

Chapter 4

BASIC RELATIONS IN TIME AVERAGING

The importance of the Eulerian time averaging in studying a single-phase turbulent flow is well known. Since the most useful information in analyzing standard fluid flow systems is the time mean values rather than the local instant responses of the fluid, its use both in experimental and analytical purposes is indispensable in turbulent flow studies. For example mean velocity, temperature and pressure or the heat transfer coefficient and the friction factor are the important mean values routinely required in standard problems. Furthermore, commonly used experimental methods and measurements are well suited for the application of the time average. Thus, a single-phase turbulent flow has been studied in great depth by using the time averaged field equations with the constitutive laws expressed by mean values. Although these models, which are based on time averaging, do not give answers to the fundamental origin, structure and transport mechanisms of turbulent flow, their applications to engineering systems are widely accepted as efficient means of solving problems.

In discussing the importance of the Eulerian time averaging applied to two-phase mixtures, we first recall that in two-phase flow the local instant fluctuations of variables are caused not only by turbulence but also by rapidly moving and deforming interfaces. Because of these complicated flow and fluctuations, the solutions from the local instant formulation are inaccessible, therefore in order to derive appropriate field and constitutive equations it is necessary to apply some averaging procedure to the original local instant formulation. In view of the above discussion on the importance and usefulness of the time average in a single-phase turbulent flow analysis, it is both natural and logical that we also apply the time averaging to two-phase flow.

It is expected that the averaged field equations distinctly exhibit macroscopic phenomena of the system from hopelessly complicated interfacial and turbulent fluctuations, since they enter the formulation only

statistically. There are two notable consequences from the time averaging when it is applied to a two-phase mixture:

1. Smoothing out of turbulent fluctuations in same sense as in a single-phase flow;
2. Bringing two phases, which are alternately occupying a volume element, into continua simultaneously existing at same point with a properly defined probability for each phase.

Furthermore, it should be recognized that the constitutive laws appearing in the averaged field equations should be expressed through the time mean values. These constitutive laws can be developed from a simple modeling of two-phase transport phenomena together with various experimental data that are commonly expressed by the mean values.

In the following chapters, we develop a detailed theory of the thermo-fluid dynamics of two-phase flow using the time averaging. First, we assume that the occupant of any particular point is alternating randomly between the two phases and that the time-averaged functions are sufficiently smooth in the new coordinates. Namely, the time coordinate having a minimum scale of Δt based on the time interval of averaging below which a time differential operator has no physical meaning.

1.1 Time domain and definition of functions

First, we recall that the singular characteristic of two-phase or of two immiscible mixture is the presence of one or several interfaces between the phases or components. Furthermore, whereas single-phase flows can be classified according to the geometry of the flow in laminar, transitional and turbulent flow, the flow of two phases or of a mixture of immiscible liquids can be classified according to the geometry of the interface into three classes: separated flow; transitional or mixed flow; and dispersed flow. These classes of structured flow are shown in Table 1-1.

In any flow regime, various properties suffer discontinuous changes at phase interfaces, if these interfaces are considered as singular surfaces with their thickness being zero and the properties having jump discontinuities. This can be illustrated more dramatically by taking a fluid density ρ , as shown in Figs.4-1 and 4-2. Since in two-phase flow systems the mass of each phase is clearly separated by the interfaces and do not mix at the molecular level, the local instant fluid density shows stepwise discontinuities between ρ_1 and ρ_2 . Figure 4-1 shows the instantaneous discontinuities of ρ in space, whereas Fig.4-2 exhibits the discontinuities in time at some fixed point x_0 .

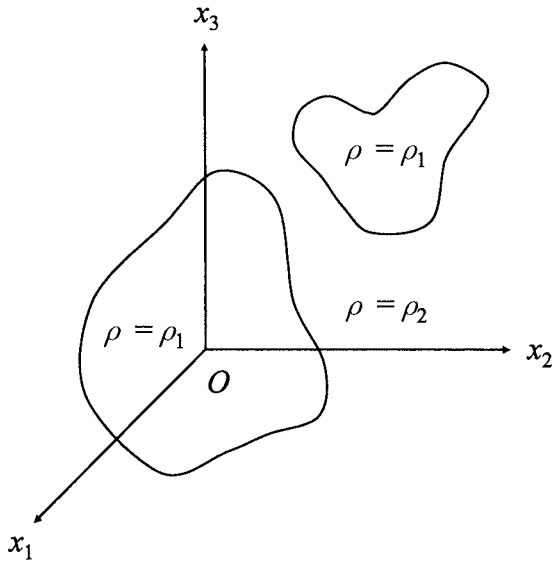


Figure 4-1. Fluid density in space at $t=t_0$ (Ishii, 1975)

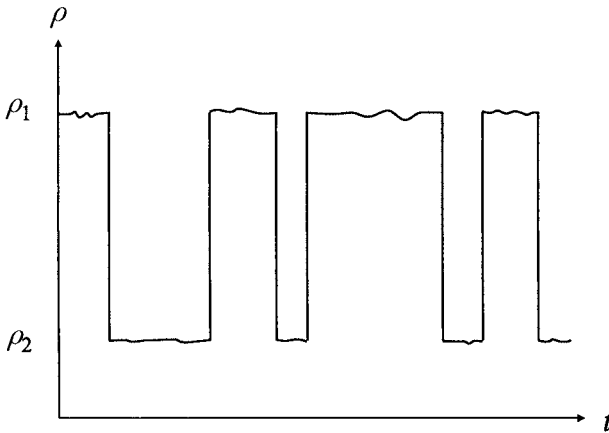


Figure 4-2. Fluid density in time at $x=x_0$ (Ishii, 1975)

For the purpose of time averaging, the observation from the time coordinate gives a more accurate picture of the problem. It can easily be seen that four distinct processes in terms of ρ may occur at any fixed point, which we can classify as follows:

1. $\rho = \rho_2(t)$ for all t ; always phase 2 at \mathbf{x}_0
2. $\rho = \rho_1(t)$ for all t ; always phase 1 at \mathbf{x}_0
3. ρ alternates between ρ_1 and ρ_2 ; phase 1 and 2 alternate at \mathbf{x}_0
4. ρ is neither ρ_1 and ρ_2 ; interface at \mathbf{x}_0 for some finite time

It is evident that, following a change of ρ from ρ_1 to ρ_2 or vice versa, all properties may change drastically because the phase occupying the point will be different. For the case of (1) and (2), since the time averaging at that point is trivial, we eliminate such cases. Furthermore, as case (4) is a rather singular configuration of case (3), it will be considered separately later. Hence, we examine the case in which the phase alternates stepwisely between 1 and 2.

