

# Introduction to Multi-step Forecast of Multivariate Volatility with Dynamic Factor Model

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Initial Version: December 2023, This Version: December 2025.

## **Abstract**

This paper introduces volatility forecasts made by Dynamic Factor Model (DFM). Advantages of DFM-based volatility model over other volatility models include capability to make multi-step forecasts of multivariate volatilities of large number of time-series. Volatility components forecasted by DFM-based model include those separately attributable to (1) common or (2) idiosyncratic (a) variances or (b) autocovariances or (c) prediction errors. Common volatility components are forecasted using DFM by spectral PCA and conjugate 2D DFT. Idiosyncratic volatility components are forecasted using AR model by Yule-Walker-PCA estimates.

**Keywords:** Volatility model, volatility forecasts, DFM-based volatility model, multi-step forecasts of multivariate volatilities, dynamic volatility attributions, forecasting volatility components, common or idiosyncratic components forecasts, variance or autocovariance components forecasts, volatility components prediction errors.

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## **1. Introduction**

Making volatility forecasts using Dynamic Factor Model (DFM) offers a number of attractive advantages over other volatility forecasting methods. Specifically, a DFM volatility model is capable

- to make multi-step forecasts of multivariate volatilities (i.e. uncertainties, risks) of a large number time-series (e.g. forecasting risks of numerous investable assets in various markets),
- to estimate source-specific components of forecasted volatilities which are attributable to
  - variances of dynamic common factors,
  - vector autocovariances of dynamic common factors,
  - variances of idiosyncratic UDCs (unobserved dynamic components),

- autocovariances (or serial-correlations) of idiosyncratic UDCs,
- common / idiosyncratic / residual prediction errors

(e.g. portions of volatilities caused by common economic or market conditions; volatility jumps in panic, and drops in euphoria, markets; risks due to specific dynamics of individual stocks; random error components),

- to make multi-step forecasts of multiple time-series themselves (i.e. forecasting conditional expectations instead of volatilities of the time-series) using the same dynamic factor model (e.g., economic or market forecasts made by government agencies or research institutions).

In short, a volatility model based on dynamic factor model (DFM) is capable to make all of the estimates and forecasts listed above simultaneously using the same integrated and dimension-reduced multivariate modeling and analysis process.

Multivariate GARCH models and static factor risk models are among popular classes of volatility models, with each class making some of the above estimates or forecasts. Meanwhile, it had been traditionally difficult to estimate large scale DFMs.

The advent of new machine learning algorithms able to solve for large DFMs (such as many long-memory dynamic factors on large number of time-series), DFM-based volatility models can now become powerful tools at your disposal to attack real-world volatility problems in real-time.

Academic literature on dynamic factor models (DFMs) is voluminous. Doz and Fuleky (2020) is among the latest best overviews on the history of DFM research and development. Alessi, Barigozzi and Capasso (2007) discusses benefits and performances of volatility forecasts by DFM vs. multivariate GARCH.

This paper discusses formulation details how to use dynamic factor model (DFM) to make multi-step forecasts of multivariate volatilities, and of various dynamic volatility components, of large number of time-series.

## **2. Model Estimation**

We denote observed multiple time-series as vector time-series  $y_t$ , which is time-series of  $n \times 1$  vector, representing  $n$  individual time-series.

To make multi-step forecasts of various components of multivariate volatilities of the vector time-series  $y_t$ , we first estimate all coefficients and attributes of a dynamic factor model representation of  $y_t$  expressed as follows:

$$y_t = \mu_t + X_{t,0} f_t + u_t, \quad (2.1)$$

$$f_t = A_{t,1} f_{t-1} + A_{t,2} f_{t-2} + \cdots + A_{t,p} f_{t-p} + v_t, \quad (2.2)$$

$$u_t = g_t + r_t, \quad (2.3)$$

$$g_t = D_{t,1} g_{t-1} + D_{t,2} g_{t-2} + \cdots + D_{t,q} g_{t-q} + e_t, \quad (2.4)$$

where (random and non-random) variables

- $y_t$  is time-series of  $n \times 1$  vector of observed data,
- $\mu_t$  is time-series of  $n \times 1$  vector of mean values of  $y_t$ ,
- $f_t$  is time-series of  $m \times 1$  vector of common dynamic factor scores,
- $X_{t,0}$  is  $n \times m$  matrix of factor loadings of dynamic factors  $f_t$ ,
- $A_{t,j}$  are  $m \times m$  matrixes of VAR (vector autoregressive) coefficients of factors  $f_t$ , here  $n > or \gg m$ ,
- $v_t$  is time-series of  $m \times 1$  vector of VAR prediction errors of factors  $f_t$ ,

- $u_t$  is time-series of  $n \times 1$  vector of idiosyncratic components of individual observed time-series,
- $g_t$  is time-series of  $n \times 1$  vector of unobserved dynamic components (UDCs) of idiosyncratic components  $u_t$ ,
- $D_{t,k}$  are  $n \times n$  diagonal matrixes of AR (autoregressive) coefficients of UDCs  $g_t$ ,
- $e_t$  is time-series of  $n \times 1$  vector of AR prediction errors of UDCs  $g_t$ , and
- $r_t$  is time-series of  $n \times 1$  vector of residual random errors.

