Chapter 9

TWO-FLUID MODEL

The two-fluid model (Ishii, 1975, Ishii and Mishima, 1984) is formulated by considering each phase separately. Thus, the model is expressed in terms of two sets of conservation equations governing the balance of mass, momentum and energy in each phase. However, since the averaged fields of one phase are not independent of the other phase, we have interaction terms appearing in these balance equations. The terms denoted by Γ_k, $\boldsymbol{M}_k$ and E_k are the mass, momentum and energy transfers to the k^{th}-phase from the interfaces. As these quantities also should obey the balance laws at the interfaces, we have derived the interfacial transfer conditions from the local jump conditions. Consequently six differential field equations with three interfacial transfer conditions govern the macroscopic two-phase flow systems.

In the two-fluid model formulation, the transfer processes of each phase are expressed by their own balance equations, thus it is anticipated that the model can predict more detailed changes and phase interactions than the drift-flux (or mixture) model. However, this means that the two-fluid model is far more complicated not only in terms of the number of field equations involved but also in terms of the necessary constitutive equations. It is evident that these constitutive equations should be accurate to display the usefulness of the model. This is particularly true with the interaction terms Γ_k, $\boldsymbol{M}_k$ and E_k since, without these interfacial exchanges in the field equations, the two phases are essentially independent. These interaction terms decide the degree of coupling between the phases, thus the transfer processes in each phase are greatly influenced by these terms.

The real importance of the two-fluid model is that it can take into account the dynamic and non-equilibrium interactions between phases. This is accomplished by using the momentum equations for each phase and two independent velocity fields as well as the two energy equations in the formulation. Thus, it is expected that two-fluid model can be useful to the

analyses of transient phenomena, wave propagations and of the flow regime changes. Particularly if the two phases are weakly coupled such that the inertia of each phase changes rapidly, the two-fluid model should be used to study these phenomena.

However, if the two phases are coupled strongly (in which the responses of phases are simultaneous such that two phases are close to mechanical and thermal equilibrium or the wave propagations are firmly interlocked), the two-fluid model brings into the system unnecessary complications for practical applications. Furthermore, it can be said that the two-fluid model is well suited to the studies of the local wave propagations and related stability problems. However, if one is concerned with the total response of the two-phase mixture in a system rather than the local behaviors of each phase, the drift-flux model is simpler and in most cases effective for solving problems. For general three-dimensional flow, the two-fluid model is better than the mixture model because the relative velocity correlation is extremely difficult to develop in a general three-dimensional form.

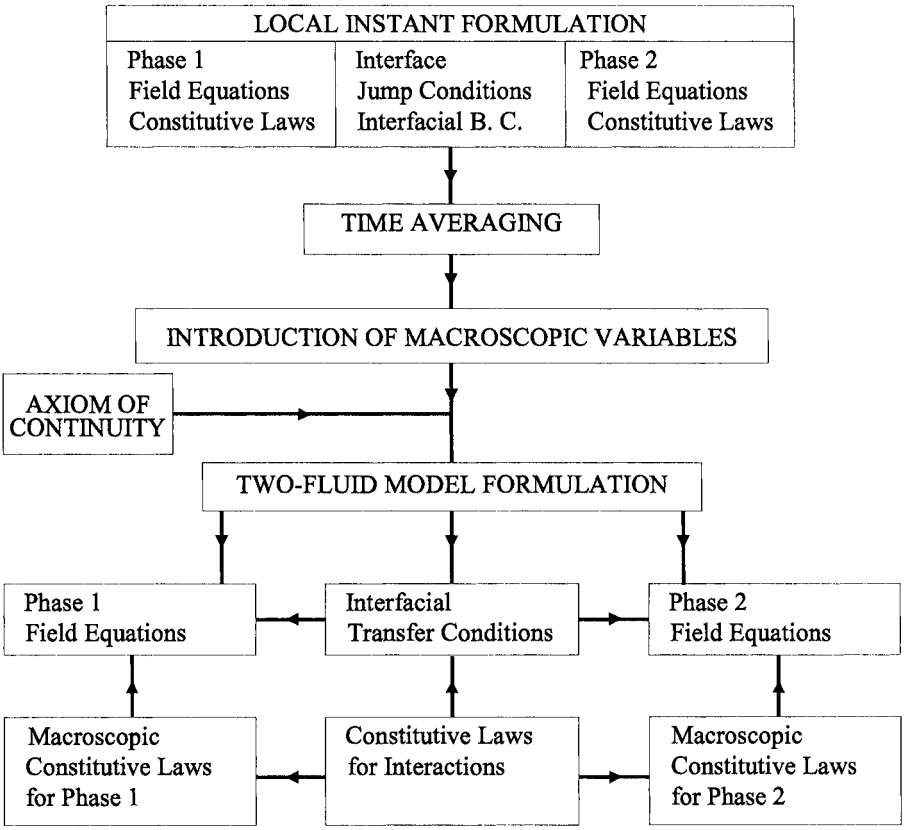
In what follows, we study a general formulation of the two-fluid model as well as various constitutive equations that are necessary to close the system of equations. It should be noted, however, that closing the system of differential equations by making the number of unknowns and equations the same does not imply the existence of a solution nor guarantee its uniqueness. However, it is a necessary condition for a properly set mathematical model that represents the physical systems to be analyzed.

It should also be remembered that mathematical models of two-phase flow systems are in no sense firmly established and some additional research is required to complete the three-dimensional model for a general two-phase flow. In order to appreciate the difficulties confronting us, we recall that even in a single-phase turbulent flow, the general constitutive equations for turbulent fluxes are not developed yet. In view of the present state of the art on the three dimensional two-fluid model formulation, the necessary form of the model is discussed in general terms first. This chapter, therefore, should be considered as a framework and guidance to formulate the constitutive equations from them. The following diagram summarizes the establishment of the two-fluid model formulation.

1.1 Two-fluid model field equations

Two-Fluid Model Continuity Equations

The two-fluid model is characterized by two independent velocity fields which specify the motions of each phase. The most natural choice of velocity fields is obviously the mass-weighted mean phase velocities $\widehat{v_k}$.



Thus the suitable form of the continuity equations to be used in the model should be Eq.(5-21)

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}}}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{v}_k} \right) = \Gamma_k \tag{9-1}$$

with the interfacial mass transfer condition from Eq.(5-22)

$$\sum_{k=1}^{2} \Gamma_k = 0. \tag{9-2}$$

Thus, the term denoted by Γ_k represents the rate of production of k^{th}-phase mass from the phase changes at the interfaces per unit volume. It appears because the local continuity equation has been integrated in time to obtain a macroscopic field equation. Furthermore, it can be said that Eqs.(9-1) and (9-2) are the general statements of the conservation of mass in the

macroscopic field, as a result they hold regardless of the mechanism of phase changes.

In terms of the convective derivative of Eq.(7-11) the continuity equation becomes

$$\frac{D_k}{Dt}\left(\alpha_k \overline{\overline{\rho_k}}\right) + \alpha_k \overline{\overline{\rho_k}} \nabla \cdot \widehat{\boldsymbol{v}_k} = \Gamma_k. \tag{9-3}$$

And thus we have

$$\alpha_k \frac{D_k \overline{\overline{\rho_k}}}{Dt} + \overline{\overline{\rho_k}} \frac{D_k \alpha_k}{Dt} + \alpha_k \overline{\overline{\rho_k}} \nabla \cdot \widehat{\boldsymbol{v}_k} = \Gamma_k. \tag{9-4}$$

For a steady state flow, the time derivative of Eq.(9-1) drops, hence we have

$$\nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{v}_k}\right) = \Gamma_k. \tag{9-5}$$

If each phase is originally incompressible, then the mean density $\overline{\overline{\rho_k}}$ is constant. Thus we have

$$\frac{\partial \alpha_k}{\partial t} + \nabla \cdot \left(\alpha_k \widehat{\boldsymbol{v}_k}\right) = \frac{\Gamma_k}{\overline{\overline{\rho_k}}}. \tag{9-6}$$

And furthermore, if there is no change of phases, the continuity equation reduces to

$$\frac{\partial \alpha_k}{\partial t} + \nabla \cdot \left(\alpha_k \widehat{\boldsymbol{v}_k}\right) = 0 \tag{9-7}$$

which can be used in a low speed two-phase flow without phase changes. Under these conditions the kinematics of the two-phase system is completely governed by the phase redistribution, namely, by the convection and diffusion. The form of the above equation is analogous to that of a single-phase compressible flow.

The general form of the phase continuity equation has been given in the vector notation by Eq.(9-1). In view of a practical importance, we express the equation in rectangular and cylindrical coordinate systems. Thus, in *rectangular coordinates* (x, y, z) we have

$$\frac{\partial}{\partial t}\left(\alpha_k \overline{\overline{\rho_k}}\right) + \frac{\partial}{\partial x}\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{xk}}\right) + \frac{\partial}{\partial y}\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{yk}}\right) + \frac{\partial}{\partial z}\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{zk}}\right) = \Gamma_k. \tag{9-8}$$

If the flow is restricted to two dimensions, then it represents a plane flow. In this case, the partial derivative with respect to x can be dropped from Eq.(9-8). We also note here that, for a steady plane flow with no phase changes, it is possible to introduce a stream function.

The continuity equation in *cylindrical coordinates* becomes

$$\frac{\partial}{\partial t}\left(\alpha_k \overline{\overline{\rho_k}}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{rk}}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{\theta k}}\right) + \frac{\partial}{\partial z}\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{zk}}\right) = \Gamma_k. \tag{9-9}$$

Flow is said to be axisymmetric, if there is no dependences on θ -direction, thus for such flows we have

$$\frac{\partial}{\partial t}\left(r\alpha_k \overline{\overline{\rho_k}}\right) + \frac{\partial}{\partial r}\left(r\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{rk}}\right) + \frac{\partial}{\partial z}\left(r\alpha_k \overline{\overline{\rho_k}}\,\widehat{v_{zk}}\right) = r\Gamma_k. \tag{9-10}$$

A stream function can also be introduced for a steady axisymmetric flow with no phase changes, making it is possible to eliminate the continuity equation from the formulation.

Two-Fluid Model Momentum Equations

In the two-fluid model formulation, the conservation of momentum is expressed by two momentum equations with the interfacial momentum transfer condition. As it was mentioned before, the appropriate field equations should be expressed by the center of mass or the barycentric velocity of each phase $\widehat{\boldsymbol{v}_k}$, thus from Eq.(5-26) we have two momentum equations given by

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}}\,\widehat{\boldsymbol{v}_k}}{\partial t} + \nabla\cdot\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{\boldsymbol{v}_k}\widehat{\boldsymbol{v}_k}\right) = -\nabla\left(\alpha_k \overline{\overline{p_k}}\right) + \nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{\tau}}}_k + \boldsymbol{\tau}_k^T\right)\right] + \alpha_k \overline{\overline{\rho_k}}\,\widehat{\boldsymbol{g}_k} + \boldsymbol{M}_k. \tag{9-11}$$

The interfacial transfer condition (8-23) has the form

$$\sum_{k=1}^{2} \boldsymbol{M}_k = \boldsymbol{M}_m \tag{9-12}$$

with

$$\boldsymbol{M}_m = 2\overline{\overline{H_{21}}}\,\overline{\sigma}\nabla\alpha_2 + \boldsymbol{M}_m^H. \tag{9-13}$$

We note here that the momentum equation for each phase has an interfacial source term $\boldsymbol{M}_k$ that couples the motions of two phases through Eq.(9-12). Here, $\overline{\overline{H_{21}}}$ and $\overline{\sigma}$ are the average mean curvature of interfaces and the surface tension, whereas the term given by $\boldsymbol{M}_m$ takes account for the effect of the changes in the mean curvature.

In view of the Section 1.2 of Chapter 8, Eq.(9-11) can be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_k\overline{\overline{\rho_k}}\widehat{\boldsymbol{v}_k}\right) + \nabla\cdot\left(\alpha_k\overline{\overline{\rho_k}}\widehat{\boldsymbol{v}_k}\widehat{\boldsymbol{v}_k}\right) = -\nabla\left(\alpha_k\overline{\overline{p_k}}\right)\\
&+\nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{\mathcal{T}}_k}} + \boldsymbol{\mathcal{T}}_k^{T}\right)\right] + \alpha_k\overline{\overline{\rho_k}}\widehat{\boldsymbol{g}_k}\\
&+\left(\widehat{\boldsymbol{v}_{ki}}\Gamma_k + \overline{\overline{p_{ki}}}\nabla\alpha_k + \boldsymbol{M}_{ik} - \nabla\alpha_k\cdot\overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}\right).
\end{aligned} \tag{9-14}$$

Hence, by using the convective derivative of Eq.(7-11) the k^{th}-phase *equation of motion* becomes

$$\begin{aligned}
&\alpha_k\overline{\overline{\rho}}_k\frac{D_k\widehat{\boldsymbol{v}_k}}{Dt} = -\alpha_k\nabla\overline{\overline{p_k}} + \nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{\mathcal{T}}_k}} + \boldsymbol{\mathcal{T}}_k^{T}\right)\right] + \alpha_k\overline{\overline{\rho_k}}\widehat{\boldsymbol{g}_k}\\
&+\left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}}\right)\nabla\alpha_k + \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)\Gamma_k + \boldsymbol{M}_{ik} - \nabla\alpha_k\cdot\overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}
\end{aligned} \tag{9-15}$$

or by substituting Eq.(8-20) into Eq.(9-15) we have

$$\begin{aligned}
&\alpha_k\overline{\overline{\rho}}_k\frac{D_k\widehat{\boldsymbol{v}_k}}{Dt} = -\alpha_k\nabla\overline{\overline{p_k}} + \nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{\mathcal{T}}_k}} + \boldsymbol{\mathcal{T}}_k^{T}\right)\right] + \alpha_k\overline{\overline{\rho_k}}\widehat{\boldsymbol{g}_k}\\
&+\left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}}\right)\nabla\alpha_k + \left(\widehat{\boldsymbol{v}_i} - \widehat{\boldsymbol{v}_k} + \frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}a_i^2}\nabla\alpha_k\right)\Gamma_k + \boldsymbol{M}_{ik}\\
&-\nabla\alpha_k\cdot\overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}.
\end{aligned} \tag{9-16}$$

In general the equation of motion is a vectorial equation, thus we have three components or three corresponding scalar equations. In what follows, we express them in two coordinate systems of practical importances.

The equation of motion in *rectangular coordinates* (x, y, z) can be given as follows

x -component

$$\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{v_{xk}}}{\partial t} + \widehat{v_{xk}} \frac{\partial \widehat{v_{xk}}}{\partial x} + \widehat{v_{yk}} \frac{\partial \widehat{v_{xk}}}{\partial y} + \widehat{v_{zk}} \frac{\partial \widehat{v_{xk}}}{\partial z} \right) = -\alpha_k \frac{\partial \overline{\overline{p_k}}}{\partial x} \\
&+\alpha_k \overline{\overline{\rho_k}} \widehat{g_{xk}} + \left\{ \frac{\partial}{\partial x} \alpha_k \left(\overline{\overline{\tau_{xxk}}} + \tau_{xxk}^T \right) + \frac{\partial}{\partial y} \alpha_k \left(\overline{\overline{\tau_{yxk}}} + \tau_{yxk}^T \right) \right. \\
&\left. + \frac{\partial}{\partial z} \alpha_k \left(\overline{\overline{\tau_{zxk}}} + \tau_{zxk}^T \right) \right\} + \left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}} \right) \frac{\partial \alpha_k}{\partial x} + \left(\widehat{v_{xki}} - \widehat{v_{xk}} \right) \Gamma_k \\
&+M_{ixk} - \left(\frac{\partial \alpha_k}{\partial x} \overline{\overline{\tau_{xxki}}} + \frac{\partial \alpha_k}{\partial y} \overline{\overline{\tau_{yxki}}} + \frac{\partial \alpha_k}{\partial z} \overline{\overline{\tau_{zxki}}} \right)
\end{aligned} \tag{9-17}$$

y -component

$$\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{v_{yk}}}{\partial t} + \widehat{v_{xk}} \frac{\partial \widehat{v_{yk}}}{\partial x} + \widehat{v_{yk}} \frac{\partial \widehat{v_{yk}}}{\partial y} + \widehat{v_{zk}} \frac{\partial \widehat{v_{yk}}}{\partial z} \right) = -\alpha_k \frac{\partial \overline{\overline{p_k}}}{\partial y} \\
&+\alpha_k \overline{\overline{\rho_k}} \widehat{g_{yk}} + \left\{ \frac{\partial}{\partial x} \alpha_k \left(\overline{\overline{\tau_{xyk}}} + \tau_{xyk}^T \right) + \frac{\partial}{\partial y} \alpha_k \left(\overline{\overline{\tau_{yyk}}} + \tau_{yyk}^T \right) \right. \\
&\left. + \frac{\partial}{\partial z} \alpha_k \left(\overline{\overline{\tau_{zyk}}} + \tau_{zyk}^T \right) \right\} + \left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}} \right) \frac{\partial \alpha_k}{\partial y} + \left(\widehat{v_{yki}} - \widehat{v_{yk}} \right) \Gamma_k \\
&+M_{iyk} - \left(\frac{\partial \alpha_k}{\partial x} \overline{\overline{\tau_{xyki}}} + \frac{\partial \alpha_k}{\partial y} \overline{\overline{\tau_{yyki}}} + \frac{\partial \alpha_k}{\partial z} \overline{\overline{\tau_{zyki}}} \right)
\end{aligned} \tag{9-18}$$

z -component

$$\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{v_{zk}}}{\partial t} + \widehat{v_{xk}} \frac{\partial \widehat{v_{zk}}}{\partial x} + \widehat{v_{yk}} \frac{\partial \widehat{v_{zk}}}{\partial y} + \widehat{v_{zk}} \frac{\partial \widehat{v_{zk}}}{\partial z} \right) = -\alpha_k \frac{\partial \overline{\overline{p_k}}}{\partial z}$$

$$
\begin{aligned}
&+\alpha_k \overline{\overline{\rho_k}}\,\widehat{g_{zk}} + \left\{ \frac{\partial}{\partial x}\alpha_k \left(\overline{\overline{\tau_{xzk}}} + \tau^T_{xzk} \right) + \frac{\partial}{\partial y}\alpha_k \left(\overline{\overline{\tau_{yzk}}} + \tau^T_{yzk} \right) \right. \\
&\left. + \frac{\partial}{\partial z}\alpha_k \left(\overline{\overline{\tau_{zzk}}} + \tau^T_{zzk} \right) \right\} + \left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}} \right) \frac{\partial \alpha_k}{\partial z} + \left(\widehat{v_{zki}} - \widehat{v_{zk}} \right) \Gamma_k \qquad (9\text{-}19) \\
&+ M_{izk} - \left(\frac{\partial \alpha_k}{\partial x} \overline{\overline{\tau_{xzki}}} + \frac{\partial \alpha_k}{\partial y} \overline{\overline{\tau_{yzki}}} + \frac{\partial \alpha_k}{\partial z} \overline{\overline{\tau_{zzki}}} \right).
\end{aligned}
$$

We note here that for a plane flow, the x-component of the equation of motion drops. Furthermore, all the partial derivatives with respect to x should be eliminated from the y and z-components of the equation of motion, namely, Eqs.(9-18) and (9-19).

