Applying Upwind Godunov Methods to Calculate Two-Phase Mixture Conservation Laws

ARTICLE · SEPTEMBER 2010

DOI: 10.1063/1.3498104

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Abstract. This paper continues a previous work (ICNAAM 2009; AIP Conference Proceedings, 1168, 601–604) on solving a hyperbolic conservative model for compressible gas–solid mixture flow using upwind Godunov methods. The numerical resolution of the model from Godunov first–order upwind and MUSCL–Hancock methods are reported. Both methods are based on the HLL Riemann solver in the framework of finite volume techniques. Calculation results are presented for a series of one–dimensional test problems. The results show that upwind Godunov methods are accurate and robust enough for two–phase mixture conservation laws.

Keywords: thermodynamically compatible systems, compressible gas–solid mixture, Riemann solver, upwinding
PACS: 47.40.Nm, 47.11.+j,47.55.–t

INTRODUCTION

A significant progress has been made in both mathematical and numerical investigations of the differential equations governing two–phase flows. This is due to a broad range of disciplines and applications of two–phase flows. Typically, these investigations have mostly been based on the two–fluid models type. Such models suffer certain mathematical difficulties related to the non–hyperbolicity character of the system and they are not written in conservative form. This paper is to continue a previous work in solving a hyperbolic conservative two–phase flow model for gas–solid mixture using Godunov methods of upwind–type [2]. The aim is to extend such methods to the solution of a one–dimensional two–phase flow model based on the theory thermodynamically compatible systems [3]. To solve the model numerically, the MUSCL–Hancock scheme is considered. Further, the HLL Riemann solver is implemented in the framework of MUSCL–Hancock scheme. The resulting MUSCL strategy is simple to implement, efficient and suitable for solving the general initial–boundary value problem for the two–phase flow model. Consequently, numerical results are given to show the capabilities of the approach for two–phase gas–solid mixture.

CONSERVATION LAWS FOR A GAS–SOLID MIXTURE

Within the framework of thermodynamically compatible systems of hyperbolic conservation laws, the equations governing the one–dimensional two–phase flow of gas–solid mixture consists of balance laws for mass, momentum and energy with a gas volume concentration, mass gas concentration and gas entropy concentration balance equations, respectively. In a standard conservation form, they can be written as

$$\frac{\partial \mathbb{U}}{\partial t} + \frac{\partial \mathbb{F}(\mathbb{U})}{\partial x} = \mathbb{S}, \quad t \in [0,\infty), \;\; x \in (-\infty,\infty), \tag{1}$$

where $\mathbb{U}$, $\mathbb{F}(\mathbb{U})$ and $\mathbb{S}$ are the conservative variables, the conservative fluxes and the source terms respectively, given by

$$\mathbb{U} = \begin{pmatrix} \rho \\ \rho\alpha \\ \rho u \\ \rho c \\ \rho\chi \\ \rho E \end{pmatrix}, \quad \mathbb{F}(\mathbb{U}) = \begin{pmatrix} \rho u \\ \rho u\alpha \\ \rho u^2 + P \\ \rho uc \\ \rho u\chi \\ \rho uE + Pu \end{pmatrix} \quad \text{and} \quad \mathbb{S}(\mathbb{U}) = \begin{pmatrix} 0 \\ \frac{-\rho}{\tau^{(P)}} e_\alpha \\ 0 \\ \frac{-\rho}{\tau^{(c)}} e_c \\ \frac{-\rho}{s\ \tau^{(\chi)}} e_\chi \\ 0 \end{pmatrix}. \tag{2}$$

CP1281, *ICNAAM, Numerical Analysis and Applied Mathematics, International Conference 2010*
edited by T. E. Simos, G. Psihoyios, and Ch. Tsitouras

Here α, c and χ are the gas volume, gas mass concentration and gas entropy concentration, and ρ, u, P and E are the corresponding variables for the gas–solid mixture. In (2), the source terms accounts for phase interaction processes. Further, a detailed derivation and description of the model equations can be found in [4, 5]. The model was shown to be unconditionally hyperbolic and fully hyperbolic conservative for two–phase gas–solid mixture. Based on the observations presented in [1, 5], it is desirable to extend theoretical aspects and numerical approximations to a higher level.

