Chapter 10

INTERFACIAL AREA TRANSPORT

The interfacial transfer terms are strongly related to the interfacial area and to the local transfer mechanisms, such as the degree of turbulence near the interfaces and the driving potential. Basically, the interfacial transport of mass, momentum and energy is proportional to the interfacial area concentration and to a driving force. This area concentration, defined as the interfacial area per unit volume of the mixture, characterizes the kinematic effects; therefore, it must be related to the structure of the two-phase flow. The driving forces for the interphase transport characterize the local transport mechanism and they must be modeled separately.

Since the interfacial transfer rates can be considered as the product of the interfacial flux and the available interfacial area, the modeling of the interfacial area concentration is essential. In two-phase flow analysis the void fraction and the interfacial area concentration represent the two fundamental first-order geometrical parameters. Therefore, they are closely related to two-phase flow regimes. However, the concept of the two-phase flow regimes is difficult to quantify mathematically at a local point, because it is often defined at the scale close to the system scale. This may indicate that the modeling of the changes of the interfacial area concentration directly by a transport equation, namely interfacial area transport equation. This is a better approach than the conventional method using the flow-regime transition criteria and regime-dependent constitutive relations for interfacial area concentration. This is particularly true for a three-dimensional formulation of multiphase flow.

In this chapter, the detailed derivation and the necessary constitutive relations of the interfacial area transport equation is presented to establish the dynamic closure relation for the interfacial area concentration in the two-fluid model. Accounting for the substantial differences in the transport mechanisms in small spherical and large cap bubbles, the two-group transport equation is derived.

1.1   Three-dimensional interfacial area transport equation

The Boltzmann transport equation describes the particle transport by an integro-differential equation of the particle-distribution function. Since the interfacial area of the fluid particle is closely related to the particle number, the interfacial area transport equation can be formulated based on the Boltzmann transport equation (Kocamustafaogullari and Ishii, 1995; Ishii and Kim, 2004).

Consider a system of fluid particles in a continuous medium, where the source and sink of the fluid particle exist due to the particle interactions such as the coalescence and disintegration. Let $f(V, x, v, t)$ be the particle number density distribution function per unit mixture and bubble volume. This is assumed to be continuous and specifies the probable number density of fluid particles moving with particle velocity v, at a given time t, in a spatial range δx with its center-of-volume located at x with particle volumes between V and $V + \delta V$. Assuming that the change of particle velocity within the time interval t to $t + \delta t$ is small, the particle number density distribution function per unit mixture and bubble volume can be simplified to be $f(V, x, t)$. This assumption of a uniform particle velocity for a given particle size is practical for most two-phase flow. However, for neutron transport, the energy of neutrons spans over many orders of magnitudes and is the essence of the transport theory. Therefore, the velocity dependence cannot be neglected. Then, we can write for a two-phase flow system

$$
\begin{aligned}
&f\left(V + \delta V, x + \delta x, t + \delta t\right)\delta\mu - f\left(V, x, t\right)\delta\mu \\
&= \left(\sum_j S_j + S_{ph}\right)\delta\mu\,\delta t
\end{aligned}
\tag{10-1}
$$

where $\delta\mu$ is a volume element in μ space. In the right-hand side of the equation, the S_j and S_{ph} are the particle source and sink rates per unit mixture volume due to j-th particle interactions (such as the disintegration or coalescence and that due to phase change, respectively). Expanding the first term on the left-hand side of Eq.(10-1) in a Taylor series in δt and dividing it by $\delta\mu\,\delta t$, Eq. (10-1) reduces to

$$
\frac{\partial f}{\partial t} + \nabla \cdot (fv) + \frac{\partial}{\partial V}\left(f\frac{dV}{dt}\right) = \sum_j S_j + S_{ph}
\tag{10-2}
$$

which is analogous to the Boltzmann transport equation of particles with the distribution function $f(V,x,t)$. Here, d/dt denotes the substantial derivative. In the following sections, we present the detailed derivations on the transport equations for fluid particle number (n), volume fraction (α_g), and interfacial area concentration (a_i).

1.1.1   Number transport equation

In two-phase flow applications, the particle transport equation given by Eq.(10-2) is much too detailed to be employed in practice, therefore, more macroscopic formulation is desirable. This can be done by integrating Eq.(10-2) over the volume of all sizes of particles from V_{min} to V_{max} and applying the Leibnitz rule of integration. We obtain the particle number transport equation as

$$\frac{\partial n}{\partial t} + \nabla \cdot \left(n v_{pm} \right) = \sum_{j} R_j + R_{ph} \tag{10-3}$$

where the distribution function for the bubbles of volume V_{min} and V_{max} are assumed to be approximately zero. Here, the left-hand side of the equation represents the time rate of change of the total particle number density and its convection. The two terms in the right-hand side represent the number source and sink rates due to particle interaction (such as particle disintegration or coalescence) and the number source rate due to the phase change, respectively.

In Eq.(10-3), the total number of particles of all sizes per unit-mixture volume and the number source and sink rates are defined respectively by

$$n(x,t) = \int_{V_{min}}^{V_{max}} f(V,x,t) dV \tag{10-4}$$

and

$$R(x,t) = \int_{V_{min}}^{V_{max}} S(V,x,t) dV. \tag{10-5}$$

Also, v_{pm} is the average local particle velocity weighted by the particle number and is defined by

$$v_{pm}\left(x,t\right) \equiv \frac{\int_{V_{min}}^{V_{max}} f\left(V,x,t\right)v\left(V,x,t\right)dV}{\int_{V_{min}}^{V_{max}} f\left(V,x,t\right)dV}.$$

(10-6)

1.1.2   Volume transport equation

The particle volume (or void fraction) transport equation can be obtained by multiplying Eq.(10-2) by particle volume V and integrating it over the volume of all sizes of particles. Then, considering that the two-phase flow of interest consists of the dispersed bubbles in a continuous liquid medium, the void fraction transport equation is given by

$$\frac{\partial \alpha_g}{\partial t} + \nabla \cdot \left(\alpha_g v_g\right) + \int_{V_{min}}^{V_{max}} V \frac{\partial \left(f\dot{V}\right)}{\partial V} dV$$
$$= \int_{V_{min}}^{V_{max}} \left(\sum_j S_j V + S_{ph} V\right) dV$$

