

## Chapter 6

# CONNECTION TO OTHER STATISTICAL AVERAGES

### 1.1 Eulerian statistical average (ensemble average)

The basic concept of the Eulerian statistical averaging has been explained in Chapter 3. By considering a set of  $N$  similar samples or systems, a statistical mean value is defined by a simple arithmetic mean among them, Eq.(3-7). Thus, the mathematical operation of integration for the time averaging should be replaced by that of summation in the statistical averaging. It is evident that the entire derivation of the field theory based on the statistical averaging can be carried out in parallel with Chapters 4 and 5 by simply substituting the finite statistical mean operator, Eq.(3-7), in the place of the time averaging, Eq.(3-2).

The most important parameters are  $N_k$  and  $N_s$  which represent the number of occurrences of the  $k^{\text{th}}$ -phase and the interfaces if the interfacial thickness is  $\delta$ . Then the void fraction can be defined as a ratio of  $N_k$  to  $N$  and taking the limit  $\delta \rightarrow 0$ . The general function  $(F)_n$  and  $(F_k)_n$  can be defined in space-time domain in analogy with Eq.(4-8).

Since the derivation of the statistically averaged equations follows exactly the same steps as in the case of the time averaging, we only list the most important and characteristic relations between them. In Table 6-1 we see four basic parameters of averaging, namely: the void fraction  $\alpha_k$ ; the average of the two-phase general function  $F$ ; the average of the  $k^{\text{th}}$ -phase general function  $F_k$ ; and the interfacial area concentration in a unit volume.

It is evident that the general balance equations of the form of Eqs.(5-16) and (5-17) can be obtained from the statistical averaging applied to a set of  $N$  similar samples. We note here that important differences between the time and statistical averaging exist not in the resulting form of the balance

Table 6-1. Relations between time and statistical averages (Ishii, 1975)

Variable	Time Average	Statistical Average
$\alpha_k$	$\frac{\Delta t_k}{\Delta t}$	$\lim_{\delta \rightarrow 0} \frac{N_k}{N}$
$\overline{F}$	$\lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_r} F dt$	$\lim_{\delta \rightarrow 0} \frac{1}{N} \sum_n (F)_n$
$\overline{F_k}$	$\lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_r} F_k dt$	$\lim_{\delta \rightarrow 0} \frac{1}{N} \sum_n (F_k)_n$
$a_i$	$\lim_{\delta \rightarrow 0} \frac{1}{\Delta t} \int_{[\Delta t]_s} \frac{1}{\delta} dt$	$\lim_{\delta \rightarrow 0} \frac{1}{N} \sum_{N_s} \left( \frac{1}{\delta} \right)$

equations but in the interpretation of the variables with respect to an actual flow, as it has been discussed in detail in the Section 1.3 of Chapter 3.

## 1.2 Boltzmann statistical average

Because of its unique characteristic among various averaging procedures, we discuss in detail the Boltzmann statistical averaging applied to two-phase flow systems. First we recall that  $f(\mathbf{x}, t, \boldsymbol{\xi})$  is the particle density function where  $\mathbf{x}$ ,  $t$  and  $\boldsymbol{\xi}$  represents the position, time and velocity of a particle, respectively. In the standard analysis of the kinetic theory of gases, the particle mass of each component is considered to be constant because it represents the molecular mass. However, in applications to two-phase highly dispersed flows, it may be necessary to assume that the particle mass varies. Thus, by taking into account the existence of variable particle mass, we define

$$f_{kn} = f_{kn}(\mathbf{x}, t, \boldsymbol{\xi}) \quad (6-1)$$

where  $f_{kn}$  is the particle density function of  $k^{\text{th}}$ -phase particles having  $m_{kn}$  mass. It can be said that  $m_{kn}$  is a multiple of the single molecular mass. The total number of  $k^{\text{th}}$ -phase particles in the phase space element  $d\mathbf{x}d\boldsymbol{\xi}$  at  $\mathbf{x}$  and  $\boldsymbol{\xi}$  is given by

$$\sum_n f_{kn}(\mathbf{x}, t, \boldsymbol{\xi}) d\mathbf{x}d\boldsymbol{\xi}. \quad (6-2)$$