NUMERICAL SOLUTION METHODS

Taking into consideration the conservative form of system (1) this paper considers the extension of Godunov methods of upwind–type for single–phase flow to two–phase gas–solid mixture using finite volume techniques [2]. Upwind Godunov methods require the solution of the Riemann problem which may be an exact solution or an approximate solution. At this time only approximate solutions are considered. Furthermore, the approximate solution to the Riemann problem for the model equations is based on the HLL Riemann solver [2]. This solution is then employed in the construction of MUSCL–Hancock scheme to solve the general initial–boundary value problem for the two–phase gas–solid mixture. The MUSCL–Hancock scheme is a TVD algorithm based on the concept of a slope limiter and consists of three steps: data reconstruction, evolution step and calculation of the fluxes of the Riemann problem using the HLL Riemann solver. Applying the MUSCL–Hancock scheme to system (1) gives the values of the unknown conservative variables related to the mixture and phase quantities for four different test problems. All the results for the homogeneous part of system (1) are displayed in the computational domain $[-10, 10]$ with a discontinuity located at $x_0 = 0$ and the solution evolves for $t > 0$. These results are obtained using the TVD MUSCL–Hancock and the Godunov first–order upwind methods on a coarse mesh of 100 cells. The reference solution is obtained by the TVD MUSCL–Hancock scheme using a finite mesh of 3000 cells. All the results are compared with the TVD SLIC scheme, a previously developed centred scheme for system (1), see [5] for further details. In all the figures, the different combinations of symbols refer to Godunov (star sign), MUSCL (dotted sign) and SLIC (squared sign) whereas the solid line refers to the reference solution. Figure 1 show the solutions for Test 1: propagation of a volume wave within a gas–solid mixture. The results in figure 2 are concerned with Test 2: gas–solid mixture collision. Figure 3 present the results calculated for Test 3: rarefaction waves propagation within the mixture. Finally, figure 4 correspond to Test 4: symmetric rarefaction waves within a gas–solid mixture. All the results compare well with the reference solution. It is observed that the major flow structures (rarefaction waves, shock waves and contact discontinuities) are all noticeably better resolved with the TVD MUSCL–Hancock scheme. Moreover, the resolution afforded by the TVD MUSCL–Hancock scheme is superior and smear discontinuities better than the Godunov first–order upwind and TVD SLIC methods.

CONCLUDING REMARKS

The hyperbolic two–phase mixture conservation laws have been solved numerically using upwind Godunov methods. These mixture conservation laws together with existing upwind Godunov methods provides physically meaningful oscillation–free numerical results. The key point is that these methods make use of the HLL Riemann solver for calculating the numerical fluxes that solves the general initial–boundary value problem. Upwind Godunov methods yields satisfactory results and match with those provided by centred Godunov methods.

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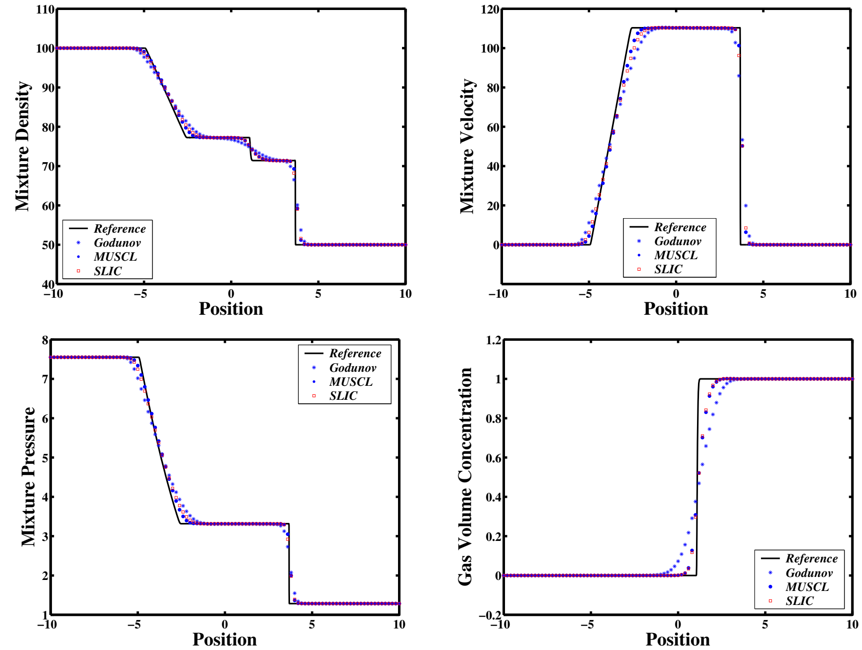

