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Hyperbolic Systems of Thermodynamically Compatible Conservation Laws in Continuum Mechanics

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Abstract—The new class of hyperbolic systems of conservation laws is proposed. Every system belonging to this class is thermodynamically compatible, i.e., is generated by one thermodynamic potential alone and has an additional conservation law as a consequence. Besides, every system can be reduced to a symmetric hyperbolic form. Many well-known systems of continuum mechanics equations belong to a formulated class. Some new well-posed systems of conservation equations for complex media models are considered. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Conservation laws, Symmetric hyperbolic system, Continuous media.

1. INTRODUCTION

The problem of classification of differential equations of mathematical physics is very important in view of modeling new industrial processes and creating advanced materials. Models based on the fundamental physical principles and allowing convenient use of the mathematical methods of investigation have a natural advantage.

This article prolongs the work [1] in which a new class of systems of differential equations of continuum mechanics is proposed. Every representative of the class consists of the equations—conservation laws being reduced to the symmetric hyperbolic system and possesses the set of stationary conservation laws as the consequences that are compatible with the original system. Besides, every system (variation rates and fluxes of all variables in the equations) is defined only by a thermodynamic potential alone.

The work in [1] is the result of prolonged researches in mathematical physics equations classification. Friedrichs article [2], in which was given the definition of symmetric hyperbolic systems, is an essential progress in the development of hyperbolic equations theory. The systems of such a type allow the use of conventional instrument of energy integrals for the analysis of solvability of different problems. For example, the matter of the existence of quasi-linear symmetric hyperbolic system's solution is considered in [3,4]. Important questions about the relation between thermodynamics in models of continuum media and the well-posedness of corresponding differential equations were first touched on in Godunov's work [5]. In this work, it formulated such a

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class of systems of quasi-linear conservation laws that every closed system owns the additional conservation law (the analog of the first law of thermodynamics) and is being reduced to the symmetric hyperbolic form. Further history of the development of research in the area of connection between thermodynamics and well-posedness of differential equations of the model and the using of the symmetric form in numerical algorithms are in many works [6–10].

In [1], there were adduced only the examples of the equation systems that fall into the formulated classes—that are the equations of nonlinear elasticity theory and magnetohydrodynamics equations. The present work contains another set of examples of continuum mechanics equation systems from this class. These are systems of slowly moving dielectrics, superfluid helium equations. Besides the new variants of locally two-phase two-velocity media equations, heat transfer with finite velocity equations and elastic conductor-superconductor equations are proposed.

2. CLASS OF THERMODYNAMICALLY COMPATIBLE CONSERVATION LAWS

Limiting ourselves by the minimum set of equations and unknown functions, we shall consider the following class of formalized conservation laws proposed in [1]:

$$
\begin{aligned}
&\text{(a)} && \frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial (u_k L)_{q_\gamma}}{\partial x_k} = 0,\\
&\text{(b)} && \frac{\partial L_{u_l}}{\partial t} + \frac{\partial\left[(u_k L)_{u_l} - r_{l\alpha} L_{r_{k\alpha}} - b_l L_{b_k} - d_l L_{d_k} + j_k L_{j_l} - \delta_{lk} j_\alpha L_{j_\alpha}\right]}{\partial x_k} = 0,\\
&\text{(c)} && \frac{\partial L_{r_{il}}}{\partial t} + \frac{\partial[u_k L_{r_{il}} - u_i L_{r_{kl}}]}{\partial x_k} = 0,\\
&\text{(d)} && \frac{\partial L_{d_i}}{\partial t} + \frac{\partial[u_k L_{d_i} - u_i L_{d_k} - e_{ikl} b_l]}{\partial x_k} = 0,\\
&\text{(e)} && \frac{\partial L_{b_i}}{\partial t} + \frac{\partial\left[u_k L_{b_i} - u_i L_{b_k} + e_{ikl} d_l\right]}{\partial x_k} = 0,\\
&\text{(f)} && \frac{\partial L_n}{\partial t} + \frac{\partial[u_k L_n + j_k]}{\partial x_k} = 0,\\
&\text{(g)} && \frac{\partial L_{j_k}}{\partial t} + \frac{\partial[u_\alpha L_{j_\alpha} + n]}{\partial x_k} = 0,
\end{aligned}
\tag{1}
$$

where e_{ikl} is the unit pseudoscalar (the fully antisymmetric unit tensor). It is easy to observe that unknown functions are divided into the groups with the different physical sense

$$
q_\gamma, \gamma = 0, \omega, 1, 2, \dots; \qquad u_i, r_{il}, i, l = 1,2,3; \qquad d_i, b_i, i = 1,2,3; \qquad n, j_k, k = 1,2,3. \tag{2}
$$

The variation rate of all these variables and their fluxes are determined only by "generating" potential L alone, depending on all unknown functions (2). To guarantee that system (1) is hyperbolic, we assume that L is strictly convex function of its arguments.

In [1], it is mentioned that implications of system (1) are the additional stationary conservation laws that are compatible with (1):

$$
\operatorname{div} L_{\mathbf{d}} = 0, \qquad \operatorname{div} L_{\mathbf{b}} = 0, \qquad \operatorname{rot} L_{\mathbf{j}} = 0, \qquad \operatorname{div} L_{\mathbf{r}_i} = 0, \tag{3}
$$

where

$$
\begin{aligned}
L_{\mathbf{d}} &= \left(L_{d_1}, L_{d_2}, L_{d_3}\right)^\top, \qquad & L_{\mathbf{b}} &= \left(L_{b_1}, L_{b_2}, L_{b_3}\right)^\top,\\
L_{\mathbf{j}} &= \left(L_{j_1}, L_{j_2}, L_{j_3}\right)^\top, \qquad & L_{\mathbf{r}_i} &= \left(L_{r_{i1}}, L_{r_{i2}}, L_{r_{i3}}\right)^\top.
\end{aligned}
$$

It is a simple matter to show that equations (3) are resulting from system (1) owing to the flux structure in its equations. For example, applying operator div to the equation for d_i from system (1), we shall obtain

$$\frac{\partial}{\partial t}\operatorname{div} L_{\mathbf{d}} = 0,$$

and if the equality $\operatorname{div} L_{\mathbf{d}} = 0$ is fulfilled at initial time $t = 0$, then it is fulfilled for all $t > 0$.

Stationary conservation laws (3) ensure the fulfillment of the additional energy conservation law for system (1) and also give an opportunity to reduce (1) to symmetric form.