(10-7)

where $\dot{V}$ denotes the time derivative of volume V. Here, the void fraction and the average center-of-volume velocity of the dispersed (or gas) phase are defined respectively by

$$\alpha_g\left(x,t\right) = \int_{V_{min}}^{V_{max}} f\left(V,x,t\right)V dV$$

(10-8)

and

$$v_g\left(x,t\right) \equiv \frac{\int_{V_{min}}^{V_{max}} f\left(V,x,t\right)V v\left(V,x,t\right)dV}{\int_{V_{min}}^{V_{max}} f\left(V,x,t\right)V dV}.$$

(10-9)

In Eq.(10-7), the third term on the left-hand side attributes to the change in the particle volume (by expansion or contraction) due to the change in the pressure along the flow field. To better represent this term, we assume

$$\frac{\dot{V}}{V} \neq f\left(V\right)$$

(10-10)

such that the time-rate of change in relative particle volume is assumed to be independent of its volume. If evaporation effect is small compared to the compressibility effect, the dominant contribution in the change in a particle volume attributes to the changes in the pressure. Therefore, this assumption is valid in most two-phase flow conditions. It should be noted, however, that the evaporation effect is not completely neglected in the average transport equation development as will be shown later in Eq.(10-14). Then, the third term on the left-hand side of Eq.(10-7) reduces to

$$\int_{V_{min}}^{V_{max}} V \frac{\partial\left(f\dot{V}\right)}{\partial V} dV \cong -\left(\frac{\dot{V}}{V}\right)\alpha_g\left(\boldsymbol{x},t\right). \qquad (10\text{-}11)$$

Furthermore, since the mass transfer by the evaporation process is given by

$$\frac{d\rho_g V}{dt} = \frac{\left(\Gamma_g - \eta_{ph}\rho_g\right)V}{\alpha_g} \qquad (10\text{-}12)$$

where Γ_g is the total rate of change of mass-per-unit mixture volume and η_{ph} is the rate of volume generated by nucleation source per unit mixture volume, defined by

$$\eta_{ph} \equiv \int_{V_{min}}^{V_{max}} S_{ph} V dV. \qquad (10\text{-}13)$$

The volume source can be written as

$$\begin{aligned}
\frac{1}{V}\frac{dV}{dt} &= \frac{1}{\rho_g}\left(\frac{\Gamma_g - \eta_{ph}\rho_g}{\alpha_g} - \frac{d\rho_g}{dt}\right) \\
&= \frac{1}{\alpha_g}\left\{\frac{\partial\alpha_g}{\partial t} + \nabla\cdot\left(\alpha_g\boldsymbol{v}_g\right) - \eta_{ph}\right\}.
\end{aligned} \qquad (10\text{-}14)$$

Thus, combining Eq.(10-14) with Eq.(10-11), and substituting them into Eq.(10-7), the final form of the void fraction transport equation is obtained as

$$\frac{\partial \alpha_g}{\partial t} + \nabla \cdot \left(\alpha_g \boldsymbol{v}_g \right) - \frac{\alpha_g}{\rho_g} \left(\frac{\Gamma_g - \eta_{ph}\rho_g}{\alpha_g} - \frac{d\rho_g}{dt} \right)$$

$$= \int_{V_{min}}^{V_{max}} \left(\sum_j S_j V + S_{ph} V \right) dV \qquad (10\text{-}15)$$

where the first two terms on the left-hand side of the equation represents the time rate of change and convection of α_g, and the rest of the terms represent the change rates in α_g due to volume change, particle interactions and phase change, respectively.

By rearranging Eq.(10-15), it is interesting to note that we have

$$\frac{1}{\rho_g} \left\{ \frac{\partial \alpha_g \rho_g}{\partial t} + \nabla \cdot \left(\alpha_g \rho_g \boldsymbol{v}_g \right) - \Gamma_g \right\} = \int_{V_{min}}^{V_{max}} \sum_j S_j V dV. \qquad (10\text{-}16)$$

Here, it is noted that conservation of mass requires

$$\frac{\partial \alpha \rho_g}{\partial t} + \nabla \cdot \left(\alpha \rho_g \boldsymbol{v}_g \right) - \Gamma_g = 0. \qquad (10\text{-}17)$$

Therefore, from Eqs.(10-16) and (10-17), we obtain the identity

$$\int_{V_{min}}^{V_{max}} \sum_j S_j V dV = 0. \qquad (10\text{-}18)$$

Equation (10-18) satisfies both the volume and mass conservation, simultaneously.

1.1.3   Interfacial area transport equation

The transport equation for the interfacial area concentration can be obtained through a similar approach applied in the previous formulations. Hence, multiplying Eq.(10-2) by the surface area of particles of volume V, $A_i(V)$, (which is independent of the coordinate system) and integrating it over the volume of all particles, we obtain

$$\frac{\partial a_i}{\partial t} + \nabla \cdot (a_i \boldsymbol{v}_i) - \left(\frac{\dot{V}}{V}\right) \int_{V_{min}}^{V_{max}} fV dA_i$$

$$= \int_{V_{min}}^{V_{max}} \left(\sum_j S_j + S_{ph}\right) A_i dV$$

(10-19)

where the average a_i of all fluid particles of volumes between V_{min} and V_{max}, and the interfacial velocity are given respectively by

$$a_i(\boldsymbol{x},t) = \int_{V_{min}}^{V_{max}} f(V,\boldsymbol{x},t) A_i(V) dV$$

(10-20)

and

$$\boldsymbol{v}_i(\boldsymbol{x},t) \equiv \frac{\int_{V_{min}}^{V_{max}} f(V,\boldsymbol{x},t) A_i(V) \boldsymbol{v}(V,\boldsymbol{x},t) dV}{\int_{V_{min}}^{V_{max}} f(V,\boldsymbol{x},t) A_i(V) dV}.$$

(10-21)

Now, in view of furnishing the third term on the left-hand side of Eq.(10-19), we define the volume-equivalent diameter, D_e, and surface-equivalent diameter, D_s, of a fluid particle with surface area A_i and volume V, as

$$V \equiv \frac{\pi}{6} D_e^3 \text{ and } A_i \equiv \pi D_s^2.$$

(10-22)

Therefore, combining them with Eq.(10-19) and recalling the volume source, given by Eq.(10-14), the interfacial area transport equation can be obtained as

$$\frac{\partial a_i}{\partial t} + \nabla \cdot (a_i \boldsymbol{v}_i) - \frac{2}{3} \left(\frac{a_i}{\alpha_g}\right) \left\{\frac{\partial \alpha_g}{\partial t} + \nabla \cdot (\alpha_g \boldsymbol{v}_g) - \eta_{ph}\right\}$$

$$= \int_{V_{min}}^{V_{max}} \left(\sum_j S_j + S_{ph}\right) A_i dV$$

(10-23)

where the third term on the left-hand side represents the change in the interfacial area concentration due to the particle volume change. In deriving Eq.(10-23), the ratio $\left(D_s/D_e\right)$ is assumed to be constant in view of simplifying the equation. While this approximation may not be appropriate

