# Numerical modelling of the Riemann problem for a mathematical two-phase flow model

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## Abstract

We report preliminary results obtained using Godunov methods of a centred type for hyperbolic systems of conservation laws for which the analytical solution of the Riemann problem is too difficult to develop. For this purpose, we consider a mathematical model used for modelling an unsteady compressible non-equilibrium mixture two-phase flow which results in a well-posed initial value problem in a conservative form. The mathematical two-phase flow model consists of six first-order partial differential equations that represent one-dimensional mass, momentum and energy balances for a mixture of gas-liquid and gas volume concentration, gas mass concentration and the relative velocity balances. This system of six partial differential equations has the mathematical property that its six characteristic roots are all real with a complete set of independent eigenvectors which form the basic structure of the solution of the Riemann problem. The construction of the solution of the Riemann problem for the model poses several difficulties. Since the model possesses a large number of non-linear waves, it is not easy to consider each wave separately to derive a single non-linear algebraic equation for the unknown region, the star region, between the left and right waves. We propose numerical techniques of a centred type specifically developed for high speed single-phase gas flows. A main feature of centred methods is that they do not explicitly require the solution of the Riemann problem. This is a desirable property which guarantees that the methods can handle the solution of the Riemann problem numerically and resolve both rarefaction and shock waves for the model in a simple way with good accuracy. Finally, we present some numerical simulations for the gas-liquid two-phase flow Riemann problem to illustrate the efficiency of the proposed schemes.
*Keywords: gas-liquid two-phase flow, thermodynamically compatible system, mathematical model, well-posed initial value problem, Riemann problem, Godunov methods, TVD SLIC scheme.*

 
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# 1 Introduction

Mathematical and numerical development of two-phase fluid flow models are of considerable practical importance for scientific and industrial problems. Yet, little work has been dedicated to the specific study of the Riemann problem when modelling two-phase flow. The Riemann problem is very useful for a better understanding of the physical and mathematical meanings of the two-phase flow models. In this paper we are concerned with numerical solutions for a class of systems of conservation laws for which the analytical solution of the Riemann problem is too difficult to develop. Systems of this class are thermodynamically compatible, i.e., are generated by one thermodynamic potential alone and have an additional conservation law as a consequence [1, 2, 3]. Furthermore, these systems can be reduced to symmetric hyperbolic systems of balance equations in conservative form [4]. This results in fully hyperbolic conservative systems of governing equations which provide a well-posed initial value problem as in the case of time-dependent two-phase flow equations. The system of equations is a mathematical model which describes gas-liquid two-phase flow as a whole. This model is derived using an approach initiated by Romensky [4] and developed in detail [7] for two-phase media with different velocities of motion of phases and interphase friction. Desirable features of the hyperbolic conservative model include the existence of real eigenvalues and a complete set of linearly independent eigenvectors in a process without dissipation. This allows the application of numerical methods originally developed for high speed single-phase gas flows such as Godunov-type methods which make use of the wave propagation information to construct numerical schemes. To achieve this, one either solves the Riemann problems exactly or provides a minimum amount of information on wave propagation directions. Due to these features, one can formulate a single wave solution such as shocks, contacts and rarefactions to solve the Riemann problem exactly for the hyperbolic conservative model. Thus, depending on the type of non-linear wave, there can be up to sixteen wave patterns involved in the construction of the solution of the Riemann problem.

In general the hyperbolic conservative two-fluid model has a complex mathematical structure due to the fact that it contains a large number of waves in addition to the closure relations. The complexity of the structure of the model makes it very difficult to develop an analytical solution for the Riemann problem or to use Godunov numerical methods of upwind-type. This leads us to consider numerical approximation to the model by using methods that do not require the knowledge of any analytical results of the Riemann problem. We propose to use Godunov methods of centred-type such as the First-Order Centred (FORCE) scheme and the second-order Slope Limiter Centred (SLIC) scheme [6]. Centred methods do not explicitly require the condition of wave propagation information to construct the schemes which make them easy to understand and implement. A comprehensive presentation of Riemann solvers and upwind and centred methods is given in [5]. The numerical methods will give insight into the Riemann problem for the hyperbolic conservative model and will resolve both rarefactions and shocks as we shall see through the numerical results presented in this paper.

Explicit details about the mathematical and numerical analysis of the mathematical two-phase flow model described in this paper can be found in the PhD thesis of the author [7].