FIGURE 1. Test 1. Comparison of numerical and reference solutions at $t = 0.01$ for CFL $= 0.9$ with 100 cells. The results of the Godunov first–order upwind scheme are compared with results from the TVD MUSCL–Hancock and TVD SLIC methods using SUPERBEE limiter function. As one can see, the mixture density jump across the multiple contact discontinuity. Also the MUSCL–Hancock scheme yields a sharper representation of all the discontinuities.

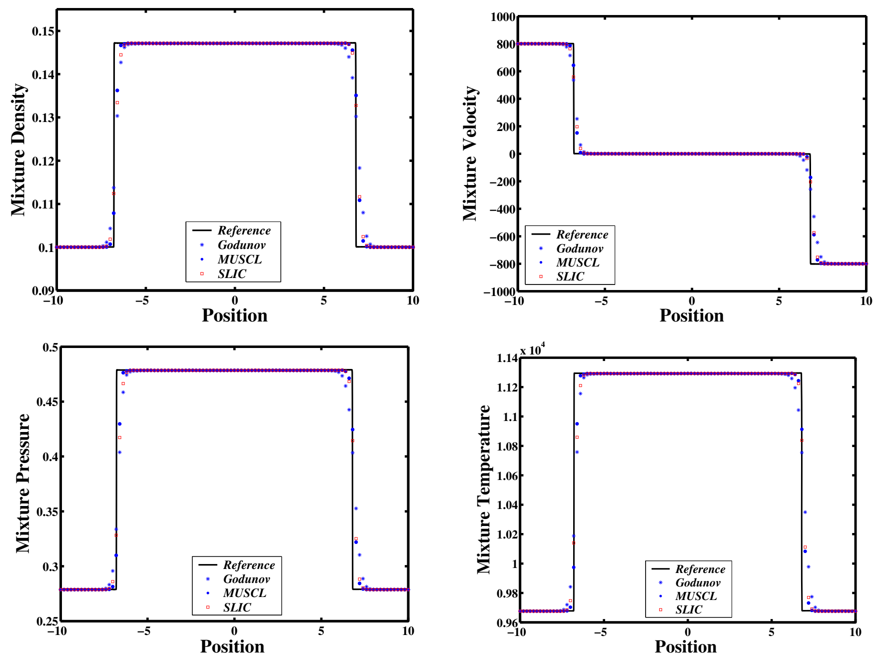

FIGURE 2. Test 2. Numerical solutions obtained with three different numerical methods (Godunov, MUSCL and SLIC) are compared at $t = 0.004$ with the reference solution. Computations were performed on 100 cells with CFL $= 0.9$ using SUPERBEE limiter function for the MUSCL–Hancock and SLIC methods. The major difference between the three methods is the resolution of the multiple contact discontinuity.

