Chapter 2

LOCAL INSTANT FORMULATION

The singular characteristic of two-phase or of two immiscible mixtures is the presence of one or several interfaces separating the phases or components. Examples of such flow systems can be found in a large number of engineering systems as well as in a wide variety of natural phenomena. The understanding of the flow and heat transfer processes of two-phase systems has become increasingly important in nuclear, mechanical and chemical engineering, as well as in environmental and medical science.
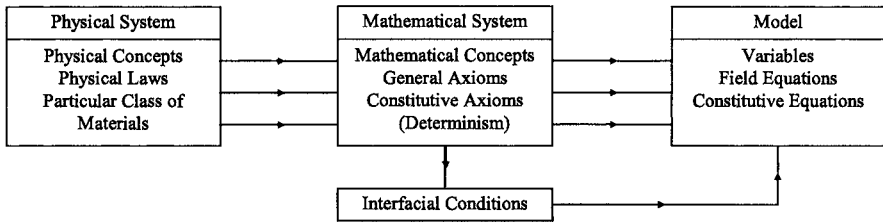
In analyzing two-phase flow, it is evident that we first follow the standard method of continuum mechanics. Thus, a two-phase flow is considered as a field that is subdivided into single-phase regions with moving boundaries between phases. The standard differential balance equations hold for each subregion with appropriate jump and boundary conditions to match the solutions of these differential equations at the interfaces. Hence, in theory, it is possible to formulate a two-phase flow problem in terms of the local instant variable, namely, $F = F(x, t)$. This formulation is called a *local instant formulation* in order to distinguish it from formulations based on various methods of averaging.

Such a formulation would result in a multiboundary problem with the positions of the interface being unknown due to the coupling of the fields and the boundary conditions. Indeed, mathematical difficulties encountered by using this local instant formulation can be considerable and, in many cases, they may be insurmountable. However, there are two fundamental importances in the local instant formulation. The first importance is the *direct application* to study the separated flows such as film, stratified, annular and jet flow, see Table 1-1. The formulation can be used there to study pressure drops, heat transfer, phase changes, the dynamic and stability of an interface, and the critical heat flux. In addition to the above applications, important examples of when this formulation can be used

include: the problems of single or several bubble dynamics, the growth or collapse of a single bubble or a droplet, and ice formation and melting.

The second importance of the local instant formulation is as a *fundamental base of the macroscopic two-phase flow models* using various averaging. When each subregion bounded by interfaces can be considered as a continuum, the local instant formulation is mathematically rigorous. Consequently, two-phase flow models should be derived from this formulation by proper averaging methods. In the following, the general formulation of two-phase flow systems based on the local instant variables is presented and discussed. It should be noted here that the balance equations for a single-phase one component flow were firmly established for some time (Truesdell and Toupin, 1960; Bird et al, 1960). However, the axiomatic construction of the general constitutive laws including the equations of state was put into mathematical rigor by specialists (Coleman, 1964; Bowen, 1973; Truesdell, 1969). A similar approach was also used for a single-phase diffusive mixture by Muller (1968).

Before going into the detailed derivation and discussion of the local instant formulation, we review the method of mathematical physics in connection with the continuum mechanics. The next diagram shows the basic procedures used to obtain a mathematical model for a physical system.

Physical System		Mathematical System		Model
Physical Concepts Physical Laws Particular Class of Materials		Mathematical Concepts General Axioms Constitutive Axioms (Determinism)		Variables Field Equations Constitutive Equations

Interfacial Conditions

As it can be seen from the diagram, a physical system is first replaced by a mathematical system by introducing mathematical concepts, general axioms and constitutive axioms. In the continuum mechanics they correspond to variables, field equations and constitutive equations, whereas at the singular surface the mathematical system requires the interfacial conditions. The latter can be applied not only at the interface between two phases, but also at the outer boundaries which limit the system. It is clear from the diagram that the continuum formulation consists of three essential parts, namely: the derivations of field equations, constitutive equations, and interfacial conditions.

Now let us examine the basic procedure used to solve a particular problem. The following diagram summarizes the standard method. Using the continuum formulation, the physical problem is represented by idealized boundary geometries, boundary conditions, initial conditions, field and
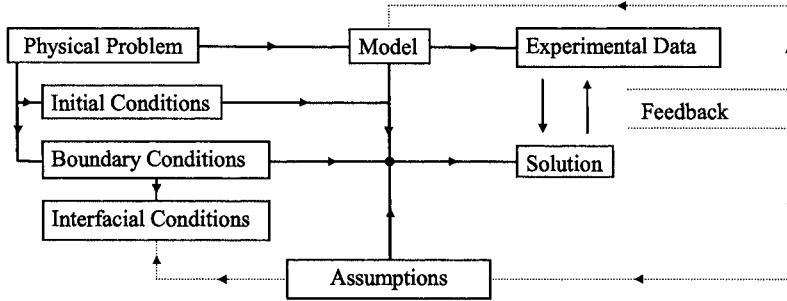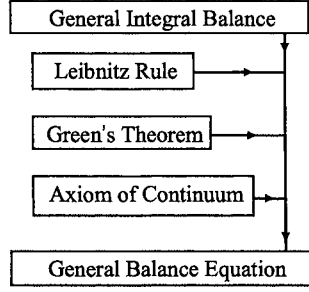
constitutive equations. It is evident that in two-phase flow systems, we have interfaces within the system that can be represented by general interfacial conditions. The solutions can be obtained by solving these sets of differential equations together with some idealizing or simplifying assumptions. For most problems of practical importance, experimental data also play a key role. First, experimental data can be taken by accepting the model, indicating the possibility of measurements. The comparison of a solution to experimental data gives feedback to the model itself and to the various assumptions. This feedback will improve both the methods of the experiment and the solution. The validity of the model is shown in general by solving a number of simple physical problems.

The continuum approach in single-phase thermo-fluid dynamics is widely accepted and its validity is well proved. Thus, if each subregion bounded by interfaces in two-phase systems can be considered as continuum, the validity of local instant formulation is evident. By accepting this assumption, we derive and discuss the field equations, the constitutive laws, and the interfacial conditions. Since an interface is a singular case of the continuous field, we have two different conditions at the interface. The balance at an interface that corresponds to the field equation is called a jump condition. Any additional information corresponding to the constitutive laws in space, which are also necessary at interface, is called an interfacial boundary condition.

1.1   Single-phase flow conservation equations

1.1.1   General balance equations

The derivation of the differential balance equation is shown in the following diagram. The general integral balance can be written by introducing the fluid density ρ_k, the efflux $\mathcal{J}_k$ and the body source ϕ_k of any quantity ψ_k defined for a unit mass. Thus we have

General Integral Balance

| Leibnitz Rule |

| Green's Theorem |

| Axiom of Continuum |

| General Balance Equation |

$$\frac{d}{dt}\int_{V_m}\rho_k\psi_k dV = -\oint_{A_m}\boldsymbol{n}_k\cdot J_k dA + \int_{V_m}\rho_k\phi_k dV \qquad (2\text{-}1)$$

where V_m is a material volume with a material surface A_m. It states that the time rate of change of $\rho_k\psi_k$ in V_m is equal to the influx through A_m plus the body source. The subscript k denotes the kth-phase. If the functions appearing in the Eq.(2-1) are sufficiently smooth such that the Jacobian transformation between material and spatial coordinates exists, then the familiar differential form of the balance equation can be obtained. This is done by using the Reynolds transport theorem (Aris, 1962) expressed as

$$\frac{d}{dt}\int_{V_m}F_k dV = \int_{V_m}\frac{\partial F_k}{\partial t}dV + \oint_{A_m}F_k\boldsymbol{v}_k\cdot\boldsymbol{n}\ dA \qquad (2\text{-}2)$$

where $\boldsymbol{v}_k$ denotes the velocity of a fluid particle. The Green's theorem gives a transformation between a certain volume and surface integral, thus

$$\int_V\nabla\cdot F_k dV = \oint_A\boldsymbol{n}\cdot F_k dA. \qquad (2\text{-}3)$$

Hence, from Eqs.(2-2) and (2-3) we obtain

$$\frac{d}{dt}\int_{V_m}F_k dV = \int_{V_m}\left[\frac{\partial F_k}{\partial t} + \nabla\cdot(\boldsymbol{v}_k F_k)\right]dV. \qquad (2\text{-}4)$$

Furthermore, we note that the Reynolds transport theorem is a special case of Leibnitz rule given by

$$\frac{d}{dt}\int_V F_k dV = \int_V\frac{\partial F_k}{\partial t}dV + \int_A F_k\boldsymbol{u}\cdot\boldsymbol{n}dA \qquad (2\text{-}5)$$

where $V(t)$ is an arbitrary volume bounded by $A(t)$ and $\boldsymbol{u} \cdot \boldsymbol{n}$ is the surface displacement velocity of $A(t)$.

In view of Eqs.(2-1), (2-3) and (2-4) we obtain a differential balance equation

$$\frac{\partial \rho_k \psi_k}{\partial t} + \nabla \cdot (\boldsymbol{v}_k \rho_k \psi_k) = -\nabla \cdot \mathcal{J}_k + \rho_k \phi_k. \tag{2-6}$$

The first term of the above equation is the time rate of change of the quantity per unit volume, whereas the second term is the rate of convection per unit volume. The right-hand side terms represent the surface flux and the volume source.

1.1.2 Conservation equation

Continuity Equation
The conservation of mass can be expressed in a differential form by setting

$$\psi_k = 1, \quad \phi_k = 0, \quad \mathcal{J}_k = 0 \tag{2-7}$$

since there is no surface and volume sources of mass with respect to a fixed mass volume. Hence from the general balance equation we obtain

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \boldsymbol{v}_k) = 0. \tag{2-8}$$

Momentum Equation
The conservation of momentum can be obtained from Eq.(2-6) by introducing the surface stress tensor $\mathcal{T}_k$ and the body force $\boldsymbol{g}_k$, thus we set

$$\psi_k = \boldsymbol{v}_k$$
$$\mathcal{J}_k = -\mathcal{T}_k = p_k \boldsymbol{I} - \mathcal{T}_k \tag{2-9}$$
$$\phi_k = \boldsymbol{g}_k$$

where $\boldsymbol{I}$ is the unit tensor. Here we have split the stress tensor into the pressure term and the viscous stress $\mathcal{T}_k$. In view of Eq.(2-6) we have

$$\frac{\partial \rho_k \boldsymbol{v}_k}{\partial t} + \nabla \cdot (\rho_k \boldsymbol{v}_k \boldsymbol{v}_k) = -\nabla p_k + \nabla \cdot \mathcal{T}_k + \rho_k \boldsymbol{g}_k. \tag{2-10}$$

Conservation of Angular Momentum

If we assume that there is no body torque or couple stress, then all torques arise from the surface stress and the body force. In this case, the conservation of angular momentum reduces to

$$T_k = T_k^+ \tag{2-11}$$

where T_k^+ denotes the transposed stress tensor. The above result is correct for a non-polar fluid, however, for a polar fluid we should introduce an intrinsic angular momentum. In that case, we have a differential angular momentum equation (Aris, 1962).