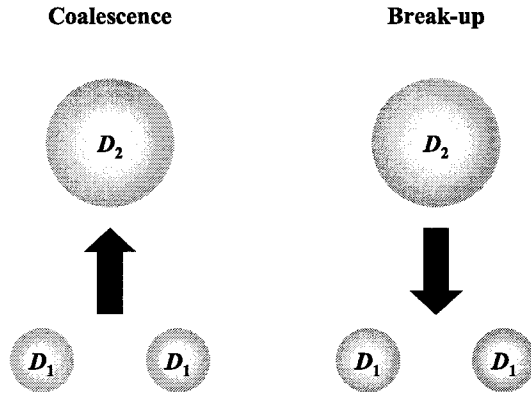**Coalescence**   **Break-up**

Figure 10-1. Illustration of fluid particle coalescence and disintegration process in view of ΔA_i (Ishii and Kim, 2004)

for the bubbles in distorted or slug shape, it is a good approximation for the spherical and cap bubbles. Essentially, this diameter ratio is a shape factor and for similar particle shapes this factor can be considered as constant.

To close the system of equations, the right-hand side of Eq.(10-23), which represents the source and sink rates of the interfacial area concentration, must be specified by the constitutive relations. In view of this, we define

$$\int_{V_{min}}^{V_{max}} \sum_j S_j dV$$

$$= \sum_j R_j: \text{ particle number source and sink rate} \tag{10-24}$$

and

$$\int_{V_{min}}^{V_{max}} \sum_j S_j A_i dV$$

$$= \sum_j \phi_j: \text{ source and sink rate for } a_i. \tag{10-25}$$

Furthermore, noting that ϕ_j can be expressed in terms of the change in surface area of a fluid particle after a certain particle interaction process, we can write

$$\phi_j = R_j \Delta A_i \tag{10-26}$$

where R_j can be mechanistically modeled for each interaction mechanism, and ΔA_i depends on the given interaction mechanism; such as the disintegration or coalescence processes.

In order to specify ΔA_i, consider a coalescence and a break-up processes as illustrated in Fig.10-1 for the spherical particles. Here, it was assumed that the given process is a binary process between the particles of same size. Then, since the total volume of the particles should be conserved, we can write

$$V_2 = 2V_1 \text{ or } D_2 = 2^{1/3} D_1 \tag{10-27}$$

where the subscripts 1 and 2 indicates the particles of smaller and bigger volumes, respectively. Hence, by assuming that the interaction process is a binary process, the change of surface area after one interaction process can be obtained for near spherical particles as

$$\Delta A_i = -0.413 A_i: \text{ for a coalescence process} \tag{10-28}$$

and

$$\Delta A_i = +0.260 A_i: \text{ for a break-up process} \tag{10-29}$$

where the minus and plus signs are used to indicate the reduction and gain of the surface area after one interaction process, respectively. Furthermore, recalling the definition given by Eq.(10-4), the particle number density n can be specified through a_i and α_g by

$$a_i = nA_i \text{ and } \alpha_g = nV \tag{10-30}$$

such that

$$n = \psi \frac{a_i^3}{\alpha_g^2} \tag{10-31}$$

with a shape factor defined by

$$\psi = \frac{1}{36\pi} \left(\frac{D_{Sm}}{D_e} \right)^3 \tag{10-32}$$

where the bubble Sauter mean diameter is given by

$$D_{Sm} = \frac{6\alpha_g}{a_i}. \tag{10-33}$$

Thus, combining these with Eq.(10-26), the surface source and sink rate, ϕ_j can be given by

$$\phi_j = \frac{1}{3\psi} \left(\frac{\alpha_g}{a_i} \right)^2 R_j. \tag{10-34}$$

Similarly, for the nucleation process, ϕ_{ph} can be given by

$$\phi_{ph} = \pi D_{bc}^2 R_{ph} \tag{10-35}$$

where D_{bc} is the critical bubble size. This should be determined depending on the given nucleation process; namely, the critical cavity size for the bulk boiling or condensation process, and the bubble departure size for the wall nucleation. For most two-phase flow, wall nucleation is the dominant mechanism.

After combining the constitutive relations given above and substituting them into Eq.(10-23), we obtain the interfacial area transport equation as

$$\frac{\partial a_i}{\partial t} + \nabla \cdot (a_i \boldsymbol{v}_i) = \frac{2}{3} \left(\frac{a_i}{\alpha_g} \right) \left\{ \frac{\partial \alpha_g}{\partial t} + \nabla \cdot (\alpha_g \boldsymbol{v}_g) - \eta_{ph} \right\}$$
$$+ \frac{1}{3\psi} \left(\frac{\alpha_g}{a_i} \right)^2 \sum_j R_j + \pi D_{bc}^2 R_{ph} \tag{10-36}$$

where the left-hand side represents the time rate of change and convection of the interfacial area concentration. Each term on the right-hand side represents the rates of change in the interfacial area concentration due to the particle volume change caused by the change in pressure, various particle interactions and phase change, respectively. As can be seen in Eq.(10-36), R_j 's should be modeled independently, based on the given particle

interaction mechanisms. Hence, the mechanistic models of the number source and sink rates for the coalescence and disintegration mechanisms, or those for the bubble nucleation and condensation phenomena, should be established as constitutive relations to solve the transport equation.

1.2   One-group interfacial area transport equation

When the transport phenomena of the fluid particles of interest do not vary significantly in a given two-phase flow system, and the particles remain similar in shape after the particle interactions, their characteristic transport phenomenon is similar and can be described by one transport equation. However, when fluid particles of various shapes and size present simultaneously, their transport mechanisms can be significantly different. In such cases, it may be necessary to employ multiple transport equations to describe the fluid particle transport.

