Introduction to Multi-step Forecast of Multivariate Volatility with Dynamic Factor Model

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Table of Content

1. Introduction

Making volatility forecasts using Dynamic Factor Models (DFMs) comes up with several sought-after advantages over other volatility modeling methods. Especially, a DFM volatility model

- can make multi-step forecasts of multivariate volatilities (i.e. forecasted uncertainty levels) of a large number time-series (e.g. risk levels of numerous investable assets in many markets),
- can quantify components of forecasted volatilities attributable to variances, and vector autocovariances, of dynamic common factors (e.g. portions of volatilities caused by common economic or market conditions; and volatility jumps in panic, and drops in euphoria, markets),
- can quantify volatility components attributable to idiosyncratic variance, and serial correlations, of individual time-series (e.g. risk components due to specific trajectories of individual equity shares),
- can make multi-step forecasts of multivariate values (i.e. forecasted conditional expectations themselves, instead of uncertainties) of the time-series, with the same dynamic factor model (e.g., economic or market forecasts made by government agencies or research institutions).

In short, a volatility model based on DFM (dynamic factor model) can offer all the estimates and forecasts listed above simultaneously under the same integrated and dimension-reduced multivariate analysis framework.

Multivariate GARCH models and static factor risk models are among popular classes of volatility models, with each class making some of the above estimates or forecasts. Meanwhile, it has been traditionally difficult to estimate large scale DFMs.

The advent of new machine learning algorithms able to solve for large scale DFMs (such as many long-memory dynamic factors on large number of time-series), DFM-based volatility models can now become a powerful tool at your disposal to attack real-world problems in real-time.

Academic literature on dynamic factor models (DFMs) is voluminous. Doz and Fuleky (2020) is among the latest best overviews on the history of DFM research and development. Alessi, Barigozzi and Capasso (2007) discusses benefits and performances of volatility forecasts by DFM vs. multivariate GARCH.

Following is a brief introduction to DFM-based "Dynamic Factor Variance-Covariance Model" (DFVCM, https://aws.amazon.com/marketplace/pp/prodview-yvaulquatt3v2?sr=0-3&ref_=beagle&applicationId=AWSMPContessa) for making multi-step forecasts in multivariate volatilities.

2. Model Estimation

We denote observed multivariate time-series as vector time-series y_t, which is time-series of $n \times 1$ vector, representing n individual time-series.

To make multi-step forecasts of multivariate volatilities of the vector time-series y_t, we first estimate all coefficients and attributes of a dynamic factor model representation of y_t expressed as follows:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t\,, \tag{2.1}$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t\,, \tag{2.2}$$

$$u_t = g_t + r_t\,, \tag{2.3}$$

$$g_t = D_{t,1}\, g_{t-1} + D_{t,2}\, g_{t-2} + \cdots + D_{t,q}\, g_{t-q} + e_t\,, \tag{2.4}$$

where (random and non-random) variables

- y_t is time-series of $n \times 1$ vector of observed data,

- μ_t is time-series of $n \times 1$ vector of mean values of y_t ,
- f_t is time-series of $m \times 1$ vector of common dynamic factor scores,
- $X_{t,0}$ is $n \times m$ matrix of factor loadings of dynamic factors f_t ,
- $A_{t,j}$ are $m \times m$ matrixes of VAR (vector autoregressive) coefficients of factors f_t , here $n > \ or \ \gg m$,
- v_t is time-series of $m \times 1$ vector of VAR prediction errors of factors f_t ,
- u_t is time-series of $n \times 1$ vector of idiosyncratic components of individual observed time-series,
- g_t is time-series of $n \times 1$ vector of unobserved dynamic components (UDCs) of idiosyncratic components u_t ,
- $D_{t,k}$ are $n \times n$ diagonal matrixes of AR (autoregressive) coefficients of UDCs g_t ,
- e_t is time-series of $n \times 1$ vector of AR prediction errors of UDCs g_t , and
- r_t is time-series of $n \times 1$ vector of residual random errors.