The equation of motion in *cylindrical coordinates* (r, θ, z) becomes

r-component

$$
\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{v_{rk}}}{\partial t} + \widehat{v_{rk}} \frac{\partial \widehat{v_{rk}}}{\partial r} + \frac{\widehat{v_{\theta k}}}{r} \frac{\partial \widehat{v_{rk}}}{\partial \theta} - \frac{\widehat{v_{\theta k}}^2}{r} + \widehat{v_{zk}} \frac{\partial \widehat{v_{rk}}}{\partial z} \right) \\
&= -\alpha_k \frac{\partial \overline{\overline{p_k}}}{\partial r} + \alpha_k \overline{\overline{\rho_k}}\,\widehat{g_{rk}} + \left\{ \frac{1}{r}\frac{\partial}{\partial r} r \alpha_k \left(\overline{\overline{\tau_{rrk}}} + \tau^T_{rrk} \right) \right. \\
&+ \frac{1}{r}\frac{\partial}{\partial \theta}\alpha_k \left(\overline{\overline{\tau_{r\theta k}}} + \tau^T_{r\theta k} \right) - \frac{1}{r}\alpha_k \left(\overline{\overline{\tau_{\theta\theta k}}} + \tau^T_{\theta\theta k} \right) \qquad (9\text{-}20) \\
&\left. + \frac{\partial}{\partial z}\alpha_k \left(\overline{\overline{\tau_{rzk}}} + \tau^T_{rzk} \right) \right\} + \left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}} \right) \frac{\partial \alpha_k}{\partial r} + \left(\widehat{v_{rki}} - \widehat{v_{rk}} \right) \Gamma_k \\
&+ M_{irk} - \left(\frac{\partial \alpha_k}{\partial r} \overline{\overline{\tau_{rrki}}} + \frac{1}{r}\frac{\partial \alpha_k}{\partial \theta} \overline{\overline{\tau_{\theta rki}}} + \frac{\partial \alpha_k}{\partial z} \overline{\overline{\tau_{zrki}}} \right)
\end{aligned}
$$

θ-component

$$
\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{v_{\theta k}}}{\partial t} + \widehat{v_{rk}} \frac{\partial \widehat{v_{\theta k}}}{\partial r} + \frac{\widehat{v_{\theta k}}}{r} \frac{\partial \widehat{v_{\theta k}}}{\partial \theta} + \frac{\widehat{v_{rk}}\,\widehat{v_{\theta k}}}{r} + \widehat{v_{zk}} \frac{\partial \widehat{v_{\theta k}}}{\partial z} \right) \\
&= -\frac{\alpha_k}{r}\frac{\partial \overline{\overline{p_k}}}{\partial \theta} + \alpha_k \overline{\overline{\rho_k}}\,\widehat{g_{\theta k}} + \left\{ \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \alpha_k \left(\overline{\overline{\tau_{r\theta k}}} + \tau^T_{r\theta k} \right) \right.
\end{aligned}
$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\alpha_k\left(\overline{\overline{\tau_{\theta\theta k}}}+\tau^T_{\theta\theta k}\right)+\frac{\partial}{\partial z}\alpha_k\left(\overline{\overline{\tau_{\theta z k}}}+\tau^T_{\theta z k}\right)\Bigg\}$$

$$+\left(\overline{\overline{p_{ki}}}-\overline{\overline{p_k}}\right)\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}+\left(\widehat{v_{\theta ki}}-\widehat{v_{\theta k}}\right)\Gamma_k+M_{i\theta k} \tag{9-21}$$

$$-\left(\frac{\partial\alpha_k}{\partial r}\overline{\overline{\tau_{r\theta ki}}}+\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}\overline{\overline{\tau_{\theta\theta ki}}}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau_{z\theta ki}}}\right)$$

z-component

$$\alpha_k\overline{\overline{\rho_k}}\left(\frac{\partial\widehat{v_{zk}}}{\partial t}+\widehat{v_{rk}}\frac{\partial\widehat{v_{zk}}}{\partial r}+\frac{\widehat{v_{\theta k}}}{r}\frac{\partial\widehat{v_{zk}}}{\partial\theta}+\widehat{v_{zk}}\frac{\partial\widehat{v_{zk}}}{\partial z}\right)$$

$$=-\alpha_k\frac{\partial\overline{\overline{p_k}}}{\partial z}+\alpha_k\overline{\overline{\rho_k}}\widehat{g_{zk}}+\Bigg\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\alpha_k\left(\overline{\overline{\tau_{rzk}}}+\tau^T_{rzk}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\alpha_k\left(\overline{\overline{\tau_{\theta zk}}}+\tau^T_{\theta zk}\right)+\frac{\partial}{\partial z}\alpha_k\left(\overline{\overline{\tau_{zzk}}}+\tau^T_{zzk}\right)\Bigg\} \tag{9-22}$$

$$+\left(\overline{\overline{p_{ki}}}-\overline{\overline{p_k}}\right)\frac{\partial\alpha_k}{\partial z}+\left(\widehat{v_{zki}}-\widehat{v_{zk}}\right)\Gamma_k+M_{izk}$$

$$-\left(\frac{\partial\alpha_k}{\partial r}\overline{\overline{\tau_{rzki}}}+\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}\overline{\overline{\tau_{\theta zki}}}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau_{zzki}}}\right).$$

For an axisymmetric flow, the terms with partial derivative with respect to θ drop from the equations. Furthermore, if the flow is free from the circulatory motion around the z-axis, then the velocity in θ-direction is also zero, thus the entire θ-components of the equation can be eliminated. For many practical problems of two-phase pipe flows, this is a sufficiently accurate model to be used.

Two-Fluid Model Energy Equations

The most fundamental form of the conservation of energy is expressed by considering the balance of the total energy. For the two-fluid model formulation, we have two *total energy equations* with the interfacial energy transfer condition. Thus from Eq.(5-32) we have

$$\frac{\partial}{\partial t}\left[\alpha_k \overline{\overline{\rho_k}}\left(\widehat{e_k}+\frac{\widehat{v_k}^2}{2}\right)\right]+\nabla\cdot\left[\alpha_k \overline{\overline{\rho_k}}\left(\widehat{e_k}+\frac{\widehat{v_k}^2}{2}\right)\widehat{\boldsymbol{v}_k}\right]$$
$$=-\nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{q}_k}}+\boldsymbol{q}_k^T\right)\right]+\nabla\cdot\left(\alpha_k \overline{\overline{T_k}}\cdot\widehat{\boldsymbol{v}_k}\right) \tag{9-23}$$
$$+\alpha_k \overline{\overline{\rho_k}}\widehat{\boldsymbol{g}_k}\cdot\widehat{\boldsymbol{v}_k}+E_k.$$

Here we have assumed that $\boldsymbol{g}_k$ is constant, namely, $\boldsymbol{g}_k=\widehat{\boldsymbol{g}_k}$. The interfacial transfer condition (8-48) couples the energy transport processes of two phases, thus we have

$$\sum_{k=1}^{2} E_k = E_m \tag{9-24}$$

where

$$E_m=\overline{\overline{T_i}}\left(\frac{d\sigma}{dT}\right)\frac{D_i a_i}{Dt}+2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\,\frac{\partial\alpha_1}{\partial t}+E_m^H. \tag{9-25}$$

These relations show that the sum of the interfacial energy transfer terms E_k for each phase balances with the time rate of change of surface energy and the work done by the surface tension. We note here that the term given by E_m^H takes account for the effect of the changes in the mean curvature. If we use the detailed expression for E_k in terms of the interfacial variables given by Eq.(8-48), the total energy equation can be rewritten as

$$\frac{\partial}{\partial t}\left[\alpha_k \overline{\overline{\rho_k}}\left(\widehat{e_k}+\frac{\widehat{v_k}^2}{2}\right)\right]+\nabla\cdot\left[\alpha_k \overline{\overline{\rho_k}}\left(\widehat{e_k}+\frac{\widehat{v_k}^2}{2}\right)\widehat{\boldsymbol{v}_k}\right]$$
$$=-\nabla\cdot\left[\alpha_k\left(\overline{\overline{\boldsymbol{q}_k}}+\boldsymbol{q}_k^T\right)\right]+\nabla\cdot\left(\alpha_k \overline{\overline{T_k}}\cdot\widehat{\boldsymbol{v}_k}\right)$$
$$+\alpha_k \overline{\overline{\rho_k}}\widehat{\boldsymbol{v}_k}\cdot\widehat{\boldsymbol{g}_k}+\left\{\Gamma_k\left(\widehat{h_{ki}}+\widehat{\boldsymbol{v}_{ki}}\cdot\widehat{\boldsymbol{v}_k}-\frac{\widehat{v_k}^2}{2}\right)+a_i\overline{\overline{q_{ki}''}}\right. \tag{9-26}$$
$$\left.-\overline{\overline{p_{ki}}}\frac{\partial\alpha_k}{\partial t}+\boldsymbol{M}_{ik}\cdot\widehat{\boldsymbol{v}_{ki}}-\left(\nabla\alpha_k\cdot\overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}\right)\cdot\widehat{\boldsymbol{v}_{ki}}+W_{ki}^T\right\}.$$

By using the transformation on the convective derivative, Eq.(7-13), we can write

$$\alpha_k \overline{\overline{\rho_k}} \frac{D_k}{Dt}\left(\widehat{e_k} + \frac{\widehat{v_k}^2}{2}\right) = -\nabla \cdot \alpha_k \left(\overline{\overline{q_k}} + \boldsymbol{q}_k^T\right) + \nabla \cdot \left(\alpha_k \overline{\overline{T_k}} \cdot \widehat{\boldsymbol{v}_k}\right)$$
$$+\alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{v}_k} \cdot \widehat{\boldsymbol{g}_k} + \Gamma_k \left\{\left(\widehat{e_{ki}} - \widehat{e_k}\right) + \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) \cdot \widehat{\boldsymbol{v}_k}\right\} + a_i \overline{\overline{q_{ki}''}} \qquad (9\text{-}27)$$
$$-\overline{\overline{p_{ki}}}\left(\frac{\partial \alpha_k}{\partial t} - \frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}}\right) + \boldsymbol{M}_{ik} \cdot \widehat{\boldsymbol{v}_{ki}} - \left(\nabla \alpha_k \cdot \overline{\overline{\boldsymbol{T}_{ki}}}\right) \cdot \widehat{\boldsymbol{v}_{ki}} + W_{ki}^T.$$

Equation (9-23) describes the transfer of energy seen from the observer at a fixed point, and Eq.(9-27) expresses the energy transfer by following the fluid with the barycentric velocity $\widehat{\boldsymbol{v}_k}$.

In many practical heat transfer problems, it is convenient to use the thermal energy equation instead of the total energy equation. This is particularly true for low-speed, two-phase flows with heat additions where the mechanical terms are insignificant in comparison with the high heat transfer rates. Thus, by recalling Eq.(5-38) and Eq.(5-39), *the thermal energy equation is* given by

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}} \widehat{h_k}}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{h_k} \widehat{\boldsymbol{v}_k}\right) = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T\right)$$
$$+\frac{D_k}{Dt}\left(\alpha_k \overline{\overline{p_k}}\right) - \widehat{\boldsymbol{v}_k} \cdot \nabla \cdot \left(\alpha_k \boldsymbol{T}_k^T\right) + \alpha_k \overline{\overline{\boldsymbol{T}_k}} : \nabla \widehat{\boldsymbol{v}_k} + \Lambda_k. \qquad (9\text{-}28)$$

Substituting the expression for Λ_k of Eq.(8-40) into the above equation, we get

$$\frac{\partial}{\partial t}\left(\alpha_k \overline{\overline{\rho_k}} \widehat{h_k}\right) + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{h_k} \widehat{\boldsymbol{v}_k}\right) = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T\right)$$
$$+\frac{D_k}{Dt}\left(\alpha_k \overline{\overline{p_k}}\right) - \widehat{\boldsymbol{v}_k} \cdot \nabla \cdot \left(\alpha_k \boldsymbol{T}_k^T\right) + \alpha_k \overline{\overline{\boldsymbol{T}_k}} : \nabla \widehat{\boldsymbol{v}_k}$$
$$+\left(\Gamma_k \widehat{h_{ki}} + a_i \overline{\overline{q_{ki}''}}\right) - \overline{\overline{p_{ki}}} \frac{D_k \alpha_k}{Dt} + \boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) \qquad (9\text{-}29)$$
$$-\nabla \alpha_k \cdot \overline{\overline{\boldsymbol{T}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) + W_{ki}^T.$$

This equation can also be transformed in terms of the convective derivatives as

$$\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{h_k}}{Dt} = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T \right) - \widehat{\boldsymbol{v}_k} \cdot \nabla \cdot \left(\alpha_k \boldsymbol{\mathcal{T}}_k^T \right) \\
&+ W_{ki}^T + \alpha_k \frac{D_k \overline{\overline{p_k}}}{Dt} + \alpha_k \overline{\overline{\boldsymbol{\mathcal{T}}_k}} : \nabla \widehat{\boldsymbol{v}_k} + \Gamma_k \left(\widehat{h_{ki}} - \widehat{h_k} \right) \\
&+ a_i \overline{\overline{q_{ki}''}} + \left(\overline{\overline{p_k}} - \overline{\overline{p_{ki}}} \right) \frac{D_k \alpha_k}{Dt} + \boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right) \\
&- \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right)
\end{aligned} \tag{9-30}$$

which is the equation describing the exchanges of thermal energy as seen from the observer moving with the mass center of the k^{th}-phase.

For simplicity we denote the turbulent energy source by Φ_k^T and the viscous dissipation term by Φ_k^μ, thus

$$\Phi_k^T \equiv -\widehat{\boldsymbol{v}_k} \cdot \nabla \cdot \left(\alpha_k \boldsymbol{\mathcal{T}}_k^T \right) + W_{ki}^T \tag{9-31}$$

$$\Phi_k^\mu \equiv \alpha_k \overline{\overline{\boldsymbol{\mathcal{T}}_k}} : \nabla \widehat{\boldsymbol{v}_k}. \tag{9-32}$$

Then Eq.(9-30) reduces to

$$\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{h_k}}{Dt} = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T \right) + \alpha_k \frac{D_k \overline{\overline{p_k}}}{Dt} + \Phi_k^T + \Phi_k^\mu \\
&+ \Gamma_k \left(\widehat{h_{ki}} - \widehat{h_k} \right) + a_i \overline{\overline{q_{ki}''}} + \left(\overline{\overline{p_k}} - \overline{\overline{p_{ki}}} \right) \frac{D_k \alpha_k}{Dt} \\
&+ \boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right) - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right).
\end{aligned} \tag{9-33}$$

Now we expand the above thermal energy equation in two coordinate systems of practical importance. Thus, in the *rectangular coordinates* (x, y, z), Eq.(9-33) becomes

$$\begin{aligned}
&\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{h_k}}{\partial t} + \widehat{v_{xk}} \frac{\partial \widehat{h_k}}{\partial x} + \widehat{v_{yk}} \frac{\partial \widehat{h_k}}{\partial y} + \widehat{v_{zk}} \frac{\partial \widehat{h_k}}{\partial z} \right) \\
&= -\frac{\partial}{\partial x} \left[\alpha_k \left(\overline{\overline{q_{xk}}} + q_{xk}^T \right) \right] - \frac{\partial}{\partial y} \left[\alpha_k \left(\overline{\overline{q_{yk}}} + q_{yk}^T \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\left[\alpha_k\left(\overline{\overline{q_{zk}}}+q_{zk}^T\right)\right]+\alpha_k\left(\frac{\partial\overline{\overline{p_k}}}{\partial t}+\widehat{v_{xk}}\frac{\partial\overline{\overline{p_k}}}{\partial x}+\widehat{v_{yk}}\frac{\partial\overline{\overline{p_k}}}{\partial y}\right.\\
&\left.+\widehat{v_{zk}}\frac{\partial\overline{\overline{p_k}}}{\partial z}\right)+\Phi_k^T+\Phi_k^\mu+\Gamma_k\left(\widehat{h_{ki}}-\widehat{h_k}\right)\\
&+a_i\overline{\overline{q_{ki}''}}+\left(\overline{\overline{p_k}}-\overline{\overline{p_{ki}}}\right)\left(\frac{\partial\alpha_k}{\partial t}+\widehat{v_{xk}}\frac{\partial\alpha_k}{\partial x}+\widehat{v_{yk}}\frac{\partial\alpha_k}{\partial y}+\widehat{v_{zk}}\frac{\partial\alpha_k}{\partial z}\right)\\
&+M_{ixk}\left(\widehat{v_{xki}}-\widehat{v_{xk}}\right)+M_{iyk}\left(\widehat{v_{yki}}-\widehat{v_{yk}}\right)+M_{izk}\left(\widehat{v_{zki}}-\widehat{v_{zk}}\right) \qquad (9\text{-}34)\\
&-\left(\frac{\partial\alpha_k}{\partial x}\overline{\overline{\tau_{xxki}}}+\frac{\partial\alpha_k}{\partial y}\overline{\overline{\tau_{yxki}}}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau_{zxki}}}\right)\left(\widehat{v_{xki}}-\widehat{v_{xk}}\right)\\
&-\left(\frac{\partial\alpha_k}{\partial x}\overline{\overline{\tau_{xyki}}}+\frac{\partial\alpha_k}{\partial y}\overline{\overline{\tau_{yyki}}}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau_{zyki}}}\right)\left(\widehat{v_{yki}}-\widehat{v_{yk}}\right)\\
&-\left(\frac{\partial\alpha_k}{\partial x}\overline{\overline{\tau_{xzki}}}+\frac{\partial\alpha_k}{\partial y}\overline{\overline{\tau_{yzki}}}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau_{zzki}}}\right)\left(\widehat{v_{zki}}-\widehat{v_{zk}}\right).
\end{aligned}
$$

For a plane flow, the partial derivative with respect to x drops and the x-component of the velocity is zero.