The Boltzmann equation for the  $m_{kn}$  particles can be obtained by introducing  $\mathbf{g}_k$ , the external force field, and taking the balance on the number of particles, thus

$$\frac{\partial f_{kn}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\boldsymbol{\xi} f_{kn}) + \frac{\partial}{\partial \boldsymbol{\xi}} \cdot (\mathbf{g}_{kn} f_{kn}) = C_{kn}^{+} - C_{kn}^{-} \quad (6-3)$$

where  $C_{kn}^{+}$  and  $C_{kn}^{-}$  represent the source and sink terms, namely, the gain and loss of the  $m_{kn}$  particles caused by the changes in the particle mass and by the collisions that throw the particles in and out of the phase element  $d\boldsymbol{\xi}$ . If  $\mathbf{g}_{kn}$  is independent of the velocity, then we have

$$\frac{\partial f_{kn}}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f_{kn}}{\partial \mathbf{x}} + \mathbf{g}_{kn} \cdot \frac{\partial f_{kn}}{\partial \boldsymbol{\xi}} = C_{kn}^{+} - C_{kn}^{-}. \quad (6-4)$$

The above Boltzmann transport equation with the collision terms expressed by the simple model of Maxwell's binary collision integral is called the *Maxwell-Boltzmann equation*. It is the foundation of the kinetic theory of gases. A similar approach can be used for neutron transport.

The partial density of the  $m_{kn}$  particles is given by

$$\rho_{kn} = \int m_{kn} f_{kn} d\boldsymbol{\xi}. \quad (6-5)$$

Thus, the expectations based upon the probability  $f_{kn}$  can be defined as

$$\widehat{\psi}_{kn}(\mathbf{x}, t) = \frac{\int m_{kn} f_{kn} \psi_{kn} d\boldsymbol{\xi}}{\int m_{kn} f_{kn} d\boldsymbol{\xi}}. \quad (6-6)$$

Since the total mass is the sum of the mass of each particle, the partial density of the  $k^{\text{th}}$ -phase is given by

$$\bar{\rho}_k = \sum_n \rho_{kn} = \sum_n \int m_{kn} f_{kn} d\boldsymbol{\xi}. \quad (6-7)$$

It follows that the mass weighted mean value is defined by

$$\widehat{\psi}_k(\mathbf{x}, t) = \frac{\sum_n \int m_{kn} f_{kn} \psi_{kn} d\xi}{\sum_n \int m_{kn} f_{kn} d\xi} = \frac{\sum_n \rho_{kn} \widehat{\psi}_{kn}}{\overline{\rho}_k}. \quad (6-8)$$

Denoting the mean velocity by  $\widehat{\mathbf{v}}_k$ , the peculiar velocity of each particle is given by

$$\mathbf{V}_{kn} = \boldsymbol{\xi}_{kn} - \widehat{\mathbf{v}}_k. \quad (6-9)$$

If we multiply the Maxwell-Boltzmann transport equation, i.e. Eq.(6-4), by  $m_{kn} \psi_{kn}$ , integrate it over the phase velocity  $\boldsymbol{\xi}$  and then sum it up for all kinds of particles of the  $k^{\text{th}}$ -phase, we obtain *Maxwell's equation of transfer* in terms of mean values. Hence,