In view of this, we first consider the two-phase flow system of the dispersed bubbles in a continuous liquid medium (namely, bubbly flow), where all the present bubbles can be categorized as *one group*. In such flow conditions, it is assumed that the bubbles are spherical in their shapes, and they are subject to the similar characteristic drag on their transport phenomena. Hence, accounting for the spherical shape in the one-group transport, ψ defined in Eq.(10-32) can be approximated by

$$\psi \approx \frac{1}{36\pi} = 8.85 \times 10^{-3}: \quad \text{for dispersed bubbles} \qquad (10\text{-}37)$$

because the bubble Sauter mean diameter is approximately equal to the volume-equivalent diameter. Furthermore, noting that critical bubble size due to nucleation is much smaller compared to the average bubble Sauter mean diameter, we may assume

$$\left(\frac{D_{bc}}{D_{Sm}} \right) \approx 0. \qquad (10\text{-}38)$$

Also, since η_{ph} can be approximated as

$$\eta_{ph} \equiv \int_{V_{min}}^{V_{max}} S_{ph} V dV \approx R_{ph} \frac{\pi}{6} D_{bc}^3 \qquad (10\text{-}39)$$

the interfacial area transport equation for the dispersed bubbles, or the one-group interfacial area transport equation, is given by

$$\frac{\partial a_i}{\partial t} + \nabla \cdot (a_i \boldsymbol{v}_i) \approx \frac{2}{3} \left(\frac{a_i}{\alpha_g}\right) \left\{\frac{\partial \alpha_g}{\partial t} + \nabla \cdot (\alpha_g \boldsymbol{v}_g)\right\}$$

$$+ \frac{1}{3\psi} \left(\frac{\alpha_g}{a_i}\right)^2 \sum_j R_j + \pi D_{bc}^2 R_{ph}. \tag{10-40}$$

The left-hand side represents the total rate of change in the interfacial area concentration, whereas the right-hand side represents the rates of change in the interfacial area concentration due to the change in particle volume, various particle interactions and phase change, respectively. It is noted that the effect of η_{ph} can be neglected because the departure size is smaller than the Sauter mean diameter.

1.3   Two-group interfacial area transport equation

In a gas-liquid two-phase flow system, a wide range of bubble shape and size exists depending on the given flow regime. Therefore, to develop the interfacial area transport equation describing the bubble transport in a wide range of two-phase flow regimes, the model must account for the differences in the transport characteristics of different types of bubbles. These variations in shape and size of bubbles cause substantial differences in their transport mechanisms due to the drag forces. Furthermore, the bubble interaction mechanisms in such flow conditions can be quite different compared to those in the one-group transport.

In most two-phase flow conditions, bubbles can be categorized into five types: spherical; distorted; cap; Taylor; and churn-turbulent bubbles. However, in view of their transport characteristics, they can be classified into two major groups, such that Group 1 includes the spherical and distorted bubbles, while Group 2 includes the cap, Taylor and churn-turbulent bubbles. Thus, in the present analysis, the approach employing two transport equations is given in describing the bubble transport over a wide range of two-phase flow conditions. That is, Group-1 transport equation describes the transport of spherical and distorted bubbles, and Group-2 transport equation describes the transport of cap, Taylor and churn-turbulent bubbles.

In the one-group formulation, the transport equation was averaged by the integration process over the volumes of all sizes of particles because the shape of the particles and their transport phenomena were assumed to be similar over the given range of particle volume. In the two-group formulation, however, the integration limit for each transport equation should be bounded by the bubble volume, by which the bubble group is determined. In view of this, we define V_c as the critical bubble volume

given by $\pi D_{d,\max}/6$ with the maximum distorted bubble limit, $D_{d,\max}$, specified by Ishii and Zuber (1979) as

$$D_{d,\max} = 4\sqrt{\frac{\sigma}{g\Delta\rho}}: \quad \text{maximum distorted bubble limit} \quad (10\text{-}41)$$

over which the bubble becomes cap in shape and the drag effect starts to deviate from that on the smaller bubbles due to the large wake region. Therefore, the Group 1 bubbles exist in the range of V_{min} to V_c, whereas the Group 2 bubbles exist in the range of V_c to V_{max}.

1.3.1 Two-group particle number transport equation

The two-group particle number transport equation can be readily obtained by integrating Eq.(10-2) over the different ranges of integration limit bounded by V_c; namely, from V_{min} to V_c, for Group 1 and from V_c to V_{max} for Group 2. In two-group formulation, as in the one-group formulation, $f(V,x,t)$ describes the particle-number density distribution function per unit mixture and bubble volume. This is assumed to be continuous, specifying the probable number density of fluid particles moving at a velocity v, at a given time t, in a spatial range δx with its center-of-volume located at x with particle volumes between V and $V+dV$. Then, the number transport equations for Group 1 and Group 2 are given by

$$\frac{\partial n_1}{\partial t} + \nabla \cdot \left(n_1 v_{pm1} \right) = -f_c V_c \left(\frac{\dot{V}}{V} \right) + \sum_j R_{j1} + R_{ph} \qquad (10\text{-}42)$$

and

$$\frac{\partial n_2}{\partial t} + \nabla \cdot \left(n_2 v_{pm2} \right) = f_c V_c \left(\frac{\dot{V}}{V} \right) + \sum_j R_{j2} \qquad (10\text{-}43)$$

where the subscripts 1 and 2 in the equations denote Group 1 and Group 2, respectively, and v_{pm1} and v_{pm2} are the average local particle velocity weighted by the particle number for each bubble group, such that they are defined by

$$v_{pm1}(x,t) \equiv \frac{\int_{V_{min}}^{V_c} f(V,x,t)v(V,x,t)\,dV}{\int_{V_{min}}^{V_c} f(V,x,t)\,dV}$$

$$v_{pm2}(x,t) \equiv \frac{\int_{V_c}^{V_{max}} f(V,x,t)v(V,x,t)\,dV}{\int_{V_c}^{V_{max}} f(V,x,t)\,dV}.$$

(10-44)

In Eqs.(10-42) and (10-43), the left-hand side of the equations represent the time rate of change and convection of fluid particle number for each bubble group. Each term on the right-hand side represents the rates of change of particle number through inter-group transfer by particle volume change, particle interaction and phase change for each bubble group. Here, it is interesting to note that in the two-group formulation, there are terms accounting for the inter-group transfer caused by particle volume change that did not appear in the one-group formulation. This is due to the fact that when two groups of bubbles present, the change in the particle volume in one group may serve as the number source in another due to the changes in bubble distribution function. These inter-group transfer terms disappear when the two equations are added together to obtain the total fluid particle number transport equation.

