Chapter 8

INTERFACIAL TRANSPORT

The exact forms of the interfacial transport terms I_k and I_m for mass, momentum and energy interchanges have been given in the Section 1.2 of Chapter 5. However, they are expressed by the local instant variables, thus it is not possible to use them as the constitutive laws in the averaged field equations. It is evident that we need to understand the physical meaning of these terms in detail before constructing any particular constitutive equations for two-phase flow systems. With this in mind we clarify different physical mechanisms controlling these terms as well as to identify important parameters on which they depend. Furthermore, it is important to accept that not all the characteristics inherent to the local instant two-phase flow can be brought into the time-averaged model. We consider that the averaged field equations express general physical principles governing the macroscopic two-phase flows while the constitutive equations approximate the material responses of a particular group of systems with simple mathematical models. In this connection, we make a number of assumptions in the interfacial transfer terms in order to both distinguish the dominant transfer mechanisms and also eliminate some of the complicated terms that have insignificant effects in the macroscopic field.

1.1 Interfacial mass transfer

The interfacial mass transfer term Γ_k has been given by Eq.(5-19), thus in view of Eqs.(2-70) and (5-57) we have

$$\Gamma_k = -\sum_j \frac{1}{\Delta t}\frac{1}{v_{ni}}\left\{\boldsymbol{n}_k \cdot \rho_k\left(\boldsymbol{v}_k - \boldsymbol{v}_i\right)\right\} = -\sum_j a_{ij}\dot{m}_k \tag{8-1}$$

where $\dot{m}_k$ is the rate of mass loss per interfacial area in unit time from k^{th}-

phase, and $a_{ij}\left(\equiv 1/L_j\right)$ is the surface area concentration for the j^{th}-interface.

We define the surface mean value as

$$\overline{\overline{F}}_{(i)} \equiv \left(\sum_j \frac{F}{L_j}\right) L_S = \frac{\sum_j a_{ij} F}{a_i}. \tag{8-2}$$

Hereafter, we use the subscript (i) only for the variables that may be confused with the bulk fluid properties, and we omit it for the variables that appear only at the interface, for example σ and H_{21}, etc. The mean value defined by Eq.(8-2) corresponds to the phase average of Eq.(4-23), thus we use the same symbol with the subscript i.

Similarly it is also possible to define a surface mean value in analogy with the phase mass weighted mean value of Eq.(4-28). However, at the interfaces these variables of the extensive characteristic appear always in the flux terms, thus it is more convenient to define a mean value weighted by the mass transfer rate $\dot{m}_k$ rather than by the density. Hence we have

$$\widehat{\psi_{ki}} \equiv \frac{\sum_j \frac{1}{\Delta t}\frac{1}{v_{ni}}\dot{m}_k\psi_k}{\sum_j \frac{1}{\Delta t}\frac{1}{v_{ni}}\dot{m}_k} = \frac{\sum a_{ij}\dot{m}_k\psi_k}{\sum a_{ij}\dot{m}_k}. \tag{8-3}$$

By using the definition of Eq.(8-2), the mean mass transfer per unit surface area becomes

$$\overline{\overline{\dot{m}_k}} \equiv \frac{\sum_j a_{ij}\dot{m}_k}{a_i}. \tag{8-4}$$

From Eqs.(8-1) and (8-4), the interfacial mass transfer condition can be rewritten as

$$\sum_{k=1}^{2} \Gamma_k = 0 \quad \text{with} \quad \Gamma_k = -a_i\overline{\overline{\dot{m}_k}} \tag{8-5}$$

1.2 Interfacial momentum transfer

We recall that the macroscopic interfacial momentum transfer term $\boldsymbol{M}_k$ has been obtained in the Section 1.2 of Chapter 5. Thus, in view of Eqs.(2-9), (2-70), (5-23) and (5-57), we have

$$\boldsymbol{M}_k = -\sum_j a_{ij}\left(\dot{m}_k\boldsymbol{v}_k + p_k\boldsymbol{n}_k - \boldsymbol{n}_k\cdot\mathcal{T}_k\right) \tag{8-6}$$

where the term inside the bracket is the rate of the interfacial momentum loss per area from the k^{th}-phase. Since $\boldsymbol{M}_k$ represents the net interfacial momentum gain, it is weighted by the surface area concentration a_{ij}. Similarly, the mixture momentum source from the interfaces is given by

$$\boldsymbol{M}_m = \sum_j a_{ij}\left\{A^{\alpha\beta}\left(\boldsymbol{t}_\alpha\right)_{,\beta}\sigma + A^{\alpha\beta}\boldsymbol{t}_\alpha\left(\sigma\right)_{,\beta}\right\} \tag{8-7}$$

in which the two terms on the right-hand side of the above equation represent the effects of the mean curvature and of the surface-tension gradient.