Our purpose here is to average the fluid properties and field equations in order to treat two-phase flow as a mixture of continua. First, we take a fixed time interval Δt of the averaging and assume that it is large enough to smooth out the local variations of properties yet small compared to the macroscopic time constant of the unsteadiness of the bulk flow. This assumption is identical to that made in analyzing turbulent single-phase flow. After choosing any particular reference point and time (\mathbf{x}_0, t_0) , we have definite times, $t_1, t_2, \dots, t_j, \dots$ referring to the interfaces which pass the point \mathbf{x}_0 from time $(t_0 - \Delta t/2)$ to $(t_0 + \Delta t/2)$. By using the arbitrarily small interfacial thickness δ of the Section 1.2 of Chapter 2, the time intervals associated with each interface can be defined as

$$2\varepsilon = \frac{\delta}{v_{ni}} = \frac{\delta_1 + \delta_2}{v_{ni}} \quad (4-1)$$

which can vary among interfaces, thus we use ε_j for the j^{th} - interface. Since we are going to treat the interfaces as a shell whose position is represented by a mathematical surface, we may take ε_j as a corresponding time interval for both δ_1 and δ_2 . Then the assumption that an interface is a singular surface, or the interface thickness $\delta \rightarrow 0$, corresponds to

$$\lim_{\delta \rightarrow 0} \varepsilon_j = 0 \text{ for all } j \text{ if } |\mathbf{v}_{ni}| \neq 0. \quad (4-2)$$

In subsequent analyses, we frequently use this relation in order to derive macroscopic field equations. Now we define the set of time intervals, in which the characteristic of the interface dominates, as

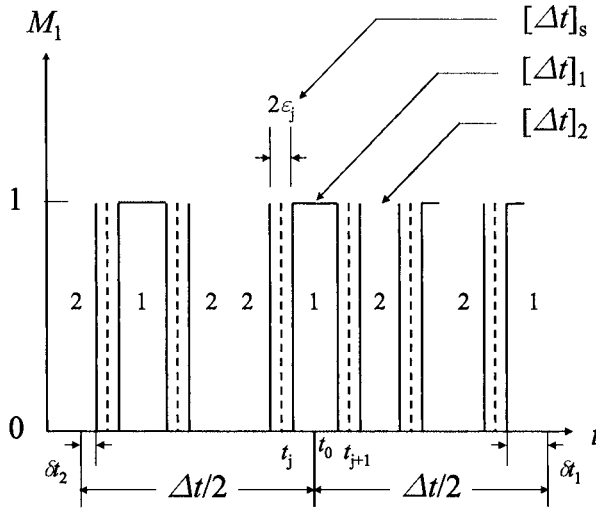


Figure 4-3. Various time intervals (Ishii, 1975)

$$[\Delta t]_s; \quad t \in [t_j - \epsilon_j; t_j + \epsilon_j] \quad \text{for } j = 1, \dots, n \quad (4-3)$$

The remaining part of the time interval is given by $[\Delta t]_T$, which can be separated into intervals of phase 1 and 2. Thus

$$[\Delta t]_T = [\Delta t]_1 + [\Delta t]_2. \quad (4-4)$$

By introducing

$$[\Delta t]; \quad t \in \left[t_0 - \frac{\Delta t}{2}; t_0 + \frac{\Delta t}{2} \right] \quad (4-5)$$

we have

$$[\Delta t] = [\Delta t]_s + [\Delta t]_T = [\Delta t]_s + \sum_{k=1}^2 [\Delta t]_k. \quad (4-6)$$

These relations are shown in Fig.4-3.

Since in the course of analyses it becomes necessary to distinguish three states, namely, phase 1, phase 2 or interface, we assign state density functions M_1 , M_2 , and M_s which are defined as:

$$M_k(\mathbf{x}, t) = 1, \quad M_s(\mathbf{x}, t) = 0 \quad (k = 1 \text{ or } 2);$$

A point occupied by k^{th} -phase

$$M_s(\mathbf{x}, t) = 1, \quad M_k(\mathbf{x}, t) = 0 \quad (k = 1 \text{ and } 2);$$

A point occupied by interface

(4-7)

A general function F associated with two phases is considered to be continuously differentiable everywhere except in the interfacial regions of thickness δ . Then a general function of the k^{th} -phase F_k at the point of averaging \mathbf{x}_0 is defined as

$$F_k(\mathbf{x}_0, t) = M_k(\mathbf{x}_0, t) F \quad \begin{cases} = F(\mathbf{x}_0, t) & \text{if } t \in [\Delta t]_k \\ = 0 & \text{if } t \notin [\Delta t]_k \end{cases} \quad (4-8)$$

This function F_k represents variables of each phase in a local instant formulation given in Chapter 2.

1.2 Local time fraction - Local void fraction

The time interval occupied by each phase is defined by taking the limit $\delta \rightarrow 0$ as (see Fig.4-3)

$$\begin{aligned} \Delta t_1 &= \lim_{\delta \rightarrow 0} \left\{ \sum_j [(t_{j+1} - \varepsilon_{j+1}) - (t_j + \varepsilon_j)] + \delta t_1 \right\}; \quad j = 2m - 1 \\ \Delta t_2 &= \lim_{\delta \rightarrow 0} \left\{ \sum_j [(t_{j+1} - \varepsilon_{j+1}) - (t_j + \varepsilon_j)] + \delta t_2 \right\}; \quad j = 2m. \end{aligned} \quad (4-9)$$

Hence, from the assumption (4-2) we have

$$\Delta t = \Delta t_1 + \Delta t_2. \quad (4-10)$$

By recalling the previous assumption that the interfaces are not stationary and do not occupy a location \mathbf{x}_0 for finite time intervals, we can find the time averaged phase density function α_k as

$$\alpha_k(\mathbf{x}_0, t_0) \equiv \lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]} M_k(\mathbf{x}_0, t) dt. \quad (4-11)$$