For  $j = 0, 1, \dots$  and  $k = 1, 2, \dots$ , variances and autocovariances of common dynamic factors and idiosyncratic UDCs are estimated as:

$$V_{t-j} = Var(f_{t-j}) = E(f_{t-j} f_{t-j}^T), \quad (2.5)$$

$$V_{(t-j),(t-k)} = Cov(f_{t-j}, f_{t-k}) = E(f_{t-j} f_{t-k}^T), \quad (2.6)$$

$$W_{t-j} = Var(g_{t-j}) = E(g_{t-j} g_{t-j}^T), \quad (2.7)$$

$$W_{(t-j),(t-k)} = Cov(g_{t-j}, g_{t-k}) = E(g_{t-j} g_{t-k}^T), \quad (2.8)$$

where estimated matrixes

- $V_{t-j}$  are  $m \times m$  diagonal matrixes of variance of dynamic factors  $f_t$ ,
- $V_{(t-j),(t-k)}$  are  $m \times m$  matrixes of autocovariance of factors  $f_t$ ,
- $W_{t-j}$  are  $n \times n$  diagonal matrixes of variance of UDCs  $g_t$ , and
- $W_{(t-j),(t-k)}$  are  $n \times n$  diagonal matrixes of autocovariance of UDCs  $g_t$ .

Variances of random errors are estimated as:

$$R_t^{(v)} = Var(v_t), \quad (2.9)$$

$$R_t^{(e)} = Var(e_t), \quad (2.10)$$

$$R_t^{(r)} = \text{Var}(r_t), \quad (2.11)$$

where estimated matrixes

- $R_t^{(v)}$  is time-series of  $m \times m$  diagonal matrix of variances of VAR prediction errors  $v_t$ ,
- $R_t^{(e)}$  is time-series of  $n \times n$  diagonal matrix of variances of AR prediction errors  $e_t$ , and
- $R_t^{(r)}$  is time-series of  $n \times n$  diagonal matrix of variances of residual errors  $r_t$ .

The DFM estimations summarized above can be performed jointly by two models. The first one is common dynamic factor (CDF) model estimated by spectral principal components analysis (spectral PCA) with conjugate two-dimensional discrete Fourier transform (C2D-DFT), formulated and detailed in Ling (2025). An actual implementation of this method by Ling (2025) and summarized in appendix A.1, LMDFM algorithm

(<https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlogp?sr=0>) can be used to estimate common components, including  $X_{t,0}$ ,  $f_t$ ,  $A_{t,j}$ ,  $R_t^{(v)}$ ,  $V_{t-j}$ ,  $V_{(t-j),(t-k)}$  and  $u_t$ . The second one is autoregressive (AR) model estimated by applying principal components analysis (PCA) to Yule-Walker (YW) equation of AR process, to be published soon by i4cast LLC. An actual implementation of this method by i4cast and summarized in appendix A.2, YWpcAR algorithm (<https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0>) can be used to estimate idiosyncratic components, including  $g_t$ ,  $D_{t,k}$ ,  $R_t^{(e)}$ ,  $W_{t-j}$ ,  $W_{(t-j),(t-k)}$  and  $R_t^{(r)}$ .

In estimation of all of the above coefficients and attributes, we applied following assumptions widely proposed, accepted and practiced in dynamic factor model research literature:

- mean value (of observed vector time-series)  $\mu_t = 0$ ,
- covariance matrix  $V_{t-j}$  (of dynamic factor scores) is diagonal matrix,
- covariance matrix  $W_{t-j}$  (of idiosyncratic components) is approximated by diagonal matrix, ignoring allowed but “mild” cross-correlation,
- assuming all cross- and serial-correlations between common factors  $f_t$  and idiosyncratic UDCs  $g_t$  are zeros.