Conservation of Energy

The balance of energy can be written by considering the total energy of the fluid. Thus we set

$$
\begin{aligned}
\psi_k &= u_k + \frac{v_k^2}{2} \\
J_k &= q_k - T_k \cdot v_k \\
\phi_k &= g_k \cdot v_k + \frac{\dot{q}_k}{\rho_k}
\end{aligned}
\tag{2-12}
$$

where u_k, q_k and $\dot{q}_k$ represent the internal energy, heat flux and the body heating, respectively. It can be seen here that both the flux and the body source consist of the thermal effect and the mechanical effect. By substituting Eq.(2-12) into Eq.(2-6) we have the total energy equation

$$
\frac{\partial \rho_k \left(u_k + \frac{v_k^2}{2} \right)}{\partial t} + \nabla \cdot \left[\rho_k \left(u_k + \frac{v_k^2}{2} \right) v_k \right] \tag{2-13}
$$
$$
= -\nabla \cdot q_k + \nabla \cdot \left(T_k \cdot v_k \right) + \rho_k g_k \cdot v_k + \dot{q}_k.
$$

These four local equations, namely, Eqs.(2-8), (2-10), (2-11) and (2-13), express the four basic physical laws of the conservation of mass, momentum, angular momentum and energy. In order to solve these equations, it is necessary to specify the fluxes and the body sources as well as the fundamental equation of state. These are discussed under the constitutive laws. Apart from these constitutive laws, we note that there are several important transformations of above equations. A good review of

transformed equations can be found in Bird et al. (1960). The important ones are given below.

The Transformation on Material Derivative
In view of the continuity equation we have

$$\frac{\partial \rho_k \psi_k}{\partial t} + \nabla \cdot \left(\rho_k \psi_k \boldsymbol{v}_k \right) = \rho_k \left(\frac{\partial \psi_k}{\partial t} + \boldsymbol{v}_k \cdot \nabla \psi_k \right) \equiv \rho_k \frac{D_k \psi_k}{Dt}. \quad (2\text{-}14)$$

This special time derivative is called the material or substantial derivative, since it expresses the rate of change with respect to time when an observer moves with the fluid.

Equation of Motion
By using the above transformation the momentum equation becomes the equation of motion

$$\rho_k \frac{D_k \boldsymbol{v}_k}{Dt} = -\nabla p_k + \nabla \cdot \mathcal{T}_k + \rho_k \boldsymbol{g}_k. \quad (2\text{-}15)$$

Here it is noted that $D_k \boldsymbol{v}_k / Dt$ is the fluid acceleration, thus the equation of motion expresses Newton's Second Law of Motion.

Mechanical Energy Equation
By dotting the equation of motion by the velocity we obtain

$$\frac{\partial}{\partial t} \left(\rho_k \frac{v_k^2}{2} \right) + \nabla \cdot \left(\rho_k \frac{v_k^2}{2} \boldsymbol{v}_k \right)$$
$$= -\boldsymbol{v}_k \cdot \nabla p_k + \boldsymbol{v}_k \cdot \left(\nabla \cdot \mathcal{T}_k \right) + \rho_k \boldsymbol{v}_k \cdot \boldsymbol{g}_k. \quad (2\text{-}16)$$

For a symmetrical stress tensor

$$\mathcal{T}_k : \nabla \boldsymbol{v}_k \equiv \left(\mathcal{T}_k \cdot \nabla \right) \cdot \boldsymbol{v}_k = \nabla \cdot \left(\mathcal{T}_k \cdot \boldsymbol{v}_k \right) - \boldsymbol{v}_k \cdot \left(\nabla \cdot \mathcal{T}_k \right). \quad (2\text{-}17)$$

Thus, Eq.(2-16) may be written as

$$\frac{\partial}{\partial t}\left(\rho_k \frac{v_k^2}{2}\right) + \nabla \cdot \left(\rho_k \frac{v_k^2}{2} \boldsymbol{v}_k\right)$$

$$= -\boldsymbol{v}_k \cdot \nabla p_k + \nabla \cdot (\mathcal{T}_k \cdot \boldsymbol{v}_k) - \mathcal{T}_k : \nabla \boldsymbol{v}_k + \rho_k \boldsymbol{v}_k \cdot \boldsymbol{g}_k.$$

(2-18)

This mechanical energy equation is a scalar equation, therefore it represents only some part of the physical law concerning the fluid motion governed by the momentum equation.

Internal Energy Equation

By subtracting the mechanical energy equation from the total energy equation, we obtain the internal energy equation

$$\frac{\partial \rho_k u_k}{\partial t} + \nabla \cdot (\rho_k u_k \boldsymbol{v}_k) = -\nabla \cdot \boldsymbol{q}_k - p_k \nabla \cdot \boldsymbol{v}_k + \mathcal{T}_k : \nabla \boldsymbol{v}_k + \dot{q}_k. \quad (2\text{-}19)$$

Enthalpy Equation

By introducing the enthalpy defined by

$$i_k \equiv u_k + \frac{p_k}{\rho_k}$$

(2-20)

the enthalpy energy equation can be obtained as

$$\frac{\partial \rho_k i_k}{\partial t} + \nabla \cdot (\rho_k i_k \boldsymbol{v}_k) = -\nabla \cdot \boldsymbol{q}_k + \frac{D_k p_k}{Dt} + \mathcal{T}_k : \nabla \boldsymbol{v}_k + \dot{q}_k. \quad (2\text{-}21)$$

1.1.3   Entropy inequality and principle of constitutive law

The constitutive laws are constructed on three different bases. The *entropy inequality* can be considered as a restriction on the constitutive laws, and it should be satisfied by the proper constitutive equations regardless of the material responses. Apart from the entropy inequality there is an important group of *constitutive axioms* that idealize in general terms the responses and behaviors of all the materials included in the theory. The principles of determinism and local action are frequently used in the continuum mechanics.

The above two bases of the constitutive laws define the general forms of the constitutive equations permitted in the theory. The third base of the constitutive laws is the *mathematical modeling* of material responses of a

certain group of fluids based on the experimental observations. Using these three bases, we obtain specific constitutive equations that can be used to solve the field equations. It is evident that the balance equations and the proper constitutive equations should form a mathematically closed set of equations.

Now we proceed to the discussion of the entropy inequality. In order to state the second law of thermodynamics, it is necessary to introduce the concept of a temperature T_k and of the specific entropy s_k. With these variables the second law can be written as an inequality

$$\frac{d}{dt}\int_{V_m}\rho_k s_k dV + \oint_{A_m}\frac{\boldsymbol{q}_k}{T_k}\cdot\boldsymbol{n}_k dA - \int_{V_m}\frac{\dot{q}_k}{T_k}dV \geq 0. \qquad (2\text{-}22)$$

Assuming the sufficient smoothness on the variables we obtain

$$\frac{\partial}{\partial t}\left(\rho_k s_k\right) + \nabla\cdot\left(\rho_k s_k \boldsymbol{v}_k\right) + \nabla\cdot\left(\frac{\boldsymbol{q}_k}{T_k}\right) - \frac{\dot{q}_k}{T_k} \equiv \Delta_k \geq 0 \qquad (2\text{-}23)$$

where Δ_k is the rate of entropy production per unit volume. In this form it appears that Eq.(2-23) yields no clear physical or mathematical meanings in relation to the conservation equations, since the relations of s_k and T_k to the other dependent variables are not specified. In other words, the constitutive equations are not given yet. The inequality thus can be considered as a restriction on the constitutive laws rather than on the process itself.

As it is evident from the previous section, the number of dependent variables exceed that of the field equations, thus the balance equations of mass, momentum, angular momentum and total energy with proper boundary conditions are insufficient to yield any specific answers. Consequently, it is necessary to supplement them with various constitutive equations that define a certain type of ideal materials. Constitutive equations, thus, can be considered as a mathematical model of a particular group of materials. They are formulated on experimental data characterizing specific behaviors of materials together with postulated principles governing them.

From their physical significances, it is possible to classify various constitutive equations into three groups:

1. Mechanical constitutive equations;
2. Energetic constitutive equations;
3. Constitutive equation of state.

The first group specifies the stress tensor and the body force, whereas the second group supplies the heat flux and the body heating. The last equation gives a relation between the thermodynamic properties such as the entropy, internal energy and density of the fluid with the particle coordinates as a parameter. If it does not depend on the particle, it is called thermodynamically homogenous. It implies that the field consists of same material.

As it has been explained, the derivation of a general form of constitutive laws follows the postulated principles such as the entropy inequality, determinism, frame indifference and local action. The most important of them all is the principle of determinism that roughly states the predictability of a present state from a past history. The principle of material frame-indifference is the realization of the idea that the response of a material is independent of the frame or the observer. And the entropy inequality requires that the constitutive equations should satisfy inequality (2-23) unconditionally. Further restrictions such as the equipresence of the variables are frequently introduced into the constitutive equations for flux, namely, $\mathcal{T}_k$ and q_k.

1.1.4   Constitutive equation

We restrict our attention to particular type of materials and constitutive equations which are most important and widely used in the fluid mechanics.

Fundamental Equation of State
The standard form of the fundamental equation of state for thermodynamically homogeneous fluid is given by a function relating the internal energy to the entropy and the density, hence we have

$$u_k = u_k\left(s_k, \rho_k\right). \tag{2-24}$$

And the temperature and the thermodynamic pressure are given by

$$T_k \equiv \frac{\partial u_k}{\partial s_k}, \quad -p_k \equiv \frac{\partial u_k}{\partial\left(1/\rho_k\right)}. \tag{2-25}$$

Thus in a differential form, the fundamental equation of state becomes

$$du_k = T_k ds_k - p_k d\left(\frac{1}{\rho_k}\right). \tag{2-26}$$

The Gibbs free energy, enthalpy and Helmholtz free energy function are defined by

$$g_k \equiv u_k - T_k s_k + \frac{p_k}{\rho_k} \tag{2-27}$$

$$i_k \equiv u_k + \frac{p_k}{\rho_k} \tag{2-28}$$

$$f_k \equiv u_k - T_k s_k \tag{2-29}$$

respectively. These can be considered as a Legendre transformation[*] (Callen, 1960) which changes independent variables from the original ones to the first derivatives. Thus in our case we have

$$g_k = g_k\left(T_k, p_k\right) \tag{2-30}$$

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[*] If we have

$$y = y\left(x_1, x_2, \cdots, x_n\right); \; P_i \equiv \frac{\partial y}{\partial x_i}$$

then the Legendre transformation is given by

$$Z = y - \sum_{i=1}^{j} P_i x_i$$

and in this case Z becomes

$$Z = Z\left(P_1, P_2, \cdots, P_j, x_{j+1}, \cdots, x_n\right).$$

Thus, we have

$$dZ = -\sum_{i=1}^{j} x_i dP_i + \sum_{i=j+1}^{n} P_i dx_i.$$

$$i_k = i_k\left(s_k, p_k\right) \tag{2-31}$$

$$f_k = f_k\left(T_k, \rho_k\right) \tag{2-32}$$

which are also a fundamental equation of state.