Additional conservation law is obtained by multiplying the equations of system (1), respectively, by

$$q_\gamma, u_i, r_{il}, d_i, b_i, n, j_k,$$

summing them and adding to this sum the equality

$$(\mathbf{u},\mathbf{d})\operatorname{div} L_{\mathbf{d}} + (\mathbf{u},\mathbf{b})\operatorname{div} L_{\mathbf{b}} + ([\mathbf{u}\times\mathbf{j}], \operatorname{rot} L_{\mathbf{j}}) + (\mathbf{u},\mathbf{r}_i)\operatorname{div} L_{\mathbf{r}_i} = 0.$$

The energy conservation law obtained has the form

$$\frac{\partial \Pi_0}{\partial t} + \frac{\partial \Pi_k}{\partial x_k} = 0,$$

where

$$\begin{aligned}
\Pi_0 &= q_\gamma L_{q_\gamma} + u_l L_{u_l} + r_{il} L_{r_{il}} + d_i L_{d_i} + b_i L_{b_i} + n L_n + j_l L_{j_l} - L,\\
\Pi_k &= u_k\left(q_\gamma L_{q\gamma} + u_l L_{u_l} + r_{il} L_{r_{il}} + d_i L_{d_i} + b_i L_{b_i} + n L_n\right)\\
&\quad + j_k(u_l L_{j_l} + n) - u_l r_{l\beta} L_{r_{k\beta}} - u_l d_l L_{d_k} - u_l b_l L_{b_k} + e_{k\alpha\beta} d_\alpha b_\beta.
\end{aligned}$$

The symmetric form of (1) is obtained by adding

$$r_{i\alpha}\frac{\partial L_{r_{k\alpha}}}{\partial x_k} + d_i\frac{\partial L_{d_k}}{\partial x_k} + b_i\frac{\partial L_{b_k}}{\partial x_k} + j_k\left(\frac{\partial L_{j_k}}{\partial x_i} - \frac{\partial L_{j_i}}{\partial x_k}\right) = 0$$

to equation (1b),

$$u_i\frac{\partial L_{r_{kl}}}{\partial x_k} = 0$$

to equation (1c),

$$u_i\frac{\partial L_{d_k}}{\partial x_k} = 0, \qquad u_i\frac{\partial L_{b_k}}{\partial x_k} = 0$$

to equations (1d),(1e), respectively, and

$$u_i\left(\frac{\partial L_{j_k}}{\partial x_i} - \frac{\partial L_{j_i}}{\partial x_k}\right) = 0$$

to equation (1f).

As the result, we obtain the system that in quasi-linear form has symmetric coefficient matrices and which is equivalent to (1a)–(1f), as well as (3), is fulfilled

$$\begin{gathered}
\frac{\partial L_{q_\gamma}}{\partial t} + \frac{\partial (u_k L)_{q_\gamma}}{\partial x_k} = 0,\\
\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} - L_{r_{k\alpha}}\frac{\partial r_{i\alpha}}{\partial x_k} - L_{d_k}\frac{\partial d_i}{\partial x_k} - L_{b_k}\frac{\partial b_i}{\partial x_k} + L_{j_i}\frac{\partial j_k}{\partial x_k} - L_{j_\alpha}\frac{\partial j_\alpha}{\partial x_k} = 0,\\
\frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial (u_k L)_{r_{ij}}}{\partial x_k} - L_{r_{kl}}\frac{\partial u_i}{\partial x_k} = 0,\\
\frac{\partial L_{d_i}}{\partial t} + \frac{\partial (u_k L)_{d_i}}{\partial x_k} - L_{d_k}\frac{\partial u_i}{\partial x_k} - e_{ikl}\frac{\partial b_l}{\partial x_k} = 0,\\
\frac{\partial L_{b_i}}{\partial t} + \frac{\partial (u_k L)_{b_i}}{\partial x_k} - L_{b_k}\frac{\partial u_i}{\partial x_k} + e_{ikl}\frac{\partial d_l}{\partial x_k} = 0,\\
\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L)_n}{\partial x_k} + \frac{\partial j_k}{\partial x_k} = 0,\\
\frac{\partial L_{j_l}}{\partial t} + \frac{\partial (u_k L)_{j_l}}{\partial x_k} + L_{j_\alpha}\frac{\partial u_\alpha}{\partial x_l} + L_{j_l}\frac{\partial u_k}{\partial x_k} + \frac{\partial n}{\partial x_l} = 0.
\end{gathered} \tag{4}$$

If the matrix of the second derivative of L is positively defined with the respect to all arguments, then system (4) is symmetric hyperbolic.

As an example of equations from the class (1) in [1], there are considered nonlinear elasticity theory equations that fit in adduced formal scheme, if one selects equations (1a)–(1c) from system (1), take the potential $L = \rho^2 E_\rho = -E_V$, and consider as the variables $q_0 = E - VE_V - SE_S - c_{ij}E_{c_{ij}} - (1/2)u_i u_i$, $q_\omega = E_S$, u_i, $r_{ij} = E_{c_{ij}}$, where $E(V, S, c_{11}, \dots, c_{33})$ is the specific internal energy, $\rho = 1/V = \rho_0 \det(c_{ij})$ is the mass density, ρ_0 is the medium density in unstressed state, S is the entropy, c_{ij} is the distortion tensor (gradient deformation tensor) that is expressed through the Jacobi matrix of the transformation from Euler x_i to Lagrange ξ_i coordinates: $c_{ij} = \frac{\partial x_i}{\partial \xi_j}$.

In [1], it also discussed, the problem of possibility to choose convex potential L in elasticity theory and related arbitrariness that is not changing smooth solutions of system (1).

The following sections contain a number of examples of the continuum mechanics equations, getting into equations class described in this section.

3. EQUATIONS OF ELECTRODYNAMICS OF MOVING DIELECTRICS

In this section, we shall show how to reduce the equations of electrodynamics of slowly moving dielectrics to the scheme described in preceding section.

As it is known (see [11]), Maxwell equations in medium in which movement is defined by the velocity vector $\mathbf{u}$ have the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\mathbf{H}\right) - \operatorname{rot}\left(\mathbf{H} + \frac{1}{c}\mathbf{D}\times\mathbf{u}\right) = 0, \\ &\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{B} + \frac{1}{c^2}\mathbf{E}\times\mathbf{u}\right) + \operatorname{rot}\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) = 0, \end{aligned} \tag{5}$$

where c is the velocity of light in vacuum, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic induction vectors, electric and magnetic fields strength vectors connected by the formulas

$$\begin{aligned} &\frac{1}{c}\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\mathbf{H} = \varepsilon\left(\frac{1}{c}\mathbf{E} + \frac{1}{c^2}\mathbf{u}\times\mathbf{B}\right), \\ &\frac{1}{c}\mathbf{B} + \frac{1}{c^2}\mathbf{E}\times\mathbf{u} = \mu\left(\frac{1}{c}\mathbf{H} + \frac{1}{c^2}\mathbf{D}\times\mathbf{u}\right), \end{aligned} \tag{6}$$

where ε, μ are the dielectric constant and magnetic permeability, respectively. In the case of low velocity in comparison with velocity of light $|\mathbf{u}|/c \ll 1$, equations (5) and formulas (6) can be replaced by the equations

