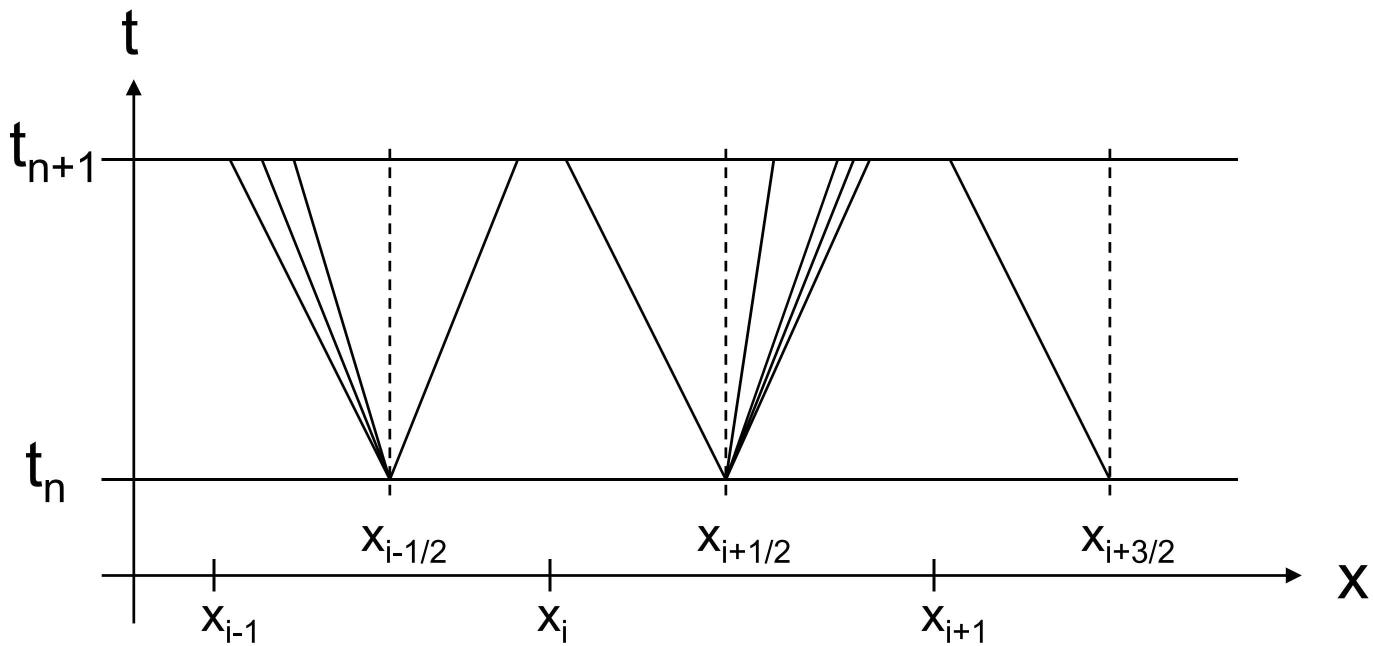
Godunov‘s Finite Volume Method



$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(g_{i+1/2}^n - g_{i-1/2}^n\right)$$

FV-scheme

$$\boxed{g_{i+1/2}^n = f(u_{RP}(0; u_i, u_{i+1}))}$$

numerical flux at $x_{i+1/2}$

Flux of the solution of the Riemann problem at the center x=0

Godunov-method

Properties

- nonlinear wave propagation incorporated
- exact conservation due to FV approach

CFL-Condition

To use the solution of the Riemann problem we assume that the waves generated at different interfaces do not interact. This means

$$\frac{\Delta t}{\Delta x} < \frac{1}{|a|_{\max}} \quad \Leftrightarrow \quad \frac{\Delta t}{\Delta x}|a|_{\max} < 1$$

consistency condition

maximal wave speed

Approximate Riemann Solution

Definition: $w=w(x/t;u_l,u_r)$ is defined to be an *approximate Riemann solution*, if the following is valid:

1. a_l smallest, a_r largest signal velocity

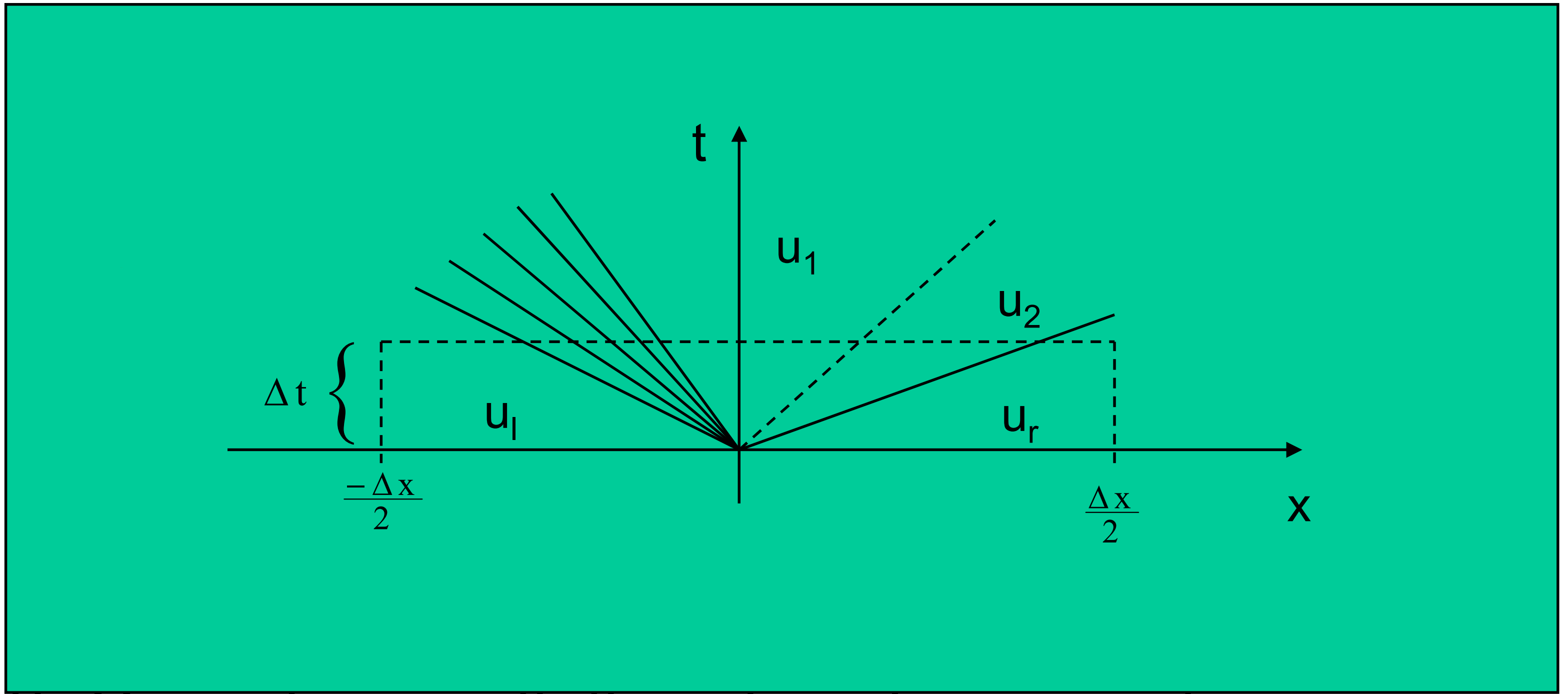
2. Consistency with the integral conservation

$$\int_{-\Delta x/2}^{\Delta x/2} w(x/\Delta t; u_l, u_r)\,dx = \frac{\Delta x}{2}(u_l + u_r) - \Delta t(f(u_r) - f(u_l))$$

3. Consistency with the integral entropy condition

$$\int_{-\Delta x/2}^{\Delta x/2} U(w(x/\Delta t; u_l, u_r))\,dx \le \frac{\Delta x}{2}(U_l + U_r) - \Delta t(F(u_r) - F(u_l))$$

Δt small enough, U,F(U) is an entropy pair



2. Consistency with the integral conservation

$$\int_{-\Delta x/2}^{\Delta x/2} w(x/\Delta t; u_l, u_r)\, dx = \frac{\Delta x}{2}(u_l + u_r) - \Delta t\big(f(u_r) - f(u_l)\big)$$

3. Consistency with the integral entropy condition

$$\int_{-\Delta x/2}^{\Delta x/2} U\big(w(x/\Delta t; u_l, u_r)\big)\, dx \le \frac{\Delta x}{2}(U_l + U_r) - \Delta t\big(F(u_r) - F(u_l)\big)$$

Δt small enough, U,F(U) is an entropy pair

The Flux of a Godunov-type Method

Definition: A method is called **Godunov-type scheme**, if it satisfies

$$u_i^{n+1} = \frac{1}{\Delta x}\int_0^{\Delta x/2} w\left(x/\Delta t\,; u_{i-1}^n, u_i^n\right)dx + \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} w\left(x/\Delta t\,; u_i^n, u_{i+1}^n\right)dx$$

where w is an approximative Riemann solution.

Theorem: A Godunov-type scheme may be written in the conservation form with the numerical flux:

$$g_{i+1/2} = f\left(u_i\right) - \frac{1}{\Delta t}\int_{-\Delta x/2}^{0} w\left(x/\Delta t\,; u_i^n, u_{i+1}^n\right)dx + \frac{\Delta x}{2\Delta t}u_i$$

or

$$g_{i+1/2} = f\left(u_{i+1}\right) + \frac{1}{\Delta t}\int_0^{\Delta x/2} w\left(x/\Delta t\,; u_i^n, u_{i+1}^n\right)dx - \frac{\Delta x}{2\Delta t}u_{i+1}$$

The Roe Linearization

P. Roe (1981): Replace the exact solution by the exact solution of the following linearized Riemann problem

$$u_t + A_{lr} u_x = 0, \qquad u(x,0) = \begin{cases} u_l & \text{für } x < 0 \\ u_r & \text{für } x > 0 \end{cases}$$

$A_{lr} = A_{lr}(u_l, u_r)$ is called a Roe matrix, if

1. $A_{lr}(u,u) = A(u)$
2. A_{lr} diagonalizable
3. mean value property:

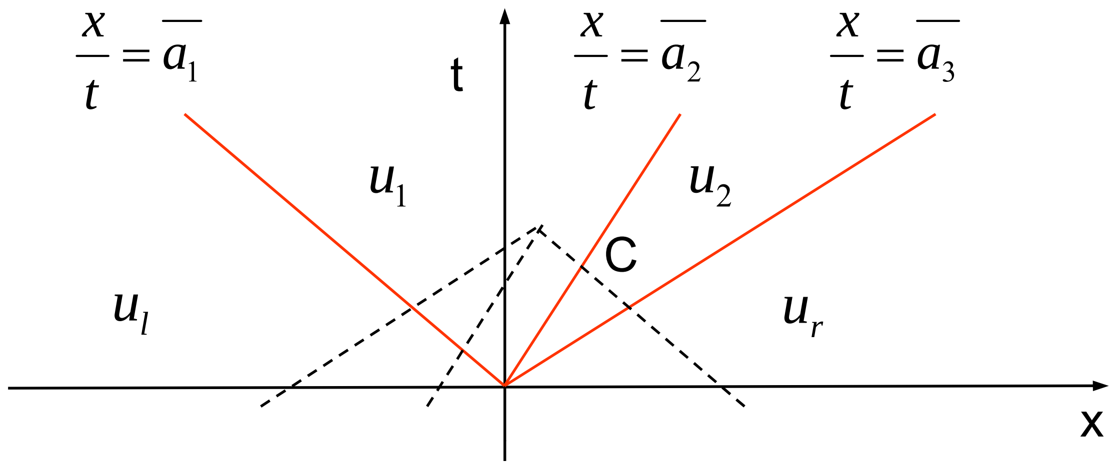
$$f(u_r) - f(u_l) = A_{lr}(u_r - u_l)$$

Solution of the Linearized Riemann Problem

1.Transformation:

$$\mathbf{u}_r - \mathbf{u}_l = R(\mathbf{w}_r - \mathbf{w}_l), \qquad \mathbf{u}_r - \mathbf{u}_l = \sum_{i=1}^{3} \gamma_j \mathbf{r}_j$$

2. The four constant states



$$\mathbf{u}_1 = \mathbf{u}_l + \gamma_1 \mathbf{r}_1$$

$$\mathbf{u}_2 = \mathbf{u}_l + \gamma_1 \mathbf{r}_1 + \gamma_2 \mathbf{r}_2$$

Roe mean values for a perfect gas

$$A_{lr}(u_l, u_r) = A(\bar{u})$$

with

$$\bar{v} = \frac{\sqrt{\rho_r}\, v_r + \sqrt{\rho_l}\, v_l}{\sqrt{\rho_r} + \sqrt{\rho_l}}, \qquad \bar{H} = \frac{\sqrt{\rho_r}\, H_r + \sqrt{\rho_l}\, H_l}{\sqrt{\rho_r} + \sqrt{\rho_l}},$$

$$\bar{c}^2 = (\gamma - 1)\left(\bar{H} - \frac{1}{2}\bar{v}^2\right)$$

How looks the numerical flux for this approximate Riemann solver?

$g(u_l, u_r) := f(u_{Roe}(x/t = 0, u_l, u_r))$ works not, because the Roe approximate solution is not a local approximation, but consistent with the integral equation only.

