Chapter 3

VARIOUS METHODS OF AVERAGING

1.1 Purpose of averaging

The design of engineering systems and the ability to predict their performance depend on the availability of experimental data and conceptual models that can be used to describe a physical process with a required degree of accuracy. From both a scientific and a practical point of view, it is essential that the various characteristics and properties of such conceptual models and processes are clearly formulated on rational bases and supported by experimental data. For this purpose, specially designed experiments are required which must be conducted in conjunction with and in support of analytical investigations. It is well established in continuum mechanics that the conceptual models for single-phase flow of a gas or of a liquid are formulated in terms of field equations describing the conservation laws of mass, momentum, energy, charge, etc. These field equations are then complemented by appropriate constitutive equations such as the constitutive equations of state, stress, chemical reactions, etc., which specify the thermodynamic, transport and chemical properties of a given constituent material, namely, of a specified solid, liquid or gas.

It is to be expected, therefore, that the conceptual models describing the steady state and dynamic characteristics of multiphase or multi-component media should also be formulated in terms of the appropriate field and constitutive equations. However, the derivation of such equations for the flow of structured media is considerably more complicated than for strictly continuous homogeneous media for single-phase flow. In order to appreciate the difficulties in deriving balance equations for structured, namely, inhomogeneous media with interfacial discontinuities, we recall that in continuum mechanics the field theories are constructed on integral balances of mass, momentum and energy. Thus, if the variables in the region of integration are continuously differentiable and the Jacobian transformation between material and spatial coordinates exists, then the Euler-type differential balance can be obtained by using the Leibnitz's rule; more specifically, however, the Reynolds's transport theorem allows us to interchange differential and integral operations.

In multi-phase or multi-component flows, the presence of interfacial surfaces introduces great difficulties in the mathematical and physical formulation of the problem. From the mathematical point of view, a multi-phase flow can be considered as a field that is subdivided into single-phase regions with moving boundaries separating the constituent phases. The differential balance holds for *each* sub-region. It cannot be applied, however, to the *set* of these sub-regions in the normal sense without violating the above conditions of continuity. From the point of view of physics, the difficulties encountered in deriving the field and constitutive equations appropriate to multi-phase flow systems stem from the presence of the interface. It also stems from the fact that both the steady and dynamic characteristics of multi-phase flows depend upon the interfacial structure of the flow. For example, the steady state and the dynamic characteristics of dispersed two-phase flow systems depend upon the collective dynamics of solid particles, bubbles or droplets interacting with each other and with the surrounding continuous phase; whereas, in the case of separated flows, these characteristics depend upon the structure and wave dynamics of the interface. In order to determine the collective interaction of particles and the dynamics of the interface, it is necessary to describe first the local properties of the flow and then to obtain a macroscopic description by means of appropriate averaging procedures. For dispersed flows, for example, it is necessary to determine the rates of nucleation, evaporation or condensation, motion and disintegration of single droplets (bubbles) as well as the collisions and coalescence processes of several droplets (or bubbles).

For separated flow, the structure and the dynamics of the interface greatly influence the rates of mass, heat and momentum transfer as well as the stability of the system. For example, the performance and flow stability of a condenser for space application depend upon the dynamics of the vapor interface. Similarly, the rate of droplet entrainment from a liquid film, and therefore, the effectiveness of film cooling, depend upon the stability of the vapor liquid interface.

It can be concluded from this discussion that in order to derive the field and constitutive equations appropriate to structured multiphase flow, it is necessary to describe the local characteristics of the flow. From that flow, the macroscopic properties should be obtained by means of an appropriate averaging procedure. It is evident also that the design, performance and, very often, the safe operation of a great number of important technological systems (which were enumerated in the preceding sections) depend upon the availability of realistic and accurate field and constitutive equations.

The formulation based on the local instant variables of Chapter 2 shows that, in general, it results in a multi-boundary problem with the positions of the interfaces being unknown. In such a case the mathematical difficulties encountered in obtaining solutions are prohibitively great and in many practical problems they are beyond our present computational capability. In order to appreciate these difficulties we recall that even in single-phase turbulent flow without moving interfaces, it has not been possible to obtain exact solutions expressing local instant fluctuations. It can be said that overwhelming difficulties encountered in the local instant formulations stem from:

1. Existence of the multiple deformable moving interfaces with their motions being unknown;
2. Existence of the fluctuations of variables due to turbulences and to the motions of the interfaces;
3. Significant discontinuities of properties at interface.