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\mathbf{H}\right) - \operatorname{rot}\left[\mathbf{H} + \left(\frac{1}{c}\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\mathbf{H}\right)\times\mathbf{u}\right] = 0, \\ &\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{B} + \frac{1}{c^2}\mathbf{E}\times\mathbf{u}\right) + \operatorname{rot}\left[\mathbf{E} + \mathbf{u}\times\left(\frac{1}{c}\mathbf{B} + \frac{1}{c^2}\mathbf{E}\times\mathbf{u}\right)\right] = 0, \end{aligned} \tag{7}$$

and relationships that close equations (7)

$$\begin{aligned} &\frac{1}{c}\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\mathbf{H} = \frac{\varepsilon}{c}\left(\mathbf{E} + \frac{\mu}{c}\mathbf{u}\times\mathbf{H}\right), \\ &\frac{1}{c}\mathbf{B} + \frac{1}{c^2}\mathbf{E}\times\mathbf{u} = \frac{\mu}{c}\left(\mathbf{H} + \frac{\varepsilon}{c}\mathbf{E}\times\mathbf{u}\right). \end{aligned} \tag{8}$$

Namely, equations (7) and (8) will be researched below.

Assuming, for the present, that the velocity vector $\mathbf{u}$ is constant, we can represent equations (7),(8) as the subsystem of the formal system (1). We shall denote by d_i and b_i the components of the vectors $\mathbf{E}$ and $\mathbf{H}$, respectively, and introduce the potential L by the formula

$$L = \frac{\varepsilon}{c}\frac{d_i d_i}{2} + \frac{\mu}{c}\frac{b_i b_i}{2} - \frac{\varepsilon\mu}{c^2}\begin{bmatrix} u_1 & d_1 & b_1 \\ u_2 & d_2 & b_2 \\ u_3 & d_3 & b_3 \end{bmatrix}. \tag{9}$$

In addition,

$$\begin{aligned} L_{d_i} &= \frac{\varepsilon}{c}\, d_i - \frac{\varepsilon\mu}{c^2}\, e_{ijk} u_j b_k, \\ L_{b_i} &= \frac{\mu}{c}\, b_i + \frac{\varepsilon\mu}{c^2}\, e_{ijk} u_j\, d_k. \end{aligned} \tag{10}$$

Now, taking into account (8),(10), we can represent equations (7) in required form

$$\begin{aligned} \frac{\partial L_{d_i}}{\partial t} + \frac{\partial\left[u_k L_{d_i} - u_i L_{d_k} - e_{ikl} b_l\right]}{\partial x_k} &= 0, \\ \frac{\partial L_{b_i}}{\partial t} + \frac{\partial\left[u_k L_{b_i} - u_i L_{b_k} + e_{ikl} d_l\right]}{\partial x_k} &= 0. \end{aligned} \tag{11}$$

In an anisotropic dielectric case, when dielectric constants and magnetic permabilities are forming tensors ε_{ij}, μ_{ij}, formulas (6) have to be replaced by

$$\begin{aligned} \frac{1}{c} D_i + \frac{1}{c^2}\left[\mathbf{u}\times\mathbf{H}\right]_i &= \varepsilon_{ij}\left(\frac{1}{c}\, E_j + \frac{1}{c^2}\left[\mathbf{u}\times\mathbf{B}\right]_j\right), \\ \frac{1}{c} B_i + \frac{1}{c^2}\left[\mathbf{E}\times\mathbf{u}\right]_i &= \mu_{ij}\left(\frac{1}{c}\, H_j + \frac{1}{c^2}\left[\mathbf{D}\times\mathbf{u}\right]_j\right). \end{aligned}$$

Further, as in an isotropic case, by eliminating the terms of higher multiplicity of a smallness, we come to the relationships

$$\begin{aligned} \frac{1}{c} D_i + \frac{1}{c^2}\left[\mathbf{u}\times\mathbf{H}\right]_i &= \varepsilon_{ij}\left(\frac{1}{c}\, E_j + \frac{1}{c^2}\, e_{jmn} u_m \mu_{nk} H_k\right), \\ \frac{1}{c} B_i + \frac{1}{c^2}\left[\mathbf{E}\times\mathbf{u}\right]_i &= \mu_{ij}\left(\frac{1}{c}\, H_j + \frac{1}{c^2}\, e_{jmn} \varepsilon_{mk} E_k u_n\right). \end{aligned} \tag{12}$$

Now, defining a generating function L in the form

$$L = \frac{1}{2c}\,\varepsilon_{ij}\, d_i\, d_j + \frac{1}{2c}\,\mu_{ij} b_i b_j - \frac{1}{c^2}\begin{vmatrix} u_1 & \varepsilon_{1j} d_j & \mu_{1j} b_j \\ u_2 & \varepsilon_{2j} d_j & \mu_{2j} b_j \\ u_3 & \varepsilon_{3j} d_j & \mu_{3j} b_j \end{vmatrix}, \tag{13}$$

we can reproduce equations (7) in collection with the relationships (12) in the form of system (11).

Based on this construction, we can take into consideration the elastic strain of moving dielectric medium also, and as well the alteration with time of its velocities field. For this, we have to use the sum of generating potentials $L = L^d + L^e$ in the capacity of generating potential L, where

$$L^d\left(u_1, u_2, u_3, q_0, q_\omega, q_1, q_2, \ldots, r_{11}, \ldots, r_{33}\right)$$

coincides with generating potential of the elastic medium mentioned in previous section, and in the capacity of L^e to use formula (13) . In this case, we can consider that ε_{ij}, μ_{ij} are functions of the parameters $r_{ij}, q_\omega, q_0, q_\omega, q_1, q_2, \ldots$, describing strain, thermal state, and chemical structure. The complete equation system of elastic dielectric movement in electromagnetic field consists of equations (1a)–(1e).

4. EQUATIONS OF SUPERFLUID HELIUM

Now we shall show how to formalize the equations of superfluid helium movement, described for example in [12], according to the scheme from Section 2. Later, we shall also consider the variant of this equation offered in [3]. Symmetrization of superfluid helium equations from [12], by the method described, for example in [6], was made in [14].

The state of the element of superfluid medium at every time is characterized by the mass density ρ, the pressure p, the temperature T, and the entropy S. Besides we have to include the components of the velocity vectors u_i, u_i^s of normal and superfluid components and also the density ρ_s of superfluid component into the number of state parameters.