In the *cylindrical coordinates* (r,θ,z), the thermal energy equation becomes

$$
\begin{aligned}
&\alpha_k\overline{\overline{\rho_k}}\left(\frac{\partial\widehat{h_k}}{\partial t}+\widehat{v_{rk}}\frac{\partial\widehat{h_k}}{\partial r}+\frac{\widehat{v_{\theta k}}}{r}\frac{\partial\widehat{h_k}}{\partial\theta}+\widehat{v_{zk}}\frac{\partial\widehat{h_k}}{\partial z}\right)\\
&=-\frac{1}{r}\frac{\partial}{\partial r}\left[r\alpha_k\left(\overline{\overline{q_{rk}}}+q_{rk}^T\right)\right]-\frac{1}{r}\frac{\partial}{\partial\theta}\left[\alpha_k\left(\overline{\overline{q_{\theta k}}}+q_{\theta k}^T\right)\right]\\
&-\frac{\partial}{\partial z}\left[\alpha_k\left(\overline{\overline{q_{zk}}}+q_{zk}^T\right)\right]+\alpha_k\left(\frac{\partial\overline{\overline{p_k}}}{\partial t}+\widehat{v_{rk}}\frac{\partial\overline{\overline{p_k}}}{\partial r}+\frac{\widehat{v_{\theta k}}}{r}\frac{\partial\overline{\overline{p_k}}}{\partial\theta}\right. \qquad (9\text{-}35)\\
&\left.+\widehat{v_{zk}}\frac{\partial\overline{\overline{p_k}}}{\partial z}\right)+\Phi_k^T+\Phi_k^\mu+\Gamma_k\left(\widehat{h_{ki}}-\widehat{h_k}\right)\\
&+a_i\overline{\overline{q_{ki}''}}+\left(\overline{\overline{p_k}}-\overline{\overline{p_{ki}}}\right)\left(\frac{\partial\alpha_k}{\partial t}+\widehat{v_{rk}}\frac{\partial\alpha_k}{\partial r}+\frac{\widehat{v_{\theta k}}}{r}\frac{\partial\alpha_k}{\partial\theta}+\widehat{v_{zk}}\frac{\partial\alpha_k}{\partial z}\right)
\end{aligned}
$$

$$+M_{irk}\left(\widehat{v_{rki}}-\widehat{v_{rk}}\right)+M_{i\theta k}\left(\widehat{v_{\theta ki}}-\widehat{v_{\theta k}}\right)+M_{izk}\left(\widehat{v_{zki}}-\widehat{v_{zk}}\right)$$
$$-\left(\frac{\partial\alpha_k}{\partial r}\overline{\overline{\tau}}_{rrki}+\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}\overline{\overline{\tau}}_{\theta rki}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau}}_{zrki}\right)\left(\widehat{v_{rki}}-\widehat{v_{rk}}\right)$$
$$-\left(\frac{\partial\alpha_k}{\partial r}\overline{\overline{\tau}}_{r\theta ki}+\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}\overline{\overline{\tau}}_{\theta\theta ki}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau}}_{z\theta ki}\right)\left(\widehat{v_{\theta ki}}-\widehat{v_{\theta k}}\right)$$
$$-\left(\frac{\partial\alpha_k}{\partial r}\overline{\overline{\tau}}_{rzki}+\frac{1}{r}\frac{\partial\alpha_k}{\partial\theta}\overline{\overline{\tau}}_{\theta zki}+\frac{\partial\alpha_k}{\partial z}\overline{\overline{\tau}}_{zzki}\right)\left(\widehat{v_{zki}}-\widehat{v_{zk}}\right).$$

For an axisymmetric flow, the partial derivative with respect to θ drops from the equation. Moreover, if the flow is free from the circulatory motion around the z-axis, the θ-component of velocity is zero. This is a good approximation for many two-phase flows in pipes, particularly for vertical pipe flows.

It can be seen that both the total energy equation and the thermal energy equation are quite complicated in their full forms, and thus several simplifications are important for solving practical problems. We study several special cases below. If the heat transfer and phase changes dominate the energy exchanges, then we may neglect the terms arisen from the mechanical effects. Under this condition, Eq.(9-30) can be reduced to

$$\alpha_k\overline{\overline{\rho}}_k\frac{D_k\widehat{h_k}}{Dt}=-\nabla\cdot\alpha_k\left(\overline{\overline{q}}_k+q_k^T\right)+\Gamma_k\left(\widehat{h_{ki}}-\widehat{h_k}\right)+a_i\overline{\overline{q}}_k''. \qquad (9\text{-}36)$$

The above equation suffices for many two-phase flow analyses except the problems of compressible wave propagations and/or at high speed flow conditions.

In the rectangular coordinates (x,y,z) Eq.(9-36) becomes

$$\alpha_k\overline{\overline{\rho}}_k\left(\frac{\partial\widehat{h_k}}{\partial t}+\widehat{v_{xk}}\frac{\partial\widehat{h_k}}{\partial x}+\widehat{v_{yk}}\frac{\partial\widehat{h_k}}{\partial y}+\widehat{v_{zk}}\frac{\partial\widehat{h_k}}{\partial z}\right)=\Gamma_k\left(\widehat{h_{ki}}-\widehat{h_k}\right)$$
$$+a_i\overline{\overline{q}}_{ki}''-\left\{\frac{\partial}{\partial x}\alpha_k\left(\overline{\overline{q}}_{xk}+q_{xk}^T\right)+\frac{\partial}{\partial y}\alpha_k\left(\overline{\overline{q}}_{yk}+q_{yk}^T\right)\right. \qquad (9\text{-}37)$$
$$\left.+\frac{\partial}{\partial z}\alpha_k\left(\overline{\overline{q}}_{zk}+q_{zk}^T\right)\right\}.$$

If we use the cylindrical coordinates (r,θ,z) we have

$$\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{h_k}}{\partial t} + \widehat{v_{rk}} \frac{\partial \widehat{h_k}}{\partial r} + \frac{\widehat{v_{\theta k}}}{r} \frac{\partial \widehat{h_k}}{\partial \theta} + \widehat{v_{zk}} \frac{\partial \widehat{h_k}}{\partial z} \right) = \Gamma_k \left(\widehat{h_{ki}} - \widehat{h_k} \right)$$
$$+ a_i \overline{\overline{q''_{ki}}} - \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \alpha_k \left(\overline{\overline{q_{rk}}} + q^T_{rk} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \alpha_k \left(\overline{\overline{q_{\theta k}}} + q^T_{\theta k} \right) \right. \tag{9-38}$$
$$\left. + \frac{\partial}{\partial z} \alpha_k \left(\overline{\overline{q_{zk}}} + q^T_{zk} \right) \right\}.$$

Furthermore, if the flow is axisymmetric with negligible heat transfers in the axial direction it reduces to the following form

$$\alpha_k \overline{\overline{\rho_k}} \left(\frac{\partial \widehat{h_k}}{\partial t} + \widehat{v_{rk}} \frac{\partial \widehat{h_k}}{\partial r} + \widehat{v_{zk}} \frac{\partial \widehat{h_k}}{\partial z} \right) = \Gamma_k \left(\widehat{h_{ki}} - \widehat{h_k} \right) + a_i \overline{\overline{q''_{ki}}}$$
$$- \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \alpha_k \left(\overline{\overline{q_{rk}}} + q^T_{rk} \right) \right\}. \tag{9-39}$$

It is a much simplified form of Eq.(9-35), yet the important heat transfer mechanisms are preserved in the above equation.

1.2 Two-fluid model constitutive laws

1.2.1 Entropy inequality

The general scheme of constructing the two-fluid model has been discussed at the beginning of this chapter. It is evident that the macroscopic field equations (9-1), (9-11) and (9-23) and the interfacial transfer conditions (9-2), (9-12) and (9-24) are insufficient to describe any particular system, since the number of the variables exceeds that of the available equations. Additional information which specifies the material and response characteristics of a particular group of materials is necessary. These are commonly called as constitutive equations, as explained in detail in Chapter 2.

The purpose of this section is to examine the necessary constitutive equations to close the system of equations. It is always possible to introduce more detailed mechanisms and variable to differentiate various effects of material and transfer mechanisms and then to complicate the set of equations. Consequently, we will discuss the most important aspect of the constitutive laws, namely the principle of the determinism, with *the simplest and the reasonably general set of the equations*. For this purpose we consider two

sets of the macroscopic conservation equations of mass, momentum and energy given by Eqs.(9-1), (9-11) and (9-23) and the interfacial transfer conditions of mass, momentum and energy, Eqs.(9-2), (9-12) and (9-24).

In analogy with Chapter 2, we proceed to study entropy inequality in the macroscopic field. Thus, by applying the averaging procedures of Eqs.(5-8) and (5-10) to the inequality (2-23) and (2-85), we obtain

$$\frac{\partial}{\partial t}\alpha_k\overline{\overline{\rho_k}}\widehat{s_k}+\nabla\cdot\left(\alpha_k\overline{\overline{\rho_k}}\widehat{s_k}\widehat{\boldsymbol{v}_k}\right)+\nabla\cdot\left\{\alpha_k\overline{\overline{\left(\frac{\boldsymbol{q}_k}{T_k}\right)}}+\alpha_k\overline{\overline{\rho_k s_k'\boldsymbol{v}_k'}}\right\}$$
$$+\frac{1}{\Delta t}\sum_j\frac{1}{v_{ni}}\left\{\dot{m}_k s_k+\boldsymbol{n}_k\cdot\left(\frac{\boldsymbol{q}_k}{T_k}\right)\right\}=\overline{\Delta_k}\geq 0 \tag{9-40}$$

and

$$\frac{1}{\Delta t}\sum_j\frac{1}{v_{ni}}\left\{\frac{d_s}{dt}s_a+s_a\nabla_s\cdot v_i-\sum_{k=1}^{2}\left[\dot{m}_k s_k+\boldsymbol{n}_k\cdot\left(\frac{\boldsymbol{q}_k}{T_k}\right)\right]\right\} \tag{9-41}$$
$$=\overline{\Delta_a}\geq 0$$

where we have taken the internal body heating $\dot{q}_k$ to be zero.

We recall that in Chapter 2 the interfacial entropy generation Δ_a has been assumed to be zero in order to obtain simple boundary conditions at the interfaces. We follow exactly the same approach here, thus we have

$$\overline{\Delta_a}=0. \tag{9-42}$$

Consequently, we obtain

$$\overline{\overline{T_{1i}}}=\overline{\overline{T_i}}=\overline{\overline{T_{2i}}}$$
$$\widehat{\boldsymbol{v}_{t1}}=\widehat{\boldsymbol{v}_{ti}}=\widehat{\boldsymbol{v}_{t2}} \tag{9-43}$$
$$\sum_j a_{ij}\sum_{k=1}^{2}\frac{\dot{m}_k}{T_i}\left\{g_k+\frac{\left|(\boldsymbol{v}_k-\boldsymbol{v}_i)\right|^2}{2}-\frac{\tau_{nnk}}{\rho_k}\right\}=0.$$

However, the last condition can be approximated for most practical problems by

$$\overline{\overline{p_{1i}}} - p^{sat}\left(\overline{\overline{T_i}}\right) = 2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\left(\frac{\overline{\overline{\rho_{1i}}}}{\overline{\overline{\rho_{2i}}} - \overline{\overline{\rho_{1i}}}}\right) \text{ or}$$
$$\overline{\overline{p_{2i}}} - p^{sat}\left(\overline{\overline{T_i}}\right) = 2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\left(\frac{\overline{\overline{\rho_{2i}}}}{\overline{\overline{\rho_{2i}}} - \overline{\overline{\rho_{1i}}}}\right) \tag{9-44}$$

which is the macroscopic form of Eq.(2-107).

The first and second conditions of Eq.(9-43) can be used to replace the fluid temperatures and the fluid tangential velocities by two parameters $\overline{\overline{T_i}}$ and $\widehat{v_{ti}}$, whereas the *last condition remains very important in the macroscopic formulation to set the energy level of the interfaces.* Since at the lower reduced pressure, the density ratio is large and at the higher reduced pressure, surface tension effect is small, Eq.(9-44) may be approximated by

$$\overline{\overline{p_{gi}}} - p^{sat}\left(\overline{\overline{T_i}}\right) \doteq 0. \tag{9-45}$$

Thus, the vapor is almost always very close to the saturation condition at the interfaces. Equation (9-45) is simple enough and it is widely used even in the local instant formulation of two-phase flow problems. From the above discussion, it is seen that the result of Eq.(9-42) can be represented by a single equation (9-45) because the other conditions of Eq.(9-43) are satisfied by simply replacing the interfacial fluid temperatures and the tangential velocities by those of the interfaces, namely, $\overline{\overline{T_i}}$ and $\widehat{v_{ti}}$.

Now we study the entropy inequality of each phase given by Eq.(9-40). If the fluctuations of the interfacial temperature are not important, then we have in analogy with Eq.(8-9) the following approximation

$$T_i \approx \overline{\overline{T_i}}. \tag{9-46}$$

Then Eq.(9-40) can be expressed with the interfacial macroscopic variables of the Section 1.1 of Chapter 8 as

$$\frac{\partial}{\partial t}\alpha_k \overline{\overline{\rho_k}}\,\widehat{s_k} + \nabla\cdot\left(\alpha_k \overline{\overline{\rho_k}}\,\widehat{s_k}\,\widehat{\boldsymbol{v}_k}\right) + \nabla\cdot\left[\alpha_k\overline{\overline{\left(\frac{\boldsymbol{q}_k}{T_k}\right)}} + \alpha_k\overline{\overline{\rho_k s_k' \boldsymbol{v}_k'}}\right]$$
$$-\left(\Gamma_k\widehat{s_{ki}} + a_i\overline{\overline{q_{ki}''}}\,\frac{1}{\overline{\overline{T_i}}}\right) = \overline{\Delta_k} \geq 0. \tag{9-47}$$

Although the above equation can be satisfied by the local instant formulation with the positive viscosity and the thermal conductivity, it still imposes some restrictions on the macroscopic constitutive equations. In other words, the exact form of $\overline{\Delta_k}$ which is obtainable from the local instant formulation satisfies Eq.(9-47), however it does not ensure that the left-hand side of the equation with various constitutive equations is always positive. It can be said that the formulation is consistent only if a set of constitutive equations with constraints imposed by the continuity, momentum and energy equations satisfy the inequality (9-47) trivially. Before we apply the above inequality, we will discuss one of the most important characteristics of the macroscopic model which appears only after the averaging.

1.2.2 Equation of state

It can be said that even if the original local instant formulation has simple linear constitutive laws with a standard equation of state as given in Chapter 2, the macroscopic model obtained by averaging may not have such simple constitutive equations. This is because the statistical effects of local instant fluctuations appear in the formulation. In general, these statistical effects depend not only on the present state in terms of the macroscopic variables, but also on the processes in which the present state has been reached. For example, fluid particles having the same values for the energy $\widehat{u_k}$ and the density $\overline{\overline{\rho_k}}$ can have entirely different values for the average temperature or the pressure. All these suggest that the macroscopic field has the characteristics of the materials with memory (Truesdell 1969). Thus the constitutive equations in general are given by the *functionals* of the past processes. This makes the analysis on the macroscopic constitutive equations extremely complicated and difficult. It is evident that the formulation will result in a set of coupled integro-differential equations. In order to avoid these difficulties, we have to make several assumptions at the expense of the accuracy of the model. We know that all materials show the characteristic of fading memory (Coleman and Noll, 1960). Thus, the importance of the effects of memory in a formulation depends on the ratio of the time span of the effective memory to the time constant of macroscopic processes.

Let us now examine the averaged equation of state corresponding to Eq.(2-24). We have

$$\frac{\partial \rho_k u_k}{\partial t} + \nabla \cdot \left(\rho_k u_k \boldsymbol{v}_k \right) \tag{9-48}$$

$$= T_k \left\{ \frac{\partial \rho_k s_k}{\partial t} + \nabla \cdot \left(\rho_k s_k \boldsymbol{v}_k \right) \right\} - p_k \nabla \cdot \boldsymbol{v}_k .$$

By averaging the above equation we obtain

$$\begin{aligned}
&\left\{ \alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{u_k}}{Dt} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k u_k' \boldsymbol{v}_k'}} \right) + \Gamma_k \left(\widehat{u_k} - \widehat{u_{ki}} \right) \right\} \\
&= \overline{\overline{T_k}} \left\{ \alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{s_k}}{Dt} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k s_k' \boldsymbol{v}_k'}} \right) + \Gamma_k \left(\widehat{s_k} - \widehat{s_{ki}} \right) \right\} \\
&+ \alpha_k \overline{\overline{T_k' \rho_k \left(\frac{\partial s_k}{\partial t} + \boldsymbol{v}_k \cdot \nabla s_k \right)}} \\
&- \left\{ -\alpha_k \overline{\overline{p_k}} \overline{\overline{\frac{1}{\rho_k}}} \frac{D_k \overline{\overline{\rho_k}}}{Dt} + \overline{\overline{p_k}} \nabla \cdot \left(\alpha_k \overline{\overline{\boldsymbol{v}_k'}} \right) + \Gamma_k \left(\overline{\overline{\frac{p_k}{\rho_k}}} - \frac{\overline{\overline{p_k}}}{\overline{\overline{\rho_{ki}}}} \right) \right\} \\
&- \alpha_k \overline{\overline{p_k' \frac{1}{\rho_k} \left(\frac{\partial \rho_k}{\partial t} + \boldsymbol{v}_k \cdot \nabla \rho_k \right)}} .
\end{aligned} \tag{9-49}$$

Here we have used the identities

$$\overline{\overline{A_k \frac{\partial B_k}{\partial t}}} = \overline{\overline{A_k}} \frac{\partial}{\partial t} \alpha_k \overline{\overline{B_k}} - \overline{\overline{A_k}} \sum_j a_{ij} \boldsymbol{n}_k \cdot \boldsymbol{v}_i B_k + \alpha_k \overline{\overline{A_k' \left(\frac{\partial B_k}{\partial t} \right)}}$$

and (9-50)

$$\overline{\overline{A_k \nabla B_k}} = \overline{\overline{A_k}} \nabla \alpha_k \overline{\overline{B_k}} + \overline{\overline{A_k}} \sum_j a_{ij} \boldsymbol{n}_k B_k + \alpha_k \overline{\overline{A_k' \nabla B_k}}$$

with

$$A_k' = A_k - \overline{\overline{A_k}} . \tag{9-51}$$

Equation (9-49) shows that, in general, we do not have simple equation of state in terms of averaged variables. The relation between the internal energy $\widehat{u_k}$, the entropy $\widehat{s_k}$ and the density $\overline{\overline{\rho_k}}$ is influenced by both the interfacial transfers and the statistical effects of the fluctuations of the variables.