For $j = 0,1,2, \ldots$ and $k = 1,2, \ldots$, variances and autocovariances of common dynamic factors and idiosyncratic UDCs are estimated as:

$$V_{(t-j),0} = Var(\, f_{t-j} \,) = E(\, f_{t-j}\, f_{t-j}^T \,) \,, \tag{2.5}$$

$$V_{(t-j),k} = Cov(\, f_{t-j} \,,\, f_{t-j-k} \,) = E(\, f_{t-j}\, f_{t-j-k}^T \,) \,, \tag{2.6}$$

$$W_{(t-j),0} = Var(\, g_{t-j} \,) = E(\, g_{t-j}\, g_{t-j}^T \,) \,, \tag{2.7}$$

$$W_{(t-j),k} = Cov(\, g_{t-j} \,,\, g_{t-j-k} \,) = E(\, g_{t-j}\, g_{t-j-k}^T \,) \,, \tag{2.8}$$

where estimated matrixes

- $V_{(t-j),0}$ are $m \times m$ diagonal matrixes of variance of dynamic factors f_t ,

- $V_{(t-j),k}$ are $m \times m$ matrixes of k-lag autocovariance of factors f_t ,
- $W_{(t-j),0}$ are $n \times n$ diagonal matrixes of variance of UDCs g_t , and
- $W_{(t-j),k}$ are $n \times n$ diagonal matrixes of k-lag autocovariance of UDCs g_t .

Variances of random errors are estimated as:

$$R_t^{(v)} = Var(\, v_t \,) \,, \tag{2.9}$$

$$R_t^{(e)} = Var(\, e_t \,) \,, \tag{2.10}$$

$$R_t^{(r)} = Var(\, r_t \,) \,, \tag{2.11}$$

where estimated matrixes

- $R_t^{(v)}$ is time-series of $m \times m$ diagonal matrix of variances of VAR prediction errors v_t ,
- $R_t^{(e)}$ is time-series of $n \times n$ diagonal matrix of variances of AR prediction errors e_t , and
- $R_t^{(r)}$ is time-series of $n \times n$ diagonal matrix of variances of residual errors r_t .

The DFM estimations summarized above can be performed jointly by two models: (1) LMDFM (long-memory dynamic factor model, https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg?sr=0-1&ref_=beagle&applicationId=AWSMPContessa) estimated by an implementation of dynamic principal components analysis (DPCA) with 2-dimensional discrete Fourier transforms (2D-DFTs) (detailed in Appendix A) for estimates of common components, including $X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, $V_{(t-j),0}$, $V_{(t-j),k}$ and u_t ; and (2) YWpcAR (Yule-Walker-PCA autoregression, https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0-4&ref_=beagle&applicationId=AWSMPContessa) estimated by an implementation of principal components analysis (PCA) on Yule-Walker (YW) equation (detailed in Appendix B) for estimates of idiosyncratic components, including g_t , $D_{t,k}$, $R_t^{(e)}$, $W_{(t-j),0}$, $W_{(t-j),k}$ and $R_t^{(r)}$.

In estimation of all of the above coefficients and attributes, we applied following assumptions widely proposed, accepted and practiced in dynamic factor model research literature:

- mean value (of observed vector time-series) $\mu_t = 0$,
- covariance matrix (of dynamic factor scores) $V_{(t-j),0}$ is diagonal matrix,
- covariance matrix (of idiosyncratic components) $W_{(t-j),0}$ is approximated by diagonal matrix, ignoring allowed but "mild" cross-correlation,
- assuming all cross- and serial-correlations between common factors f_t and idiosyncratic UDCs g_t are $0's$.

In order to make multi-step forecast of multivariate volatility in the next section, we need past autocovariance matrix of dynamic factor scores f_t , and diagonal matrix of past autocovariance of UDC g_t , with two separate time lags, j and k. When $0 \leq j \leq k$, the past autocovariance matrix of f_t ,

$$E(\, f_{t-j}\, f_{t-k}^T\,) = V_{[t-Min(j,k)],Abs(j-k)}\,, \tag{2.12}$$

When $j \geq k \geq 0$, the past autocovariance of f_t ,

$$E(\, f_{t-j}\, f_{t-k}^T\,) = E(\, f_{t-k}\, f_{t-j}^T\,)^T = V_{[t-Min(j,k)],Abs(j-k)}^T\,. \tag{2.13}$$