In the capacity of thermodynamic potential, we choose chemical potential

$$\mu = \mu\left(p, T, \frac{(u_i^s - u_i)(u_i^s - u_i)}{2}\right),$$

satisfying thermodynamic identity

$$d\mu = -S\,dT + \frac{1}{\rho}\,dp - \left(1 - \frac{\rho_s}{\rho}\right) d\left[\frac{(u_i^s - u_i)(u_i^s - u_i)}{2}\right]. \tag{14}$$

Equations of superfluid helium movement are contained in the last chapter of [12]. We use them only changing the form of notation

$$\begin{gathered}
\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0,\\
\frac{\partial \rho}{\partial t} + \frac{\partial\left[\rho u_k + \rho_s\left(u_k^s - u_k\right)\right]}{\partial x_k} = 0,\\
\frac{\partial\left[\rho u_k + \rho_s\left(u_k^s - u_k\right)\right]}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0,\\
\frac{\partial u_k^s}{\partial t} + \frac{\partial\left[u_\alpha u_\alpha^s + (\mu + (1/2)\left(u_\alpha^s - u_\alpha\right)\left(u_\alpha^s - u_\alpha\right) - (1/2) u_\alpha u_\alpha)\right]}{\partial x_k} = 0.
\end{gathered} \tag{15}$$

Flux tensor Π_{ik} is defined by the following formula:

$$\Pi_{ik} = \delta_{ik} p + u_k\left[\rho u_i + \rho_s\left(u_i^s - u_i\right)\right] + u_i^s \rho^s\left(u_k^s - u_k\right). \tag{16}$$

In the capacity of variables defining the medium state, we choose

$$q = T,\ u_i, \qquad n = \mu + \frac{1}{2}\left(u_\alpha^s - u_\alpha\right)\left(u_\alpha^s - u_\alpha\right) - \frac{1}{2} u_\alpha u_\alpha, \qquad j_k = \rho_s\left(u_k^s - u_k\right). \tag{17}$$

If now, we choose potential

$$L = p + \rho_s u_\alpha^s\left(u_\alpha^s - u_\alpha\right) \tag{18}$$

as the function of state parameters (17), then with help of formula (14) we can determine that

$$dL = \rho S\,dq + \left[\rho u_i + \rho_s\left(u_i^j - u_i\right)\right] du_i + \rho\,dn + u_k^s\,dj_k, \tag{19}$$

and hence,

$$L_q = \rho S, \qquad L_{u_i} = \rho u_i + \rho_s\left(u_i^s - u_i\right), \qquad L_n = \rho, \qquad L_{j_k} = u_k^s. \tag{20}$$

Equalities (17)–(20) make it possible to write equations (15) in the form of a subsystem, consisting of equations (1a), (1b), (1f), (1g) of system (1).

$$\begin{gathered}
\frac{\partial L_q}{\partial t} + \frac{\partial (u_k L)_q}{\partial x_k} = 0,\\
\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\left[(u_k L)_{u_i} + j_k L_{j_i} - \delta_{ik}\left(j_\alpha L_{j_\alpha}\right)\right]}{\partial x_k} = 0,\\
\frac{\partial L_n}{\partial t} + \frac{\partial\left[(u_k L)_n + j_k\right]}{\partial x_k} = 0,\\
\frac{\partial L_{j_k}}{\partial t} + \frac{\partial\left(u_\alpha L_{j_\alpha} + n\right)}{\partial x_k} = 0.
\end{gathered} \tag{21}$$

Assumption (14) about independent state variables, defining thermodynamic potential, leads to the fact that the value of the mass density ρ_s of superfluid component can be defined if p, T, u_i, u_i^s are determined. In [13], it is considered a variant of the superfluid helium model in which ρ_s is an independent parameter. But it is more convenient to choose the chemical potential μ_s of superfluid component as independent parameter than ρ_s. In this case, instead of the last equations of system (15) for velocities of superfluid component, we have to take the equation

$$\frac{\partial u_k^s}{\partial t} + \frac{\partial \left[u_\alpha u_\alpha^s + (\mu + \mu_s + (1/2)(u_\alpha^s - u_\alpha)(u_\alpha^s - u_\alpha) - (1/2)\, u_\alpha u_\alpha)\right]}{\partial x_k} = 0; \tag{22}$$

besides, it is necessary to add one more evolutional equation for ρ_s

$$\frac{\partial \rho_s}{\partial t} + \frac{\partial \left[u_k \rho_s + \rho_s (u_k^s - u_k)\right]}{\partial x_k} = 0. \tag{23}$$

Instead of the thermodynamic identity (14), we have to use its generalization

$$d\mu + \frac{\rho_s}{\rho}\, d\mu_s = -S\, dT + \frac{1}{\rho}\, dp - \left(1 - \frac{\rho_s}{\rho}\right) d\left[\frac{(u_i^s - u_i)(u_i^s - u_i)}{2}\right]. \tag{24}$$

Schematization of the equation in the considered model leads to a system somewhat more complicated than (21):

$$\begin{gathered}
\frac{\partial L_q}{\partial t} + \frac{\partial (u_k L)_q}{\partial x_k} = 0, \\
\frac{\partial L_{u_l}}{\partial t} + \frac{\partial \left[(u_k L)_{u_l} + j_l L_{j_l} - \delta_{lk} \left(j_\alpha L_{j_\alpha}\right)\right]}{\partial x_k} = 0, \\
\frac{\partial L_{n_1}}{\partial t} + \frac{\partial \left[u_k L_{n_1} + j_k\right]}{\partial x_k} = 0, \\
\frac{\partial L_{n_2}}{\partial t} + \frac{\partial \left[u_k L_{n_2} + j_k\right]}{\partial x_k} = 0, \\
\frac{\partial L_{j_k}}{\partial t} + \frac{\partial \left[u_i L_{j_i} + n_1 + n_2\right]}{\partial x_k} = 0.
\end{gathered} \tag{25}$$

The difference between this system and (21) is that this system contains two unknown scalar functions n_1, n_2 whose equations have the same form. Besides, in flux contained in the equation for vector $\mathbf{j}$ the sum $n_1 + n_2$ is situated. To obtain this formal writing, it is necessary to choose generating function

$$L = p + \rho_s \mu_s + \rho_s u_\alpha^s (u_\alpha^s - u_\alpha),$$

and independent variables

$$\begin{gathered}
q = T, \qquad n_1 = \mu + \frac{1}{2}(u_\alpha^s - u_\alpha)(u_\alpha^s - u_\alpha) - \frac{1}{2} u_\alpha u_\alpha, \qquad n_2 = \mu_s, \\
j_l = \rho_s (u_l^s - u_l), \qquad u_1, u_2, u_3.
\end{gathered}$$

In this case, with help of thermodynamic identity (24), we can determine that

$$\begin{gathered}
L_q = \rho S, \qquad L_{u_l} = \rho u_l + \rho_s (u_l^s - u_l), \\
L_{n_1} = \rho, \qquad L_{n_2} = \rho_s, \qquad L_{j_l} = u_l^s.
\end{gathered}$$

By direct substitution of these formulas into (25), we can verify that they coincide with the system composed of (22),(23) and first three equations of system (15).