## 2 Mathematical two-phase flow model-hyperbolic conservative model

The mathematical hyperbolic conservative two-fluid model considered in this paper is given by the following set of equations in one-space dimension [7]

$$\frac{\partial \mathbb{U}}{\partial t} + \frac{\partial \mathbb{F}}{\partial x} = \mathbb{S}. \tag{1}$$

Here $x \in \mathbb{R}$, $t > 0$ and $\mathbb{U}$, $\mathbb{F}$ and $\mathbb{S}$ are a vector-valued function, the flux vector and the source term vector, respectively. The components of $\mathbb{U}$ are the conserved quantities

$$\mathbb{U} = \Big[\rho, \rho\alpha, \rho u, \rho c, u_r, \rho\Big(\mathcal{E} + \frac{u^2}{2}\Big)\Big]^T. \tag{2}$$

These represent the mass for the mixture, the volume concentration for the gas phase, the momentum for the mixture, the conservation law for the mass concentration of the gas phase with no phase transition, the balance law for the relative velocity and the energy conservation law for the mixture. The flux vector $\mathbb{F}$ is a function of $\mathbb{U}$

$$\mathbb{F}(\mathbb{U}) = \left( \begin{array}{c} \rho u, \;\; \rho u\alpha, \;\; \rho u^2 + \mathcal{P} + \rho u_r \mathcal{E}_{u_r}, \;\; \rho u c + \rho \mathcal{E}_{u_r}, \;\; u u_r + \mathcal{E}_c, \\ \rho u\Big(\mathcal{E} + \frac{u^2}{2}\Big) + \mathcal{P} u + \rho \mathcal{E}_{u_r}\Big(u u_r + \mathcal{E}_c\Big) \end{array} \right). \tag{3}$$

In the above expressions, the variables denote the usual quantities: $\rho$ is the mixture density, $\alpha$ and $c$ are the volume fraction and the mass concentration of the gas phase, $u$, $\mathcal{P}$, $u_r$ and $\mathcal{E}$ are the mixture velocity, pressure, relative velocity between the two phases and the mixture internal energy, respectively. The source term vector is a function of $\mathbb{U}$

$$\mathbb{S}(\mathbb{U}) = \Big[0, -\frac{\rho}{\tau}, 0, 0, -\kappa c(1-c)u_r, 0\Big]^T. \tag{4}$$

Additionally, several physical closure laws must be given which involve the source terms, the pressure, the equations of state for the mixture and for the phases and the mixture relations. We do not state these closure laws here (see instead [7] for details).

The mathematical structure of the model is more clearly revealed and the nature of the characteristic analysis is simplified in terms of primitive variables [7]. One possible primitive formulation would be

$$\mathbb{W} = \Big[\rho, \alpha, u, c, u_r, \mathcal{S}\Big]^T. \tag{5}$$

Thus, in quasi-linear form we have

$$\frac{\partial \mathbb{W}}{\partial t} + \mathbb{A}(\mathbb{W})\frac{\partial \mathbb{W}}{\partial x} = \mathbb{Q}(\mathbb{W}). \tag{6}$$

For a more comprehensive discussion of the mathematical properties of the hyperbolic conservative two-fluid model we refer to [7]. The model was shown to be hyperbolic (using perturbation methods) in processes without dissipation with six real eigenvalues. Four of these eigenvalues correspond to the left and right sound waves while two of these eigenvalues are equal to the mixture velocity corresponding to the convective waves. In addition the characteristic fields corresponding to the acoustic waves were shown to be genuinely non-linear while the last two are linearly degenerate.

## 3 Numerical modelling of the Riemann problem

The Riemann problem for the hyperbolic conservative two-fluid model is the initial value problem for the conservation laws in processes without dissipation

$$\left.\begin{array}{l} \partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = 0, \\ \mathbb{U}(x,0) = \begin{cases} \mathbb{U}_{\mathcal{L}} & \text{if } x < 0, \\ \mathbb{U}_{\mathcal{R}} & \text{if } x > 0, \end{cases} \end{array}\right\} \tag{7}$$