Hence, in view of Eq.(4-9), we get

$$\alpha_k = \frac{\Delta t_k}{\Delta t} \text{ for } k = 1 \text{ and } 2. \quad (4-12)$$

And from Eqs.(4-10) and (4-12) we obtain the following relation

$$\alpha_1 + \alpha_2 = 1 \quad (4-13)$$

which is the consequence that the averaged interface density function α_s is zero. We note here that α_s is defined in parallel with Eqs.(4-7) and (4-11) as

$$\alpha_s \equiv \lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]} M_s(\mathbf{x}_0, t) dt. \quad (4-14)$$

The function α_k , which appears only after the integral operation, is a fundamental parameter in studying the time averaged field equations. Physically α_k represents a probability of finding the k^{th} -phase, thus it expresses the geometrical (static) importance of that phase. Hereafter we call α_k as a *local time fraction* or a *local void fraction* of the k^{th} -phase.

1.3 Time average and weighted mean values

In this section, we define the time average and weighted mean values of functions associated with two-phase flow fields.

Time Average

The Eulerian time average of the general function F is defined by

$$\bar{F}(\mathbf{x}_0, t_0) \equiv \lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_r} F(\mathbf{x}_0, t) dt. \quad (4-15)$$

Hereafter, the symbol \bar{F} denotes the mathematical operation defined in the right-hand side of Eq.(4-15). Similarly, the mean value of the k^{th} -phase general function F_k is given by

$$\bar{F}_k(\mathbf{x}_0, t_0) \equiv \lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_r} F_k(\mathbf{x}_0, t) dt. \quad (4-16)$$

Because of the special property associated with M_k it can be shown that

$$\alpha_k = \overline{M_k}. \quad (4-17)$$

In view of Eqs.(4-8), (4-15) and (4-16) we have

$$\overline{F} = \lim_{\delta \rightarrow 0} \left\{ \frac{1}{\Delta t} \int_{[\Delta t]_T} F_1(\mathbf{x}_0, t) dt + \frac{1}{\Delta t} \int_{[\Delta t]_T} F_2(\mathbf{x}_0, t) dt \right\}. \quad (4-18)$$

Hence we obtain an important relation

$$\overline{F} = \overline{F_1} + \overline{F_2}. \quad (4-19)$$

The functions of F_1 and F_2 are directly related to instant, local physical or flow variables of each phase; however, F_1 and F_2 are averaged over the total time interval Δt . Thus, they can be considered as superficially averaged values. From this point of view, we introduce various weighted mean values that preserve some of the important characteristics of original variables.

We start from a general case, then proceed to special cases. Hence by taking a non-zero scalar weight function w , we define the general weighted mean value of F as

$$\overline{F}^w \equiv \frac{\overline{wF}}{\overline{w}} \quad (4-20)$$

where the function w also belongs to the group of a general function F defined in the previous section. Then it follows that the weighting function for each phase can also be defined through Eq.(4-8).

Consequently a general phase weighted mean value in parallel with Eq.(4-20) should be

$$\overline{F_k}^{w_k} \equiv \frac{\overline{w_k F_k}}{\overline{w_k}}. \quad (4-21)$$

It follows that

$$\overline{F}^w = \frac{\sum_{k=1}^2 \overline{w_k F_k}}{\overline{w}} = \frac{\sum_{k=1}^2 \overline{w_k} \overline{F}^{w_k}}{\sum_{k=1}^2 \overline{w_k}} \quad (4-22)$$

which relates the mixture and the phase mean values. Since these formulas are too general, we discuss in below some of the important special cases.

Phase Average $\overline{\overline{F_k}}$

The most natural mean value associated with each phase can be defined by taking the phase density function M_k as a weighting function in Eq.(4-21), hence we have

$$\overline{\overline{F_k}} \equiv \frac{\overline{M_k F_k}}{\overline{M_k}} = \frac{\overline{F_k}}{\alpha_k} = \frac{1}{\Delta t_k} \int_{[\Delta t]_k} F_k dt \quad (4-23)$$

where we used Eqs.(4-7), (4-8) and (4-17). As it is evident from the definition, the phase average denoted by $\overline{\overline{F_k}}$ represents the simple average in the time interval $[\Delta t]_k$ of the phase. Hence, we have

$$\overline{F} = \sum_{k=1}^2 \overline{F_k} = \sum_{k=1}^2 \alpha_k \overline{\overline{F_k}}. \quad (4-24)$$

Mass Weighted Mean Value $\widehat{\psi}$ and $\widehat{\psi_k}$

In general, the volume, momentum, energy and entropy, etc. are considered to be extensive variables (Callen, 1960). If the function F is taken as a quantity per unit volume of the extensive characteristic, then they can also be expressed in terms of the variable per unit mass ψ as

$$F = \rho \psi \quad (4-25)$$

where ρ is the local instant fluid density. Hence, the properties for each phase F_k are given by

$$F_k = \rho_k \psi_k \quad k = 1 \text{ or } 2. \quad (4-26)$$

Here ρ_k and ψ_k denote k^{th} -phase local instant density and a quantity per unit mass, respectively. Then the appropriate mean values for ψ and ψ_k should be weighted by the densities as

$$\widehat{\psi} = \frac{\overline{\rho\psi}}{\bar{\rho}} \quad (4-27)$$

and

$$\widehat{\psi}_k = \frac{\overline{\rho_k\psi_k}}{\bar{\rho}_k} = \frac{\overline{\overline{\rho_k\psi_k}}}{\overline{\overline{\rho_k}}} \quad (4-28)$$

The most important reason of weighting ψ by the density ρ is that the quantities represented by ψ are an additive set function of mass.