Numerical Flux of the Roe Scheme

Use the theorem and obtain

$$g_{i+\frac{1}{2}} = f(u_i) + \sum_{j=1}^{3} a_j^- \gamma_j r_j$$

$$g_{i+\frac{1}{2}} = f(u_{i+1}) - \sum_{j=1}^{3} a_j^+ \gamma_j r_j$$

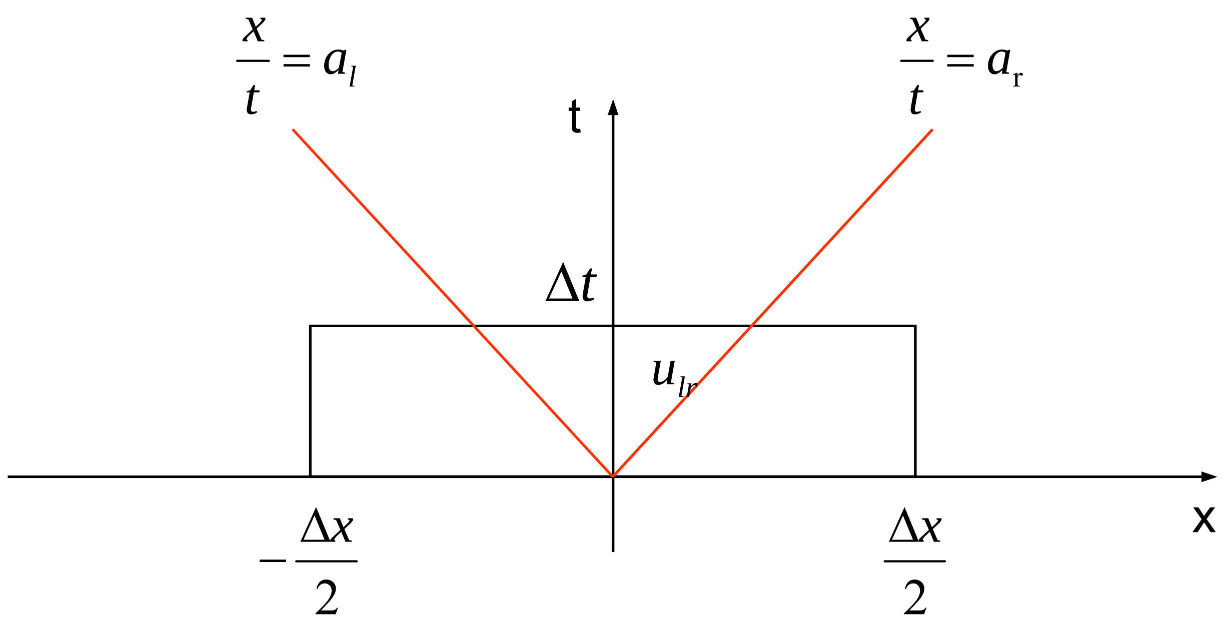
Usually the average is taken

$$g_{Roe}(u_i, u_{i+1}) = \frac{1}{2}\left(f(u_{i+1}) + f(u_i)\right) - \frac{1}{2}\sum_{j=1}^{3} \left|a_j\right| \gamma_j r_j$$

central differencing correction according to wave propagation

The Method of Harten, Lax and Van Leer

Simplest Godunov-type scheme



HLL Approximate Riemann Solution

$$w\left(\frac{x}{t}; u_l; u_r\right) = \begin{cases} a_l & \text{for} \quad \dfrac{x}{t} < a_l \\ a_{lr} & \text{for} \quad a_l \le \dfrac{x}{t} \le a_r \\ a_r & \text{for} \quad \dfrac{x}{t} > a_r \end{cases}$$

a_l, a_r smallest and largest wave velocity

a_{lr} intermediate state

Consistency with integral conservation

$$u_{lr} = \frac{a_r u_r - a_l u_l - f(u_r) + f(u_l)}{a_r - a_l}$$

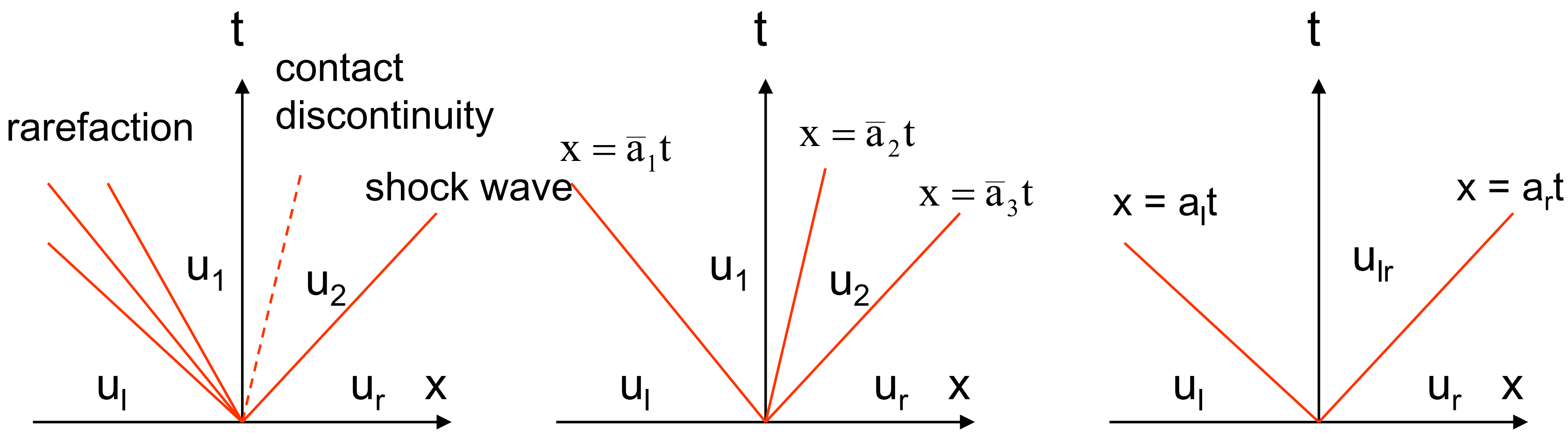
Numerical flux

$$g_{HLL}(u_l, u_r) = \frac{a_r^+ \, f(u_l) - a_l^- \, f(u_r)}{a_r^+ - a_l^-} + \frac{a_r^+ \, a_l^-}{a_r^+ - a_l^-}(u_r - u_l)$$

Calculation of signal velocities

$$a_l = \min\left\{v_l - c_l, \overline{v} - \overline{c}\right\}$$
$$a_r = \max\left\{v_r - c_r, \overline{v} + \overline{c}\right\}$$

Riemann solver



Godunov-scheme	**Roe-scheme**	**HLL-scheme**
Exact solution of Riemann problem, fixed point problem	Exact solution of lin. Riemann problem, theory of characteristics	A priori estimates of the fastest wave speeds,

Flux-Vector Splitting Methods

Splitting of the flux

$$f(u) = \underbrace{f^-}_{\text{left}}(u) + \underbrace{f^+}_{\text{right}}(u)$$

with

EW

$$A^+(u) = \frac{d f^+(u)}{du} \qquad \text{non-negative}$$

$$A^-(u) = \frac{d f^-(u)}{du} \qquad \text{non-positive EW}$$

Numerical flux $\quad g(u_i, u_{i+1}) = f^+(u_i) + f^-(u_{i+1})$

Conclusion: Flux Calculation

Upwind differencing

Godunov-type schemes

Godunov, Roe, HLL, Osher,

Flux-vector Splitting schemes

Steger-Warming, van Leer, AUSM, ...

Reconstruction piecewise constant ⇒

first order accurate - similiar results

4. FV-Schemes in Multi Dimensions

Two-dimensional Euler equations

$$u_t + f_1(u)_x + f_2(u)_y = 0$$

$$u = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ e \end{pmatrix} \qquad f_1(u) = \begin{pmatrix} \rho v_1 \\ \rho v_1^{\,2} + p \\ \rho v_1 v_2 \\ v_1(e+p) \end{pmatrix} \qquad f_2(u) = \begin{pmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^{\,2} + p \\ v_2(e+p) \end{pmatrix}$$

equation of state

$$p = (\gamma - 1)\rho\varepsilon \;, \qquad e = \rho\varepsilon + \tfrac{1}{2}\rho\left(v_1^{\,2} + v_2^{\,2}\right)$$

Quasilinear form

$$u_t + Au_x + Bu_y = 0$$

with $$A(u) = \frac{df_1(u)}{du} \quad , \qquad B(u) = \frac{df_2(u)}{du}$$

Eigenvalues

$$a_1 = v_1 - c\,, \quad a_2 = a_3 = v_1\,, \quad a_4 = v_1 + c$$
$$b_1 = v_2 - c\,, \quad b_2 = b_3 = v_2\,, \quad b_4 = v_2 + c$$

FV-Schemes in General Grids

Conservation equation

$$u_t + \nabla \cdot f(u) = 0 \ \text{ in } D \times [0, T]$$

Grid

$$D = \bigcup \overline{C}_j, \quad C_j \cap C_k = \Phi \ \text{ für } \ j \neq k$$

$$\text{Boundary } \ \partial C_j \ \text{ piecewise smooth}$$

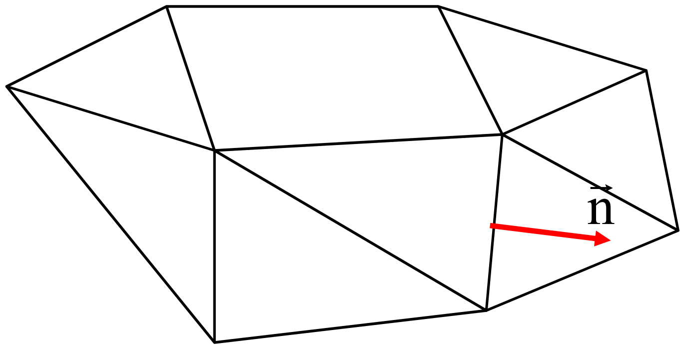
Integration over $C_j \times [t_n, t_{n+1}]$

$$|C_j| u_j^{n+1} = |C_j| u_j^n - \int_{t_n}^{t_{n+1}} \int_{\partial C_j} f(u(x,t)) \cdot \vec{n} \, dS \, dt$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{|C_j|} \sum_k \int_{\partial C_j^k} f_n(C_j, C^{k*})\, dS$$

Gauß quadrature

numerical flux into normal direction



Flux calculation into normal direction:

The rotational symmetry of the Euler equations enables to use the one-dimensional flux calculations

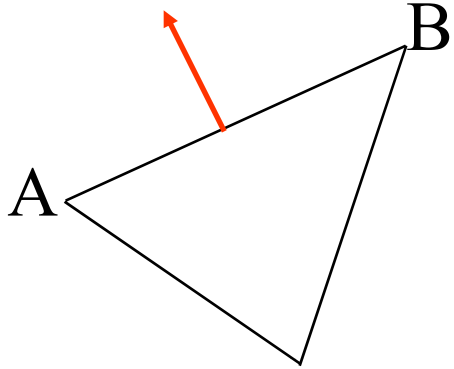
Local coordinate system: normal and tangential velocity components

Calculation of the Flux into Normal Direction

$$F_{AB} = \int_{t_n}^{t_{n+1}} \int_A^B \left(n_1 f_1(u) + n_2 f_2(u) \right) ds\, dt$$

Use rotational symmetry of Euler equations

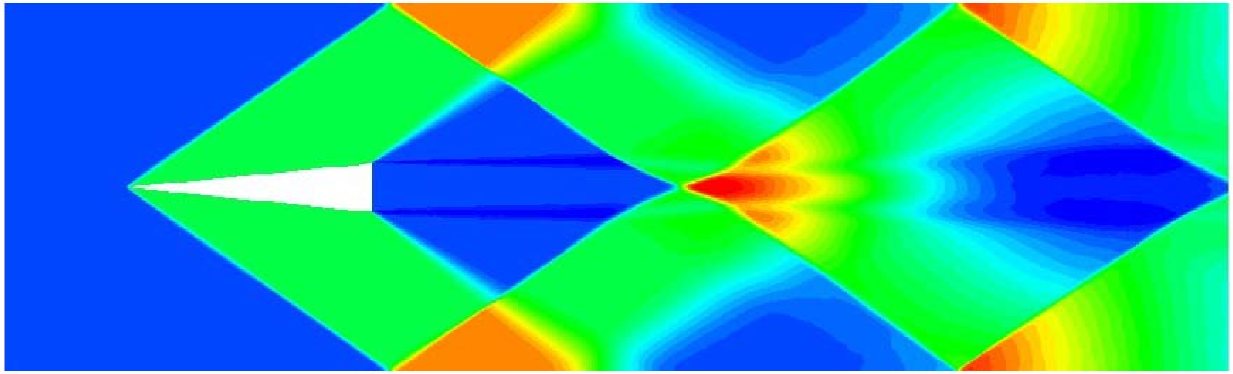
$$\boxed{n_1 f_1(u) + n_2 f_2(u) = T^{-1} f_1(Tu)}$$

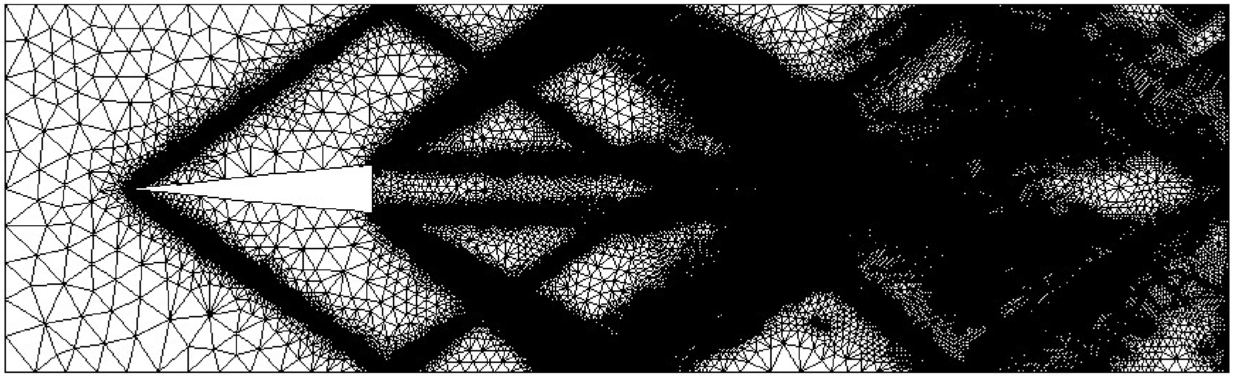


$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & 0 \\ 0 & -n_2 & n_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

rotation

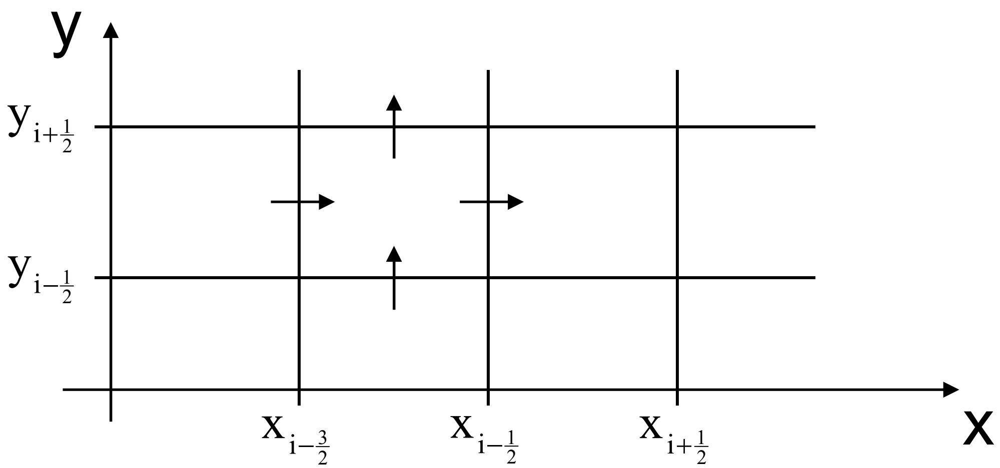
Adaptive grid refinement





Supersonic flow in a channel over a wedge - h-refinement

Example: FV – Scheme in a Cartesian Grid



$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(g_{i+\frac{1}{2},j} - g_{i-\frac{1}{2},j}\right) - \frac{\Delta t}{\Delta y}\left(h_{i,j+\frac{1}{2}} - h_{i,j-\frac{1}{2}}\right)$$