5. TWO-VELOCITY CONTINUUM DYNAMYCS EQUATIONS

Let us consider now the equations of the so-called two-velocity media. Such media are locally two-phase, but their description is being reduced to the description of the medium with homogeneous element stress (pressure) and temperature, but the velocities of the movement of phases are different.

The approach of phenomenological thermodynamics for modeling such media, applied by Landau while creating superfluid helium model (described in preceding section), was used for development and analysis of the group of models of two-velocity media in works [15–17]. In particular, there were created the models of fluid flowing through the elastic skeleton. The shortcomings of this models are nondivergence of the arising differential equations and missing of the direct proof of hyperbolicity. The variant of the model proposed below is free from mentioned drawbacks.

Let us start from the consideration of the formal description of model's equation. Let us suppose that the density of internal energy of medium is defined in the form of functional relation

$$E = E(\rho, S, \theta, w_i), \tag{26}$$

where ρ is the mass density, S is the entropy. Physical sense of the parameters θ and w_i will become clear below. We shall assume that the medium does not undergo resistance to the shear strains, and the possibility of including the shear strains in the model we shall discuss below.

Let us suppose that the medium movement is defined by the field of velocity u_i and that mass, momentum, and entropy conservation laws are satisfied in the form

$$\begin{gathered} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \\ \frac{\partial \rho u_i}{\partial t} + \frac{(\partial \rho u_i u_k + \delta_{ik} p + \rho w_i E_{w_k})}{\partial x_k} = 0, \\ \frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0, \end{gathered} \tag{27}$$

where $p = \rho^2 E_\rho$.

Let the medium parameters θ and w_k evolution to be described by the equations

$$\begin{gathered} \frac{\partial \rho\theta}{\partial t} + \frac{\partial\left(\rho\theta u_k + \rho E_{w_k}\right)}{\partial x_k} = 0, \\ \frac{\partial w_k}{\partial t} + \frac{(\partial u_\alpha w_\alpha + E_\theta)}{\partial x_k} = 0. \end{gathered} \tag{28}$$

Equation systems (27),(28) that are closed by formula (26) have an additional conservation law—the energy conservation law:

$$\frac{\partial \rho\left(E + (1/2) u_\alpha u_\alpha\right)}{\partial t} + \frac{\partial\left[\rho u_k\left(E + (1/2) u_\alpha u_\alpha\right) + u_k p + u_i \rho w_i E_{w_k} + \rho E_\theta E_{w_k}\right]}{\partial x_k} = 0. \tag{29}$$

To be sure of this we have to sum equations (27),(28) multiplied, respectively, by

$$q_0 = E + \rho E_\rho - S E_S - \theta E_\theta - \frac{1}{2} u_\alpha u_\alpha, \qquad u_i,\ q_\omega = E_S, \qquad n = E_\theta, \qquad j_k = \rho E_{w_k}, \tag{30}$$

and to add the equality

$$\rho E_{w_k} u_\alpha \left(\frac{\partial w_k}{\partial x_\alpha} - \frac{\partial w_\alpha}{\partial x_k}\right) = 0,$$

that is the consequence of the equation for w_k of system (28).

Choosing further generating potential in the form

$$L = \rho^2 E_\rho + \rho w_k E_{w_k}, \tag{31}$$

we can write out systems (27),(28) that are closed by the state equation (26) in schematic form from Section 2 with help of equations (1a), (1b), (1f), (1g) of system (1).

Let us fix one's attention on the possibility of including of the low terms into equations as they are the sources that are required for the description of irreversible processes taking place into the medium.

For description of interphase friction processes in two-velocity medium model, we have to introduce the relaxation term into the equations for w_k of system (28). Let us choose it in the simplest way

$$\frac{\partial w_k}{\partial t} + \frac{\partial (u_\alpha w_\alpha + E_\theta)}{\partial x_k} = -\frac{w_k}{\tau}. \tag{32}$$

Here $\tau =$ const > 0 is the relaxation time for the components of the vector w_k. We have fixed on such a definition of relaxation only because it remains the presence of integral of system (28)

$$\frac{\partial w_k}{\partial x_j} - \frac{\partial w_j}{\partial x_k} = 0.$$

Naturally, we can define the relaxation w_k using more complicated methods considering, for example, that relaxation time depends on the medium state parameters. However, this leads to necessity of more detailed analysis of the symmetrizableness of the equations, namely, consideration of the extended system, including equations for the unknown functions derivatives. The problem of this type was discussed in [18] while researching the symmetrization problem for the relaxation equations of nonlinear Maxwell model of inelastic medium.

If governing equation for w_k is chosen in the form (32), then we have to include the source-generation of the entropy into equation for the entropy of system (27)

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = \frac{\rho w_k E_{w_k}}{E_S \tau}.$$

It is necessary for the energy conservation law (29) to remain satisfied. We have to note that state equation have to be chosen in such a way that the entropy does not decrease, i.e., it is necessary that inequality ($\rho > 0$, $E_S = T > 0$, $\tau > 0$)

$$w_k E_{w_k} > 0$$

is fulfilled. Thus, the formal writing of the system with sources that uses potential (31) and variables (30) is the following:

$$\begin{gathered}
\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0, \\
\frac{\partial L_{q_\omega}}{\partial t} + \frac{\partial (u_k L)_{q_\omega}}{\partial x_k} = \frac{j_k L_{j_k}}{q_\omega \tau}, \\
\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left[(u_k L)_{u_i} + j_k L_{j_i} - \delta_{ik} \left(j_\alpha L_{j_\alpha}\right)\right]}{\partial x_k} = 0, \\
\frac{\partial L_n}{\partial t} + \frac{\partial \left[(u_k L)_n + j_k\right]}{\partial x_k} = 0, \\
\frac{\partial L_{j_k}}{\partial t} + \frac{\partial \left(u_\alpha L_{j_\alpha} + n\right)}{\partial x_k} = -\frac{L_{j_k}}{\tau}.
\end{gathered} \tag{33}$$

In this case, the integrals of the system—additional conservation laws take place

$$\frac{\partial L_{j_k}}{\partial x_l} - \frac{\partial L_{j_l}}{\partial x_k} = 0,$$

if they take place in $t = 0$.