where $\mathbb{U}$ and $\mathbb{F}(\mathbb{U})$ are vectors of conserved variables and fluxes given by (2) and (3), respectively. As anticipated in the previous section there are six real eigenvalues. Two of these eigenvalues are equal to the mixture velocity. Thus there are six waves separating six constant states. In terms of a choice of primitive variables given by (5), these six constant states are: $\mathbb{W}_{\mathcal{L}}$ (left data), $\mathbb{W}^{\star}_{\mathcal{L}1}$, $\mathbb{W}^{\star}_{\mathcal{L}2}$, $\mathbb{W}^{\star}_{\mathcal{R}2}$, $\mathbb{W}^{\star}_{\mathcal{R}1}$, $\mathbb{W}_{\mathcal{R}}$ (right data). The states $\mathbb{W}^{\star}_{\mathfrak{s}1,2}$ ($\mathfrak{s} = \mathcal{L}, \mathcal{R}$) are the unknowns of the problem and they cover the star region and emerge from the interaction of the data states $\mathbb{W}_{\mathcal{L}}$ and $\mathbb{W}_{\mathcal{R}}$. The star region is divided into four subregions. The multiple middle wave is the contact discontinuity while the two-left and two-right non-linear waves are either shock or rarefaction waves, see [7] for details. Therefore, depending on the type of the non-linear waves there could be sixteen waves patterns to construct for the solution of the Riemann problem in the star region. The main step in solving the Riemann problem is the determination of the solution in the star region. This depends on suitable relations through the left and right waves connecting $\mathbb{W}^{\star}_{\mathcal{L}1}$, $\mathbb{W}^{\star}_{\mathcal{L}2}$ to $\mathbb{W}_{\mathcal{L}}$ and $\mathbb{W}^{\star}_{\mathcal{R}2}$, $\mathbb{W}^{\star}_{\mathcal{R}1}$ to $\mathbb{W}_{\mathcal{R}}$, respectively. In practice the determination of the solution in the star region is too difficult to develop because it typically requires iterations for the non-linear equations.

Since the model possesses a large number of non-linear waves it is not easy to analyse the star region mathematically. In addition, the number of non-linear equations will depend on the number of the non-linear waves considered. These facts motivate us to consider numerical approximations by using modern Riemann-problem shock-capturing methods of centred-type. Determining the analytical explicit solution of the Riemann problem is very useful for a better understanding of the physical and mathematical meaning of the model and to develop an efficient numerical method.

As mentioned before, we considered two methods, namely, a FORCE scheme and a SLIC scheme. Both methods advance the solution via the conservative formula

$$\mathbb{U}_i^{n+1} = \mathbb{U}_i^n - \frac{\Delta t}{\Delta x}\Big[\mathbb{F}_{i+\frac{1}{2}} - \mathbb{F}_{i-\frac{1}{2}}\Big], \tag{8}$$

where $\mathbb{F}_{i+\frac{1}{2}}$ is the inter-cell numerical flux corresponding to the inter-cell boundary at $x = x_{i+\frac{1}{2}}$ between $i$ and $i+1$. The flux of the FORCE scheme is written as

$$\mathbb{F}^{\text{FORCE}}_{i+\frac{1}{2}} = \frac{1}{2}\Big[\mathbb{F}^{\text{LF}}_{i+\frac{1}{2}} + \mathbb{F}^{\text{RI}}_{i+\frac{1}{2}}\Big], \tag{9}$$

where $\mathbb{F}^{\text{LF}}_{i+\frac{1}{2}}$ is the Lax–Friedrichs flux and $\mathbb{F}^{\text{RI}}_{i+\frac{1}{2}}$ is the Richtmyer flux. The FORCE scheme is a first-order centred method and has half the numerical viscosity of the Lax–Friedrichs method. The SLIC scheme is a second-order TVD scheme that is an extension of FORCE scheme. The scheme is of the slope limiter type and results from replacing the Godunov flux by the FORCE flux in the MUSCL–Hancock Method [5]. The TVD version results from limiting the slopes in data construction. We refer to [7] for further detail on the centred methods for two-phase fluid flow problems.

## 4 Numerical simulations

Numerical results for the gas-liquid mixture two-phase flow Riemann problem (7) are now presented. They were obtained using centred methods with no source terms in (7) (i.e. no influence for dissipation). Physically this means that we look at the experiment at a time when the relaxation of volume concentration and interfacial friction are negligible.

The results illustrate typical wave patterns: two-left rarefactions and two-right shocks separated by contact discontinuities with high pressure in the left side and with low pressure on the right side resulting from the solution of the Riemann problem for the hyperbolic conservative two-fluid model. The left and right states of the Riemann problem are: $\mathcal{P}_{\mathcal{L}} = 0.1$, $\mathcal{P}_{\mathcal{R}} = 0.0001$, $\alpha_{\mathcal{L}} = 0.4$, $\alpha_{\mathcal{R}} = 0.6$, $u_{\mathcal{L}} = 0.0 = u_{\mathcal{R}}$, $u_{r\mathcal{L}} = 0.0 = u_{r\mathcal{R}}$, $c_{\mathcal{L}} = 0.08175$, $c_{\mathcal{R}} = 0.00150$ and $\mathcal{S}_{\mathcal{L}} = 0.0 = \mathcal{S}_{\mathcal{R}}$.