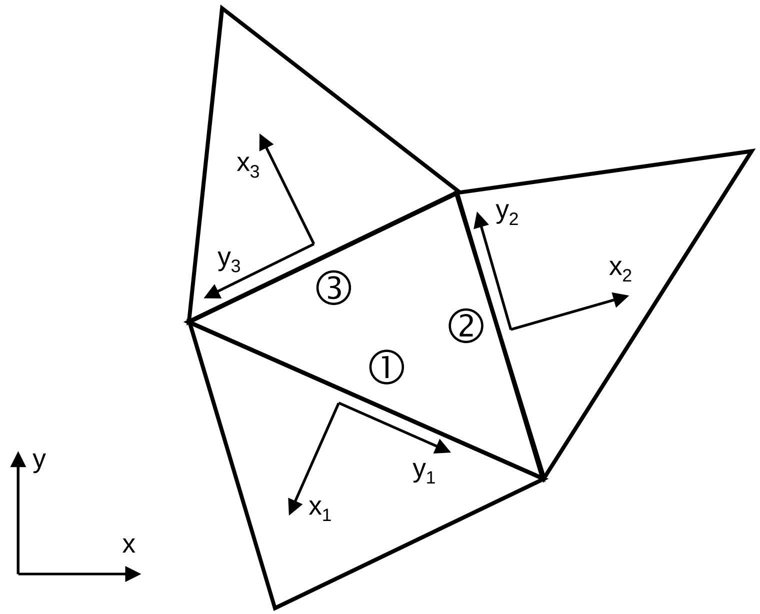
$$g_{i+\frac{1}{2},j} = g(u_{i,j}, u_{i+1,j}) \qquad \text{flux in x - direction}$$

$$h_{i,j+\frac{1}{2}} = g(u_{i,j}, u_{i,j+1}) \qquad \text{flux in y - direction}$$

2nd order accuracy: MUSCL - scheme

More General Grid Cells

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{|C_j|} \sum_k \int_{\partial C_j^k} f_n(C_j, C^{k*}) dS$$



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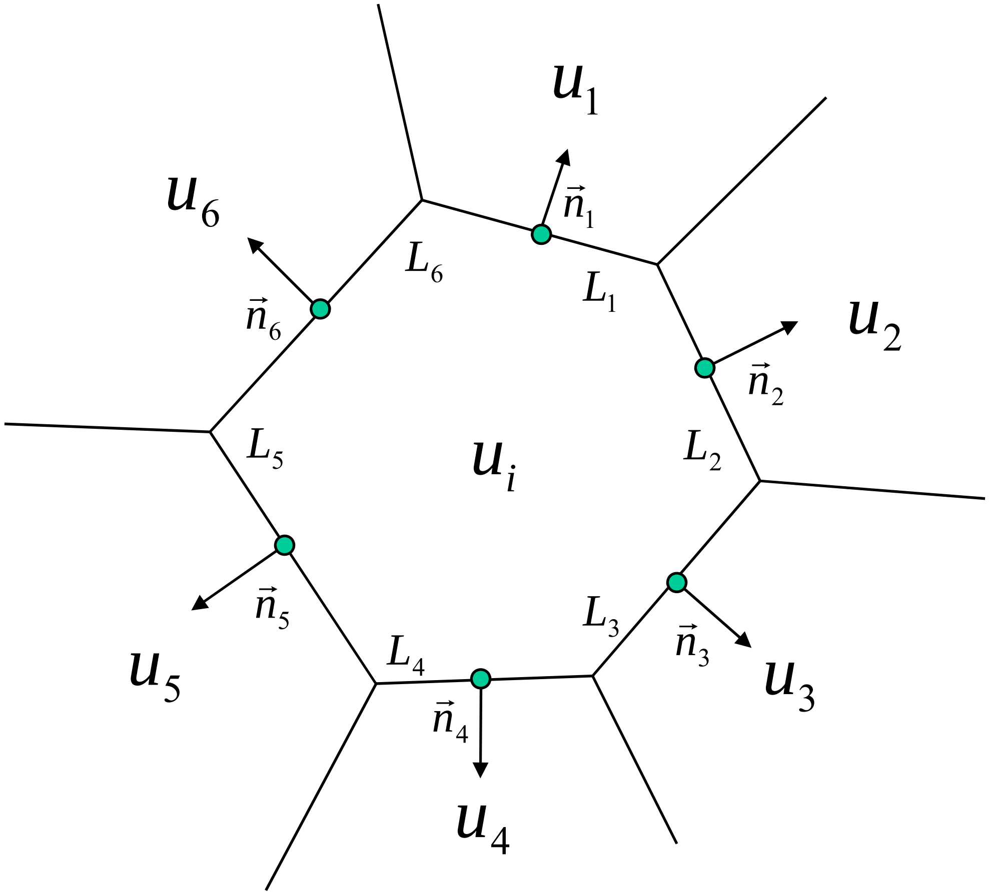
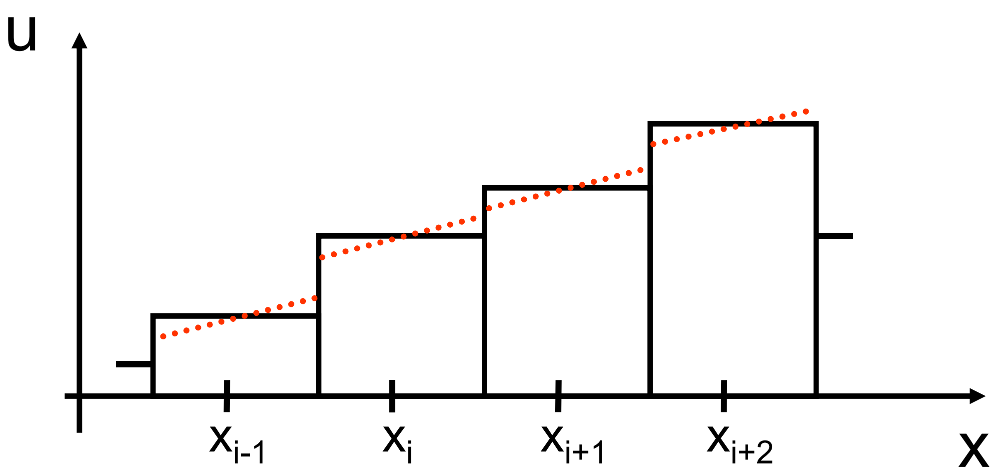


5. Second Order Accuracy: MUSCL

MUSCL-Idea (van Leer, 1979)



Piecewise linear reconstruction

Slope calculation from the integral mean values by a monotonicity preserving interpolation. (TVD property – Total Variation Diminishing)

MUSCL- Scheme

Idea: Calculate better approximate values for the numerical flux

$$u(x) = u_i + s(x - x_i)$$

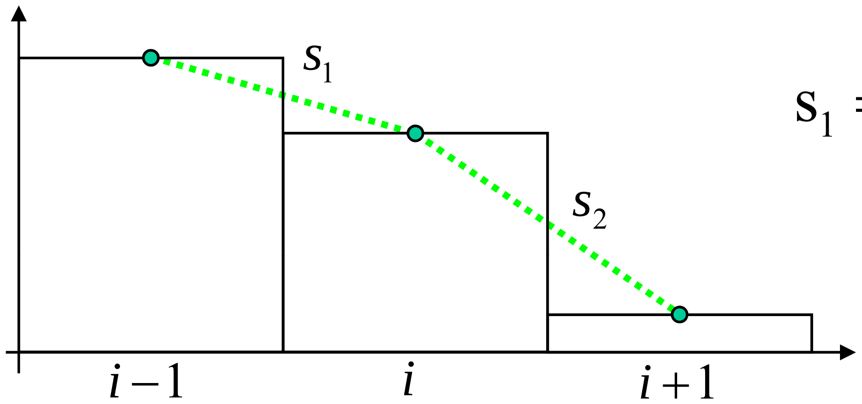
piecewise linear reconstruction

$$u_{i+} = \bar{u}_i + s\frac{\Delta x}{2}$$

$$u_{i-} = u_i - s\frac{\Delta x}{2}$$

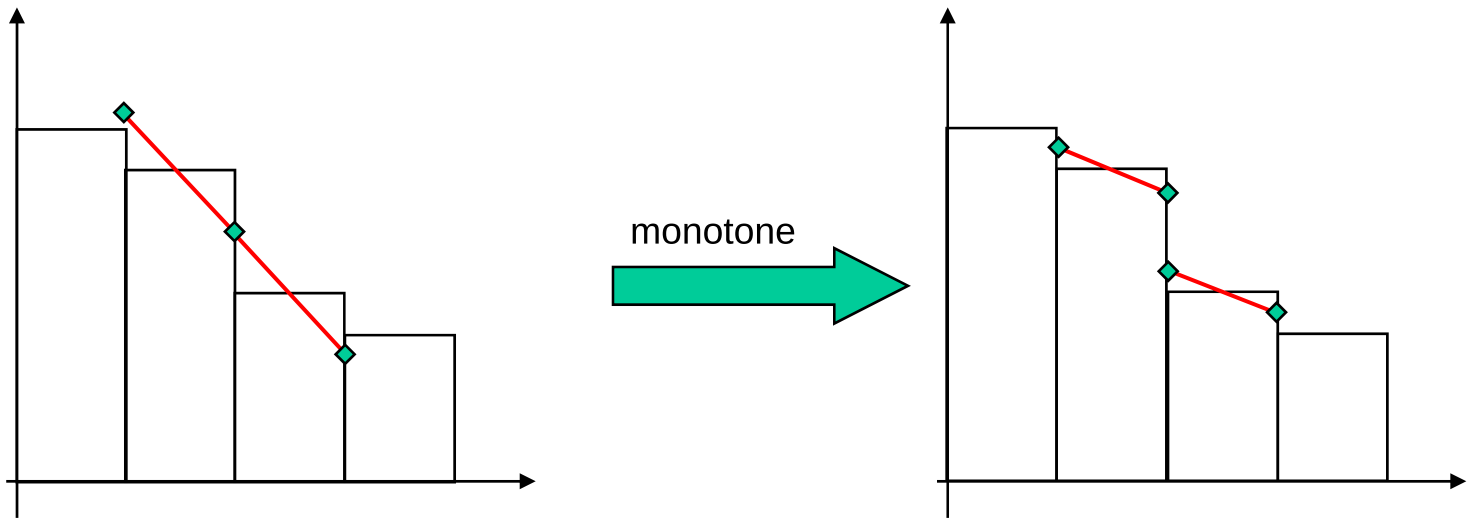
more accurate values at the grid cell interface

candidates:



$$s_1 = \frac{u_i - u_{i-1}}{\Delta x} \qquad s_2 = \frac{u_{i+1} - u_i}{\Delta x}$$

Monotonicity preserving interpolation



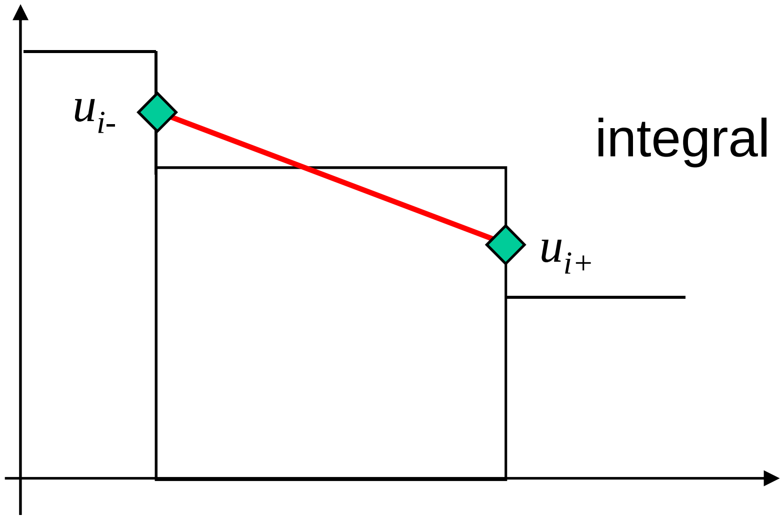
Example

$$s_i = \frac{1}{\Delta x} \min\operatorname{mod}\left(u_{i+1} - u_i \ , u_i - u_{i-1}\right)$$

$$\min\operatorname{mod}\left(a \ , b\right) = \begin{cases} a & \text{for} \quad |a| < |b| \ , ab > 0 \\ b & \text{for} \quad |a| \geq |b| \ , ab > 0 \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

MUSCL – Procedure 2nd Order in Space

1. Calculate interface values $u^n_{i\pm} = u^n_i \pm \frac{\Delta x}{2} s^n_i$ ← Slope



integral values are preserved

2. Insert the better values into the flux

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x}\left(g^n_{i+1/2} - g^n_{i-1/2}\right) \quad \text{with} \quad g^n_{i+1/2} = g\left(u^n_{i+}, u^n_{(i+1)-}\right)$$

Explicit MUSCL – Procedure, 2nd Order in Time

Explicit first order accurate in time is not stable: 2nd order upwind can not stabilize the first order time approximation

Runge Kutta time discretization

Approximation of time integral by midpoint rule and use of Taylor expansion in time:

$$u(x, t + \frac{\Delta t}{2}) = u(x,t) + \frac{\Delta t}{2} u_t(x,t) + O(\Delta t^2)$$

Replace time derivative by space derivative:

$$u(x, t + \frac{\Delta t}{2}) = u(x,t) - \frac{\Delta t}{2} f(u(x,t))_x + O(\Delta t^2)$$

using $u_t + f(u)_x = 0 \Rightarrow u_t = -f(u)_x$ and approximate

MUSCL – 2nd Order in Space and Time

Boundary values at t_n

$$u_{i\pm}^{n} = u_{i}^{n} \pm \frac{\Delta x}{2} s_{i}^{n}$$

slope

$t_n \to t_{n+1/2}$

$$u_{i\pm}^{n+1/2} = u_{i\pm}^{n} - \frac{\Delta t}{2\Delta x}\left(f\left(u_{i+}^{n}\right) - f\left(u_{i-}^{n}\right)\right)$$

FV - scheme

$$u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x}\left(g_{i+1/2}^{n+1/2} - g_{i-1/2}^{n+1/2}\right)$$

$$\text{with} \quad g_{i+1/2}^{n+1/2} = g\left(u_{i+}^{n+1/2}, u_{(i+1)-}^{n+1/2}\right)$$

Conclusion Second Order - MUSCL

Second order accuracy gives much better results
Computational effort about 2 times
Standard scheme since 20 years