A simple static relation between these mean values follows if the fluctuating components are sufficiently smaller than the macroscopic changes of these variables in question and thus the linear expansion of the equation of state is a good approximation. In this case we have

$$\widehat{u_k} = \widehat{u_k}\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) \doteq u_k\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) \tag{9-52}$$

with

$$T_k\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) = \frac{\partial u_k}{\partial s_k}\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) \doteq \overline{\overline{T_k}} \tag{9-53}$$

and

$$p_k\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) = -\frac{\partial u_k}{\partial\left(1/\rho_k\right)}\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) \doteq \overline{\overline{p_k}}. \tag{9-54}$$

These relations hold for a two-phase flow with each phase itself being in near equilibrium state in the time interval of Δt. Hereafter we assume that each phase obeys the static equation of state, Eq.(9-52) in the macroscopic field. It is a rather significant and practical assumption that enables us to construct the two-fluid model and its constitutive equations in parallel with the standard single-phase flow formulation. Under the above conditions, we have following relations in analogy with Eqs.(2-24), (2-25) and (2-26)

$$\begin{aligned}
&\widehat{u_k} = \widehat{u_k}\left(\widehat{s_k}, \overline{\overline{\rho_k}}\right) \\
&\overline{\overline{T_k}} = \frac{\partial \widehat{u_k}}{\partial \widehat{s_k}}, \quad -\overline{\overline{p_k}} = \frac{\partial \widehat{u_k}}{\partial\left(1/\overline{\overline{\rho_k}}\right)} \\
&\widehat{u_k} = \overline{\overline{T_k}}\widehat{s_k} - \frac{\overline{\overline{p_k}}}{\overline{\overline{\rho_k}}} + \widehat{g_k}
\end{aligned} \tag{9-55}$$

and

$$d\widehat{u_k} = \overline{\overline{T_k}} d\widehat{s_k} - \overline{\overline{p_k}} d\left(\frac{1}{\overline{\overline{\rho_k}}}\right).$$

The fundamental equation of state can also be represented by the combination of *the caloric and the thermal equations of state*, hence

$$\widehat{u_k} = \widehat{u_k}\left(\overline{\overline{T_k}}, \overline{\overline{\rho_k}}\right) \tag{9-56}$$

$$\overline{\overline{p_k}} = \overline{\overline{p_k}}\left(\overline{\overline{T_k}}, \overline{\overline{\rho_k}}\right) \tag{9-57}$$

or if we take the enthalpy as a variable, it becomes

$$\widehat{i_k} = \widehat{i_k}\left(\overline{\overline{T_k}}, \overline{\overline{p_k}}\right) \tag{9-58}$$

$$\overline{\overline{\rho_k}} = \overline{\overline{\rho_k}}\left(\overline{\overline{T_k}}, \overline{\overline{p_k}}\right). \tag{9-59}$$

In view of their great practical importance, we now study several thermodynamic second derivatives. The specific heats at constant pressure c_{pk} and at constant density c_{vk} are defined by

$$c_{pk} \equiv \frac{\partial \widehat{i_k}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{p_k}}} = \overline{\overline{T_k}} \frac{\partial \widehat{s_k}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{p_k}}} \tag{9-60}$$

and

$$c_{vk} \equiv \frac{\partial \widehat{u_k}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{\rho_k}}} = \overline{\overline{T_k}} \frac{\partial \widehat{s_k}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{\rho_k}}}. \tag{9-61}$$

Similarly, the thermal expansivity β_k and the isothermal compressibility κ_{Tk} are defined by

$$\beta_k \equiv -\frac{1}{\overline{\overline{\rho_k}}} \frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{p_k}}} \tag{9-62}$$

$$\kappa_{Tk} \equiv \frac{1}{\overline{\overline{\rho_k}}} \frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{p_k}}}\bigg)_{\overline{\overline{T_k}}} = \frac{1}{\overline{\overline{\rho_k}} \left(a_{Tk}\right)^2} \tag{9-63}$$

where a_{Tk} is the isothermal sound velocity. Among these four derivatives we have the following identity

$$c_{pk} - c_{vk} = \frac{\overline{\overline{T_k}} \left(\beta_k\right)^2}{\kappa_{Tk} \overline{\overline{\rho_k}}}. \tag{9-64}$$

It is known that if Eq.(9-55) holds, then only three of the thermodynamic second derivatives are independent and others can be obtained from these three. Let us introduce the ratio of the specific heat

$$\gamma_k \equiv \frac{c_{pk}}{c_{vk}} \tag{9-65}$$

and the isentropic compressibility κ_{sk}

$$\kappa_{sk} = \frac{1}{\overline{\overline{\rho_k}}} \frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{p_k}}}\bigg)_{\widehat{s_k}} = \frac{1}{\overline{\overline{\rho_k}} \left(a_{Sk}\right)^2} \tag{9-66}$$

where a_{Sk} is the isentropic sound velocity. Then we have

$$\left(a_{Tk}\right)^2 = \frac{\left(a_{Sk}\right)^2}{\gamma_k}. \tag{9-67}$$

It shows that the isentropic sound velocity is always larger than the isothermal sound velocity, since from the stability of the system $\kappa_{Tk} \geq 0$, thus $\gamma_k \geq 1$. The importance of the thermodynamic second derivatives or the thermal and the caloric equations of state are related to the possibilities of measurements. For example, the fluid pressure and temperature are relatively easy to measure, thus the equations of state in the form of Eqs.(9-58) and (9-59) can be constructed experimentally.

Saturation Condition

The classical saturation condition is given by

$$\begin{cases} \overline{\overline{p_1}} = \overline{\overline{p_2}} = p^{sat} \\ \overline{\overline{T_1}} = \overline{\overline{T_2}} = T^{sat} \end{cases} \tag{9-68}$$

then

$$\widehat{g_1}\left(T^{sat}, p^{sat}\right) = \widehat{g_2}\left(T^{sat}, p^{sat}\right) = g^{sat}. \tag{9-69}$$

Thus we have the relation

$$p^{sat} = p^{sat}\left(T^{sat}\right) \tag{9-70}$$

which is assumed to be identical to Eq.(2-99).

1.2.3 Determinism

In the present analysis, we have assumed the existence of the static equation of state, Eq.(9-55). From the principle of determinism, we should be able to predict the present state from the past history. The necessary condition is that the system of equations is closed, or the number of unknown being same as that of equations. We see that this condition is not satisfied by the field equations (9-1), (9-11) and (9-23), the interfacial conditions (9-2), (9-12) and (9-24) and the equations of state (9-55). Consequently, it is necessary to add several constitutive equations that express the transfer mechanisms of average molecular diffusion, turbulent transfer and interfacial exchanges.

By taking the thermal and caloric equations of state, Eqs.(9-56) and (9-57), the variables appearing in the two-fluid model formulation are:

1. Conservation of Mass $\alpha_k, \overline{\overline{\rho_k}}, \widehat{\boldsymbol{v}_k}, \Gamma_k$;
2. Conservation of Momentum $\overline{\overline{p_k}}, \overline{\overline{\boldsymbol{\mathcal{T}}_k}}, \boldsymbol{\mathcal{T}}_k^T, \widehat{\boldsymbol{g}_k}, \boldsymbol{M}_k, \boldsymbol{M}_m$;
3. Conservation of Energy $\widehat{e_k}, \overline{\overline{\boldsymbol{q}_k}}, \boldsymbol{q}_k^T, E_k, E_m$;
4. Equations of State $\widehat{u_k}, \overline{\overline{T_k}}, \overline{\overline{T_i}}$.

where k = 1 and 2. Hence, the total number of the variables is thirty three. For a properly set model we should have also the same number of equations. These can be classified into the following groups.

	Equations		*Number of Equations*
1)	Field equations		
	mass	Eq.(9-1)	2
	momentum	Eq.(9-11)	2
	energy	Eq.(9-23)	2
2)	Interfacial transfer conditions		
	mass	Eq.(9-2)	1
	momentum	Eq.(9-12)	1
	energy	Eq.(9-24)	1
3)	Axiom of continuity		
	$\alpha_1 = 1 - \alpha_2$		1
4)	Average molecular diffusion fluxes		
	viscous stress $\overline{\overline{\mathcal{T}_k}}$		2
	conduction heat transfer $\overline{\overline{\boldsymbol{q}_k}}$		2
5)	Turbulent fluxes		
	turbulent stress $\mathcal{T}_k^T$		2
	turbulent energy transfer $\boldsymbol{q}_k^T$		2
6)	Body force fields $\widehat{\boldsymbol{g}_k}$		2
7)	Interfacial transfers		
	mass Γ_1		1
	momentum $\boldsymbol{M}_1$		1
	energy E_1		1
8)	Interfacial sources		
	momentum $\boldsymbol{M}_m$		1
	energy E_m		1
9)	Equations of State		
	thermal equation of state		2
	caloric equation of state		2
10)	Turbulent kinetic energy		
	$\widehat{e_k} - \widehat{u_k}$		2
11)	Phase change condition specifying the interfacial temperature $\overline{\overline{T_i}}$		1

12) Mechanical condition at interface specifying
the relation between $\overline{\overline{p_1}}$ and $\overline{\overline{p_2}}$
(Average normal momentum jump condition) 1

This shows that we also have thirty three equations, thus the formulation is consistent. However, it should be noted that the constitutive equations shown above are expressed in the most primitive forms, thus it is quite possible that these equations are coupled with each other through some additional parameters with the same number of supplemental constitutive equations. Furthermore, if one is to use the entropy inequality then Eq.(9-55) should be introduced in the formulation.

1.2.4 Average molecular diffusion fluxes

Viscous Stress Tensor

The constitutive equations for $\overline{\overline{\mathcal{T}_k}}$ and $\overline{\overline{q_k}}$ can be studied by using the identity (9-50). For simplicity, we assume that the fluid is Newtonian and

$$\begin{cases} \rho_k \approx \overline{\overline{\rho_k}} \\ \mu_k \approx \overline{\overline{\mu_k}} \end{cases} \quad \text{in } t \in [\Delta t]_T . \tag{9-71}$$

Then we obtain from Eqs.(2-38) and (9-50)

$$\overline{\overline{\mathcal{T}_k}} = \overline{\overline{\mu_k}} \left\{ \left[\nabla \widehat{\boldsymbol{v}_k} + (\nabla \widehat{\boldsymbol{v}_k})^+ \right] + \frac{1}{\alpha_k} \sum_j a_{ij} \left(\boldsymbol{n}_k \boldsymbol{v}_k' + \boldsymbol{v}_k' \boldsymbol{n}_k \right) \right\} \tag{9-72}$$

where $\boldsymbol{v}_k'$ is the fluctuating component of the k^{th}-phase velocity with respect to $\widehat{\boldsymbol{v}_k}$. It is easy to see that the second part of the stress tensor becomes important when the difference between the interfacial fluid velocity and the mean velocity is large. Thus it takes account for the effects of the interfacial motions and the mass transfers on the average deformation. Let us define the *interfacial extra deformation tensor* by

$$\begin{aligned} \mathbb{D}_{ki} &\equiv \frac{1}{2\alpha_k} \sum_j a_{ij} \left\{ \boldsymbol{n}_k \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k} \right) + \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k} \right) \boldsymbol{n}_k \right\} \\ &= \frac{a_i}{2\alpha_k} \left\{ \overline{\overline{\boldsymbol{n}_k \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k} \right)}} + \overline{\overline{\left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k} \right) \boldsymbol{n}_k}} \right\}. \end{aligned} \tag{9-73}$$

The *bulk deformation tensor* is given by

$$\mathbb{D}_{kb} \equiv \frac{1}{2}\left[\nabla \widehat{\boldsymbol{v}}_k + \left(\nabla \widehat{\boldsymbol{v}}_k\right)^+\right]. \tag{9-74}$$

Consequently, we have

$$\overline{\overline{\boldsymbol{\mathcal{T}}}}_k = 2\overline{\overline{\mu}}_k\left(\mathbb{D}_{kb} + \mathbb{D}_{ki}\right). \tag{9-75}$$

If the effect of the extra deformation tensor is included in the formulation, then a constitutive equation specifying $\mathbb{D}_{ki}$ for each phase should be given. In general, it is considered to be quite complex due to various mechanisms affecting $\mathbb{D}_{ki}$, however, under *special conditions* it can be reduced to a simple form. For example, if phase c is a continuous phase in a dispersed flow and the motions of interfaces are quite regular with little effects from the phase changes, then Eq.(9-73) with Eq.(4-62) can be approximated by

$$\mathbb{D}_{ci} \doteq -\frac{1}{2\alpha_c}\left\{\left(\nabla\alpha_c\right)\left(\widehat{\boldsymbol{v}}_d - \widehat{\boldsymbol{v}}_c\right) + \left(\widehat{\boldsymbol{v}}_d - \widehat{\boldsymbol{v}}_c\right)\left(\nabla\alpha_c\right)\right\}$$

and (9-76)

$$\mathbb{D}_{di} \doteq 0.$$

For a more general case, we may approximate Eq.(9-73) by

$$\mathbb{D}_{ci} = -\frac{a_c^i}{2\alpha_c}\left\{\left(\nabla\alpha_c\right)\left(\widehat{\boldsymbol{v}}_d - \widehat{\boldsymbol{v}}_c\right) + \left(\widehat{\boldsymbol{v}}_d - \widehat{\boldsymbol{v}}_c\right)\left(\nabla\alpha_c\right)\right\} \tag{9-77}$$

where a_c^i represents the mobility of phase c.

Conduction Heat Transfer

The average heat flux $\overline{\overline{\boldsymbol{q}_k}}$ for a fluid obeying Fourier's Law of Heat Conduction, Eq.(2-41), can be given by

$$\overline{\overline{\boldsymbol{q}_k}} = -\overline{\overline{K_k}}\left\{\nabla\overline{\overline{T_k}} + \frac{1}{\alpha_k}\sum_j a_{ij}\boldsymbol{n}_{kj}\left(T_k - \overline{\overline{T_k}}\right)_j\right\} \tag{9-78}$$

in which we have used the identity (9-50) and assumed

$$K_k \approx \overline{\overline{K_k}} \quad \text{in } t \in [\Delta t]_T . \tag{9-79}$$

Furthermore, if we assume the thermal equilibrium at the interface, $T_{ki} = T_i \doteq \overline{\overline{T_i}}$,

$$T_i \approx \overline{\overline{T_i}} \quad \text{at } t \in [\Delta t]_S \tag{9-80}$$

then Eq.(9-78) reduces to

$$\overline{\overline{q_k}} = -\overline{\overline{K_k}} \left\{ \nabla \overline{\overline{T_k}} - \frac{\nabla \alpha_k}{\alpha_k} \left(\overline{\overline{T_i}} - \overline{\overline{T_k}} \right) \right\} \tag{9-81}$$

where we have used Eq.(4-62). It is interesting to note here that the second term represents the heat flux due to the concentration gradient and somehow it resembles the Dufour effect in the single-phase mixtures (Hirschfelder et al., 1954).

1.2.5 Turbulent fluxes

Turbulent Stress Tensor - Mixing Length Model

The difficulties encountered in writing the constitutive equations for turbulent fluxes, even in a single-phase flow, are quite considerable. The essential problem in turbulent flow analyses is to formulate a closure scheme for the averaged field equations. There are two different methods that have been used extensively in studying the transport mechanisms of turbulent flows. The first approach is based on the phenomenological construction of the constitutive equations for the turbulent fluxes. It is best represented by the mixing-length hypothesis of Prandtl who proposed a turbulent model by analogy with the kinematic theory of gases.

The second method is to use more accurate dynamical equations describing the turbulent transports. This can be done by taking the higher moments of the momentum equation. In this way, the number of dynamical equations can be increased as desired. This set is not closed, however, because a turbulent correlation term that arises as an additional flux in the moment equation is always one order higher than the other terms. Thus, these equations can never be closed mathematically. Consequently, it is necessary to make some approximations and use only limited number of the dynamical equations. In contrast to the statistical theories based on higher

moment equations, the phenomenological approach is simple because it supplies directly the turbulent stress.

For many engineering problems, the mixing length model still remains as the primary means to obtain a solution particularly for the wall-induced turbulence. Inclusion of coupled higher-order moment equations almost always requires extensive computer calculations, whereas in many cases even the integral method suffices for engineering requirements.

Even in a single-phase flow, the statistical theories for turbulent flows are not firmly established and the method very often involves a system of quite complicated equations. Consequently, we do not discuss their applications to two-phase flow systems except for dispersed two-phase flow in the Section 1.4 of Chapter 12. Because of its simplicity, we now study the phenomenological approach for the turbulent fluxes in the two-fluid model formulation.

Following the standard analysis on the stress tensor (Truesdell and Toupin, 1960; Aris, 1962; Slattery, 1972), we assume that the local turbulent stress $\mathcal{T}_k^T$ can be decided if we know the phase velocity at the point and the deformation of the phase around it. The above assumption satisfies the constitutive *principle of local action*. Furthermore, if we use the *principle of material frame indifference*, we arrive to the conclusion that the stress tensor depends only on the deformation tensor

$$\mathcal{D}_k \equiv \mathcal{D}_{kb} + \mathcal{D}_{ki} \tag{9-82}$$

where the bulk deformation tensor $\mathcal{D}_{kb}$ and the interfacial extra deformation tensor are given by Eqs.(9-74) and (9-73), respectively. The turbulent stress which is caused by the bulk deformation can be called as the shear-induced turbulence, whereas the one that is caused by the interfacial extra deformation may be called as the bubble-induced turbulence.

In reality, these two constitutive principles may not be fulfilled in a strict sense even in single-phase turbulent flows as it has been discussed by Lumley (1970). Since there are very few experimental facts to depend on, we consider the simple case when above assumptions is valid. Consequently, the most general form permitted under the conditions is

$$\mathcal{T}_k^T = a_{k0}\mathcal{I} + a_{k1}\mathcal{D}_k + a_{k2}\mathcal{D}_k \cdot \mathcal{D}_k \tag{9-83}$$

where the coefficients a_{k0}, a_{k1}, and a_{k2} are functions of the three invariants of the deformation tensor $\mathcal{D}_k$ given by $\mathrm{tr}\,\mathcal{D}_k$, $\mathcal{D}_k : \mathcal{D}_k$ and $\det \mathcal{D}_k$. These are, namely, the trace, double dot product on itself and determinant of $\mathcal{D}_k$, respectively. Hence, we have

$$a_{kn} = a_{kn}\left(\overline{\overline{\rho_k}}, \overline{\overline{\mu_k}}, \alpha_k, \ell, a_i, \overline{\overline{H_{21}}}, \mathrm{tr}\, \mathbb{D}_k, \mathbb{D}_k : \mathbb{D}_k, \det \mathbb{D}_k\right). \tag{9-84}$$

In addition to the three invariants, the arguments of the coefficients are: the fluid density $\overline{\overline{\rho_k}}$; viscosity $\overline{\overline{\mu_k}}$; void fraction α_k ; the distance from the wall ℓ ; the interfacial area concentration a_i ; and the mean curvature $\overline{\overline{H_{21}}}$. The expression for the turbulent stress tensor given by Eq.(9-83) with Eq.(9-84) is still very complicated. However, if we use the mixing length hypothesis similar to the one made in a single-phase flow, the result reduces to a simple form.

First, we assume that the stress tensor of Eq.(9-83) depends only on the second term, which is the Newtonian assumption. Then we have

$$\mathbb{T}_k^T = a_{k1}\mathbb{D}_{kb} \equiv 2\mu_k^T \mathbb{D}_{kb} \tag{9-85}$$

where μ_k^T is the turbulent viscosity. Furthermore, here the coefficient a_{k1} is taken to be

$$a_{k1} = a_{k1}\left(\overline{\overline{\rho_k}}, \overline{\overline{\mu_k}}, \alpha_k, \ell, a_i, \overline{\overline{H_{21}}}, \mathbb{D}_{kb} : \mathbb{D}_{kb}\right). \tag{9-86}$$

It is noted here that the bulk deformation tensor $\mathbb{D}_{kb}$ is used in the place of the total deformation $\mathbb{D}_k$, since the mixing length model is for the shear-induced turbulence. Because of its significance, it is discussed in more detail in Chapter 12. Consequently, from the dimensional analysis, we define

$$2\mu_k^{T^*} = \frac{a_{k1}}{\overline{\overline{\rho_k}}\ell^2\sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}}}. \tag{9-87}$$

Then the non-dimensional function $\mu_k^{T^*}$ should depend on four groups as

$$\mu_k^{T^*} = \mu_k^{T^*}\left(\frac{\overline{\overline{\rho_k}}\ell^2\sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}}}{\overline{\overline{\mu_k}}}, \ell a_i, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_k\right). \tag{9-88}$$

The final expression then becomes

$$\mathbb{T}_k^T = 2\left(\mu_k^{T^*}\right)\overline{\overline{\rho_k}}\ell^2\sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}}\,\mathbb{D}_{kb}. \tag{9-89}$$

This is the corresponding mixing length model for the two-fluid model formulation. The turbulent stress given by Eq.(9-89) with Eq.(9-88) is sufficiently simple to be a realistic model.