The diagonal matrix of past autocovariance of g_t ,

$$E(\, g_{t-j}\, g_{t-k}^T\,) = W_{[t-Min(j,k)],Abs(j-k)}\,. \tag{2.14}$$

<u>3. Volatility Forecasts</u>

According to the factor model representation discussed in the Section 2 above, s-step forecasts of time-series, f_t, g_t and y_t, based on data observed until time t can be made by dynamic equations as

$$f_{(t+s)|t} = A_{t,1}\, f_{t+s-1} + A_{t,2}\, f_{t+s-2} + \cdots + A_{t,p}\, f_{t+s-p} + v_{t+s} \tag{3.1}$$

$$g_{(t+s)|t} = D_{t,1}\, g_{t+s-1} + D_{t,2}\, g_{t+s-2} + \cdots + D_{t,q}\, g_{t+s-q} + e_{t+s} \tag{3.2}$$

$$y_{(t+s)|t} = X_{t,0}\, f_{(t+s)|t} + g_{(t+s)|t} + r_{t+s} \tag{3.3}$$

where $s = 1,2, \dots$. The random errors v_{t+s} , e_{t+s} and r_{t+s} cannot be forecasted, but can be characterized by assumed diagonal variance matrixes as

$$R_{t+s}^{(v)} = R_t^{(v)} \tag{3.4}$$

$$R_{t+s}^{(e)} = R_t^{(e)} \tag{3.5}$$

$$R_{t+s}^{(r)} = R_t^{(r)} \tag{3.6}$$

Therefore, s-step forecast of diagonal variance-covariance matrix of dynamic factor scores f_t is

$$\begin{aligned} &V_{[(t+s)|t],0} = Var(\, f_{(t+s)|t}\,) = E(\, f_{(t+s)|t}\, f_{(t+s)|t}^T\,) \\ &= Diag(\, E(\,(\,\textstyle\sum_{j=1}^p A_{t,j}\, f_{t+s-j}\,)(\,\sum_{k=1}^p A_{t,k}\, f_{t+s-k}\,)^T\,)\,) + R_t^{(v)} \\ &= \textstyle\sum_{j=1}^p \sum_{k=1}^p Diag(\, A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,)\, A_{t,k}^T\,) + R_t^{(v)} \end{aligned} \tag{3.7}$$

where $s = 1,2, \dots,$ and variance and autocovariance of factors, $E(\, f_{t+s-j}\, f_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by "earlier-step" forecasts Eqs. (3.7) and (3.8). Then, s-Step forecast of k-lag autocovariance matrix of dynamic factor scores f_t is

$$\begin{aligned} &V_{[(t+s)|t],k} = Cov(\, f_{(t+s)|t}\,,\, f_{t+s-k}\,) = E(\, f_{(t+s)|t}\, f_{t+s-k}^T\,) \\ &= E(\,\textstyle\sum_{j=1}^p A_{t,j}\, f_{t+s-j}\; f_{t+s-k}^T\,) = \sum_{j=1}^p A_{t,j}\, E(\, f_{t+s-j}\, f_{t+s-k}^T\,) \end{aligned} \tag{3.8}$$

where $s = 1,2, \dots,$ $k = 1,2, \dots,$ and, again, variance and autocovariance, $E(\, f_{t+s-j}\, f_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (2.5) and (2.6), or by earlier forecasts Eqs. (3.7) and (3.8). The evaluation of Eqs. (3.7) and (3.8) can be simplified by the separate time lag autocovariance Eqs. (2.12) and (2.13).

Similarly, s-Step forecast of variance of unobserved dynamic components (UDCs) g_t is

$$\begin{aligned} & W_{[(t+s)|t],0} = Var(\, g_{(t+s)|t}\,) = E(\, g_{(t+s)|t}\, g_{(t+s)|t}^T\,) \\ & = Diag(\, E(\, (\sum_{j=1}^{q} D_{t,j}\, g_{t+s-j}\,)(\sum_{k=1}^{q} D_{t,k}\, g_{t+s-k}\,)^T\,)\,) + R_t^{(e)} \\ & = \sum_{j=1}^{q} \sum_{k=1}^{q} Diag(\, D_{t,j}\, E(\, g_{t+s-j}\, g_{t+s-k}^T\,)\, D_{t,k}\,) + R_t^{(e)} \end{aligned} \tag{3.9}$$