Better slope calculation: Sweby‘s slope calculation

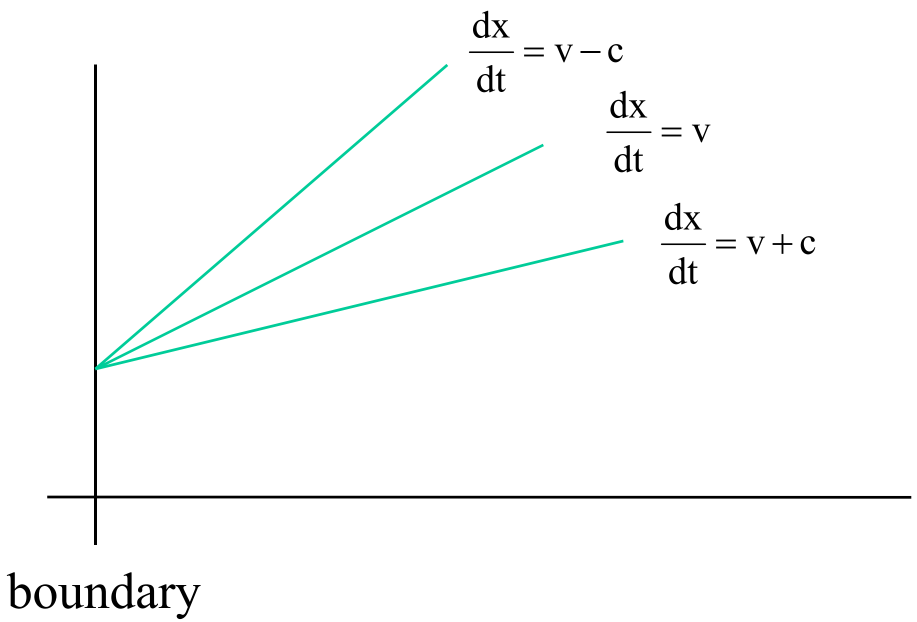
$$s_k(a,b) = \operatorname{sign} a \max\left\{\left|\operatorname{minmod}(a\,,kb)\right|,\left|\operatorname{minmod}(ka\,,b)\right|\right\}$$

$$\text{with} \qquad 1 \le k \le 2$$

Euler equations: Slope calculation in primitive variables
in characteristic variables better

Boundary Conditions for the Euler Equations

1. Supersonic Inflow



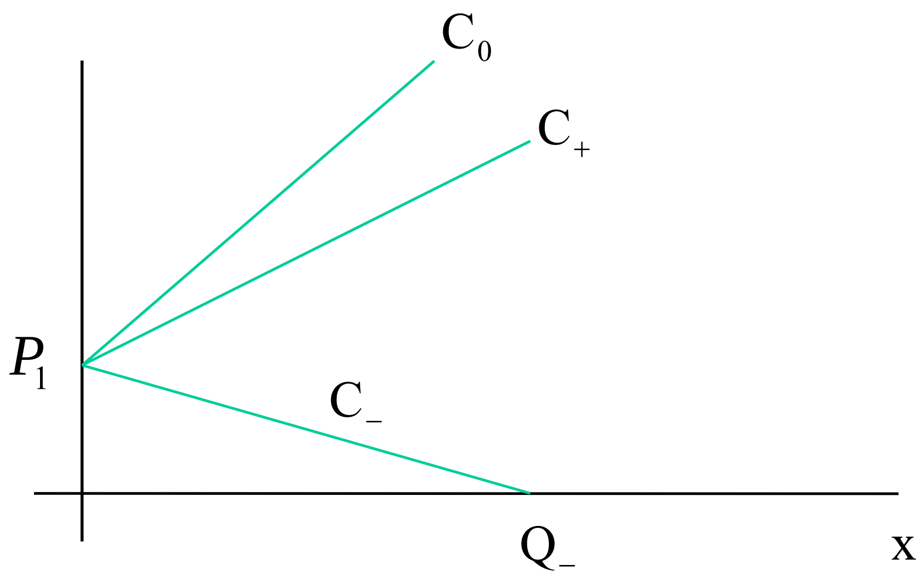
$$C_+ : \quad \frac{dx}{dt} = v + c$$

$$C_0 : \quad \frac{dx}{dt} = v$$

$$C_- : \quad \frac{dx}{dt} = v - c$$

3 boundary values are necessary

2. Subsonic Inflow

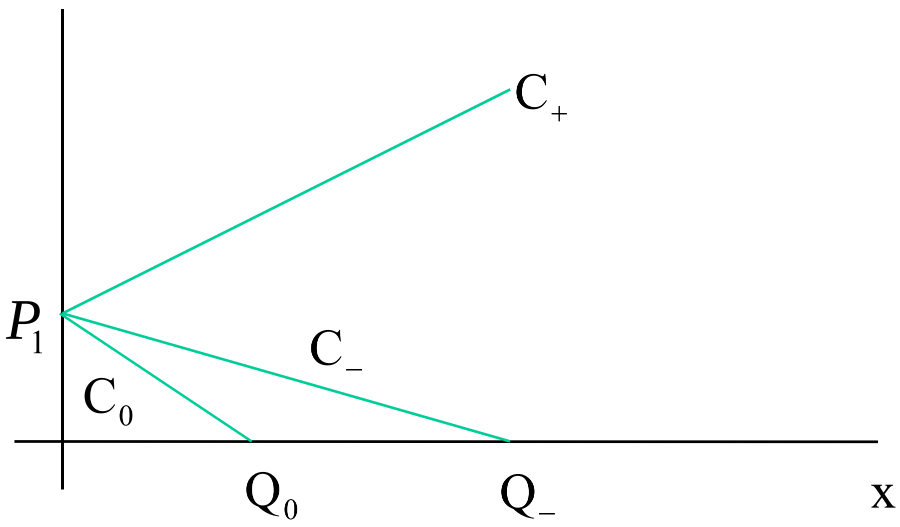


2 boundary values, e.g. ρ, v

1 compatibility condition

$$(*) \quad \left(v - \frac{2c}{\gamma - 1}\right)_{P_1} = \left(v - \frac{2c}{\gamma - 1}\right)_{Q_-} \qquad \text{characteristic theory}$$

3. Subsonic Outflow

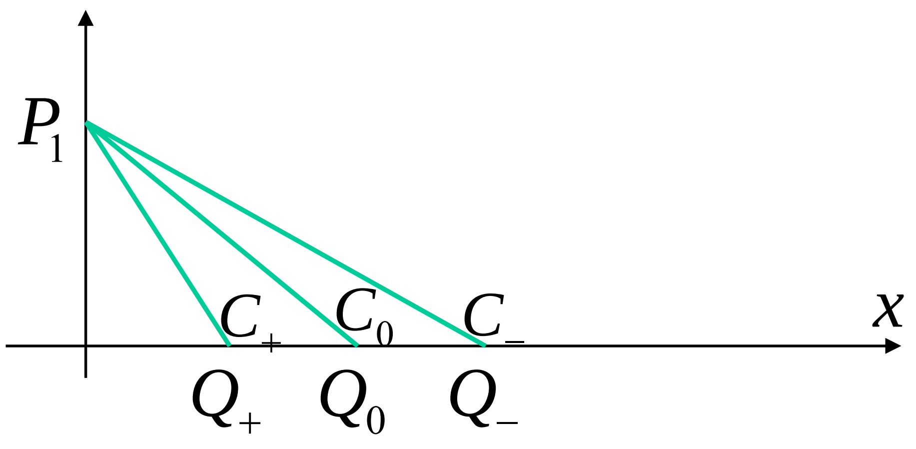


1 boundary value

2 compatibility conditions

$$(*) \text{ and } (**) \quad \left(\frac{p}{\rho\gamma}\right)_{P_1} = \left(\frac{p}{\rho\gamma}\right)_{Q_0}$$

4.Supersonic Outflow



3 Compatibility conditions (*), (**) and

$$\left(v + \frac{2c}{\gamma - 1}\right)_{P_1} = \left(v + \frac{2c}{\gamma - 1}\right)_{Q_+}$$

Remarks „Boundary Conditions“

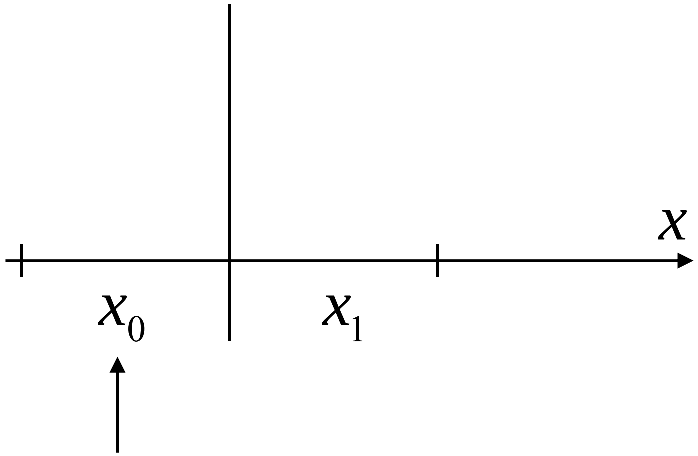
Upwind schemes usually capture the physical situation, proper information is incorporated automatically. The number of boundary conditions are given by the physics.

Non-physical or artificial boundary conditions:

Non-reflecting boundary conditions, reduction of the size of the computational domain

Characteristic boundary conditions are usually necessary for central differencing only.

Practical Implementation

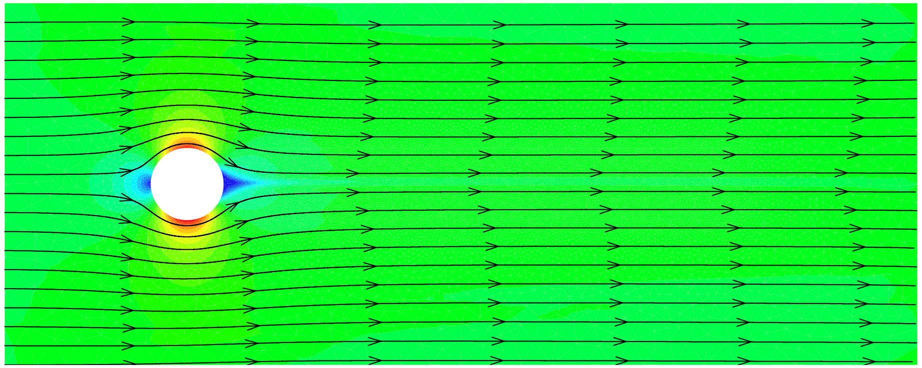


dummy grid zone

Specify appropriate values in a dummy grid zone

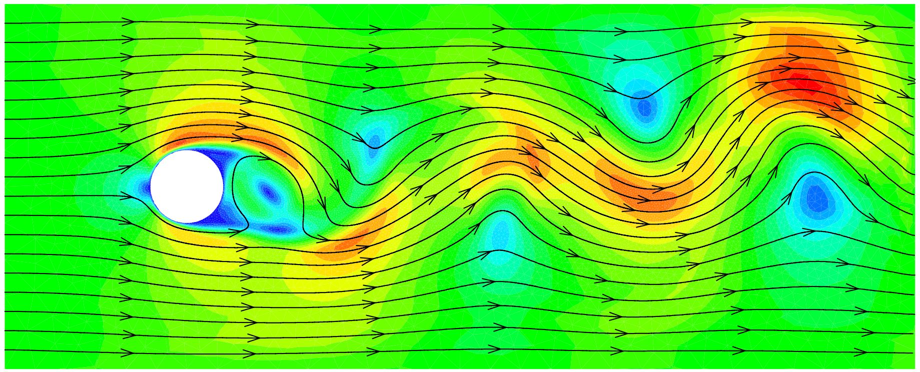
wall	non-reflecting
$\rho_1, -v_1, p_1 \mid \rho_1, v_1, p_1$	$\rho_1, v_1, p_1 \mid \rho_1, v_1, p_1$

5. Approximation of Viscous Terms



Euler equations:

Inviscid flow about a cylinder



Navier-Stokes-equations:

Viscous flow about a cylinder

Viscous effects in gases can often be neglected
They may become important for flow around obstacles
Boundary layers

Compressible Navier-Stokes Equations

$$
\begin{array}{cccc}
\rho_t & + & \nabla\cdot(\rho v) & = & 0 \\
(\rho v)_t & + & \nabla\cdot((\rho v)\circ v) + \nabla p & = & \nabla\cdot\tau \\
e_t & + & \nabla\cdot(v(e+p)) & = & \nabla\cdot(\tau v) - \nabla\cdot q
\end{array}
$$

$$\tau = 2\mu D - \frac{2}{3}\mu\nabla v \quad \text{with} \quad D = \frac{1}{2}\left[\nabla v + (\nabla v)^T\right]$$

$$q = -k\,\nabla T, \qquad k = \frac{(9\gamma-5)c_p\mu}{4\gamma}$$

$$\mu = \text{const. or e.g.,} \quad \mu = 1{,}46\cdot 10^{-6}\left[1+\frac{112\text{K}}{T}\right]^{-1}\sqrt{T}$$

Viscous terms are nonlinear and depend on derivatives

Flux Formulation

Formulation in conservation form

$$u_t + \nabla \cdot F^C = \nabla \cdot F^D \quad \text{with} \quad F^C = F^C(u), \quad F^D = F^D(u, \nabla u)$$

with the dissipation fluxes

$$F^D = \begin{pmatrix} 0 \\ \frac{1}{Re_{ref}}\left(\frac{4}{3}\mu(v_1)_x - \frac{2}{3}\mu(v_2)_y\right) \\ \frac{1}{Re_{ref}}\mu\left((v_1)_y + (v_2)_x\right) \\ \frac{1}{Re_{ref}}(v_1)\left(\frac{4}{3}\mu(v_1)_x - \frac{2}{3}\mu v_y\right) + \frac{1}{Re_{ref}}(v_2)\mu\left((v_1)_y + (v_2)_x\right) - \frac{\gamma}{(\gamma-1)Re_{ref}\,Pr_{ref}} q_1 \end{pmatrix}$$

$$F^D = \begin{pmatrix} 0 \\ \frac{1}{Re_{ref}}\mu\left((v_1)_y + (v_2)_x\right) \\ \frac{1}{Re_{ref}}\left(\frac{4}{3}\mu(v_2)_y - \frac{2}{3}\mu(v_1)_x\right) \\ \frac{1}{Re_{ref}}(v_1)\mu\left((v_1)_y + (v_2)_x\right) + \frac{1}{Re_{ref}}(v_2)\left(\frac{4}{3}\mu(v_2)_y - \frac{2}{3}\mu(v_1)_x\right) - \frac{\gamma}{(\gamma-1)Re_{ref}\,Pr_{ref}} q_2 \end{pmatrix}$$

Formulation with Dissipation Matrix

Formulation as parabolic system

$$u_t + \nabla \cdot F^C = \nabla \cdot (D(u)\nabla u)$$

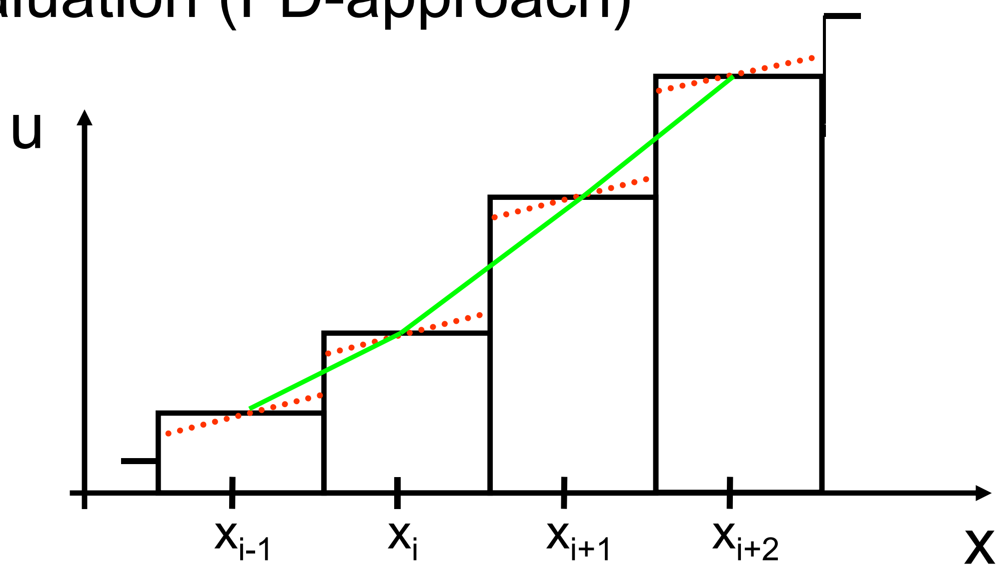
Dissipation matrix positive semi-definite