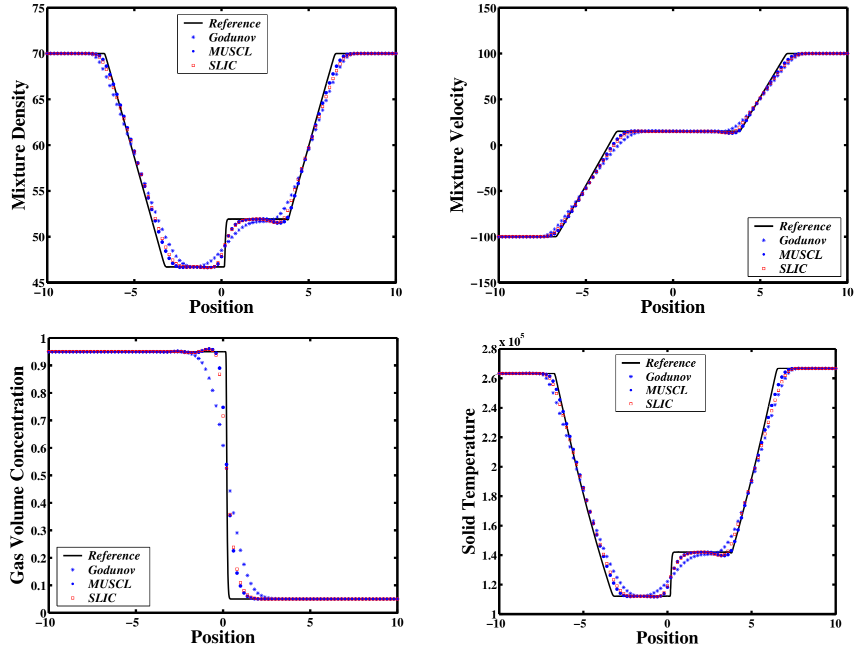

FIGURE 3. Test 3. Comparison between the Godunov first–order upwind, MUSCL–Hancock and SLIC methods for rarefaction waves propagation within the mixture at $t = 0.015$ using SUPERBEE limiter function. A CFL $= 0.9$ and 100 cells were used in the computations. The TVD MUSCL–Hancock scheme behaves quantitatively better than the Godunov and SLIC methods across the non–trivial multiple contact discontinuity and reproduce both the head and the tail of the rarefaction waves accurately.

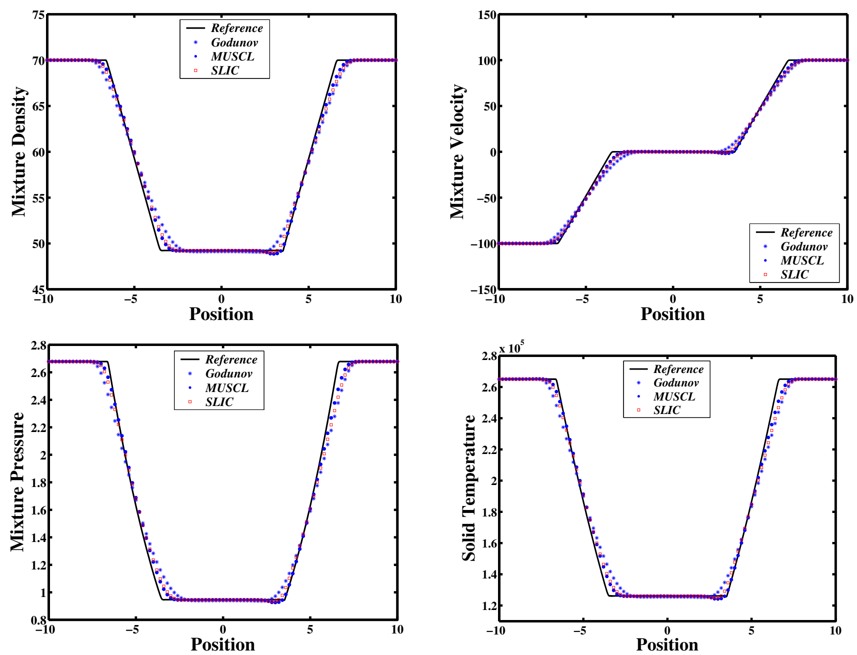

FIGURE 4. Test 4. results for symmetric rarefaction waves at $t = 0.015$ with CFL $= 0.9$ on 100 cells and using the SUPERBEE limiter. The solutions comprise a left going rarefaction wave, a trivial multiple contact discontinuity, that is $\alpha_L = \alpha_R$, and right going rarefaction wave. As in Test 3, the MUSCL–Hancock behaves better than the Godunov first–order upwind and SLIC methods across the wave structure.