Introduction to Multi-step Forecast of Multivariate Volatility with Dynamic Factor Model

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Abstract

This paper introduces volatility forecasts made by Dynamic Factor Model (DFM). Advantages of DFM-based volatility model over other volatility models include capability to make multi-step forecasts of multivariate volatilities of large number of time-series. Volatility components forecasted by DFM-based model include those separately attributable to (1) common or (2) idiosyncratic (a) variances or (b) autocovariances or (c) prediction errors. Common volatility components are forecasted using DFM by spectral PCA and conjugate 2D DFT. Idiosyncratic volatility components are forecasted using AR model by Yule-Walker-PCA estimates.

Keywords: Volatility model, volatility forecasts, DFM-based volatility model, multi-step forecasts of multivariate volatilities, dynamic volatility attributions, forecasting volatility components, common or idiosyncratic components forecasts, variance or autocovariance components forecasts, volatility components prediction errors.

List of Sections

1. Introduction

Making volatility forecasts using Dynamic Factor Model (DFM) offers a number of attractive advantages over other volatility forecasting methods. Specifically, a DFM volatility model is capable

- to make multi-step forecasts of multivariate volatilities (i.e. uncertainties, risks) of a large number time-series (e.g. forecasting risks of numerous investable assets in various markets),
- to estimate source-specific components of forecasted volatilities which are attributable to
    - variances of dynamic common factors,
    - vector autocovariances of dynamic common factors,
    - variances of idiosyncratic UDCs (unobserved dynamic components),

        - autocovariances (or serial-correlations) of idiosyncratic UDCs,
        - common / idiosyncratic / residual prediction errors

    (e.g. portions of volatilities caused by common economic or market conditions; volatility jumps in panic, and drops in euphoria, markets; risks due to specific dynamics of individual stocks; random error components),

- to make multi-step forecasts of multiple time-series themselves (i.e. forecasting conditional expectations instead of volatilities of the time-series) using the same dynamic factor model (e.g., economic or market forecasts made by government agencies or research institutions).

In short, a volatility model based on dynamic factor model (DFM) is capable to make all of the estimates and forecasts listed above simultaneously using the same integrated and dimension-reduced multivariate modeling and analysis process.

Multivariate GARCH models and static factor risk models are among popular classes of volatility models, with each class making some of the above estimates or forecasts. Meanwhile, it had been traditionally difficult to estimate large scale DFMs.

The advent of new machine learning algorithms able to solve for large DFMs (such as many long-memory dynamic factors on large number of time-series), DFM-based volatility models can now become powerful tools at your disposal to attack real-world volatility problems in real-time.

Academic literature on dynamic factor models (DFMs) is voluminous. Doz and Fuleky (2020) is among the latest best overviews on the history of DFM research and development. Alessi, Barigozzi and Capasso (2007) discusses benefits and performances of volatility forecasts by DFM vs. multivariate GARCH.

This paper discusses formulation details how to use dynamic factor model (DFM) to make multi-step forecasts of multivariate volatilities, and of various dynamic volatility components, of large number of time-series.

2. Model Estimation

We denote observed multiple time-series as vector time-series y_t, which is time-series of $n \times 1$ vector, representing n individual time-series.

To make multi-step forecasts of various components of multivariate volatilities of the vector time-series y_t, we first estimate all coefficients and attributes of a dynamic factor model representation of y_t expressed as follows:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t \ , \tag{2.1}$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t \ , \tag{2.2}$$

$$u_t = g_t + r_t \ , \tag{2.3}$$

$$g_t = D_{t,1}\, g_{t-1} + D_{t,2}\, g_{t-2} + \cdots + D_{t,q}\, g_{t-q} + e_t \ , \tag{2.4}$$

where (random and non-random) variables

- y_t is time-series of $n \times 1$ vector of observed data,
- μ_t is time-series of $n \times 1$ vector of mean values of y_t,
- f_t is time-series of $m \times 1$ vector of common dynamic factor scores,
- $X_{t,0}$ is $n \times m$ matrix of factor loadings of dynamic factors f_t,
- $A_{t,j}$ are $m \times m$ matrixes of VAR (vector autoregressive) coefficients of factors f_t, here $n > \ or \ \gg m$,
- v_t is time-series of $m \times 1$ vector of VAR prediction errors of factors f_t,

- u_t is time-series of $n \times 1$ vector of idiosyncratic components of individual observed time-series,
- g_t is time-series of $n \times 1$ vector of unobserved dynamic components (UDCs) of idiosyncratic components u_t ,
- $D_{t,k}$ are $n \times n$ diagonal matrixes of AR (autoregressive) coefficients of UDCs g_t ,
- e_t is time-series of $n \times 1$ vector of AR prediction errors of UDCs g_t , and
- r_t is time-series of $n \times 1$ vector of residual random errors.

