PCA autoregression) algorithm (see Appendix C).
3. Step Three, combine estimates and forecasts of common factors and idiosyncratic UDCs together to make multi-step forecasts of multivariate volatilities.

Appendix B. LMDFM Algorithm

Long-memory dynamic factor model (LMDFM, https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg?sr=0) is estimated by an implementation of spectral (or dynamic) principal components analysis (SPCA or DPCA), reviewed by Doz and Fuleky (2020). An SPCA on conjugate two-dimensional discrete Fourier transform (C2D-DFT) can be summarized as follows:

- Estimating variance-covariance matrixes (VCMs) and autocovariance matrixes (ACMs) of observed vector (i.e. multiple) time-series y_t , as $C(j,k) = Cov(\, y_{t-j}\, ,\, y_{t-k}\,) = E(\, y_{t-j}\, y_{t-k}^T\,)$, $j,k = 0,1,\dots,p$, assuming $E(\, y_t\,) = 0$.
- Combining VCMs, $C(j,j)$ when $k = j$, and ACMs, $C(j,k)$ when $k \neq j$, by applying conjugate two-dimensional discrete Fourier transform (C2D-DFT) on $C(j,k)$, $j,k = 0,1,\dots,p$.
- Referring resulted transform by C2D-DFT as spectral density matrixes (SDMs), $S(m,n) = DFT_{C2D}(\,\{C(j,k)\}\,)$, $m,n = 0,1,\dots,p$, of vector time-series, y_t .
- Estimating spectral principal components of "on-diagonal SDMs", $S(m,m)$, $m = n$, and SPCA representations of "off-diagonal SDMs", $S(m,n)$, $m \neq n$.

- Estimating full-spectrum principal components (FSPCs) of original VCMs by applying inverse C2D-DFT on SPCA representations of SDMs.
- This way, FSPCs of original VCMs contain dynamic information in all VCMs and ACMs.
- Dynamic factor loadings and vector autoregressive (VAR) coefficients of factor scores can be estimated by full-spectrum principal components of VCMs.

The LMDFM estimates two different simplest forms of DFM. DFM of Form I estimates:

$$y_t = \mu_t + X_{t,0}\, f_t + X_{t,1}\, f_{t-1} + \cdots + X_{t,p}\, f_{t-p} + u_t\,,$$

$$f_t = g(\, e_t\,)\,,$$

where e_t is a vector of white noises of unit variance, $g(\cdot)$ is a linear transformation, and elements of vector f_t are independent of each other and over time. Other application facts about Form I include:

- classic form of DFM for classic dynamic analysis,
- suitable for simple Monte Carlo simulation,
- estimating a whole set of matrixes of factor loadings $X_{t,k}$,
- not applicable to "big data set" due to large number of elements of loadings,
- question on time-series forecast: sizable impact by assuming unknown factor scores $f_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of f_t .

LMDFM's DFM of Form II estimates Eqs. (3.1) and (3.2) as:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t\,,$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t\,,$$

where the only loadings matrix $X_{t,0}$ in Form II is NOT the same as that in Form I (just share the same symbol). Other application facts about Form II include:

- only one matrix of factor loadings, $X_{t,0}$,
- estimating a whole set of VAR matrixes, $A_{t,k}$, of common factors,
- non-zero vector autocovariance of comm factors, $Cov(\, f_{t-j} \,, \, f_{t-j-k} \,) \neq 0$, therefore, not for simple Monte Carlo simulation,
- applicable to "big data set" due to limited number of elements of VAR matrixes,
- question on time-series forecast: smaller impact by assuming unknown errors $v_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of errors v_t .

DFM of Form II, i.e. Eqs. (3.1) and (3.2), is utilized to estimate common components, including $X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, V_{t-j} , $V_{(t-j),(t-k)}$ and u_t , shown in Section "3. DFM Volatility Model on DFM-Input TS".

<u>Appendix C. YWpcAR Algorithm</u>

Yule-Walker-PCA autoregressive model (YWpcAR, https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0) is estimated by an implementation of applying principal components analysis (PCA) on Yule-Walker (YW) equation of AR model, as summarized as follows:

- A set of autoregression (AR) coefficients is a solution to Yule-Walker (YW) equation of AR model.
- Applying principal components analysis (PCA) on symmetric (sample) variance-autocovariance matrixes (VACMs) associated with YW equation.
- Replacing VACMs by their principal components (PC) representations.

- Replacing design matrix and response vector of YW equation by their PC representations formed with elements of PC representations of VACMs.
- Estimating minimum norm solution to PC-represented YW equation to serve as PC representation of set of AR coefficients.
- Combining principal eigen score time-series into an unobserved dynamic component (UDC) time-series of observed data.
- Forecasting values of observed time-series by AR forecasts of UDC time-series.
- Forecasting variances of observed time-series by AR forecasted variances of UDC and estimated residual variance.

The YWpcAR can be utilized to estimate idiosyncratic components, including g_t, $D_{t,k}$, $R_t^{(e)}$, W_{t-j}, $W_{(t-j),(t-k)}$ and $R_t^{(r)}$, shown in Section “3. DFM Volatility Model on DFM-Input TS”.

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