# Chapter 7

# KINEMATICS OF AVERAGED FIELDS

## 1.1 Convective coordinates and convective derivatives

The time-mean values are consistently expressed by the spatial description as shown by the definitions (4-15) and (4-16), and the idea of the particle coordinates for the averaged two-phase flow fields is not clear nor trivial due to the phase changes and the diffusions. The phase change corresponds to the production or disappearance of fluid particles for each phase throughout the field. The difficulty arises because each phase itself does not apparently obey the corollary of the axiom of continuity, namely, the permanence of matter. However, the diffusion of each phase permits the penetration of mixture particles by other fluid particles. It is clear that the material coordinates, which is the base of the standard continuum mechanics, is not inherent to a general two-phase flow field obtained from the time averaging. However, it is possible to introduce mathematically special convective coordinates which are useful in studying the kinematics of each phase and of the mixture.

The path line of each phase is defined by the integral curve of the system

$$d\boldsymbol{x} = \widehat{\boldsymbol{v}_k}(\boldsymbol{x}, t)dt \tag{7-1}$$

with the initial condition

$$\boldsymbol{x} = \boldsymbol{X}_k \text{ at } t = t_0 \tag{7-2}$$

where we define $\boldsymbol{X}_k$ as the convective coordinates of the $k^{\text{th}}$-phase. Hence, upon integration of Eq.(7-1), we obtain

$$\boldsymbol{x} = \boldsymbol{x}\left(\boldsymbol{X}_k, t\right) \tag{7-3}$$

This equation gives the path line of the fixed point on the convective coordinates $\boldsymbol{X}_k$, which are moving with the particle velocity $\widehat{\boldsymbol{v}_k}$.

With the standard assumption of smoothness, or the existence of the Jacobian, we can transform Eq.(7-3) to

$$\boldsymbol{X}_k = \boldsymbol{X}_k\left(\boldsymbol{x}, t\right) \tag{7-4}$$

This equation expresses the position of the imaginary particle that moves with the local mean velocity of the $k^{\text{th}}$-phase $\widehat{\boldsymbol{v}_k}$. The formulation of problems in which $\boldsymbol{x}$ and $t$ are taken as independent variables is called the spatial description, whereas if $\boldsymbol{X}_k$ and $t$ are taken as the independent variables, it is called the convective description. In general, the contents of the particles occupying the neighborhood of $\boldsymbol{x} = \boldsymbol{x}\left(\boldsymbol{X}_k, t\right)$ can be different from the initial particles due to phase changes. Thus, it is not possible to consider the change with fixed particles. However, it is simple to observe a process with fixed $\boldsymbol{X}_k$. The velocity of the $k^{\text{th}}$-phase, for example, can be given in analogy with a single-phase flow as

$$\widehat{\boldsymbol{v}_k} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}_k} \quad k = 1 \text{ and } 2. \tag{7-5}$$

The above analysis can also be applied to the mixture center of mass, thus we define the mixture path line by

$$d\boldsymbol{x} = \boldsymbol{v}_m\left(\boldsymbol{x}, t\right) dt \tag{7-6}$$

with $\boldsymbol{x} = \boldsymbol{X}_m$ at $t = t_0$. By integrating Eq.(7-6) we obtain the path line

$$\boldsymbol{x} = \boldsymbol{x}\left(\boldsymbol{X}_m, t\right) \tag{7-7}$$

and with the inverse transformation we get

$$\boldsymbol{X}_m = \boldsymbol{X}_m\left(\boldsymbol{x}, t\right). \tag{7-8}$$

Hence, if the mixture convective coordinates are fixed in Eq.(7-7), the observer moves with the local mixture velocity $\boldsymbol{v}_m$. However, due to the diffusion of each phase with respect to the mass center, the particles at fixed $\boldsymbol{X}_m$ are continually changing along the path line.

Furthermore, it is interesting to note that if the flow field is homogeneous, or $\widehat{\boldsymbol{v}_1} = \widehat{\boldsymbol{v}_2} = \boldsymbol{v}_m$, then the mixture convective coordinates become the material coordinates regardless of the phase changes. From Eqs.(7-6) and (7-7) the mixture velocity can be given symbolically as

$$\boldsymbol{v}_m = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}_m}. \tag{7-9}$$

It is easily seen from Eqs.(7-1) and (7-6) that the path line for each phase and for the mixture can intersect each other.