$$\begin{aligned} \frac{\partial}{\partial t} (\overline{\rho}_k \widehat{\psi}_k) + \nabla \cdot (\overline{\rho}_k \widehat{\psi}_k \widehat{\mathbf{v}}_k) &= -\nabla \cdot \sum_n \rho_{kn} \widehat{\psi}_{kn} \mathbf{V}_{kn} \\ &- \sum_n \int m_{kn} \psi_{kn} \mathbf{g}_{kn} \cdot \frac{\partial f_{kn}}{\partial \boldsymbol{\xi}} d\boldsymbol{\xi} \\ &+ \sum_n \int m_{kn} f_{kn} \left[ \frac{\partial \psi_{kn}}{\partial t} + \boldsymbol{\xi} \cdot \nabla \psi_{kn} \right] d\boldsymbol{\xi} \\ &+ \sum_n \int [C_{kn}^+ - C_{kn}^-] m_{kn} \psi_{kn} d\boldsymbol{\xi}. \end{aligned} \quad (6-10)$$

We note here that each term in the right-hand side of Eq.(6-10) represents the transfer due to inter-particle diffusions, the source due to the body force field, the multi-molecular particle effect which arises when the transport property  $\psi_{kn}$  is a function of the phase velocity  $\boldsymbol{\xi}$ , and the source due to phase changes and/or collisions.

#### *Conservation of Mass*

The mass field equation can be obtained by setting  $\psi_{kn} = 1$  in Eq.(6-10), thus we have

$$\frac{\partial \overline{\rho}_k}{\partial t} + \nabla \cdot (\overline{\rho}_k \widehat{\mathbf{v}}_k) = \Gamma_k \quad (6-11)$$

where

$$\Gamma_k = \sum_n \int [C_{kn}^+ - C_{kn}^-] m_{kn} d\xi. \quad (6-12)$$

The assumptions of over all conservation of mass gives

$$\sum_{k=1}^2 \Gamma_k = 0. \quad (6-13)$$

From Eqs.(6-11) and (6-13) we have the mixture continuity equation

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m) = 0 \quad (6-14)$$

where

$$\rho_m = \sum_{k=1}^2 \overline{\rho_k} \quad \text{and} \quad \mathbf{v}_m = \sum_{k=1}^2 \frac{\overline{\rho_k} \widehat{\mathbf{v}_k}}{\rho_m}. \quad (6-15)$$

Recalling that  $\rho_k$  is a partial density of the  $k^{\text{th}}$ -phase, we find complete similarity between the results of the time averaging, Eqs.(5-21), (5-22) and (5-40), and those of the Boltzmann statistical averaging, Eqs.(6-11), (6-13) and (6-14).

#### *Conservation of Momentum*

The linear momentum equation can be obtained from Eq.(6-10) by setting  $\psi_{kn} = \xi$ , thus

$$\frac{\partial \overline{\rho_k} \widehat{\mathbf{v}_k}}{\partial t} + \nabla \cdot (\overline{\rho_k} \widehat{\mathbf{v}_k} \widehat{\mathbf{v}_k}) = -\nabla \cdot \overline{\mathbf{P}_k} + \overline{\rho_k} \widehat{\mathbf{g}_k} + \mathbf{M}_k \quad (6-16)$$

where the partial pressure tensor  $\overline{\mathbf{P}_k}$  is defined by

$$\overline{\mathbf{P}_k} \equiv \sum_n \rho_{kn} \widehat{\mathbf{V}_{kn} \mathbf{V}_{kn}} \quad (6-17)$$

and the momentum supply  $\mathbf{M}_k$  by

$$\mathbf{M}_k \equiv \sum_n \int [C_{kn}^+ - C_{kn}^-] m_{kn} \xi d\xi. \quad (6-18)$$

We note here that if the negative of the partial pressure tensor,  $-\overline{P}_k$ , is interpreted as the combined stresses acting on the  $k^{\text{th}}$ -phase, the momentum equation (6-16) has exactly the same form as that obtained from the time averaging, namely, Eq.(5-26). However, the physical meaning of the flux term and the momentum source  $\mathbf{M}_k$  in this equation can be significantly different from those of the time-averaged equation.