where $s = 1,2, \dots$, and variance and autocovariance of UDCs, $E(\, g_{t+s-j}\, g_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by "earlier-step" forecasts Eqs. (3.9) and (3.10). Then, s-step forecast of k-lag autocovariance of UDC g_t is

$$\begin{aligned} & W_{[(t+s)|t],k} = Cov(\, g_{(t+s)|t}\,,\, g_{t+s-k}\,) = E(\, g_{(t+s)|t}\, g_{t+s-k}^T\,) \\ & = Diag(\, E(\, \sum_{j=1}^{q} D_{t,j}\, g_{t+s-j}\;\, g_{t+s-k}^T\,)\,) \\ & = \sum_{j=1}^{q} Diag(\, D_{t,j}\, E(\, g_{t+s-j}\, g_{t+s-k}^T\,)\,) \end{aligned} \tag{3.10}$$

where $s = 1,2, \dots$, $k = 1,2, \dots$, and, again, variance and autocovariance, $E(\, g_{t+s-j}\, g_{t+s-k}^T\,)$, are evaluated by estimates Eqs. (2.7) and (2.8), or by earlier forecasts Eqs. (3.9) and (3.10). The evaluation of Eqs. (3.9) and (3.10) can be simplified by the separate time lag autocovariance Eq. (2.14).

Having

- estimated factor loadings matrix $X_{t,0}$ in Eq. (2.1),
- forecasted diagonal variance matrix $V_{[(t+s)|t],0}$ of dynamic factor scores f_t by Eq. (3.7),
- forecasted diagonal variance matrix $W_{[(t+s)|t],0}$ of unobserved dynamic components (UDCs) g_t by Eq. (3.9), and
- "forecasted" diagonal variance matrix of residual errors r_t by Eq. (3.6),

s-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$C_{(t+s)|t} = Var(\, y_{(t+s)|t} \,) = E(\, y_{(t+s)|t} \, y_{(t+s)|t}^T \,)$$

$$= X_{t,0} \, V_{[(t+s)|t],0} \, X_{t,0}^T + W_{[(t+s)|t],0} + R_t^{(r)} \qquad (3.11)$$

Where forecast step $s = 1,2, \dots$.

4. Simple Forecasts as Benchmarks

There are two classes of simple forecasts widely practiced for multivariate variance-covariance matrix of observed vector time-series y_t .

The simplest calculated forecast is to use a sample-based variance-covariance matrix as forecasted matrix:

$$C_t^{(Sample)} = K^{-1} \sum_{k=0}^{K} (\, y_{t-k} - \mu_t \,) \, (\, y_{t-k} - \mu_t \,)^T \qquad (4.1)$$

$$C_{(t+s)|t}^{(Sample)} = C_t^{(Sample)} \qquad (4.2)$$

where forecast step $s = 1,2, \dots$.

A factor-based "forecast" is to use a variance-covariance matrix estimated by a static factor model as forecasted matrix:

$$C_t^{(Estimate)} = X_t \, V_t \, X_t^T + W_t + R_t \qquad (4.3)$$

$$C_{(t+s)|t}^{(Estimate)} = C_t^{(Estimate)} \qquad (4.4)$$

where $s = 1,2, \dots$; and factor loadings matrix X_t , diagonal factor variance matrix V_t , diagonal UDC variance matrix W_t and diagonal residual error variance matrix R_t are estimated coefficients and attributes of a static factor model expressed as follows:

$$y_t = \mu_t + X_t \, f_t + u_t$$

$$u_t = g_t + r_t$$

$$V_t = Var(\, f_t \,) = E(\, f_t \, f_t^T \,)$$

$$W_t = Var(\, g_t \,) = E(\, g_t \, g_t^T \,)$$

$$R_t = Var(\, r_t \,) = E(\, r_t \, r_t^T \,)$$

Another widely practiced class of static factor models is “fundamental risk factor analysis”, in which factor loadings matrix is pre-determined based on certain fundamental analysis theory or framework ahead of factor model estimation. Therefore, fundamental factor analysis is able to make meaningful explanations about multivariate volatility structure, but variance-covariance matrix of fundamental factor score time-series is not diagonal.