From the definition of the mass weighted mean values, we have

$$\overline{\rho\psi} = \sum_{k=1}^2 \overline{\rho_k\psi_k} \quad (4-29)$$

Hence, we obtain the most important relation between the mass weighted mixture property and that of two phase as

$$\widehat{\psi} = \frac{\sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k\psi_k}}}{\sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k}}} = \frac{\sum_{k=1}^2 \overline{\overline{\rho_k\psi_k}}}{\sum_{k=1}^2 \overline{\overline{\rho_k}}} \quad (4-30)$$

We note here that the above result is analogous to the definitions used in the thermo-mechanical theory of diffusion (Truesdell, 1969) and in Maxwell's equation of transfer in the kinetic theory of gases (Maxwell, 1867). In particular, since the density is a property per unit volume, we have

$$\bar{\rho} = \sum_{k=1}^2 \bar{\rho}_k = \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k}} \quad (4-31)$$

Fundamental Hypothesis on Smoothness of Mean Values

Our purpose of the averaging is to transform two phases, alternately occupying a point with discontinuities at interfaces, into two simultaneous continua. Consequently, the assumption on the continuity of derivatives is required. Thus, let us introduce here a fundamental hypothesis on the smoothness of the mean values \overline{F} and \overline{F}_k .

By considering a macroscopic process in terms of the mean values, it is assumed that they are sufficiently smooth to have higher order derivatives as necessary except at some isolated singularities, if the time constant of the process is sufficiently larger than Δt . In other words any changes of mean values within the time interval Δt are considered to be infinitesimal. This can be visualized by considering the time differential operator in the average field as a finite difference operator with the time increment $\delta t \rightarrow \Delta t$ and not $\delta t \rightarrow 0$. If we apply averaging to a mean value we get

$$\overline{(\overline{F})} = \frac{1}{\Delta t} \lim_{\delta \rightarrow 0} \int_{[\Delta t]_r} \overline{F}(\mathbf{x}_0, t) dt. \quad (4-32)$$

However, since \overline{F} is continuous, we obtain from the integral mean value theorem

$$\overline{(\overline{F})} = \overline{F}(\mathbf{x}_0, \tau_0) \quad (4-33)$$

where

$$t_0 - \frac{\Delta t}{2} \leq \tau_0 \leq t_0 + \frac{\Delta t}{2}.$$

Thus, in analogy with the fundamental hypothesis, it is assumed that in the macroscopic fields we have

$$\overline{(\overline{F})} = \overline{F}(\mathbf{x}_0, t_0). \quad (4-34)$$

It states that the averaging does not alter the mean values. Then it is straightforward that we obtain

$$\begin{aligned} \overline{(\overline{F}^w)}^w &= \overline{F}^w; & \overline{(\overline{F}^w)} &= \overline{F}^w \\ \overline{(\overline{F}_k^{w_k})}^{w_k} &= \overline{F}_k^{w_k}; & \overline{\overline{(\overline{F}_k^{w_k})}} &= \overline{F}_k^{w_k}. \end{aligned} \quad (4-35)$$

And for a constant C we have

$$\overline{C} = \overline{\overline{C}} = \overline{C}_k^w = \overline{C}_k^{w_k} = \frac{\overline{C}_k}{\alpha_k} = C. \quad (4-36)$$

Fluctuating Component

As in the analyses of turbulent flows, it is a prime importance to introduce fluctuating components of variables in order to take into account these effects statistically. In general, they are defined as a difference between a local instant variable and its weighted mean value, thus we have

$$F'_k \equiv F_k - \overline{F}_k^{w_k}. \quad (4-37)$$

Since once a variable F_k is specified then the form of the weighted average will be given, the fluctuating component can be defined uniquely. From Eqs.(4-35) and (4-37) we immediately obtain

$$\overline{F'_k}^{w_k} = 0. \quad (4-38)$$

Furthermore, the mean value of the fluctuating component can be related to other parameters as

$$\overline{F}_k = \alpha_k \overline{\overline{F}_k} = \alpha_k \left(\overline{F}_k^{w_k} + \overline{F'_k} \right) \quad (4-39)$$

and

$$\overline{F'_k} = \overline{F}_k - \overline{F}_k^{w_k} = \alpha_k \overline{\overline{F'_k}} - (1 - \alpha_k) \overline{F}_k^{w_k}. \quad (4-40)$$

These relations will be used in analyzing the two-phase turbulent fluxes in the averaged field equations.

1.4 Time average of derivatives

In this section, the relation between the average of the derivative and the derivative of the average is obtained. By the time derivative of the average, we mean

$$\frac{\partial \overline{F}(\mathbf{x}_0, t_0)}{\partial t_0} = \frac{\partial}{\partial t_0} \left\{ \frac{1}{\Delta t} \lim_{\delta \rightarrow 0} \int_{[\Delta t]_t} F(\mathbf{x}_0, t) dt \right\}. \quad (4-41)$$

Since the domain of the integration is discontinuous, we subdivide it and apply the Leibnitz rule to Eq.(4-41). Thus we have

$$\begin{aligned} \frac{\partial \bar{F}}{\partial t_0} = \frac{1}{\Delta t} \lim_{\delta \rightarrow 0} & \left\{ \int_{[\Delta t]_r} \frac{\partial F(\mathbf{x}_0, t)}{\partial t} dt \right. \\ & \left. + \sum_j \left[F(\mathbf{x}_0, t_j + \varepsilon_j) - F(\mathbf{x}_0, t_j - \varepsilon_j) \right] \right\}. \end{aligned} \quad (4-42)$$

In view of Eq.(4-2) we define

$$\lim_{\varepsilon_j \rightarrow 0} t_j \pm \varepsilon_j \equiv t_j^\pm \quad (4-43)$$

and correspondingly

$$\lim_{\varepsilon_j \rightarrow 0} F(\mathbf{x}_0, t_j \pm \varepsilon_j) = F(\mathbf{x}_0, t_j^\pm) \equiv F^\pm(\mathbf{x}_0, t_j). \quad (4-44)$$

Hence, from Eq.(4-42) we get

$$\frac{\partial \bar{F}}{\partial t} = \frac{\partial \bar{F}(\mathbf{x}_0, t_0)}{\partial t_0} - \sum_j \frac{1}{\Delta t} \left\{ F^+(\mathbf{x}_0, t_j) - F^-(\mathbf{x}_0, t_j) \right\}. \quad (4-45)$$

The average of the space derivative at $\mathbf{x} = \mathbf{x}_0$ can be written as

$$\overline{\nabla F} = \lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_r} \nabla F(\mathbf{x}_0, t) dt. \quad (4-46)$$