The first effect causes complicated coupling between the field equations of each phase and the interfacial conditions, whereas the second effect inevitably introduces a statistical characteristic originated from the instability of the Navier-Stokes equation and of the interfacial waves. The third effect introduces huge local jumps in various variables in space and time. Since these difficulties exist in almost all two-phase flow systems, an application of the local instant formulation to obtain a solution is severely limited. For a system with a simple interfacial geometry, however, as in the case of a single or several bubble problem or of a separated flow, it has been used extensively and very useful information have been obtained. As most two-phase flow observed in practical engineering systems have extremely complicated interfacial geometry and motions, it is not possible to solve for local instant motions of the fluid particles. Such microscopic details of the fluid motions and of other variables are rarely needed for an engineering problem, but rather macroscopic aspects of the flow are much more important.

By proper averaging, we can obtain the mean values of fluid motions and properties that effectively eliminate local instant fluctuations. The averaging procedure can be considered as low-pass filtering, excluding unwanted high frequency signals from local instant fluctuations. However, it is important to note that the statistical properties of these fluctuations influencing the macroscopic phenomena should be taken into account in a formulation based on averaging.

1.2 Classification of averaging

The importance and the necessity of averaging procedures in order to derive macroscopic field and constitutive equations for structured two-phase media have been discussed in the Section 1.1 of Chapter 3 In this section we study various methods of averaging that can be applied to thermo-fluid dynamics in general and to two-phase flow in particular. Depending on the basic physical concepts used to formulate thermal-hydraulic problems, averaging procedures can be classified into three main groups: the Eulerian averaging; the Lagrangian averaging; and the Boltzmann statistical averaging. They can be further divided into sub-groups based on a variable with which a mathematical operator of averaging is defined. The summary of the classifications and the definitions of various averaging are given below.

i) *Eulerian Average - Eulerian Mean Value*

Function: $F = F(t, \boldsymbol{x})$ (3-1)

Time (Temporal) mean value: $\dfrac{1}{\Delta t}\displaystyle\int_{\Delta t} F\left(t, \boldsymbol{x}\right) dt$ (3-2)

Spatial mean value: $\dfrac{1}{\Delta R}\displaystyle\int_{\Delta R} F\left(t, \boldsymbol{x}\right) dR\left(\boldsymbol{x}\right)$ (3-3)

Volume: $\dfrac{1}{\Delta V}\displaystyle\int_{\Delta V} F\left(t, \boldsymbol{x}\right) dV$ (3-4)

Area: $\dfrac{1}{\Delta A}\displaystyle\int_{\Delta A} F\left(t, \boldsymbol{x}\right) dA$ (3-5)

Line: $\dfrac{1}{\Delta C}\displaystyle\int_{\Delta C} F\left(t, \boldsymbol{x}\right) dC$ (3-6)

Statistical mean value: $\dfrac{1}{N}\sum_{n=1}^{N} F_n\left(t, \boldsymbol{x}\right)$ (3-7)

Mixed mean value: combination of above operations

ii) *Lagrangian Average - Lagrangian Mean Value*

Function: $F = F(t, \boldsymbol{X});\;\; \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$ (3-8)

Time (Temporal) mean value: $\dfrac{1}{\Delta t}\int_{\Delta t} F\left(t, \boldsymbol{X}\right) dt$ (3-9)

Statistical mean value: $\dfrac{1}{N}\sum_{n=1}^{N} F_n\left(t, \boldsymbol{X}\right)$ (3-10)

iii) *Boltzmann Statistical Average*

Particle density function: $f = f\left(\boldsymbol{x}, \xi, t\right)$ (3-11)

Transport properties: $\psi\left(t, \boldsymbol{x}\right) = \dfrac{\int \psi(\xi) f d\xi}{\int f d\xi}$ (3-12)

Here we note that $\boldsymbol{x}$ and $\boldsymbol{X}$ are the spatial and the material coordinates, respectively, whereas ξ is the phase velocity or kinetic energy of particles. Furthermore, we point out that the true time or statistical averaging is defined by taking the limit $\Delta t \to \infty$ or $N \to \infty$, which is only possible in concept. The material coordinates can be considered as the initial positions of all the particles, thus if $\boldsymbol{X}$ is fixed it implies the value of a function following a particle.