Let us consider now the physical example of two-velocity medium which equations fit in the formal scheme (33). Let us assume that the medium consists of two phases with the mass densities ρ_1, ρ_2 and the velocities $u_i^{(1)}$, $u_i^{(2)}$. Let us take as the parameters of the state of two-phase medium $\rho = \rho_1 + \rho_2$ is the density of the mixture, $u_i = (\rho_1/\rho)u_i^{(1)} + (\rho_2/\rho)u_i^{(2)}$ are the average velocities, $\theta = (\rho_2/\rho)$ is the concentration of the second phase, $w_i = u_i^{(2)} - u_i^{(1)}$ the vector of the difference between phase velocities.

If now we define the specific internal energy by the formula

$$E(\rho, S, \theta, w_i) = E^0(\rho, S, \theta) + \frac{1}{2}\theta(1-\theta)w_i w_i, \tag{34}$$

then system (33) is represented in the form

$$\begin{gathered}
\frac{\partial(\rho_1+\rho_2)}{\partial t} + \frac{\partial\left(\rho_1 u_k^{(1)} + \rho_2 u_k^{(2)}\right)}{\partial x_k} = 0, \\
\frac{\partial\left(\rho_1 u_i^{(1)} + \rho_2 u_i^{(2)}\right)}{\partial t} + \frac{\partial\left(\rho_1 u_i^{(1)} u_k^{(1)} + \rho_2 u_i^{(2)} u_k^{(2)} + \delta_{ik}\rho^2 E_\rho^0\right)}{\partial x_k} = 0, \\
\frac{\partial \rho S}{\partial t} + \frac{\partial \rho u_k S}{\partial x_k} = \frac{\rho_1\rho_2\left(u_i^{(2)} - u_i^{(1)}\right)\left(u_i^{(2)} - u_i^{(1)}\right)}{T\tau}, \\
\frac{\partial \rho_2}{\partial t} + \frac{\partial \rho_2 u_k^{(2)}}{\partial x_k} = 0, \\
\frac{\partial\left(u_k^{(2)} - u_k^{(1)}\right)}{\partial t} + \frac{\partial\left(u_i^{(2)} u_i^{(2)} - u_i^{(1)} u_i^{(1)} + E_\theta^0\right)}{\partial x_k} = -\frac{u_k^{(2)} - u_k^{(1)}}{\tau}.
\end{gathered} \tag{35}$$

The system of equations given describes the two-velocity medium in case when its state is described by the spherical stress tensor-pressure. Let us consider the possible way of generalization of such two-velocity model in case when the medium undergoes the shear strains. This way is in selection of state equation in the form generalizing (26)

$$E = E^0(\rho, S, \theta, c_{11}, \dots, c_{33}) + \frac{1}{2}\theta(1-\theta)w_i w_i,$$

where ρ, S, θ, w_i have the same sense as in system (35), and c_{ij} is the distortion tensor, satisfying to the equation (1c)

$$\frac{\partial \rho c_{ij}}{\partial t} + \frac{\partial\left(\rho c_{ij} u_k - \rho c_{kj} u_i\right)}{\partial x_k} = 0.$$

Selecting potential L by formula (31), we obtain the closed equations system for the movement of two-velocity medium that can be used for research of fluid movement through the elastic skeleton.

6. THE EQUATIONS OF HEAT TRANSFER WITH FINITE VELOCITY

Let us consider now the way to write the hyperbolic heat conductivity equations, modeling the processes of heat transfer with finite velocity, in our schematization. Highly vast number of the literature is devoted to the research of this subject. Let us consider only some of this works in which while formulating the model was used the approach of phenomenological thermodynamics [19–21]. This works appeared simultaneously and they are independent from each other, we have only note that equations from [19,20] are not the conservation laws. Further in [22,23] it was carried on an attempt to include the processes of the hyperbolic heat conductivity into the equations of the movement of deformable media. In [23], the modification of relations from [19] is adduced, taking into consideration the invariance with respect to a change of coordinates.

The formal scheme that is under investigation allows us to obtain all the equations possessing the useful properties of invariance. Let us consider the first medium which deformations are described only by a density variation. Let us use equation (28) as constitutive relations, in which we shall take the entropy S of the medium as the parameter θ. In this case, the state equation is given in the form

$$E = E(\rho, S, w_1, w_2, w_3), \tag{36}$$

where ρ is the mass density, S is the entropy, w_i is the heat flow velocity vector. The simplest choice of the dependence E of w_i is the following:

$$E = E^0(\rho, S) + \frac{1}{2}\alpha_{ij} w_i w_j, \tag{37}$$

where $E^0(\rho, S)$ is the specific internal elastic energy, α_{ij} are the parameters that can depend on medium state—the pressure and the temperature. The complete equations system can be represented by the analogy with the equations of the preceding section:

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial(\rho u_i u_k + \delta_{ik} p + \rho w_i E_{w_k})}{\partial x_k} = 0, \\
\frac{\partial \rho S}{\partial t} + \frac{\partial(\rho S u_k + \rho E_{w_k})}{\partial x_k} = \frac{\rho w_k E_{w_k}}{E_S \tau}, \\
\frac{\partial w_k}{\partial t} + \frac{\partial(u_\alpha w_\alpha + T)}{\partial x_k} = -\frac{w_k}{\tau}.
\end{gathered} \tag{38}$$

Here $p = \rho^2 E_\rho$ is the pressure, $T = E_S$ is the temperature. The first equation of (38) expresses the mass conservation law, the second, the momentum conservation law, the third, the entropy balance law, the fourth, constitutive formula, generalizing Fourier law for heat transfer. In system (38), we suppose again $\tau =$ const for simplicity because in this case as in the case of preceding section conserves the integral of the system

$$\frac{\partial w_k}{\partial x_j} - \frac{\partial w_j}{\partial x_k} = 0.$$

The presence of this additional integral provides the symmetrization of system (38) according to the scheme from Section 2. More complicated forms of the relaxation terms, as it was mentioned in the preceding section lead to a necessity to use the widened system for symmetrization of the equations.

For system (38), it is a valid additional conservation law—the energy conservation law:

$$\frac{\partial \rho\left(E + (1/2)u_\alpha u_\alpha\right)}{\partial t} + \frac{\partial\left[\rho u_k\left(E + (1/2)u_\alpha u_\alpha\right) + u_k p + u_i \rho w_i E_{w_k} + \rho T E_{w_k}\right]}{\partial x_k} = 0. \tag{39}$$

System (38), by the choice of the generating potential

$$L = \rho^2 E_\rho + \rho w_k E_{w_k} = \rho^2 E^0_\rho + \rho \alpha_{ij} w_i w_j,$$

and the variables

$$q_0 = E + \rho E_\rho - S E_S - \frac{1}{2} u_\alpha u_\alpha, \qquad u_i,\ n = E_S, \qquad j_k = \rho E_{w_k}$$

can be represented in the schematic form with help of equations (1a), (1b), (1f), (1g) of system (1).