1.3.2   Two-group void fraction transport equation

The two-group void fraction transport equation can be obtained in a similar manner. Multiplying Eq.(10-2) by particle volume V, and integrating it over the specified limits for each group, we obtain

$$\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1} v_{g1}\right) + \int_{V_{min}}^{V_c} \left\{ V \frac{\partial}{\partial V}\left(f \frac{dV}{dt}\right)\right\} dV$$

$$= \int_{V_{min}}^{V_c} \left(\sum_j S_j + S_{ph}\right) V dV$$

(10-45)

and

$$\frac{\partial \alpha_{g2}}{\partial t} + \nabla \cdot \left(\alpha_{g2} \boldsymbol{v}_{g2}\right) + \int_{V_c}^{V_{max}} \left\{V \frac{\partial}{\partial V}\left(f \frac{dV}{dt}\right)\right\} dV$$

$$= \int_{V_c}^{V_{max}} \left(\sum_j S_j + S_{ph}\right) V dV. \tag{10-46}$$

For Group 1 and Group 2, respectively, the third term on the left-hand side of the equations represents the rate of change in void fraction due to particle volume change. They are given by

$$\int_{V_{min}}^{V_c} \left\{V \frac{\partial}{\partial V}\left(f \frac{dV}{dt}\right)\right\} dV = \left(\frac{\dot{V}}{V}\right)\left\{-\alpha_{g1} + V_c\left(f_c V_c\right)\right\} \tag{10-47}$$

and

$$\int_{V_c}^{V_{max}} \left\{V \frac{\partial}{\partial V}\left(f \frac{dV}{dt}\right)\right\} dV = \left(\frac{\dot{V}}{V}\right)\left\{-\alpha_{g2} - V_c\left(f_c V_c\right)\right\} \tag{10-48}$$

for Group 1 and Group 2, respectively, where f_c is the distribution function of a bubble with critical volume V_c, or $f(V_c)$.

Here, the volume source $\left(\dot{V}/V\right)$ can be expressed by the total mass transfer rate as shown previously in the one-group formulation. However, due to the presence of the two groups of bubbles and their interactions, the rate of mass transfer between the two groups must be considered. Hence, by denoting the subscripts ij as the inter-group transfer from group i to group j, the volume sources for each group are given by

$$\frac{1}{V_1}\left(\frac{dV_1}{dt}\right) = \frac{1}{\rho_g}\left(\frac{\Gamma_{g1} - \Delta \dot{m}_{12}}{\alpha_{g1}} - \frac{d\rho_g}{dt}\right) - \frac{\eta_{ph}}{\alpha_{g1}}$$

$$= \frac{1}{\alpha_{g1}}\left\{\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1} \boldsymbol{v}_{g1}\right) - \eta_{ph}\right\} \tag{10-49}$$

and

$$\frac{1}{V_2}\left(\frac{dV_2}{dt}\right) = \frac{1}{\rho_g}\left(\frac{\Gamma_{g2} + \Delta\dot{m}_{12}}{\alpha_{g2}} - \frac{d\rho_g}{dt}\right)$$

$$= \frac{1}{\alpha_{g2}}\left\{\frac{\partial\alpha_{g2}}{\partial t} + \nabla\cdot\left(\alpha_{g2}\boldsymbol{v}_{g2}\right)\right\}$$

(10-50)

for Group 1 and Group 2, respectively, where $\Delta\dot{m}_{12}$ represents the inter-group mass transfer rates from Group 1 to Group 2. The constitutive relation for the mass transfer between groups $\Delta\dot{m}_{12}$ will be discussed later. Hence, Eqs.(10-49) and (10-50) require the following identities

$$\frac{\partial\alpha_{g1}\rho_g}{\partial t} + \nabla\cdot\alpha_{g1}\rho_g\boldsymbol{v}_{g1}$$

$$= \Gamma_{g1} - \Delta\dot{m}_{12}: \quad \text{Mass Balance for Group 1}$$

(10-51)

and

$$\frac{\partial\alpha_{g2}\rho_g}{\partial t} + \nabla\cdot\alpha_{g2}\rho_g\boldsymbol{v}_{g2}$$

$$= \Gamma_{g2} + \Delta\dot{m}_{12}: \quad \text{Mass Balance for Group 2.}$$

(10-52)

Furthermore, by adding the two equations, we obtain the continuity equation for the gas phase as

$$\frac{\partial\alpha_g\rho_g}{\partial t} + \nabla\cdot\left(\alpha_g\rho_g\boldsymbol{v}_g\right) = \Gamma_g$$

(10-53)

with the following constitutive relations

$$\alpha_g = \alpha_{g1} + \alpha_{g2}$$

(10-54)

$$\Gamma_g = \Gamma_{g1} + \Gamma_{g2}$$

(10-55)

and

$$\boldsymbol{v}_g = \frac{\alpha_{g1}\boldsymbol{v}_{g1} + \alpha_{g2}\boldsymbol{v}_{g2}}{\alpha_{g1} + \alpha_{g2}}.$$

(10-56)

The term $f_c V_c$ on the right-hand side of Eqs.(10-47) and (10-48) represents the rate of change in the void fraction due to inter-group transfer, and it can be expressed in terms of other two-phase flow parameters. For convenience, the detailed discussion on this term will be presented in the following section. The two-group void fraction transport equation for each group is then given by

$$\frac{1}{\rho_g} \left\{ \frac{\partial \alpha_{g1} \rho_g}{\partial t} + \nabla \cdot \left(\alpha_{g1} \rho_g \boldsymbol{v}_{g1} \right) - \Gamma_{g1} + \Delta \dot{m}_{12} \right\}$$

$$= - \left\{ \frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1} \boldsymbol{v}_{g1} \right) - \eta_{ph} \right\} \chi \left(\frac{D_{sc}}{D_{Sm1}} \right)^3 \qquad (10\text{-}57)$$

$$+ \int_{V_{\min}}^{V_c} \sum_j S_j V dV$$

and

$$\frac{1}{\rho_g} \left\{ \frac{\partial \alpha_{g2} \rho_g}{\partial t} + \nabla \cdot \left(\alpha_{g2} \rho_g \boldsymbol{v}_{g2} \right) - \Gamma_{g2} - \Delta \dot{m}_{12} \right\}$$

$$= \left\{ \frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1} \boldsymbol{v}_{g1} \right) - \eta_{ph} \right\} \chi \left(\frac{D_{sc}}{D_{Sm1}} \right)^3 \qquad (10\text{-}58)$$

$$+ \int_{V_c}^{V_{\max}} \sum_j S_j V dV$$

for Group 1 and Group 2, respectively. Here, D_{sc} is the critical bubble size for the group boundary with surface area and volume of A_{ic} and V_c. Also, χ is the coefficient accounting for the contribution from the inter-group transfer, which will be discussed in detail in the following section.