Before we study the vectorial form of the interfacial momentum transfer equation, let us examine the normal component of the momentum jump condition, Eq.(2-91). This is because the original jump condition, Eq.(2-72), contains two distinct pieces of information; one in normal direction and the other in the tangential direction. We should pay special attention in order to preserve this characteristic in the interfacial transfer equation. By dotting the normal jump condition, Eq.(2-91), by a unit normal vector $\boldsymbol{n}_1$ and taking the time average we obtain

$$\begin{gathered}\sum_j a_{ij}\left\{\left(\frac{\dot{m}_1^2}{\rho_1}-\frac{\dot{m}_2^2}{\rho_2}\right)+\left(p_1-p_2\right)\right.\\ \left.-\left(\tau_{nn1}-\tau_{nn2}\right)+2H_{21}\sigma\right\}=0.\end{gathered} \tag{8-8}$$

Now we express this equation with the surface mean values defined by Eq.(8-2). In order to simplify the result, we assume

$$\dot{m}_k \approx \overline{\overline{\dot{m}_k}};\quad \sigma \approx \overline{\overline{\sigma}};\quad \rho_k \approx \overline{\overline{\rho_{ki}}}\quad \text{at } t\in\left[\Delta t\right]_S. \tag{8-9}$$

Thus the mass transfer rate $\dot{m}_k$, the surface tension σ and the density at the interfaces remain approximately constant during the time interval of averaging Δt. Then by neglecting the normal stress terms, we obtain

$$\frac{\Gamma_1^2}{a_i^2}\left(\frac{1}{\overline{\overline{\rho_{1i}}}}-\frac{1}{\overline{\overline{\rho_{2i}}}}\right)+\left(\overline{\overline{p_{1i}}}-\overline{\overline{p_{2i}}}\right)+2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\doteq 0. \tag{8-10}$$

We note here that under similar assumptions we should be able to recover Eq.(8-10) from the vectorial interfacial transfer equation. By using the surface mean values, the k^{th}-phase interfacial momentum gain $\boldsymbol{M}_k$ becomes

$$\boldsymbol{M}_k=\boldsymbol{M}_k^{\Gamma}+\boldsymbol{M}_k^{n}+\overline{\overline{p_{ki}}}\nabla\alpha_k+\boldsymbol{M}_k^{t}-\nabla\alpha_k\cdot\overline{\overline{\mathcal{T}_{ki}}} \tag{8-11}$$

where

$$\begin{aligned}
\boldsymbol{M}_k^{\Gamma}&=\Gamma_k\widehat{\boldsymbol{v}_{ki}}\\
\boldsymbol{M}_k^{n}&\doteq\sum_j a_{ij}\left(\overline{\overline{p_{ki}}}-p_k\right)\boldsymbol{n}_k\\
\boldsymbol{M}_k^{t}&\doteq\sum_j a_{ij}\boldsymbol{n}_k\cdot\left(\mathcal{T}_k-\overline{\overline{\mathcal{T}_{ki}}}\right).
\end{aligned} \tag{8-12}$$

It is noted here that the shear at the interface can be decomposed into the normal and tangential components, thus $\boldsymbol{n}_k\cdot\mathcal{T}_k=\boldsymbol{\tau}_{nk}+\boldsymbol{\tau}_{tk}$. However, the normal stress is negligibly small, therefore it can be assumed that $\boldsymbol{n}_k\cdot\mathcal{T}_k=\boldsymbol{\tau}_{tk}$. The first three terms on the right-hand side of Eq.(8-11) are originally the normal components, whereas the last two terms are essentially tangential components. Here the concentration gradient appears because of Eq.(4-62).