Since we have already estimated a dynamic factor model, we can replace estimates of a static factor model by nowcasts (not forecasts) of our dynamic factor model as follows:

$$C_t^{(Nowcast)} = X_{t,0} \, V_{t,0} \, X_{t,0}^T + W_{t,0} + R_t^{(r)} \tag{4.5}$$

$$C_{(t+s)|t}^{(Nowcast)} = C_t^{(Nowcast)} \tag{4.6}$$

where $s = 1,2, \dots$.

The above simple forecasts, Eq. (4.2), (4.4) or (4.6), can serve as benchmarks in evaluation of our multi-step forecast in multivariate volatility by dynamic factor model.

5. Evaluation of Volatility Forecast

Directly examining quality or accuracy of the forecasted variance-covariance matrix $C_{(t+s)|t}$ itself is a complicated undertaking in theory and difficult (for a large number of time-series) task in practice.

A widely practiced evaluation technique is to measure quality or accuracy of forecasted variance of a weighted aggregation of the time-series, with forecasted variance of the aggregate made by the forecasted variance-covariance matrix, as

$$(\sigma^{(w)}_{(t+s)|t})^2 = w^T\, C_{(t+s)|t}\, w. \quad (5.1)$$

where w is a $n \times 1$ vector of weights for aggregation and $(\sigma^{(w)}_{(t+s)|t})^2$ is forecasted variance of aggregated time-series

$$y^{(w)}_t = w^T\, y_t = y^T_t\, w, \quad (5.2)$$

The vector of weights, w, is set according to relevant application(s) of business or research. If one of the elements in the vector w is 1 and all others are $0s$, the weighted aggregate, $y^{(w)}_t$, is essentially a selected individual time-series.

To measure the accuracy of forecasted variance $(\sigma^{(w)}_{(t+s)|t})^2$ by Eq. (5.1), a "realized z-score squared of the forecasts" defined by

$$(z^{(w)}_{(t+s)|t})^2 = (\, y^{(w)}_{t+s} - \mu^{(w)}_{t+s}\,)^2 \,/\, (\sigma^{(w)}_{(t+s)|t})^2, \quad (5.3)$$

is handy, where observation $y^{(w)}_{t+s}$ and estimate $\mu^{(w)}_{t+s}$ are made at time $t+s$, while forecast $(\sigma^{(w)}_{(t+s)|t})^2$ is made at earlier time t. According to Litterman and Winkelmann (1998), Patton (2011), Menchero, Morozov and Pasqua (2013) and Fan, Furger and Xiu (2015), the accuracy of portfolio volatility forecasts over a given time period $(t+1) \in [t_1, t_2]$ can be measured by bias statistic $BS^{(w)}_{t_1,t_2}$, log-likelihood $LL^{(w)}_{t_1,t_2}$, and Q-statistic $QS^{(w)}_{t_1,t_2}$ defined as

$$BS^{(w)}_{t_1,t_2} = [\, \frac{1}{t_2 - t_1} \sum_{t=t_1-1}^{t_2-1} (z^{(w)}_{t+1|t})^2\,]^{1/2}, \quad (5.4)$$

$$LL^{(w)}_{t_1,t_2} = -\, \frac{1/2}{t_2 - t_1 + 1} \sum_{t=t_1-1}^{t_2-1} [\, \ln(2\pi) + (z^{(w)}_{t+1|t})^2 + \ln(\sigma^{(w)}_{t+1|t})^2\,], \quad (5.5)$$

$$QS^{(w)}_{t_1,t_2} = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1-1}^{t_2-1} [\, (z^{(w)}_{t+1|t})^2 - \ln(z^{(w)}_{t+1|t})^2\,], \quad (5.6)$$

where $sd^{t_2}_{t=t_1}(\cdot)$ denotes sample standard deviation and $w^{(QP)}_{t+1|t}$ is time-varying stock weights of the minimum variance portfolio obtained by quadratic programing (QP) using the predicted covariance matrix $C_{t+1|t}$. A bias statistic $BS_{t_1,t_2}^{(w)} > 1$ or $BS_{t_1,t_2}^{(w)} < 1$ shows an under- or over-prediction of volatility. A higher log-likelihood $LL_{t_1,t_2}^{(w)}$ or a lower Q-statistic $QS_{t_1,t_2}^{(w)}$ indicates more accurate forecasts.