Since the Eulerian time mean values are in spatial description, the time rate of change at fixed point is denoted by

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}}. \tag{7-10}$$

However, the rate of change seen from the observer moving with the fluid velocity is called the convective or substantial derivative. It is given by

$$\frac{D_k}{Dt} = \left.\frac{\partial}{\partial t}\right)_{\boldsymbol{X}_k} = \frac{\partial}{\partial t} + \widehat{\boldsymbol{v}_k} \cdot \nabla \tag{7-11}$$

and

$$\frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right)_{\boldsymbol{X}_m} = \frac{\partial}{\partial t} + \boldsymbol{v}_m \cdot \nabla . \tag{7-12}$$

The convective derivatives of Eq.(7-11) and Eq.(7-12) are taken by following the center of mass of the $k^{\text{th}}$-phase and that of the mixture, respectively.

If the phase convective derivative is applied to express the left-hand side of the field equation (5-17), we obtain

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}} \widehat{\psi_k}}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \widehat{\psi_k} \widehat{\boldsymbol{v}_k}\right) = \alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{\psi_k}}{Dt} + \Gamma_k \widehat{\psi_k} \tag{7-13}$$

where we have used the continuity equation (5-21). Similarly for the mixture, we get from Eqs.(7-12) and (5-40) the following result

$$\frac{\partial \rho_m \psi_m}{\partial t} + \nabla \cdot \left(\rho_m \psi_m \boldsymbol{v}_m\right) = \rho_m \frac{D\psi_m}{Dt}. \tag{7-14}$$

We note here that the contribution of the mass source term appears in Eq.(7-13), since the amount of mass within a volume having the surface velocity of $\widehat{\boldsymbol{v}_k}$ is not constant. By combining the corollary of the fundamental identity, Eq.(4-111), and above two relations, we have an important transformation between the mixture and phase convective derivatives, thus

$$\rho_m \frac{D\psi_m}{Dt} + \nabla \cdot \mathcal{J}^D = \sum_{k=1}^{2} \left( \alpha_k \overline{\overline{\rho_k}} \frac{D_k \widehat{\psi_k}}{Dt} + \Gamma_k \widehat{\psi_k} \right). \tag{7-15}$$

## 1.2 Streamline

The stagnation point is defined as a point where all velocities vanish, thus

$$\widehat{\boldsymbol{v}_1} = \widehat{\boldsymbol{v}_2} = \boldsymbol{v}_m = 0. \tag{7-16}$$

And the point where $\widehat{\boldsymbol{v}_k}$ is zero for one of the phases is called the $k^{\text{th}}$-phase stagnation point. If the mixture velocity $\boldsymbol{v}_m$ is zero at a point, we call it as a pseudo-stagnation point. At such a point the motions of two phases are pure diffusions. The flow is completely steady if each of the phase velocities is independent of time as

$$\widehat{\boldsymbol{v}_k} = \widehat{\boldsymbol{v}_k}(\boldsymbol{x}) \text{ for both } k = 1 \text{ and } 2. \tag{7-17}$$

The mixture motion is steady if $\boldsymbol{v}_m = \boldsymbol{v}_m(\boldsymbol{x})$, however it does not correspond to the complete steady motion because the diffusion velocities can be a function of time.

The vector line of a vector field is a curve that is everywhere tangent to that vector. In particular, the vector line of the velocity field $\widehat{\boldsymbol{v}_k}$ is called the streamline of the $k^{\text{th}}$-phase. Thus it can be given by an integral curve of the simultaneous equations

$$d\boldsymbol{x} = \widehat{\boldsymbol{v}_k} dl \text{ at } t = t_0 \tag{7-18}$$