In order to make multi-step forecasts of multivariate volatility in the next section, we need past autocovariance matrix of dynamic factor scores  $f_t$ , and diagonal matrix of past autocovariance of UDC  $g_t$ , with two separate time lags,  $j$  and  $k$ . When  $0 \leq j \leq k$ , the past autocovariance matrix of  $f_t$ ,

$$E(f_{t-j} f_{t-k}^T) = V_{[t-\text{Min}(j,k)], \text{Abs}(j-k)}, \quad (2.12)$$

When  $j \geq k \geq 0$ , the past autocovariance of  $f_t$ ,

$$E(f_{t-j} f_{t-k}^T) = E(f_{t-k} f_{t-j}^T)^T = V_{[t-\text{Min}(j,k)], \text{Abs}(j-k)}^T. \quad (2.13)$$

The diagonal matrix of past autocovariance of  $g_t$ ,

$$E(g_{t-j} g_{t-k}^T) = W_{[t-\text{Min}(j,k)], \text{Abs}(j-k)}. \quad (2.14)$$

### **3. Volatility Forecasts**

According to the factor model representation discussed in the Section 2 above,  $s$ -step forecasts of time-series,  $f_t$ ,  $g_t$  and  $y_t$ , based on data observed until time  $t$  can be made by dynamic equations as

$$f_{(t+s)|t} = A_{t,1} f_{t+s-1} + A_{t,2} f_{t+s-2} + \cdots + A_{t,p} f_{t+s-p} + v_{t+s}, \quad (3.1)$$

$$g_{(t+s)|t} = D_{t,1} g_{t+s-1} + D_{t,2} g_{t+s-2} + \cdots + D_{t,q} g_{t+s-q} + e_{t+s}, \quad (3.2)$$

$$y_{(t+s)|t} = X_{t,0} f_{(t+s)|t} + g_{(t+s)|t} + r_{t+s}, \quad (3.3)$$

where  $s = 1, 2, \dots$ . The random errors  $v_{t+s}$ ,  $e_{t+s}$  and  $r_{t+s}$  cannot be forecasted, but can be characterized by assumed diagonal variance matrixes as

$$R_{t+s}^{(v)} = R_t^{(v)}, \quad (3.4)$$

$$R_{t+s}^{(e)} = R_t^{(e)}, \quad (3.5)$$

$$R_{t+s}^{(r)} = R_t^{(r)}. \quad (3.6)$$

Therefore,  $s$ -step forecast of diagonal variance matrix of dynamic factor scores  $f_t$  is

$$\begin{aligned} V_{(t+s)|t} &= \text{Var}(f_{(t+s)|t}) = E(f_{(t+s)|t} f_{(t+s)|t}^T) \\ &= \text{Diag}(E((\sum_{j=1}^p A_{t,j} f_{t+s-j})(\sum_{k=1}^p A_{t,k} f_{t+s-k})^T)) + R_t^{(v)} \\ &= \sum_{j=1}^p \sum_{k=1}^p \text{Diag}(A_{t,j} E(f_{t+s-j} f_{t+s-k}^T) A_{t,k}^T) + R_t^{(v)} \\ &= \sum_{j=1}^p \sum_{k=1}^p \text{Diag}(A_{t,j} V_{(t+s-j),(t+s-k)} A_{t,k}^T) + R_t^{(v)}, \end{aligned} \quad (3.7)$$

where  $s = 1, 2, \dots$ , and autocovariance of factors,  $E(f_{t+s-j} f_{t+s-k}^T)$ , are evaluated by estimates Eqs. (2.5) and (2.6), or by “earlier-step” forecasts Eqs. (3.7) and (3.8). Then,  $s$ -step forecast of  $k$ -lag autocovariance matrix of dynamic factor scores  $f_t$  is

$$\begin{aligned} V_{(t+s),(t+s-k)} &= \text{Cov}(f_{(t+s)|t}, f_{t+s-k}) \\ &= E(f_{(t+s)|t} f_{t+s-k}^T) = E(\sum_{j=1}^p A_{t,j} f_{t+s-j} f_{t+s-k}^T) \\ &= \sum_{j=1}^p A_{t,j} E(f_{t+s-j} f_{t+s-k}^T) = \sum_{j=1}^p A_{t,j} V_{(t+s-j),(t+s-k)}, \end{aligned} \quad (3.8)$$

where  $s = 1, 2, \dots$ ,  $k = 1, 2, \dots$ , and, again, autocovariance,  $E(f_{t+s-j} f_{t+s-k}^T)$ , are evaluated by estimates Eqs. (2.5) and (2.6), or by earlier forecasts Eqs. (3.7) and (3.8). The evaluation of Eqs. (3.7) and (3.8) can be simplified by the separate time lag autocovariance Eqs. (2.12) and (2.13).

Similarly,  $s$ -step forecast of variance of idiosyncratic unobserved dynamic components (UDCs)  $g_t$  is

$$W_{(t+s)|t} = \text{Var}(g_{(t+s)|t}) = E(g_{(t+s)|t} g_{(t+s)|t}^T)$$



$$\begin{aligned}
&= \text{Diag}(E((\sum_{j=1}^q D_{t,j} g_{t+s-j})(\sum_{k=1}^q D_{t,k} g_{t+s-k})^T)) + R_t^{(e)} \\
&= \sum_{j=1}^q \sum_{k=1}^q \text{Diag}(D_{t,j} E(g_{t+s-j} g_{t+s-k}^T) D_{t,k}) + R_t^{(e)} \\
&= \sum_{j=1}^q \sum_{k=1}^q D_{t,j} W_{(t+s-j),(t+s-k)} D_{t,k} + R_t^{(e)}, \tag{3.9}
\end{aligned}$$