It is an easy matter to check that the last two equations of system (38) in case of an immovable nondeformable medium coincide with the equations of the hyperbolic heat transfer model from [21]. Really, they shall have the form

$$\frac{\partial S}{\partial t}+\frac{\partial E_{w_k}}{\partial x_k}=\frac{w_k E_{w_k}}{T\tau},$$
$$\frac{\partial w_k}{\partial t}+\frac{\partial T}{\partial x_k}=-\frac{w_k}{\tau},$$

and they are closed by the special variant of the state equation (37)

$$E=E^0(S)+\frac{1}{2}\alpha_{ij}w_i w_j.$$

It is possible to show the invariance of (38) with respect to the change of coordinates. For this, we have to replace the equation for w_k in the system on equivalent equation

$$\frac{\partial w_k}{\partial t}+u_\alpha\frac{\partial w_k}{\partial x_\alpha}+w_\alpha\frac{\partial u_\alpha}{\partial x_k}+\frac{\partial T}{\partial x_k}=-\frac{w_k}{\tau},$$

obtained from the original with help of the integral of the system

$$\frac{\partial w_k}{\partial x_j}-\frac{\partial w_j}{\partial x_k}=0.$$

Let us note that defining the dependence of the potential L of distortion tensor as it is said in Section 2, we can obtain the equations of the heat transfer with the finite velocity in the elastic medium.

7. EQUATIONS OF PHENOMENOLOGICAL THEORY OF SUPERCONDUCTING DEFORMABLE MEDIA

Using the experience of preceding section where we considered equations of different media, taking into consideration flowing of fields of different physical sence through the element of the medium and also the equations of electromagnetic processes, we can formulate the equations of superconducting media movement. These equations after some modification can be used also for the analysis of processes in media with ordinary conductivity.

As a foundation, we shall take London's classical model [24] of phenomenological theory of superconductivity, which equations except Maxwell equations contain also equations of superconducting electrons movement:

$$\begin{aligned}&\frac{\partial \rho_e}{\partial t}+\frac{\partial \rho_e v_k}{\partial x_k}=0,\\ &\frac{\partial v_k}{\partial t}+\frac{\partial\left((1/2)v_\alpha v_\alpha+\mu\right)}{\partial x_k}=\frac{e}{m}d_k,\\ &\frac{\partial \mathbf{D}}{\partial t}-\operatorname{rot}\mathbf{b}=-\mathbf{J},\\ &\frac{\partial \mathbf{B}}{\partial t}+\operatorname{rot}\mathbf{d}=0.\end{aligned}\tag{40}$$

Here ρ_e is the density of electrons, v_k is their velocity, μ is the chemical potential of an electron gas, e and m are the charge and the mass of an electron, $\mathbf{D}$, $\mathbf{d}$ are the vectors of the electric field displacement and strength, $\mathbf{B}$, $\mathbf{b}$ are the vectors of the magnetic field displacement and strength,

$$J_k=\frac{e}{m}\rho_e v_k\tag{41}$$

is the density of the electric current.

Similarly with the superfluid helium model (see Section 3), it follows from system (40) that the electron velocity vector possess the property that for $t > 0$ the equality

$$\operatorname{rot} \mathbf{v} = -\frac{e}{m} \mathbf{B} \tag{42}$$

is valid if it takes place for $t = 0$. Besides from system (40) with the help of (41), it follows the equality

$$\operatorname{div} \mathbf{D} = \frac{e}{m} \rho_e, \tag{43}$$

that is valid for $t > 0$ if it is for $t = 0$.

It is easy to show that system (40)–(43) fits in the schematization of Section 2.

We shall turn our attention in more detailed way to the generalization of the model in the case of a moving elastic medium. Let us suppose that the medium state is completely described by the velocities field u_i, the mass density ρ, the elastic distortion tensor c_{ij}, the fields of the electric and magnetic displacement $\mathbf{D}$, $\mathbf{B}$, the electron density ρ_e, and the relative velocity of the electron movement w_i.

Let us suppose that the medium state equation is given in the form

$$E = E^0 (\rho, c_{ij}, S, \rho_e) + \frac{\rho_e}{\rho} \frac{w_i w_i}{2}. \tag{44}$$

Then we consider that the medium is isotropic, and hence, the electric and magnetic displacement vectors are connected with the vectors of the electric and magnetic field strength by the formula

$$\mathbf{B} = \frac{\mu}{c} \left(\mathbf{b} + \frac{\varepsilon}{c} \mathbf{d} \times \mathbf{u} \right), \qquad \mathbf{D} = \frac{\varepsilon}{c} \left(\mathbf{d} + \frac{\mu}{c} \mathbf{u} \times \mathbf{b} \right). \tag{45}$$

Let us suppose also that the complete density of the electric current is connected with the medium velocity and with the relative velocity of the charge movement by the formula that is similar with (41)

$$J_k = \frac{e}{m} \rho_e (u_k + w_k). \tag{46}$$

The complete system of the equations of the moving elastic superconductor will have the form

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \\
\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0, \\
\frac{\partial \rho u_i}{\partial t} + \frac{\left(\partial \rho u_i u_k + \delta_{ik} \rho^2 E_\rho - \rho c_{ij} E_{c_{kj}} + \rho w_i E_{w_k} \right)}{\partial x_k} = 0, \\
\frac{\partial \Pi_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \\
\frac{\partial \rho c_{ij}}{\partial t} + \frac{\partial \left(\rho c_{ij} u_k - \rho c_{kj} u_i \right)}{\partial x_k} = 0, \\
\frac{\partial \mathbf{D}}{\partial t} - \operatorname{rot} (\mathbf{b} + \mathbf{D} \times \mathbf{u}) = -\mathbf{J}, \\
\frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} (\mathbf{d} + \mathbf{u} \times \mathbf{B}) = 0, \\
\frac{\partial \rho_e}{\partial t} + \frac{\partial \left(\rho_e u_k + \rho_e w_k \right)}{\partial x_k} = 0, \\
\frac{\partial w_k}{\partial t} + \frac{\partial \left(u_\alpha w_\alpha + E_{\rho_e} \right)}{\partial x_k} = \frac{e}{m} (\mathbf{d} + \mathbf{u} \times \mathbf{B})_k ,
\end{gathered} \tag{47}$$

where Π_i is the momentum vector and Π_{ik} is the momentum flux tensor which will be determined further.

The first four equations from system (47) are the mass, entropy, momentum, and elastic distortion conservation laws. The next pair of the equations are Maxwell electromagnetic field equations and the last pair of the equations are the electric charge and electric current conservation laws.