The most important and widely used group of averaging in continuum mechanics is the Eulerian averaging, because it is closely related to human observations and most instrumentations. The basic concept underlining this method is the time-space description of physical phenomena. In a so-called Eulerian description, the time and space coordinates are taken as independent variables and various dependent variables express their changes with respect to these coordinates. Since the standard field equations of continuum mechanics developed in Chapter 2 adapt to this description, it is natural to consider averaging with respect to these independent variables, namely, the time and the space. Furthermore, these averaging processes are basically integral operators, therefore, they have an effect of smoothing out instant or local variations within a domain of integration.

The Lagrangian mean values are directly related to the Lagrangian description of mechanics. As the particle coordinate $\boldsymbol{X}$ displaces the spatial variable $\boldsymbol{x}$ of the Eulerian description, this averaging is naturally fitted to a study of the dynamics of a particle. If our interest is focused on a behavior of an individual particle rather than on the collective mechanics of a group of particles, the Lagrangian average is important and useful for analyses. The Lagrangian time average is taken by following a certain particle and observing it over some time interval. A simple example is the average speed of a particular vehicle such as a car, a train or an airplane. Furthermore, the Eulerian temporal mean values can be exemplified by an average velocity of all cars passing at a point on a road over some time interval.

In contrast to the mean values explained above, the Eulerian and the Lagrangian statistical mean values are based on a statistical assumption, since they involve a collection of N similar samples denoted by F_n with $n = 1, \cdots, N$. A fundamental question arises as we ask, "What are the similar samples for a system with fluctuating signals?" To visualize a group of similar samples, it is useful to consider a time averaging as a filtering process that eliminates unwanted fluctuations. The similar samples may then be considered as a group of samples which have time mean values of all the important variables within certain ranges of deviations. In this case, the time interval of the averaging and the ranges of deviations define the unwanted fluctuations, thus the statistical averaging depends on them. For a steady-state flow based on time averaging, random sampling over a time domain can constitute a proper set of samples as it is often done in experimental measurements. In this case, the time averaging and the statistical averaging are equivalent. There are many other factors to consider, however it is also possible to leave it as an abstract concept. The difficulties arise when the constitutive equations are studied in connection with experimental data. The true statistical averaging involving an infinite number of similar samples is only possible in concept, and it cannot be realized. Thus, if it is considered alone, the ensemble averaging faces two difficulties, namely, choosing a group of similar samples and connecting the experimental data to a model.

The Boltzmann statistical averaging with a concept of the particle number density is important when the collective mechanics of a large number of particles are in question. As the number of particles and their interactions between them increase, the behavior of any single particle becomes so complicated and diversified, it is not practical to solve for each particle. In such a case, the behavior of a group of many particles increasingly exhibits some particular characteristics that are different from a single particle as the collective particle mechanics becomes a governing factor. It is well known that the Boltzmann statistical averaging applied to a large number of molecules with an appropriate mean-free path can lead to field equations that closely resemble that of the continuum mechanics. It can also be applied to subatomic particles, such as neutrons, to obtain a transport theory for them. This can be done by first writing the balance equation for the particle density function, which is known as the Boltzmann transport equation. Then it is necessary to assume a form of the particle interaction term as well as stochastic characteristics of the particle density function. A simple model using two-molecular interaction was developed by Maxwell, thus the Boltzmann transport equation with the collision integral of Maxwell was called the Maxwell-Boltzmann equation. This equation became the foundation of the kinetic theory of gases. We recall that if the Maxwell-Boltzmann equation is multiplied by 1, particle velocity, or the kinetic energy $(1/2)\xi^2$ then averaged over the particle velocity field, it can be reduced to a form similar to the standard conservation equations of mass, momentum and energy in the continuum mechanics.