Therefore, by applying the Leibnitz rule we obtain

$$\begin{aligned} \overline{\nabla F} = \lim_{\delta \rightarrow 0} & \left[\nabla \left\{ \frac{1}{\Delta t} \int_{[\Delta t]_r} F(\mathbf{x}_0, t) dt \right\} \right. \\ & + \frac{1}{\Delta t} \sum_j \left\{ -\nabla(t_j - \varepsilon_j) F(\mathbf{x}_0, t_j - \varepsilon_j) \right. \\ & \left. \left. + \nabla(t_j + \varepsilon_j) F(\mathbf{x}_0, t_j + \varepsilon_j) \right\} \right]. \end{aligned} \quad (4-47)$$

The physical significance of the last term of the right-hand side of the above equation is not clear and should be examined in more detail. For this purpose we introduce the equation of a surface given by

$$f(x, y, z, t) = 0 \quad (4-48)$$

which passes the point \mathbf{x}_0 at $t = t_j$. Then

$$df = (\nabla f) \cdot d\mathbf{x}_0 + \frac{\partial f}{\partial t} dt_j = 0 \quad (4-49)$$

Thus, in view of Eq.(4-43), we have

$$\nabla t_j = -\frac{\nabla f}{\frac{\partial f}{\partial t}}. \quad (4-50)$$

However, the normal vector and the displacement velocity (Truesdell and Toupin, 1960) are given by

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} \quad (4-51)$$

and

$$\mathbf{v}_i \cdot \mathbf{n} = v_{ni} = -\frac{\frac{\partial f}{\partial t}}{|\nabla f|}. \quad (4-52)$$

Hence, by eliminating the surface function in Eq.(4-50) we obtain

$$\nabla t_j = \frac{\mathbf{n}}{\mathbf{v}_i \cdot \mathbf{n}}. \quad (4-53)$$

Then Eq.(4-47) becomes

$$\begin{aligned} \overline{\nabla F} &= \nabla \overline{F}(\mathbf{x}_0, t_0) \\ &+ \sum_j \frac{1}{\Delta t} \frac{1}{v_{ni}} \left\{ \mathbf{n}^+ F(\mathbf{x}_0, t_j^+) + \mathbf{n}^- F(\mathbf{x}_0, t_j^-) \right\}. \end{aligned} \quad (4-54)$$

The unit normal vector of the interface is defined such that

$$\mathbf{n} \cdot \mathbf{v}_i \equiv v_{ni} \geq 0. \quad (4-55)$$

Then

$$\mathbf{n}^+ \cdot \mathbf{v}_i \geq 0; \quad \mathbf{n}^- \cdot \mathbf{v}_i \leq 0 \quad (4-56)$$

where \mathbf{n}^+ and \mathbf{n}^- correspond to the limit outward normal vector of the fluid at each side of the interface.

Using the simplified notations, Eqs.(4-45) and (4-54) become

$$\frac{\partial \overline{F}}{\partial t} = \frac{\partial \overline{F}(\mathbf{x}_0, t_0)}{\partial t_0} - \frac{1}{\Delta t} \sum_j \frac{1}{v_{ni}} (F^+ \mathbf{n}^+ \cdot \mathbf{v}_i + F^- \mathbf{n}^- \cdot \mathbf{v}_i) \quad (4-57)$$

$$\overline{\nabla F} = \nabla \overline{F}(\mathbf{x}_0, t_0) + \frac{1}{\Delta t} \sum_j \frac{1}{v_{ni}} (\mathbf{n}^+ F^+ + \mathbf{n}^- F^-). \quad (4-58)$$

We note here that the function F can be a scalar, a vector or a tensor, and ∇ operator can be a divergence or a gradient operator with proper tensorial operation between \mathbf{n}^\pm and F^\pm . These above two transformations and the definitions of various mean values are the basic tools to be used to obtain the macroscopic field equations in terms of mean values. In contrast to the case without discontinuities in the function F , the above transformations show the important contributions made by the moving interfaces in relating the average of the derivatives to the derivatives of the average.

As corollaries of Eqs.(4-57) and (4-58) we have

$$\frac{\partial \overline{F_k}}{\partial t} = \frac{\partial \overline{F_k}}{\partial t_0} - \frac{1}{\Delta t} \sum_j \frac{1}{v_{ni}} (F_k \mathbf{n}_k \cdot \mathbf{v}_i) \quad (4-59)$$

$$\overline{\nabla F_k} = \nabla \overline{F_k} + \frac{1}{\Delta t} \sum_j \frac{1}{v_{ni}} (\mathbf{n}_k F_k). \quad (4-60)$$

The special case of the above equations is for the time fraction α_k . In this case, directly from the original definition Eq.(4-11), we obtain

$$\frac{\partial \alpha_k}{\partial t} = \frac{1}{\Delta t} \sum_j \frac{1}{v_{ni}} (\mathbf{n}_k \cdot \mathbf{v}_i) \quad (4-61)$$

$$\nabla \alpha_k = -\frac{1}{\Delta t} \sum_j \frac{\mathbf{n}_k}{v_{ni}}. \quad (4-62)$$

These equations clearly demonstrate the existence of *microscopic singularities* explained in connection with the fundamental hypothesis of smoothness in the Section 1.3 of Chapter 4.

1.5 Concentrations and mixture properties

The local time fraction α_k has been defined in the Section 1.2 of Chapter 4. The parameter α_k signifies the physical events and the structures of the two-phase flow at any particular point. Therefore, it is anticipated that the local time fraction α_k appears in all field equations. Furthermore, as the two-phase constitutive laws should also depend on the physical structures of the flow, its importance in deriving these laws is expected.