Since the temperature and the pressure are the first order derivatives of u_k of the fundamental equation of state, Eq.(2-24) can be replaced by a combination of *thermal and caloric equations of state* (Bird et al., 1960; Callen, 1960) given by

$$p_k = p_k\left(\rho_k, T_k\right) \tag{2-33}$$

$$u_k = u_k\left(\rho_k, T_k\right). \tag{2-34}$$

The temperature and pressure are easily measurable quantities; therefore, it is more practical to obtain these two equations of state from experiments as well as to use them in the formulation. A simple example of these equations of state is for an incompressible fluid

$$\rho_k = \text{constant}$$
$$u_k = u_k\left(T_k\right). \tag{2-35}$$

And in this case the pressure cannot be defined thermodynamically, thus we use the hydrodynamic pressure which is the average of the normal stress. Furthermore, an ideal gas has the equations of state

$$p_k = R_M T_k \rho_k$$
$$u_k = u_k\left(T_k\right) \tag{2-36}$$

where R_M is the ideal gas constant divided by a molecular weight.

Mechanical Constitutive Equation

The simplest rheological constitutive equation is the one for an inviscid fluid expressed as

$$\mathcal{T}_k = 0. \tag{2-37}$$

For most fluid, Newton's Law of Viscosity apples. The generalized linearly viscous fluid of Navier-Stokes has a constitutive equation (Bird et al., 1960)

$$\mathcal{T}_k = \mu_k \left[\nabla v_k + \left(\nabla v_k \right)^+ \right] - \left(\frac{2}{3} \mu_k - \lambda_k \right) \left(\nabla \cdot v_k \right) I \qquad (2\text{-}38)$$

where μ_k and λ_k are the viscosity and the bulk viscosity of the kth-phase, respectively.

The body forces arise from external force fields and from mutual interaction forces with surrounding bodies or fluid particles. The origins of the forces are Newtonian gravitational, electrostatic, and electromagnetic forces. If the mutual interaction forces are important the body forces may not be considered as a function only of the independent variables x and t. In such a case, the principle of local actions cannot be applied. For most problems, however, these mutual interaction forces can be neglected in comparison with the gravitational field force g. Thus we have

$$g_k = g. \qquad (2\text{-}39)$$

Energetic Constitutive Equation

The contact heat transfer is expressed by the heat flux vector q_k, and its constitutive equation specifies the nature and mechanism of the contact energy transfer. Most fluids obey the generalized Fourier's Law of Heat Conduction having the form

$$q_k = -K_k \cdot \nabla T_k. \qquad (2\text{-}40)$$

The second order tensor K_k is the conductivity tensor which takes account for the anisotropy of the material. For an isotropic fluid the constitutive law can be expressed by a single coefficient as

$$q_k = -K_k \left(T_k \right) \nabla T_k. \qquad (2\text{-}41)$$

This is the standard form of Fourier's Law of Heat Conduction and the scalar K_k is called the thermal conductivity.

The body heating $\dot{q}_k$ arises from external energy sources and from mutual interactions. Energy can be generated by nuclear fission and can be transferred from distance by radiation, electric conduction and magnetic induction. The mutual interaction or transfer of energy is best exemplified by the mutual radiation between two parts of the fluid. In most cases these

interaction terms are negligibly small in comparison with the contact heating. The radiation heat transfer becomes increasingly important at elevated temperature and in that case the effects are not local. If the radiation effects are negligible and the nuclear, electric or magnetic heating are absent, then the constitutive law for body heating is simply

$$\dot{q}_k = 0 \tag{2-42}$$

which can be used in a wide range of practical problems.

Finally, we note that the entropy inequality requires the transport coefficients μ_k, λ_k and K_k to be non-negative. Thus, viscous stress works as a resistance of fluid motions and it does not give out work. Furthermore, the heat flows only in the direction of higher to lower temperatures.

1.2   Interfacial balance and boundary condition

1.2.1   Interfacial balance (Jump condition)

The standard differential balance equations derived in the previous sections can be applied to each phase up to an interface, but not across it. A particular form of the balance equation should be used at an interface in order to take into account the singular characteristics, namely, the sharp changes (or discontinuities) in various variables. By considering the interface as a singular surface across which the fluid density, energy and velocity suffer jump discontinuities, the so-called jump conditions have been developed. These conditions specify the exchanges of mass, momentum, and energy through the interface and stand as matching conditions between two phases, thus they are indispensable in two-phase flow analyses. Furthermore since a solid boundary in a single-phase flow problem also constitutes an interface, various simplified forms of the jump conditions are in frequent use without much notice. Because of its importances, we discuss in detail the derivation and physical significance of the jump conditions.

The interfacial jump conditions without any surface properties were first put into general form by Kotchine (1926) as the dynamical compatibility condition at shock discontinuities, though special cases had been developed earlier by various authors. It can be derived from the integral balance equation by assuming that it holds for a material volume with a surface of discontinuity. Various authors (Scriven, 1960; Slattery, 1964; Standart, 1964; Delhaye, 1968; Kelly, 1964) have attempted to extend the Kotchine's theorem. These include the introduction of interfacial line fluxes such as the surface tension, viscous stress and heat flux or of surface material properties. There are several approaches to the problem and the results of the above

authors are not in complete agreement. The detailed discussion on this subject as well as a comprehensive analysis which shows the origins of various discrepancies among previous studies have been presented by Delhaye (1974). A particular emphasis is directed there to the correct form of the energy jump condition and of the interfacial entropy production.

Since it will be convenient to consider a finite thickness interface in applying time average to two-phase flow fields, we derive a general interfacial balance equation based on the control volume analyses. Suppose the position of an interface is given by a mathematical surface $f(x,t) = 0$. The effect of the interface on the physical variables is limited only to the neighborhood of the surface, and the domain of influence is given by a thin layer of thickness δ with δ_1 and δ_2 at each side of the surface. Let's denote the simple connected region on the surface by A_i, then the control volume is bounded by a surface Σ_i which is normal to A_i and the intersection of A_i and Σ_i is a closed curve C_i. Thus Σ_i forms a ring with a width δ, whereas the boundaries of the interfacial region at each side are denoted by A_1 and A_2. Our control volume V_i is formed by Σ_i, A_1 and A_2.

Since the magnitude of δ is assumed to be much smaller than the characteristic dimension along the surface A_i, we put

$$\boldsymbol{n}_1 = -\boldsymbol{n}_2 \qquad (2\text{-}43)$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the outward unit normal vectors from the bulk fluid of phase 1 and 2, respectively. The outward unit vector normal to Σ_i is denoted by $\boldsymbol{N}$, then the extended general integral balance equation for the control volume V_i is given by

$$\frac{d}{dt}\int_{V_i} \rho\psi\, dV = \sum_{k=1}^{2}\int_{A_k} \boldsymbol{n}_k \cdot \left[(\boldsymbol{v}_k - \boldsymbol{v}_i)\rho_k\psi_k + \boldsymbol{J}_k\right] dA$$
$$-\int_{C_i}\int_{-\delta_2}^{\delta_1} \boldsymbol{N}\cdot\left[(\boldsymbol{v} - \boldsymbol{v}_i)\rho\psi + \boldsymbol{J}\right] d\delta\, dC + \int_{V_i}\rho\phi\, dV. \qquad (2\text{-}44)$$

The first two integrals on the right-hand side take account for the fluxes from the surface A_1, A_2 and Σ_i. In order to reduce the volume integral balance to a surface integral balance over A_i, we should introduce surface properties defined below.

The surface mean particle velocity $\boldsymbol{v}_s$ is given by

$$\rho_s \boldsymbol{v}_s \delta \equiv \int_{-\delta_2}^{\delta_1} \rho\, \boldsymbol{v}\, d\delta \qquad (2\text{-}45)$$
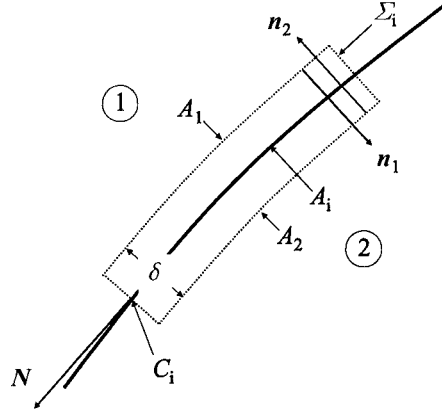
Figure 2-1. Interface (Ishii, 1975)

where the mean density ρ_s and the mean density per unit surface area ρ_a are defined as

$$\rho_a = \rho_s \delta \equiv \int_{-\delta_2}^{\delta_1} \rho \, d\delta. \qquad (2\text{-}46)$$

Then the weighted mean values of ψ and ϕ are given by

$$\rho_a \psi_s \equiv \int_{-\delta_2}^{\delta_1} \rho \psi \, d\delta \qquad (2\text{-}47)$$

and

$$\rho_a \phi_s \equiv \int_{-\delta_2}^{\delta_1} \rho \phi \, d\delta. \qquad (2\text{-}48)$$

The notation here is such that a quantity per unit interface mass and per unit surface area is denoted by the subscript s and a, respectively.

The control surface velocity can be split into the tangential and normal components, thus

$$v_i = v_{ti} + v_{ni} \qquad (2\text{-}49)$$

where

$$v_{ti} = v_{ts}$$

$$v_i \cdot n = -\frac{\frac{\partial f}{\partial t}}{|\nabla f|}. \tag{2-50}$$

Hence the normal component is the surface displacement velocity and the tangential component is given by the mean tangential particle velocity v_{ts}. Since the unit vector N is in the tangential plane and normal to C_i, we have

$$N \cdot v_i = N \cdot v_s. \tag{2-51}$$

Thus, from Eqs.(2-45) and (2-51) we obtain

$$\int_{-\delta_2}^{\delta_1} \rho N \cdot (v_i - v) d\delta = 0 \tag{2-52}$$

and

$$\int_{-\delta_2}^{\delta_1} \rho\psi N \cdot (v_i - v) d\delta = \int_{-\delta_2}^{\delta_1} \rho\psi N \cdot (v_s - v) d\delta. \tag{2-53}$$

In view of Eqs.(2-44) and (2-53) we define the average line efflux along C_i by

$$\mathcal{J}_a \equiv \int_{-\delta_2}^{\delta_1} \left\{ \mathcal{J} - (v_s - v)\rho\psi \right\} d\delta. \tag{2-54}$$

Using the above definitions the integral balance at the interfacial region becomes

$$\frac{d}{dt} \int_{A_i} \rho_a \psi_s \, dA$$

$$= \sum_{k=1}^{2} \int_{A_k} n_k \cdot \left[(v_k - v_i)\rho_k\psi_k + \mathcal{J}_k \right] dA - \int_{C_i} N \cdot \mathcal{J}_a \, dC \tag{2-55}$$

$$+ \int_{A_i} \rho_a \phi_s \, dA.$$

As in the case for the derivation of the field equation, here we need two mathematical transformations, namely, the surface transport theorem and the surface Green's theorem (Weatherburn, 1927; McConnell, 1957; Aris 1962). The surface transport theorem is given by

$$\frac{d}{dt}\int_{A_i} F\, dA = \int_{A_i} \left\{ \frac{d_s}{dt}(F) + F\nabla_s \cdot v_i \right\} dA \qquad (2\text{-}56)$$

where d_s/dt denotes the convective derivative with the surface velocity v_i defined by Eq.(2-50), and ∇_s denotes the surface divergence operator. The surface Green's theorem is given by

$$\int_{C_i} N \cdot J_a\, dC = \int_{A_i} A^{\alpha\beta} g_{\ln} \left(t_\alpha^n J_a^{l\cdot} \right)_{,\beta} dA. \qquad (2\text{-}57)$$

Here, $A^{\alpha\beta}$, $g_{\ln}$, t_α^n and $(\quad)_{,\beta}$ denote the surface metric tensor, the space metric tensor, the hybrid tensor, and the surface covariant derivative, respectively (Aris, 1962).