Approximation of parabolic terms in a central way, but the simple expression (one-dimensional)

$$g^D_{i+1/2} = \frac{1}{2}(F^D(u_i, \nabla u_i) + F^D(u_{i+1}, \nabla u_{i+1}))$$

does not work, because the jumps are not taken into account

1. Smooth reconstruction for viscous terms and central evaluation (FD-approach)



one-dimensional heat conduction equation $u_t = k u_{xx}$

$$g^D_{i+1/2} = \frac{1}{2\Delta x}(u_{i+1} - u_{i-1})$$

central difference for derivative

2. FE-approximation of dissipation terms

3. Self-consistent treatment by using local solutions
Solution of the diffusive generalized Riemann problem (dGRP): Initial value problem with piecewise linear data

Example: Riemann problem for pure diffusion equation

$$u_t = k u_{xx}, \qquad u(x,0) = \begin{cases} u_i^n + \dfrac{\Delta x}{2} s_i^n & , x < 0 \\ u_{i+1}^n - \dfrac{\Delta x}{2} s_{i+1}^n & , x > 0 \end{cases}$$

Heat flux of the local solution

$$u_x(0,t) = \underbrace{\frac{1}{2}\{u_x\}_{i+1/2}}_{\text{arithmetic mean}} + \frac{1}{2\sqrt{\pi k t}} \underbrace{\left(u_{(i+1)-} - u_{i+}\right)}_{\text{jump}}$$

Conclusion: Approximation of Viscous Terms

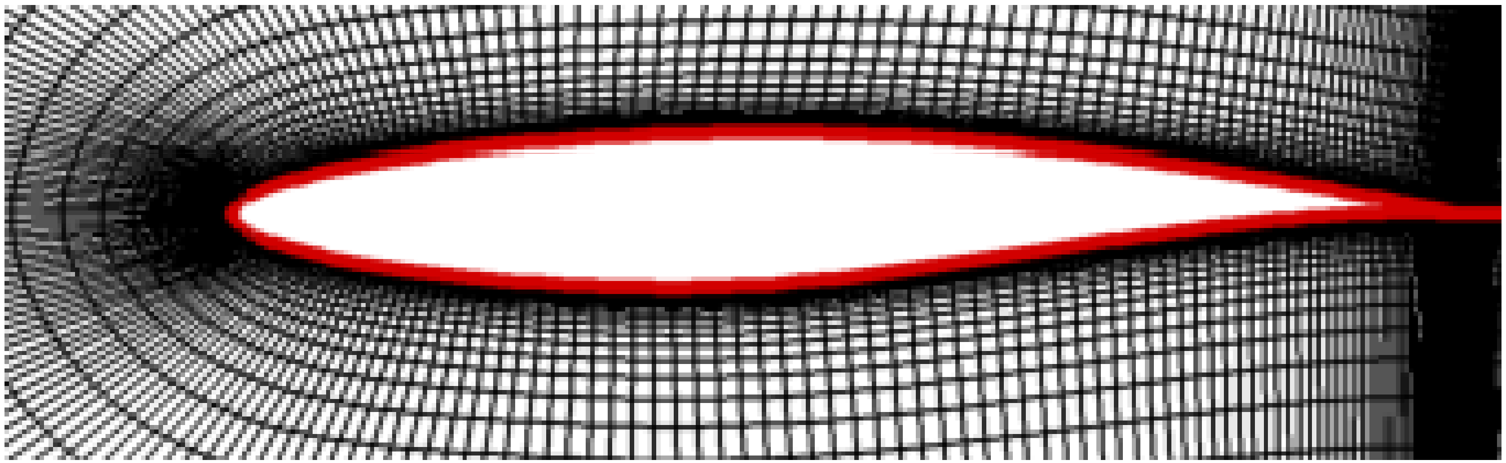
Parabolic Terms: Usual approximation as for FD: central differencing in flux formulation $\mathrm{f} = \mathrm{f}_{\mathrm{eul}} + \mathrm{f}_{\mathrm{vis}}$

explicit => parabolic time step restriction

Self-consistent treatment: dGRP

Implicit treatment after linearization: Linear system of equations

Viscous terms simplify shock-capturing, but complicate the numerical computation (resolution of boundary layers)



6. Implicit Methods

Unsteady solutions

The time step should be chosen such that during one time step the solution moves through one grid zone only – otherwise loss of information

=> CFL-condition

Steady solutions

Implicit methods are used to calculate stationary solutions

Two possibilities to choose the mathematical model

1. Solve stationary equations directly
2. Solve non-stationary equations up to a stationary state

usual approach

Stationary Equations

$$\cancel{\rho_t} + \nabla \cdot (\rho v) = 0$$

$$\cancel{(\rho v)_t} + \nabla \cdot ((\rho v) \circ v) + \nabla p = \nabla \cdot \tau + f$$

$$\cancel{e_t} + \nabla \cdot (v(e+p)) = \nabla(\tau \cdot v) - \nabla \cdot q + f \cdot v + Q$$

M<1: elliptic system
M>1: hyperbolic system in space

Equations that change type are difficult to approximate.
In elliptic region: Central differences, iterative solution
In hyperbolic region: Characteristic based methods, marching schemes

Shock-Fitting: The shock curve is fitted and considered as an interior boundary. In subsonic and supersonic region different methods are used and coupled by the shock relations

This approach becomes difficult, if complex shock configurations occur, for non-stationary problems with moving shock waves not efficient.

The usual approach now-a-days is Shock-Capturing for steady problems, too

Solve Non-stationary Equations for Large Times

$$\rho_t + \nabla \cdot (\rho v) = 0$$

$$(\rho v)_t + \nabla \cdot ((\rho v) \circ v) + \nabla p = \nabla \cdot \tau + f$$

$$e_t + \nabla \cdot (v(e + p)) = \nabla(\tau \cdot v) - \nabla \cdot q + f \cdot v + Q$$

Stationary state: $\lim\limits_{t \to \infty} u$

CFL-condition is a strong restriction in this case

Better: unconditionally stable implicit method, first order accurate in time is enough

Implicit Schemes

FD-Methods:

implicit time approximation gives more stability

FV-Methods:

implicit formulation of the shock-capturing schemes

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(g_{i+1/2}^{n+1} - g_{i-1/2}^{n+1}\right)$$ implicit FV-scheme

$$g_{i+1/2}^{n+1} = g(u_i^{n+1}, u_{i+1}^{n+1})$$ numerical flux

$\Rightarrow$ numerical solution of a nonlinear system

Linearization in the Scalar Case

$$g(u_i^{n+1}, u_{i+1}^{n+1}) = g(u_i^n, u_{i+1}^n) + \partial_1 g(u_i^n, u_{i+1}^n)(u_i^{n+1} - u_i^n) \\ + \partial_2 g(u_i^n, u_{i+1}^n)(u_{i+1}^{n+1} - u_{i+1}^n) + O(\Delta t^2)$$

with

$$\partial_1 g(u_i^n, u_{i+1}^n) := \frac{\partial g(u_i^n, u_{i+1}^n)}{\partial u_i^n}, \quad \partial_2 g(u_i^n, u_{i+1}^n) := \frac{\partial g(u_i^n, u_{i+1}^n)}{\partial u_{i+1}^n}$$

Hence, we get the system of linear equations

$$-\partial_1 g_{i-1/2}^n \delta u_{i-1}^{n+1} + \left(\frac{\Delta x}{\Delta t} + \partial_1 g_{i+1/2}^n - \partial_2 g_{i-1/2}^n\right)\delta u_i^{n+1} + \partial_2 g_{i+1/2}^n \delta u_{i+1}^{n+1} = 0$$

numerical flux should be continuously differentiable

General Case

Implicit scheme

$$u^{n+1} = u^n - R\left(u^{n+1}\right)$$

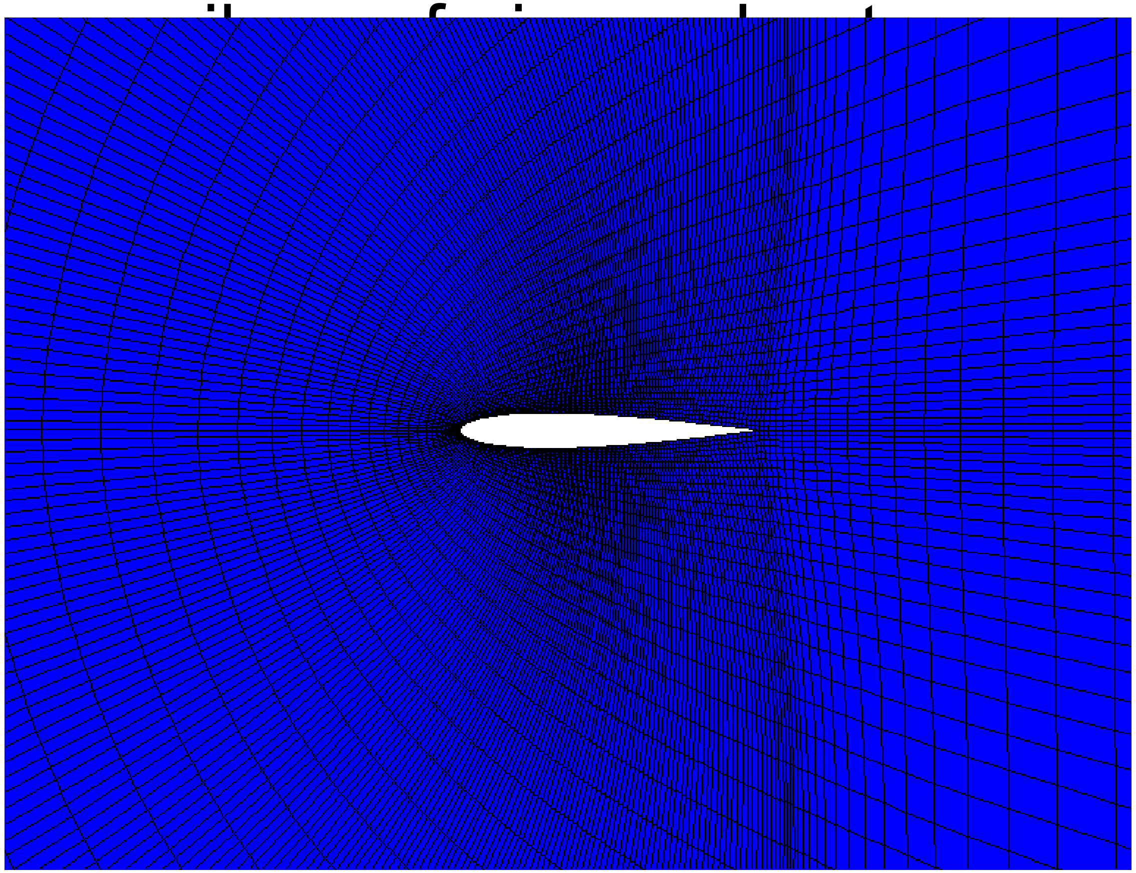
Linearization – Taylor expansion

$$R\left(u^{n+1}\right) = R\left(u^n\right) + \left(u^{n+1} - u^n\right)\frac{\partial R\left(u^n\right)}{\partial u} + Fehler$$

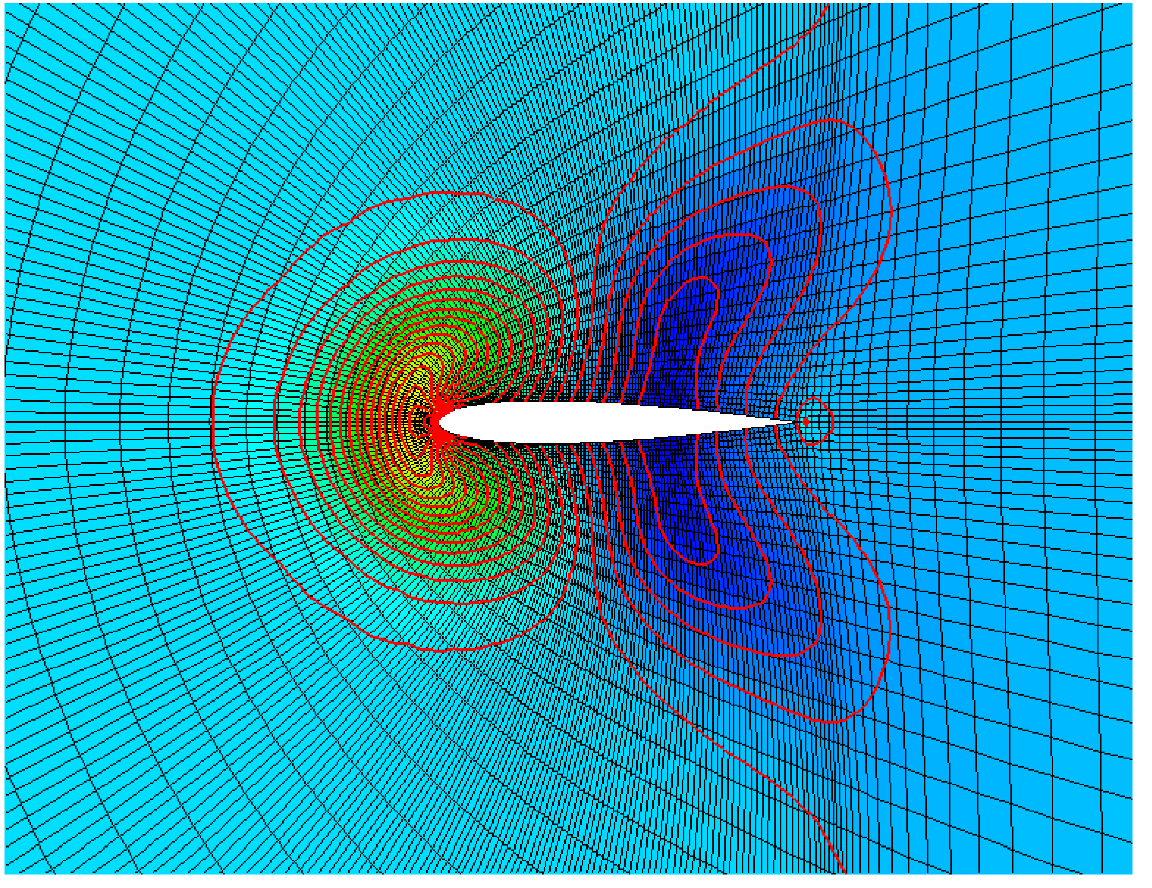
Linear system of equations

$$\left(I + \frac{\partial R\left(u^n\right)}{\partial u}\right)\Delta u = R\left(u^n\right) \qquad \text{with} \qquad \Delta u := u^{n+1} - u^n$$