Apart from the local time fraction α_k , another concentration based on mass can be defined. In analogy with the theory of diffusion, the mass fraction c_k is given by

$$c_k \equiv \frac{\overline{\rho_k}}{\overline{\rho}} = \frac{\overline{\alpha_k \rho_k}}{\overline{\alpha_1 \rho_1 + \alpha_2 \rho_2}} = \frac{\overline{\alpha_k \rho_k}}{\overline{\rho_m}}. \quad (4-63)$$

It is the measure of the relative significance of the k^{th} -phase mass with respect to the mixture mass. Since the momentum and energy are an additive set function of mass, it is expected that the mixture properties of these variables can be expressed by those of each phase with the mass fraction c_k as a weighting function. From Eq.(4-63) we have

$$\sum_{k=1}^2 c_k = 1 \quad (4-64)$$

and

$$\frac{1}{\rho_m} = \sum_{k=1}^2 \frac{c_k}{\rho_k} . \quad (4-65)$$

The above two parameters, namely, α_k and c_k , are static concentrations, because they represent the events, structures or masses in a two-phase flow. Furthermore, kinematic concentrations are defined through various mean velocity fields, thus they represent the relative importance of the amount of flows or fluxes. Because of this basic characteristic of the kinematic variables, they cannot generally be defined in a 3-dimensional formulation, since the flows and fluxes are vector and not scalar quantities. However, they can easily be defined for a one-dimensional model. For example, the quality x has been frequently used in the literature. In what follows, we define important mixture properties.

1) The mixture density

$$\rho_m = \sum_{k=1}^2 \alpha_k \overline{\rho_k} \quad (4-66)$$

where

$$\overline{\overline{\rho_k}} = \frac{\overline{\rho_k}}{\alpha_k} . \quad (4-67)$$

2) The mixture center of mass velocity

$$\mathbf{v}_m = \frac{\sum_{k=1}^2 \alpha_k \overline{\rho_k} \widehat{\mathbf{v}}_k}{\rho_m} = \sum_{k=1}^2 c_k \widehat{\mathbf{v}}_k \quad (4-68)$$

where

$$\widehat{\mathbf{v}}_k = \frac{\overline{\overline{\rho_k \mathbf{v}_k}}}{\overline{\overline{\rho_k}}} = \frac{\overline{\rho_k \mathbf{v}_k}}{\overline{\rho_k}}. \quad (4-69)$$

3) The mixture energy

$$u_m = \frac{\sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \widehat{u}_k}}}{\rho_m} = \sum_{k=1}^2 c_k \widehat{u}_k \quad (4-70)$$

where

$$\widehat{u}_k = \frac{\overline{\overline{\rho_k u_k}}}{\overline{\overline{\rho_k}}} = \frac{\overline{\rho_k u_k}}{\overline{\rho_k}}. \quad (4-71)$$

4) The mixture pressure

$$p_m = \sum_{k=1}^2 \alpha_k \overline{\overline{p_k}} \quad (4-72)$$

where

$$\overline{\overline{p_k}} = \frac{\overline{p_k}}{\alpha_k}. \quad (4-73)$$

5) The mixture enthalpy

$$\dot{i}_m = \frac{\sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \dot{i}_k}}}{\rho_m} = \sum_{k=1}^2 c_k \widehat{\dot{i}}_k \quad (4-74)$$

where

$$\widehat{\dot{i}}_k = \frac{\overline{\overline{\rho_k \dot{i}_k}}}{\overline{\overline{\rho_k}}} = \frac{\overline{\rho_k \dot{i}_k}}{\overline{\rho_k}}. \quad (4-75)$$

Then it can be shown that

$$\dot{i}_m = u_m + \frac{p_m}{\rho_m} \quad (4-76)$$

and

$$\hat{i}_k = \hat{u}_k + \frac{\overline{\overline{p_k}}}{\rho_k}. \quad (4-77)$$

6) The mixture entropy

$$s_m = \frac{\sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \hat{s}_k}}}{\rho_m} = \sum_{k=1}^2 c_k \hat{s}_k \quad (4-78)$$

where

$$\hat{s}_k = \frac{\overline{\overline{\rho_k s_k}}}{\rho_k} = \frac{\overline{\overline{\rho_k s_k}}}{\rho_k}. \quad (4-79)$$

7) The general mixture flux $\overline{\overline{J}}$

We recall here the general balance equation, Eq.(2-6), the generalized flux and the volume source defined in the Section 1.1 of Chapter 2. From the form of the balance equation, it is natural to define the mixture molecular diffusion flux $\overline{\overline{J}}$ as

$$\overline{\overline{J}} = \sum_{k=1}^2 \alpha_k \overline{\overline{J_k}} \quad (4-80)$$

where

$$\overline{\overline{J_k}} = \frac{\overline{\overline{J_k}}}{\alpha_k}. \quad (4-81)$$

8) The mixture general source term ϕ_m

Since the source term ϕ is defined as the variable per unit mass, it should be weighed by the density. Hence, we have

$$\phi_m = \frac{\sum_{k=1}^2 \alpha_k \overline{\rho_k \hat{\phi}_k}}{\rho_m} = \sum_{k=1}^2 c_k \hat{\phi}_k \quad (4-82)$$

where

$$\hat{\phi}_k = \frac{\overline{\rho_k \phi_k}}{\overline{\rho_k}} = \frac{\overline{\rho_k \phi_k}}{\overline{\rho_k}}. \quad (4-83)$$

It can be seen that the variables based on unit mass are weighted by the mass concentrations, whereas the ones based on unit volume or surface are weighted by the time fractions.

1.6 Velocity field

In general, two-phase flow systems with transport of mass, momentum and energy are characterized by the existence of two different densities and velocities. Thus it is necessary to introduce two properly defined mean velocity fields in the formulation in order to take into account the effects of the relative motion between the phases, namely, the diffusion of mass, momentum, and energy. However, there are several velocity fields that are useful in analyzing various aspects of a two-phase flow problem. A selection of velocity fields for a particular problem depends upon characteristics and nature of the flow as well as on the forms of available constitutive laws. In what follows, we present these velocity fields that are important in studying various aspects of two-phase flow systems.

As it has been explained in the previous section, the definition of *the center of mass velocities* is based on the fundamental characteristic of linear momentum. First, we recall that it is an additive set function of mass. In other words, as it is well known as a fundamental theorem on the center of mass, the total momentum of a body is given by the momentum of the center of mass with the same mass as the body. It is the direct extension of the above idea into the averaging procedure that we obtain the mass-weighted mixture and phase velocities as proper mean velocities.