where  $s = 1, 2, \dots$ , and autocovariance of UDCs,  $E(g_{t+s-j} g_{t+s-k}^T)$ , are evaluated by estimates Eqs. (2.7) and (2.8), or by “earlier-step” forecasts Eqs. (3.9) and (3.10). Then,  $s$ -step forecast of  $k$ -lag autocovariance of UDC  $g_t$  is

$$\begin{aligned}
W_{(t+s),(t+s-k)} &= \text{Cov}(g_{(t+s)|t}, g_{t+s-k}) \\
&= E(g_{(t+s)|t} g_{t+s-k}^T) = \text{Diag}(E(\sum_{j=1}^q D_{t,j} g_{t+s-j} g_{t+s-k}^T)) \\
&= \sum_{j=1}^q \text{Diag}(D_{t,j} E(g_{t+s-j} g_{t+s-k}^T)) \\
&= \sum_{j=1}^q D_{t,j} W_{(t+s-j),(t+s-k)}, \tag{3.10}
\end{aligned}$$

where  $s = 1, 2, \dots$ ,  $k = 1, 2, \dots$ , and, again, autocovariance,  $E(g_{t+s-j} g_{t+s-k}^T)$ , are evaluated by estimates Eqs. (2.7) and (2.8), or by earlier forecasts Eqs. (3.9) and (3.10). The evaluation of Eqs. (3.9) and (3.10) can be simplified by the separate time lag autocovariance Eq. (2.14).

Having

- estimated factor loadings matrix  $X_{t,0}$  in Eq. (2.1),
- forecasted diagonal variance matrix  $V_{(t+s)|t}$  of dynamic factor scores  $f_t$  by Eq. (3.7),
- forecasted diagonal variance matrix  $W_{(t+s)|t}$  of unobserved dynamic components (UDCs)  $g_t$  by Eq. (3.9), and
- “forecasted” diagonal variance matrix of residual errors  $r_t$  by Eq. (3.6),

$s$ -step forecast of variance-covariance matrix of the observed vector time-series  $y_t$  is

$$\begin{aligned}
C_{(t+s)|t} &= \text{Var}(y_{(t+s)|t}) = E(y_{(t+s)|t} y_{(t+s)|t}^T) \\
&= X_{t,0} V_{(t+s)|t} X_{t,0}^T + W_{(t+s)|t} + R_t^{(r)}
\end{aligned}$$

$$\begin{aligned}
&= X_{t,0} \sum_{j=1}^p \sum_{k=1}^p \text{Diag}(A_{t,j} V_{(t+s-j),(t+s-k)} A_{t,k}^T) X_{t,0}^T \\
&+ \sum_{j=1}^q \sum_{k=1}^q D_{t,j} W_{(t+s-j),(t+s-k)} D_{t,k} \\
&+ X_{t,0} R_t^{(v)} X_{t,0}^T + R_t^{(e)} + R_t^{(r)} .
\end{aligned} \tag{3.11}$$

Where forecast step  $s = 1, 2, \dots$ .

#### **4. Simple Forecasts as Benchmarks**

There are two classes of simple forecasts widely practiced for multivariate variance-covariance matrix of observed vector time-series  $y_t$ .

The simplest calculated forecast is to use a sample-based variance-covariance matrix as forecasted matrix:

$$C_t^{(Sample)} = K^{-1} \sum_{k=0}^K (y_{t-k} - \mu_t) (y_{t-k} - \mu_t)^T, \tag{4.1}$$

$$C_{(t+s)|t}^{(Sample)} = C_t^{(Sample)}, \tag{4.2}$$

where forecast step  $s = 1, 2, \dots$ .

A factor-based “forecast” is to use a variance-covariance matrix estimated by a static factor model as forecasted matrix:

$$C_t^{(Estimate)} = X_t V_t X_t^T + W_t + R_t, \tag{4.3}$$

$$C_{(t+s)|t}^{(Estimate)} = C_t^{(Estimate)}, \tag{4.4}$$

where  $s = 1, 2, \dots$ ; and factor loadings matrix  $X_t$ , diagonal factor variance matrix  $V_t$ , diagonal UDC variance matrix  $W_t$  and diagonal residual error variance matrix  $R_t$  are estimated coefficients and attributes of a static factor model expressed as follows:

$$y_t = \mu_t + X_t f_t + u_t,$$

$$u_t = g_t + r_t,$$

$$V_t = \text{Var}(f_t) = E(f_t f_t^T),$$

$$W_t = \text{Var}(g_t) = E(g_t g_t^T),$$

$$R_t = \text{Var}(r_t) = E(r_t r_t^T).$$

Another widely practiced class of static factor models is “fundamental risk factor analysis”, in which factor loadings matrix is pre-determined based on certain fundamental analysis theory or framework ahead of factor model estimation. Therefore, fundamental factor analysis is able to make meaningful explanations about multivariate volatility structure, but variance-covariance matrix of fundamental factor score time-series is not diagonal.