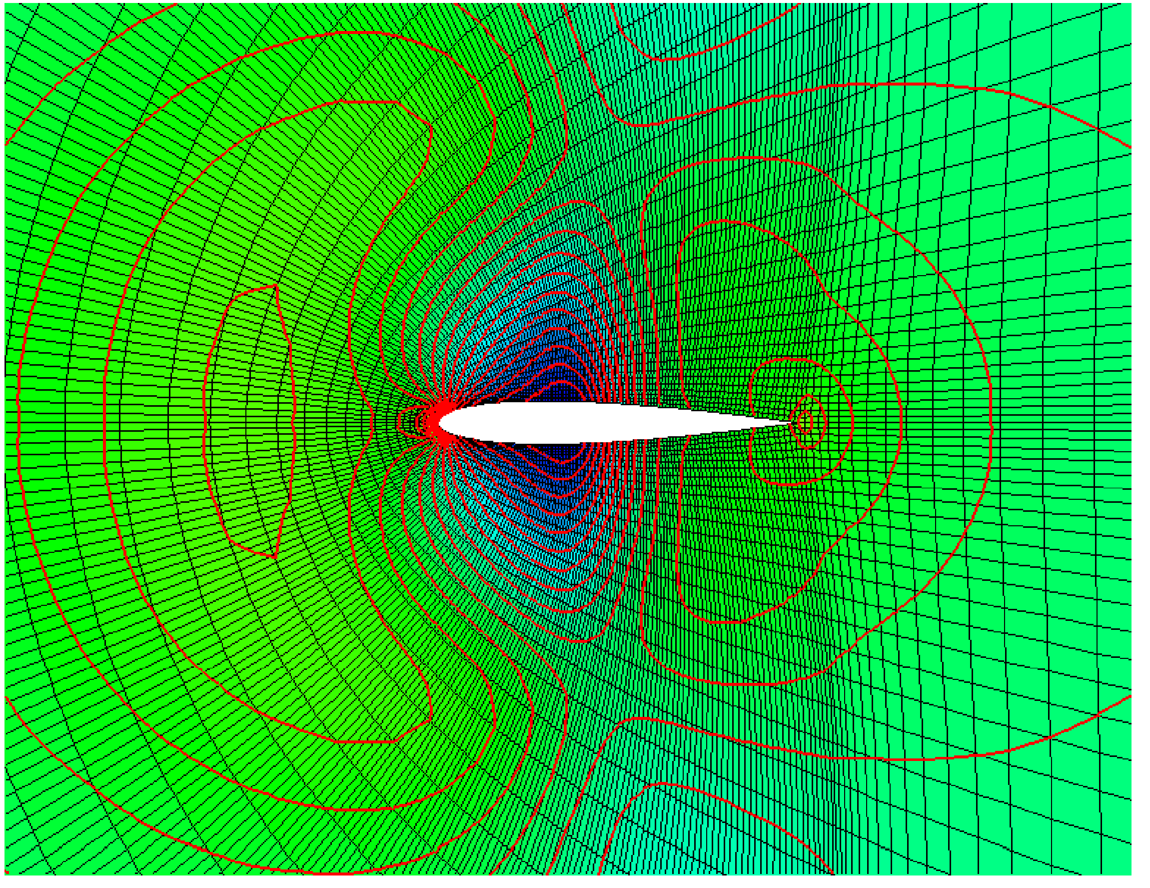
NACA 0012, Anstellwinkel 0°, M = 0,



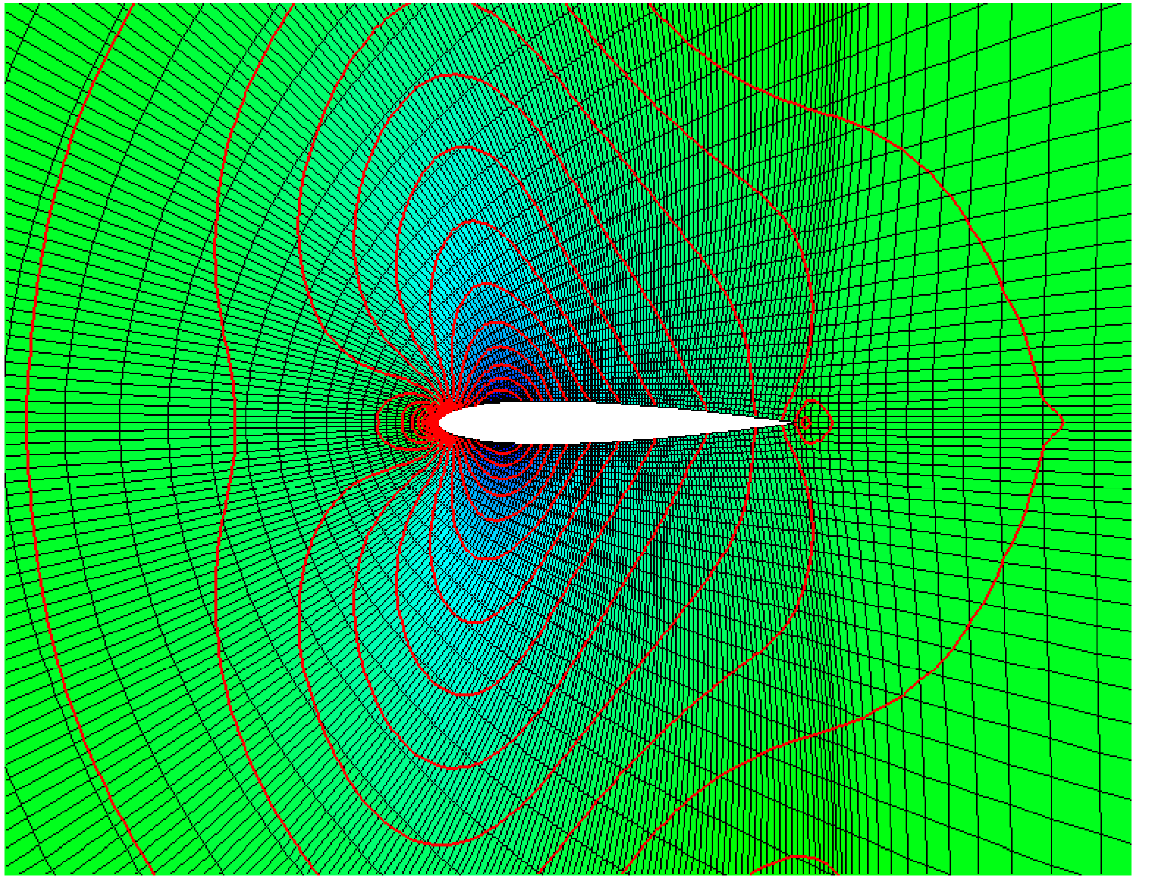
Anfangszustand: Das Profil erscheint in der ungestörten Strömung