The concept of the center of mass in the time averaging is trivial and it has the form

$$\mathbf{x} = \frac{\lim_{\delta \rightarrow 0} \int_{[\Delta t]_T} \rho \mathbf{x}_0 dt}{\lim_{\delta \rightarrow 0} \int_{[\Delta t]_T} \rho dt}. \quad (4-84)$$

However \mathbf{x}_0 is kept constant during the integration, thus the center of mass is \mathbf{x}_0 . Then the definition of the mixture density, Eq.(4-66), naturally follows. The fundamental theorem on the center of mass can be extended as

$$\lim_{\delta \rightarrow 0} \int_{[\Delta t]_T} \rho_m \mathbf{v}_m dt = \lim_{\delta \rightarrow 0} \int_{[\Delta t]_T} \rho \mathbf{v} dt. \quad (4-85)$$

Hence in view of the definitions of weighted mean values, it is straightforward to show that the center of mass velocities of the mixture and of each phase have been given correctly by Eqs.(4-68) and (4-69), respectively.

We define the relative velocity by

$$\mathbf{v}_r \equiv \widehat{\mathbf{v}}_2 - \widehat{\mathbf{v}}_1. \quad (4-86)$$

And the *volumetric flux* of each phase are given by

$$\mathbf{j}_k \equiv \alpha_k \widehat{\mathbf{v}}_k \quad (4-87)$$

which can be considered as the velocity when one of the phases superficially occupies the entire interval Δt with the total amount of the flow fixed, and therefore it is also called the superficial velocity. Accordingly the mixture volumetric flux, namely, the velocity of the center of volume, is defined by

$$\mathbf{j} = \sum_{k=1}^2 \mathbf{j}_k = \sum_{k=1}^2 \alpha_k \widehat{\mathbf{v}}_k. \quad (4-88)$$

If the relative velocity between the phases exists, the velocities \mathbf{v}_m and \mathbf{j} are not equal because of the differences of the densities of the two phases. The *diffusion velocity* of each phase, namely, the relative velocity with respect to the mass center of the mixture, is defined by

$$\mathbf{V}_{km} = \widehat{\mathbf{v}}_k - \mathbf{v}_m \quad (4-89)$$

which are frequently used in the analyses of heterogeneous chemically reacting single-phase systems. The diffusion velocities can also be

expressed by the relative velocity, though the symmetry between phases cannot be kept because of the definition, Eq.(4-86). Thus we have

$$\mathbf{V}_{1m} = -\frac{\alpha_2 \overline{\rho_2}}{\rho_m} \mathbf{v}_r = -c_2 \mathbf{v}_r$$

and

(4-90)

$$\mathbf{V}_{2m} = \frac{\alpha_1 \overline{\rho_1}}{\rho_m} \mathbf{v}_r = c_1 \mathbf{v}_r.$$

In a two-phase flow system, the drift velocity of each phase, namely the relative velocity with respect to the center of volume, is important because the constitutive equations for these velocities in the mixture formulation is relatively simple and well developed (Zuber et al, 1964; Ishii, 1977). By definition, the *drift velocity* is given by

$$\mathbf{V}_{kj} = \widehat{\mathbf{v}}_k - \mathbf{j}. \quad (4-91)$$

In terms of the relative velocity, it becomes

$$\mathbf{V}_{1j} = -\alpha_2 \mathbf{v}_r$$

and

(4-92)

$$\mathbf{V}_{2j} = \alpha_1 \mathbf{v}_r.$$

Several important relations between the above velocities can be obtained directly from the definitions. For example, from Eqs.(4-88), (4-89) and (4-90) we get

$$\mathbf{j} = \mathbf{v}_m + \alpha_1 \alpha_2 \frac{(\overline{\rho_1} - \overline{\rho_2})}{\rho_m} \mathbf{v}_r$$

or

(4-93)

$$= \mathbf{v}_m - \alpha_1 \frac{(\overline{\rho_1} - \overline{\rho_2})}{\rho_m} \mathbf{V}_{1j}.$$

From Eqs.(4-90) and (4-92) we have

$$\sum_{k=1}^2 c_k \mathbf{V}_{km} = 0 \quad (4-94)$$

and

$$\sum_{k=1}^2 \alpha_k \mathbf{V}_{kj} = 0. \quad (4-95)$$

Finally, we note that if the relative velocity is zero, then

$$\mathbf{V}_{1m} = \mathbf{V}_{2m} = \mathbf{V}_{1j} = \mathbf{V}_{2j} = \mathbf{v}_r = 0 \quad (4-96)$$

and thus

$$\widehat{\mathbf{v}}_1 = \widehat{\mathbf{v}}_2 = \mathbf{v}_m = \mathbf{j} \quad (4-97)$$

which characterizes the homogenous velocity field.

Generally speaking, the velocities based on the center of mass are important for dynamic analyses because of the fundamental theorem of center of mass. However, the velocities based on the center of volume, namely, volumetric fluxes, are useful for kinematic analyses. This is particularly true if each phase has constant properties such as the constant densities, internal energies or enthalpies, etc.

1.7 Fundamental identity

In developing a drift-flux model based on the mixture properties, it is necessary to express an average convective flux by various mean values. In this section, we derive this relation directly from the definitions. From Eq.(4-29), the convective flux of the mixture becomes

$$\overline{\rho \psi \mathbf{v}} = \sum_{k=1}^2 \overline{\rho_k \psi_k \mathbf{v}_k} = \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \psi_k \mathbf{v}_k}}. \quad (4-98)$$

Our purpose here is to split the right-hand side of the equation into the terms expressed by the mean values and the ones representing the statistical effects of the fluctuating components. Since the mean values of ψ and \mathbf{v} are weighted by mass, the fluctuating components are given by Eq.(4-37) as

$$\rho_k = \overline{\overline{\rho_k}} + \rho'_k, \quad \psi_k = \widehat{\psi_k} + \psi'_k, \quad \mathbf{v}_k = \widehat{\mathbf{v}_k} + \mathbf{v}'_k \quad (4-99)$$

with

$$\overline{\overline{\rho'_k}} = 0, \quad \overline{\overline{\rho_k \psi'_k}} = 0, \quad \overline{\overline{\rho_k \mathbf{v}'_k}} = 0. \quad (4-100)$$

By substituting Eqs.(4-99) and (4-100) into Eq.(4-98) we obtain

$$\overline{\rho \psi \mathbf{v}} = \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k}} \widehat{\psi_k} \widehat{\mathbf{v}_k} + \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \psi'_k \mathbf{v}'_k}}. \quad (4-101)$$