1.3 Various Averaging in Connection with Two-Phase Flow Analysis

In order to study two-phase flow systems, many of above averaging methods have been used by various researchers. The applications of averaging can be divided into two main categories

1. To define properties and then to correlate experimental data.
2. To obtain usable field and constitutive equations that can be used to predict macroscopic processes.

The most elementary use is to define mean properties and motions that include various kinds of concentrations, density, velocity and energy of each phase or of a mixture. These properly defined mean values then can be used for various experimental purposes and for developments of empirical correlations. The choice of averaging and instrumentations are closely coupled since, in general, measured quantities represent some kinds of mean values themselves.

Both the Eulerian time and spatial averaging are frequently in use, because experimenters incline to consider two-phase mixtures as quasi-continuum. Furthermore, they are usually the easiest mean values to measure in fluid flow systems. However, when a particular fluid particle can be distinguishable and traceable, as in the case of a bubbly or droplet flow, the Lagrangian mean values are also used. It is only natural that these mean values are obtained for stationary systems that can be considered to have steady-state characteristics in terms of mean values. Various correlations are then developed by further applying the statistical averaging among different data. This is the standard method of experimental physics to minimize errors.

Before we proceed to the second application of averaging, we discuss briefly two fundamentally different formulations of the macroscopic field equations; namely the two-fluid model and the drift-flux (mixture) model. The two-fluid model is formulated by considering each phase separately. Consequently, it is expressed by two sets of conservation equations of mass, momentum and energy. Each of these six field equations has invariably an interaction term coupling the two phases through jump conditions. The mixture model is formulated by considering the mixture as a whole. Thus, the model is expressed in terms of three-mixture conservation equations of mass, momentum, and energy with one additional diffusion (continuity) equation which takes account of the concentration changes. A mixture conservation equation can be obtained by adding two corresponding conservation equations for each phase with an appropriate jump condition. However, it should be noted that a proper mixture model should be formulated in terms of correctly defined mixture quantities. It can be said that the drift-flux model is an example of a mixture model that includes diffusion model, slip flow model and homogeneous flow model. However, for most practical applications, the drift-flux model is the best mixture model that is highly developed for normal gravity (Ishii, 1977) as well as micro-gravity conditions (Hibiki and Ishii, 2003b; Hibiki et al., 2004).

Now we proceed to a discussion of the second and more important application of these averaging. That is to obtain the macroscopic two-phase flow field equations and the constitutive equations in terms of mean values. Here, again, the Eulerian spatial and time averaging have been used extensively by various authors, though the Eulerian or Boltzmann statistical averaging have also been applied.

Using the *Eulerian volumetric averaging*, important contributions for an establishment of a three-dimensional model of highly dispersed flows has been made by Zuber (1964a), Zuber et al. (1964), Wundt (1967), Delhaye (1968) and Slattery (1972). These analyses were based on a volume element that included both phases at the same moment. Moreover, it was considered to be much smaller than the total system in interest, thus main applications were for highly dispersed flows.

It has long been realized that the *Eulerian area averaging* over a cross section of a duct is very useful for engineering applications, since field equations reduce to a one-dimensional model. By area averaging, the information on changes of variables in the direction normal to the main flow is basically lost. Therefore, the transfer of momentum and energy between the wall and the fluid should be expressed by empirical correlations or by simplified models which replace the exact interfacial conditions. We note that even in single-phase flow problems, the area-averaging method has been widely used because its simplicity is highly desirable in many practical engineering applications. For example, the use of the wall friction factor or the heat transfer coefficient is closely related to the concept of the area averaging. We also mention here its extensive use in compressible fluid flow analyses. A good review of single-phase flow area averaging as well as macroscopic equations that correspond to the open-system equations in thermodynamics can be found in Bird et al. (1960), Whitaker (1968) and Slattery (1972). The boundary layer integral method of von Karman is also an ingenious application of the area averaging. Furthermore, numerous examples of area averaging can be found in the literature on lubricating films, open channel flow and shell theories in mechanics.