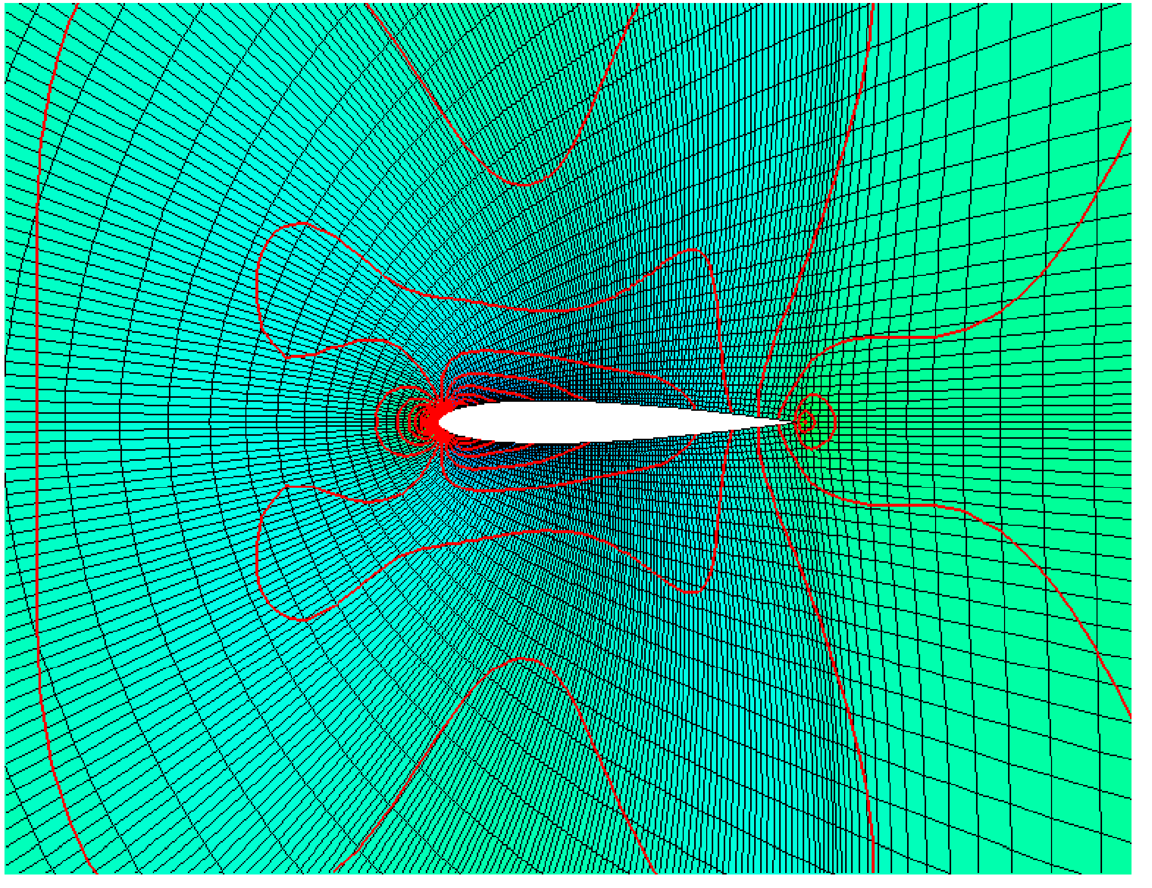
nach 200 Iterationen



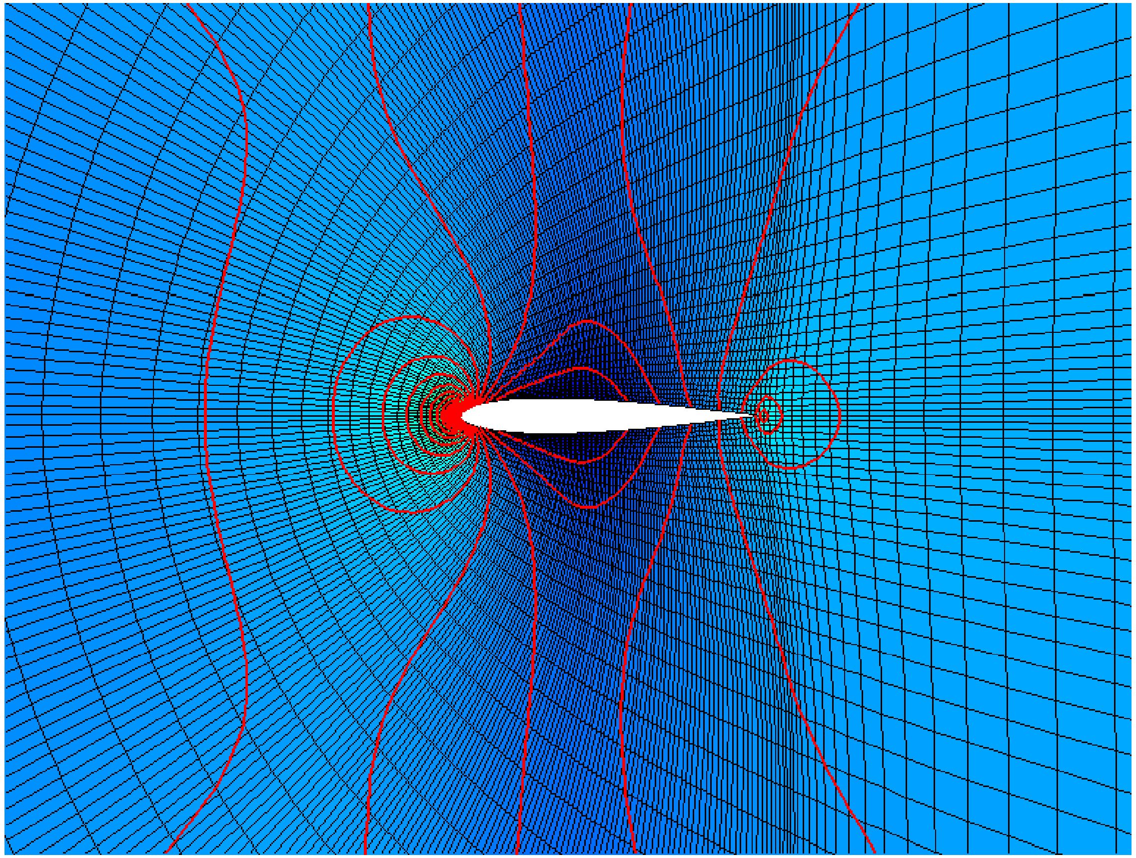
nach 400 Iterationen



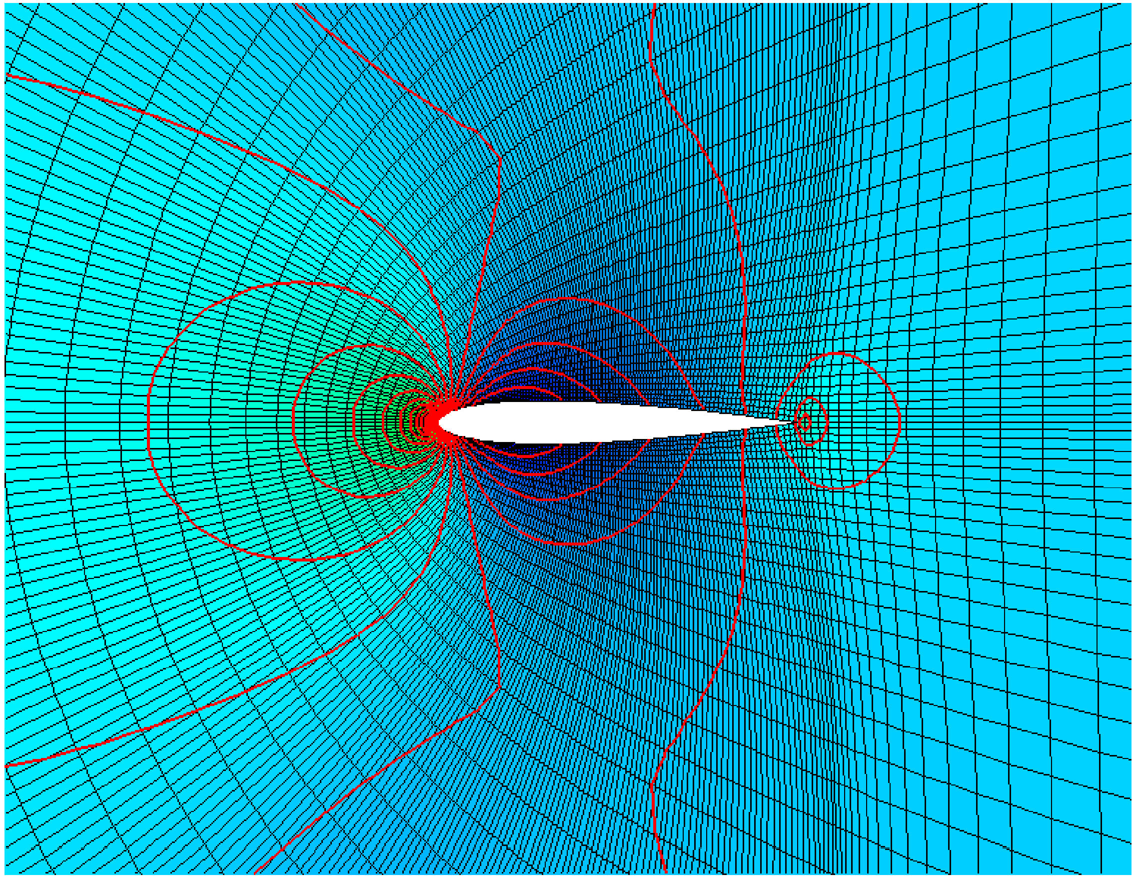
nach 600 Iterationen



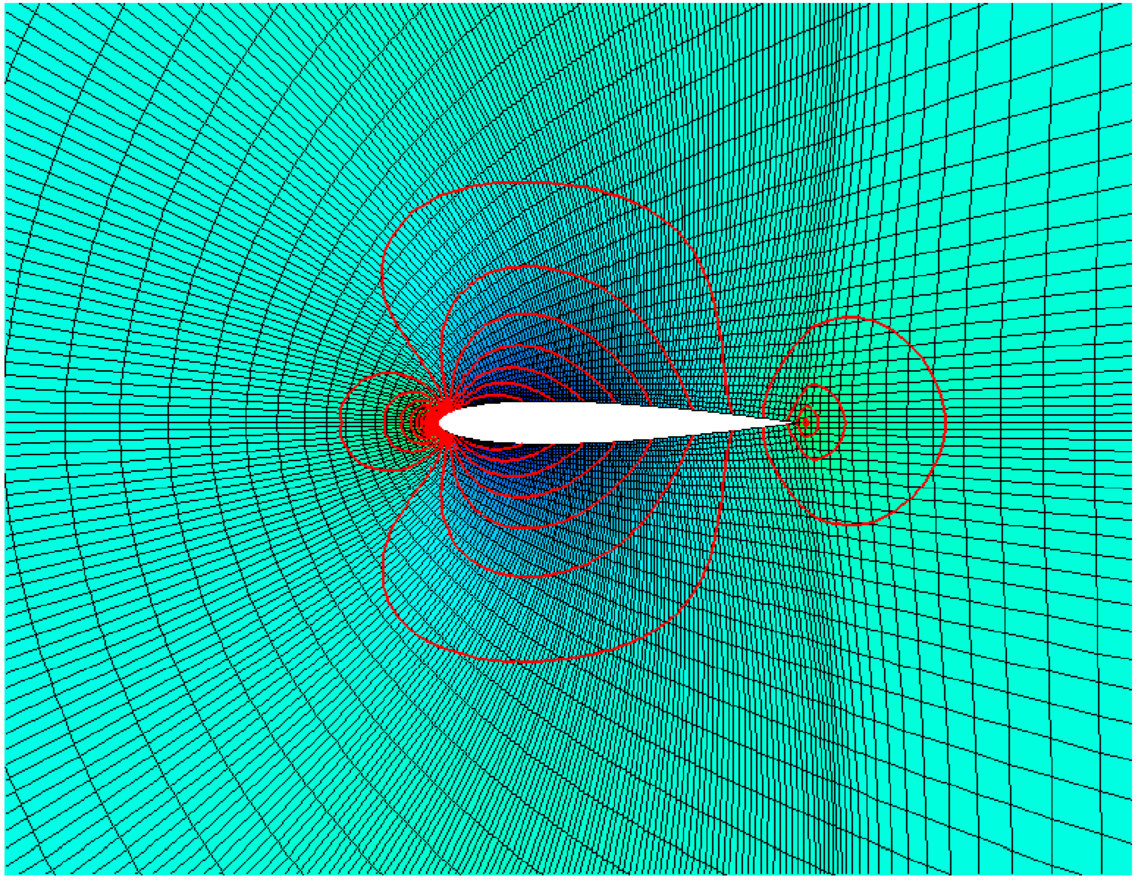
nach 800 Iterationen



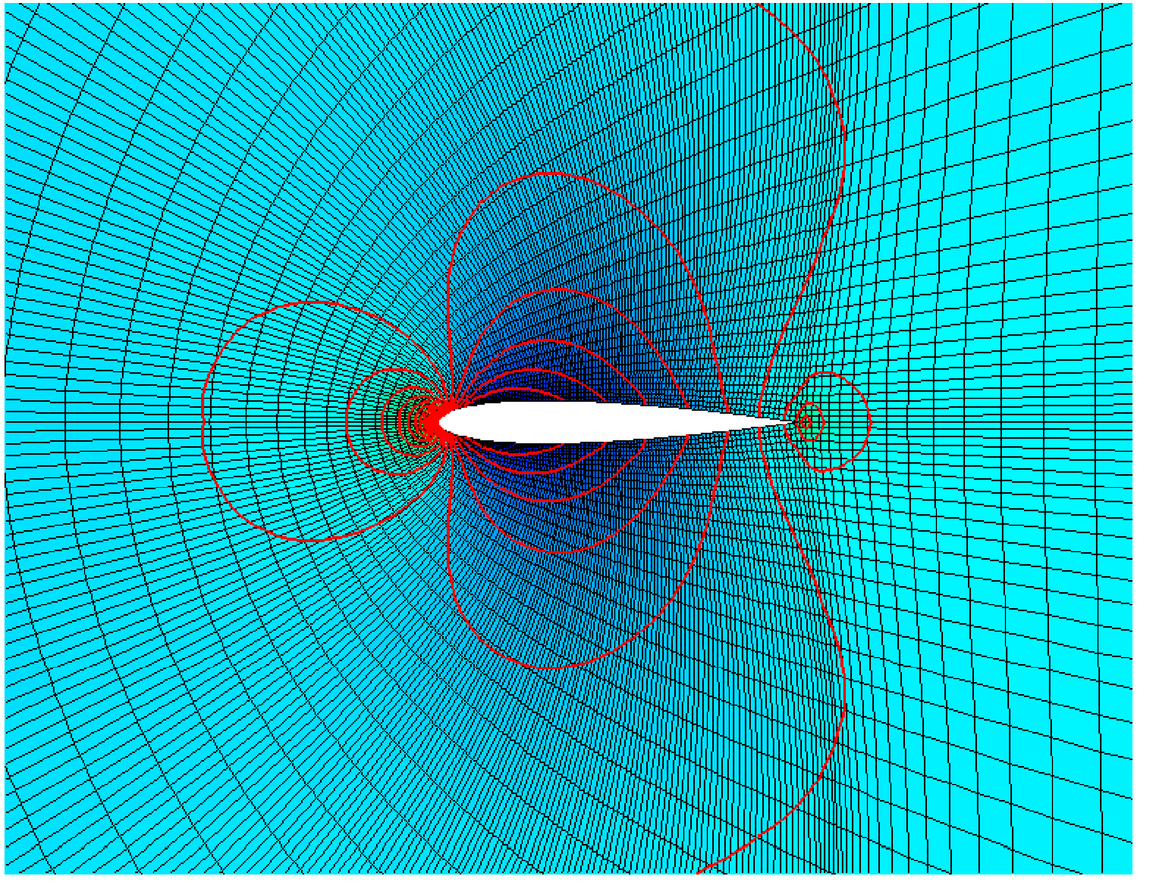
nach 1000 Iterationen



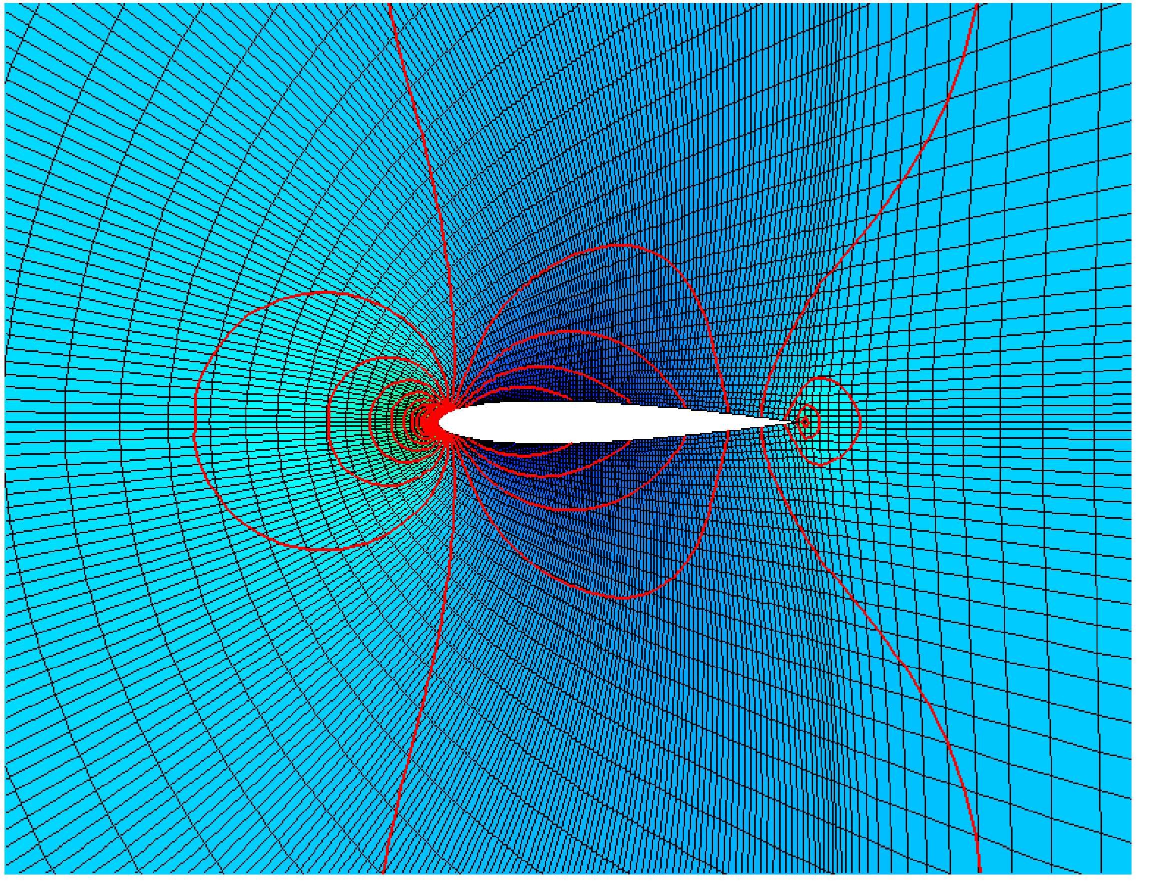
nach 1200 Iterationen



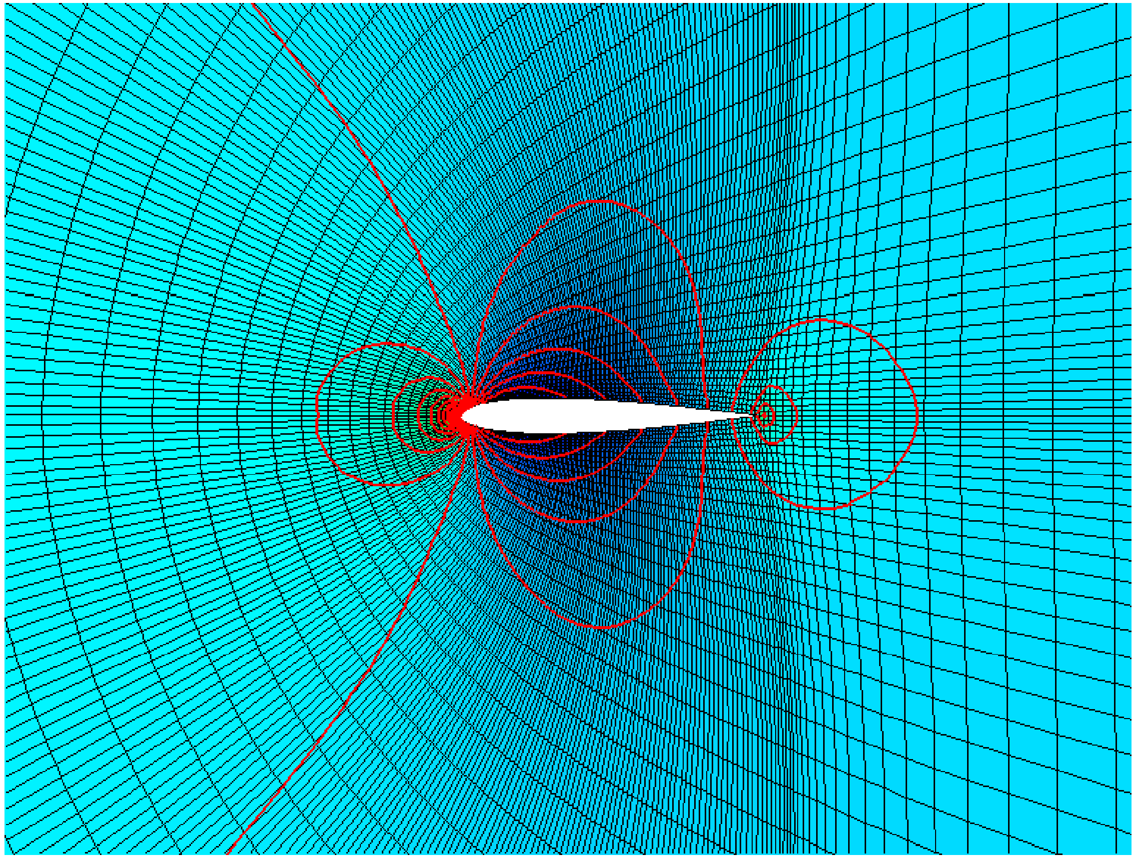
nach 1400 Iterationen



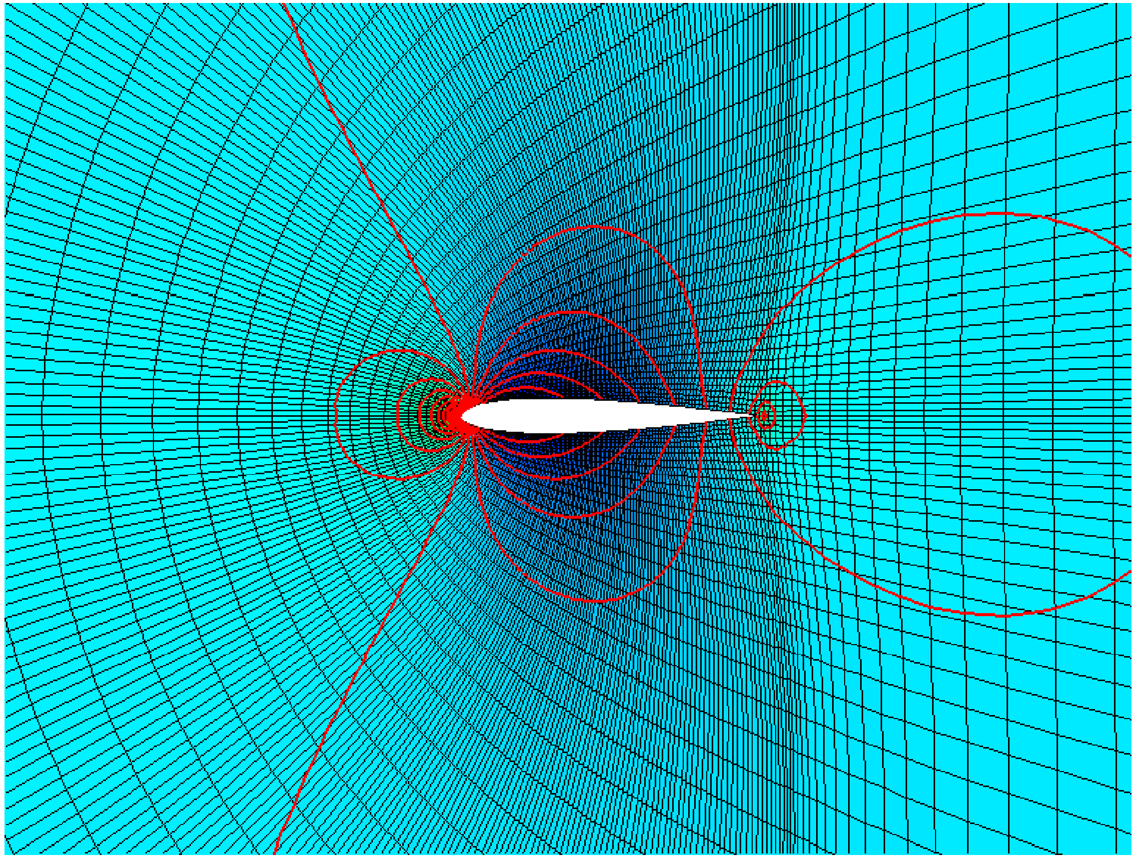
nach 1600 Iterationen



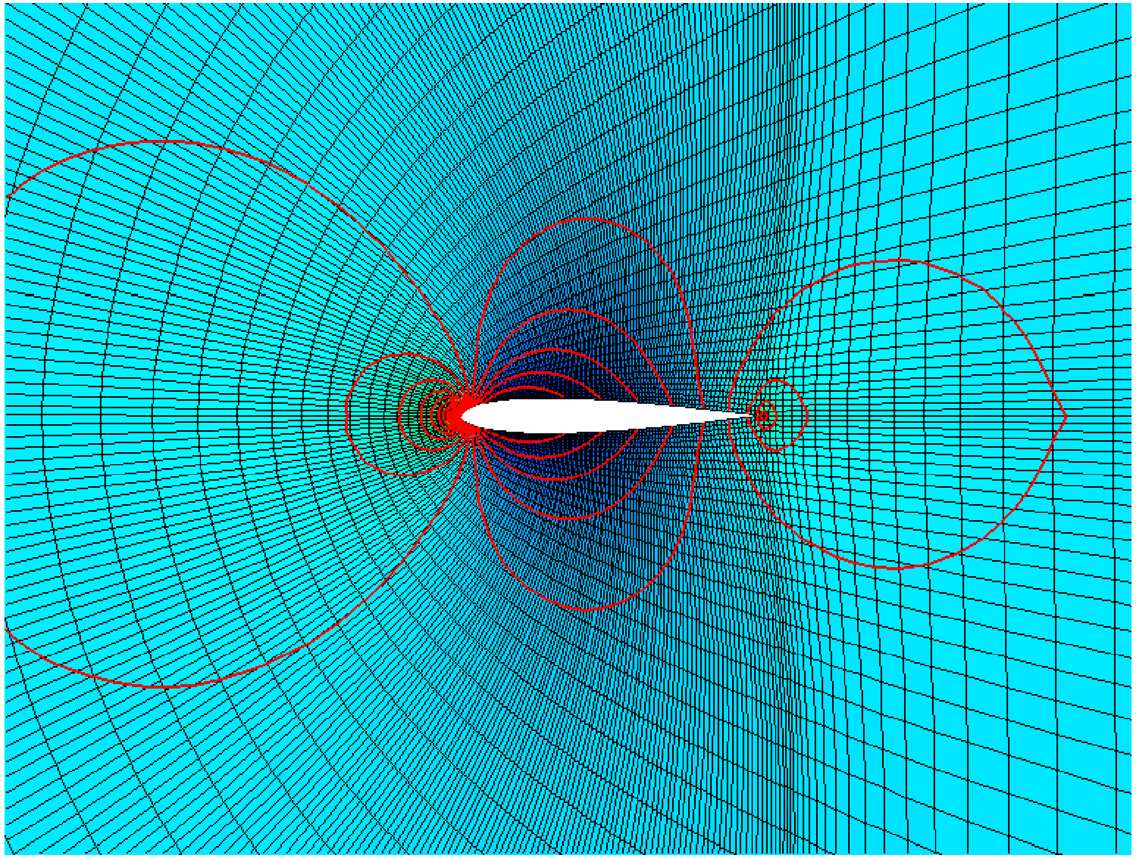
nach 1800 Iterationen



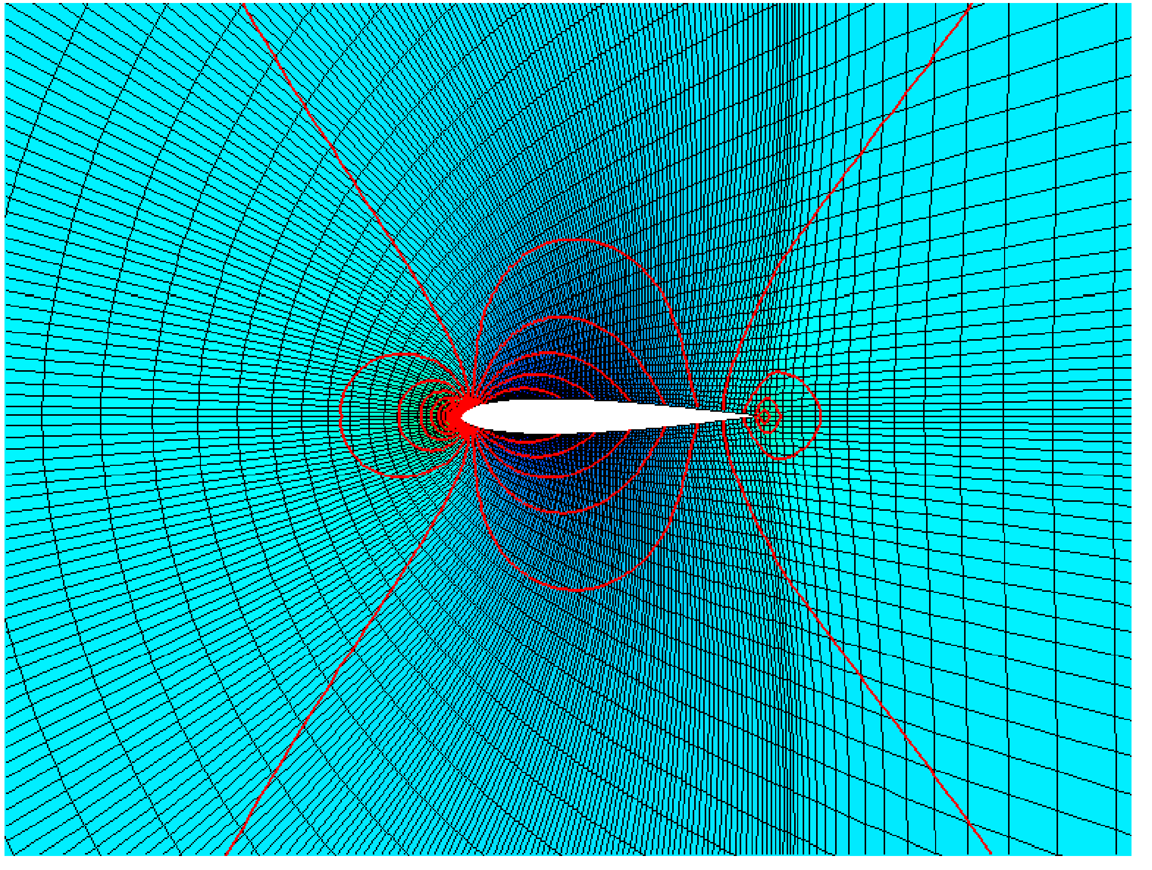
nach 2000 Iterationen



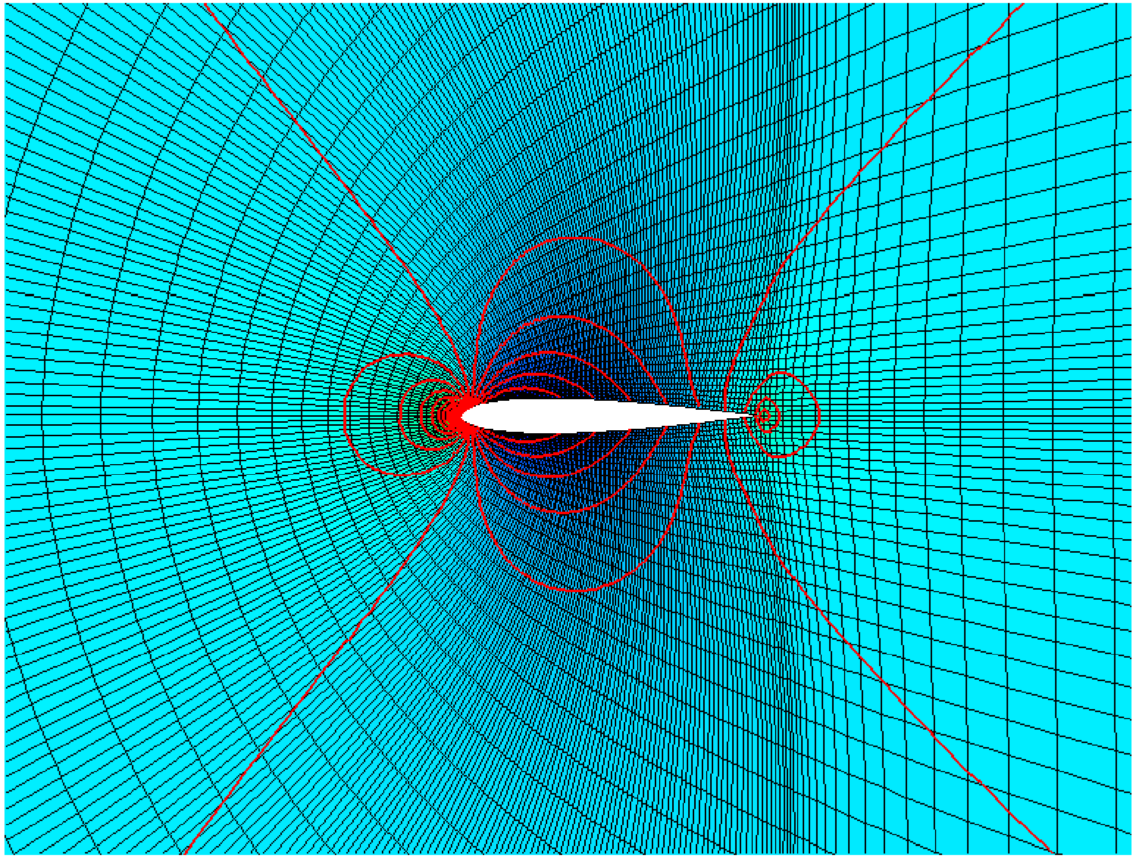
nach 3000 Iterationen



nach 4000 Iterationen



nach 5000 Iterationen



nach 9000 Iterationen

Conclusions: Numerical Gasdynamics

$$\rho_t + \nabla \cdot (\rho v) = 0$$

$$(\rho v)_t + \nabla \cdot ((\rho v) \circ v) + \nabla p = \nabla \cdot \tau + f$$

$$e_t + \nabla \cdot (v(e + p)) = \nabla(\tau \cdot v) - \nabla \cdot q + f \cdot v + Q$$

convection terms
hyperbolic - upwind

dissipation terms
parabolic - central

The choice of the numerical method depends on the problem

Choice of the Numerical Method

Unsteady Problems

Convection terms dominate:

- Explicit method for Euler equations with small parabolic perturbation
- Low Mach number ($0.1<M<1.4$): FV-method of Jameson type
- Otherwise: FV-shock-capturing scheme, MUSCL