For $j = 0, 1, \cdots$ and $k = 1, 2, \cdots$, variances and autocovariances of common dynamic factors and idiosyncratic UDCs are estimated as:

$$V_{t-j} = Var(\, f_{t-j} \,) = E(\, f_{t-j}\, f_{t-j}^T \,) \,, \tag{2.5}$$

$$V_{(t-j),(t-k)} = Cov(\, f_{t-j} \,,\, f_{t-k} \,) = E(\, f_{t-j}\, f_{t-k}^T \,) \,, \tag{2.6}$$

$$W_{t-j} = Var(\, g_{t-j} \,) = E(\, g_{t-j}\, g_{t-j}^T \,) \,, \tag{2.7}$$

$$W_{(t-j),(t-k)} = Cov(\, g_{t-j} \,,\, g_{t-k} \,) = E(\, g_{t-j}\, g_{t-k}^T \,) \,, \tag{2.8}$$

where estimated matrixes

- V_{t-j} are $m \times m$ diagonal matrixes of variance of dynamic factors f_t ,
- $V_{(t-j),(t-k)}$ are $m \times m$ matrixes of autocovariance of factors f_t ,
- W_{t-j} are $n \times n$ diagonal matrixes of variance of UDCs g_t , and
- $W_{(t-j),(t-k)}$ are $n \times n$ diagonal matrixes of autocovariance of UDCs g_t .

Variances of random errors are estimated as:

$$R_t^{(v)} = Var(\, v_t \,) \,, \tag{2.9}$$

$$R_t^{(e)} = Var(\, e_t \,) \,, \tag{2.10}$$

$$R_t^{(r)} = Var(\, r_t \,) \,, \tag{2.11}$$

where estimated matrixes

- $R_t^{(v)}$ is time-series of $m \times m$ diagonal matrix of variances of VAR prediction errors v_t ,
- $R_t^{(e)}$ is time-series of $n \times n$ diagonal matrix of variances of AR prediction errors e_t , and
- $R_t^{(r)}$ is time-series of $n \times n$ diagonal matrix of variances of residual errors r_t .

The DFM estimations summarized above can be performed jointly by two models. The first one is common dynamic factor (CDF) model estimated by spectral principal components analysis (spectral PCA) with conjugate two-dimensional discrete Fourier transform (C2D-DFT), formulated and detailed in Ling (2025). An actual implementation of this method by Ling (2025) and summarized in appendix A.1, LMDFM algorithm (https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg?sr=0) can be used to estimate common components, including $X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, V_{t-j} , $V_{(t-j),(t-k)}$ and u_t . The second one is autoregressive (AR) model estimated by applying principal components analysis (PCA) to Yule-Walker (YW) equation of AR process, to be published soon by i4cast LLC. An actual implementation of this method by i4cast and summarized in appendix A.2, YWpcAR algorithm (https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0) can be used to estimate idiosyncratic components, including g_t , $D_{t,k}$, $R_t^{(e)}$, W_{t-j} , $W_{(t-j),(t-k)}$ and $R_t^{(r)}$.

In estimation of all of the above coefficients and attributes, we applied following assumptions widely proposed, accepted and practiced in dynamic factor model research literature:

- mean value (of observed vector time-series) $\mu_t = 0$,
- covariance matrix V_{t-j} (of dynamic factor scores) is diagonal matrix,
- covariance matrix W_{t-j} (of idiosyncratic components) is approximated by diagonal matrix, ignoring allowed but "mild" cross-correlation,
- assuming all cross- and serial-correlations between common factors f_t and idiosyncratic UDCs g_t are zeros.