The mixture momentum source $\boldsymbol{M}_m$ becomes

$$\begin{aligned}
\boldsymbol{M}_m&=2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\nabla\alpha_2+\sum_j 2a_{ij}\left(H_{21}-\overline{\overline{H_{21}}}\right)\overline{\overline{\sigma}}\boldsymbol{n}_1\\
&+\sum_j a_{ij}A^{\alpha\beta}\boldsymbol{t}_\alpha(\sigma)_{,\beta}.
\end{aligned} \tag{8-13}$$

The second term takes into account the effect of the changes of the mean curvature. However, the gradient of σ in the microscopic scale is considered to be small and its vectorial direction is quite random, thus the last term may be neglected. Hence we approximate

$$\boldsymbol{M}_m = 2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\nabla\alpha_2 + \boldsymbol{M}_m^H \tag{8-14}$$

where $\boldsymbol{M}_m^H$ is the effect of the change of the mean curvature on the mixture momentum source.

It is easy to show in view of Eq.(8-11) and Eq.(8-14) that the normal component of the interfacial momentum transfer condition, Eq.(5-27), can be given by

$$\sum_{k=1}^{2}\left\{\left(\frac{\Gamma_k^2}{\overline{\overline{\rho_{ki}}}a_i^2} + \overline{\overline{p_{ki}}}\right)\nabla\alpha_k + \boldsymbol{M}_k^n\right\} - 2\overline{\overline{H_{21}}}\,\overline{\overline{\sigma}}\nabla\alpha_2 - \boldsymbol{M}_m^H = 0. \tag{8-15}$$

Hence, by comparing the scalar form of the normal component Eq.(8-10) to the vectorial form Eq.(8-15), we obtain

$$\sum_{k=1}^{2}\boldsymbol{M}_k^n = \boldsymbol{M}_m^H. \tag{8-16}$$

Here $\boldsymbol{M}_k^n$ represents the form drag and lift force arising from the pressure imbalance at the interface. $\boldsymbol{M}_k^t$ represents the skin drag due to the imbalance of shear forces. The shear components, thus, should satisfy

$$\sum_{k=1}^{2}\boldsymbol{M}_k^t = 0 \quad \text{with} \quad \overline{\overline{\mathcal{T}_{1i}}} \doteq \overline{\overline{\mathcal{T}_{2i}}} = \overline{\overline{\mathcal{T}_i}}. \tag{8-17}$$

Equation (8-17) shows that there exists an action-reaction relation between the skin drag forces of each phase as well as between the interfacial shear forces.

For simplicity, we combine these two drag forces and define the total generalized drag forces $\boldsymbol{M}_{ik}$ by

$$\boldsymbol{M}_{ik} = \boldsymbol{M}_k^n + \boldsymbol{M}_k^t \tag{8-18}$$

where

$$\boldsymbol{M}_k^n = \sum_j a_{ij}\left(\overline{\overline{p_{ki}}} - p_k\right)\boldsymbol{n}_k$$

$$\boldsymbol{M}_k^t = \sum_j a_{ij}\boldsymbol{n}_k \cdot \left(\mathcal{T}_k - \overline{\overline{\mathcal{T}_{ki}}}\right).$$

Hence,

$$\sum_{k=1}^{2} \boldsymbol{M}_{ik} = \boldsymbol{M}_m^H. \tag{8-19}$$

Furthermore, from the straightforward analysis on the mass transfer rate $\overline{\overline{\dot{m}_k}}$ with the relations given by Eqs.(4-61) and (4-62), we can show

$$\widehat{\boldsymbol{v}_{ki}} = \widehat{\boldsymbol{v}_i} + \frac{\Gamma_k}{\overline{\overline{\rho_{ki}}} a_i^2} \nabla \alpha_k \tag{8-20}$$

which enables us to replace $\widehat{\boldsymbol{v}_{1i}}$ and $\widehat{\boldsymbol{v}_{2i}}$ by a single parameter $\widehat{\boldsymbol{v}_i}$.