where $l$ is a parameter along the streamline. In general, the streamline is a

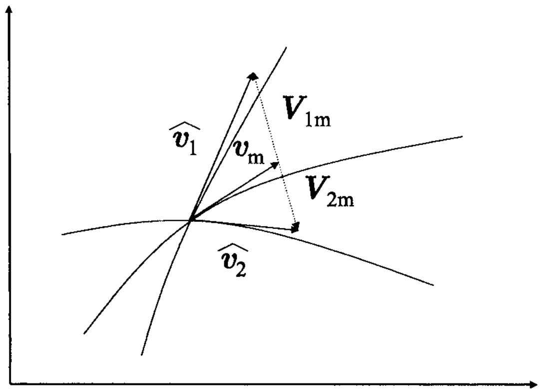


*Figure 7-1.* Streamlines (Ishii, 1975)

function of time and does not coincide with the path line. The two streamlines for each phase are also different because two velocity fields are not parallel. The streamline of the mixture can be defined similarly as

$$d\boldsymbol{x} = \widehat{\boldsymbol{v}_m} dl \quad \text{at } t = t_0. \tag{7-19}$$

The relations between various streamlines are illustrated in Figure 7-1. We also note here that the coincidence of the streamlines of each phase does not signify the homogenous flow field.

## 1.3 Conservation of mass

*Formulation Based on Center of Mass Velocities*

The continuity equations for each phase have been derived in the Section 1.2 of Chapter 5, thus we have

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}}}{\partial t} + \nabla \cdot \left( \alpha_k \overline{\overline{\rho_k}} \widehat{\boldsymbol{v}_k} \right) = \Gamma_k \quad k = 1 \text{ and } 2 \tag{7-20}$$

and the interfacial mass transfer condition is given by

$$\sum_{k=1}^{2} \Gamma_k = 0. \tag{7-21}$$

Equation (7-20) simply states that the local time rate of change of the partial density $\alpha_k \overline{\overline{\rho_k}}$ per unit volume equals the net mass influx of the $k^{\text{th}}$-phase

$-\nabla\cdot\left(\alpha_k\overline{\overline{\rho}}_k\widehat{\boldsymbol{v}}_k\right)$ plus the mass source due to the phase changes. As it has been shown in the Section 1.2 of Chapter 5, by adding these three equations we obtain

$$\frac{\partial\rho_m}{\partial t}+\nabla\cdot\left(\rho_m\boldsymbol{v}_m\right)=0 \tag{7-22}$$

which is the equation of continuity for the mixture.

In order to specify the conservation of mass in two-phase mixtures, it is necessary to employ two continuity equations. We have expressed these relations through the *center of mass velocity of each phase* in Eq.(7-20), however, it is interesting now to consider alternative forms of the continuity equations by introducing different views of observations. If the observer moves with the mixture center of mass, the diffusion terms explicitly appear in the phase continuity equations, thus we have

$$\frac{\partial\alpha_k\overline{\overline{\rho}}_k}{\partial t}+\nabla\cdot\left(\alpha_k\overline{\overline{\rho}}_k\boldsymbol{v}_m\right)=\Gamma_k-\nabla\cdot\left(\alpha_k\overline{\overline{\rho}}_k\boldsymbol{V}_{km}\right). \tag{7-23}$$

*Formulation Based on Mass Fractions*

Instead of using the time fraction $\alpha_k$, we may also express the above equation in terms of the mass fraction $c_k$ defined by Eq.(4-63) as

$$\frac{\partial c_k}{\partial t}+\boldsymbol{v}_m\cdot\nabla c_k=\frac{\Gamma_k}{\rho_m}-\frac{1}{\rho_m}\nabla\cdot\left(c_k\rho_m\boldsymbol{V}_{km}\right) \tag{7-24}$$

in which we have used the mixture continuity equation. Furthermore, the diffusion coefficient $D_k$ may be used to express the diffusion flux in analogy with a heterogeneous single-phase mixture, hence we set

$$c_k\rho_m\boldsymbol{V}_{km}=-\rho_m D_k\nabla c_k. \tag{7-25}$$

We note here that Eq.(7-25) is correct only when the diffusion is due to the concentration gradient and it can be expressed by a linear constitutive law. However, it is expected that Fick's Law of Diffusion may not hold for a general two-phase flow system because the interfacial geometry, the body force field and the interfacial momentum transfer term are the significant factors affecting the diffusion of phases. The linear constitutive law given by Eq.(7-25) is in complete analogy with Newton's Law of Viscosity and with Fourier's Law of Heat Conduction. These linear constitutive laws are applicable for the molecular transport phenomena. However, it has to be remembered that the latter two constitutive equations express the *microscopic* molecular diffusions of momentum and energy, whereas the diffusion of phases in two-phase flow is *macroscopic*.