Since we have already estimated a dynamic factor model, we can replace estimates of a static factor model by nowcasts (not forecasts) of our dynamic factor model as follows:

$$C_t^{(Nowcast)} = X_{t,0} V_{t,0} X_{t,0}^T + W_{t,0} + R_t^{(r)}, \quad (4.5)$$

$$C_{(t+s)|t}^{(Nowcast)} = C_t^{(Nowcast)}, \quad (4.6)$$

where  $s = 1, 2, \dots$ .

The above simple forecasts, Eq. (4.2), (4.4) or (4.6), can serve as benchmarks in evaluation of our multi-step forecast in multivariate volatility by dynamic factor model.

## **5. Evaluation of Volatility Forecast**

Directly examining quality or accuracy of the forecasted variance-covariance matrix  $C_{(t+s)|t}$  itself is a complicated undertaking in theory and difficult (for a large number of time-series) task in practice.

A widely practiced evaluation technique is to measure quality or accuracy of forecasted variance of a weighted aggregation of the time-series, with forecasted variance of the aggregate made by the forecasted variance-covariance matrix, as

$$(\sigma_{(t+s)|t}^{(w)})^2 = w^T C_{(t+s)|t} w, \quad (5.1)$$

where  $w$  is a  $n \times 1$  vector of weights for aggregation and  $(\sigma_{(t+s)|t}^{(w)})^2$  is forecasted variance of aggregated time-series

$$y_t^{(w)} = w^T y_t = y_t^T w. \quad (5.2)$$

The vector of weights,  $w$ , is set according to relevant application(s) of business or research. If one of the elements in the vector  $w$  is 1 and all others are zeros, the weighted aggregate time-series,  $y_t^{(w)}$ , is essentially a selected individual time-series.

To measure the accuracy of forecasted variance  $(\sigma_{(t+s)|t}^{(w)})^2$  by Eq. (5.1), a “realized z-score squared of the forecasts” defined by

$$(z_{(t+s)|t}^{(w)})^2 = (y_{t+s}^{(w)} - \mu_{t+s}^{(w)})^2 / (\sigma_{(t+s)|t}^{(w)})^2, \quad (5.3)$$

is handy, where observation  $y_{t+s}^{(w)}$  and estimate  $\mu_{t+s}^{(w)}$  are made at time  $t + s$ , while forecast  $(\sigma_{(t+s)|t}^{(w)})^2$  is made at earlier time  $t$ . According to Litterman and Winkelmann (1998), Patton (2011), Menchero, Morozov and Pasqua (2013) and Fan, Furger and Xiu (2015), the accuracy of  $s$ -step volatility forecasts given an evaluation dataset in time period  $(t + s) \in [t_1, t_2]$  can be measured by bias statistic  $BS_{s|[t_1, t_2]}^{(w)}$ , log-likelihood  $LL_{s|[t_1, t_2]}^{(w)}$ , and  $Q$ -statistic  $QS_{s|[t_1, t_2]}^{(w)}$  defined as

$$BS_{s|[t_1, t_2]}^{(w)} = \left[ \frac{1}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} (z_{(t+s)|t}^{(w)})^2 \right]^{1/2}, \quad (5.4)$$

$$LL_{s|[t_1, t_2]}^{(w)} = - \frac{1/2}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} \left[ \ln(2\pi) + (z_{(t+s)|t}^{(w)})^2 + \ln(\sigma_{(t+s)|t}^{(w)})^2 \right], \quad (5.5)$$

$$QS_{s|[t_1, t_2]}^{(w)} = \frac{1}{t_2 - t_1} \sum_{t=(t_1-s)}^{t_2-s} \left[ (z_{(t+s)|t}^{(w)})^2 - \ln(z_{(t+s)|t}^{(w)})^2 \right]. \quad (5.6)$$

A bias statistic  $BS_{s|[t_1, t_2]}^{(w)} > 1$  or  $BS_{s|[t_1, t_2]}^{(w)} < 1$  shows an under- or over-prediction of volatility.

A higher log-likelihood  $LL_{s|[t_1, t_2]}^{(w)}$  or a lower  $Q$ -statistic  $QS_{s|[t_1, t_2]}^{(w)}$  indicates more accurate

forecasts.