In Eqs.(10-57) and (10-58), the left-hand side of the equations represents the time rate of change and convection of void fraction for each group and the right-hand side represents the rates of change in the void fraction due to the volume change. This includes the inter-group transfer and various particle interactions. Furthermore, since the left-hand side of the equations corresponds to the continuity equations for each bubble group, it requires the following identities;

$$-\left\{\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1}\boldsymbol{v}_{g1}\right) - \eta_{ph}\right\} \chi \left(\frac{D_{sc}}{D_{Sm1}}\right)^3 + \int_{V_{\min}}^{V_c} \sum_j S_j V dV \quad (10\text{-}59)$$
$$= 0$$

and

$$\left\{\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1}\boldsymbol{v}_{g1}\right) - \eta_{ph}\right\} \chi \left(\frac{D_{sc}}{D_{Sm1}}\right)^3 + \int_{V_c}^{V_{\max}} \sum_j S_j V dV \quad (10\text{-}60)$$
$$= 0$$

for Group 1 and Group 2 respectively, which indicates the conservation in bubble volume. Here, the first terms in Eqs.(10-59) and (10-60) represent the inter-group transfer at the bubble group boundary, and the second terms represent sources and sinks due to various fluid particle interaction for the given bubble group.

1.3.3   Two-group interfacial area transport equation

Similarly, as in the one-group interfacial area transport equation formulation, multiply Eq.(10-2) by the surface area of particles with volume V, which is independent of the coordinate system. Then, after integrating it over the volume within which each bubble group is defined, we obtain

$$\frac{\partial a_{i1}}{\partial t} + \nabla \cdot \left(a_{i1}\boldsymbol{v}_{i1}\right) + \int_{V_{\min}}^{V_c} \left\{A_i \frac{\partial}{\partial V}\left(f\frac{dV}{dt}\right)\right\} dV$$
$$= \int_{V_{\min}}^{V_c} \left(\sum_j S_j + S_{ph}\right) A_i dV \quad \text{for Group 1} \quad (10\text{-}61)$$

and

$$\frac{\partial a_{i2}}{\partial t} + \nabla \cdot \left(a_{i2}\boldsymbol{v}_{i2}\right) + \int_{V_c}^{V_{\max}} \left\{A_i \frac{\partial}{\partial V}\left(f\frac{dV}{dt}\right)\right\} dV$$
$$= \int_{V_c}^{V_{\max}} \sum_j S_j A_i dV \quad \text{for Group 2} \quad (10\text{-}62)$$

where the average interfacial velocity for each bubble group is defined by

$$v_{i1}(x,t) \equiv \frac{\int_{V_{min}}^{V_c} f(V,x,t)A_i v(V,x,t)\,dV}{\int_{V_{min}}^{V_c} f(V,x,t)A_i\,dV}$$

$$v_{i2}(x,t) \equiv \frac{\int_{V_c}^{V_{max}} f(V,x,t)A_i v(V,x,t)\,dV}{\int_{V_c}^{V_{max}} f(V,x,t)A_i\,dV}.$$

(10-63)

In Eqs.(10-61) and (10-62), the third terms on the left-hand side of the equations represent the changes in the interfacial area concentration due to the particle volume change, such that

$$\int_{V_{min}}^{V_c} \left\{ A_i \frac{\partial}{\partial V}\left(f \frac{dV}{dt} \right) \right\} dV = \left(\frac{\dot{V}}{V} \right)\left(-\frac{2}{3} a_{i1} + A_{ic} f_c V_c \right)$$

(10-64)

and

$$\int_{V_c}^{V_{max}} \left\{ A_i \frac{\partial}{\partial V}\left(f \frac{dV}{dt} \right) \right\} dV = \left(\frac{\dot{V}}{V} \right)\left(-\frac{2}{3} a_{i2} - A_{ic} f_c V_c \right).$$

(10-65)

where $A_{ic} f_c V_c$ is attributed to the inter-group transfer as a result of bubble interactions between the two groups of bubbles. Hence, when $f_c \to 0$, there is no contribution due to the inter-group interaction. In reality, however, when two bubble groups present f_c is finite, and this inter-group transfer term plays an important role as sources or sinks of the interfacial area concentration for each group.

In order to incorporate the contributions from this inter-group transfer, the particle distribution function should be specified. However, the accurate mathematical description for the particle distribution function in two-phase flows with various sizes of bubbles requires the use of the original Boltzmann transport equation and statistical mechanics. For our purpose, we need to develop a simple integrated transport equation. Hence, in the present analysis, a linear profile or a uniform profile is assumed in the particle distribution for simplicity as shown in Fig.10-2. In this, V_{1p} is the peak bubble volume in Group 1 bubbles specifying the value f_1, and is defined by

$$V_{1p} = \xi V_c \quad \text{where} \quad \frac{V_{min}}{V_c} < \xi < 1.$$
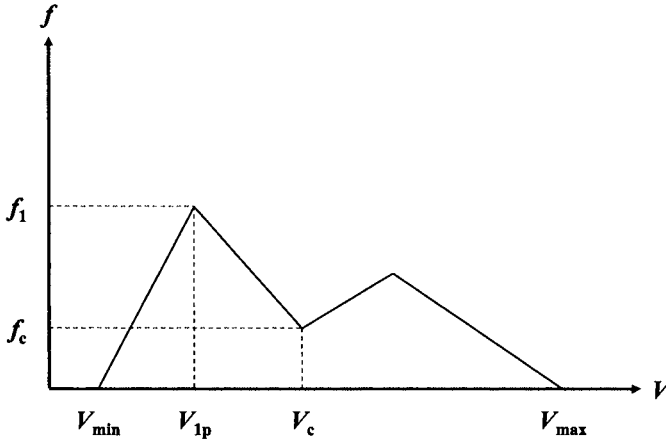
(10-66)

Figure 10-2. Linear approximation on profile of fluid particle distribution function (Ishii and Kim, 2004)

Then, the number density for Group 1 bubble can be expressed as

$$n_1 = \frac{1}{2} f_1 \left(V_c - V_{\min} \right) + \frac{1}{2} f_c V_c \left(1 - \xi \right) \tag{10-67}$$

which yields

$$f_c V_c A_{ic} = \left[\frac{2}{1 - \xi} - \frac{\left(V_c - V_{\min} \right)}{n_1 \left(1 - \xi \right)} f_1 \right] n_1 A_{ic}. \tag{10-68}$$

Now, consider three limiting cases as shown in Figs.10-3(a) through 10-3(c) such that