By considering a very simplified form of Eq.(7-24), it is possible to show that the diffusion equation with the linear constitutive law of Eq.(7-25) exhibits the *diffusive characteristic* of the concentration $c_k$ due to the second order derivative of $c_k$ in the equation. This is in direct contrast with the formulation based on the kinematic wave velocity in the later part of this section, which exhibits the *characteristic of propagations*.

*Formulation Based on Volumetric Flux*

The continuity relations also can be expressed through the volumetric flux $\boldsymbol{j}$ and the drift velocities $\boldsymbol{V}_{kj}$. Thus, from Eqs.(7-20) and (4-91) we have

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}}}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \boldsymbol{j}\right) = \Gamma_k - \nabla \cdot \left(\alpha_k \overline{\overline{\rho_k}} \boldsymbol{V}_{kj}\right). \tag{7-26}$$

The last term on the right-hand side of the above equation represents the drift of $k^{\text{th}}$-phase mass with respect to the mixture volume center. By differentiating by part of the left-hand side of Eq.(7-20), we get

$$\frac{\partial \alpha_k}{\partial t} + \nabla \cdot \left(\alpha_k \widehat{\boldsymbol{v}_k}\right) = \frac{\Gamma_k}{\overline{\overline{\rho_k}}} - \frac{\alpha_k}{\overline{\overline{\rho_k}}} \frac{D_k \overline{\overline{\rho_k}}}{Dt} \tag{7-27}$$

where the substantial derivative is defined by Eq.(7-11). The above equation can be considered as the continuity equation in terms of the time fraction or the void fraction, therefore it represents the volumetric transport. Thus, from the point of view of $\alpha_k$, the continuity equation has a source term due to the mass transfer and a sink term due to the true compressibility of the phase. Furthermore, if we use Eq.(4-87), we obtain

$$\frac{\partial \alpha_k}{\partial t} + \nabla \cdot \boldsymbol{j}_k = \frac{\Gamma_k}{\overline{\overline{\rho_k}}} - \frac{\alpha_k}{\overline{\overline{\rho_k}}} \frac{D_k \overline{\overline{\rho_k}}}{Dt} \tag{7-28}$$

By adding these two equations for each phase, we get

$$\nabla \cdot \boldsymbol{j} = \sum_{k=1}^{2} \left\{ \frac{\Gamma_k}{\overline{\overline{\rho_k}}} - \frac{\alpha_k}{\overline{\overline{\rho_k}}} \frac{D_k \overline{\overline{\rho_k}}}{Dt} \right\} \tag{7-29}$$

which describes the divergence of the center of volume velocity. The first term of the right-hand side is the volume source due to the phase changes and the second term is the volume sink due to the compressibility.

The formulation based on the volumetric fluxes is important if each phase undergoes the incompressible or isochoric process defined by

$$\frac{D_k \overline{\overline{\rho_k}}}{Dt} = 0 \quad k = 1 \text{ and } 2. \tag{7-30}$$

In this case Eq.(7-29) reduces to

$$\nabla \cdot \boldsymbol{j} = \sum_{k=1}^{2} \frac{\Gamma_k}{\overline{\overline{\rho_k}}} \text{ (isochoric)} \tag{7-31}$$

which simply states that the divergence of the volumetric flux is proportional to the amount of phase changes and to the difference between the specific volumes of each phase. We recall that for an incompressible single-phase flow, the divergence of the velocity is zero. Thus, the two-phase flow equivalence is expressed by the velocity of a center of volume instead of that of mass and, furthermore, it has a source due to phase changes. It is seen that in the absence of the mass transfer, the above equation reduces to $\nabla \cdot \boldsymbol{j} = 0$. Since in many practical two-phase flow problems the incompressible fluid assumption is valid and the rate of phase change $\Gamma_k$ can be expressed as a known function of position and time, Eq.(7-31) can play an important role in solving these problems.