As a summary of the interfacial momentum transfer condition, we have the following relations

$$\boldsymbol{M}_k = \boldsymbol{M}_k^{\Gamma} + \overline{\overline{p_{ki}}} \nabla \alpha_k + \boldsymbol{M}_{ik} - \nabla \alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \tag{8-21}$$

where $\boldsymbol{M}_{ik}$ includes the effects of form drag, lift force and skin drag.

$$\boldsymbol{M}_m = 2\overline{\overline{H_{21}}}\, \overline{\overline{\sigma}} \nabla \alpha_2 + \boldsymbol{M}_m^H \tag{8-22}$$

$$\sum_{k=1}^{2} \boldsymbol{M}_k = \boldsymbol{M}_m \tag{8-23}$$

with

$$\boldsymbol{M}_k^{\Gamma} = \widehat{\boldsymbol{v}_{ki}} \Gamma_k = \left(\widehat{\boldsymbol{v}_i} + \frac{\Gamma_k}{\overline{\overline{\rho_{ki}}} a_i^2} \nabla \alpha_k \right) \Gamma_k \tag{8-24}$$

$$\sum_{k=1}^{2} \boldsymbol{M}_{ik} = \boldsymbol{M}_m^H. \tag{8-25}$$

If we assume that $\overline{\overline{p_{ki}}}$, $\overline{\overline{\sigma}}$, $\boldsymbol{M}_m^H$, a_i, Γ_k, and $\overline{\overline{\rho_{ki}}}$ are known, then three constitutive laws should be specified for $\overline{\overline{H_{21}}}$, $\widehat{\boldsymbol{v}_i}$ and $\boldsymbol{M}_{i1}$ identifying the interfacial geometry, motion and generalized drag forces. Furthermore, we note that the total generalized drag force consists of the form drag, the skin drag as well as the lift force.

1.3 Interfacial energy transfer

The macroscopic interfacial total energy transfer for the k^{th}-phase is denoted by E_k which appears only after the phase energy equation has been averaged, whereas the mixture interfacial energy source term is E_m. These three terms should satisfy the interfacial energy transfer condition that is a balance equation at the interfaces. Since the relations for E_k and E_m given by Eqs.(5-28) and (5-29) are expressed by the local instant variables, they cannot be used in the macroscopic formulation in their original forms. Now we transform these relations in terms of the macroscopic variables as a first step to establish the constitutive laws at the interfaces.

Because of its practical importance, we start from the analysis on the interfacial thermal energy transfer term Λ_k, then we proceed to the study of E_k. From the definition of Eq.(5-39) and Eqs.(5-19), (5-23) and (5-28), we have

$$\begin{aligned} \Lambda_k &= \frac{\widehat{v_k}^2}{2}\Gamma_k - \boldsymbol{M}_k \cdot \widehat{\boldsymbol{v}_k} + E_k \\ &= \sum_j \frac{1}{\Delta t}\frac{1}{v_{ni}}\left\{-\dot{m}_k\left(u_k + \frac{v_k^2}{2} - \boldsymbol{v}_k \cdot \widehat{\boldsymbol{v}_k} + \frac{\widehat{v_k}^2}{2}\right)\right. \\ &\left. + \boldsymbol{n}_k \cdot \mathcal{T}_k \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) - \boldsymbol{n}_k \cdot \boldsymbol{q}_k\right\}. \end{aligned} \tag{8-26}$$

We define the virtual internal energy at the interfaces in analogy with Eq.(5-31), thus

$$\widehat{e_{ki}} \equiv \frac{\sum_j \left\{a_{ij}\dot{m}_k\left(u_k + \dfrac{\left|\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right|^2}{2}\right)\right\}}{\sum_j a_{ij}\dot{m}_k}. \tag{8-27}$$

And the heat input per unit interfacial area is defined by

$$\overline{\overline{q''_{ki}}} = -\left(\sum_j a_{ij}\boldsymbol{n}_k \cdot \boldsymbol{q}_k\right)\frac{1}{a_i}. \tag{8-28}$$

Then Eq.(8-26) can be rewritten as

$$\Lambda_k = \left(\Gamma_k \widehat{e_{ki}} + a_i \overline{\overline{q''_{ki}}}\right) + \sum_j \left\{a_{ij} \boldsymbol{n}_k \cdot \mathcal{T}_k \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right)\right\}. \tag{8-29}$$

In order to examine the second group on the right-hand side of the above equation, we introduce fluctuating components defined by

$$p'_{ki} = p_k - \overline{\overline{p_{ki}}};\ \boldsymbol{v}'_{ki} = \boldsymbol{v}_k - \widehat{\boldsymbol{v}_{ki}}. \tag{8-30}$$