Case I : $f = f_c = \text{constant}$, hence

$$n_1 = f_1 \left(V_c - V_{\min} \right) = f_c \left(V_c - V_{\min} \right) \tag{10-69}$$

Case II : $\xi \to V_{\min}/V_c$, or $V_{1p} \to V_{\min}$, hence

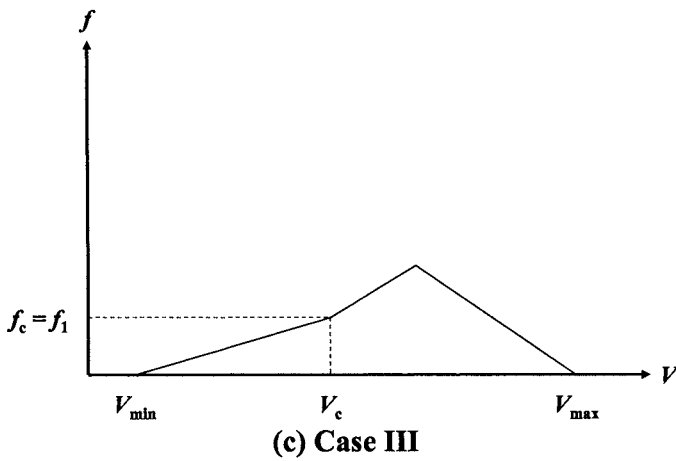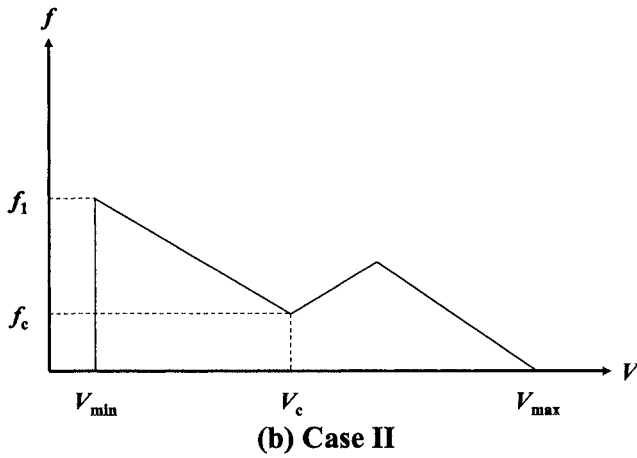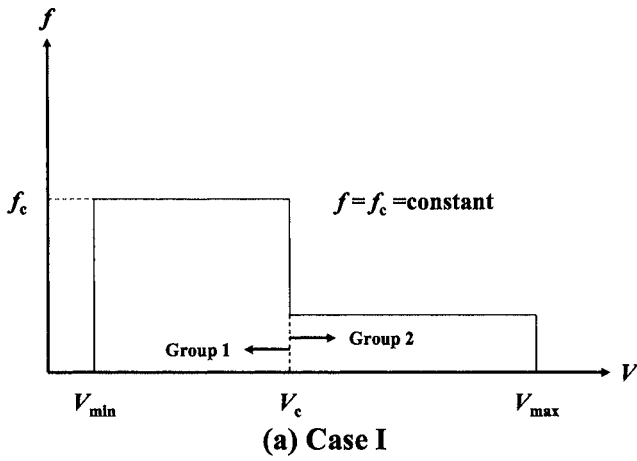$$n_1 = \frac{1}{2} \left(f_1 + f_c \right) \left(V_c - V_{\min} \right) \tag{10-70}$$

Figure 10-3. Limiting conditions for fluid Particle distribution (a) f=constant, (b) $\xi \rightarrow V_{min}/V_c$, (c) $\xi \rightarrow 1$ (Ishii and Kim, 2004)

and

   Case III : $\xi \rightarrow 1$; $V_{1p} \rightarrow V_c$; $f_1 \rightarrow f_c$, hence

$$n_1 = \frac{1}{2} f_1 \left(V_c - V_{min} \right) \cong \frac{1}{2} f_c \left(V_c - V_{min} \right). \tag{10-71}$$

Therefore, assuming that $V_c \gg V_{min}$, and defining an arbitrary coefficient χ as

$$A_{ic} f_c V_c = \chi n_1 A_{ic} \tag{10-72}$$

we obtain, for each limiting case

$$\chi = \begin{cases} 1 & \text{for Case I} \\[2mm] 2 - \dfrac{V_c}{n_1} f_1 & \text{for Case II} \\[2mm] 2 & \text{for Case III} \end{cases} \tag{10-73}$$

and from Eq.(10-68)

$$\chi = \left\{ \frac{2}{1-\xi} - \frac{V_c}{n_1 (1-\xi)} f_1 \right\} \text{ in general.} \tag{10-74}$$

Thus, recalling that there is no inter-group transfer contribution when $f_c \rightarrow 0$, the constant χ should be bounded by

$$0 \le \chi \le 2. \tag{10-75}$$

In providing the analytical solution for χ, we have three unknowns with three equations, such that

$$\begin{aligned} n_1 &= f\left(f_1, f_c, V_{1p} \right) \\ \alpha_{g1} &= f\left(f_1, f_c, V_{1p} \right) \\ a_{i1} &= f\left(f_1, f_c, V_{1p} \right) \end{aligned} \tag{10-76}$$

where α_{g1} and a_{i1}, defined by Eqs.(10-8) and (10-20), can be obtained through experiment, and n_1 can be furnished by Eqs.(10-31) and (10-67).

Therefore, if necessary, the analytical solution for χ can be obtained. Furthermore, since the a_i can be written in terms of the average surface area of the particles A_i, and the particle number density, n as

$$a_{i1} = n_1 A_{i1} \tag{10-77}$$

and since $D_{s1} \approx D_{Sm1}$

$$\frac{A_{ic}}{A_{i1}} = \left(\frac{D_{sc}}{D_{s1}}\right)^2 \approx \left(\frac{D_{sc}}{D_{Sm1}}\right)^2 \tag{10-78}$$

we can rewrite Eqs.(10-64) and (10-65) as

$$\int_{V_{\min}}^{V_c} \left\{ A_i \frac{\partial}{\partial V}\left(f\frac{dV}{dt}\right) \right\} dV = \left(\frac{\dot{V}}{V}\right) a_{i1} \left\{ -\frac{2}{3} + \chi\left(\frac{D_{sc}}{D_{Sm1}}\right)^2 \right\} \tag{10-79}$$

and

$$\int_{V_c}^{V_{\max}} \left\{ A_i \frac{\partial}{\partial V}\left(f\frac{dV}{dt}\right) \right\} dV$$

$$= \left(\frac{\dot{V}}{V}\right) a_{i2} \left\{ -\frac{2}{3} - \chi\frac{a_{i1}}{a_{i2}}\left(\frac{D_{sc}}{D_{Sm1}}\right)^2 \right\} \tag{10-80}$$

respectively.