*Kinematic Wave Velocity and Void Propagation Equation*

As in the thermomechanical theory of diffusion for single-phase mixtures, it is one of the basic assumptions of drift-flux (or mixture) model formulation that the relative motions between two phases can be expressed by *a constitutive law* rather than by two momentum equations. In this connection we already discussed Fick's Law of Diffusion which effectively eliminates one of the two momentum equations. It is quite clear from Eq.(7-25) that the constitutive law gives special kinematic relation between $\widehat{\boldsymbol{v}_k}$ and $\boldsymbol{v}_m$, thus the $k^{\text{th}}$-phase momentum equation becomes redundant in the presence of the mixture momentum equation.

However, we have noted there that in general, the use of Fick's Law for two-phase mixtures is *not correct* and thus a different type of constitutive laws for diffusion should be used as in the drift-flux (or mixture) model formulation. One of the more useful constitutive laws for the relative motions between phases is to express it in terms of the drift velocity $\boldsymbol{V}_{kj}$ (Zuber et al., 1964, Ishii, 1977, Kataoka and Ishii, 1987, Hibiki and Ishii, 2003a, 2003b; Hibiki et al., 2003, Goda et al., 2003; Hibiki et al., 2004).

In particular if the drift velocity is a function only of the time concentration $\alpha_k$, one of the very important theories in fluid mechanics, namely, the theory of kinematic waves (Kynch, 1952; Lighthill and Whitham, 1955; Hayes, 1970), can be applied to the two-phase flow systems. It was shown by Zuber (1964b) that such was the case for many flow regimes of practical interest and it was particularly useful for a dispersed flow regime.

Under the incompressible fluid assumption, Eq.(7-27) can be expressed by $\boldsymbol{j}$ and $\boldsymbol{V}_{kj}$ in the following form

$$\frac{\partial \alpha_k}{\partial t} + \boldsymbol{j} \cdot \nabla \alpha_k + \nabla \cdot \left(\alpha_k \boldsymbol{V}_{kj}\right) = \frac{\rho_m \Gamma_k}{\overline{\overline{\rho_1}}\,\overline{\overline{\rho_2}}}. \tag{7-32}$$

And if the drift velocity can be approximated as a function of $\alpha_k$ only, then

$$\boldsymbol{V}_{kj} \approx \boldsymbol{V}_{kj}\left(\alpha_k\right). \tag{7-33}$$

Substituting Eq.(7-33) into Eq.(7-32) we obtain the *void propagation equation*

$$\frac{\partial \alpha_k}{\partial t} + \boldsymbol{C}_K \cdot \nabla \alpha_k = \frac{\rho_m \Gamma_k}{\overline{\overline{\rho_1}}\,\overline{\overline{\rho_2}}} \tag{7-34}$$

where the kinematic wave velocity $\boldsymbol{C}_K$ is defined by

$$\boldsymbol{C}_K \equiv \boldsymbol{j} + \frac{\partial}{\partial \alpha_k}\left(\alpha_k \boldsymbol{V}_{kj}\right). \tag{7-35}$$

Hence, denoting the special convective derivative following the kinematic wave by

$$\frac{D_c}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{C}_K \cdot \nabla \tag{7-36}$$

the void propagation equation reduces to

$$\frac{D_c}{Dt}(\alpha_k) = \frac{\rho_m \Gamma_k}{\overline{\overline{\rho_1}}\,\overline{\overline{\rho_2}}}. \tag{7-37}$$

Thus if we observe the time rate of change of $\alpha_k$ by moving with the kinematic wave velocity, it is proportional to the source term due to phase changes. In the absence of mass transfer between the phases, the disturbance of $\alpha_k$ propagates with the kinematic wave velocity. Furthermore, under the condition of *constant* $\overline{\overline{\rho_1}}$ *and* $\overline{\overline{\rho_2}}$, we can express Eq.(7-34) in terms of the mixture density as follows

$$\frac{\partial \rho_m}{\partial t} + \boldsymbol{C}_K \cdot \nabla \rho_m = \frac{\rho_m}{\overline{\overline{\rho_1}}\,\overline{\overline{\rho_2}}} \sum_{k=1}^{2} \Gamma_k \overline{\overline{\rho_k}}$$

or (7-38)