The surface flux, J_a in space coordinates is expressed by $J_a^{l\cdot}$ which represents the space vector for mass and energy balance and the space tensor for momentum balance. The essential concepts of the tensor symbols are given below. First the Cartesian space coordinates are denoted by $\left(y_1, y_2, y_3\right)$ and a general coordinates by $\left(x_1, x_2, x_3\right)$, then the space metric tensor is defined by

$$g_{\ln} \equiv \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^l} \frac{\partial y^k}{\partial x^n} \qquad (2\text{-}58)$$

which relates the distance of the infinitesimal coordinate element between these two systems. As shown in Fig.2-2, if the Cartesian coordinates y^k give a point of a surface with the surface coordinates of $\left(u^1, u^2\right)$ as $y^k = y^k\left(u^1, u^2\right)$, then the surface metric tensor is defined by

$$A^{\alpha\beta} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial u^\alpha} \frac{\partial y^k}{\partial u^\beta} \qquad (2\text{-}59)$$
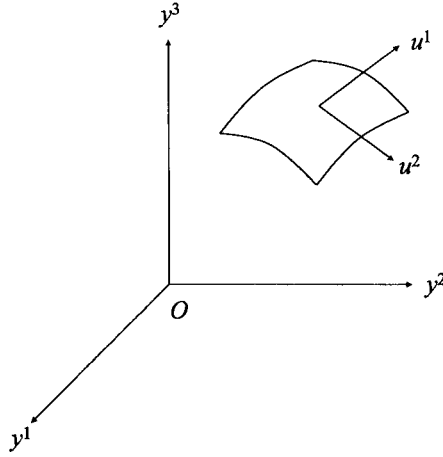
and the small distance ds is given by

Figure 2-2. Relationship between Cartesian coordinates and surface coordinates

$$(ds)^2 = \left(dy^1\right)^2 + \left(dy^2\right)^2 + \left(dy^3\right)^2 = A^{\alpha\beta} du^\alpha du^\beta. \tag{2-60}$$

By introducing the general space coordinates, the surface position is given by $x^i = x^i\left(u^1, u^2\right)$. The hybrid tensor is then defined by

$$t_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}. \tag{2-61}$$

The covariance surface derivative $\left(\quad\right)_{,\beta}$ is similar to the space derivative but it also takes into account for the curved coordinate effects. Furthermore, if $\boldsymbol{N} \cdot \mathcal{J}_a$ has only a tangential component as in the case of surface tension force, $A^{\alpha\beta} g_{\ln} t_\alpha^n \mathcal{J}_a^{lm} = t_\alpha^m \mathcal{J}_a^{\alpha\beta}$. Hence, the surface flux contribution can be written as $\left(t_\alpha^m \mathcal{J}_a^{\alpha\beta}\right)_{,\beta}$ or $\left(\boldsymbol{t}_\alpha \mathcal{J}_a^{\alpha\beta}\right)_{,\beta}$ where $\boldsymbol{t}_\alpha$ denotes the hybrid tensor in vector notation. It is noted that for the momentum transfer, the dominant interfacial momentum flux is the isotropic surface tension σ . Then, $\mathcal{J}_a^{\alpha\beta} = \sigma A^{\alpha\beta}$. In this case, the surface flux contribution becomes as follows

$$\left(\boldsymbol{t}_\alpha \sigma A^{\alpha\beta}\right)_{,\beta} = 2H\sigma\boldsymbol{n} + \boldsymbol{t}_\alpha A^{\alpha\beta}\left(\sigma\right)_{,\beta}. \tag{2-62}$$

The first term represents the net effect of the curved surface and gives the normal component force with the mean curvature H, whereas the second term represents the tangential force due to surface tension gradient.

Since we assumed that δ is sufficiently small, the surface A_1 and A_2 coincide with A_i geometrically. Thus, Eq.(2-55) reduces to

$$\int_{A_i} \left\{ \frac{d_s}{dt}(\rho_a \psi_s) + \rho_a \psi_s \nabla_s \cdot \boldsymbol{v}_i \right\} dA$$

$$= \int_{A_i} \left\{ \sum_{k=1}^{2} \left[\rho_k \psi_k \boldsymbol{n}_k \cdot (\boldsymbol{v}_k - \boldsymbol{v}_i) + \boldsymbol{n}_k \cdot \boldsymbol{J}_k \right] \right. \tag{2-63}$$

$$\left. - A^{\alpha\beta} g_{\ln} \left(t_\alpha^n J_a^{l\cdot} \right)_{,\beta} + \rho_a \phi_s \right\} dA.$$

This balance equation holds for any arbitrary portion of an interface with $A_i \gg \delta^2$, thus we obtain a differential balance equation

$$\frac{d_s}{dt}(\rho_a \psi_s) + \rho_a \psi_s \nabla_s \cdot \boldsymbol{v}_i$$

$$= \sum_{k=1}^{2} \left\{ \rho_k \psi_k \boldsymbol{n}_k \cdot (\boldsymbol{v}_k - \boldsymbol{v}_i) + \boldsymbol{n}_k \cdot \boldsymbol{J}_k \right\} \tag{2-64}$$

$$- A^{\alpha\beta} g_{\ln} \left(t_\alpha^n J_a^{l\cdot} \right)_{,\beta} + \rho_a \phi_s.$$

We note here this result has exactly the same form as the one derived by Delhaye (1974), although the method used and the definition of the surface velocity $\boldsymbol{v}_i$ is different. Let's define a surface quantity and a source per surface area as

$$\psi_a \equiv \rho_a \psi_s \tag{2-65}$$

and

$$\phi_a \equiv \rho_a \phi_s. \tag{2-66}$$

Then the surface balance equation becomes

$$\frac{d_s}{dt}(\psi_a) + \psi_a \nabla_s \cdot \boldsymbol{v}_i = \sum_{k=1}^{2} \left\{ \rho_k \psi_k \boldsymbol{n}_k \cdot (\boldsymbol{v}_k - \boldsymbol{v}_i) + \boldsymbol{n}_k \cdot \boldsymbol{J}_k \right\}$$

$$- A^{\alpha\beta} g_{\ln} \left(t_\alpha^n J_a^{l\cdot} \right)_{,\beta} + \phi_a. \tag{2-67}$$

The left-hand side represents the time rate of change of ψ_a from the observer moving at v_i, plus the effect of the surface dilatation. Whereas the three terms on the right-hand side are the fluxes from the bulk phases, the line flux along the surface, and the surface source respectively. We note that Eq.(2-6) and Eq.(2-67) govern the physical laws in the bulk phases and at an interface.

In order to obtain a simpler expression for interfacial jump of quantities, we make further assumptions which are consistent with our thin layer assumption given by

$$\delta^2 << A_i. \tag{2-68}$$

First the mass density of interface ρ_a is negligibly small so that its momentum and mechanical energy can also be neglected. Secondly, all the molecular diffusion fluxes along the line are neglected, namely, no surface viscous stress or surface heat flux. Furthermore all the surface sources are neglected, namely, no particular body force other than the gravity and no radiation effect.

The thermodynamic tension and hence the interfacial energy are *included* in the following analysis, consequently from the principle of determinism we should postulate the existence of the surface equation of state. Under these assumptions we obtain

Interfacial Mass Balance

$$\sum_{k=1}^{2} \rho_k \boldsymbol{n}_k \cdot \left(\boldsymbol{v}_k - \boldsymbol{v}_i\right) = 0. \tag{2-69}$$

By defining the interfacial mass efflux from the kth-phase as

$$\dot{m}_k \equiv \rho_k \boldsymbol{n}_k \cdot \left(\boldsymbol{v}_k - \boldsymbol{v}_i\right) \tag{2-70}$$

we have from Eq.(2-69)

$$\sum_{k=1}^{2} \dot{m}_k = 0. \tag{2-71}$$

This equation simply states that there is no capacity of mass at the interface, hence phase changes are pure exchanges of mass between the two phases.

Interfacial Momentum Balance

$$\sum_{k=1}^{2}\left\{\rho_k \boldsymbol{n}_k \cdot \left(\boldsymbol{v}_k - \boldsymbol{v}_i\right)\boldsymbol{v}_k - \boldsymbol{n}_k \cdot \boldsymbol{T}_k\right\} + \left(\boldsymbol{t}_\alpha A^{\alpha\beta}\sigma\right)_{,\beta} = 0. \qquad (2\text{-}72)$$

Equation (2-72) is a balance between the momentum fluxes from the bulk fluids and the interfacial tension.

Interfacial Energy Balance
Substituting the interfacial energy u_a per unit surface area for ψ_a, we obtain from Eq.(2-67)

$$\frac{d_s u_a}{dt} + u_a \nabla_s \cdot \boldsymbol{v}_i$$

$$= \sum_{k=1}^{2}\left[\rho_k \boldsymbol{n}_k \cdot \left(\boldsymbol{v}_k - \boldsymbol{v}_i\right)\left(u_k + \frac{v_k^2}{2}\right) + \boldsymbol{n}_k \cdot \left(-\boldsymbol{T}_k \cdot \boldsymbol{v}_k + \boldsymbol{q}_k\right)\right] \qquad (2\text{-}73)$$

$$+ \left(\boldsymbol{t}_\alpha A^{\alpha\beta}\sigma \cdot \boldsymbol{v}_i\right)_{,\beta}.$$

The left-hand side represents the rate change of the surface energy, whereas the right-hand side accounts for the energy transfer from the bulk at each side and for the work done by the surface tension.