6. Dynamic Volatility Attribution

One of the primary objectives of volatility modeling and analysis is "volatility attribution": evaluating various (static or dynamic) sources of volatility that contribute to estimated or forecasted volatility values. A simplest classic source of volatility is an ideal random walk – assuming zero values of multivariate/individual serial-correlations. They are, however, almost non-zeros in real world. Multivariate volatility values in general are sum of contributions from two sources: variance-covariance and vector/individual autocovariances. Positive serial-correlations increase volatility levels, while negative ones decrease them. Non-zero autocovariances also make term-structure of volatility different from that of random walks which is volatility levels proportional to square root of time horizon.

Volatility analysis based on dynamic factor models jointly estimate both variance-covariance and vector autocovariances of a large number of time-series. Traditional static factor model volatility analyses, however, do not estimate autocovariances at all. Therefore, dynamic volatility attributions provide more and deeper insights than static volatility attributions.

Details in comprehensive discussions on volatility forecasts demonstrate that equations for immediate (as opposed to "forward") one-step volatility forecasts present all volatility sources quantified by the latest data values and model estimates. According to Eq. (3.11), 1-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$C_{(t+1)|t} = X_{t,0}\, V_{(t+1)|t}\, X_{t,0}^T + W_{(t+1)|t} + R_t^{(r)}. \tag{6.1}$$

Here, by Eq. (3.7), 1-step forecast of variances of dynamic factor scores f_t is

$$
\begin{aligned}
V_{(t+1)|t} &= \sum_{j=1}^{p} \sum_{k=1}^{p} Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T \,) + R_t^{(v)} \\
&= \sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T \,) \\
&+ \sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T \,) \\
&+ R_t^{(v)} \qquad (6.2)
\end{aligned}
$$

where variance/autocovariance $V_{(t+1-j),(t+1-k)} = E(\, f_{t+1-j}\, f_{t+1-k}^T \,)$ is evaluated by estimates Eqs. (2.5) and (2.6), and δ_{ij} is Kronecker delta. The first term of $k = j$, with multiplier δ_{ij} , represents aggregate contribution from estimated variances of dynamic factors f_t ; the second term of $k \neq j$, with $(1-\delta_{ij})$, from estimated vector autocovariances of f_t ; and the third term, from prediction error of f_t . Similarly, by Eq. (3.9), 1-step forecast of variance of idiosyncratic UDCs g_t is

$$
\begin{aligned}
W_{(t+1)|t} &= \sum_{j=1}^{q} \sum_{k=1}^{q} D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} + R_t^{(e)} \\
&= \sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ \sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ R_t^{(e)} \qquad (6.3)
\end{aligned}
$$

where $W_{(t+1-j),(t+1-k)} = Diag(\, E(\, g_{t+1-j}\, g_{t+1-k}^T \,)\,)$ is evaluated by estimates Eqs. (2.7) and (2.8), and δ_{ij} is Kronecker delta. The first term $k = j$, with multiplier δ_{ij} , represents aggregate contribution from estimated variances of UDCs g_t ; the second term of $k \neq j$, with $(1-\delta_{ij})$, from autocovariances of g_t ; and the third term, from prediction error of g_t .

Therefore, the 1-step forecast of variance-covariance matrix of the observed vector time-series y_t is

$$
\begin{aligned}
&C_{(t+1)|t} \\
&= X_{t,0} \sum_{j=1}^{p} \sum_{k=1}^{p} \delta_{ij}\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^T \,)\, X_{t,0}^T
\end{aligned}
$$

$$
\begin{aligned}
&+ X_{t,0} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} (1-\delta_{ij})\, Diag(\, A_{t,j}\, V_{(t+1-j),(t+1-k)}\, A_{t,k}^{T}\,)\, X_{t,0}^{T} \\
&+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} \delta_{ij}\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ \textstyle\sum_{j=1}^{q} \sum_{k=1}^{q} (1-\delta_{ij})\, D_{t,j}\, W_{(t+1-j),(t+1-k)}\, D_{t,k} \\
&+ X_{t,0}\, R_t^{(v)}\, X_{t,0}^{T} + R_t^{(e)} + R_t^{(r)}\,. \qquad (6.4)
\end{aligned}
$$

Here, each of the five terms represents an aggregate dynamic source of volatility, measured by contribution to variance-covariance forecast from the source:

- the first term, can be labeled as “common volatility”, is from variances of dynamic common factors;
- the second term, as “common serial-correlation”, is from vector autocovariances of dynamic common factors;
- the third term, as “idiosyncratic volatility”, is from variance of idiosyncratic UDCs;
- the fourth term, as “idiosyncratic serial-correlation”, is from autocovariance of idiosyncratic UDCs; and
- the fifth term, as “prediction error”, is from (a) dynamic factor prediction error, (b) UDC prediction error, and (c) residual error.