In order to make multi-step forecasts of multivariate volatility in the next section, we need past autocovariance matrix of dynamic factor scores f_t , and diagonal matrix of past autocovariance of UDC g_t , with two separate time lags, j and k. When $0 \leq j \leq k$, the past autocovariance matrix of f_t ,

$$E(\, f_{t-j}\, f_{t-k}^T\,) = V_{[t-Min(j,k)],Abs(j-k)} \,, \tag{2.12}$$

When $j \geq k \geq 0$, the past autocovariance of f_t ,

$$E(\, f_{t-j}\, f_{t-k}^T\,) = E(\, f_{t-k}\, f_{t-j}^T\,)^T = V_{[t-Min(j,k)],Abs(j-k)}^T \,. \tag{2.13}$$

The diagonal matrix of past autocovariance of g_t ,

$$E(\, g_{t-j}\, g_{t-k}^T\,) = W_{[t-Min(j,k)],Abs(j-k)} \,. \tag{2.14}$$

3. Volatility Forecasts

According to the factor model representation discussed in the Section 2 above, s-step forecasts of time-series, f_t , g_t and y_t , based on data observed until time t can be made by dynamic equations as

$$f_{(t+s)|t} = A_{t,1}\, f_{t+s-1} + A_{t,2}\, f_{t+s-2} + \cdots + A_{t,p}\, f_{t+s-p} + v_{t+s} \,, \tag{3.1}$$

$$g_{(t+s)|t} = D_{t,1}\, g_{t+s-1} + D_{t,2}\, g_{t+s-2} + \cdots + D_{t,q}\, g_{t+s-q} + e_{t+s} \,, \tag{3.2}$$

$$y_{(t+s)|t} = X_{t,0}\, f_{(t+s)|t} + g_{(t+s)|t} + r_{t+s} \,, \tag{3.3}$$

where $s = 1, 2, \cdots$. The random errors v_{t+s}, e_{t+s} and r_{t+s} cannot be forecasted, but can be characterized by assumed diagonal variance matrixes as

$$R_{t+s}^{(v)} = R_t^{(v)} , \tag{3.4}$$

$$R_{t+s}^{(e)} = R_t^{(e)} , \tag{3.5}$$

$$R_{t+s}^{(r)} = R_t^{(r)} . \tag{3.6}$$

Therefore, s-step forecast of diagonal variance matrix of dynamic factor scores f_t is

$$\begin{aligned} & V_{(t+s)|t} = Var(\, f_{(t+s)|t}\,) = E(\, f_{(t+s)|t}\, f_{(t+s)|t}^T\,) \\ &= Diag(\, E(\,(\, \textstyle\sum_{j=1}^p A_{t,j}\, f_{t+s-j}\,)(\, \sum_{k=1}^p A_{t,k}\, f_{t+s-k}\,)^T\,)\,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^p \sum_{k=1}^p Diag(\, A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,)\, A_{t,k}^T\,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^p \sum_{k=1}^p Diag(\, A_{t,j}\, V_{(t+s-j),(t+s-k)}\, A_{t,k}^T\,) + R_t^{(v)} , \end{aligned} \tag{3.7}$$

where $s = 1, 2, \cdots$, and autocovariance of factors, $E(\, f_{t+s-j}\, f_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by “earlier-step” forecasts Eqs. (3.7) and (3.8). Then, s-step forecast of k-lag autocovariance matrix of dynamic factor scores f_t is

$$\begin{aligned} & V_{(t+s),(t+s-k)} = Cov(\, f_{(t+s)|t}\,,\, f_{t+s-k}\,) \\ &= E(\, f_{(t+s)|t}\, f_{t+s-k}^T\,) = E(\, \textstyle\sum_{j=1}^p A_{t,j}\, f_{t+s-j}\ f_{t+s-k}^T\,) \\ &= \textstyle\sum_{j=1}^p A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,) = \sum_{j=1}^p A_{t,j}\, V_{(t+s-j),(t+s-k)} , \end{aligned} \tag{3.8}$$

where $s = 1, 2, \cdots$, $k = 1, 2, \cdots$, and, again, autocovariance, $E(\, f_{t+s-j}\, f_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by earlier forecasts Eqs. (3.7) and (3.8). The evaluation of Eqs. (3.7) and (3.8) can be simplified by the separate time lag autocovariance Eqs. (2.12) and (2.13).