## **6. Dynamic Volatility Attribution**

One of the primary objectives of volatility modeling and analysis is “volatility attribution”: evaluating various (static or dynamic) sources of volatility that contribute to the estimated or forecasted volatility levels. A simplest classic source of volatility is ideal random walks – assuming zero values of multivariate and individual autocovariances or, in another word, serial-correlations. They are, however, almost never zeros in the real world. Multivariate volatility values in general are sum of contributions from two sources: synchronous variances-covariances and asynchronous (vector and individual) autocovariances. Positive serial-correlations increase volatility levels, while negative ones decrease them. Non-zero autocovariances also make term-structure of volatility different from that of random walks, which is “volatility levels proportional to square root of lengths of time period”.

Volatility analysis based on DFMs (dynamic factor models) jointly estimate both synchronous variances-covariances and asynchronous vector autocovariances of a large number of time-series. Traditional static factor model volatility analyses, however, do not estimate autocovariances at all. Therefore, dynamic volatility attributions provide more and deeper insights than static volatility attributions.

Details in comprehensive discussions on volatility forecasts demonstrate that equations for immediate one-step volatility forecasts present all volatility sources quantified by the latest data observations and model estimates. According to Eq. (3.11), one-step forecast of variance-covariance matrix of the observed vector time-series  $y_t$  is

$$C_{(t+1)|t} = X_{t,0} V_{(t+1)|t} X_{t,0}^T + W_{(t+1)|t} + R_t^{(r)}. \quad (6.1)$$

Here, by Eq. (3.7), one-step forecast of variances of dynamic factor scores  $f_t$  is

$$\begin{aligned}
V_{(t+1)|t} &= \sum_{j=1}^p \sum_{k=1}^p \text{Diag}(A_{t,j} V_{(t+1-j),(t+1-k)} A_{t,k}^T) + R_t^{(v)} \\
&= \sum_{j=1}^p \sum_{k=1}^p \delta_{ij} \text{Diag}(A_{t,j} V_{(t+1-j),(t+1-k)} A_{t,k}^T) \\
&\quad + \sum_{j=1}^p \sum_{k=1}^p (1 - \delta_{ij}) \text{Diag}(A_{t,j} V_{(t+1-j),(t+1-k)} A_{t,k}^T) \\
&\quad + R_t^{(v)}, \tag{6.2}
\end{aligned}$$

where autocovariance  $V_{(t+1-j),(t+1-k)} = E(f_{t+1-j} f_{t+1-k}^T)$  is evaluated by estimates Eqs. (2.5) and (2.6), and  $\delta_{ij}$  is Kronecker delta. The first term of  $k = j$ , with multiplier  $\delta_{ij}$ , is aggregate contribution from estimated variances of dynamic factors  $f_t$ ; the second term of  $k \neq j$ , with multiplier  $(1 - \delta_{ij})$ , is from estimated vector autocovariances of  $f_t$ ; and the third term is from prediction error of  $f_t$ . Similarly, by Eq. (3.9), one-step forecast of variance of idiosyncratic UDCs  $g_t$  is

$$\begin{aligned}
W_{(t+1)|t} &= \sum_{j=1}^q \sum_{k=1}^q D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} + R_t^{(e)} \\
&= \sum_{j=1}^q \sum_{k=1}^q \delta_{ij} D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} \\
&\quad + \sum_{j=1}^q \sum_{k=1}^q (1 - \delta_{ij}) D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} \\
&\quad + R_t^{(e)}, \tag{6.3}
\end{aligned}$$

where  $W_{(t+1-j),(t+1-k)} = \text{Diag}(E(g_{t+1-j} g_{t+1-k}^T))$  is evaluated by estimates Eqs. (2.7) and (2.8), and  $\delta_{ij}$  is Kronecker delta. The first term  $k = j$ , with multiplier  $\delta_{ij}$ , is aggregate contribution from estimated variances of UDCs  $g_t$ ; the second term of  $k \neq j$ , with multiplier  $(1 - \delta_{ij})$ , is from autocovariances of  $g_t$ ; and the third term is from prediction errors of  $g_t$ .

Therefore, the one-step forecast of variance-covariance matrix of the observed vector time-series  $y_t$  is

$$\begin{aligned}
C_{(t+1)|t} &= X_{t,0} \sum_{j=1}^p \sum_{k=1}^p \delta_{ij} \text{Diag}(A_{t,j} V_{(t+1-j),(t+1-k)} A_{t,k}^T) X_{t,0}^T \\
&\quad + X_{t,0} \sum_{j=1}^q \sum_{k=1}^q (1 - \delta_{ij}) D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} X_{t,0}^T \\
&\quad + R_t^{(e)}
\end{aligned}$$

$$\begin{aligned}
& + X_{t,0} \sum_{j=1}^p \sum_{k=1}^p (1 - \delta_{ij}) \text{Diag}(A_{t,j} V_{(t+1-j),(t+1-k)} A_{t,k}^T) X_{t,0}^T \\
& + \sum_{j=1}^q \sum_{k=1}^q \delta_{ij} D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} \\
& + \sum_{j=1}^q \sum_{k=1}^q (1 - \delta_{ij}) D_{t,j} W_{(t+1-j),(t+1-k)} D_{t,k} \\
& + X_{t,0} R_t^{(v)} X_{t,0}^T + R_t^{(e)} + R_t^{(r)} .
\end{aligned} \tag{6.4}$$