In order to visualize the model, let us consider a very simple two-phase pipe flow. By taking a fully developed flow with no phase changes, we have

$$\begin{aligned}
&\left(\overline{\overline{\tau_d}} + \tau_d^T\right)_{rz} = \left\{\overline{\overline{\mu_d}} + \left(\mu_d^{T*}\right)\overline{\overline{\rho_d}}(R-r)^2\left|\frac{d\widehat{v_{zd}}}{dr}\right|\right\}\frac{d\widehat{v_{zd}}}{dr} \\
&\left(\overline{\overline{\tau_c}} + \tau_c^T\right)_{rz} \\
&= \left\{\overline{\overline{\mu_c}} + \left(\mu_c^{T*}\right)\overline{\overline{\rho_c}}(R-r)^2\left|\frac{d\widehat{v_{zc}}}{dr} - \frac{1}{\alpha_c}\frac{d\alpha_c}{dr}\left(\widehat{v_{zd}} - \widehat{v_{zc}}\right)\right|\right\} \\
&\times\left[\frac{d\widehat{v_{zc}}}{dr} - \frac{1}{\alpha_c}\frac{d\alpha_c}{dr}\left(\widehat{v_{zd}} - \widehat{v_{zc}}\right)\right]
\end{aligned} \tag{9-90}$$

where the coefficient μ_k^{T*} is given by

$$\begin{aligned}
\mu_d^{T*} &= \mu_d^{T*}\left(\frac{\overline{\overline{\rho_d}}(R-r)^2\left|\frac{d\widehat{v_{zd}}}{dr}\right|}{\overline{\overline{\mu_d}}}, a_i(R-r), \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_d\right) \\
\mu_c^{T*} &= \mu_c^{T*}\left(\frac{\overline{\overline{\rho_c}}(R-r)^2\left|\frac{d\widehat{v_{zc}}}{dr} - \frac{1}{\alpha_c}\frac{d\alpha_c}{dr}\left(\widehat{v_{zd}} - \widehat{v_{zc}}\right)\right|}{\overline{\overline{\mu_c}}},\right. \\
&\left. a_i(R-r), \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_c\right).
\end{aligned} \tag{9-91}$$

If we exclude the region very near to the wall, the first non-dimensional group, which is a local Reynolds number, may be dropped from the arguments of the function μ_k^{T*}. Thus in this case, the constitutive equation for the turbulent flux, Eq.(9-88), depends only on the static parameters that express the mean geometrical configurations at a point in a flow.

Turbulent Heat Transfer - Mixing Length Model

The turbulent energy flux has been defined by Eq.(5-46). As we can see from the equation, it consists of three parts, namely the turbulent transfers of internal energy, of kinetic energy and the work done by the turbulences. For many practical two-phase flow systems, the latter two effects have less significant roles than the first effect as in the case of a single-phase flow. Thus, we construct a turbulent heat flux model by considering mainly the effect of the thermal energy transport, namely, the first and the last terms of Eq.(5-46) which give *enthalpy transport*. In analogy with Eq.(9-81) we assume

$$\begin{aligned} \boldsymbol{q}_k^T &= -K_k^T \left\{ \nabla \overline{\overline{T}}_k - \frac{\nabla \alpha_k}{\alpha_k} \left(\overline{\overline{T}}_i - \overline{\overline{T}}_k \right) \right\} \\ &= -\overline{\overline{\rho}}_k \kappa_k^T \left\{ \nabla \overline{\overline{i}}_k - \frac{\nabla \alpha_k}{\alpha_k} \left(\overline{\overline{i}}_{ki} - \overline{\overline{i}}_k \right) \right\} \end{aligned} \tag{9-92}$$

where the turbulent energy transport coefficient is expressed by

$$K_k^T = K_k^T \left(\overline{\overline{\rho}}_k, \overline{\overline{K}}_k, \alpha_k, c_{pk}, \ell, a_i, \overline{\overline{H}}_{21}, \sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}} \right). \tag{9-93}$$

From the dimensional analysis, we introduce

$$K_k^{T*} \equiv \frac{K_k^T}{\overline{\overline{\rho}}_k c_{pk} \ell^2 \sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}}}. \tag{9-94}$$

Here c_{pk} and ℓ are the specific heat and the distance from the wall or the mixing length, respectively. Then the non-dimensional parameter K_k^{T*} is a function of four similarity groups as

$$K_k^{T*} = K_k^{T*} \left(\frac{\overline{\overline{\rho}}_k c_{pk} \ell^2 \sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}}}{\overline{\overline{K}}_k}, \ell a_i, \frac{\overline{\overline{H}}_{21}}{a_i}, \alpha_k \right). \tag{9-95}$$

Thus, the turbulent heat flux can be given by

$$\boldsymbol{q}_k^T = -K_k^{T*} \overline{\overline{\rho}}_k c_{pk} \ell^2 \sqrt{2\mathbb{D}_{kb} : \mathbb{D}_{kb}} \left\{ \nabla \overline{\overline{T}}_k - \frac{\nabla \alpha_k}{\alpha_k} \left(\overline{\overline{T}}_i - \overline{\overline{T}}_k \right) \right\}. \tag{9-96}$$

It can be seen from Eqs.(9-73) and (9-82) that if a two-phase system undergoes a changing of phases, then the second invariant of the deformation tensor can be quite complicated. This effect, due to the extra interfacial deformation tensor, promotes the heat transfers in two-phase flow systems.

1.2.6 Interfacial transfer constitutive laws

From the entropy inequality (9-47), thermal energy equation (9-28) and the equation of state (9-55), it can be shown that the entropy productions associated with the interfacial transfer of mass Γ_k, generalized drag force $\boldsymbol{M}_{ik}$ and heat transfer $a_i\overline{\overline{q_k''}}$ become

$$\Gamma_k\left\{\widehat{i_{ki}}\left(\frac{1}{\overline{\overline{T_k}}}-\frac{1}{\overline{\overline{T_i}}}\right)-\left(\frac{\widehat{g_k}}{\overline{\overline{T_k}}}-\frac{\widehat{g_{ki}}}{\overline{\overline{T_i}}}\right)\right\}+\frac{\boldsymbol{M}_{ik}\cdot\left(\widehat{\boldsymbol{v}_{ki}}-\widehat{\boldsymbol{v}_k}\right)}{\overline{\overline{T_k}}}$$
$$+a_i\overline{\overline{q_{ki}''}}\left(\frac{1}{\overline{\overline{T_k}}}-\frac{1}{\overline{\overline{T_i}}}\right)\geq 0. \tag{9-97}$$

Here we have based our analysis on the assumption that these effects satisfy the entropy inequality independently. The standard theory on irreversible thermodynamics (De Groot and Mazur, 1962) gives a simple method to obtain linear constitutive equations. For this purpose, first we should arrange the terms in the entropy inequality into suitable combinations of fluxes and potentials (the fluxes are expanded linearly in terms of the potentials). We should pay special attention here because we have two inequalities from Eq.(9-97) for each phase; The mass transfer term Γ_k and the generalized drag force $\boldsymbol{M}_{ik}$ should satisfy the jump condition (8-5) and (8-19). Since in many practical problems the order of magnitude of $\boldsymbol{M}_m^H$ and $\boldsymbol{M}_k^t$ are much smaller than the drag force itself, we may approximate $\sum_{k=1}^{2}\boldsymbol{M}_{ik}\approx 0$ in Eq.(8-19). By taking into account these effects, we have the following inequality

$$\Gamma_1\left\{\overline{\overline{T_1}}\left[\widehat{i_{1i}}\left(\frac{1}{\overline{\overline{T_2}}}-\frac{1}{\overline{\overline{T_i}}}\right)-\left(\frac{\widehat{g_1}}{\overline{\overline{T_1}}}-\frac{\widehat{g_{1i}}}{\overline{\overline{T_i}}}\right)\right]\right. \tag{9-98}$$

$$-\overline{\overline{T}}_2\left[\widehat{i}_{2i}\left(\frac{1}{\overline{\overline{T}}_2}-\frac{1}{\overline{\overline{T}}_i}\right)-\left(\frac{\widehat{g}_2}{\overline{\overline{T}}_2}-\frac{\widehat{g}_{2i}}{\overline{\overline{T}}_i}\right)\right]\Bigg\}$$

$$+\boldsymbol{M}_{i1}\cdot\left\{\left(\widehat{\boldsymbol{v}}_2-\widehat{\boldsymbol{v}}_1\right)+\left(\widehat{\boldsymbol{v}}_{1i}-\widehat{\boldsymbol{v}}_{2i}\right)\right\}+\sum_{k=1}^{2}a_i\overline{\overline{q}}''_{ki}\left(1-\frac{\overline{\overline{T}}_k}{\overline{\overline{T}}_i}\right)\geq 0.$$

Furthermore, if we neglect the thrust forces due to mass transfer and the normal stresses at interfaces, then from Eqs.(2-104) and (9-44) we have

$$\widehat{g}_{1i}-g^{sat}\left(\overline{\overline{T}}_i\right)=\widehat{g}_{2i}-g^{sat}\left(\overline{\overline{T}}_i\right)\doteq-\left(\frac{2\overline{\overline{H}}_{21}\,\overline{\overline{\sigma}}}{\overline{\overline{\rho}}_{1i}-\overline{\overline{\rho}}_{2i}}\right). \tag{9-99}$$

And the total momentum flux at the interfaces given by Eq.(8-11) can be simplified to

$$\boldsymbol{M}_k\doteq\overline{\overline{p}}_{ki}\nabla\alpha_k+\boldsymbol{M}_{ik}+\Gamma_k\widehat{\boldsymbol{v}}_i-\nabla\alpha_k\cdot\overline{\overline{\boldsymbol{\mathcal{T}}}}_{ki}. \tag{9-100}$$

Thus the pressures at the interfaces should be related by

$$\overline{\overline{p}}_{1i}-\overline{\overline{p}}_{2i}=-2\overline{\overline{H}}_{21}\,\overline{\overline{\sigma}} \tag{9-101}$$

which is automatically satisfied by Eq.(9-100) with Eqs.(8-22), (8-23) and (8-25) as the normal component of the interfacial momentum transfer condition.

In what follows, we assume that the effects of the differences between the phase mean values in the bulk fluid and at the interfaces are negligible for the densities and pressures, but not for the temperatures. Thus we take

$$\overline{\overline{\rho}}_{ki}\doteq\overline{\overline{\rho}}_k \tag{9-102}$$

$$\overline{\overline{p}}_{ki}\doteq\overline{\overline{p}}_k\quad\text{(for most cases)}. \tag{9-103}$$

Under these assumptions, simple linear constitutive equations for interfacial transfer terms may be put into the following forms

$$\Gamma_1 = b_1^{\Gamma}\left(\overline{\overline{T_i}} - \overline{\overline{T_1}}\right) - b_2^{\Gamma}\left(\overline{\overline{T_i}} - \overline{\overline{T_2}}\right) \tag{9-104}$$

$$\boldsymbol{M}_{i1} = b_1^{M}\left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right) \tag{9-105}$$

$$a_i \overline{\overline{q_k''}} = b_k^{E}\left(\overline{\overline{T_i}} - \overline{\overline{T_k}}\right) \quad (k = 1 \text{ or } 2) \tag{9-106}$$

in which the transport coefficients b_k^{Γ}, b_1^{M} and b_k^{E} are considered to be positive scalars.

Interfacial Mass Transfer Term

We assume that the transfer coefficient b_k^{Γ} in Eq.(9-104) is a function of following parameters

$$b_k^{\Gamma} = b_k^{\Gamma}\left(\overline{\overline{\rho_1}}, \overline{\overline{\rho_2}}, \widehat{i_k} - \widehat{i_{ki}}, \overline{\overline{K_k}} + K_k^T, \widehat{i_{1i}} - \widehat{i_{2i}}, \overline{\overline{H_{21}}}, a_i, \alpha_k\right). \tag{9-107}$$

In order to simplify the above equation, we first introduce a non-dimensional parameter

$$b_k^{\Gamma^*} \equiv \frac{b_k^{\Gamma}\left|\widehat{i_{1i}} - \widehat{i_{2i}}\right|}{\left(\overline{\overline{K_k}} + K_k^T\right)a_i^2}. \tag{9-108}$$

And the Jakob numbers are defined by

$$N_{J1} \equiv \frac{\overline{\overline{\rho_1}}\left(\widehat{i_1} - \widehat{i_{1i}}\right)}{\overline{\overline{\rho_2}}\left(\widehat{i_{2i}} - \widehat{i_{1i}}\right)} \tag{9-109}$$

$$N_{J2} \equiv \frac{\overline{\overline{\rho_2}}\left(\widehat{i_2} - \widehat{i_{2i}}\right)}{\overline{\overline{\rho_1}}\left(\widehat{i_{1i}} - \widehat{i_{2i}}\right)}. \tag{9-110}$$

Consequently, the interfacial mass transfer term can be rewritten as

$$\Gamma_1 = a_i^2 \left\{ b_1^{\Gamma^*} \frac{\left(\overline{\overline{K_1}} + K_1^T\right)}{\left|\widehat{i_{1i}} - \widehat{i_{2i}}\right|} \left(\overline{\overline{T_i}} - \overline{\overline{T_1}}\right) - b_2^{\Gamma^*} \frac{\left(\overline{\overline{K_2}} + K_2^T\right)}{\left|\widehat{i_{1i}} - \widehat{i_{2i}}\right|} \left(\overline{\overline{T_i}} - \overline{\overline{T_2}}\right) \right\} \tag{9-111}$$

where the non-dimensional function $b_k^{\Gamma^*}$ can be expressed by four similarity groups as

$$b_k^{\Gamma^*} = b_k^{\Gamma^*}\left(\frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}, N_{Jk}, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_k\right). \tag{9-112}$$

The Jakob number, defined by Eq.(9-109), is the scale of the available energy. It is known to be an important parameter in the analyses of bubble growth.

Now let us examine some special cases in which the constitutive equation for Γ_1 can be reduced to a simple form. In many practical engineering problems, we may assume that the vapor phase is in saturation condition, thus we may take

$$\overline{\overline{p_g}} = \overline{\overline{p_g}}\left(\overline{\overline{T_i}}\right) \quad \text{and} \quad \overline{\overline{T_g}} = \overline{\overline{T_i}}. \tag{9-113}$$

Under this condition Eq.(9-111) reduces to

$$\Gamma_g = -\Gamma_f = \frac{\left(\overline{\overline{K_f}} + K_f^T\right)}{\left(\widehat{i_{gi}} - \widehat{i_{fi}}\right)} \left(\overline{\overline{T_f}} - \overline{\overline{T_i}}\right) b_f^{\Gamma^*} a_i^2. \tag{9-114}$$

For example, the analyses on the bubble growth in a laminar flow suggest that for such flow $b_f^{\Gamma^*}$ can be approximated by

$$b_f^{\Gamma^*} \doteq b_f^{\Gamma^*}\left(N_{Jf}, \frac{\overline{\overline{H_{21}}}}{a_i}\right) = C\left(1 + \frac{2N_{Jf}}{\pi}\right)\frac{\overline{\overline{H_{21}}}}{a_i}. \tag{9-115}$$

Here, C is a parameter that takes into account the thickness of the boundary layer. It varies approximately from 1 to 0.6 as the size of bubble increases. The form of the function $b_k^{\Gamma*}$ for more general case should be obtained from the analyses on a single bubble dynamics as well as from experimental data.

Interfacial Drag Force

The general expression for $\boldsymbol{M}_{ik}$ has been postulated by Eq.(9-105). Now we further assume that the coefficient b_1^M depends on the following parameters.

$$b_1^M = b_1^M\left(a_i, \overline{\overline{H_{21}}}, \overline{\overline{\rho_k}}, |\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}|, \overline{\overline{\mu_k}} + \mu_k^T, \alpha_k, \Gamma_1\right). \tag{9-116}$$

Then from a dimensional analysis, we can rewrite Eq.(9-105) as

$$\boldsymbol{M}_{i1} = \left(\overline{\overline{\rho_1}} + \overline{\overline{\rho_2}}\right)|\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}|\left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right) b_1^{M*} a_i \tag{9-117}$$

where

$$b_1^{M*} \equiv \frac{b_1^M}{\left(\overline{\overline{\rho_1}} + \overline{\overline{\rho_2}}\right)|\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}|\, a_i}. \tag{9-118}$$

The dimensionless function b_1^{M*} depends on the following similarity groups

$$b_1^{M*} = b_1^{M*}\left(\frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_1, N_{\mathrm{Re}1}^i, N_{\mathrm{Re}2}^i, N_{pch}^i\right). \tag{9-119}$$

Here we have defined the interfacial Reynolds number by

$$N_{\mathrm{Re}k}^i \equiv \frac{\overline{\overline{\rho_k}}|\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}|}{\left(\overline{\overline{\mu_k}} + \mu_k^T\right) a_i} \tag{9-120}$$

and the phase change effect number by

$$N_{pch}^i \equiv \frac{\Gamma_1}{\left(\overline{\overline{\rho_1}} + \overline{\overline{\rho_2}}\right)|\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}|\, a_i}. \tag{9-121}$$

The phase change effect number takes account for the mass transfer effect on the drag forces. If this number is large, then the mass transfer effect significantly alters the standard drag correlations. This is exemplified in the field of aerodynamics by variations in the drag forces induced by changes in the rates of boundary-layer suction or blowing.

For a dispersed flow regime, there are numerous researches on the drag forces. The analysis is relatively easy for a flow of a dilute suspension with constant diameter solid spherical particles. However, the problem becomes increasingly complex as the void fraction of the dispersed phase increases or as the wall effect becomes important. It is evident that the drag correlations should depend extensively on experimental data, for a flow with deformable interfaces, interfacial mass transfer and the turbulences.

Some of the results on the dispersed flow drag law can be found in Brodkey (1967) , Soo (1967), Wallis (1969), Schlichting (1979), Happel and Brenner (1965). We discuss important special cases in Section 1.4 of Chapter 9. More general and complete modeling and discussion are presented in Chapter 12.

Interfacial Heat Flux

The constitutive equation for heat transfer at the interface has been postulated by Eq.(9-106). First, we introduce a non-dimensional heat transfer coefficient b_1^E as

$$b_1^{E*} \equiv \frac{b_1^E}{\left(\overline{\overline{K_1}} + K_1^T\right) a_i^2}. \tag{9-122}$$

Then we have

$$a_i \overline{\overline{q_{1i}''}} = \left(\overline{\overline{K_1}} + K_1^T\right) b_1^{E*} \left(\overline{\overline{T_i}} - \overline{\overline{T_1}}\right) a_i^2. \tag{9-123}$$

It is expected that b_1^{E*} depends on the following parameters

$$b_1^{E*} = b_1^{E*}\left(N_{\mathrm{Pr1}}^T, N_{\mathrm{Re1}}^i, N_{pch}^i, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_1, \frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}\right) \tag{9-124}$$

where the Prandtl number is defined by

$$N_{\mathrm{Pr}k}^{T} = \frac{c_{pk}\left(\overline{\overline{\mu_k}} + \mu_k^T\right)}{\overline{\overline{K_k}} + K_k^T}. \tag{9-125}$$

The interfacial Reynolds number $N_{\mathrm{Re}k}^{i}$ and the phase change effect number N_{pch}^{i} have been given by Eqs.(9-120) and (9-121), respectively. It is interesting to note here that actually the non-dimensional parameter b_1^{E*} is an interfacial Nusselt number. Furthermore, if we give a constitutive equation for Γ_1, then it is sufficient to supply a constitutive law for only *one* of the interfacial heat transfer, $a_i\overline{\overline{q_1''}}$. However, it is also possible to give a constitutive relation for $a_i\overline{\overline{q_1''}}$ and for $a_i\overline{\overline{q_2''}}$, then it is equivalent that Γ_1 is known due to the macroscopic energy jump condition.