Similarly, s-step forecast of variance of idiosyncratic unobserved dynamic components (UDCs) g_t is

$$W_{(t+s)|t} = Var(\, g_{(t+s)|t}\,) = E(\, g_{(t+s)|t}\, g_{(t+s)|t}^T\,)$$

$$= Diag(E((\sum_{j=1}^{q} D_{t,j}\, g_{t+s-j})(\sum_{k=1}^{q} D_{t,k}\, g_{t+s-k})^T)) + R_t^{(e)}$$

$$= \sum_{j=1}^{q} \sum_{k=1}^{q} Diag(D_{t,j}\, E(g_{t+s-j}\, g_{t+s-k}^T)\, D_{t,k}) + R_t^{(e)}$$

$$= \sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j}\, W_{(t+s-j),(t+s-k)}\, D_{t,k} + R_t^{(e)} , \tag{3.9}$$

where $s = 1, 2, \cdots$, and autocovariance of UDCs, $E(g_{t+s-j}\, g_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by "earlier-step" forecasts Eqs. (3.9) and (3.10). Then, s-step forecast of k-lag autocovariance of UDC g_t is

$$W_{(t+s),(t+s-k)} = Cov(g_{(t+s)|t} , g_{t+s-k})$$

$$= E(g_{(t+s)|t}\, g_{t+s-k}^T) = Diag(E(\sum_{j=1}^{q} D_{t,j}\, g_{t+s-j}\, g_{t+s-k}^T))$$

$$= \sum_{j=1}^{q} Diag(D_{t,j}\, E(g_{t+s-j}\, g_{t+s-k}^T))$$

$$= \sum_{j=1}^{q} D_{t,j}\, W_{(t+s-j),(t+s-k)} , \tag{3.10}$$

where $s = 1, 2, \cdots$, $k = 1, 2, \cdots$, and, again, autocovariance, $E(g_{t+s-j}\, g_{t+s-k}^T)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by earlier forecasts Eqs. (3.9) and (3.10). The evaluation of Eqs. (3.9) and (3.10) can be simplified by the separate time lag autocovariance Eq. (2.14).

Having

- estimated factor loadings matrix $X_{t,0}$ in Eq. (2.1),
- forecasted diagonal variance matrix $V_{(t+s)|t}$ of dynamic factor scores f_t by Eq. (3.7),
- forecasted diagonal variance matrix $W_{(t+s)|t}$ of unobserved dynamic components (UDCs) g_t by Eq. (3.9), and
- "forecasted" diagonal variance matrix of residual errors r_t by Eq. (3.6),

s-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$C_{(t+s)|t} = Var(y_{(t+s)|t}) = E(y_{(t+s)|t}\, y_{(t+s)|t}^T)$$

$$= X_{t,0}\, V_{(t+s)|t}\, X_{t,0}^T + W_{(t+s)|t} + R_t^{(r)}$$

$$= X_{t,0} \sum_{j=1}^{p} \sum_{k=1}^{p} Diag(A_{t,j} V_{(t+s-j),(t+s-k)} A_{t,k}^{T}) X_{t,0}^{T}$$

$$+ \sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j} W_{(t+s-j),(t+s-k)} D_{t,k}$$

$$+ X_{t,0} R_t^{(v)} X_{t,0}^{T} + R_t^{(e)} + R_t^{(r)} . \qquad (3.11)$$

Where forecast step $s = 1, 2, \cdots$.

4. Simple Forecasts as Benchmarks

There are two classes of simple forecasts widely practiced for multivariate variance-covariance matrix of observed vector time-series y_t .

The simplest calculated forecast is to use a sample-based variance-covariance matrix as forecasted matrix:

$$C_t^{(Sample)} = K^{-1} \sum_{k=0}^{K} (y_{t-k} - \mu_t) (y_{t-k} - \mu_t)^T , \qquad (4.1)$$

$$C_{(t+s)|t}^{(Sample)} = C_t^{(Sample)} , \qquad (4.2)$$

where forecast step $s = 1, 2, \cdots$.