$$\frac{\partial(\ln \rho_m)}{\partial t} + \boldsymbol{C}_K \cdot \nabla(\ln \rho_m) = \frac{1}{\overline{\overline{\rho_1}}\,\overline{\overline{\rho_2}}} \sum_{k=1}^{2} \Gamma_k \overline{\overline{\rho_k}}$$

which is called the *density propagation equation.*

*Kinematic Shock Wave*

It has been shown that if the drift velocity is a function only of $\alpha_k$, the void fraction equation can be transformed into the void propagation equation. In contrast, if the diffusion of phases can be expressed by the constitutive equation having the form of Fick's Law of Diffusion, the field exhibits the characteristic of diffusive media and the clear-cut void propagation cannot be observed due to the second-order derivative in space.

The former phenomenon of the void propagation is known for several types of mixtures. For example, they are important in open channel, bubbly two-phase and highway traffic flows. In such systems, it is observed that under certain conditions these kinematic wave propagations lead to a formation of a concentration shock. Because of its origin and a necessity to differentiate it from a shock wave due to compressibility effects, we refer it as a *kinematic shock wave.* As it has been shown by Lighthill and Whitham (1955) and Kynch (1952), this phenomenon can be analyzed by a kinematic consideration with a simple constitutive law for the flux of matter which depends on the concentration. However, without going into a detailed discussion, it is possible to write conditions that should be satisfied at the kinematic shock wave. This can be done by utilizing the macroscopic jump conditions of the Section 1.5 of Chapter 5. Thus we apply Eqs.(5-64) and (5-65) to the balance of mass at the shock front, then

$$\sum_{+,-} \rho_m \boldsymbol{n} \cdot \left(\boldsymbol{v}_m - \boldsymbol{U}\right) = 0 \tag{7-39}$$

and

$$\sum_{+,-} \alpha_k \overline{\overline{\rho_k}} \boldsymbol{n} \cdot \left(\widehat{\boldsymbol{v}_k} - \boldsymbol{U}\right) + \Gamma_{ka} = 0. \tag{7-40}$$

It is evident that Eq.(7-39) expresses the conservation of total mass, whereas Eq.(7-40) gives the balance of $k^{\text{th}}$-phase mass across the shock. Here $\Gamma_{ka}$ denotes the amount of phase changes within the shock layer. Thus Eq.(7-40) states that the $k^{\text{th}}$-phase mass fluxes from each side of the shock wave balance with the mass production due to phase changes in the shock.

By solving Eq.(7-40) for the displacement velocity of the shock, we obtain

$$\boldsymbol{n}^{+} \cdot \boldsymbol{U} = \frac{\left(\alpha_k^{+} \overline{\overline{\rho_k}}^{+} \widehat{\boldsymbol{v}_k}^{+} - \alpha_k^{-} \overline{\overline{\rho_k}}^{-} \widehat{\boldsymbol{v}_k}^{-}\right) \cdot \boldsymbol{n}^{+} + \Gamma_{ka}}{\left(\alpha_k^{+} \overline{\overline{\rho_k}}^{+} - \alpha_k^{-} \overline{\overline{\rho_k}}^{-}\right)} \tag{7-41}$$

where + and - denote each side of the shock layer. It should be remembered that the condition given by Eq.(7-41) is applicable not only to a kinematic shock wave but also to a dynamic shock wave due to compressibility effects.

By limiting our case to a strictly kinematic phenomenon, we assume here that the phase densities are continuous across the shock and there is no change of phase in the layer. Thus we have

$$\overline{\overline{\rho_k}}^{+} = \overline{\overline{\rho_k}}^{-} \tag{7-42}$$

and

$$\Gamma_{ka} = 0. \tag{7-43}$$

Hence Eq.(7-40) reduces to

$$\sum_{+,-} \alpha_k \boldsymbol{n} \cdot \left( \widehat{\boldsymbol{v}_k} - \boldsymbol{U} \right) = 0. \tag{7-44}$$