- Viscosity and heat conduction small => explicit approximation possible, boundary layers?

  implicit approximation with splitting

Dissipation terms dominate

- Implicit approximation of viscosity and heat conduction
- Approximation of convection terms is simpler

Choice of the Numerical Method

Stationary Problems

Convection terms dominate:

- Implicit FV-shock-capturing scheme, first order accurate in time, large time steps
- Low Mach numbers: Implicit FD-scheme, upwind?
- Viscosity and heat conduction implicit
- Explicit FV-shock-capturing scheme with local time
- stepping, not longer time accurate, iteration method to get the stationary state

Dissipation terms dominate

- Implicit approximation of viscosity and heat conduction
- central approximation of convection terms

Some Remarks to FE-Methods

Classical FE-Methods: Similar to FD-methods, because they require continuity

Stabilization at strong gradients, artificial viscosity

Upwinding of convection terms: Petrov-Galerkin

Low Mach numbers

New: Discontinuous Galerkin Schemes

still topic of research

Books

Numerical Gasdynamics

C.Hirsch: Numerical Computation of internal and external flow, Vol I,II, John Wiley and Sons 1988, 2nd edition 2007

Standardwerk

K. A. Hoffmann, S. T. Chiang: Computational Fluid Dynamics for Engineers, Vol. I,II

C. B. Laney: Computational Gasdynamics, Cambridge University Press 1998

E.F. Toro: Godunov Methods: Theory and Applications Springer Verlag