A factor-based "forecast" is to use a variance-covariance matrix estimated by a static factor model as forecasted matrix:

$$C_t^{(Estimate)} = X_t V_t X_t^T + W_t + R_t , \qquad (4.3)$$

$$C_{(t+s)|t}^{(Estimate)} = C_t^{(Estimate)} , \qquad (4.4)$$

where $s = 1, 2, \cdots$; and factor loadings matrix X_t , diagonal factor variance matrix V_t , diagonal UDC variance matrix W_t and diagonal residual error variance matrix R_t are estimated coefficients and attributes of a static factor model expressed as follows:

$$y_t = \mu_t + X_t f_t + u_t ,$$

$$u_t = g_t + r_t ,$$

$$V_t = Var(f_t) = E(f_t f_t^T) ,$$

$$W_t = Var(\, g_t \,) = E(\, g_t\, g_t^T \,)\ ,$$

$$R_t = Var(\, r_t \,) = E(\, r_t\, r_t^T \,)\ .$$

Another widely practiced class of static factor models is "fundamental risk factor analysis", in which factor loadings matrix is pre-determined based on certain fundamental analysis theory or framework ahead of factor model estimation. Therefore, fundamental factor analysis is able to make meaningful explanations about multivariate volatility structure, but variance-covariance matrix of fundamental factor score time-series is not diagonal.

Since we have already estimated a dynamic factor model, we can replace estimates of a static factor model by nowcasts (not forecasts) of our dynamic factor model as follows:

$$C_t^{(Nowcast)} = X_{t,0}\, V_{t,0}\, X_{t,0}^T + W_{t,0} + R_t^{(r)}\ , \tag{4.5}$$

$$C_{(t+s)|t}^{(Nowcast)} = C_t^{(Nowcast)}\ , \tag{4.6}$$

where $s = 1, 2, \cdots$.

The above simple forecasts, Eq. (4.2), (4.4) or (4.6), can serve as benchmarks in evaluation of our multi-step forecast in multivariate volatility by dynamic factor model.

5. Evaluation of Volatility Forecast

Directly examining quality or accuracy of the forecasted variance-covariance matrix $C_{(t+s)|t}$ itself is a complicated undertaking in theory and difficult (for a large number of time-series) task in practice.

A widely practiced evaluation technique is to measure quality or accuracy of forecasted variance of a weighted aggregation of the time-series, with forecasted variance of the aggregate made by the forecasted variance-covariance matrix, as

$$(\sigma_{(t+s)|t}^{(w)})^2 = w^T\, C_{(t+s)|t}\, w\ , \tag{5.1}$$

where w is a $n \times 1$ vector of weights for aggregation and $(\sigma_{(t+s)|t}^{(w)})^2$ is forecasted variance of aggregated time-series

$$y_t^{(w)} = w^T y_t = y_t^T w \, . \tag{5.2}$$

The vector of weights, w, is set according to relevant application(s) of business or research. If one of the elements in the vector w is 1 and all others are zeros, the weighted aggregate time-series, $y_t^{(w)}$, is essentially a selected individual time-series.

To measure the accuracy of forecasted variance $(\sigma_{(t+s)|t}^{(w)})^2$ by Eq. (5.1), a "realized z-score squared of the forecasts" defined by

$$(z_{(t+s)|t}^{(w)})^2 = (\, y_{t+s}^{(w)} - \mu_{t+s}^{(w)})^2 \, / \, (\sigma_{(t+s)|t}^{(w)})^2 \, , \tag{5.3}$$

is handy, where observation $y_{t+s}^{(w)}$ and estimate $\mu_{t+s}^{(w)}$ are made at time $t+s$, while forecast $(\sigma_{(t+s)|t}^{(w)})^2$ is made at earlier time t. According to Litterman and Winkelmann (1998), Patton (2011), Menchero, Morozov and Pasqua (2013) and Fan, Furger and Xiu (2015), the accuracy of s-step volatility forecasts given an evaluation dataset in time period $(t+s) \in [t_1, t_2]$ can be measured by bias statistic $BS_{s|[t_1,t_2]}^{(w)}$, log-likelihood $LL_{s|[t_1,t_2]}^{(w)}$, and Q-statistic $QS_{s|[t_1,t_2]}^{(w)}$ defined as

$$BS_{s|[t_1,t_2]}^{(w)} = [\, \frac{1}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} (z_{(t+s)|t}^{(w)})^2 \,]^{1/2} \, , \tag{5.4}$$

$$LL_{s|[t_1,t_2]}^{(w)} = -\, \frac{1/2}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} [\, \ln(2\pi) + (z_{(t+s)|t}^{(w)})^2 + \ln(\sigma_{(t+s)|t}^{(w)})^2 \,] \, , \tag{5.5}$$

$$QS_{s|[t_1,t_2]}^{(w)} = \frac{1}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} [\, (z_{(t+s)|t}^{(w)})^2 - \ln(z_{(t+s)|t}^{(w)})^2 \,] \, . \tag{5.6}$$

A bias statistic $BS_{s|t_1,t_2}^{(w)} > 1$ or $BS_{s|t_1,t_2}^{(w)} < 1$ shows an under- or over-prediction of volatility. A higher log-likelihood $LL_{s|t_1,t_2}^{(w)}$ or a lower Q-statistic $QS_{s|t_1,t_2}^{(w)}$ indicates more accurate forecasts.