As an example, let us consider a dispersed two-phase flow system with phase 1 as a dispersed phase. If the particle sizes are considerably small and the continuous phase is a dilute gas, then the formulation essentially reduces to that of reacting gas mixtures. In this case, the effects of the collisions of particles with the molecules and/or particles that lie outside of a particular volume element in real space can be neglected. Thus, the total collision term  $\sum_{k=1}^2 \mathbf{M}_k$  can be taken as zero. Hence, as we can see from Eq.(6-18), the collision term of each phase consists of the momentum source due to mass transfer and the drag forces resulting from the momentum exchange during collisions.

Furthermore, if the particles have definite volumes in contrast to the previous point mass assumption, the multi-collisions of particles with molecules and with other particles whose centers lie outside of the volume element become important. In this case, the collision term for the dispersed phase can be split to three different parts: the internal momentum transfer due to collisions; the effects of the changes of phases; and the external collision effect. Consequently, the introduction of the void fraction  $\alpha_k$  in the formulation is necessary, where  $\alpha_1$  is the ratio of the volume occupied by the particles to the total volume element and thus  $\alpha_2 = 1 - \alpha_1$ . Nevertheless, in order to integrate the momentum collision term, it is necessary to introduce models of molecule-particle and particle-particle collision processes. Because of the multi-collisions between a particle and a cloud of molecules and of the effects of the phase changes, these models will be extremely complicated. Thus we do not go into the detail of the collision integrals here. If the standard fluid mechanic viewpoint is introduced, it is possible to interpret the momentum source  $\mathbf{M}_k$  into relevant physical terms. From the above discussion, we expand the total collision term of each phase in the following form

$$\mathbf{M}_k = \mathbf{M}_k^i + \mathbf{M}_k^\Gamma + \mathbf{M}_k^e. \quad (6-19)$$

Each term on the right-hand side represents the internal collision force interaction, the momentum source due to phase changes and the external collision force interaction because of the finite particle size, respectively. The first term can be considered as the standard drag force and the pressure

effect due to the void fraction gradient, whereas the last term gives extra flux from the particles having their centers outside of a volume element.

For the mixture, the total collision term  $\sum_{k=1}^2 \mathbf{M}_k$  is not zero due to the external effects  $\mathbf{M}_k^e$ . Thus, we can write approximately

$$\sum_{k=1}^2 \mathbf{M}_k \doteq \mathbf{M}_m - \left( \nabla \cdot \frac{\alpha_1}{\alpha_2} \overline{P_2} \right) \quad (6-20)$$

where  $\mathbf{M}_m$  can be considered as the inter-particle collision effect. The second term is the effect of the continuous phase on particles which lie on the boundary of a volume element. The real pressure tensor for the continuous phase 2 should be

$$\overline{\overline{P_2}} = \frac{\overline{P_2}}{\alpha_2}. \quad (6-21)$$

Then the mixture momentum equation becomes

$$\begin{aligned} & \frac{\partial \rho_m \mathbf{v}_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m \mathbf{v}_m) \\ &= -\nabla \cdot \left( \overline{P_1} + \overline{\overline{P_2}} + \sum_{k=1}^2 \overline{\rho_k \mathbf{V}_{km} \mathbf{V}_{km}} \right) + \rho_m \mathbf{g}_m + \mathbf{M}_m. \end{aligned} \quad (6-22)$$

Thus, the mixture total stress consists of the partial stress of the dispersed phase, the real stress of the continuous phase and the diffusion stress due to the relative motion between two phases. Here, the definition of the average diffusion velocity  $\mathbf{V}_{km}$  takes the form of Eq.(4-89). It is evident that significant differences exist between the physical meaning of the stress tensor term of the present mixture momentum equation based on the Boltzmann transport equation and that of the one obtained from the time averaging, namely, Eq.(5-42). This basic difference arises because, in the former approach, the stresses are defined from the motions of the particles. Thus, the stress inside the particles has no place in the analysis. In the latter approach, however, the stresses are defined everywhere in the system. We may add here that the Boltzmann statistical average can be easily extended to include the turbulent fluctuations in the continuous phase by considering the eddy transports as the particle transports.