1.2.2   Boundary conditions at interface

As in the case of the three-dimensional field equations the surface balance equations should be supplemented by various constitutive laws. In order to establish the principle of determinism, first we introduce a simple equation of state. Since the mass of interface is negligible, we have

$$u_a = u_a\left(s_a\right) \qquad (2\text{-}74)$$

where u_a and s_a are the specific internal energy and the specific entropy per unit surface area, respectively.
The thermodynamic tension is given by

$$\sigma \equiv -T_i s_a + u_a \qquad (2\text{-}75)$$

where the temperature T_i is defined by

$$T_i \equiv \frac{du_a}{ds_a}.$$

(2-76)

Thus, in a differential form, Eq.(2-74) becomes

$$du_a = T_i ds_a$$

(2-77)

and the Gibbs-Duhem relation is given by

$$s_a dT_i + d\sigma = 0.$$

(2-78)

The interfacial enthalpy is defined by

$$i_a = u_a - \sigma.$$

(2-79)

From Eq.(2-78) we have

$$\frac{d\sigma}{dT_i} = -s_a.$$

(2-80)

Hence, from Eqs.(2-77) and (2-80) we obtain

$$du_a = -T_i d\left(\frac{d\sigma}{dT_i}\right).$$

(2-81)

By combining Eqs.(2-75), (2-79) and (2-80) we get

$$u_a = -T_i\left(\frac{d\sigma}{dT_i}\right) + \sigma; \ i_a = -T_i\left(\frac{d\sigma}{dT_i}\right).$$

(2-82)

Thus the thermal equation of state

$$\sigma = \sigma\left(T_i\right)$$

(2-83)

supplies all the necessary information to interrelate the thermodynamic properties. By substituting Eq.(2-81) into Eq.(2-73) we obtain an energy jump condition in terms of the surface tension as

$$-T_i \left\{ \frac{d_s}{dt}\left(\frac{d\sigma}{dT_i}\right) + \left(\frac{d\sigma}{dT_i}\right)\nabla_s \cdot \boldsymbol{v}_i \right\}$$

$$= \left(\boldsymbol{t}_\alpha A^{\alpha\beta}\sigma\right)_{,\beta}\cdot\boldsymbol{v}_i \tag{2-84}$$

$$+ \sum_{k=1}^{2}\left\{\dot{m}_k\left(u_k + \frac{v_k^2}{2}\right) + \boldsymbol{n}_k\cdot\left(-\mathcal{T}_k\cdot\boldsymbol{v}_k + \boldsymbol{q}_k\right)\right\}.$$

Interfacial Entropy Inequality

Following the above discussion, we assume the existence of the surface temperature T_i which enables us to write an entropy inequality at the interface. Thus, in the absence of surface heat flux and source terms, we have

$$\Delta_a = \frac{d_s s_a}{dt} + s_a\nabla_s\cdot\boldsymbol{v}_i - \sum_{k=1}^{2}\left(\dot{m}_k s_k + \frac{\boldsymbol{n}_k\cdot\boldsymbol{q}_k}{T_k}\right) \geq 0. \tag{2-85}$$

The entropy s_a in above inequality can be eliminated by using the energy balance equation, Eq.(2-73), and the equation of state, Eq.(2-77), hence we obtain

$$T_i\Delta_a = \sum_{k=1}^{2}\left\{\dot{m}_k\left[u_k - s_k T_i + \frac{|\boldsymbol{v}_k - \boldsymbol{v}_i|^2}{2} + \frac{p_k}{\rho_k}\right]\right.$$

$$\left. -\boldsymbol{n}_k\cdot\mathcal{T}_k\cdot\left(\boldsymbol{v}_k - \boldsymbol{v}_i\right) + \boldsymbol{n}_k\cdot\boldsymbol{q}_k\left(1 - \frac{T_i}{T_k}\right)\right\} \geq 0. \tag{2-86}$$

We note here that this expression has the same form as the one obtained by Delhaye (1974). Also a similar result was derived by Standart (1968) without considering the surface properties and the surface tension term, but including the effect of chemical reactions.

In general, the interfacial jump conditions, Eqs.(2-69), (2-72) and (2-84), do not constitute sufficient matching conditions which are necessary to define the problem uniquely. Consequently, they should be supplemented by various boundary conditions that restrict the kinematical, dynamical and thermal relations between two phases. These relations can also be considered as interfacial constitutive laws, satisfying the restriction imposed by the entropy inequality (2-86). They may be obtained from the standard

argument of the irreversible thermodynamics. In order to do so, first suitable combinations of fluxes and potentials should be postulated in the inequality (2-86), and then the fluxes were expanded linearly in terms of the potentials. Here, the principle of equipresence and the symmetric relations between the expansion coefficients are normally used. The standard procedure for a general system is discussed in detail by De Groot and Mazur (1962) among others, and it has been applied to an interface by Standart (1968), and Bornhorst and Hatsopoulos (1967). Standart based his argument on the correct jump conditions and the entropy inequality and obtained the interfacial constitutive laws with great care, though he neglected from the beginning all the surface properties and the surface tension that are generally important in a two-phase system. The results of Bornhorst are limited to particular cases and the argument is based on the classical thermodynamic tools of piston, reservoir, homogeneous system, etc.

The analysis based on the constitutive laws of the interface may be important for a detailed study of a two-phase system. However, they are generally too complicated to apply as boundary conditions. Furthermore, the effects of the potentials, namely, the discontinuities of temperature, chemical potential, tangential velocity, etc., as driving forces of transfer of quantities, or resulting interfacial resistances to heat, momentum and mass transfer are relatively insignificant in the total system.

Consequently a much simpler theory for providing the necessary boundary conditions is desirable. As a limiting case, it is possible to consider the case when entropy production of the interface Δ_a becomes zero. This means that there are no resistances to interfacial transfer of quantities. Hence, the exchanges between two phases are governed by the conditions of the bulk fluid at each side, but not by the interface itself. Furthermore, from the classical thermodynamic point of view, the transfer at the interface is said to be reversible. This is not so for a shock discontinuity in a single-phase flow.

By setting the entropy production of Eq.(2-86) to be zero we obtain

$$
\sum_{k=1}^{2} \frac{\dot{m}_k}{T_i} \left(g_k + \frac{|\boldsymbol{v}_k - \boldsymbol{v}_i|^2}{2} - \frac{T_{nnk}}{\rho_k} \right) - \sum_{k} \frac{\boldsymbol{\tau}_{tk}}{T_i} \cdot \left(\boldsymbol{v}_{tk} - \boldsymbol{v}_{ti} \right)
$$
$$
+ \sum_{k=1}^{2} \left(\boldsymbol{n}_k \cdot \boldsymbol{q}_k + \dot{m}_k s_k T_k \right) \left(\frac{1}{T_i} - \frac{1}{T_k} \right) = 0.
$$

(2-87)

Moreover, we assume that the three terms in Eq.(2-87) are independently zero for all combinations of the mass flux, the tangential stresses and the heat fluxes.

Thermal Boundary Condition

Thus, from the last term of Eq.(2-87), we obtain a thermal equilibrium condition at the interface

$$T_{1i} = T_{2i} = T_i \tag{2-88}$$

that is consistent with the assumption of the existence of the equation of state at the interface, Eqs.(2-74) and (2-83). In view of Eqs.(2-82) and (2-84) this thermal boundary condition sets the energy level of the interface. In contrast to the above equation, the energy jump condition, Eq.(2-73), specifies the relation between the energy transfers to the interface. Furthermore, the thermal equilibrium condition, Eq.(2-88), eliminates a variable T_i, and it stands as a matching condition for the temperature of each phase at the interface. We note here that, in reality, the discontinuity of the temperature at the interface exists and can be estimated from the kinetic theories (Hirschfelder et al., 1954). However, its value in comparison with the absolute temperature is very small for most materials with few exceptions, such as for liquid metals (Brodkey, 1971). Thus, the influence on the interfacial transfer is negligible under the standard conditions.

No-Slip Condition

In view of the definition of the interfacial surface velocity v_i, Eq.(2-50), the tangential velocity v_{ti} is an unknown parameter, whereas the normal component is directly related to the position of the interface. Furthermore, it appears in the dissipation term in the entropy inequality (2-86) and Eq.(2-87). Thus, it is natural to supply a constitutive relation between the tangential stress τ_{tk} and the tangential relative velocity $v_{tk} - v_{ti}$, as it has been discussed previously. However, in the present analysis we have assumed that the interfacial entropy production is identically zero. By taking the second term of Eq.(2-87) to be zero independently, we obtain a no-slip condition

$$v_{t1} = v_{t2} = v_{ti}. \tag{2-89}$$

The no-slip condition for a moving viscous fluid in contact with a solid wall is well established (Goldstein, 1938; Serrin, 1959). It is called a classical adherence condition and it has been verified experimentally and also analytically from kinetic theories. The relation given by Eq.(2-89) can be used to eliminate the interfacial tangential particle velocity and then it can be utilized as a velocity boundary condition at an interface.

However, it should be noted here that for an inviscid fluid the no-slip condition (2-89) is not necessary and cannot be satisfied generally, due to the tangential component of the momentum jump condition, Eq.(2-72). This is in complete agreement with our analysis, since the viscous dissipation term in Eq.(2-87) is identically zero for an inviscid fluid and does not appear in the entropy inequality. Consequently, Eq.(2-89) cannot be obtained. Furthermore, under the condition of no-slip, the momentum jump condition, Eq.(2-72), in the tangential and the normal directions becomes

$$\sum_{k=1}^{2} \tau_{tk} = A^{\alpha\beta} t_\beta \left(\sigma \right)_{,\alpha} \tag{2-90}$$

and

$$\sum_{k=1}^{2} \left(n_k \frac{\dot{m}_k^2}{\rho_k} + n_k p_k - \tau_{nk} \right) = -2H_{21} n_1 \sigma \tag{2-91}$$

where the normal and the tangential viscous stress is given by

$$n_k \cdot \mathcal{T}_k = \tau_{nk} + \tau_{tk} = n_k \tau_{nnk} + \tau_{tk}. \tag{2-92}$$

And the mean curvature H_{21} is taken from phase 2 to 1, namely, $H_{21} > 0$ if the interface makes a convex surface in phase 1.

Chemical (Phase Change) Boundary Condition

In analogy with the preceding discussion, the chemical (or phase change) boundary condition can be obtained by setting the first term of Eq.(2-87) to be independently zero for all values of $\dot{m}_k$. This implies that the entropy production due to a phase transition is zero, and hence the phase change is considered not as a transfer due to non-equilibrium forces, but rather as an equilibrium transformation of state.

Substituting the thermal equilibrium condition, Eq.(2-88), into the first term of Eq.(2-87) and equating it to zero, we obtain

$$\left(g_1 - g_2 \right) = \left(\frac{\left| v_2 - v_i \right|^2}{2} - \frac{\left| v_1 - v_i \right|^2}{2} \right) - \left(\frac{\tau_{nn2}}{\rho_2} - \frac{\tau_{nn1}}{\rho_1} \right). \tag{2-93}$$

The phase change condition given by the above equation shows that the difference in the chemical potential compensates for the mechanical effects

of the relative kinetic energy difference and of the normal stresses. Here it should be noted that this phase change condition is only applicable to the case when the transfer of mass across the interface is possible. In other words, if the transfer of mass is identically zero for all conditions as in the case of two immiscible non-reacting liquids, the boundary condition should be

$$\dot{m}_k = 0 \tag{2-94}$$

which replaces the condition on the chemical potentials.