It follows that the kinematic shock wave velocity $\boldsymbol{U}$ should satisfy

$$\boldsymbol{n}^+ \cdot \boldsymbol{U} = \frac{\boldsymbol{n}^+ \cdot \left[ \alpha_k^+ \widehat{\boldsymbol{v}_k}^+ - \alpha_k^- \widehat{\boldsymbol{v}_k}^- \right]}{\left( \alpha_k^+ - \alpha_k^- \right)}. \tag{7-45}$$

If we use the definition of the volumetric flux $\boldsymbol{j}_k$ defined by Eq.(4-87), we have

$$\boldsymbol{n}^+ \cdot \boldsymbol{U} = \frac{\boldsymbol{n}^+ \cdot \left( \boldsymbol{j}_k^+ - \boldsymbol{j}_k^- \right)}{\left( \alpha_k^+ - \alpha_k^- \right)}. \tag{7-46}$$

From Eq.(7-46) it can be shown

$$\boldsymbol{n}^+ \cdot \boldsymbol{j}^+ + \boldsymbol{n}^- \cdot \boldsymbol{j}^- = 0. \tag{7-47}$$

This means that across the simple kinematic shock, namely, the phase densities being continuous and no phase changes in the shock, the total volumetric flux $\boldsymbol{j}$ is conserved. In view of Eq.(7-47), we can transform Eq.(7-46) into the following form

$$\boldsymbol{n}^+ \cdot \boldsymbol{U} = \boldsymbol{n}^+ \cdot \boldsymbol{j} + \frac{\boldsymbol{n}^+ \cdot \left( \alpha_k^+ \boldsymbol{V}_{kj}^+ - \alpha_k^- \boldsymbol{V}_{kj}^- \right)}{\left( \alpha_k^+ - \alpha_k^- \right)}. \tag{7-48}$$

Here we see the close connection between the kinematic wave velocity given by Eq.(7-35) and the displacement velocity of the shock given by Eq.(7-48).

## 1.4 Dilatation

The Jacobian of the convective and spatial coordinates of the Section 1.1 of Chapter 7 is given by

$$J_k = \frac{\partial (\boldsymbol{x})}{\partial \left( \boldsymbol{X}_k \right)} = \frac{\partial \left( x_1, x_2, x_3 \right)}{\partial \left( X_{1k}, X_{2k}, X_{3k} \right)}. \tag{7-49}$$

Since $\boldsymbol{X}_k$ denotes the initial position, the Jacobian $J_k$ gives the relation between the initial and the present volumes if the surface of the volume element moves with the center of mass velocity $\widehat{\boldsymbol{v}_k}$. Hence we have

$$dV_k = J_k dV_{k0} \tag{7-50}$$

where $V_k$ and $V_{k0}$ denotes the present volume and its initial volume, respectively. In view of Eq.(7-5) we have

$$\frac{1}{J_k}\frac{D_k J_k}{Dt} = \nabla \cdot \widehat{\boldsymbol{v}_k} \tag{7-51}$$

which gives an important physical interpretation of the divergence of the phase velocity. Recalling the continuity equation for the $k^{\text{th}}$-phase, it also can be expressed as

$$\nabla \cdot \widehat{\boldsymbol{v}_k} = \frac{1}{\alpha_k \overline{\overline{\rho_k}}}\left(\Gamma_k - \frac{D_k \alpha_k \overline{\overline{\rho_k}}}{Dt}\right). \tag{7-52}$$

These two equations show that the divergence of $\widehat{\boldsymbol{v}_k}$ is directly related to the dilatation of a volume element rather than the density changes within the volume. Furthermore, we see that the dilatation is cause by three effects, namely, the phase change $\Gamma_k$, the phase redistribution $D_k\alpha_k/Dt$ and the real compressibility of the fluid $D_k\overline{\overline{\rho_k}}/Dt$.

For a mixture as a whole, the characteristic of the dilatation and of $\nabla \cdot \boldsymbol{v}_m$ basically reduces to that of a single-phase flow. Thus we have

$$\nabla \cdot \boldsymbol{v}_m = \frac{1}{J_m}\frac{DJ_m}{Dt} = -\frac{1}{\rho_m}\frac{D\rho_m}{Dt} \tag{7-53}$$

where $J_m$ is the Jacobian between $\boldsymbol{x}$ and $\boldsymbol{X}_m$ given by

$$J_m = \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{X}_m)}. \tag{7-54}$$