Dynamic volatility attribution is to tabulate portions, or weights, of total variance-covariance matrix, or total variances, attributed to these dynamic sources of volatility.

As a comparison, traditional static volatility attribution to static sources of volatility based on static factor model of volatility, is unable to provide dynamic information. In a static volatility model, nowcast (estimated current values) of volatility serves as forecast. Volatility is attributed to only two sources:

- “common”, contributions from static common factors; and

- "idiosyncratic", from idiosyncratic randomness (or residual error).

7. Examples

To test DFVCM model, an example of input data table is tabulated with publicly available real data sets. The table contains multiple weekly time-series for several years ending as of the last Friday of the most recent calendar year. Each row is of an individual time-series. There are more than 50 weekly time-series of logarithmic changes in values of investment funds publicly traded in the U.S. exchanges. The funds invested in equities, fixed incomes, financial indexes, and physical commodities.

DFVCM model estimated and forecasted variance-covariances, vector autocovariances and individual volatility levels of these input time-series.

8. Discussion

The numeric example described above demonstrates that the DFM-based "Dynamic Factor Variance-Covariance Model (DFVCM)" can generate statistically good and consistent multi-step forecasts of multivariate volatilities for large number of time-series. With several statistic measures, the performance of multi-step DFVCM forecasts is consistently better than various objective benchmarks: performances of (a) sample volatility as forecast, (b) estimate, based on static factors, as forecast, and (c) nowcast (or estimate), based on dynamic factors, as forecast. Various statistic measures include (a) bias statistic, (b) log-likelihood, and (c) Q-statistic.

Dynamic attributions of forecasts by DFVCM can reveal useful information in terms of common vs. idiosyncratic factors, and cross-sectional vs. serial correlations.

9. Further Development

Many real-world large sets of time-series are nonstationary. In general, a filtering approach could be among the best for analysis and forecasts on nonstationary time-series. Bayesian filters (BFs) are more adaptive filters: more powerful due to fewer restrictive assumptions and/or conditions. A variational Bayesian filtering (VBF) is the fastest one among BFs.

Out team, i4cast LLC, is an advanced developer of variational Bayesian filtering, demonstrated by our VBfFA (Variational Bayesian filtering Factor Analysis, https://aws.amazon.com/marketplace/pp/prodview-vdwcbntcsnu72?sr=0-2&ref_=beagle&applicationId=AWSMPContessa) algorithm published on AWS.

We are now working on developing a long memory dynamic factor model (LMDFM) estimated by a variational Bayesian filter, and a Yule-Walker-PCA autoregressive model (YWpcAR) estimated by a VBF as well.

Appendix A. LMDFM Algorithm

Long-memory dynamic factor model (LMDFM, https://aws.amazon.com/marketplace/pp/prodview-da6ffrp4mlopg?sr=0-1&ref_=beagle&applicationId=AWSMPContessa) is estimated by an implementation of spectral (or dynamic) principal components analysis (SPCA or DPCA), reviewed by Doz and Fuleky (2020). Here the SPCA can be described with a two-dimensional discrete Fourier transform (2D-DFT) summarized as follows:

- Estimating variance-covariance matrixes (VCMs) and autocovariance matrixes

(ACMs) of observed vector (i.e. multiple) time-series y_t , $C_{j,k} = Cov(y_{t-j}, y_{t-k}) = E(y_{t-j}\, y_{t-k}^T)$, $j, k = 0,1, \dots, p$, assuming $E(y_t) = 0$.