Interfacial Shear Stress

Since bubbles are dispersed in continuous phase shear layer, the interfacial shear stress is approximated by the shear stress in continuous phase. Thus, we have

$$\overline{\overline{\mathcal{T}_{ki}}} \doteq \overline{\overline{\mathcal{T}_c}}. \tag{9-126}$$

Interfacial Momentum Source

In the original momentum jump condition there are two distinct pieces of information; the normal jump and the tangential jump balances. Since we preserved this special characteristic in the averaged formulation, we obtained the drag force balance (8-19) in addition to the interfacial momentum transfer condition given by Eq.(9-12). Furthermore, by neglecting the mass thrust effect and using the assumption, Eq.(9-103), we obtain from Eq.(8-21)

$$\begin{aligned}\sum_{k=1}^{2} \boldsymbol{M}_k &= \sum_{k=1}^{2}\left(\overline{\overline{p_k}}\nabla\alpha_k + \boldsymbol{M}_k^n + \boldsymbol{M}_k^t - \nabla\alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}}\right) \\ &= \boldsymbol{M}_m = \overline{\overline{2H_{21}}}\,\overline{\overline{\sigma}}\nabla\alpha_2 + \boldsymbol{M}_m^H.\end{aligned} \tag{9-127}$$

Thus, in view of Eqs.(8-19), (9-12) and (9-13), we have

$$\overline{\overline{p_{1i}}} - \overline{\overline{p_{2i}}} = -2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}} \tag{9-128}$$

and

$$\sum_{k=1}^{2} \boldsymbol{M}_{ik} = \sum_{k=1}^{2} \left(\boldsymbol{M}_k^n + \boldsymbol{M}_k^t \right) = \boldsymbol{M}_m^H . \tag{9-129}$$

Here we have the thermal equation of state for the interfaces

$$\overline{\overline{\sigma}} = \overline{\overline{\sigma}} \left(\overline{\overline{T_i}} \right). \tag{9-130}$$

Since the normal component of the interfacial momentum transfer equation (9-129) specifies the mechanical equilibrium condition between two phases, it is necessary to specify the mean curvature $\overline{\overline{H_{21}}}$ by a constitutive equation. A simple case is to assume that the interfacial geometries are completely irregular, thus we may take $\overline{\overline{H_{21}}} = 0$. For dispersed two-phase flow, the mean curvature is the inverse of the radius of a particle. If the fluid particle size is uniform, the radius is given by $3\alpha/a_i$. Therefore, the ratio of α and a_i is in general an important length scale. $6\alpha/a_i$ is known as the Sauter mean diameter, thus the mean curvature is essentially proportional to the inverse of the Sauter mean diameter.

The importance of the parameter a_i remains in the macroscopic formulation, however, since it represents the available area of contact between two phases. It is evident that the interfacial transports of mass, momentum and energy are significantly influenced by the surface area concentration per unit volume a_i. In general, the constitutive equations for $\overline{\overline{H_{21}}}$ and a_i are extremely complicated because these are the parameters that decide the local geometric configuration in the macroscopic field. It is evident that we may supply this information directly or indirectly.

The direct information means that we have a prior knowledge of the flow structures. Then it is not very difficult to obtain a relation for $\overline{\overline{H_{21}}}$ and a_i in terms of various variables and initial and boundary conditions. For example, this can be done easily for a bubbly or droplet flow without phase changes, coalescences or disintegration of bubbles (or droplets). It is also possible to give indirect information on the flow structures through the constitutive equations for $\overline{\overline{H_{21}}}$ and a_i in terms of various parameters such as α_k, $\overline{\overline{\sigma}}$, $\widehat{v_2} - \widehat{v_1}$, $\overline{\overline{\mu_k}}$, μ_k^T, Γ_k etc. Then we may solve the whole set of equations to find a local geometrical configuration. This is difficult because the geometrical configuration has a long-lasting memory and it does not obey the principle of local action in most cases. This means that the initial conditions as well as the wall effects on the flow geometries are very important.

Because of the difficulties encountered in the general case, let us start our discussion on the above constitutive equations from a simple case. Now let

us suppose that phase 2 is dispersed in phase 1. Then we may assume that the volume occupied by phase 2 in a total volume V can be given as a function of the mean curvature $\overline{\overline{H_{21}}}$. Thus we have

$$V_2 = F_{V2}\left(\overline{\overline{H_{21}}}\right) = \alpha_2 V. \tag{9-131}$$

The surface area of phase 2 in the volume V is given by

$$A_2 = F_{A2}\left(\overline{\overline{H_{21}}}\right) = a_i V. \tag{9-132}$$

Exactly the same argument can be carried out in the time domain, hence we have

$$\Delta t_2 = f_{V2}\left(\overline{\overline{H_{21}}}\right) = \alpha_2 \Delta t \tag{9-133}$$

and

$$\sum_j \frac{1}{v_{ni}} = f_{A2}\left(\overline{\overline{H_{21}}}\right) = a_i \Delta t. \tag{9-134}$$

Then we may assume that

$$\frac{f_{V2}\left(\overline{\overline{H_{21}}}\right)}{f_{A2}\left(\overline{\overline{H_{21}}}\right)} = \frac{\alpha_S}{a_i} \doteq \frac{F_{V2}\left(\overline{\overline{H_{21}}}\right)}{F_{A2}\left(\overline{\overline{H_{21}}}\right)} = \frac{1}{3C^i \overline{\overline{H_{21}}}} \tag{9-135}$$

where C^i is a factor to take into account for the shapes and sizes of dispersed phase. Thus we can write the constitutive equation for $\overline{\overline{H_{21}}}$ as

$$\overline{\overline{H_{21}}} = \frac{a_i}{3C^i \alpha_2}. \tag{9-136}$$

Here the factor C^i is 1 for fairly uniform spherical droplets or bubbles and it does not vary much unless the dispersed phase has quite elongated shapes. The relation given by Eq.(9-136) is a static or geometric relation and we may take a more general form given by

$$\overline{\overline{H_{21}}} = \overline{\overline{H_{21}}}\left(\alpha_2, a_i, |\nabla\alpha_2|\right)$$

or (9-137)

$$\frac{\overline{\overline{H_{21}}}}{a_i} = \overline{\overline{H_{21}}}^{*} = \overline{\overline{H_{21}}}^{*}\left(\alpha_2, \frac{|\nabla\alpha_2|}{a_i}\right).$$

We call this relation as the geometric equation of state. From Eq.(9-137) we see that the mean curvature depends on the void fraction, surface area concentration and the void fraction gradient.

However, we still should have one more essential constitutive equation for a_i. It is considered that the information on a_i in terms of other parameters is really a part of the solutions for a local instant formulation. The most general method to include a_i in the two-fluid formulation would be to introduce one more transport equation for the interfacial area concentration as

$$\frac{\partial a_i}{\partial t} + \nabla\cdot\left(a_i\widehat{\boldsymbol{v}}_i\right) = \phi_L. \tag{9-138}$$

With this method, the source term takes account for the bubble or droplet expansions or collapses, coalescences, disintegration and the interfacial instabilities. It is evident that the constitutive equations for $\widehat{\boldsymbol{v}}_i$ and ϕ_L should be supplied. The interfacial area transport equation is a fundamental equation describing the change of surface area between phases. Because of its significance, it is discussed in detail in Chapter 10. In some cases the balance equation (9-138) may be replaced by a simpler algebraic constitutive relation such as

$$a_i = a_i\left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}, \overline{\overline{\rho_k}}, \overline{\overline{\mu_k}}, \alpha_1, |\nabla\alpha_1|, \overline{\overline{\sigma}}, g\right). \tag{9-139}$$

The constitutive equation for $\boldsymbol{M}_m^H$ can be given for a dispersed flow by

$$\boldsymbol{M}_m^H \doteq \alpha_2\nabla\left(2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\right) \tag{9-140}$$

where phase 2 is the dispersed phase. However, in many practical problems the order of magnitude of this term is small in comparison with $\boldsymbol{M}_{i1}$ or $\boldsymbol{M}_{i2}$. In such a case, we may set $\boldsymbol{M}_m^H$ is to be zero.

Interfacial Energy Source

The interfacial energy source E_m is given by Eq.(8-45). It is clear that the inclusion of the interfacial thermal energy term, namely, the terms given by Eq.(8-45), complicates the formulation significantly. Except for very few cases, this term can be neglected with respect to the large-energy exchanges that involve the latent heat at the changing of phases. Consequently, we approximate Eq.(8-45) by

$$E_m = \overline{\overline{T}}_i \left(\frac{d\sigma}{dT}\right) \frac{D_i a_i}{Dt} + 2\overline{\overline{H}}_{21} \overline{\overline{\sigma}} \frac{\partial \alpha_1}{\partial t} + E_m^H \approx 0 \tag{9-141}$$

which does not require any additional constitutive equations.

In order to complete the model for the interfacial energy transfer condition given by Eq.(8-48), we should supply the constitutive equations for the turbulent kinetic energies, the difference between the mean velocity and the average interfacial phase velocity, and the interfacial turbulent flux from the drag force W_{ki}^T. As in two-phase flow with phase changes, the order of magnitude of these terms compared to the thermal terms is relatively small, therefore, we may assume

$$\widehat{h_{ki}} - \widehat{i_{ki}} = \frac{\overline{\left(v'_{ki}\right)^2}}{2} \doteq 0 \tag{9-142}$$

$$\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \doteq 0 \tag{9-143}$$

$$W_{ki}^T \doteq 0. \tag{9-144}$$

In analogy with Eq.(9-142), we take for the bulk phases

$$\widehat{h_k} \doteq \widehat{i_k}. \tag{9-145}$$

Then, Eq.(8-48) can be reduced to the following form

$$E_k \doteq \Gamma_k \left(\widehat{i_{ki}} + \frac{\widehat{v_k}^2}{2}\right) + a_i \overline{\overline{q_k''}} - \overline{\overline{p_k}} \frac{\partial \alpha_k}{\partial t} + \boldsymbol{M}_{ik} \cdot \widehat{\boldsymbol{v}_{ki}} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}}}_{ki} \cdot \widehat{\boldsymbol{v}_{ki}} \tag{9-146}$$

where we have used Eq.(9-103). And the interfacial total energy transfer condition becomes

$$\sum_{k=1}^{2} E_k = E_m \approx 0. \tag{9-147}$$

Then, the thermal energy transfer condition (8-49) can be approximated by

$$\Lambda_k \doteq \left(\Gamma_k \hat{i}_{ki} + a_i \overline{\overline{q''_{ki}}}\right) - \overline{\overline{p_k}} \frac{D_k \alpha_k}{Dt} \tag{9-148}$$

and

$$\begin{aligned} \sum_{k=1}^{2} \Lambda_k &= \overline{\overline{T_i}} \left(\frac{d\sigma}{dT}\right) \frac{D_i a_i}{Dt} + E_m^H - \sum_{k=1}^{2} \overline{\overline{p_k}} \frac{D_k \alpha_k}{Dt} \\ &- \Gamma_1 \left(\frac{\hat{v}_1^{\,2}}{2} - \frac{\hat{v}_2^{\,2}}{2}\right) - \sum_{k=1}^{2} \boldsymbol{M}_{ik} \cdot \hat{\boldsymbol{v}}_k + \sum_{k=1}^{2} \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \cdot \hat{\boldsymbol{v}}_{ki} \end{aligned} \tag{9-149}$$

in which we have used Eq.(9-128) in order to eliminate the surface tension term. By combing Eq.(9-148) with Eq.(9-149) we get

$$\begin{aligned} \sum_{k=1}^{2} \left(\Gamma_k \hat{i}_{ki} + a_i \overline{\overline{q''_{ki}}} + \Gamma_k \frac{\hat{v}_k^{\,2}}{2}\right) &= \left\{\overline{\overline{T_i}} \left(\frac{d\sigma}{dT}\right) \frac{D_i a_i}{Dt} + E_m^H\right\} \\ - \sum_{k=1}^{2} \boldsymbol{M}_{ik} \cdot \hat{\boldsymbol{v}}_{ki} + \sum_{k=1}^{2} \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \cdot \hat{\boldsymbol{v}}_{ki}. \end{aligned} \tag{9-150}$$

The first group on the right-hand side of the above equation is the effect of the surface tension, the second group arises from the interfacial drag work, and the third term is related to the work done by interfacial shear. Thus, we may assume for relatively low speed flow that

$$\sum_{k=1}^{2} \left(\Gamma_k \hat{i}_{ki} + a_i \overline{\overline{q''_{ki}}}\right) \approx 0 \tag{9-151}$$

which is a well-known relation for a local instant formulation. We have shown here the conditions under which we can apply this important and useful formula to the macroscopic two-phase flow problems.

1.3 Two-fluid model formulation

The most general case of the two-fluid model formulation has been discussed in the Section 1.2.3 of Chapter 9 in connection with the principle of determinism. We will now set up a realistic formulation by combining the results of the previous two sections. We already have made a number of assumptions on the interfacial variables and the constitutive equations, thus the present analysis is not a complete mapping of the microscopic field in terms of the local instant variables into the macroscopic field. Rather, it should be considered as an approximate theory based on various constitutive assumptions. The results presented in this section are simplified to an extent of being realistic, yet it is general enough for most engineering problems encountered in two-phase flow system analyses.

First, we list all the important assumptions that have been made to obtain the model.

- Fundamental hypothesis on smoothness of mean values — Section 1.3 in Chapter 4
- Existence of the equation of state — Eq.(9-55)
- Transport properties μ_k and K_k are constant in the interval of time average — Eq.(9-71) Eq.(9-79)
- Interfacial variables are approximated by
  - $\rho_{ki} \approx \overline{\overline{\rho_{ki}}},\ \sigma \approx \overline{\overline{\sigma}},\ \dot{m}_k \approx \overline{\overline{\dot{m}_k}}$ — Eq.(8-9)
  - $T_i \approx \overline{\overline{T_i}}$ — Eq.(9-80)
  - $\overline{\overline{\rho_{ki}}} \approx \overline{\overline{\rho_k}}$ — Eq.(9-102)
  - $\overline{\overline{p_{ki}}} \approx \overline{\overline{p_k}}$ — Eq.(9-103)
- Interfacial normal stress and thrust due to mass transfer are neglected — Eq.(9-127)
- Negligible turbulent kinetic energy or energy transfer — Eq.(9-145)
- Mechanical interaction terms in the interfacial energy transfer condition are neglected — Eq.(9-148)
- Uniform body force — Eq.(5-50)

Under these conditions we have the following field equations

The continuity equations from Eq.(9-1)

$$\frac{\partial}{\partial t}\left(\alpha_k \overline{\overline{\rho_k}}\right) + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{v}_k}\right) = \Gamma_k \quad (k=1 \text{ and } 2) \tag{9-152}$$

The equations of motion from Eq.(9-15)

$$\alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{\boldsymbol{v}_k}}{Dt} = -\alpha_k \nabla \overline{\overline{p_k}} + \nabla \cdot \left[\alpha_k \left(\overline{\overline{\boldsymbol{\mathcal{T}}_k}} + \boldsymbol{\mathcal{T}}_k^T \right) \right] + \alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{g}_k}$$
$$+ \boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} + \Gamma_k \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right) + \left(\overline{\overline{p_{ki}}} - \overline{\overline{p_k}} \right) \nabla \alpha_k \qquad (9\text{-}153)$$
(k=1 and 2)

It is noted that the last term in the above equation is retained though for most cases it is very small, because for some cases such as horizontal flow it can be important.

The equations of thermal energy from Eq.(9-30)

$$\alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{i_k}}{Dt} = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T \right) + \alpha_k \frac{D_k \overline{\overline{p_k}}}{Dt}$$
$$+ \alpha_k \overline{\overline{\boldsymbol{\mathcal{T}}_k}} : \nabla \widehat{\boldsymbol{v}_k} + \Gamma_k \left(\widehat{i_{ki}} - \widehat{i_k} \right) + a_i \overline{\overline{q''_{ki}}} \qquad (9\text{-}154)$$
$$+ \left(\boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \right) \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k} \right) \quad (k\text{=}1 \text{ and } 2)$$

Here we have neglected the turbulent work term $\widehat{\boldsymbol{v}_k} \cdot \nabla \cdot \left(\alpha_k \boldsymbol{\mathcal{T}}_k^T \right)$, since it is considered to contribute mainly for the turbulent kinetic energy changes which have been neglected in the formulation. These two sets of three balance equations describe the physical laws of conservation of mass, momentum and energy in the macroscopic field.

Two phases that are governed by their own field equations are coupled by three interfacial transfer conditions given below.

The interfacial mass transfer condition from Eq.(9-2)

$$\sum_{k=1}^{2} \Gamma_k = 0 \qquad (9\text{-}155)$$

The interfacial momentum transfer condition from Eqs.(9-127) and (9-128)

$$\sum_{k=1}^{2} \left(\overline{\overline{p_k}} \nabla \alpha_k + \boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \right) = 2 \overline{\overline{H_{21}}}\, \overline{\overline{\sigma}} \nabla \alpha_2 + \boldsymbol{M}_m^H \qquad (9\text{-}156)$$

with the normal component satisfying

$$\overline{\overline{p_{1i}}} - \overline{\overline{p_{2i}}} = -2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}} \tag{9-157}$$

The interfacial thermal energy transfer condition from Eq.(9-151)

$$\sum_{k=1}^{2}\left(\Gamma_k \widehat{i_{ki}} + a_i \overline{\overline{q''_{ki}}}\right) = 0. \tag{9-158}$$

Then, from the axiom of continuity we have

$$\alpha_1 = 1 - \alpha_2. \tag{9-159}$$

The equation of state for each phase is given by Eq.(9-55) or by Eqs.(9-58) and (9-59), thus we have the *caloric equations of state*

$$\widehat{i_k} = \widehat{i_k}\left(\overline{\overline{T_k}}, \overline{\overline{p_k}}\right) \quad (k=1 \text{ and } 2) \tag{9-160}$$

and the *thermal equations of state*

$$\overline{\overline{\rho_k}} = \overline{\overline{\rho_k}}\left(\overline{\overline{T_k}}, \overline{\overline{p_k}}\right) \quad (k=1 \text{ and } 2) \tag{9-161}$$

whereas the *equation of state for the surface* is given by Eq.(9-130)

$$\overline{\overline{\sigma}} = \overline{\overline{\sigma}}\left(\overline{\overline{T_i}}\right). \tag{9-162}$$

The interfacial temperature is given by the phase change condition (9-44) as

$$\overline{\overline{p_2}} - p^{sat}\left(\overline{\overline{T_i}}\right) = 2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\left(\frac{\overline{\overline{\rho_2}}}{\overline{\overline{\rho_2}} - \overline{\overline{\rho_1}}}\right) \tag{9-163}$$

where

$$p^{sat} = p^{sat}\left(\overline{\overline{T_i}}\right) \tag{9-164}$$

is the classical saturation condition. For many practical cases we may approximate Eq.(9-163) by

$$\overline{\overline{p_g}} = p^{sat}\left(\overline{\overline{T_i}}\right) \tag{9-165}$$

where $\overline{\overline{p_g}}$ is the vapor phase pressure.