Similarly with system (40), equations (47) have the consequences—the additional conservation laws

$$\begin{aligned}\operatorname{rot}\mathbf{w} &= -\frac{e}{m}\,\mathbf{B},\\ \operatorname{div}\mathbf{D} &= \frac{e}{m}\,\rho_e,\end{aligned} \tag{48}$$

if they are fulfilled for initial values.

We can verify that if we shall choose the generating potential L in the form

$$L = \rho^2 E_\rho + \rho w_k E_{w_k} + \frac{\varepsilon}{c}\frac{d_i d_i}{2} + \frac{\mu}{c}\frac{b_i b_i}{2} - \frac{\varepsilon\mu}{c^2}\begin{vmatrix} u_1 & d_1 & b_1\\ u_2 & d_2 & b_2\\ u_3 & d_3 & b_3\end{vmatrix},$$

and take as the variables

$$q_0 = E + \rho E_\rho - c_{ij}E_{c_{ij}} - SE_S - \theta E_\theta - \frac{1}{2}u_\alpha u_\alpha, \qquad q_\omega = E_S,$$
$$u_i, r_{ij} = E_{c_{ij}}, \qquad d_i, b_i, n = E_{\rho_e}, \qquad j_k = \rho E_{w_k},$$

then system (47) can be represented as the subsystem of system (1) with the source terms

$$\begin{aligned}
\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} &= 0,\\
\frac{\partial L_{q_\omega}}{\partial t} + \frac{\partial (u_k L)_{q_\omega}}{\partial x_k} &= 0,\\
\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\left[(u_k L)_{u_i} - r_{i\alpha}L_{r_{k\alpha}} - b_i L_{b_k} - d_i L_{d_k} + j_k L_{j_i} - \delta_{ik}(j_\alpha L_{j_\alpha})\right]}{\partial x_k} &= 0,\\
\frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial\left[u_k L_{r_{ij}} - u_i L_{r_{ik}}\right]}{\partial x_k} &= 0,\\
\frac{\partial L_{d_i}}{\partial t} + \frac{\partial[u_k L_{d_i} - u_i L_{d_k} - e_{ikl}b_l]}{\partial x_k} &= -\frac{e}{m}(u_i L_n + j_i),\\
\frac{\partial L_{b_i}}{\partial t} + \frac{\partial[u_k L_{b_i} - u_i L_{b_k} + e_{ikl}d_l]}{\partial x_k} &= 0,\\
\frac{\partial L_n}{\partial t} + \frac{\partial[(u_k L)_n + j_k]}{\partial x_k} &= 0,\\
\frac{\partial L_{j_k}}{\partial t} + \frac{\partial(u_\alpha L_{j_\alpha} + n)}{\partial x_k} &= \frac{e}{m}\left(d_k + e_{k\alpha\beta}u_\alpha L_{b_\beta}\right).
\end{aligned} \tag{49}$$

Equations (48) will have the form

$$\begin{aligned}\frac{\partial L_{d_i}}{\partial x_i} &= \frac{e}{m}\,L_n,\\ \frac{\partial L_{j_l}}{\partial x_k} - \frac{\partial L_{j_k}}{\partial x_l} &= -\frac{e}{m}\,L_{b_i}e_{ilk}.\end{aligned} \tag{50}$$

Besides, the system also has the additional conservation laws

$$\frac{\partial L_{b_i}}{\partial x_i} = 0, \qquad \frac{\partial L_{r_{ki}}}{\partial x_k} = 0. \tag{51}$$

The appearance of the right sides in the equations of the formalized system (49) leads to the appearance of the right sides in the conservation laws (50)—the consequences of system (49).

Nevertheless, this does not create obstacles to obtaining of the additional energy conservation law. For this it is necessary to add to the sum of the equations of system (49) multiplied by the corresponding factors the following equality:

$$r_{ik}u_i \operatorname{div} L_{\mathbf{r}_k} + (\mathbf{u},\mathbf{d})\left(\operatorname{div} L_{\mathbf{d}} - \frac{e}{m} L_n\right) + (\mathbf{u},\mathbf{b}) \operatorname{div} L_{\mathbf{b}} + \left([\mathbf{u}\times\mathbf{j}], \operatorname{rot} L_{\mathbf{j}} + \frac{e}{m} L_{\mathbf{b}}\right) = 0,$$

that is fulfilled in the consequence of the equations (50),(51). Let us also note that the right sides do not make difficulties while the reduction of the system to the symmetric form according to the scheme of the Section 3 because they are the low terms of the equations.

System (47) describes only superconductive media. For the description of the passage to the conductive state, it is necessary to modify the equation for L_{j_k} of system (47). Using that the rotor of the vector L_{j_k} is equal to zero we can transform this equation to the form

$$\frac{\partial L_{j_k}}{\partial t} + u_\alpha \frac{\partial L_{j_k}}{\partial x_k} + \frac{\partial n}{\partial x_k} + L_{j_\alpha} \frac{\partial u_\alpha}{\partial x_k} = \frac{e}{m} d_k.$$

After this we add one more term to the right side

$$\frac{\partial L_{j_k}}{\partial t} + u_\alpha \frac{\partial L_{j_k}}{\partial x_k} + \frac{\partial n}{\partial x_k} + L_{j_\alpha} \frac{\partial u_\alpha}{\partial x_k} = \frac{e}{m} d_k - \frac{1}{\sigma} j_k. \tag{52}$$

In this case for $\sigma \to \infty$, we obtain the equation from (52), while for finite values of σ we can approximately assume that for the small gradients all values and for $t \gg \sigma$ the following equation (Ohm law) takes place:

$$j_l = \frac{e}{m} \sigma\, d_l.$$

Besides, for the energy conservation law not to be disturbed we have to add the right side (Joule heat source) to the entropy equation

$$\frac{\partial L_{q_w}}{\partial t} + \frac{\partial (u_k L_{q_w})}{\partial x_k} = \frac{j_\alpha j_\alpha}{\sigma}. \tag{53}$$

It is necessary to note that the relaxation of the electron velocity that leads to the modified Ohm law as to the defining formula of the superconductivity model was introduced into the nondivergent variant of the equation. It is connected with the fact that the introduction of the relaxation leads to the violation of the equality

$$\operatorname{rot} \mathbf{w} = -\frac{e}{m} \mathbf{B}.$$

That is why the rejection from the divergent form is motivated by the requirement that the energy conservation law would not be violated while using the equations (48). Appearing symmetrization problems of the systems have to be resolved in a similar way to the work of [18], introducing the new variables—the rotors of the vector $\mathbf{w}$ and considering the equations for the derivatives of unknown functions with the original equations. This is the subject for a separate research.

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