### Conservation of Energy

In contrast to the simple kinetic theory, the energy transfer by multi-molecular particles is considerably complicated due to the internal freedom of the energy state of the particles. It is obvious that the kinetic theory translational temperature based on the diffusion kinetic energy is not useful if each particle consists of a large number of molecules.

Now let us suppose that  $\psi_{kn}$  is the total energy carried by the  $k^{\text{th}}$ -phase particles having mass  $m_{kn}$ , thus

$$\psi_{kn} = \frac{1}{2} \xi_{kn}^2 + u_{kn} \quad (6-23)$$

where the first term is the translational kinetic energy and  $u_{kn}$  the internal energy contained by molecules in the particles. Since the term  $u_{kn}$  is not a function of  $\xi_{kn}$ , the transport equation (6-10) is not a velocity moment of the Boltzmann transport equation.

We introduce the average energy and the flux of the  $k^{\text{th}}$ -phase as

$$\widehat{e}_k = \frac{\sum_n \rho_{kn} \left( \frac{1}{2} \widehat{V}_{kn}^2 + \widehat{u}_{kn} \right)}{\overline{\rho}_k} \quad (6-24)$$

and

$$\overline{\mathbf{q}}_k = \sum_n \rho_{kn} \overline{\left( \frac{1}{2} V_{kn}^2 + u_{kn} \right) \mathbf{V}_{kn}} \quad (6-25)$$

where the definitions of the right-hand side averages follow Eq.(6-8). Substituting Eqs.(6-23), (6-24) and (6-25) into Eq.(6-10), we obtain the total energy equation for the  $k^{\text{th}}$ -phase

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \overline{\rho}_k \left( \widehat{e}_k + \frac{1}{2} \widehat{v}_k^2 \right) \right\} + \nabla \cdot \left\{ \overline{\rho}_k \left( \widehat{e}_k + \frac{1}{2} \widehat{v}_k^2 \right) \widehat{\mathbf{v}}_k \right\} \\ &= -\nabla \cdot \overline{\mathbf{q}}_k - \nabla \cdot \left( \overline{\mathbf{P}}_k \cdot \widehat{\mathbf{v}}_k \right) + \overline{\rho}_k \widehat{\mathbf{g}}_k \cdot \widehat{\mathbf{v}}_k \\ &+ \sum_n \int m_{kn} f_{kn} \left( \frac{\partial u_{kn}}{\partial t} + \xi \cdot \nabla u_{kn} \right) d\xi + E_k. \end{aligned} \quad (6-26)$$



The average internal energy  $\widehat{e}_k$  is the sum of the random thermal translational energy and the true internal energy of the particles. This definition is in complete analogy with Eq.(5-31) of the time averaging. The most important characteristic appears in the last two terms of Eq.(6-26). It is evident that the change of the individual particle internal energy given by the second term from the last is coupled with the collision term  $E_k$  since, in the absence of the long range energy exchanges, the particle internal energy changes only by the interactions with the surrounding molecules and particles. Thus, in analogy with the momentum exchange given by Eq.(6-20), the total energy interaction can be given as

$$\begin{aligned} & \sum_{k=1}^2 \left\{ E_k + \sum_n \int m_{kn} f_{kn} \left( \frac{\partial u_{kn}}{\partial t} + \xi \cdot \nabla u_{kn} \right) d\xi \right\} \\ & \sim -\nabla \cdot \left\{ \frac{\alpha_1}{\alpha_2} (\overline{\mathbf{q}}_2 + \overline{\mathbf{P}}_2 \cdot \widehat{\mathbf{v}}_2) \right\} + E_m. \end{aligned} \quad (6-27)$$

Here the first term of the right-hand side of Eq.(6-27) takes into account the particle-molecular collisions for finite volume particles. The term denoted by  $E_m$  represents such effects as the inter-particle collision transport of energy. By adding two energy equations for each phase, the mixture equation can be obtained