Then we have

$$\begin{aligned} &\sum_j a_{ij} \boldsymbol{n}_k \cdot \mathcal{T}_k \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) \\ &= \sum_j a_{ij} \left\{-\overline{\overline{p_{ki}}} \boldsymbol{n}_k \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right)\right\} + \sum_j a_{ij} \left\{\boldsymbol{n}_k \cdot \overline{\overline{\mathcal{T}_{ki}}}\right. \\ &\left. + \boldsymbol{n}_k \cdot \left(\mathcal{T}_k - \overline{\overline{\mathcal{T}_{ki}}}\right) + \left(\overline{\overline{p_{ki}}} - p_k\right) \boldsymbol{n}_k\right\} \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right). \end{aligned} \tag{8-31}$$

Since we have

$$\sum_j a_{ij} \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) \cdot \boldsymbol{n}_k = \frac{D_k \alpha_k}{Dt} - \frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}} \tag{8-32}$$

the first term on the right-hand side of Eq.(8-31) becomes

$$\sum_j a_{ij} \left\{-\overline{\overline{p_{ki}}} \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) \cdot \boldsymbol{n}_k\right\} = \overline{\overline{p_{ki}}} \left(\frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}} - \frac{D_k \alpha_k}{Dt}\right). \tag{8-33}$$

The second term can be rearranged to the following form

$$\begin{aligned} &\sum_j a_{ij} \left\{\boldsymbol{n}_k \cdot \overline{\overline{\mathcal{T}_{ki}}} + \boldsymbol{n}_k \cdot \left(\mathcal{T}_k - \overline{\overline{\mathcal{T}_{ki}}}\right) + \left(\overline{\overline{p_{ki}}} - p_k\right) \boldsymbol{n}_k\right\} \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) \\ &= \boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) - \nabla \alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) \\ &+ \sum_j a_{ij} \left\{\left(\boldsymbol{M}_{ik}\right)' + \left(\boldsymbol{\tau}_i\right)'\right\} \cdot \widehat{\boldsymbol{v}'_{ki}} \end{aligned} \tag{8-34}$$

where $(\boldsymbol{M}_{ik})'$ and $(\boldsymbol{\tau}_i)'$ are defined by

$$\begin{aligned}
&(\boldsymbol{M}_{ik})' = (\boldsymbol{M}_k^n)' + (\boldsymbol{M}_k^t)' \\
&(\boldsymbol{M}_k^n)' \equiv -\left(p_k - \overline{\overline{p_{ki}}}\right)\boldsymbol{n}_k - \frac{\boldsymbol{M}_k^n}{a_i} \\
&(\boldsymbol{M}_k^t)' \equiv \boldsymbol{n}_k \cdot \left(\mathcal{T}_k - \overline{\overline{\mathcal{T}_{ki}}}\right) - \frac{\boldsymbol{M}_k^t}{a_i} \\
&(\boldsymbol{\tau}_i)' = \boldsymbol{n}_k \cdot \overline{\overline{\mathcal{T}_{ki}}} - \frac{\left(-\nabla\alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}}\right)}{a_i}.
\end{aligned} \tag{8-35}$$

Thus, it represents the fluctuating component of the total drag force. Consequently, we define the turbulent flux of work due to drag force W_{ki}^T as

$$W_{ki}^T \equiv \sum_j a_{ij}\left\{(\boldsymbol{M}_{ik})' + (\boldsymbol{\tau}_i)'\right\} \cdot \widehat{\boldsymbol{v}_{ki}'}. \tag{8-36}$$

Substituting Eqs.(8-33) and (8-34) with Eq.(8-36) into Eq.(8-31), we obtain

$$\begin{aligned}
&\sum_j a_{ij}\boldsymbol{n}_k \cdot \mathcal{T}_k \cdot \left(\boldsymbol{v}_k - \widehat{\boldsymbol{v}_k}\right) = \overline{\overline{p_{ki}}}\left(\frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}} - \frac{D_k\alpha_k}{Dt}\right) \\
&+\boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) - \nabla\alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) + W_{ki}^T.
\end{aligned} \tag{8-37}$$

In view of Eqs.(8-29) and (8-37), the macroscopic interfacial thermal energy transfer Λ_k becomes

$$\begin{aligned}
&\Lambda_k = \left(\Gamma_k\widehat{e_{ki}} + a_i\overline{\overline{q_{ki}''}}\right) + \overline{\overline{p_{ki}}}\left(\frac{\Gamma_k}{\overline{\overline{\rho_{ki}}}} - \frac{D_k\alpha_k}{Dt}\right) \\
&+\boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) - \nabla\alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) + W_{ki}^T.
\end{aligned} \tag{8-38}$$