1.2.3   Simplified boundary condition

In the preceding sections the interfacial jump conditions and supplementary boundary conditions have been given. It is important to realize that the thermal equilibrium condition, Eq.(2-88), normal component of the momentum jump condition, Eq.(2-91), and the phase change boundary condition, Eq.(2-93), correspond to the standard thermal, mechanical and chemical equilibrium conditions of the thermostatics (Gibbs, 1948). The difference is that the present analysis takes into account the dynamic effects of mass transfer and of the normal stresses in the mechanical and phase change boundary conditions. These interesting properties between the results of dynamical analysis and of the thermostatic theory can be summarized in the following table.

It can be seen from the table that except the thermal condition these interfacial relations are still very complicated for many practical applications. This is mainly due to the terms arisen from the mass transfer and from the normal stresses. The former contributes as a thrust force due to the density change in the mechanical boundary condition and also as an impact kinetic energy change in the chemical (phase) boundary condition. The latter introduces complicated coupling effects of the flow fields with the thermodynamic properties at the interface. Under standard conditions, however, the normal stresses may be neglected with respect to the pressure terms, which greatly simplify the mechanical boundary condition, Eq.(2-91). The same argument can be applied to the chemical boundary condition, since the order of magnitude of the term $\rho_k g_k$ is p_k, thus the normal stress terms can be neglected also in Eq.(2-93). Similarly the mass transfer terms are negligibly small in most practical problems, though they can be important for problems with large mass transfer rate or with vapor film boiling.

Since in the standard formulation of field equations the Gibbs free energy g_k does not appear explicitly, it is desirable to transform the variable g_k in the chemical boundary condition, Eq.(2-93), into other variables which have

Table 2-1. Interfacial relations of thermodynamic potentials (Ishii, 1975)

Analysis / Condition	Thermostatics	Present Dynamical Analysis
Thermal	$T_1 - T_2 = 0$	$T_1 - T_2 = 0$
Mechanical	$p_1 - p_2 = 0$	$p_1 - p_2 = -2H_{21}\sigma - \dot{m}_1^2 \left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2} \right) + \left(\tau_{nn1} - \tau_{nn2} \right)$
Chemical (phase change)	$g_1 - g_2 = 0$	$g_1 - g_2 = -\dfrac{\dot{m}_1^2}{2} \left(\dfrac{1}{\rho_1^2} - \dfrac{1}{\rho_2^2} \right) + \left(\dfrac{\tau_{nn1}}{\rho_1} - \dfrac{\tau_{nn2}}{\rho_2} \right)$

already been used in the field equations. For this purpose, we recall here that the Gibbs free energy expressed as a function of the temperature and pressure is a fundamental equation of state, Eq.(2-30), thus we have

$$g_k = g_k \left(T_k, p_k \right) \tag{2-95}$$

and

$$dg_k = -s_k \, dT_k + \frac{1}{\rho_k} dp_k. \tag{2-96}$$

The thermostatic phase equilibrium condition is then given by

$$T_1 = T_2 = T^{sat}; \; p_1 = p_2 = p^{sat}; \; \text{and} \; g_1 = g_2. \tag{2-97}$$

Hence from Eqs.(2-95) and (2-97) we obtain

$$g_1 \left(T^{sat}, p^{sat} \right) = g_2 \left(T^{sat}, p^{sat} \right) \tag{2-98}$$

which reduces to the classical saturation condition

$$p^{sat} = p^{sat} \left(T^{sat} \right). \tag{2-99}$$

This relation shows that the thermostatic equilibrium condition uniquely relates the thermodynamic potentials of each phase. Furthermore, the

differential form of Eq.(2-99) known as the Clausius-Clapeyron equation can be obtained from Eqs.(2-27) and (2-28) and Eqs.(2-96) and (2-97)

$$\frac{dp^{sat}}{dT^{sat}} = \frac{i_1 - i_2}{T^{sat}\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right)}$$

(2-100)

where all values of the right-hand side are calculated on the saturation line given by Eq.(2-99).

If we assume that the deviations of the interface pressures of each phase from the saturation pressure corresponding to the interfacial temperature T_i are sufficiently small in comparison with the pressure level, the Gibbs free energy function can be expanded around the static saturation point. Thus we have

$$g_k\left(p^{sat}, T_i\right) \doteq g_k\left(p_k, T_i\right) - \frac{\delta p_k}{\rho_k\left(p_k^{sat}, T_i\right)}$$

(2-101)

where δp_k is defined by

$$\delta p_k \equiv p_k - p^{sat}\left(T_i\right).$$

(2-102)

Since we have

$$g_1\left(p^{sat}\left(T_i\right), T_i\right) = g_2\left(p^{sat}\left(T_i\right), T_i\right).$$

(2-103)

Equation (2-93) can be reduced to

$$\frac{\delta p_1}{\rho_1} - \frac{\delta p_2}{\rho_2} \doteq -\frac{\dot{m}_1^2}{2}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right) + \left(\frac{\tau_{nn1}}{\rho_1} - \frac{\tau_{nn2}}{\rho_2}\right)$$

(2-104)

whereas the mechanical boundary condition, Eq.(2-91), with the definition of δp_k becomes

$$\delta p_1 - \delta p_2 = -2H_{21}\sigma - \dot{m}_1^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \left(\tau_{nn1} - \tau_{nn2}\right).$$

(2-105)

These above two equations can be solved for the pressure deviation from the saturation pressure as

$$\delta p_1 = -2H_{21}\sigma\left(\frac{\rho_1}{\rho_1 - \rho_2}\right) + \frac{\left(\dot{m}_1\right)^2}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + \tau_{nn1}$$

and   (2-106)

$$\delta p_2 = -2H_{21}\sigma\left(\frac{\rho_2}{\rho_1 - \rho_2}\right) + \frac{\left(\dot{m}_1\right)^2}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \tau_{nn2}.$$

This result shows that neither phase is in the saturation condition given by Eq.(2-99). The amount of deviation of pressure from p^{sat} depends on the mean curvature, the surface tension, the mass transfer rate and the normal stress. An interesting result follows if we take into account only the effect of the surface tension and drop the other terms which are generally negligibly small. In this case, we can approximate

$$\delta p_g = 2H_{fg}\sigma\left(\frac{\rho_g}{\rho_f - \rho_g}\right) \text{ and } \delta p_f = 2H_{fg}\sigma\left(\frac{\rho_f}{\rho_f - \rho_g}\right). \qquad (2\text{-}107)$$

Since the mean curvature of the liquid phase H_{fg} is positive for a droplet and negative for a bubble, the phase pressures at the interface are both over the saturation pressure for a droplet flow, and they are both under it for a bubbly flow.

Now we recall the existence of the limits on heating of liquid or cooling of vapor beyond the saturation condition in terms of the pressure deviation at fixed temperature, namely, the instability points of the equation of state in the thermostatics. Thus, we write

$$\delta p_g \leqq \delta p_{g\max}(T_i)$$

                                                                                            (2-108)

$$\delta p_f \geqq \delta p_{f\min}(T_i)$$

which are shown in Fig.2-3.

Figure 2-3 shows the saturation line corresponding to the Clausius-Clapeyron equation or Eq.(2-99) and the limits of the metastable liquid and vapor phases. These two limits can be obtained from the van der Waals equation of state given by
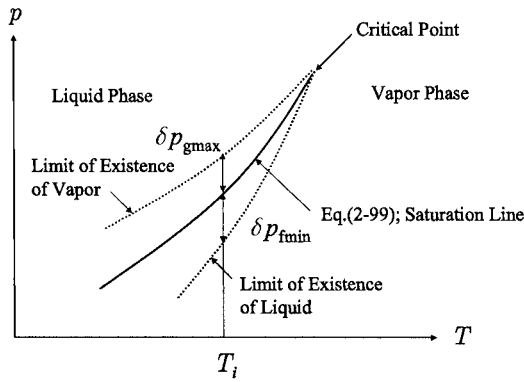
Figure 2-3. p-T diagram (Ishii, 1975)

$$\left\{ p + \frac{a}{\left(M/\rho\right)^2} \right\} \left(\frac{M}{\rho} - b \right) = RT \qquad (2\text{-}109)$$

where R and M are the gas constant and the molecular weight, respectively. a and b are empirical constants. The thermodynamic theory states that the intrinsic thermodynamic stability requires

$$\left. \frac{\partial p}{\partial\left(1/\rho\right)} \right|_T < 0. \qquad (2\text{-}110)$$

Therefore, by using the van der Waals equation, the loci of $\partial p \big/ \partial\left(1/\rho\right) = 0$ can be found. These loci actually represent two limits, namely the superheated liquid limit and subcooled vapor limit. These two loci are shown by the broken curves in Fig.2-3.

It is interesting to note that Eq.(2-107) with the limiting condition of Eq.(2-108) gives the smallest droplet and the bubble sizes. In other words, these sizes are the lowest natural level of the disturbances in the statistical sense. Beyond these limits the liquid or the vapor phase cannot stay without changing the phase, because the statistical fluctuations create a core which can grow to a bubble or a droplet.