However, in applications to two-phase flow systems, many authors used phenomenological approach rather than mathematically exact area averaging, thus the results of Martinelli and Nelson (1948), Kutateladze (1952), Brodkey (1967), Levy (1960) and Wallis (1969) were in disagreement with each other and none of them are complete (Kocamustafaogullari, 1971). The rational approach to obtain a one-dimensional model is to integrate single-phase differential field equations over the cross sectional area. Meyer (1960) was an early user of this method to obtain mixture equations, but his definitions of various mixture properties as well as the lack of a diffusion (continuity) equation were objectionable (Zuber, 1967).

A rigorous derivation of one-dimensional mixture field equations with an additional diffusion (continuity) equation, namely, the drift-flux model, was carried out by Zuber et al. (1964) and Zuber (1967). The result shows a significant similarity with the field equations for heterogeneous chemically reacting single-phase systems. The latter had been developed as the thermomechanical theory of diffusion based on the interacting continua occupying the same point at the same time but having two different velocities. Numerous authors have contributed in this theory, thus we only recall those of Fick (1855), Stefan (1871), von Karman (1950), Prigogine and Mazur (1951), Hirschfelder et al. (1954), Truesdell and Toupin (1960), and Truesdell (1969). A similar result obtained from an entirely different method of the kinetic theory of gas mixtures by Maxwell (1867) should also be noted here.

In contrast to the analysis of Zuber, the analysis of Delhaye (1968) and Vernier and Delhaye (1968) was directed to two-fluid model based on three field equations for each phase with three jump conditions that couple the two fields. A very systematic method was employed in deriving field equations from three different Eulerian spatial averaging as well as the statistical and the temporal averaging in Vernier and Delhaye (1968). This is apparently the first publication which shows important similarities as well as differences between the various averaging methods. The effect of surface tension, which is important for the analysis of interfacial stability and of flow regimes, has been included in the study of Kocamustafaogullari (1971). This study highlighted that the area averaged model is particularly suited for studying a separated flow regime and interfacial wave instabilities. However, it can be used in any type of flow regimes, provided the constitutive equations can be supplied (Bouré and Réocreux, 1972). Furthermore, in the former reference a clear separation of analytical methods between the drift-flux model and the two-fluid model has been given. Although this distinction had been well known for other kind of mixtures, for example in study of the super fluidity of helium II of Landau (1941), of the plasma dynamics of Pai (1962), and of the diffusion theory of Truesdell (1969), in two-phase flow analysis it had been vague. This shortcoming of traditional two-phase flow formulation was first pointed out by Zuber and Dougherty (1967). In subsequent analyses of Ishii (1971) using time averaging and of Kocamustafaogullari (1971) using area averaging a clear distinction between the two models has been made. This point was also discussed by Bouré and Réocreux (1972) in connection with the problem of the two-phase sound wave propagation and of choking phenomena.

The Eulerian time averaging, which has been widely applied in analyzing a single-phase turbulent flow, is also used for two-phase flow. In applying the time averaging method to a mixture, many authors coupled it with other space averaging procedures. Important contributions were made by Russian researchers (Teletov, 1945; Frankl, 1953; Teletov, 1957; Diunin, 1963) who have used the Eulerian time-volume mean values and obtained three-dimensional field equations.

An analysis based on the Eulerian time averaging alone was apparently initiated by Vernier and Delhaye (1968), however, a detailed study leading to a mathematical formulation was not given there. Furthermore, Panton (1968) obtained the mixture model by first integrating in a time interval then in a volume element. His analysis was more explicit in integration procedures than the works by the Russian researchers, but both results were quite similar. In Ishii (1971), a two-fluid model formulation including the surface source terms was obtained by using the time averaging alone, then the area averaging over a cross section of a duct was carried out. There, all the constitutive equations as well as boundary conditions, which should be specified in a standard one-dimensional two-phase flow model, were identified. We also note the extensive study by Drew (1971) who has used an Eulerian multiple mixed averaging procedure. In his analysis, two integrals over both space and time domains have been taken in order to smooth out higher order singularities. These multiple integral operations are equivalent to the continuum assumption, therefore, they are not necessary. The averaging should not be considered as a pure mathematical transformation, since the constitutive model can be only developed based on the continuum assumption. Here, readers should also refer to Delhaye (1969; 1970), where various models based on Eulerian space averaging as well as a comprehensive review on the subject could be found.