Now we introduce the virtual enthalpy of the k^{th}-phase at interfaces in analogy with Eq.(8-27) and Eq.(5-37), thus

$$\widehat{h_{ki}} = \widehat{e_{ki}} + \frac{\overline{\overline{p_{ki}}}}{\overline{\overline{\rho_{ki}}}}. \tag{8-39}$$

Then we have

$$\Lambda_k = \left(\Gamma_k \widehat{h_{ki}} + a_i \overline{\overline{q''_{ki}}}\right) - \overline{\overline{p_{ki}}} \frac{D_k \alpha_k}{Dt} + \boldsymbol{M}_{ik} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)$$
$$-\nabla \alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \cdot \left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) + W_{ki}^T. \tag{8-40}$$

It is straightforward to obtain E_k from the relations for Λ_k, $\boldsymbol{M}_k$ and Γ_k, therefore we have from Eqs.(8-21) and (8-40) the following result

$$E_k = \Gamma_k \left(\widehat{h_{ki}} + \widehat{\boldsymbol{v}_{ki}} \cdot \widehat{\boldsymbol{v}_k} - \frac{\widehat{v_k}^2}{2}\right) + a_i \overline{\overline{q''_{ki}}} - \overline{\overline{p_{ki}}} \frac{\partial \alpha_k}{\partial t}$$
$$+\boldsymbol{M}_{ik} \cdot \widehat{\boldsymbol{v}_{ki}} - \nabla \alpha_k \cdot \overline{\overline{\mathcal{T}_{ki}}} \cdot \widehat{\boldsymbol{v}_{ki}} + W_{ki}^T. \tag{8-41}$$

The expressions for Λ_k and E_k give the k^{th}-phase interfacial fluxes of thermal energy and the total energy in terms of the mean values at the interfaces.

Now we proceed to the analysis of the mixture energy source term E_m. By assuming that

$$\frac{d\sigma}{dT} \approx \text{constant} \tag{8-42}$$

Eq.(5-29) can be approximated by

$$E_m = \sum_j a_{ij} \left\{T_i \left(\frac{d\sigma}{dT}\right) \nabla_s \cdot v_i + \left(\boldsymbol{t}_\alpha A^{\alpha\beta} \sigma\right)_{,\beta} \cdot \boldsymbol{v}_i\right\}. \tag{8-43}$$

We recall here that the surface divergence of the interfacial velocity is the surface area dilatation (Aris, 1962). Therefore, we have

$$\frac{1}{\left(dA\right)} \frac{d_s}{dt} (dA) = \nabla_s \cdot \boldsymbol{v}_i. \tag{8-44}$$

Hence, together with the assumption that the surface tension gradient is small, we may approximate Eq.(8-43) by

$$E_m \doteq \overline{\overline{T}}_i \left(\frac{d\sigma}{dT}\right)\frac{D_i}{Dt}(a_i) + 2\overline{\overline{H}}_{21}\,\overline{\overline{\sigma}}\frac{\partial \alpha_1}{\partial t} + E_m^H \tag{8-45}$$

where the convective derivative D_i/Dt is defined by

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \widehat{\boldsymbol{v}_i}\cdot\nabla. \tag{8-46}$$

We note that the first term on the right-hand side of Eq.(8-45) takes into account the effects of the surface energy change associated with the changes in area, whereas the second and the last terms stand for the average work done by the surface tension. The last term represents the effect of the changes of the mean curvature on the mixture energy source. By combining Eqs.(8-21) and (8-26) we have

$$\begin{aligned}\sum_{k=1}^{2}\Lambda_k &= \sum_{k=1}^{2}\Gamma_k\left(\frac{\widehat{\boldsymbol{v}_k}^2}{2} - \widehat{\boldsymbol{v}_{ki}}\cdot\widehat{\boldsymbol{v}_k}\right) - \sum_{k=1}^{2}\boldsymbol{M}_{ik}\cdot\widehat{\boldsymbol{v}_k}\\ &+\sum_{k=1}^{2}\left(\overline{\overline{\mathcal{T}_{ki}}}\cdot\widehat{\boldsymbol{v}_k} - \overline{\overline{p_{ki}}}\widehat{\boldsymbol{v}_k}\right)\cdot\nabla\alpha_k + E_m.\end{aligned} \tag{8-47}$$

As a summary on interfacial energy transfer we have the following relations.