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \rho_m \left[ e_m + \left( \frac{v^2}{2} \right)_m \right] \right\} + \nabla \cdot \left\{ \rho_m \left[ e_m + \left( \frac{v^2}{2} \right)_m \right] \mathbf{v}_m \right\} \\ & = -\nabla \cdot \left[ \overline{\mathbf{q}}_1 + \overline{\mathbf{q}}_2 + \sum_{k=1}^2 \overline{\rho}_k \left( \widehat{e}_k + \frac{\widehat{v}_k^2}{2} \right) \mathbf{V}_{km} \right] + \rho_m \mathbf{g}_m \cdot \mathbf{v}_m \\ & -\nabla \cdot \left[ \overline{\mathbf{P}}_1 \cdot \widehat{\mathbf{v}}_1 + \overline{\mathbf{P}}_2 \cdot \widehat{\mathbf{v}}_2 \right] + \sum_k \overline{\rho}_k \mathbf{g}_k \cdot \mathbf{V}_{km} + E_m. \end{aligned} \quad (6-28)$$

Here, we again note that phase 1 is the dispersed phase. The physical significance of each term of the total energy equation parallels that of the momentum equation. The partial fluxes  $\overline{\mathbf{q}}_1$  and  $\overline{\mathbf{P}}_1$  for the dispersed phase, and the real fluxes  $\overline{\mathbf{q}}_2$  and  $\overline{\mathbf{P}}_2$  for the continuous phase appear because of the finite volume occupied by the particles of phase 1.

As in the case of the mixture momentum equation, significant differences between the Boltzmann statistical model and the time-averaged model appear in the total flux terms of heat and work. Since the heat transfer and work inside the particles are not considered in the former model, the partial

heat flux and the partial pressure tensor for the dispersed phase represent the transfer due to inter-particle diffusions. The molecules of the continuous phase 2 are treated as the point mass, thus *the fluxes of each phase do not appear symmetrically*.

The interest of the Boltzmann statistical average applied to two-phase flow systems lies mainly on the study of constitutive equations from a simple model for the collision terms together with stochastic assumptions. It is highly improbable that this can be a general model for a dispersed two-phase flow system, since the inclusion of the effects of particle shapes and deformations in the Boltzmann transport equation brings considerable difficulties in the analysis. Even without these effects, it is anticipated that the collision terms for each phase are very complicated due to three effects: 1) the inter-particle collisions, coalescences and disintegrations; 2) the multi-collisions between a particle and a large number of molecules; 3) the existence of phase changes. Furthermore as it can be seen from Eq.(6-27), the dispersed phase energy transfer term requires a special constitutive law for the heat transfer between particles and fluid.

In summarizing the section, it can be said that the Boltzmann statistical averaging is useful for a highly dispersed flow where each particle is considered as a lumped entity rather than as a distributed system itself. For example, the mixture stress tensor for a particle flow has a more natural form in the present model than the one from the time averaging. This is because in such two-phase flow it is not practical to introduce stresses inside the particles. However, the number of particles in a volume element should be significantly large for a statistical treatment of the number density to be realistic. Furthermore, if the deformations of the interfaces and the changes of properties within the particles are important, the Boltzmann statistical method cannot be used. In contrast to the case of dilute gases, the collision integral terms for a dispersed two-phase system with finite particle sizes are extremely complicated, thus to obtain constitutive laws from the statistical mechanics is very difficult. In many cases these collision terms should be supplied from the continuum mechanics considerations. It is, therefore, also possible to construct a model on a combination of the continuum theory and the Boltzmann statistical method. For example, we take the statistical average only for the dispersed phase with the drag forces included in the body force. Then we use the standard volume averaged field equations for the other phase with the corresponding interaction terms. Such a formulation is useful to dispersed two-phase systems as well as to fluidized beds. Furthermore, the Boltzmann statistical method can be very useful for obtaining some constitutive laws as demonstrated in Chapter 10.