By using the definition of the mixture properties and that of the diffusion velocities, the above equation reduces to

$$\overline{\rho \psi \mathbf{v}} = \rho_m \psi_m \mathbf{v}_m + \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k}} \widehat{\psi_k} \mathbf{V}_{km} + \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \psi'_k \mathbf{v}'_k}}. \quad (4-102)$$

It shows that the average convective flux can be split into three parts according to the different transport mechanisms: the mixture transport based on the mixture properties; the diffusion transport of $\widehat{\psi_k}$ due to the difference of the phase velocities; and the transport due to the two-phase and turbulent fluctuations. In order to distinguish these last two transport mechanisms we introduce special fluxes associated with them. Hence, we define the diffusion flux \mathcal{J}^D as

$$\mathcal{J}^D \equiv \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k}} \widehat{\psi_k} \mathbf{V}_{km} = \frac{\overline{\overline{\rho_1}} \overline{\overline{\rho_2}}}{\rho_m} \sum_{k=1}^2 \alpha_k \widehat{\psi_k} \mathbf{V}_{kj} \quad (4-103)$$

whereas the covariance or the turbulent flux \mathcal{J}_k^T is defined as

$$\mathcal{J}_k^T \equiv \overline{\overline{\rho_k \psi'_k \mathbf{v}'_k}}. \quad (4-104)$$

Thus, the mixture turbulent flux should be

$$\mathcal{J}^T = \sum_{k=1}^2 \alpha_k \mathcal{J}_k^T = \sum_{k=1}^2 \alpha_k \overline{\overline{\rho_k \psi'_k \mathbf{v}'_k}}. \quad (4-105)$$

By substituting Eqs.(4-103) and (4-105) into Eq.(4-101) we obtain a fundamental identity

$$\overline{\rho\psi\mathbf{v}} = \rho_m\psi_m\mathbf{v}_m + \mathcal{J}^D + \mathcal{J}^T = \sum_{k=1}^2 \alpha_k \overline{\rho_k} \widehat{\psi_k} \widehat{\mathbf{v}_k} + \sum_{k=1}^2 \alpha_k \mathcal{J}_k^T. \quad (4-106)$$

Hence, from the definitions of the mean values we have

$$\begin{aligned} \frac{\partial \overline{\rho\psi}}{\partial t} + \nabla \cdot (\overline{\rho\psi\mathbf{v}}) &= \frac{\partial \rho_m\psi_m}{\partial t} + \nabla \cdot (\rho_m\psi_m\mathbf{v}_m) \\ &+ \nabla \cdot (\mathcal{J}^D + \mathcal{J}^T) \end{aligned} \quad (4-107)$$

and for each phase we get

$$\begin{aligned} \frac{\partial \overline{\rho_k\psi_k}}{\partial t} + \nabla \cdot (\overline{\rho_k\psi_k\mathbf{v}_k}) &= \frac{\partial \alpha_k \overline{\rho_k} \widehat{\psi_k}}{\partial t} + \nabla \cdot (\alpha_k \overline{\rho_k} \widehat{\psi_k} \widehat{\mathbf{v}_k}) \\ &+ \nabla \cdot (\alpha_k \mathcal{J}_k^T). \end{aligned} \quad (4-108)$$

We note that Eq.(4-108) shows a simple analogy with a single-phase turbulent flow averaging, therefore, the last term is called the Reynolds flux.

In view of Eq.(4-107), it can be seen that the left-hand side of the equation is not expressed by the averages of the derivatives, but by the derivatives of the averages. However, when we apply the Eulerian temporal averaging to the local formulation of two-phase flows, we first encounter with the averages of the derivatives. Now we recall that the important transformations between these two operations have been derived in the Section 1.4 of Chapter 4. Thus, by substituting Eqs.(4-57) and (4-58) into Eq.(4-107) we obtain

$$\begin{aligned} \left(\frac{\partial \overline{\rho\psi}}{\partial t} \right) + \overline{\nabla \cdot (\rho\psi\mathbf{v})} &= \frac{\partial \rho_m\psi_m}{\partial t} + \nabla \cdot (\rho_m\psi_m\mathbf{v}_m) \\ &+ \nabla \cdot (\mathcal{J}^D + \mathcal{J}^T) + \frac{1}{\Delta t} \sum_j \left[\frac{1}{v_{ni}} \sum_{k=1}^2 \{ \mathbf{n}_k \cdot \rho_k (\mathbf{v}_k - \mathbf{v}_i) \psi_k \} \right]. \end{aligned} \quad (4-109)$$

A similar relation for an individual phase follows, thus we have

$$\begin{aligned}
& \left(\frac{\partial \rho_k \psi_k}{\partial t} \right) + \nabla \cdot (\rho_k \psi_k \mathbf{v}_k) = \frac{\partial \alpha_k \overline{\rho_k} \widehat{\psi_k}}{\partial t} + \nabla \cdot (\alpha_k \overline{\rho_k} \widehat{\psi_k} \widehat{\mathbf{v}}_k) \\
& + \nabla \cdot (\alpha_k \mathcal{J}_k^T) + \frac{1}{\Delta t} \sum_j \left\{ \frac{1}{v_{ni}} \mathbf{n}_k \cdot \rho_k (\mathbf{v}_k - \mathbf{v}_i) \psi_k \right\}.
\end{aligned} \tag{4-110}$$

These above two equations show important contributions made by the interfacial transfer in addition to the statistical effects of fluctuations. Furthermore, in the mixture average, Eq.(4-107), the diffusion term \mathcal{J}^D appears due to the differences in phase velocities.

Furthermore, from the definitions of mean values and of the diffusion flux we have

$$\begin{aligned}
& \frac{\partial \rho_m \psi_m}{\partial t} + \nabla \cdot (\rho_m \psi_m \mathbf{v}_m) + \nabla \cdot \mathcal{J}^D \\
& = \sum_{k=1}^2 \left\{ \frac{\partial \alpha_k \overline{\rho_k} \widehat{\psi_k}}{\partial t} + \nabla \cdot (\alpha_k \overline{\rho_k} \widehat{\psi_k} \widehat{\mathbf{v}}_k) \right\}.
\end{aligned} \tag{4-111}$$

This relation enables us to transform the field equations for the two-fluid model to the ones for the drift-flux model.