The relations given by Eqs.(2-107) and (2-108) at a temperature T_i are exhibited in Fig.2-4. The widely used interfacial condition that the vapor
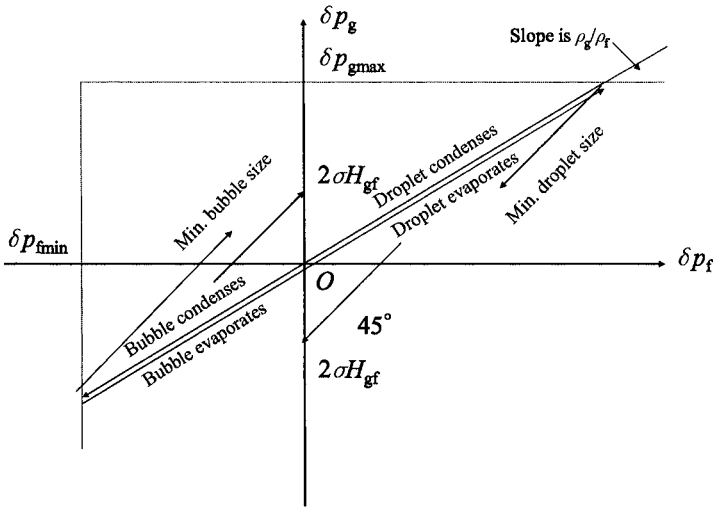
Figure 2-4. δp_g-δp_f relation (Ishii, 1975)

interfacial pressure equals the saturation pressure p^{sat} at a temperature T_i can be derived as a further approximation to Eq.(2-107). Since the density ratio between phases is very large at a small reduce pressure, namely, $p/p_c \ll 1$ where p_c is the critical pressure, Eq.(2-107) can be approximated by

$$\delta p_g \approx 0, \qquad p_g \approx p^{sat}(T_i)$$
$$\delta p_f \approx 2H_{fg}\sigma, \qquad p_f \approx p^{sat}(T_i) + 2H_{fg}\sigma.$$

(2-111)

1.2.4 External boundary condition and contact angle

The external boundary condition is a special case of the jump and the supplemental interfacial boundary conditions which have been discussed in the previous section. For a standard single-phase flow problem, these conditions become particularly simple because the mass transfer rate $\dot{m}_k$, the effect of the surface tension and the velocity of the solid-wall interface are all set to be zero. Similar simplifications could also be applied to a two-phase flow system, however, two exceptional characteristics should be taken into account here. These are:

1. The wall microstructure effect on bubble nucleations;
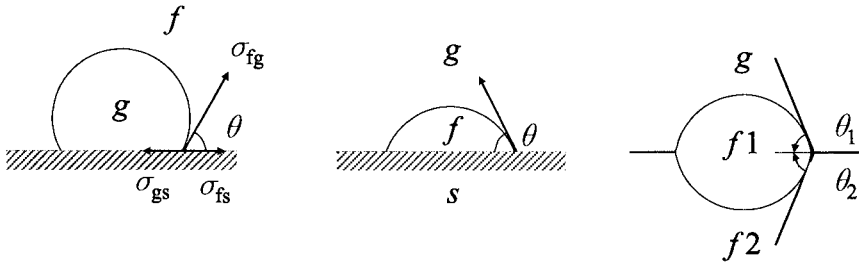2. The intersection of a phase interface with the external boundary.

Figure 2-5. Contact angle (Ishii, 1975)

The first effect characterizes the necessity to consider the existence of surface nucleation sites which have irregular geometries deviating from the standard idealized wall boundary. These microstructures and the gas content in these sites often decide the bubble nucleation conditions and the degree of thermodynamic non-equilibrium. The second case is the singularity created by meeting of two different interfaces, see Fig.2-5. As a bubble or a droplet comes in contact with the external boundary, the vapor-liquid interface attaches to the wall and forms a singular curve at the intersection. When such a contact line is formed, the angle of contact θ measured though the liquid characterizes the condition along the curve. An analysis similar to the one for the interface can be developed also for this singular line. In this case, since the area of transport from the bulk fluids is the thickness of the interface δ, the effects of the mass transfers and of the fluxes of the fluids can be neglected. Hence, only the surface fluxes and possibly the properties associated with the curve, namely, energy of the contact line, are important. By considering only the surface fluxes, we have from the force balance in the normal plane to the singular curve

$$\cos \theta = \frac{\sigma_{gs} - \sigma_{fs}}{\sigma_{fg}} \tag{2-112}$$

where σ_{fg}, σ_{gs} and σ_{fs} denote the surface tension between vapor-liquid, vapor-solid, and liquid-solid respectively.

We note here that Eq.(2-112) is consistent with the jump conditions, if we neglect the tension tangent to the singular curve and thus the thermal energy of the curve. If these effects are neglected, Eq.(2-112) is the only condition obtainable in parallel with the jump conditions. Hence, as it has been mentioned, the contact angle θ characterizes the phenomenon and an appropriate constitutive law should be supplied if σ_{gs} and σ_{fs} are not available. The static contact angle θ is well measured and tabulated for

various interfaces: in reality however it is greatly influenced by the surface roughness, the deposit of foreign materials and the purity of fluid itself.

Furthermore, the dynamic contact angle of a moving interface can be significantly different from the static values. However, in the absence of a well established constitutive law for θ under dynamic condition, the static values are frequently used in practical problems. We only note here that it is generally accepted that the apparent difference between the static and the dynamic contact angle is a function of a surface tension σ_{fg} and the normal slipping velocity of the singular curve (Schwartz and Tejada, 1972; Phillips and Raddiford, 1972).

In summarizing this section we list standard external boundary conditions at the solid wall:

The position of an external boundary

$$f_w(\boldsymbol{x}) = 0 \tag{2-113}$$

No-mass transfer condition

$$v_{nk} = v_{nw} = 0 \tag{2-114}$$

No-slip condition for a viscous fluid

$$v_{tk} = v_{tw} = 0 \tag{2-115}$$

The force balance from the momentum jump condition

$$\boldsymbol{n}_k \cdot \boldsymbol{T}_k + \boldsymbol{n}_w \cdot \boldsymbol{T}_w = 0 \tag{2-116}$$

The energy balance from the energy jump condition

$$\boldsymbol{n}_k \cdot \boldsymbol{q}_k + \boldsymbol{n}_w \cdot \boldsymbol{q}_w = 0 \tag{2-117}$$

The thermal equilibrium condition

$$T_k = T_w \tag{2-118}$$

These above conditions can be applied where a fluid is in contact with the wall. It cannot be applied however at an intersection of an interface with the solid boundary. On such a singular curve the constitutive equation for the contact angle θ should be given. Finally, we summarize the local instant formulation of a two-phase flow system in the following diagram.

```
┌─────────────────────────────────────────────────────────────────┐
│              EXTERNAL BOUNDARY CONDITIONS                         │
│                                                                   │
│   Position of the Wall                    Jump Conditions         │
│   Constraints                             Interfacial B. C.       │
│                                           Contact Angle           │
└─────────────────────────────────────────────────────────────────┘
```

PHASE 1	INTERFACE	PHASE 2
Field Equations	Jump Conditions	Field Equations
Continuity Eq.	Mass J. C.	C. E.
Momentum Eq.	Momentum J. C.	M. E.
Energy Eq.	Energy J. C.	E. E.
Constitutive Equations	Interfacial B. C.	Constitutive Equations
Equation of State	Thermal B. C.	E. S.
Mechanical C. E.	(No-slip B. C.)	M. C. E.
Energetic C. E.	(Chemical B. C.)	E. C. E.

```
┌─────────────────────────────────────────────────────────┐
│                  INITIAL CONDITIONS                       │
└─────────────────────────────────────────────────────────┘
```

1.3   Application of local instant formulation to two-phase flow problems

1.3.1   Drag force acting on a spherical particle in a very slow stream

As an example of applying local instant formulation to two-phase flow problems, let us study the drag constitutive equation of a solid sphere of radius r_d in a very slow stream of speed U_0 (creeping flow) (Stokes, 1851; Schlichting, 1979). In order to analyze this problem analytically, we assume (1) Newtonian viscous fluid with constant viscosity, (2) incompressible flow (fluid density is constant), and (3) very small Reynolds number ($Re\left(\equiv 2r_d\rho_c U_0/\mu_c\right) \ll 1$) where viscous effects dominate the flow and the inertia term can be neglected in the momentum equation. Then, the continuity equation, Eq.(2-8), and the momentum equation, Eq.(2-10), can be linearized as

$$\nabla \cdot v_c = 0 \tag{2-119}$$

$$\nabla p_c = \mu_c \nabla^2 v_c. \tag{2-120}$$

The gravity term is dropped by considering the pressure field which excludes the hydrostatic effect. The velocity components and the pressure in

spherical coordinates (r,θ) with $\theta = 0$ in the direction of U_0 can be derived under the boundary condition of no-slip on the solid sphere as

$$v_{rc} = U_0\cos\theta\left(1 - \frac{3}{2}\frac{r_d}{r} + \frac{1}{2}\frac{r_d^3}{r^3}\right)$$

(2-121)

$$v_{\theta c} = -U_0\cos\theta\left(1 - \frac{3}{4}\frac{r_d}{r} - \frac{1}{4}\frac{r_d^3}{r^3}\right)$$

(2-122)

$$p_c = p_\infty - \frac{3\mu_c r_d U_0}{2r^2}\cos\theta.$$

(2-123)

where p_∞ is the uniform freestream pressure. The shear stress acting on the solid sphere, $\tau_{r\theta c}$, is given by

$$\tau_{r\theta c}\big|_{r=a} = \mu_c\left(\frac{1}{r}\frac{\partial v_{rc}}{\partial\theta} + \frac{\partial v_{rc}}{\partial r}\right)\bigg|_{r=r_d} = \frac{3}{2}\frac{\mu_c U_0}{r_d}\sin\theta.$$

(2-124)

Thus, the total drag force, F_D, acting on the solid sphere is given by integrating the pressure and the shear stress around the surface as

$$F_D = \int_0^\pi \tau_{r\theta c}\sin\theta dA - \int_0^\pi p_c\cos\theta dA$$

$$= 4\pi r_d\mu_c U_0 + 2\pi r_d\mu_c U_0 = 6\pi r_d\mu_c U_0$$

(2-125)

where A is the surface area. This indicates that the drag consists of the pressure and shear forces even in this viscosity dominated flow. Then, we define the drag coefficient, C_D, by

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_c U_0^2 A_p}$$

(2-126)

where A_p is the projected area of a particle. Thus, we have

$$C_D = \frac{24}{Re}. \tag{2-127}$$

This analysis was extended by Rybczynski (1911) and Hadamard (1911) to creeping motion of a spherical fluid particle in an infinite Navier-Stokes fluid (Brodkey, 1967; Soo, 1967). Thus, the total force acting on a fluid particle is given by

$$F_D = 6\pi r_d \mu_c \left(v_{c\infty} - v_d\right) \left[\frac{2\mu_c + 3\mu_d}{3\left(\mu_c + \mu_d\right)} \right]. \tag{2-128}$$

Then, we define the drag coefficient, $C_{D\infty}$, by

$$C_{D\infty} \equiv \frac{F_D}{\frac{1}{2} \rho_c \left(v_{c\infty} - v_d\right) A_p} \tag{2-129}$$

and the particle Reynolds number by

$$Re_d \equiv \frac{2 r_d \rho_c \left(v_{c\infty} - v_d\right)}{\mu_c}. \tag{2-130}$$

It is evident here that $v_{c\infty}$ and v_d are the undisturbed flow velocity and the particle velocity. Thus, we have

$$C_{D\infty} = \frac{24}{Re_d} \left[\frac{2\mu_c + 3\mu_d}{3\left(\mu_c + \mu_d\right)} \right]; Re_d < 1. \tag{2-131}$$

The drag law given by Rybczynski and Hadamard is good up to a Reynolds number of about 1.