Here, each of the five terms represents an aggregate dynamic source of volatility, measured by contribution to variance-covariance forecast from the source:

- the first term, can be labeled as “common volatility”, is from variances of dynamic common factors;
- the second term, as “common serial-correlation”, is from vector autocovariances of dynamic common factors;
- the third term, as “idiosyncratic volatility”, is from variance of idiosyncratic UDCs;
- the fourth term, as “idiosyncratic serial-correlation”, is from autocovariance of idiosyncratic UDCs; and
- the fifth term, as “prediction error”, is from (a) dynamic factor prediction error, (b) UDC prediction error, and (c) residual error.

“Dynamic volatility attribution” is to tabulate portions, or weights, of total variance-covariance matrix, or total variances, attributed to these dynamic sources of volatility.

As a comparison, traditional static volatility attribution to static sources of volatility based on static factor model of volatility, is unable to provide dynamic information. In a static volatility model, nowcast (estimated current values) of volatility serves as forecast. Volatility is attributed to only two sources:

- “common”, contributions from static common factors; and

- “idiosyncratic”, from idiosyncratic randomness (or residual error).

## **7. Examples**

“Dynamic Factor Variance-Covariance Model” algorithm (DFVCM, <https://aws.amazon.com/marketplace/pp/prodview-yvaulquatt3v2?sr=0>) publicly available on AWS is an implementation of the DFM-based volatility forecasting method presented in details in this paper.

To test DFVCM model, an example of input data table is tabulated with publicly available real data sets ([https://github.com/i4cast/aws/blob/main/dynamic\\_factor\\_variance-covariance\\_model/input/Weekly\\_VTS\\_6Yr.csv](https://github.com/i4cast/aws/blob/main/dynamic_factor_variance-covariance_model/input/Weekly_VTS_6Yr.csv)). The table contains multiple weekly time-series for several years ending as of the last Friday of the most recent calendar year. Each row is of an individual time-series. There are more than 50 weekly time-series of logarithmic changes in values of investment funds publicly traded in the U.S. exchanges. The funds invested in equities, fixed incomes, financial indexes, and physical commodities.

DFVCM model estimated and forecasted variances-covariances, vector autocovariances and individual volatility levels of these input time-series.

## **8. Discussion**

The numeric example described above demonstrates that the DFM-based “Dynamic Factor Variance-Covariance Model (DFVCM)” can generate statistically good and consistent multi-step forecasts of multivariate volatilities of large number of time-series. With several statistic measures, the performance of multi-step DFVCM forecasts is consistently better than various objective benchmark performances by (a) sample volatility as forecast, (b) estimate



(based on static factors) as forecast and (c) nowcast (or estimate, based on dynamic factors) as forecast. Various statistic performance measures include (1) bias statistic, (2) log-likelihood and (3)  $Q$ -statistic.

Dynamic attributions of forecasts by DFVCM algorithm quantifies volatility contributions from (1) common or (2) idiosyncratic (a) synchronous variances or (b) asynchronous autocovariances or (c) prediction/residual errors.

## **9. Further Development**

Many real-world large sets of time-series are non-stationary. In general, a filtering approach could be among the best for analysis and forecasts on non-stationary time-series. Bayesian filters (BFs) are more adaptive filters: more powerful due to fewer restrictive assumptions and/or conditions. A variational Bayesian filtering (VBF) is the fastest one among BFs.

Our team, i4cast LLC, is an advanced developer of variational Bayesian filtering, demonstrated by our VBfFA (Variational Bayesian filtering Factor Analysis, <https://aws.amazon.com/marketplace/pp/prodview-vdwcbntcsnu72?sr=0>) algorithm publicly available on AWS.

We are now working on developing a long memory dynamic factor model (LMDFM) estimated by a variational Bayesian filter, and a Yule-Walker-PCA autoregressive model (YWpcAR) estimated by a VBF as well.

### **A.1. LMDFM Algorithm**

The LMDFM algorithm (<https://aws.amazon.com/marketplace/pp/prodview->

[da6ffrp4mlogp?sr=0](#)) is an implementation of DFM estimation by spectral principal components analysis (spectral PCA) with conjugate two-dimensional discrete Fourier transform (C2D-DFT), formulated and detailed in Ling (2025). This algorithm can be summarized as follows:

- Estimating variance-covariance matrixes (VCMs) and autocovariance matrixes (ACMs) of observed vector (i.e. multiple) time-series  $y_t$ , as  $C(j, k) = Cov(y_{t-j}, y_{t-k}) = E(y_{t-j} y_{t-k}^T)$ ,  $j, k = 0, 1, \dots, p$ , assuming  $E(y_t) = 0$ .
- Combining VCMs,  $C(j, j)$  when  $k = j$ , and ACMs,  $C(j, k)$  when  $k \neq j$ , by applying conjugate two-dimensional discrete Fourier transform (C2D-DFT) on  $C(j, k)$ ,  $j, k = 0, 1, \dots, p$ .
- Referring resulted transform by C2D-DFT as spectral density matrixes (SDMs),  $S(m, n) = DFT_{C2D}(\{C(j, k)\})$ ,  $m, n = 0, 1, \dots, p$ , of vector time-series,  $y_t$ .
- Estimating spectral principal components of “on-diagonal SDMs”,  $S(m, m)$ ,  $m = n$ , and SPCA representations of “off-diagonal SDMs”,  $S(m, n)$ ,  $m \neq n$ .
- Estimating full-spectrum principal components (FSPCs) of original VCMs by applying inverse C2D-DFT on SPCA representations of SDMs.
- This way, FSPCs of original VCMs contain dynamic information in all VCMs and ACMs.
- Dynamic factor loadings and vector autoregressive (VAR) coefficients of factor scores can be estimated by full-spectrum principal components of VCMs.

The LMDFM estimates two different simplest forms of DFM. DFM of Form I estimates:

$$y_t = \mu_t + X_{t,0} f_t + X_{t,1} f_{t-1} + \dots + X_{t,p} f_{t-p} + u_t,$$

$$f_t = g(e_t),$$

where  $e_t$  is a vector of white noises of unit variance,  $g(\cdot)$  is a linear transformation, and

elements of vector  $f_t$  are independent of each other and over time. Other application facts about Form I include:

- classic form of DFM for classic dynamic analysis,
- suitable for simple Monte Carlo simulation,
- estimating a whole set of matrixes of factor loadings  $X_{t,k}$ ,
- not applicable to “big data set” due to large number of elements of loadings,
- question on time-series forecast: sizable impact by assuming unknown factor scores  $f_{t+s} = 0$  ?
- for variance-covariance forecast: assuming stable variance of  $f_t$ .

LMDFM’s DFM of Form II estimates Eqs. (2.1) and (2.2) as:

$$y_t = \mu_t + X_{t,0} f_t + u_t,$$

$$f_t = A_{t,1} f_{t-1} + A_{t,2} f_{t-2} + \cdots + A_{t,p} f_{t-p} + v_t,$$

where the only loadings matrix  $X_{t,0}$  in Form II is NOT the same as that in Form I (just share the same symbol). Other application facts about Form II include:

- only one matrix of factor loadings,  $X_{t,0}$ ,
- estimating a whole set of VAR matrixes,  $A_{t,k}$ , of common factors,
- non-zero vector autocovariance of comm factors,  $Cov(f_{t-j}, f_{t-j-k}) \neq 0$ ,  
therefore, not for simple Monte Carlo simulation,
- applicable to “big data set” due to limited number of elements of VAR matrixes,
- question on time-series forecast: smaller impact by assuming unknown errors  $v_{t+s} = 0$  ?
- for variance-covariance forecast: assuming stable variance of errors  $v_t$ .

DFM of Form II, i.e. Eqs. (2.1) and (2.2), is utilized to estimate common components, including

$X_{t,0}$ ,  $f_t$ ,  $A_{t,j}$ ,  $R_t^{(v)}$ ,  $V_{t-j}$ ,  $V_{(t-j),(t-k)}$  and  $u_t$ , shown in Section “2. Model Estimation”.

## **A.2. YWpcAR Algorithm**

The YWpcAR algorithm (<https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0>) is an implementation of autoregressive (AR) model estimation by applying principal components analysis (PCA) to Yule-Walker (YW) equation of AR process, to be published soon by i4cast LLC. This algorithm can be as summarized as follows:

- A set of autoregression (AR) coefficients is a solution to Yule-Walker (YW) equation of AR model.
- Applying principal components analysis (PCA) on symmetric (sample) variance-autocovariance matrixes (VACMs) associated with YW equation.
- Replacing VACMs by their principal components (PC) representations.
- Replacing design matrix and response vector of YW equation by their PC representations formed with elements of PC representations of VACMs.
- Estimating minimum norm solution to PC-represented YW equation to serve as PC representation of set of AR coefficients.
- Combining principal eigen score time-series into an unobserved dynamic component (UDC) time-series of observed data.
- Forecasting values of observed time-series by AR forecasts of UDC time-series.
- Forecasting variances of observed time-series by AR forecasted variances of UDC and estimated residual variance.

The YWpcAR can be utilized to estimate idiosyncratic components, including  $g_t$ ,  $D_{t,k}$ ,  $R_t^{(e)}$ ,  $W_{t-j}$ ,  $W_{(t-j),(t-k)}$  and  $R_t^{(r)}$ , shown in Section “2. Model Estimation”.

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