6. Dynamic Volatility Attribution

One of the primary objectives of volatility modeling and analysis is "volatility attribution": evaluating various (static or dynamic) sources of volatility that contribute to the estimated or forecasted volatility levels. A simplest classic source of volatility is ideal random walks – assuming zero values of multivariate and individual autocovariances or, in another word, serial-correlations. They are, however, almost never zeros in the real world. Multivariate volatility values in general are sum of contributions from two sources: synchronous variances-covariances and asynchronous (vector and individual) autocovariances. Positive serial-correlations increase volatility levels, while negative ones decrease them. Non-zero autocovariances also make term-structure of volatility different from that of random walks, which is "volatility levels proportional to square root of lengths of time period".

Volatility analysis based on DFMs (dynamic factor models) jointly estimate both synchronous variances-covariances and asynchronous vector autocovariances of a large number of time-series. Traditional static factor model volatility analyses, however, do not estimate autocovariances at all. Therefore, dynamic volatility attributions provide more and deeper insights than static volatility attributions.

Details in comprehensive discussions on volatility forecasts demonstrate that equations for immediate one-step volatility forecasts present all volatility sources quantified by the latest data observations and model estimates. According to Eq. (3.11), one-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$C_{(t+1)|t} = X_{t,0}\, V_{(t+1)|t}\, X_{t,0}^T + W_{(t+1)|t} + R_t^{(r)}\ . \qquad (6.1)$$

Here, by Eq. (3.7), one-step forecast of variances of dynamic factor scores f_t is

$$
\begin{aligned}
V_{(t+1)|t} &= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} Diag(A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T) + R_t^{(v)} \\
&= \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, Diag(A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T) \\
&+ \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, Diag(A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T) \\
&+ R_t^{(v)} ,
\end{aligned} \tag{6.2}
$$

where autocovariance $V_{(t+1-j),(t+1-k)} = E(f_{t+1-j}\, f_{t+1-k}^T)$ is evaluated by estimates Eqs. (2.5) and (2.6), and δ_{ij} is Kronecker delta. The first term of $k = j$, with multiplier δ_{ij} , is aggregate contribution from estimated variances of dynamic factors f_t ; the second term of $k \neq j$, with multiplier $(1-\delta_{ij})$, is from estimated vector autocovariances of f_t ; and the third term is from prediction error of f_t . Similarly, by Eq. (3.9), one-step forecast of variance of idiosyncratic UDCs g_t is

$$
\begin{aligned}
W_{(t+1)|t} &= \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} + R_t^{(e)} \\
&= \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ R_t^{(e)} ,
\end{aligned} \tag{6.3}
$$

where $W_{(t+1-j),(t+1-k)} = Diag(E(g_{t+1-j}\, g_{t+1-k}^T))$ is evaluated by estimates Eqs. (2.7) and (2.8), and δ_{ij} is Kronecker delta. The first term $k = j$, with multiplier δ_{ij} , is aggregate contribution from estimated variances of UDCs g_t ; the second term of $k \neq j$, with muliplier $(1-\delta_{ij})$, is from autocovariances of g_t ; and the third term is from prediction errors of g_t .

Therefore, the one-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$
\begin{aligned}
&C_{(t+1)|t} \\
&= X_{t,0} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, Diag(A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T)\, X_{t,0}^T
\end{aligned}
$$

$$
\begin{aligned}
&+ X_{t,0} \sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^{T}\,)\, X_{t,0}^{T} \\
&+ \sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ \sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ X_{t,0}\, R_t^{(v)}\, X_{t,0}^{T} + R_t^{(e)} + R_t^{(r)}\, . \qquad (6.4)
\end{aligned}
$$

Here, each of the five terms represents an aggregate dynamic source of volatility, measured by contribution to variance-covariance forecast from the source:

- the first term, can be labeled as “common volatility”, is from variances of dynamic common factors;
- the second term, as “common serial-correlation”, is from vector autocovariances of dynamic common factors;
- the third term, as “idiosyncratic volatility”, is from variance of idiosyncratic UDCs;
- the fourth term, as “idiosyncratic serial-correlation”, is from autocovariance of idiosyncratic UDCs; and
- the fifth term, as “prediction error”, is from (a) dynamic factor prediction error, (b) UDC prediction error, and (c) residual error.