Total energy transfer condition

$$\begin{aligned}E_k &= \Gamma_k\left(\widehat{h_{ki}} + \widehat{\boldsymbol{v}_{ki}}\cdot\widehat{\boldsymbol{v}_k} - \frac{\widehat{\boldsymbol{v}_k}^2}{2}\right) + a_i\overline{\overline{q_{ki}''}} - \overline{\overline{p_{ki}}}\frac{\partial\alpha_k}{\partial t}\\ &+\boldsymbol{M}_{ik}\cdot\widehat{\boldsymbol{v}_{ki}} - \nabla\alpha_k\cdot\overline{\overline{\mathcal{T}_{ki}}}\cdot\widehat{\boldsymbol{v}_{ki}} + W_{ki}^T \quad \text{with}\\ E_m &= \sum_{k=1}^{2}E_k = \overline{\overline{T}}_i\left(\frac{d\sigma}{dT}\right)\frac{D_i}{Dt}(a_i) + 2\overline{\overline{H}}_{21}\,\overline{\overline{\sigma}}\frac{\partial\alpha_1}{\partial t} + E_m^H\end{aligned} \tag{8-48}$$

Thermal energy transfer condition

$$\begin{aligned}\Lambda_k &= \left(\Gamma_k\widehat{h_{ki}} + a_i\overline{\overline{q_{ki}''}}\right) - \overline{\overline{p_{ki}}}\frac{D_k\alpha_k}{Dt} + \boldsymbol{M}_{ik}\cdot\left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right)\\ &-\nabla\alpha_k\cdot\overline{\overline{\mathcal{T}_{ki}}}\cdot\left(\widehat{\boldsymbol{v}_{ki}} - \widehat{\boldsymbol{v}_k}\right) + W_{ki}^T \quad \text{with}\end{aligned} \tag{8-49}$$

$$\sum_{k=1}^{2} \Lambda_k = \overline{\overline{T}}_i \left(\frac{d\sigma}{dT} \right) \frac{D_i}{Dt}(a_i) + 2\overline{\overline{H}}_{21}\, \overline{\overline{\sigma}} \frac{\partial \alpha_1}{\partial t}$$

$$+E_m^H + \sum_{k=1}^{2} \left(\overline{\overline{\mathcal{T}}}_{ki} \cdot \widehat{\boldsymbol{v}_k} - \overline{\overline{p}}_{ki} \widehat{\boldsymbol{v}_{ki}} \right) \cdot \nabla \alpha_k$$

$$+\sum_{k=1}^{2} \left\{ \Gamma_k \left(\frac{\widehat{\boldsymbol{v}_k}^2}{2} - \widehat{\boldsymbol{v}_{ki}} \cdot \widehat{\boldsymbol{v}_k} \right) - \boldsymbol{M}_{ik} \cdot \widehat{\boldsymbol{v}_k} \right\}$$

Since these relations are now expressed by the mean values of the bulk fluid and of the interfaces, they can be considered as having the macroscopic forms. The constitutive equations can be obtained by relating the interfacial variables to the bulk fluid mean values and other characteristics parameters such as a_i.

In view of Eqs.(8-5), (8-21) and (8-48) we recognize considerable differences between the necessary interfacial constitutive laws for the two-fluid model and those for the drift-flux (mixture) model. For the former model it is necessary to specify Γ_1, $\boldsymbol{M}_1$, E_1, $\boldsymbol{M}_m$ and E_m by constitutive equations, whereas for the latter model it is sufficient to supply only Γ_1, $\boldsymbol{M}_m$ and E_m (or $\sum_{k=1}^{2} \Lambda_k$). Indeed this makes the drift-flux model quite simpler than the two-fluid model. In the diffusion or drift-flux model we supply the relation between the velocities of each phase, thus only one momentum equation is required. However, in the two-fluid model we specify the momentum exchange term $\boldsymbol{M}_k$ and then solve two momentum equations simultaneously. We also note that the sum of Λ_k for two phases does not reduce to a simple form as E_m without making assumptions, thus it is expected that special attention should be paid in using the thermal energy equation in the drift-flux model.