It can be said that the Eulerian time averaging is particularly useful for a turbulent two-phase flow or for a dispersed two-phase flow (Ishii, 1975; Ishii, 1977; Ishii and Mishima, 1984). In these flows, since the transport processes are highly dependent on the local fluctuations of variables about the mean, the constitutive equations are best obtainable for a time averaged model from experimental data. This is supported by the standard single-phase turbulent flow analysis.

An extensive study using *Eulerian statistical averaging* was carried out by Vernier and Delhaye (1968) in which they obtained an important conclusion. Under stationary flow condition, they concluded, the field equations from the true time averaging, namely, the temporal averaging with $\Delta t \to \infty$, and the ones from the statistical averaging are identical. Furthermore, the statistical averaging was combined with the spatial averaging, then supplemented with various constitutive assumptions to yield a practical two-dimensional model. The Boltzmann statistical averaging has also been used by several authors (Murray, 1954; Buevich, 1969; Buyevich, 1972; Culick, 1964; Kalinin, 1970; Pai, 1971) for a highly dispersed two-phase flow systems. In general, the particle density functions are considered, then the Boltzmann transport equation for the functions is written. Kalinin (1970) assumed that the particle density functions represent the expected number of particles of a particular mass and velocity, whereas Pai (1971) considered the radius, velocity and temperature as the arguments of the functions. Then a simplified version of Maxwell's equation of transfer for each phase has been obtained from the Maxwell-Boltzmann equation by integrating over the arguments of the particle density function except time and space variables. Since it involves assumptions on the distributions as well as on inter-particle and particle-gas interaction terms, the results are not general, but represent a special kind of continuum.

It is interesting, however, to note that three different methods and views of mechanics of mixtures in a local sense are represented: the Eulerian time or statistical averaging applied to two-phase mixtures; the thermomechanical theory of diffusion based on two continua; and the Boltzmann statistical averaging applied to gas mixtures or to highly dispersed flows. The first theory considers the mixture to be essentially a group of single-phase regions bounded by interfaces, whereas in the second theory the two components coexist at the same point and time. In contrast to the above two theories, which are established on the foundation of the continuum mechanics, the last theory is based on the statistical expectations and the probability. The importance, however, is that if each transfer terms of above models are correctly interpreted, the resulting field equations have very similar forms.

A preliminary study using *ensemble cell averaging* was carried out by Arnold et al. (1989) where they derived turbulent stress and interfacial pressure forces due to pressure variations over the surface of non-distorting bubbles for an idealized inviscid bubbly flow. They discussed deficiencies inherent in spatial averaging techniques, and recommended ensemble averaging for the formulation of two-fluid models of two-phase flows. Zhang and Prosperetti (1994a) derived averaged equations governing a mixture of equal spherical compressible bubbles in an inviscid liquid by the ensemble-averaging method. They concluded that the method was systematic and general because of no *ad hoc* closure relations required, and suggested the possibility that the method might be applied to a variety of thermo-fluid and solid mechanics situations. Zhang and Prosperetti (1994b) extended this method to the case of spheres with a variable radius. Zhang (1993) summarized the other applications to heat conduction and convection, Stokes flow, and thermocapillary process. Here, readers should also refer to Prosperetti (1999), in which some considerations on the modeling of disperse multiphase flows by averaged equations can be found. Kolev (2002) presented a two-phase flow formulation mostly for development of safety analysis codes based on multi-field approach.

Finally, we briefly discuss the application of the Lagrangian averaging to two-phase flow systems. This approach is useful for particulate flow, however, in general it encounters considerable difficulties and impracticabilities due to the diffusion and phase changes. For the particulate flow without phase changes, the Lagrangian equation of the mean particle motion can be obtained in detail for many practical cases. Thus, we note that the Lagrangian description of a single particle dynamics is frequently in use as a momentum equation for a particulate phase in a highly dispersed flow (Carrier, 1958; Zuber, 1964a). Many analyses on the bubble rise and terminal velocity use the Lagrangian time averaging implicitly, particularly in a case when the continuum phase is in the turbulent flow regime.