1.3.2   Kelvin-Helmholtz instability

As another example of application of local instant formulation to two-phase flow problems, let us study the Kelvin-Helmholtz instability (Helmholtz, 1868; Kelvin; 1871; Lamb, 1945). The Kelvin-Helmholtz instability arises at the interface of two fluid layers of different densities ρ_1 and ρ_2 flowing with average velocities v_1 and v_2 in a horizontal duct. In order to analyze this problem analytically, we assume: (1) inviscid flow

(viscous force is negligible); (2) incompressible flow (fluid density is constant); and (3) irrotational flow. It is convenient to use rectangular coordinates (x, y) where x and y indicate the coordinate in the horizontal direction and the coordinate in the vertical direction measured from the average interface of the two fluid layers, respectively. Then, the velocity components are given in terms of the velocity potential, ϕ_k, as

$$v_{xk} = -\frac{\partial \phi_k}{\partial x}, v_{yk} = -\frac{\partial \phi_k}{\partial y}. \tag{2-132}$$

Thus, the continuity equation, Eq.(2-8), is given in terms of the velocity potential as

$$\frac{\partial^2 \phi_k}{\partial x^2} + \frac{\partial^2 \phi_k}{\partial y^2} = 0 \tag{2-133}$$

and the momentum equation, Eq.(2-10), is given by

$$\frac{p_k}{\rho_k} + \frac{1}{2} v_k^2 + gy = \frac{\partial \phi}{\partial t} + F(t) \tag{2-134}$$

where $F(t)$ is the function of t, respectively. The shape of the interface between two phases are approximated by a sinusoidal wave as

$$\eta = \eta_0 \sin \left\{ k \left(x - Ct \right) \right\} \tag{2-135}$$

where η_0, k, and C are the amplitude, the wave number, and the wave velocity, respectively. Then, the velocity potentials of the upper fluid $(k = 1)$ and lower fluid $(k = 2)$ are derived under the boundary condition of no fluid penetration on the upper and lower duct surfaces and the assumption of small perturbation.

$$\phi_1 = -v_1 x + \eta_0 \left(v_1 - C \right) \frac{\cosh \left\{ k \left(h_1 - y \right) \right\}}{\sinh \left(k h_1 \right)} \cos \left\{ k \left(x - Ct \right) \right\} \tag{2-136}$$

$$\phi_2 = -v_2 x - \eta_0 \left(v_2 - C \right) \frac{\cosh \left\{ k \left(h_2 + y \right) \right\}}{\sinh \left(k h_2 \right)} \cos \left\{ k \left(x - Ct \right) \right\} \tag{2-137}$$

where h_1 and h_2 are the average thickness of the upper and lower fluid layers, respectively. Substituting Eqs.(2-136) and (2-137) into Eq.(2-134) and assuming $v_{yk} \ll v_{xk}$ yield the pressure of each phase at the interface as

$$p_{i1} = -\rho_1 \left\{ (v_1 - C)^2 \, k\coth(kh_1) + g \right\} \eta_0 \sin\{k(x - Ct)\} \qquad (2\text{-}138)$$
$$+ p_i$$

$$p_{i2} = \rho_2 \left\{ (v_2 - C)^2 \, k\coth(kh_2) - g \right\} \eta_0 \sin\{k(x - Ct)\} \qquad (2\text{-}139)$$
$$+ p_i$$

where p_i is the pressure at a smooth interface. The interfacial pressure difference between two fluid layers is due to the surface tension, and can be approximated by

$$p_{i2} - p_{i1} = -\sigma \frac{\partial^2 \eta}{\partial x^2}. \qquad (2\text{-}140)$$

Then, the wave velocity can be obtained from Eq.(2-135) and Eqs.(2-138)-to-(2-140) as

$$C = \frac{\rho_1' v_1 + \rho_2' v_2}{\rho_1' + \rho_2'} \pm \sqrt{\frac{\sigma k + (\rho_2 - \rho_1) g/k}{\rho_1' + \rho_2'} - \rho_1' \rho_2' \left(\frac{v_1 - v_2}{\rho_1' + \rho_2'}\right)^2} \qquad (2\text{-}141)$$

where $\rho_k' \equiv \rho_k \coth(kh_k)$. Under the deep water assumption of $h_1 / (2\pi/k)$, $h_2 / (2\pi/k) > 0.25$, ρ_k' can be approximated to be ρ_k. In this case, Eq.(2-141) can be simplified as

$$C = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} \pm \sqrt{C_\infty^2 - \rho_1 \rho_2 \left(\frac{v_1 - v_2}{\rho_1 + \rho_2}\right)^2} \qquad (2\text{-}142)$$

where

$$C_\infty^2 = \frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} + \frac{\sigma k}{\rho_1 + \rho_2}. \qquad (2\text{-}143)$$

When the root in the expression for the wave velocity C has a nonzero imaginary part, then the interfacial disturbance can grow exponentially. Hence, the flow is unstable if

$$\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} + \frac{\sigma k}{\rho_1 + \rho_2} < \rho_1\rho_2\left(\frac{v_1 - v_2}{\rho_1 + \rho_2}\right)^2. \tag{2-144}$$

There are several important points to be recognized in this stability criterion. First, the viscous effects of the fluids are neglected; therefore, the Reynolds number plays no role in this type of interfacial instability. The stability of the system then is governed by three effects, namely, the gravity force, surface-tension force, and relative motion. The relative-motion term is always destabilizing due to the inertia force from Bernoulli effect. The surface-tension force is always stabilizing, since the flat interface has the minimum surface area, and the surface-tension force acts to resist any deformation from the equilibrium configuration. The gravity term is stabilizing only if the upper fluid is lighter than the lower fluid ($\rho_2 > \rho_1$).

The propagation velocity C_∞ in the absence of the flows (or the left-hand side of the stability criterion) is a function of the wave number k. Therefore, as the wavelength $\lambda = 2\pi/k$ changes from zero to infinite, the wave velocity decreases to the minimum value and then increases. This minimum value of C_∞^2 is given by $C_{\infty c}^2 = 2\left[\sigma g\left(\rho_2 - \rho_1\right)/\left(\rho_2 - \rho_1\right)^2\right]^{1/2}$, which occurs at $k_c^2 = g\left(\rho_2 - \rho_1\right)/\sigma$. This corresponds to the critical wavelength of $\lambda_c = 2\pi/k_c$. This is known as Taylor wave length that is one of the most important internal length scales in two-phase flow. Then the system is stable for small disturbances of all wavelengths if the relative velocity is sufficiently small to satisfy

$$\left(v_1 - v_2\right)^2 < \frac{2\left(\rho_1 + \rho_2\right)}{\rho_1\rho_2}\sqrt{\sigma g\left(\rho_2 - \rho_1\right)}. \tag{2-145}$$

For a relative velocity larger than this limit, the system is only conditionally stable for a certain range of the wavelength. When the wavelength is large, the value of C_∞^2 in Eq.(2-143) is mainly determined by the gravity term. Conversely, if λ is sufficiently small, the capillary force governs the wave motion.

Furthermore, it is possible to develop a similar stability criterion based on the one-dimensional two-phase flow equations (Wallis, 1969; Kocamustafaogullari, 1971). It is noted (Miles, 1957) that the Kelvin-Helmholtz instability theory tends to overpredict the critical relative velocity

for the initial generation of surface waves, except in the case of highly viscous fluids. However, the Kelvin-Helmholtz instability mechanism is important in wave-propagation phenomena, particularly for flows in a confined channel (Kordyban, 1977). Based on the analysis, Kelvin proposed the word "Ripples" to describe waves having a wavelength of less than

$$\lambda_c = 2\pi\sqrt{\sigma/g(\rho_2 - \rho_1)}.$$

For a gravity dominated flow with a relatively large wave length $\lambda \gg \lambda_c$, the surface tension effect can be neglected. By considering the finite channel flow, Eq.(2-141) can give a criterion for instability as

$$\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1' + \rho_2'} < \rho_1'\rho_2'\left(\frac{v_1 - v_2}{\rho_1' + \rho_2'}\right)^2. \tag{2-146}$$

By taking a Taylor expansion and retaining only the first order term for the hyperbolic functions, a following simplified but useful criterion can be obtained.

$$\left(v_1 - v_2\right)^2 > \frac{g}{k}\frac{(\rho_2 - \rho_1)(\rho_1' + \rho_2')}{\rho_1'\rho_2'} \approx \frac{g(\rho_2 - \rho_1)h_1}{\rho_1}. \tag{2-147}$$

When this criterion is compared to experimental data for slug formation in a channel, the critical relative velocity is overpredicted by a factor close to two. This discrepancy can be explained by a theoretical analysis introducing a finite amplitude or wave front propagation method (Mishima and Ishii, 1980; Wu and Ishii, 1996).

1.3.3   Rayleigh-Taylor instability

The Rayleigh-Taylor instability is the interfacial instability between two fluids of different densities that are stratified in the gravity field or accelerated normal to the interface. It is commonly observed that the boundary between two stratified fluid layers at rest is not stable if the upper-fluid density ρ_1 is larger than the lower-fluid density ρ_2. Since the Rayleigh-Taylor instability can lead to the destruction of the single common interface, it is important in the formation of bubbles or droplets. In particular, the critical wavelength predicted by the related stability analysis is one of the most significant length scales for two-phase flow.

The Rayleigh-Taylor instability can be considered as a special case of the Kelvin-Helmholtz instability with zero flows and $\rho_1 > \rho_2$. Hence, the

propagation velocity can be obtained from Eq.(2-142) by setting $v_1 = v_2 = 0$

$$C^2 = \frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} + \frac{\sigma k}{\rho_2 + \rho_1}. \tag{2-148}$$

The system is unstable if the root of the propagation velocity has a nonzero imaginary part. Therefore, Eq.(2-148) shows that the gravitational force is destabilizing for $\rho_1 > \rho_2$, whereas the surface-tension force is stabilizing. There is a critical wavelength λ below which C^2 is always positive. This is given by $\lambda_c = 2\pi\sqrt{\sigma/g(\rho_2 - \rho_1)}$. If the wavelength of a disturbance is larger than the critical wave length ($\lambda > \lambda_c$), then C^2 becomes negative and the interface is unstable. For fluids that are unlimited laterally, the wavelength of the disturbance can be as large as desired; therefore such a system is always unstable. However, if the fluids are confined laterally, the maximum wavelength is limited to twice the system dimension. This implies that a system is stable if the lateral characteristic dimension is less than half the critical wavelength λ_c. For an air-water system, this characteristic dimension is 0.86 cm. A similar dimension can be obtained from fluids contained in a vertical cylinder by using polar coordinates in the stability analysis.

For an unstable system, any disturbance having a wavelength greater than λ_c can grow in time. However, the dominant waves are those having the maximum growth factor. Since the wave amplitude grows with $\exp(-ikCt)$, the predominant wavelength should be

$$\lambda_m = 2\pi\sqrt{\frac{3\sigma}{g(\rho_1 - \rho_2)}}. \tag{2-149}$$

These unstable waves can be observed as water droplets dripping from a wire in a rainy day, or condensed water droplets falling from a horizontal downward-facing surface. Quite regular waveforms and generation of bubbles due to the Rayleigh-Taylor instability can also be observed in film boiling. Note that this instability is not limited to the gravitational field. Any interface, and fluids that are accelerated normal to the interface, can exhibit the same instability. This can occur for example in nuclear explosion and inertia confinement of a fusion pellet. In such a case the acceleration should replace the gravity field g in the analysis.