The constitutive equation for the *viscous stress* $\overline{\overline{\mathcal{T}_k}}$ is given by

$$\overline{\overline{\mathcal{T}_k}} = 2\overline{\overline{\mu_k}}\left(\mathbb{D}_{kb} + \mathbb{D}_{ki}\right) = 2\overline{\overline{\mu_k}}\mathbb{D}_k \ (k=1 \text{ and } 2). \tag{9-166}$$

Here the bulk and interfacial extra deformation tensors $\mathbb{D}_{kb}$ and $\mathbb{D}_{ki}$ are given by Eqs.(9-74) and (9-77). Thus we have

$$\mathbb{D}_{kb} = \frac{1}{2}\left[\nabla\widehat{\boldsymbol{v}_k} + \left(\nabla\widehat{\boldsymbol{v}_k}\right)^{+}\right] \tag{9-167}$$

and

$$\mathbb{D}_{ki} \doteq -\frac{a_k^i}{2\alpha_k}\left\{\left(\nabla\alpha_k\right)\left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right) + \left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right)\left(\nabla\alpha_k\right)\right\}. \tag{9-168}$$

The coefficient a_k^i represents the mobility of the k^{th}-phase.

The constitutive equation for the *turbulent stress* $\mathcal{T}_k^T$ has been obtained from the mixing length hypothesis of Eqs.(9-85), (9-86), (9-87), (9-88) and (9-89), thus

$$\begin{aligned}\mathcal{T}_k^T &= 2\mu_k^T\mathbb{D}_{kb} \\ &= 2\mu_k^{T*}\overline{\overline{\rho_k}}\ell^2\sqrt{2\left(\mathbb{D}_{kb} : \mathbb{D}_{kb}\right)}\mathbb{D}_{kb} \ (k=1 \text{ and } 2).\end{aligned} \tag{9-169}$$

Here the non-dimensional turbulent viscosity μ_k^{T*} is a function of the following parameters

$$\mu_k^{T*} = \mu_k^{T*}\left(\frac{\overline{\overline{\rho_k}}\ell^2\sqrt{2\mathbb{D}_k : \mathbb{D}_k}}{\overline{\overline{\mu_k}}}, a_i\ell, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_k\right). \tag{9-170}$$

We recall here that ℓ and a_i are the distances from the wall and the surface area concentration, respectively. If we exclude the region very near to the wall, then the first parameter may be dropped from the arguments of μ_k^{T*}.

The constitutive law for average *conduction heat flux* $\overline{\overline{\boldsymbol{q}_k}}$ is given by Eq.(9-81), thus we have

$$\overline{\overline{q_k}} = -\overline{\overline{K_k}}\left\{\nabla\overline{\overline{T_k}} - \frac{\nabla\alpha_k}{\alpha_k}\left(\overline{\overline{T_i}} - \overline{\overline{T_k}}\right)\right\}. \tag{9-171}$$

It is interesting to note that the second term, due to the concentration gradient, represents the effect of the temperature difference between the bulk phase and the interfaces, namely, thermal non-equilibrium.

From the mixing length model for the *turbulent heat flux* $\boldsymbol{q}_k^T$, we have

$$\boldsymbol{q}_k^T = -K_k^{T*}\overline{\overline{\rho_k}}c_{pk}\ell^2\sqrt{2\mathbb{D}_{kb}:\mathbb{D}_{kb}}\left\{\nabla\overline{\overline{T_k}} - \frac{\nabla\alpha_k}{\alpha_k}\left(\overline{\overline{T_i}} - \overline{\overline{T_k}}\right)\right\} \tag{9-172}$$

where the non-dimensional conductivity K_k^{T*} is a function of the following parameters

$$K_k^{T*} = K_k^{T*}\left(\frac{\overline{\overline{\rho_k}}c_{pk}\ell^2\sqrt{2\mathbb{D}_{kb}:\mathbb{D}_{kb}}}{\overline{\overline{K_k}}}, a_i\ell, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_k\right). \tag{9-173}$$

We note here that the first dimensionless group may be dropped from the argument of the function K_k^{T*} if we exclude the region very near to the wall.

The mass transfer term Γ_1 is given by Eqs.(9-111) and (9-112), thus we have

$$\begin{aligned}\Gamma_1 = &\frac{\left(\overline{\overline{K_1}} + K_1^T\right)a_i^2}{\left|\hat{i}_{1i} - \hat{i}_{2i}\right|}b_1^{\Gamma*}\left(\overline{\overline{T_i}} - \overline{\overline{T_1}}\right)\\ &-\frac{\left(\overline{\overline{K_2}} + K_2^T\right)a_i^2}{\left|\hat{i}_{1i} - \hat{i}_{2i}\right|}b_2^{\Gamma*}\left(\overline{\overline{T_i}} - \overline{\overline{T_2}}\right)\end{aligned} \tag{9-174}$$

in which the coefficient $b_k^{\Gamma*}$ depends on four parameters

$$b_k^{\Gamma*} = b_k^{\Gamma*}\left(\frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}, N_{Jk}, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_k\right). \tag{9-175}$$

The group denoted by N_{Jk} is the Jakob number defined by Eqs.(9-109) and (9-110) and it is the most important parameter on the phase changes. A

simpler case of the above constitutive law has been discussed in the previous section.

In view of the interfacial momentum transfer condition, Eq.(9-156), we should supply two constitutive equations that specify the drag force $\boldsymbol{M}_{il}$ and the effects of surface tension. The *interfacial drag force* has been given by Eqs.(9-117) and (9-119), thus we have

$$\boldsymbol{M}_{il} = \left(\overline{\overline{\rho_1}} + \overline{\overline{\rho_2}}\right) a_i \left|\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right| \left(\widehat{\boldsymbol{v}_2} - \widehat{\boldsymbol{v}_1}\right) b_1^{M*} \tag{9-176}$$

where the coefficient b_1^{M*} is expected to be a function of the following parameters

$$b_1^{M*} = b_1^{M*}\left(\frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_1, N^i_{\mathrm{Re}1}, N^i_{\mathrm{Re}2}, N^i_{pch}\right). \tag{9-177}$$

Here, the interfacial Reynolds number $N^i_{\mathrm{Re}k}$ and the phase change effect number N^i_{pch} are defined by Eqs.(9-120) and (9-121), respectively.

Furthermore, the geometrical equation of state Eq.(9-137), specifies the mean curvature of the interfaces

$$\frac{\overline{\overline{H_{21}}}}{a_i} = \overline{\overline{H_{21}}}^{*}\left(\alpha_2, \left|\nabla\alpha_2\right|, a_i\right) \tag{9-178}$$

The interfacial thermal energy transfer condition given by Eq.(9-158) requires the *constitutive equation for the heat transfer at the interfaces.* Thus, from Eqs.(9-123) and (9-124) we have

$$a_i \overline{\overline{q''_{1i}}} = \left(\overline{\overline{K_1}} + K_1^T\right) a_i^2 b_1^{E*} \left(\overline{\overline{T_i}} - \overline{\overline{T_1}}\right) \tag{9-179}$$

where

$$b_1^{E*} = b_1^{E*}\left(N^T_{\mathrm{Pr}1}, N^i_{\mathrm{Re}1}, N^i_{pch}, \frac{\overline{\overline{H_{21}}}}{a_i}, \alpha_1, \frac{\overline{\overline{\rho_1}}}{\overline{\overline{\rho_2}}}\right). \tag{9-180}$$

The definition of the Prandtl number is given by Eq.(9-125). In many practical problems, the dispersed phase can be assumed to be in thermal equilibrium, then it follows that the constitutive equation (9-179) reduces to

a trivial form $a_i \overline{\overline{q''_{di}}} = 0$.

Finally, we should have a constitutive equation for the surface area concentration a_i. In general it should have the form of the balance equation

$$\frac{\partial a_i}{\partial t} + \nabla \cdot \left(a_i \widehat{\boldsymbol{v}_i} \right) = \phi_L \tag{9-181}$$

where the source term ϕ_L is expressed by various parameters that have already appeared. It is expected that, in general, the constitutive equation for ϕ_L is quite difficult to obtain unless the flow geometry is very simple, namely, such as the bubbly or droplet flows. The average interface velocity $\widehat{\boldsymbol{v}_i}$ can be given approximately by

$$\widehat{\boldsymbol{v}_i} \doteq \widehat{\boldsymbol{v}_d} - \frac{\Gamma_d}{\overline{\overline{\rho_d}} a_i^2} \nabla \alpha_d \doteq \widehat{\boldsymbol{v}_d} \tag{9-182}$$

in which the subscript *d* denotes the dispersed phase.

The constitutive equation for $\boldsymbol{M}_m^H$ for a dispersed flow can be given by Eq. (9-140), thus we have

$$\boldsymbol{M}_m^H \doteq \alpha_d \nabla \left(2 \overline{\overline{H_{dc}}}\, \overline{\overline{\sigma}} \right). \tag{9-183}$$

The basic variables appearing in the two-fluid formulation are

$$\alpha_k, \overline{\overline{\rho_k}}, \widehat{\boldsymbol{v}_k}, \Gamma_k$$

$$\overline{\overline{p_k}}, \overline{\overline{\boldsymbol{\mathcal{T}}_k}}, \boldsymbol{\mathcal{T}}_k^T, \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}, \boldsymbol{M}_{ik}, \boldsymbol{M}_m^H, \overline{\overline{p_{ki}}}$$

$$\widehat{i_k}, \overline{\overline{\boldsymbol{q}_k}}, \boldsymbol{q}_k^T, a_i \overline{\overline{q''_k}}, \overline{\overline{T_k}},$$

$$\overline{\overline{T_i}}, \overline{\overline{H_{21}}}, \overline{\overline{\sigma}}, a_i, p^{sat}.$$

Thus, the total number of unknown is thirty six, and we have:

- Six balance equations — Eqs.(9-152), (9-153) and (9-154);
- Three interfacial conditions — Eqs.(9-155), (9-156) and (9-158);
- Mechanical condition at interfaces — Eq.(9-157);
- Chemical condition at interfaces (Phase change) — Eq.(9-163);
- Saturation condition — Eq.(9-164);

- Axiom of continuity Eq.(9-159);
- Two caloric equations of state Eq.(9-160);
- Two thermal equations of state Eq.(9-161);
- Surface equation of state Eq.(9-162);
- Two constitutive equations for $\overline{\overline{\mathcal{T}_k}}$ Eq.(9-166);
- Two constitutive equations for $\mathcal{T}_k^T$ Eq.(9-169);
- Two constitutive equations for $\overline{\overline{\mathcal{T}_{ki}}}$ Eq.(9-126);
- Two constitutive equations for $\overline{\overline{q_k}}$ Eq.(9-171);
- Two constitutive equations for q_k^T Eq.(9-172);
- Phase change constitutive law Eq.(9-174);
  (or constitutive equation for $a_i \overline{\overline{q_{2i}''}}$ similar to Eq. (9-179))
- Drag constitutive law Eq.(9-176);
- Geometrical equation of state Eq.(9-178);
- Constitutive equation for $a_i \overline{\overline{q_{1i}''}}$ Eq.(9-179);
- Balance equation for surface area Eq.(9-181);
- Constitutive equation for M_m^H Eq.(9-183);
- Constitutive equation for $\overline{\overline{p_{ki}}}$ (for most cases) Eq.(9-103).

This shows that we have thirty six equations. Hence the total number of unknown and of equations is the same. Consequently, our description is consistent and complete in mathematical sense, although this does not guarantee the uniqueness of the solution of the model nor even the existence of the solution. These should be checked by solving various simple cases, then as we have discussed in Chapter 2, the results should be compared to the experimental data to verify and improve the model. It is very important to note that Γ_1, $\overline{\overline{q_{1i}''}}$ and $\overline{\overline{q_{2i}''}}$ are related by the interfacial thermal energy transfer condition that represents the macroscopic energy jump condition. Therefore, the constitutive relation for $\overline{\overline{q_{2i}''}}$ can be used in place of the phase change constitutive law, Eq. (9-174). Since the heat flux can be modeled more easily than the phase change, this is a much more practical approach.

1.4 Various special cases

Scaling Parameter

The general formulation of the two-fluid model has been given above. In the following analysis, we obtain some important *scaling parameters from the field equations.* Before going into the detailed study, we recall that dimensionless groups can be obtained from the conservation equations, boundary conditions and constitutive laws. The similarity of two different systems can only be discussed by including all these groups. This will be

very difficult to accomplish in the model based on the two-fluid formulation due to the large number of unknowns involved and of the complexity of the constitutive equations. For such systems, the dimensional analysis is more important for obtaining scaling parameters of various effects in the field equations than for making the similarity analysis of the entire system. The order of magnitude analysis based on these scaling parameters frequently leads to a much-simplified formulation that can be solved to yield meaningful answers to various engineering problems. It should be noted, however, that under certain conditions smaller terms cannot be neglected from the formulation. Thus, for a system of coupled nonlinear differential equations, the order of magnitude analysis should be accepted as a general trend with exceptions. Consequently, since many complicated problems can be solved only approximately, it becomes necessary to check the results with experimental data.

In the following analysis, the subscript o denotes the reference parameters chosen to be constant. The characteristic length is L_o, whereas the time constant is τ_o. For most problems it is taken as the ratio of L_o to the velocity scale, however, for oscillating flows it can be the period of oscillations. Below, we define dimensionless parameters whose order of magnitude is considered to be 1.

$$\rho_k^* = \frac{\overline{\overline{\rho_k}}}{\rho_{ko}}, \boldsymbol{v}_k^* = \frac{\widehat{\boldsymbol{v}_k}}{\boldsymbol{v}_{ko}}, t^* = \frac{t}{\tau_o}, \nabla^* = L_o \nabla,$$

$$\Gamma_k^* = \frac{\Gamma_k}{|\Gamma_{ko}|}, p_k^* = \frac{\overline{\overline{p_k}} - p_0}{\Delta p_o}, \mu_k^* = \frac{\overline{\overline{\mu_k}} + \mu_k^T}{\mu_{ko}},$$

$$\boldsymbol{M}_{ik}^* = \frac{\boldsymbol{M}_{ik}}{a_{io}(\rho_{1o} + \rho_{2o})(v_{2o} - v_{1o})^2}, i_k^* = \frac{\widehat{i_k} - i_{ko}}{\Delta i_{ko}},$$

$$K_k^* = \frac{\overline{\overline{K_k}} + K_k^T}{K_{ko}}, q_{ki}''^* = \frac{\overline{\overline{q_{ki}''}}}{a_{io} K_{ko}(T_{io} - T_{ko})},$$

$$H_{21}^* = \frac{\overline{\overline{H_{21}}}}{H_{21o}}, \sigma^* = \frac{\overline{\overline{\sigma}}}{\sigma_o}, \left(\overline{\overline{\mathbb{T}_k}} + \mathbb{T}_k^T\right)^* = \frac{\left(\overline{\overline{\mathbb{T}_k}} + \mathbb{T}_k^T\right)}{\mu_{ko} v_{ko} / L_o},$$

$$\left(\overline{\overline{\mathbb{T}_{ki}}}\right)^* = \frac{\overline{\overline{\mathbb{T}_{ki}}}}{\mu_{ko} v_{ko} / L_o}, \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T\right)^* = \frac{\left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T\right)}{K_{ko} \Delta T_{ko} / L_o},$$

$$T_k^* = \frac{\overline{\overline{T_k}} - T_o}{\Delta T_{ko}} \approx \frac{\widehat{i_k} - i_{ko}}{\Delta i_{ko}}, a_i^* = \frac{a_i}{a_{io}}, \qquad (9\text{-}184)$$

$$M_m^{H*} = \frac{M_m^H}{a_{io}\left(\rho_{1o} + \rho_{2o}\right)\left(v_{2o} - v_{1o}\right)^2}$$

Substituting these new parameters into the field equations we obtain the following results.