“Dynamic volatility attribution” is to tabulate portions, or weights, of total variance-covariance matrix, or total variances, attributed to these dynamic sources of volatility.

As a comparison, traditional static volatility attribution to static sources of volatility based on static factor model of volatility, is unable to provide dynamic information. In a static volatility model, nowcast (estimated current values) of volatility serves as forecast. Volatility is attributed to only two sources:

- “common”, contributions from static common factors; and

- “idiosyncratic”, from idiosyncratic randomness (or residual error).

7. Examples

“Dynamic Factor Variance-Covariance Model” algorithm (DFVCM, https://aws.amazon.com/marketplace/pp/prodview-yvaulquatt3v2?sr=0) publicly available on AWS is an implementation of the DFM-based volatility forecasting method presented in details in this paper.

To test DFVCM model, an example of input data table is tabulated with publicly available real data sets (https://github.com/i4cast/aws/blob/main/dynamic_factor_variance-covariance_model/input/Weekly_VTS_6Yr.csv). The table contains multiple weekly time-series for several years ending as of the last Friday of the most recent calendar year. Each row is of an individual time-series. There are more than 50 weekly time-series of logarithmic changes in values of investment funds publicly traded in the U.S. exchanges. The funds invested in equities, fixed incomes, financial indexes, and physical commodities.

DFVCM model estimated and forecasted variances-covariances, vector autocovariances and individual volatility levels of these input time-series.

8. Discussion

The numeric example described above demonstrates that the DFM-based “Dynamic Factor Variance-Covariance Model (DFVCM)” can generate statistically good and consistent multi-step forecasts of multivariate volatilities of large number of time-series. With several statistic measures, the performance of multi-step DFVCM forecasts is consistently better than various objective benchmark performances by (a) sample volatility as forecast, (b) estimate (based on static factors) as forecast and (c) nowcast (or estimate, based on dynamic factors) as forecast. Various statistic performance measures include (1) bias statistic, (2) log-likelihood and (3) Q-statistic.

Dynamic attributions of forecasts by DFVCM algorithm quantifies volatility contributions from (1) common or (2) idiosyncratic (a) synchronous variances or (b) asynchronous autocovariances or (c) prediction/residual errors.

9. Further Development

Many real-world large sets of time-series are non-stationary. In general, a filtering approach could be among the best for analysis and forecasts on non-stationary time-series. Bayesian filters (BFs) are more adaptive filters: more powerful due to fewer restrictive assumptions and/or conditions. A variational Bayesian filtering (VBF) is the fastest one among BFs.

Out team, i4cast LLC, is an advanced developer of variational Bayesian filtering, demonstrated by our VBfFA (Variational Bayesian filtering Factor Analysis, https://aws.amazon.com/marketplace/pp/prodview-vdwcbntcsnu72?sr=0) algorithm publicly available on AWS.

We are now working on developing a long memory dynamic factor model (LMDFM) estimated by a variational Bayesian filter, and a Yule-Walker-PCA autoregressive model (YWpcAR) estimated by a VBF as well.

A.1. LMDFM Algorithm

The LMDFM algorithm (https://aws.amazon.com/marketplace/pp/prodviewda6ffrp4mlopg?sr=0) is an implementation of DFM estimation by spectral principal components analysis (spectral PCA) with conjugate two-dimensional discrete Fourier transform (C2D-DFT), formulated and detailed in Ling (2025). This algorithm can be summarized as follows:

- Estimating variance-covariance matrixes (VCMs) and autocovariance matrixes (ACMs) of observed vector (i.e. multiple) time-series y_t, as $C(j,k) = Cov(y_{t-j}, y_{t-k}) = E(y_{t-j}\, y_{t-k}^T)$, $j,k = 0,1,\cdots,p$, assuming $E(y_t) = 0$.
- Combining VCMs, $C(j,j)$ when $k = j$, and ACMs, $C(j,k)$ when $k \neq j$, by applying conjugate two-dimensional discrete Fourier transform (C2D-DFT) on $C(j,k)$, $j,k = 0,1,\cdots,p$.
- Referring resulted transform by C2D-DFT as spectral density matrixes (SDMs), $S(m,n) = DFT_{C2D}(\{C(j,k)\})$, $m,n = 0,1,\cdots,p$, of vector time-series, y_t.
- Estimating spectral principal components of "on-diagonal SDMs", $S(m,m)$, $m = n$, and SPCA representations of "off-diagonal SDMs", $S(m,n)$, $m \neq n$.
- Estimating full-spectrum principal components (FSPCs) of original VCMs by applying inverse C2D-DFT on SPCA representations of SDMs.
- This way, FSPCs of original VCMs contain dynamic information in all VCMs and ACMs.
- Dynamic factor loadings and vector autoregressive (VAR) coefficients of factor scores can be estimated by full-spectrum principal components of VCMs.

The LMDFM estimates two different simplest forms of DFM. DFM of Form I estimates:

$$y_t = \mu_t + X_{t,0}\, f_t + X_{t,1}\, f_{t-1} + \cdots + X_{t,p}\, f_{t-p} + u_t\,,$$

$$f_t = g(e_t)\,,$$

where e_t is a vector of white noises of unit variance, $g(\cdot)$ is a linear transformation, and elements of vector f_t are independent of each other and over time. Other application facts about Form I include:

- classic form of DFM for classic dynamic analysis,
- suitable for simple Monte Carlo simulation,
- estimating a whole set of matrixes of factor loadings $X_{t,k}$,
- not applicable to "big data set" due to large number of elements of loadings,
- question on time-series forecast: sizable impact by assuming unknown factor scores $f_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of f_t .

LMDFM's DFM of Form II estimates Eqs. (2.1) and (2.2) as:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t \,,$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t \,,$$

where the only loadings matrix $X_{t,0}$ in Form II is NOT the same as that in Form I (just share the same symbol). Other application facts about Form II include:

- only one matrix of factor loadings, $X_{t,0}$,
- estimating a whole set of VAR matrixes, $A_{t,k}$, of common factors,
- non-zero vector autocovariance of comm factors, $Cov(\, f_{t-j} \,,\, f_{t-j-k}\,) \neq 0$, therefore, not for simple Monte Carlo simulation,
- applicable to "big data set" due to limited number of elements of VAR matrixes,
- question on time-series forecast: smaller impact by assuming unknown errors $v_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of errors v_t .

DFM of Form II, i.e. Eqs. (2.1) and (2.2), is utilized to estimate common components, including

$X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, V_{t-j} , $V_{(t-j),(t-k)}$ and u_t , shown in Section "2. Model Estimation".

A.2. YWpcAR Algorithm

The YWpcAR algorithm (https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0) is an implementation of estimating non-stationary autoregressive (AR) model by Yule-Walker dimension reduction method, formulated and detailed in Ling (2026, to be published in 2026). This algorithm can be summarized as follows:

- A set of AR coefficients of AR model on observed univariate time-series is a solution vector of linear Yule-Walker (YW) equation of the AR time-series.
- Design matrix and response vector of the YW equation are expressed in terms of variances and autocovariances of the time-series at a range of time-lags.
- The YW solution vector can be expressed with all eigenvalues and all eigenvectors of symmetric (sample) variance-autocovariance matrixes (VACMs) associated with the YW equation.
- Replacing the set of all eigenpairs with selected principal eigenvalues and principal eigenvectors of the VACMs, the YW solution vector becomes dimension-reduced set of AR coefficients.
- Combe principal eigen score time-series into an unobserved dynamic component (UDC) time-series of the observed time-series.
- The UDC time-series is of an AR process governed by the dimension-reduced AR coefficients.
- Forecast values of observed time-series by dimension-reduced AR forecasts of the UDC time-series.

- Forecast variances of observed time-series by dimension-reduced AR forecasts of variances of the UDC as well as estimated residual variance.

The YWpcAR can be utilized to estimate idiosyncratic components, including g_t, $D_{t,k}$, $R_t^{(e)}$, W_{t-j}, $W_{(t-j),(t-k)}$ and $R_t^{(r)}$, shown in Section "2. Model Estimation".

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