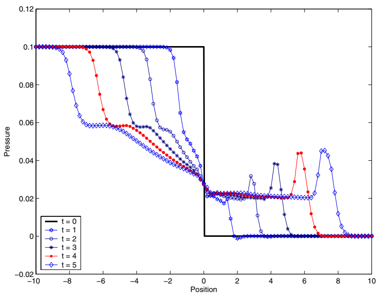


Figure 1: Time-evolution of the pressure using the TVD SLIC scheme on a mesh of 100 cells.

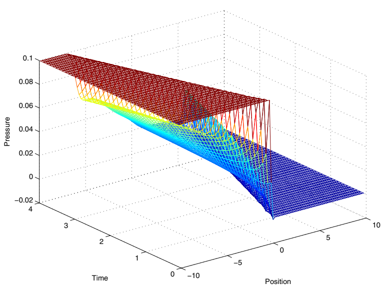


Figure 2: Pressure distribution in the three-dimensional results from the numerical modelling of the gas-liquid mixture two-phase flow Riemann problem at time $t = 4.0\ s$ on a mesh of $1000$ cells.

A precise description of the other parameters involved is given in [7]. The numerical solutions are computed with $100$ and $1000$ computational cells representing a coarse and fine mesh, respectively, on the spatial domain $-10 \leq x \leq 10$ with Courant number coefficient $C_{\text{CFL}} = 0.9$ and with transmissive boundary conditions together with the SUPERBEE limiter. As mentioned before, it is difficult to construct an exact solution for the Riemann problem in general. However, for given states on the left and right the solution is composed of six waves which are either two-rarefactions, two-shocks or multiple contact discontinuities. Since we do not have an analytical solution, numerical evidence [7] shows that there are two-left rarefactions and two-right shock waves separated by contact discontinuities for the considered gas-liquid mixture two-phase flow Riemann problem as displayed in figure 1.

Figure 1 shows the time-evolution of the pressure wave propagation profile for different time values $(t = 0, 1, 2, 3, 4, 5)$ using the TVD SLIC scheme. However, for all different times the waves move away from the origin and they need a closer look to be identified. From the results plotted in figure 1, one can exhibit the two-left rarefaction-like waves propagating to the far left in the high pressure region, a

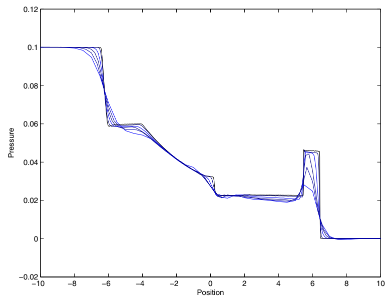


Figure 3: Grid-convergence study of the pressure profile, at time $t = 4.0\ s$, computed with the SLIC scheme with SUPERBEE limiter.

shock-like wave propagating to the far right in the low pressure region and contact-like wave in the middle of the structure which remains approximately constant. In order to demonstrate the behaviour of these waves, the results for the pressure in three-dimension are presented in figure 2 on a mesh of 1000 cells at $t = 4.0\ s$. In figure 3 we study the convergence of the SLIC scheme as the mesh is refined for the pressure profile at time $t = 4.0\ s$. The results demonstrate that the SLIC scheme approach the reference solution without introducing any spurious oscillations. Thus, numerical evaluation suggest that the TVD SLIC scheme with fine mesh (i.e. using 1000 computational cells) is the reference solution for the gas-liquid mixture two-phase flow Riemann problem.

Figure 4 displays the pressure profile at $t = 4.0\ s$ for the gas-liquid mixture two-phase flow with a mesh of 100 computational cells. The different methods are plotted for the sake of comparison, as shown in figure 4. We emphasise that these oscillations are not of a numerical nature but rather due to some existing non-linearity effects within our two-fluid model system in the absence of any source terms (which may here introduce some dissipative or regularising effect) [7].

From the plot of the pressure shown in figure 4, we see that Lax–Friedrichs, FORCE and SLIC schemes seem to produce the same wave structure in the pressure variable as the reference solution. This structure arises due to the existence of two-left rarefactions and shock-like wave separated by a contact discontinuity as mentioned earlier.