Non-dimensional continuity equations from Eq.(9-152)

$$\frac{1}{\left(N_{sl}\right)_k}\frac{\partial \alpha_k \rho_k^*}{\partial t^*} + \nabla^* \cdot \left(\alpha_k \rho_k^* v_k^*\right) = \left(N_{pch}\right)_k \Gamma_k^* \tag{9-185}$$

Non-dimensional equations of motion from Eq.(9-153)

$$\begin{aligned}
&\alpha_k \rho_k^* \left\{\frac{1}{\left(N_{sl}\right)_k}\frac{\partial \boldsymbol{v}_k^*}{\partial t^*} + \boldsymbol{v}_k^* \cdot \nabla^* \boldsymbol{v}_k^*\right\} = -\left(N_{Eu}\right)_k \alpha_k \nabla^* p_k^* \\
&+ \frac{1}{\left(N_{\mathrm{Re}}\right)_k}\nabla^* \cdot \left[\alpha_k \left(\overline{\overline{\boldsymbol{\tau}}}_k + \boldsymbol{\tau}_k^T\right)^*\right] + \frac{1}{\left(N_{Fr}\right)_k}\alpha_k \rho_k^* \frac{\widehat{\boldsymbol{g}_k}}{|\boldsymbol{g}|} \\
&+ \left(N_{drag}\right)_k \boldsymbol{M}_{ik}^* - \frac{1}{\left(N_{\mathrm{Re}}\right)_k}\nabla^* \alpha_k \cdot \left(\overline{\overline{\boldsymbol{\tau}}}_{ki}\right)^* \\
&+ \left(N_{pch}\right)_k \Gamma_k^* \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)^* + \left(N_{Eu}\right)_k \left(p_{ki}^* - p_k^*\right)\nabla^* \alpha_k
\end{aligned} \tag{9-186}$$

Non-dimensional thermal energy equations from Eq.(9-154)

$$\begin{aligned}
&\alpha_k \rho_k^* \left\{\frac{1}{\left(N_{sl}\right)_k}\frac{\partial i_k^*}{\partial t^*} + \boldsymbol{v}_k^* \cdot \nabla^* i_k^*\right\} \\
&= -\frac{1}{\left(N_{Pe}\right)_k}\nabla^* \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}}}_k + \boldsymbol{q}_k^T\right)^* \\
&+ \left(N_{Eu}\right)_k \left(N_{Ec}\right)_k \alpha_k \left\{\frac{1}{\left(N_{sl}\right)_k}\frac{\partial p_k^*}{\partial t^*} + \boldsymbol{v}_k^* \cdot \nabla^* p_k^*\right\} \\
&+ \frac{\left(N_{Ec}\right)_k}{\left(N_{\mathrm{Re}}\right)_k}\alpha_k \left(\overline{\overline{\boldsymbol{\tau}}}_k\right)^* : \nabla^* \boldsymbol{v}_k^* + \left(N_{pch}\right)_k \Gamma_k^* \left(i_{ki}^* - i_k^*\right)
\end{aligned} \tag{9-187}$$

$$+\left(N_q\right)_k a_i^* q_{ki}''^* + \left(N_{drag}\right)_k \left(N_{Ec}\right)_k \boldsymbol{M}_{ik}^* \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)^*$$
$$-\frac{\left(N_{Ec}\right)_k}{\left(N_{\mathrm{Re}}\right)_k}\left\{\nabla^* \alpha_k \cdot \left(\overline{\overline{\boldsymbol{\tau}_{ki}}}\right)^*\right\} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)^*$$

Here we have introduced several scaling parameters defined as

$$\text{Strouhal number } \left(N_{sl}\right)_k \equiv \frac{\tau_o v_{ko}}{L_o}$$

$$\text{Phase change number } \left(N_{pch}\right)_k \equiv \frac{|\Gamma_{ko}| L_o}{\rho_{ko} v_{ko}}$$

$$\text{Euler number } \left(N_{Eu}\right)_k \equiv \frac{\Delta p_o}{\rho_{ko} v_{ko}^2}$$

$$\text{Reynolds number } \left(N_{\mathrm{Re}}\right)_k \equiv \frac{\rho_{ko} v_{ko} L_o}{\mu_{ko}}$$

$$\text{Froude number } \left(N_{Fr}\right)_k \equiv \frac{v_{ko}^2}{|g| L_o}$$

$$\text{Drag number } \left(N_{drag}\right)_k \equiv \frac{\left(\rho_{1o} + \rho_{2o}\right) L_o a_{io} \left(v_{2o} - v_{1o}\right)^2}{\rho_{ko} v_{ko}^2}$$

$$\text{Peclet number } \left(N_{Pe}\right)_k \equiv \frac{\rho_{ko} v_{ko} \Delta i_{ko} L_o}{K_{ko} \Delta T_{ko}}$$

$$\text{Eckert number } \left(N_{Ec}\right)_k \equiv \frac{v_{ko}^2}{\Delta i_{ko}}$$

Interface heating number (9-188)

$$\left(N_q\right)_k \equiv \frac{K_{ko}\left(T_{io} - T_{ko}\right) L_o a_{io}^2}{\rho_{ko} v_{ko} \Delta i_{ko}}$$

The first two parameters, the Strouhal and phase change numbers, are the kinematic groups. The Euler, Reynolds, Froude and drag numbers are the dynamic groups, since they are the ratios of various forces appearing in the momentum equations. Similarly, the Peclet, Eckert and interface heating numbers are the energy groups that scale various energy transfer mechanisms. From the definitions (9-188) and the forms of the non-

dimensional field equations, the physical meanings of various scaling parameters are evident. In two-fluid model formulation, the phase change, drag and interfacial heating numbers are particularly important since they are the parameters to scale the effects of the interactions between two phases.

Before we discuss various special cases that can be obtained by considering limiting conditions in terms of the scaling parameters, let us study the non-dimensional form of the conditions of interfacial transfer, mechanical state between phases and phase change (or chemical state). Non-dimensional interfacial mass transfer condition can be obtained from Eqs.(9-155) and (9-184), thus

$$\sum_{k=1}^{2} \Gamma_k^* = 0. \tag{9-189}$$

From Eqs.(9-156), (9-157) and (9-184) the interfacial momentum transfer condition is given by

$$\sum_{k=1}^{2} \boldsymbol{M}_{ik}^* = \boldsymbol{M}_m^{H^*} \tag{9-190}$$

and the condition of the mechanical state by

$$p_1^* - p_2^* = -2N_\sigma H_{21}^* \sigma^*. \tag{9-191}$$

Similarly, the phase change condition (9-163) becomes

$$p_2^* - p^{sat^*} = 2N_\sigma \left(\frac{\rho_2^*}{\rho_2^* - \rho_1^*/N_\rho} \right) H_{21}^* \sigma^*. \tag{9-192}$$

Furthermore, the energy transfer condition (9-158) becomes

$$\Gamma_1^* \left[\left\{ (N_i)_1 i_{1i}^* - (N_i)_2 i_{2i}^* \right\} - 1 \right] + \sum_{k=1}^{2} \frac{(N_q)_k (N_i)_k}{(N_{pch})_k} a_i^* q_k''^* = 0. \tag{9-193}$$

In these equations, we have introduced the following scaling parameters

$$\text{Surface tension number } N_\sigma \equiv \frac{H_{21_o} \sigma_o}{\Delta p_o} \tag{9-194}$$

$$\text{Density ratio } N_\rho \equiv \frac{\rho_{2o}}{\rho_{1o}} \tag{9-195}$$

$$\text{Converted enthalpy ratio } (N_i)_k \equiv \frac{\Delta i_{ko}}{i_{2o} - i_{1o}}. \tag{9-196}$$

By combining the surface tension and Euler numbers we obtain the Weber number as

$$(N_{We})_k \equiv \frac{\rho_{ko} v_{ko}^2}{H_{21o}\sigma_o} = \frac{1}{N_\sigma (N_{Eu})_k}. \tag{9-197}$$

This shows that we obtain two Weber numbers in the two-fluid formulation, thus using the surface tension number is more convenient.

The converted enthalpy ratio scales the phase enthalpy change to the latent heat. This number is normally small, if the pressure is well below the critical pressure. However, the most important simplifications can be obtained by studying Eq.(9-192). If the surface tension number *or* the density ratio is very small we have

$$\overline{\overline{p_2}} \approx p^{sat}\left(\overline{\overline{T_i}}\right) \quad \text{for} \begin{cases} N_\sigma << 1 \\ \text{or} \\ N_\rho << 1. \end{cases} \tag{9-198}$$

Then from Eq.(9-157) we get

$$\overline{\overline{p_1}} \approx -2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}} + p^{sat}\left(\overline{\overline{T_i}}\right). \tag{9-199}$$

The simplest case happens, if the surface tension number is small, then

$$\overline{\overline{p_1}} \doteq \overline{\overline{p_2}} \doteq p^{sat}\left(\overline{\overline{T_i}}\right) \quad \text{for } N_\sigma << 1 \tag{9-200}$$

which indicates that the two phases are in mechanical equilibrium.

Now we study some of the important special cases.

Flow without Phase Changes

If the flow is without phase changes, we can set

$$\left(N_{pch}\right)_k = 0 \tag{9-201}$$

Then all the terms weighted by the phase change number in the field equations drop out from the formulation. In this case it is usually more convenient to transform the thermal energy equation in terms of c_{pk} and $\overline{\overline{T_k}}$. Thus, from the caloric equation of state (9-58), we have

$$d\widehat{i_k} = c_{pk}\, d\overline{\overline{T_k}} + \frac{1}{\overline{\overline{\rho_k}}}\left(1 + \frac{\overline{\overline{T_k}}}{\overline{\overline{\rho_k}}}\frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{T_k}}}\right)_{\overline{\overline{p_k}}} d\overline{\overline{p_k}}$$

or (9-202)

$$d\widehat{i_k} = c_{pk}\, d\overline{\overline{T_k}} + \frac{1}{\overline{\overline{\rho_k}}}\left(1 + \overline{\overline{T_k}}\beta_k\right) d\overline{\overline{p_k}}.$$

Consequently, we have the following set of field equations from Eqs.(9-152), (9-153) and (9-154)

$$\frac{\partial}{\partial t}\alpha_k \overline{\overline{\rho_k}} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}}\, \widehat{\boldsymbol{v}_k}\right) = 0 \tag{9-203}$$

$$\alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{\boldsymbol{v}_k}}{Dt} = -\alpha_k \nabla \overline{\overline{p_k}} + \nabla \cdot \left\{\alpha_k \left(\overline{\overline{\boldsymbol{\mathcal{T}}_k}} + \boldsymbol{\mathcal{T}}_k^T\right)\right\} + \alpha_k \overline{\overline{\rho_k}}\, \widehat{\boldsymbol{g}_k} + \boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}} \tag{9-204}$$

and

$$\alpha_k \overline{\overline{\rho_k}} c_{pk} \frac{D_k \overline{\overline{T_k}}}{Dt} = -\nabla \cdot \alpha_k \left(\overline{\overline{\boldsymbol{q}_k}} + \boldsymbol{q}_k^T\right) - \alpha_k \frac{\overline{\overline{T_k}}}{\overline{\overline{\rho_k}}}\frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{T_k}}}\Bigg)_{\overline{\overline{p_k}}} \frac{D_k \overline{\overline{p_k}}}{Dt} + \alpha_k \overline{\overline{\boldsymbol{\mathcal{T}}_k}} : \nabla \widehat{\boldsymbol{v}_k} + a_i \overline{\overline{q_{ki}''}} + \left(\boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\boldsymbol{\mathcal{T}}_{ki}}}\right) \cdot \left(\widehat{\boldsymbol{v}_i} - \widehat{\boldsymbol{v}_k}\right). \tag{9-205}$$

Here we have substituted Eq.(9-202) into Eq.(9-154).

Now we can define the Prandtl number as

$$\left(N_{\mathrm{Pr}}\right)_k \equiv \frac{c_{pko}\mu_{ko}}{K_{ko}}. \tag{9-206}$$

And the Peclet number should be modified to

$$\left(N_{Pe}\right)_k \equiv \frac{\rho_{ko}v_{ko}c_{pko}L_o}{K_{ko}} = \left(N_{\mathrm{Re}}\right)_k\left(N_{\mathrm{Pr}}\right)_k. \tag{9-207}$$

Furthermore, we note that the second term on the right-hand side of Eq.(9-205) reduces to a simple form for an ideal gas or for an incompressible fluid

$$-\alpha_k \frac{\overline{\overline{T_k}}}{\overline{\overline{\rho_k}}} \frac{\partial \overline{\overline{\rho_k}}}{\partial \overline{\overline{T_k}}}\bigg)_{\overline{\overline{p_k}}} \frac{D_k \overline{\overline{p_k}}}{Dt} = \begin{cases} \alpha_g \dfrac{D_g \overline{\overline{p_g}}}{Dt} & \text{(ideal gas)} \\ 0 & \text{(incompressible).} \end{cases} \tag{9-208}$$

A simple form of the energy equations of a practical importance can be used if the Eckert numbers are very small, or the heat transfer dominates the energy exchanges. Then we have

$$\alpha_k \overline{\overline{\rho_k}} c_{pk} \frac{D_k \overline{\overline{T_k}}}{Dt} \doteq -\nabla \cdot \left\{\alpha_k \left(\overline{\overline{q_k}} + q_k^T\right)\right\} + a_i \overline{\overline{q_{ki}''}} \tag{9-209}$$

where the compressibility effect and the viscous dissipation term have been dropped from Eq.(9-205). In addition, if the two phases are incompressible with the temperature independent transport properties, the energy equations can be decoupled from the continuity and the momentum equations.

Isothermal Flow with No Phase Changes

Under the condition, the entire energy equations may be dropped from the formulation. And we have

$$\overline{\overline{\rho_k}} = \overline{\overline{\rho_k}}\left(\overline{\overline{p_k}}\right) \tag{9-210}$$

thus the flow is called *barotropic*. Furthermore, if the change of pressure or the isothermal compressibility is small, the flow can be considered as *incompressible*. Then we have

$$\overline{\overline{\rho_k}} = \text{constant}. \tag{9-211}$$

Under this condition, the pressure $\overline{\overline{p_k}}$ is independent of the density $\overline{\overline{\rho_k}}$ and it represents the hydrodynamic pressure. Moreover if the effects of the viscous stresses can be neglected, then we have

$$\begin{aligned} &\frac{\partial \alpha_k}{\partial t} + \nabla \cdot \alpha_k \widehat{\boldsymbol{v}_k} = 0 \\ &\frac{D_k \widehat{\boldsymbol{v}_k}}{Dt} = -\frac{1}{\overline{\overline{\rho_k}}} \nabla \overline{\overline{p_k}} + \boldsymbol{g} + \frac{\boldsymbol{M}_{ik}}{\alpha_k \overline{\overline{\rho_k}}}. \end{aligned} \tag{9-212}$$

In addition, if the system has a fixed interfacial geometry, the formulation reduces to a simple form due to the geometrical constitutive laws. Equations (9-178) and (9-181), as well as the drag law Eq.(9-176), can be obtained without much difficulties. Some of the results on the dispersed flow regime given below can be applied for this case.

Dispersed Flow with Fluid Particles

In the following analysis, we use subscript *c* and *d* for the continuous and dispersed phases, respectively. Thus we set:

phase 1 → phase *c*; continuous phase;
phase 2 → phase *d*; dispersed phase.

For simplicity, we assume that the dispersed phase has spherical geometry with fairly uniform diameters at any point and time. Then, from the *geometrical equation of state* (9-136) or Eq.(9-137), we have

$$\overline{\overline{R_d}} = \frac{3 C^i \alpha_d}{a_i} \tag{9-213}$$

where

$$\begin{cases} \overline{\overline{H_{dc}}} = \dfrac{1}{\overline{\overline{R_d}}} \\ C^i \doteq 1. \end{cases} \tag{9-214}$$

Thus $\overline{\overline{R_d}}$ can be considered as the mean radius of the fluid particles. The *volume balance equation* can be put into the following form

$$\frac{\partial}{\partial t}\left(\frac{36\pi\alpha_d^2}{a_i^3}\right)+\nabla\cdot\left(\frac{36\pi\alpha_d^2}{a_i^3}\widehat{\boldsymbol{v}}_d\right)=\left(V_d^+-V_d^-\right) \tag{9-215}$$

where

$$\frac{1}{n_d}=\frac{36\pi\alpha_d^2}{a_i^3} \tag{9-216}$$

is the free volume available per each fluid particle, where n_d is the particle number density. The right-hand side represents the volume source due to coalescences and the sink due to disintegrations of particles.

We demonstrate the derivation of Eq.(9-138) for a simple case without the source or sink terms. Thus, by considering a fluid particle of average properties, we can approximate

$$\frac{D_d}{Dt}\left(\frac{4}{3}\pi\overline{\overline{R}}_d^3\overline{\overline{\rho}}_d\right)\doteq\Gamma_d\frac{4\pi\overline{\overline{R}}_d^2}{a_i}. \tag{9-217}$$

By substituting Eq.(9-136) for $\overline{\overline{R}}_d$, then using the dispersed phase continuity equation (9-152) to eliminate Γ_d, we have

$$\frac{\partial}{\partial t}\left(\frac{36\pi\alpha_d^2}{a_i^3}\right)+\nabla\cdot\left(\frac{36\pi\alpha_d^2}{a_i^3}\widehat{\boldsymbol{v}}_d\right)=0. \tag{9-218}$$

We note here that if the particle diameters vary considerably then the coefficient C^i is not a constant. In this case, we should have an additional term due to the changes in C^i because the average surface area and volume of fluid particles are not exactly the same as those calculated from the mean diameter.

Now let us study the drag constitutive equation in the fluid particle systems. The well-known Stokes' Law was extended by Hadamard to creeping motion of a *spherical fluid particle* in an infinite Navier-Stokes fluid (Brodkey, 1967; Soo, 1967). Thus, the total force acting on a fluid particle is given by

$$F=6\pi\mu_c\left(v_{c\infty}-v_d\right)R_d\left\{\frac{2\mu_c+3\mu_d}{3\left(\mu_c+\mu_d\right)}\right\}. \tag{9-219}$$

Then we define the drag coefficient $C_{D\infty}$ by

$$C_{D\infty} = \frac{F}{\frac{1}{2}\rho_c \left(v_{c\infty} - v_d\right)^2 \pi R_d^2} \tag{9-220}$$

and the particle Reynolds number by

$$(Re)_d = \frac{\rho_c \left(v_{c\infty} - v_d\right) 2R_d}{\mu_c}. \tag{9-221}$$

Here $v_{c\infty}$ and v_d are the undisturbed flow velocity and the particle velocity. From the above, we have

$$C_{D\infty} = \frac{24}{(Re)_d}\left[\frac{2\mu_c + 3\mu_d}{3(\mu_c + \mu_d)}\right];\ (Re)_d < 1. \tag{9-222}$$

The drag law given by Hadamard is good up to a Reynolds number of about 1. For higher Reynolds numbers we have the results of Levich (1962) and Chao (1962) given by

$$C_{D\infty} = \frac{48}{(Re)_d};\ (Re)_d < 100 \tag{9-223}$$

$$C_{D\infty} = \frac{32}{(Re)_d}\left\{1 + 2\frac{\mu_d}{\mu_c} - 0.314\frac{\left(1 + 4\mu_d/\mu_c\right)}{\sqrt{(Re)_d}}\right\} \tag{9-224}$$

respectively. We also note the review work done by Clift et al. (1978) in these connections. At still higher Reynolds numbers, the value of $C_{D\infty} \approx 0.44$ given by Newton can be used for droplets. For bubbles, however, the interfacial deformations lead to ellipsoidal or spherical cap bubbles.

By combining these results, we can set that the drag coefficient as a function of the Reynolds number $(Re)_d$ and the ratio of the two viscosities, thus

$$C_{D\infty} = C_{D\infty}\left((Re)_d, \frac{\mu_c}{\mu_d}\right). \tag{9-225}$$

The above relation summarizes the ideal cases of a single fluid particle in infinite media.

In general cases, we have postulated that the interfacial drag force can be given by the constitutive law having the forms of Eqs.(9-117) and (9-119). For a dispersed flow restricted by Eqs.(9-213), (9-214) and (9-218), we may simplify the general drag constitutive law by introducing a drag coefficient C_D defined by

$$C_D = \frac{|\boldsymbol{F}|}{\frac{1}{2}\overline{\overline{\rho_c}}\left|\widehat{\boldsymbol{v}_c} - \widehat{\boldsymbol{v}_d}\right|^2 \pi\left(\overline{\overline{R_d}}\right)^2}. \tag{9-226}$$

In view of Eqs.(9-221) and (9-226), we redefine the appropriate particle Reynolds number by

$$N^i_{\mathrm{Re}c} \equiv \frac{\overline{\overline{\rho_c}}\left|\widehat{\boldsymbol{v}_c} - \widehat{\boldsymbol{v}_d}\right| 2\overline{\overline{R_d}}}{\overline{\overline{\mu_c}}}. \tag{9-227}$$

And the Reynolds number for the dispersed phase is redundant if we use the viscosity ratio as a non-dimensional group.

Thus, in view of Eqs.(9-119) and (9-227), we postulate that the drag law can be given by

$$C_D = C_{D\infty}\left(N^i_{\mathrm{Re}c}, \frac{\overline{\overline{\mu_c}}}{\overline{\overline{\mu_d}}}\right) f^*\left(\frac{\overline{\overline{\rho_c}}}{\overline{\overline{\rho_d}}}, N^i_{pch}, \alpha_c\right) \tag{9-228}$$

where f^* is the correction factor which takes into account for the effects of other particles and the changes of phase. It can be said that if N^i_{pch} is large then the linear correction of Eq.(9-228) cannot be applied because of the rapid expansions or collapses of fluid particles. A detailed discussion of the drag force in multi-particle systems is given in Chapter 12.