Thus, combining above equations, the two-group interfacial area transport equation is given by

$$\frac{\partial a_{i1}}{\partial t} + \nabla \cdot \left(a_{i1}\boldsymbol{v}_{i1}\right) = \frac{2}{3}\frac{a_{i1}}{\alpha_{g1}}\left\{\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1}\boldsymbol{v}_{g1}\right) - \eta_{ph}\right\}$$

$$-\chi\left(\frac{D_{sc}}{D_{Sm1}}\right)^2 \frac{a_{i1}}{\alpha_{g1}}\left\{\frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1}\boldsymbol{v}_{g1}\right) - \eta_{ph}\right\} \tag{10-81}$$

$$+\int_{V_{\min}}^{V_c}\left(\sum_j S_j + S_{ph}\right) A_i dV$$

$$\frac{\partial a_{i2}}{\partial t} + \nabla \cdot \left(a_{i2} \boldsymbol{v}_{i2}\right) = \frac{2}{3} \frac{a_{i2}}{\alpha_{g2}} \left\{ \frac{\partial \alpha_{g2}}{\partial t} + \nabla \cdot \left(\alpha_{g2} \boldsymbol{v}_{g2}\right) \right\}$$

$$+ \chi \left(\frac{D_{sc}}{D_{Sm1}}\right)^2 \frac{a_{i1}}{\alpha_{g1}} \left\{ \frac{\partial \alpha_{g1}}{\partial t} + \nabla \cdot \left(\alpha_{g1} \boldsymbol{v}_{g1}\right) - \eta_{ph} \right\} \qquad (10\text{-}82)$$

$$+ \int_{V_c}^{V_{max}} \sum_j S_j A_i dV$$

for Group 1 and Group 2, respectively. In this, the left-hand sides of the equations represent the time-rate of change and convection of a_i for each group. Each term on the right-hand side represents the rate of change in a_i due to the particle volume change, inter-group transfer, various particle interactions and phase change for each group. The total interfacial area transport equation can be also obtained by adding the two equations, such that

$$\frac{\partial a_i}{\partial t} + \nabla \cdot \left(a_i \boldsymbol{v}_i\right) = \int_{V_{min}}^{V_{max}} \left(\sum_j S_j + S_{ph}\right) A_i dV$$

$$+ \sum_{k=1}^{2} \frac{2}{3} \frac{a_{ik}}{\alpha_{gk}} \left\{ \frac{\partial \alpha_{gk}}{\partial t} + \nabla \cdot \left(\alpha_{gk} \boldsymbol{v}_{gk}\right) - \eta_{ph} \right\} \qquad (10\text{-}83)$$

where the subscript k denotes the bubble group.

In this analysis, we demonstrate the inter-group transfer as a result of bubble interactions between the two groups of bubbles by assuming a liner profile or a uniform profile in the particle distribution. However, in two-phase flow applications, the uniform profile in the particle distribution may be assumed to be employed in practice.

1.3.4   Constitutive relations

In this section, the necessary constitutive relations in solving the interfacial area transport equation are summarized, and they are as follows:

$$\int_{V_{min}}^{V_{max}} \sum_j S_j dV = R_j: \quad \text{source and sink rate}$$

$$\qquad (10\text{-}84)$$

for particle number

$$\int_{V_{\min}}^{V_{\max}} \sum_j S_j A_i dV = \phi_j: \quad \text{source and sink rate for } a_i \qquad (10\text{-}85)$$

$$\phi_{ph} = \pi D_{bc}^2 R_{ph}: \quad \text{source and sink rate for } a_i$$
by phase-change
$$(10\text{-}86)$$

$$n = \psi \frac{a_i^3}{\alpha_g^2} \text{ where } \psi = \frac{1}{36\pi} \left(\frac{D_{Sm}}{D_e} \right)^3 \qquad (10\text{-}87)$$

$$D_{Sm} = \frac{6\alpha_g}{a_i} \qquad (10\text{-}88)$$

$$\alpha_g = \alpha_{g1} + \alpha_{g2} \qquad (10\text{-}89)$$

$$\Gamma_g = \Gamma_{g1} + \Gamma_{g2} \qquad (10\text{-}90)$$

and

$$v_g = \frac{\alpha_{g1} v_{g1} + \alpha_{g2} v_{g2}}{\alpha_{g1} + \alpha_{g2}}. \qquad (10\text{-}91)$$

For the continuity equations, the net-mass transfer rate between Group 1 and Group 2 bubbles due to bubble interactions at steady state without phase change effect can be obtained from the modeling of the two-group bubble interactions as

$$\Delta \dot{m}_{12} = \rho_g \left\{ \sum_j \eta_{j,2} + \chi \left(\frac{D_{sc}}{D_{Sm1}} \right)^3 \nabla \cdot \left(\alpha_{g1} v_{g1} \right) \right\}. \qquad (10\text{-}92)$$

In this, $\eta_{j,2}$ is the net volume transfer from Group 1 bubbles to Group 2 bubbles due to the j-th interaction between the two groups of bubbles, such as bubble coalescence and disintegration.

Among the constitutive relations given above, the number source and sink rates defined by Eq.(10-84) should be established through mechanistic modeling of the major particle interactions that contribute to the change in the interfacial area concentration. Accounting for the wide range of gas-liquid two-phase flow, the major bubble interaction mechanisms that lead to the particle coalescence or disintegration can be summarized as follows:

- Random Collision (R_{RC}): coalescence through random collision driven by turbulent eddies;
- Wake Entrainment (R_{WE}): coalescence through collision due to acceleration of the following particle in the wake of the preceding particle;
- Turbulent Impact (R_{TI}): disintegration upon impact of turbulent eddies;
- Shearing-off (R_{SO}): shearing-off around the base rim of the cap bubble;
- Surface Instability (R_{SI}): break-up of large cap bubble due to surface instability;
- Rise Velocity (R_{RV}): collision due to the difference in the bubble rise velocity;
- Laminar Shear (R_{LS}): breakup due to the laminar shear in viscous fluid, and;
- Velocity Gradient (R_{VG}): collision due to the velocity gradient.