Concerning the shock-like wave in figures 1, 2, 3 and 4, we observe that the structure of the pressure shock waves propagation depends on the value of the initial pressure on the right side. To find an explanation of this phenomenon, we have plotted the pressure contours for the Riemann problem at $t = 4.0\ s$ as shown in figure 5.

In particular, figures 4 and 5 reflect the following dynamics: as the mixture velocity is the same in both sides at $t = 4.0\ s$, the decreasing pressure in the right side leads to a shock-like wave propagating to the far right in the low pressure region. Clearly the SLIC scheme is able to handle the gas-liquid mixture

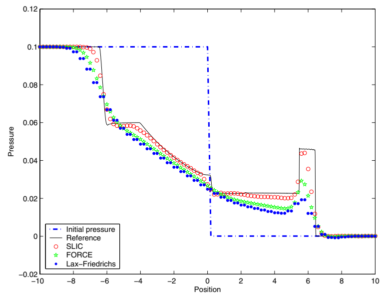


Figure 4: Comparison between the TVD SLIC, FORCE, Lax–Friedrichs scheme on a mesh of 100 cells with the reference solution computed by the TVD SLIC scheme using a mesh of 1000 cells.

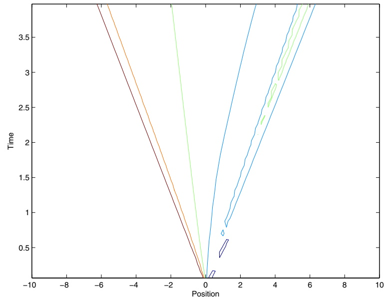


Figure 5: Effect of the initial pressure on the shock-like wave at time $t = 4.0\ s$.

two-phase flow Riemann problem without generating any oscillation and the accuracy of the coarse mesh solution is totally satisfactory. From figure 4 we observe that the results for the Lax–Friedrichs scheme look substandard to those of the FORCE scheme. The numerical results of the SLIC scheme are better to those of the FORCE scheme. In particular, we see that the coarse mesh solution is able to approximate the fine mesh solution very well.

## 5 Conclusions

Godunov methods of centred-type have been proposed for a mathematical model which describes unsteady compressible non-equilibrium mixture two-phase flow. The model is given by a non-linear hyperbolic system of conservation law for which its mathematical structure prevent us from developing an analytical solution for the Riemann problem or from using Godunov methods of upwind-type. The associated Riemann problem is solved numerically and its solution is obtained by centred methods. The concept of centred methods totally avoids the explicit solution of the Riemann problem and are not biased by the wave propagation directions. Taking advantage of these properties, the methods are applied to the simulation of the gas-liquid mixture two-phase flow Riemann problem with particular focus on the discontinuities expected to appear for the numerical modelling of the Riemann problem. Numerical results show that the approximate solutions are in good agreement with the reference solution. In particular, the SLIC scheme gives a very accurate solution of the wave propagation as demonstrated through the considered two-phase flow problem. Moreover, the scheme is robust with an ability to capture solutions accurately. Finally, Godunov methods of centred-type seem to be very useful for developing new mathematical models together with closure laws and to study Riemann problem two-phase flows.

## References

[1] Godunov, S. & Romensky, E., Thermodynamics, conservation laws and symmetric forms of differential equations in mechanics of continuous media. In *Computational Fluid Dynamics Review*, John Wiley & Sons: New York, pp. 19–31 1995.

[2] Müller, I. & Ruggeri, T., *Rational Extended Thermodynamics*, Springer–Verlag: New York, 1998.

[3] Romensky E., Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics. *Mathl. Comput. Modelling*, **28**, 115–130, 1998.

[4] Romensky E., Thermodynamics and hyperbolic systems of balance laws in continuum mechanics. In *Godunov Methods: theory and applications*, ed. E.F. Toro, Kluwer Academic/Plenum Publishers pp. 745–761, 2001.

[5] Toro E. F., *Riemann Solvers and Numerical Methods for Fluid Dynamics.* A Practical Introduction. Springer–Verlag, Berlin, 1999.

[6] Toro E. F. & Billett S., Centred TVD schemes for hyperbolic conservation laws. *IMA. J. Numerical Analysis*, **20**, 47–79, 2000.

[7] Zeidan D., *Mathematical and Numerical Study of Two–Phase Flow Models*, PhD Thesis, Manchester Metropolitan University, UK, 2003.