- Combining VCMs, $C_{j,j}$ when $k = j$, and ACMs, $C_{j,k}$ when $k \neq j$, by applying two-dimensional discrete Fourier transform (2D-DFT) on $C_{j,k}$, $j, k = 0,1, \dots, p$.
- Referring resulted transform (by 2D-DFT) as spectral density matrixes (SDMs), $\{\Omega_{q,r}\} = DFT_{2D}(\{C_{j,k}\})$, $q, r = 0,1, \dots, p$, of the vector time-series, y_t .
- Applying principal components analysis (PCA) on diagonal spectrum, $\Omega_{q,q}$, $r = q$, and off-diagonal spectrum, $\Omega_{q,r}$, $r \neq q$, of the set of SDMs.
- Estimating principal components (PCs) of original VCMs by applying inverse 2D-DFT on spectral principal components (i.e. PCs of SDMs).
- If observed vector time-series can be assumed as locally stationary, the 2D-DFT becomes simplified as weighted (1-D) DFT, with exactly the same "weights of the Bartlett window" shown by Doz and Fuleky (2020).

The LMDFM estimates two different simplest forms of DFM. DFM of Form I estimates:

$$y_t = \mu_t + X_{t,0}\, f_t + X_{t,1}\, f_{t-1} + \cdots + X_{t,p}\, f_{t-p} + u_t ,$$

$$f_t = g(e_t) ,$$

where e_t is a vector of white noises of unit variance, $g(\cdot)$ is a linear transformation, and elements of vector f_t are independent of each other and over time. Other application facts about Form I include:

- classic form of DFM for classic dynamic analysis,
- suitable for simple Monte Carlo simulation,
- estimating a whole set of matrixes of factor loadings $X_{t,k}$,

- not applicable to "big data set" due to large number of elements of loadings,
- question on time-series forecast: sizable impact by assuming unknown factor scores $f_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of f_t .

LMDFM's DFM of Form II estimates Eqs. (2.1) and (2.2) as:

$$y_t = \mu_t + X_{t,0}\, f_t + u_t \,,$$

$$f_t = A_{t,1}\, f_{t-1} + A_{t,2}\, f_{t-2} + \cdots + A_{t,p}\, f_{t-p} + v_t \,,$$

where the only loadings matrix $X_{t,0}$ in Form II is NOT the same as that in Form I (just share the same symbol). Other application facts about Form II include:

- only one matrix of factor loadings, $X_{t,0}$,
- estimating a whole set of VAR matrixes, $A_{t,k}$, of common factors,
- non-zero vector autocovariance of comm factors, $Cov(\, f_{t-j} \,,\, f_{t-j-k}\,) \neq 0$, therefore, not for simple Monte Carlo simulation,
- applicable to "big data set" due to limited number of elements of VAR matrixes,
- question on time-series forecast: smaller impact by assuming unknown errors $v_{t+s} = 0$?
- for variance-covariance forecast: assuming stable variance of errors v_t .

DFM of Form II, i.e. Eqs. (2.1) and (2.2), is utilized to estimate common components, including $X_{t,0}$, f_t , $A_{t,j}$, $R_t^{(v)}$, $V_{(t-j),0}$, $V_{(t-j),k}$ and u_t , shown in Section "2. Model Estimation".

<u>Appendix B. YWpcAR Algorithm</u>

Yule-Walker-PCA autoregression model (YWpcAR, https://aws.amazon.com/marketplace/pp/prodview-prndys7tr7go6?sr=0-

4&ref_=beagle&applicationId=AWSMPContessa) is estimated by an implementation of principal components analysis (PCA) on Yule-Walker (YW) equation summarized as follows:

- Applying principal components analysis (PCA) on sample variance-autocovariance matrix (VACM) in Yule-Walker (YW) equation.
- Replacing elements of sample VACM by PCA-based common components.
- Constructing PCA-based YW equation with replacing elements of matrix and vector in YW equation by correspondent PCA-based common components of VACM.
- Estimating AR model coefficients by PCA-based YW equation.
- Combining individual principal component score time-series into an unobserved dynamic component (UDC) time-series.
- Forecasting expected value and variance of observed time-series with UDC time-series of YW-PCA AR model.

The YWpcAR can be utilized to estimate idiosyncratic components, including g_t, $D_{t,k}$, $R_t^{(e)}$, $W_{(t-j),0}$, $W_{(t-j),k}$ and $R_t^{(r)}$, shown in Section "2. Model Estimation".

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